MAXIMAL AND TYPICAL TOPOLOGY OF REAL POLYNOMIAL SINGULARITIES

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Abstract. Given a semialgebraic set $W \subseteq J^r(S^m, \mathbb{R}^k)$ and a polynomial map $\psi : S^m \to \mathbb{R}^k$ with components of degree d, we investigate the structure of the semialgebraic set $j^r\psi^{-1}(W) \subseteq S^m$ (we call such a set a "singularity").

Concerning the upper estimate on the topological complexity of a polynomial singularity, we sharpen the classical bound $b(j^r\psi^{-1}(W)) \leq O(d^{m+1})$, proved by Milnor [21], with

$$b(j^r\psi^{-1}(W)) \leq O(d^m), \tag{0.1}$$

which holds for the generic polynomial map.

For what concerns the "lower bound" on the topology of $j^r\psi^{-1}(W)$, we prove a general semicontinuity result for the Betti numbers of the zero set of $\mathcal{C}^0$ perturbations of smooth maps – the case of $\mathcal{C}^1$ perturbations is the content of Thom's Isotopy Lemma (essentially the Implicit Function Theorem). This result is of independent interest and it is stated for general maps (not just polynomial); this result implies that small continuous perturbations of $\mathcal{C}^1$ manifolds have a richer topology than the one of the original manifold.

Keeping (0.1) in mind, we compare the extremal case with a random one and prove that on average the topology of $j^r\psi^{-1}(W)$ behaves as the "square root" of its upper bound: for a random Kostlan map $\psi : S^m \to \mathbb{R}^k$ with components of degree d and $W \subset J^r(S^m, \mathbb{R}^k)$ semialgebraic, we have:

$$\mathbb{E}b(j^r\psi^{-1}(W)) = \Theta(d^{\frac{m}{2}}).$$

This generalizes classical results of Edelman-Kostlan-Shub-Smale from the zero set of a random map, to the structure of its singularities.

1. Introduction

In this paper we deal with the problem of understanding the structure of the singularities of polynomial maps

$$\psi : S^m \to \mathbb{R}^k,$$

where each component of $\psi = (\psi_1, \ldots, \psi_k)$ is the restriction to the sphere of a homogeneous polynomial of degree d. For us "singularity" means the set of points in the sphere where the r-jet extension $j^r\psi : S^n \to J^r(S^n, \mathbb{R}^k)$ meets a given semialgebraic set $W \subseteq J^r(S^n, \mathbb{R}^k)$. Example of these type of singularities are: zero sets of polynomial functions, critical points of a given Morse index of a real valued function or the set of Whitney cusps of a planar map.

Because we are looking at *polynomial* maps, this problem has two different quantitative faces, which we both investigate in this paper.

(1) From one hand we are interested in understanding the *extremal* cases, meaning that, for fixed m, d and k we would like to know how complicated can the singularity be, at least in the generic case.

(2) On the other hand, we can ask what is the *typical* complexity of such a singularity. Here we adopt a measure-theoretic point of view and endow the space of polynomial maps with a natural Gaussian probability measure, for which it makes sense to ask about expected properties of these singularities, such as their Betti numbers.

1.1. **Quantitative bounds, the h-principle and the topology semicontinuity.** Measuring the complexity of $Z = j^r\psi^{-1}(W)$ with the sum $b(Z)$ of its Betti numbers, problem (1) above means producing a-priori upper bounds for $b(Z)$ (as a function of m, d, k) as well as trying to realize given subsets of the sphere as $j^r\psi^{-1}(W)$ for some W and some map ψ.

For the case of the zero set $Z = \psi^{-1}(0)$ of a polynomial function $\psi : S^m \to \mathbb{R}$ of degree d, the first problem is answered by a Milnor's type bound[1] $b(Z) \leq O(d^m)$ and the second problem by Seifert's theorem: every smooth hypersurface in the sphere can be realized (up to ambient diffeomorphisms) as the zero set of a polynomial function.

In the case of more general singularities, both problems are more subtle. The problem of giving a good upper bound on the complexity of $Z = j^r\psi^{-1}(W)$ will require us to develop a quantitative version of stratified Morse Theory for semialgebraic maps (Theorem 8). We use the word "good" because there is a vast literature on the subject of quantitative semialgebraic geometry, and it is not difficult to produce a bound of the form $b(Z) \leq O(d^{m+1})$; instead here (Theorem 13 and Theorem 14) we prove the following result.

Theorem 1. *For the generic polynomial map $\psi : S^m \to \mathbb{R}^k$ with components of degree d, and for $W \subseteq J^r(S^m, \mathbb{R}^k)$ semialgebraic, we have:*

$$b(j^r\psi^{-1}(W)) \leq O(d^m). \tag{1.1}$$

(The implied constant depends on W.)

In the case W is algebraic we do not need the genericity assumption on ψ for proving (1.1), but in the general semialgebraic case some additional complications arise and this assumption allows to avoid them through the use of Theorem 8. We believe, however, that (1.1) is still true even in the general case[2]. Moreover, for our scopes the genericity assumption is not restrictive, as it fits in the probabilistic point of view of the second part of the paper, where a generic property is a property holding with probability one.

[1] Milnor's bound [21] would give $b(Z) \leq O(d^{m+1})$, whereas [19, Proposition 14] gives the improvement $b(Z) \leq O(d^m)$. In the context of this paper the difference between these two bounds is relevant, especially because when switching to the probabilistic setting it will give the so called "generalized square root law".

[2] In the algebraic case in fact one can use directly Thom-Milnor bound, but in the general semialgebraic case it is necessary first to "regularize" the semialgebraic set, keeping control on its Betti numbers. In the algebraic (or even the basic semialgebraic case) this is the procedure of Milnor [21], in the general semialgebraic case it is not clear what this controlled regularization procedure would be. The nondegeneracy assumption on the jet allows us to avoid this step.

For what concerns the realizability problem, as simple as it might seem at first glance, given $W \subseteq J^r(S^m, \mathbb{R}^k)$ it is not even trivial to find a map $f : S^m \to \mathbb{R}^k$ whose jet is transversal to W and such that $b(j^r f^{-1}(W)) > 0$ (we prove this in Corollary 20).

Let us try to explain carefully what is the subtlety here. In order to produce such a map, one can certainly produce a section of the jet bundle $\sigma : S^m \to J^r(S^m, \mathbb{R}^k)$ which is transversal to W and such that $b(\sigma^{-1}(W)) > 0$ (this is easy). However, unless $r = 0$, this section needs not to be holonomic, i.e. there might not exist a function $f : S^m \to \mathbb{R}^k$ such that $\sigma = j^r f$.

We fix this first issue using an h-principle argument: the Holonomic Approximation Theorem [8, p. 22] guarantees that, after a small $\mathcal{C}^0$ perturbation of the whole picture, we can assume that there is a map $f : S^m \to \mathbb{R}^k$ whose jet $j^r f$ is $\mathcal{C}^0$ close to σ.

There is however a second issue that one needs to address. In fact, if the jet perturbation was $\mathcal{C}^1$ small (i.e. if σ and $j^r f$ were $\mathcal{C}^1$ close), Thom's Isotopy Lemma would guarantee that $\sigma^{-1}(W) \sim j^r f^{-1}(W)$ (i.e. the two sets are ambient diffeomorphic), but the perturbation that we get from the Holonomic Approximation Theorem is guaranteed to be only $\mathcal{C}^0$ small! To avoid this problem we prove the following general result on the semicontinuity of the topology of small $\mathcal{C}^0$ perturbations (see Theorem 18 below for a more precise statement).

Theorem 2. *Let S, J be smooth manifolds, $W \subseteq J$ be a closed cooriented submanifold and $\sigma \in \mathcal{C}^1(S, J)$ such that $\sigma \pitchfork W$. Then for every $\gamma \in \mathcal{C}^1(S, J)$ which is sufficiently close to σ in the $\mathcal{C}^0$-topology and such that $\gamma \pitchfork W$, we have:*

$$b(\gamma^{-1}(W)) \geq b(\sigma^{-1}(W)).$$

In particular we see that if small $\mathcal{C}^1$ perturbations of a regular equation preserve the topology of the zero set, still if we take just small $\mathcal{C}^0$ perturbations the topology of such zero set can only increase.

To apply Theorem 2 to our original question we consider $S = S^m$ and $J = J^r(S^m, \mathbb{R}^k)$, $W \subseteq J^r(S^m, \mathbb{R}^m)$ is the semialgebraic set defining the singularity and $\sigma : S^m \to J^r(S^m, \mathbb{R}^k)$ is the (possibly non-holonomic) section such that $\sigma \pitchfork W$ and $b(\sigma^{-1}(W)) > 0$. Moreover we can construct σ in such a way that its image meets only a small (relatively compact and cooriented) subset of the smooth locus of W. Then for every $f \in \mathcal{C}^{r+1}(S^m, \mathbb{R}^k)$ with $\tau = j^r f$ sufficiently close to σ and such that $j^r f \pitchfork W$, we have:

$$b(j^r f^{-1}(W)) \geq b(\sigma^{-1}(W)) > 0. \tag{1.2}$$

(We will use the content of Corollary 20 and the existence of a function f such that (1.2) holds in the second part of the paper for proving the convergence of the expected Betti numbers of a random singularity.)

1.2. The random point of view and the generalized square-root law.

Switching to the random point of view offers a new perspective on these problems: from Theorem 1 we have an extremal bound (1.1) for the complexity of polynomial singularities, but it is natural to ask how far is this bound from the typical situation. Of course, in order to start talking about randomness, we need to choose a probability distribution on the space of (homogeneous) polynomials. It is natural to require that this distribution is gaussian, centered, and that it is invariant under orthogonal changes of variables (in this way there are no preferred points or directions in the sphere). If we further assume that the monomials are independent, this distribution is unique (up to multiples), and called the *Kostlan distribution*.

To be more precise, this probability distribution is the measure on $\mathbb{R}[x_0, \dots, x_m]_{(d)}$ (the space of homogeneous polynomials of degree d) induced by the gaussian random polynomial:

$$P(x) = \sum_{|\alpha|=d} \xi_\alpha \cdot \left(\frac{d!}{\alpha_0! \cdots \alpha_m!}\right)^{1/2} x_0^{\alpha_0} \cdots x_m^{\alpha_m},$$

where $\{\xi_\alpha\}$ is a family of standard independent gaussian variables. A list of k independent Kostlan polynomials $P = (P_1, \dots, P_k)$ defines a random polynomial map:

$$\psi = P|_{S^m} \to \mathbb{R}^k.$$

In particular, it is now natural to view such a ψ as a random variable in the space $\mathcal{C}^\infty(S^m, \mathbb{R}^k)$ and to study the differential topology of this map, such as the behavior of its singularities, described a preimages of jet submanifolds $W \subseteq J^r(S^m, \mathbb{R}^k)$ in the previous section.

In this direction, it has already been observed by several authors, in different contexts, that random real algebraic geometry seems to behave as the "square root" of generic complex geometry. Edelman and Kostlan [7, 17] were the first to observe this phenomenon: a random Kostlan polynomial of degree d in one variable has $\sqrt{d}$ many real zeroes, on average[3]. Shub and Smale [25] generalized this result and proved that the expected number of zeroes of a system of m Kostlan equations of degrees $(d_1, \dots, d_m)$ in m variables is $\sqrt{d_1 \cdots d_m}$ (the bound coming from complex algebraic geometry would be $d_1 \cdots d_m$).

Moving a bit closer to topology, Bürgisser [3] and Podkorytov [23] proved that the expectation of the Euler characteristic of a random Kostlan algebraic set has the same order of the square-root of the Euler characteristic of its complex part (when the dimension is even, otherwise it is zero). A similar result for the Betti numbers has also been proved by Gayet and Welschinger [11–13], and by Fyodorov, Lerario and Lundberg [10] for invariant distributions.

Using the language of the current paper, these results correspond to the case of a polynomial map $\psi : S^m \to \mathbb{R}^k$ and to the "singularity" $Z = j^0\psi^{-1}(W)$, where

$$W = S^m \times \{0\} \subset J^0(S^m, \mathbb{R}^k) = S^m \times \mathbb{R}^k$$

and $j^0\psi(x) = (x, \psi(x))$ is the section given by the map ψ itself. Here we generalize these results and prove that a similar phenomenon is a very general fact of Kostlan polynomial maps.

[3] In the notation of the current paper this correspond to the case of $\psi : S^1 \to \mathbb{R}$ of degree d, whose expected number of zeroes is $2\sqrt{d}$. The multiplicative constant "2" appears when passing from the projective to the spherical picture

Theorem 3. *Let $W \subset J^r(S^m, \mathbb{R}^k)$ be a closed intrinsic*[4] *semialgebraic set of positive codimension. If $\psi : S^m \to \mathbb{R}^k$ is a random Kostlan polynomial map, then*

$$\mathbb{E}b(j^r\psi^{-1}(W)) = \Theta(d^{\frac{m}{2}}).^{5} \tag{1.3}$$

(The implied constants depend on W.)

We call the previous Theorem 3 the "generalized square root law" after comparing it with the extremal inequality $b(j^r\psi^{-1}(W)) \leq O(d^m)$ from Theorem 1, whose proof is ultimately based on bounds coming from complex algebraic geometry[6]. In the case W has codimension m (i.e. when we expect $j^r\psi^{-1}(W)$ to consist of points), we actually sharpen (1.3) and get the explicit asymptotic to the leading order, see Theorem 27 below. Moreover, a similar result holds for every fixed Betti number $b_i(j^r\psi^{-1}(W))$ when i is in the range $0 \leq i \leq m - \mathrm{codim}(W)$, see Theorem 29 for a detailed statement.

Remark 4. The ingredients for the proof of Theorem 3 are: Theorem 8 for the upper bound and Corollary 20 for the lower bound. The main property that we use in this context is the fact that a Kostlan map $\psi : S^m \to \mathbb{R}^k$ has a rescaling limit when restricted to a small disk $D_d = D(x, d^{-1/2})$ around any point $x \in S^m$. In other words, one can fix a diffeomorphism $a_d : \mathbb{D}^m \to D_d$ of the standard disk $\mathbb{D}^m$ with the small spherical disk $D(x, d^{-1/2}) \subset S^m$ and see that the sequence of random functions:

$$X_d = \psi \circ a_d : \mathbb{D}^m \to \mathbb{R}^k$$

converges to the Bargmann-Fock field, see Theorem 23. In a recent paper [20] we introduced a general framework for dealing with random variables in the space of smooth functions and their differential topology – again we can think of $X_d \in C^\infty(\mathbb{D}^m, \mathbb{R}^k)$ as a sequence of random variables of this type. The results from [20], applied to the setting of random Kostlan polynomial maps are collected in Theorem 23 below, which lists the main properties of the rescaled Kostlan polynomial X_d. Some of these properties are well-known to experts working on random fields, but some of them seem to have been missed. Moreover, we believe that our language is more flexible and well-suited to the setting of differential topology, whereas classical references look at these random variables from the point of view of functional analysis and stochastic calculus.

Of special interest from Theorem 23 are properties (2), (5) and (7), which are closely related. In fact (2) and (5) combined together tells that open sets $U \subset \mathcal{C}^\infty(\mathbb{D}^m, \mathbb{R}^k)$ which are defined by open conditions on the r-jet of X_d, have a positive limit probability when $d \to \infty$. Property (7), tells that the law for Betti numbers of a random singularity $Z_d = j^r X_d^{-1}(W)$ has a limit. (Even in the case of zero sets this property was not noticed before, see Example 30.)

We consider Theorem 23 as a practical tool that people interested in random algebraic geometry can directly use, and we will show how to concretely use this tool in a list of examples that we give in Appendix 3.3.

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[4]We say that $W \subset J^r(S^m, \mathbb{R}^k)$ is intrinsic if it is invariant under diffeomorphisms of S^m, see Definition 11. This property it is satisfied in all natural examples.

[5]We write $f(d) = \Theta(g(d))$ if there exist constants $a_1, a_2 > 0$ such that $a_1 g(d) \leq f(d) \leq a_2 f(d)$ for all $d \geq d_0$ sufficiently large.

[6]The reader can now appreciate the estimate $O(d^m)$ instead of $O(d^{m+1})$ from Theorem 1.

Remark 5. The current paper, and in particular the generalized square-root law Theorem 3, complement recent work of Diatta and Lerario [6] and Breiding, Keneshlou and Lerario [2], where tail estimates on the probabilities of the maximal configurations are proved.

1.3. **Structure of the paper.** In Section 2.1 we prove a quantitative semialgebraic version of stratified Morse Theory, which is a technical tool needed in the sequel, and in Section 2.2 we prove Theorem 13 and Theorem 14 (whose combination give Theorem 1). In section 2.3 we discuss the semicontinuity of topology under holonomic approximation and prove Theorem 18 (which is Theorem 2 from the Introduction). In Section 3 we introduce the random point of view and prove the generalized square-root law. Appendix 1 contains three short examples of use the random techniques.

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2. Quantitative bounds, the h-principle and the topology semicontinuity

2.1. **Stratified Morse Theory.** Let us fix a Whitney stratification $W = \sqcup_{S\in\mathscr{S}} S$ (see [14, p. 37] for the definition) of the semialgebraic subset $W \subset J^r(S^m, \mathbb{R}^k) =: J$, with each stratum $S \in \mathscr{S}$ being semialgebraic and smooth (such decomposition exists [14, p. 43]), so that, by definition a smooth map $f\colon M \to J$, is transverse to W if $f \pitchfork S$ for all strata $S \in \mathscr{S}$. When this is the case, we write $\psi \pitchfork W$ and implicitly consider the subset $\psi^{-1}(W) \subset M$ to be equipped with the Whitney stratification given by $\psi^{-1}\mathscr{S} = \{\psi^{-1}(S)\}_{S\in\mathscr{S}}$.

Definition 6. Given a Whitney stratified subset $Z = \cup_{i\in I} S_i$ of a smoooth manifold M (without boundary), we say that a function $g\colon Z \to \mathbb{R}$ is a Morse function if g is the restriction of a smooth function $\tilde{g}\colon M \to \mathbb{R}$ such that

(a) $g|_{S_i}$ is a Morse function on S_i.
(b) For every critical point $p \in S_i$ and every generalized tangent space $Q \subset T_pM$ (defined as in [14, p. 44]) we have $d_p\tilde{g}(Q) \neq 0$, except for the case $Q = T_pS_i$.

Note that the condition of being a Morse function on a stratified space $Z \subset M$ depends on the given stratification of Z.

Remark 7. The definition above is slightly different than the one given in the book [14, p. 52] by Goresky and MacPherson, where a Morse function, in addition, must be proper and have distinct critical values.

The following theorem is the quantitative version of stratified Morse theory for semialgebraic maps we need in order to prove Theorem 1.

Theorem 8. *Let $W \subset J$ be a semialgebraic subset of a real algebraic smooth manifold J, with a given semialgebraic Whitney stratification $W = \sqcup_{S\in\mathscr{S}} S$ and let M be a real algebraic smooth manifold. There exists a semialgebraic subset $\hat{W} \subset J^1(M, J \times \mathbb{R})$ having codimension larger or equal than* $\dim M$*, equipped with a semialgebraic Whitney* *stratification that satisfies the following properties with respect to any couple of smooth maps* $\psi\colon M\to J$ *and* $g\colon M\to\mathbb{R}$.

(1) *If* $\psi \pitchfork W$ *and* $j^1(\psi,g)\pitchfork \hat{W}$, *then* $g|_{\psi^{-1}(W)}$ *is a Morse function with respect to the stratification* $\psi^{-1}\mathscr{S}$ *and*

$$Crit(g|_{\psi^{-1}(W)}) = \left(j^1(\psi,g)\right)^{-1}(\hat{W}). \tag{2.1}$$

(2) *There is a constant* $N_W>0$ *depending only on* W *and* $\mathscr{S}$, *such that if* $\psi^{-1}(W)$ *is compact,* $\psi\pitchfork W$ *and* $j^1(\psi,g)\pitchfork\hat{W}$, *then*

$$b_i(\psi^{-1}(W)) \le N_W \# Crit(g|_{\psi^{-1}(W)}),$$

for all $i=0,1,2\dots$

Proof. Let $S\in\mathscr{S}$ be a stratum of W, hence $S\subset J$ is a smooth submanifold and since $\psi\pitchfork W$ implies that $\psi\pitchfork S$, we also have that $\psi^{-1}(S)$ is a submanifold of M of the same codimension which we denote by k. Define

$$\begin{aligned}\hat{S} &= \{j^1_p(F,f)\in J^1(M,J\times\mathbb{R})\colon F(p)\in S \text{ and } d_pf\in d_pF^*(T_{F(p)}S^\perp)\}\\ &= \{j^1_p(F,f)\in J^1(M,J\times\mathbb{R})\colon F(p)\in S \text{ and } \exists\lambda\in T_{F(p)}S^\perp \text{ s.t. } d_pf=\lambda\circ d_pF\}.\end{aligned}$$

Orthogonality here is meant in the sense of dual vector spaces: if $Q\subset T$ are vector spaces, then $Q^\perp=\{\xi\in T^*\colon \xi(Q)=0\}$.

It is clear, by this definition, that $\hat{S}$ is semialgebraic and its codimension is equal to the dimension of M.

Claim 9. $j^1_{p_0}(\psi,g)\in\hat{S}$ *if and only if* p_0 *is a critical point for* $g|_{\psi^{-1}(S)}$.

If $j^1_{p_0}(\psi,g)\in\hat{S}$, then of course $p_0\in\psi^{-1}(S)$ and there exists a (Lagrange multiplier) conormal covector $\lambda\in T_{\psi(p_0)}S^\perp$ such that $d_{p_0}g=\lambda\circ d_{p_0}\psi$. It follows that $d_{p_0}g$ vanishes on $T_{p_0}\psi^{-1}(S)=d_{p_0}\psi^{-1}(T_{\psi(p_0)}S)$. This proves the "only if" statement of the Claim as a consequence of the following inclusion

$$d_{p_0}\psi^*\left(T_{p_0}S^\perp\right)\subset\left(T_{p_0}\psi^{-1}(S)\right)^\perp.$$

To conclude the proof of Claim 9 we need to show the opposite inclusion. We do this by showing that the dimensions of the two spaces are equal. First observe that, since by hypotheses $\psi\pitchfork S$, the image $d_{p_0}\psi$ is a complement to $T_{\psi(p_0)}S$ in $T_{\psi(p_0)}J$ and this is equivalent (it is the dual statement) to say that the restriction of $d_{p_0}\psi^*$ to $(T_{\psi(p_0)}S)^\perp$ is injective. It follows that

$$\begin{aligned}\dim d_{p_0}\psi^*\left(T_{p_0}S^\perp\right) &= \dim\left(T_{p_0}S^\perp\right)=\\ &=\text{codim } S=\\ &=\text{codim } \psi^{-1}(S)=\\ &=\dim\left(T_{p_0}\psi^{-1}(S)\right)^\perp.\end{aligned}$$

This concludes the proof of Claim 9.

Claim 10. *Given a Whitney stratification of $\hat{S}$, and a critical point $p_0 \in M$ of the map $g|_{\psi^{-1}(S)}$, if $j^1(\psi, g) \pitchfork \hat{S}$ at p_0 then this critical point is Morse.*

Let us pass to a coordinate chart ϕ defined on a nighborhood $\mathcal{U} \subset J^1(M, J \times \mathbb{R})$ of $j^1_{p_0}(\psi, g)$:

$$\phi = \left(x = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}, y = \begin{pmatrix} y^1 \\ y^2 \end{pmatrix}, a, Y = \begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix}, A\right) : \mathcal{U} \to \mathbb{R}^m \times \mathbb{R}^{s+k} \times \mathbb{R} \times \mathbb{R}^{(s+k)\times m} \times \mathbb{R}^m$$

$$j^1_p(F, f) \mapsto \left(x(p), y(F(p)), g(p), \frac{\partial (y \circ F)}{\partial x}, \frac{\partial g}{\partial x}\right);$$

where $y^2 = 0$ is a local equation for S and $x^2 = 0$ is a local equation for $\psi^{-1}(S)$. Indeed, by the implicit function theorem (applied to the map ψ in virtue of the transversality assumption $\psi \pitchfork S$) we can assume that $y^2(\psi(x^1, x^2)) = x^2$. In this coordinate chart we have that the restriction of d_pF^* to the space $T_{\psi(p)}S^\perp$ is represented by the matrix $(Y^2)^T$, thus

$$\hat{S} \cap \mathcal{U} = \left\{y^2 = 0; A \in \operatorname{Im}\left((Y^2)^T\right)\right\} \cap \phi(\mathcal{U}).$$

Let us denote by $x \mapsto (x, \tilde{y}(x), \tilde{a}(x), \tilde{Y}(x), \tilde{A}(x))$ the local expression of the jet map $p \mapsto j^1_p(\psi, g)$ with respect to the above coordinates. By construction we have that

$$\left(\tilde{Y}^2(p_0)\right)^T = \begin{pmatrix} 0 \\ \mathbb{1}_k \end{pmatrix}.$$

In particular the image of the above matrix is a complement to the subspace spanned by the first $m-k$ coordinates and we may assume, reducing the size of the neighborhood if needed, that this property holds for every element $(x, y, a, Y, A) \in \phi(\mathcal{U})$, so that there exist unique vectors $\lambda \in \mathbb{R}^k$ and $\xi \in \mathbb{R}^{(m-k)}$ such that

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \left(Y^2\right)^T \lambda + \begin{pmatrix} \xi \\ 0 \end{pmatrix}. \tag{2.2}$$

Now, this defines a smooth function $\xi \colon \mathcal{U} \to \mathbb{R}^k$ such that the equations $y^2 = 0$; $\xi = 0$ are smooth regular equations for $\phi(\hat{S} \cap \mathcal{U})$.

Notice that this ensures that $\phi(\mathcal{U})$ intersects only the smooth locus of $\hat{S}$. Now, since by hypotheses $j^1(\psi, g)$ is transverse to all the strata of $\hat{S}$ then it must be transverse to the smooth locus in the usual sense, even if the latter is a union of strata (this follows directly from the definition of transversality). Therefore, while proving Claim 10, we are allowed to forget about the stratification of $\hat{S}$ and just assume that the map $j^1(\psi, g)$ is transverse to the smooth manifold $\hat{S} \cap \phi(\mathcal{U})$ in the usual sense.

In this setting we can see that if $j^1(\psi, g) \pitchfork \hat{S}$ at p_0, then the following matrix has to be surjective:

$$\begin{pmatrix} dy^2 \\ d\xi \end{pmatrix} \circ d_{p_0}\left(j^1(\psi, g)\right) = \begin{pmatrix} 0 & \mathbb{1}_k \\ \frac{\partial \tilde{\xi}}{\partial x^1}(p_0) & \frac{\partial \tilde{\xi}}{\partial x^2}(p_0) \end{pmatrix} \in \mathbb{R}^{(k+k)\times((m-k)+k)},$$

where $\tilde{\xi}(x) = \xi(x, \tilde{y}(x), \tilde{a}(x), \tilde{Y}(x), \tilde{A}(x))$. Therefore the lower left block $\frac{\partial \tilde{\xi}}{\partial x^1}(p_0)$ is surjective as well and hence invertible. This concludes our proof of Claim 10 since such matrix is in fact the hessian of the map $g|_{\psi^{-1}(S)}$ at the critical point p_0:

$$d_{p_0}\left(g|_{\psi^{-1}(S)}\right) = \frac{\partial}{\partial x^1}\Big|_{p_0}\left(\frac{\partial g}{\partial x^1}\right) = \frac{\partial \tilde{A}_1}{\partial x^1}(p_0) = \frac{\partial \tilde{\xi}}{\partial x^1}(p_0).$$

The last equality is due to the equation (2.2) combined with the observation that $\tilde{Y^2}$ is of the form $\begin{pmatrix} 0 & * \end{pmatrix}$ for all p in a neighborhood of p_0, since $\frac{\partial \tilde{y}^2}{\partial x_1}(p) = 0$.

At this point, Claim 9 and Claim 10 prove that, for whatever stratification of $\hat{S}$, if $j^1(\psi, g) \pitchfork \hat{S}$ and $\psi \pitchfork S$ then $g|_{\psi^{-1}(S)}$ is a Morse function and that its critical set coincide with the set $\left(j^1(\psi, g)\right)^{-1}(\hat{S})$, so that condition (a) of Definition 6 is satisifed along the stratum S. In order to establish when $g|_{\psi^{-1}(W)}$ is a Morse function along the stratum $\psi^{-1}(S)$ on the stratified manifold W, in the sense of Definition 6, we now need to prove a similar statement to ensure condition (b).

Let us consider the set D_qS of degenerate covectors at a point $q \in S$ that are conormal to S (conormal and degenerate covectors are defined as in [14, p.44]), in other words:

$$D_qS = \{\xi \in T_q^*J \colon \xi \in T_qS^\perp,\ \xi \in Q^\perp \text{ for some } Q \text{ generalized tangent space at } q\}.$$

It is proved in [14, p.44] that $DS = \cup_{q \in S} D_qS$ is a semialgebraic subset of codimension greater than 1 of the conormal bundle $TS^\perp$[7] to the stratum S. We claim that the subset $D\hat{S} \subset \hat{S}$ containing the jets that do not satisfy condition (b) of Definition 6 has the following description:

$$D\hat{S} = \{j^1_p(F, f) \in J^1(M, J \times \mathbb{R}) \colon F(p) \in S \text{ and } d_pf \in d_pF^*(D_{F(p)}S)\}.$$

In fact, since $\psi \pitchfork W$, then all the generalized tangent spaces of the stratified subset $\psi^{-1}(W) \subset M$ at a point $p \in \psi^{-1}(S)$ are of the form $d_p\psi^{-1}(Q)$. It follows that if a conormal covector $d_pg = \lambda \circ d_p\psi$ is degenerate then $\lambda \in D_{\psi(p)}S$.

Note that $D\hat{S}$ is a subset of $\hat{S}$ of codimension ≥ 1, thus the codimension of $D\hat{S}$ in $J^1(M, J \times \mathbb{R})$ is $\geq m + 1$. As a consequence we have that $j^1(\psi, g) \pitchfork D\hat{S}$ if and only if $j^1(\psi, g) \notin D\hat{S}$. Therefore if $j^1(\psi, g) \pitchfork \hat{S}$ and $j^1(\psi, g) \notin D\hat{S}$ then $\psi \pitchfork S$ and $g|_{\psi^{-1}(W)}$ is a Morse function on $\psi^{-1}(W)$ along the stratum $\psi^{-1}(S)$.

We are now ready to define $\hat{W} = \cup_{S \in \mathscr{S}} \hat{S}$. An immediate consequence of Claim 9 is that $\hat{W}$ satisfies equation (2.1). Moreover, since $\hat{S} \supset D\hat{S}$ are semialgebraic, $\hat{W}$ is semialgebraic and admits a semialgebraic Whitney stratification $\hat{\mathscr{S}}$ (refining the one of $\hat{S}$) such that all the subsets $\hat{S}$ and $D\hat{S}$ are unions of strata. With such a stratification, if the jet map $j^1(\psi, g)$ is transverse to $\hat{W}$ then, for each stratum $S \in \mathscr{S}$, it is also transverse to $\hat{S}$ and it avoids the set $D\hat{S}$, so that $g|_{\psi^{-1}(W)}$ is a Morse function, in the sense of Definition 6. This proves that $\hat{W}$ satisfies condition (1) of the Theorem.

Let us prove condition (2). Let $Z = \psi^{-1}(W) \subset M$ be compact. Without loss of generality we can assume that each of the critical values $c_1, \ldots, c_n$ of $g|_Z$ corresponds to

[7] $TS^\perp = T_S^*J$, in the notation of [14].

only one critical point (this can be obtained by makingcontaining the jets that do not satisfy condition (b) of Definition 6: a $\mathcal{C}^1$ small perturbation of g, which won't affect the number of its critical points). Consider a sequence of real numbers $a_1, \dots a_{n+1}$ such that

$$a_1 < c_1 < a_2 < c_2 < \cdots < a_n < c_n < a_{n+1}.$$

By the main Theorem of stratified Morse theory [14, p. 8, 65], there is an homeomorphism

$$Z \cap \{g \le a_{l+1}\} \cong (Z \cap \{g \le a_l\}) \sqcup_B A,$$

with

$$(A, B) = TMD_p(g) \times NMD_p(g),$$

where $TMD_p(g)$ is the tangential Morse data and $NMD_p(g)$ is the normal Morse data. A fundamental result of classical Morse theory is that the tangential Morse data is homeomorphic to a pair

$$TMD_p(g) \cong (\mathbb{D}^\lambda \times \mathbb{D}^{m-\lambda}, (\partial \mathbb{D}^\lambda) \times \mathbb{D}^{m-\lambda}),$$

while the normal Morse data is defined as the local Morse data of $g|_{N_p}$ for a normal slice (see [14, p. 65]) at p. A consequence of the transversality hypothesis $\psi \pitchfork W$ is that there is a small enough normal slice N_p such that $\psi|_{N_p} \colon N_p \to J$ is the embedding of a normal slice at $\psi(p)$ for W. Therefore the normal data $NMD_p(g)$ belongs to the set $\nu(W)$ of all possible normal Morse data that can be realized (up to homeomeorphisms) by a critical point of a Morse function on W. By Corollary 7.5.3 of [14, p. 95] it follows that the cardinality of the set $\nu(W)$ is smaller or equal than the number of connected components of the semialgebraic set $\cup_{S \in \mathscr{S}} (TS^\perp \backslash DS)$, hence finite[8]. Let

$$N_W := \max_{Y \in \nu(W),\ \lambda \in \{0,\dots,m\}} b_i \left(\left(\mathbb{D}^\lambda \times \mathbb{D}^{m-\lambda}, (\partial \mathbb{D}^\lambda) \times \mathbb{D}^{m-\lambda} \right) \times Y \right) \in \mathbb{N}.$$

From the long exact sequence of the pair $(Z \cap \{g \ge a_{l+1}\}, (Z \cap \{g \ge a_l\})$ we deduce that

$$\begin{aligned} b_i(Z \cap \{g \le a_{l+1}\}) - b_i(Z \cap \{g \le a_l\}) &\le b_i \left(Z \cap \{g \le a_{l+1}\}, Z \cap \{g \le a_l\} \right) \\ &= b_i \left(A, B \right) \\ &= b_i \left(TMD_p(g) \times NMD_p(g) \right) \\ &\le N_W. \end{aligned} \tag{2.3}$$

Since Z is compact, the set $Z \cap \{g \le a_1\}$ is empty, hence by repeating the inequality (2.3) for each critical value, we finally get

$$b_i(Z) = b_i(Z \cap g \le a_{n+1}) \le N_W n = N_W \# \mathrm{Crit} \left(g|_{\psi^{-1}(W)} \right).$$

This concludes the proof of Theorem 8. □

Below we will restrict to those semialgebraic sets $W \subset J^r(S^m, \mathbb{R}^k)$ that have a differential geometric meaning, as specified in the next definition.

[8] In the book this is proved only for any fixed point p, as a corollary of Theorem 7.5.1 [14, p.93]. However the same argument generalizes easily to the whole bundle.

Definition 11. A submanifold $W \subset J^r(M, \mathbb{R}^k)$ is said to be *intrinsic* if there is a submanifold $W_0 \subset J^r(\mathbb{D}^m, \mathbb{R}^k)$, called the *model*, such that for any embedding $\varphi\colon \mathbb{D}^m \hookrightarrow M$, one has that $j^r\varphi^*(W) = W_0$, where

$$j^r\varphi^*\colon J^r\left(\varphi(\mathbb{D}^m), \mathbb{R}^k\right) \xrightarrow{\cong} J^r\left(\mathbb{D}^m, \mathbb{R}^k\right), \qquad j^r_{\varphi(p)}f \mapsto j^r_p(f\circ\varphi).$$

Intrinsic submanifolds are, in other words, those that have the same shape in every coordinate charts, as in the following examples.

(1) $W = \{j^r_p f\colon f(p) = 0\}$;
(2) $W = \{j^r_p f\colon j^s f(p) = 0\}$ for some $s \le r$;
(3) $W = \{j^r_p f\colon \operatorname{rank}(df(p)) = s\}$ for some $s \in \mathbb{N}$.

Remark 12. In the case when $J = J^r(M, \mathbb{R}^k)$ we can consider $\hat{W}$ to be a subset of $J^{r+1}(M, \mathbb{R}^{k+1})$ taking the preimage via the natural submersion

$$J^{r+1}(M, \mathbb{R}^{k+1}) \to J^1\left(M, J^r(M, \mathbb{R}^k)\times\mathbb{R}\right), \qquad j^{r+1}(f, g) \mapsto j^1(j^r f, g).$$

In this setting Theorem 8 can be translated to a more natural statement by considering ψ of the form $\psi = j^r f$. Moreover, in this case, observe that if W is intrinsic (in the sense of Definition 11 below), then $\hat{W}$ is intrinsic as well.

2.2. **Quantitative bounds.** In this section we prove Theorem 1, which actually immediately follows by combining Theorem 13 and Theorem 14.

Next theorem gives a deterministic bound for on the complexity of $Z = j^r\psi^{-1}(W)$ when the codimension of W is m.

Theorem 13. *Let $P \in \mathbb{R}[x_0, \ldots, x_m]^k_{(d)}$ be a polynomial map and consider its restriction $\psi = P|_{S^m}$ to the unit sphere:*

$$\psi : S^m \to \mathbb{R}^k.$$

Let also $j^r\psi : S^m \to J^r(S^m, \mathbb{R}^k)$ be the associated jet map and $W \subset J^r(S^m, R^k)$ be a semialgebraic set of codimension m. There exists a constant $c > 0$ (which only depends on W, m and k) such that, if $j^r\psi \pitchfork W$, then:

$$\#j^r\psi^{-1}(W) \le c\cdot d^m. \tag{2.4}$$

Proof. Let us make the identification $J^r(\mathbb{R}^{m+1}, \mathbb{R}^k) \simeq \mathbb{R}^{m+1}\times\mathbb{R}^N$, so that the restricted jet bundle $J^r(\mathbb{R}^{m+1}, \mathbb{R}^k)|_{S^m}$ corresponds to the semialgebraic subset $S^m \times \mathbb{R}^N$. Observe that the inclusion $S^m \hookrightarrow \mathbb{R}^{m+1}$ induces a semialgebraic map:

$$J^r(\mathbb{R}^{m+1}, \mathbb{R}^k)|_{S^m} \xrightarrow{i^*} J^r(S^m, \mathbb{R}^k),$$

that, roughly speaking, forgets the normal derivatives. Notice that while the map $j^r\psi = j^r(P|_{S^m})$ is a section of $J^r(S^m, \mathbb{R}^k)$, $(j^r P)|_{S^m}$ is a section of $J^r(\mathbb{R}^{m+1}, \mathbb{R}^k)|_{S^m}$. These sections are related by the identity

$$i^*\circ(j^r P)|_{S^m} = j^r\psi.$$

Thus, defining $\overline{W} = (i^*)^{-1}(W)$, we have

$$j^r\psi^{-1}(W) = ((j_rP)|_{S^m})^{-1}(\overline{W}).$$

Since $\overline{W}$ is a semialgebraic subset of $\mathbb{R}^{m+1} \times \mathbb{R}^N$, it can be written as:

$$\overline{W} = \bigcup_{j=1}^{\ell} \left\{ f_{j,1} = 0, \dots, f_{j,\alpha_j} = 0, g_{j,1} > 0, \dots, g_{j,\beta_j} > 0 \right\},$$

where the $f_{j,i}$s and the $g_{j,i}$s are polynomials of degree bounded by a constant $b > 0$. For every $j = 1, \dots, \ell$ we can write:

$$\left\{ f_{j,1} = 0, \dots, f_{j,\alpha_j} = 0, g_{j,1} > 0, \dots, g_{j,\beta_j} > 0 \right\} = Z_j \cap A_j,$$

where Z_j is algebraic (given by the equations) and A_j is open (given by the inequalities).

Observe also that the map $(j^rP)|_{S^m}$ is the restriction to the sphere S^m of a polynomial map

$$Q : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1} \times \mathbb{R}^N$$

whose components have degree smaller than d. Therefore for every $j = 1 \dots, \ell$ the set $((j^rP)|_{S^m})^{-1}(Z_j) = (Q|_{S^m})^{-1}(Z_j)$ is an algebraic set on the sphere defined by equations of degree less than $b \cdot d$ and, by [19, Proposition 14] we have that:

$$b_0(Q|_{S^m})^{-1}(Z_j)) \leq Bd^m \tag{2.5}$$

for some constant $B > 0$ depending on b and m. The set $(Q|_{S^m})^{-1}(Z_j)$ consists of several components, some of which are zero dimensional (points):

$$(Q|_{S^m})^{-1}(Z_j) = \underbrace{\{p_{j,1}, \dots, p_{j,\nu_j}\}}_{P_j} \cup \underbrace{X_{j,1} \cup \cdots \cup X_{j,\mu_j}}_{Y_j}.$$

The inequality (2.5) says in particular that:

$$\#P_j \leq Bd^n. \tag{2.6}$$

Observe now that if $j^r\psi \pitchfork W$ then, because the codimension of W is m, the set $j^r\psi^{-1}(W) = (Q|_{S^m})^{-1}(\overline{W})$ consists of finitely many points and therefore, since $(Q|_{S^m})^{-1}(A_j)$ is open, we must have:

$$j^r\psi^{-1}(W) \subset \bigcup_{j=1}^{\ell} P_j.$$

(Otherwise $j^r\psi^{-1}(W)$ would contain an open, nonempty set of a component of codimension smaller than m.) Inequality (2.6) implies now that:

$$\#j^r\psi^{-1}(W) \leq \sum_{j=1}^{\ell} \#P_j \leq \ell bd^m \leq cd^m.$$

□

Using Theorem 8 it is now possible to improve Theorem 13 to the case of any codimension, replacing the cardinality with any Betti number.

Theorem 14. *Let $P \in \mathbb{R}[x_0, \ldots, x_m]^k_{(d)}$ be a polynomial map and consider its restriction $\psi = P|_{S^m}$ to the unit sphere:*

$$\psi : S^m \to \mathbb{R}^k.$$

Let also $j^r\psi : S^m \to J^r(S^m, \mathbb{R}^k)$ be the associated jet map and $W \subset J^r(S^m, R^k)$ be a closed semialgebraic set (of arbitrary codimension). There exists a constant $c > 0$ (which only depends on W, m and k) such that, if $j^r\psi \pitchfork W$, then:

$$b_i\left(j^r\psi^{-1}(W)\right) \leq c \cdot d^m.$$

Proof. Let $J = J^r(S^m, \mathbb{R}^k)$ and let $\hat{W}$ be the (stratified according to a chosen stratification of W) subset of $J^{r+1}(S^m, \mathbb{R}^{k+1})$ coming from Theorem 8 and Remark 12. Let g be a homogeneous polynomial of degree d such that

$$\Psi = (\psi, g) \in \mathbb{R}[x_0, \ldots, x_m]^{k+1}_{(d)}$$

satisfies the condition $j^{r+1}\Psi \pitchfork \hat{W}$ (almost every polynomial g has this property by standard arguments) and $(j^r\psi)^{-1}(W)$ is closed in S^m, hence compact. Then by Theorem 8, there is a constant N_W, such that

$$b_i\left(j^r\psi^{-1}(W)\right) \leq N_W \#\{(j^{r+1}\Psi)^{-1}(\hat{W})\}$$

and by Theorem 13, the right hand side is bounded by cd^m. □

Given $P = (P_1, \ldots, P_k)$ with each P_i a homogeneous polynomial of degree d in $m+1$ variables, we denote by

$$\psi_d : S^m \to \mathbb{R}^k$$

its restriction to the unit sphere (the subscript keeps track of the dependence on d).

Example 15 (Real algebraic sets)**.** Let us take $W = S^m \times \{0\} \subset J^0(S^m, \mathbb{R}^k)$, then $j^0\psi^{-1}(W)$ is the zero set of $\psi_d : S^m \to \mathbb{R}^k$, i.e. the set of solutions of a system of polynomial equations of degree d. In this case the inequality (2.4) follows from [19].

Example 16 (Critical points)**.** If we pick $W = \{j^1 f = 0\} \subset J^1(S^m, \mathbb{R})$, then $Z_d = j^1\psi_d^{-1}(W)$ is the set of critical points of $\psi_d : S^m \to \mathbb{R}$. In 2013 Cartwright and Sturmfels [4] proved that

$$\#Z_d \leq 2(d-1)^m + \cdots + (d-1) + 1$$

(this bounds follows from complex algebraic geometry), and this estimate was recently proved to be sharp by Kozhasov [18]. Of course one can also fix the index of a nondegenerate critical point (in the sense of Morse Theory); for example we can take $W = \{df = 0, d^2 f > 0\} \subset J^2(S^m, \mathbb{R})$, and $j^2\psi_d^{-1}(W)$ is the set of nondegenerate *minima* of $\psi_d : S^m \to \mathbb{R}$ (similar estimates of the order d^m holds for the fixed Morse index, but the problem of finding a sharp bound is very much open).

Example 17 (Whitney cusps)**.** When $W = \{\text{Whitney cusps}\} \subset J^3(S^2, \mathbb{R}^2)$, then $\psi_d^3 f^{-1}(W)$ consists of the set of points where the polynomial map $\psi_d : S^2 \to \mathbb{R}^2$ has a critical point which is a Whitney cusp. In this case (2.4) controls the number of possible Whitney cusps (the bound is of the order $O(d^2)$).

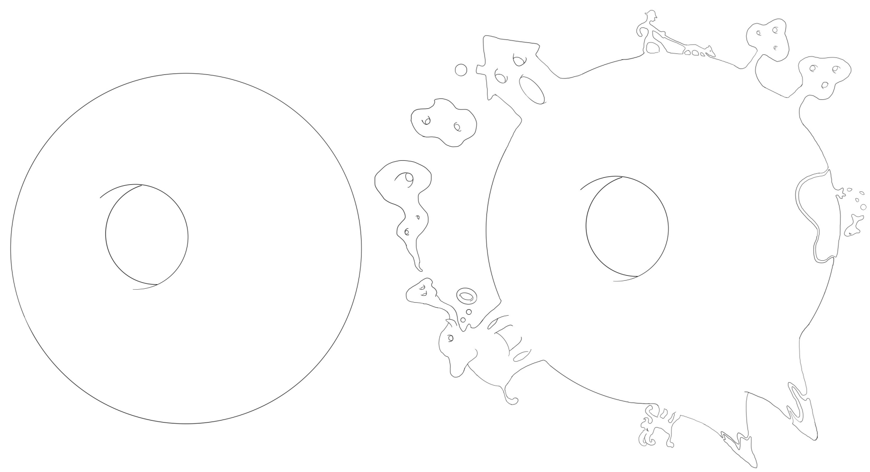

FIGURE 1. A small $\mathcal{C}^0$ perturbation of a regular equation can only increase the topology of its zero set.

2.3. **Semicontinuity of topology under holonomic approximation.** Consider the following setting: M and J are smooth manifolds, M is compact, and $W \subset J$ is a smooth cooriented submanifold. Given a smooth map $F\colon M \to J$ which is transversal to W, it follows from standard transversality arguments that there exists a small $\mathcal{C}^1$ neighborhood U_1 of F such that for every map $\tilde{F} \in U_1$ the pairs $(M, F^{-1}(W))$ and $(M, \tilde{F}^{-1}(W))$ are isotopic (in particular $F^{-1}(W)$ and $\tilde{F}^{-1}(W)$ have the same Betti numbers, this is the so-called "Thom's isotopy Lemma"). The question that we address is the behavior of the Betti numbers of $\tilde{F}^{-1}(W)$ under small $\mathcal{C}^0$ perturbations, i.e. how the Betti number can change under modifications of the map F *without* controlling its derivative. In this direction we prove the following result.

Theorem 18. *Let M, J be smooth manifolds and let $W \subset J$ be a smooth cooriented closed submanifold. Let $F\colon M \to J$ be a smooth map such that $F \pitchfork W$. If a smooth map $\tilde{F}$ is strongly*[9] *$\mathcal{C}^0$-close to F such that $\tilde{F} \pitchfork W$, then for all $i \in \mathbb{N}$ there is a group K^i such that*

$$H^i\left(\tilde{F}^{-1}(W)\right) \cong H^i\left(F^{-1}(W)\right) \oplus K^i. \tag{2.7}$$

Proof. Call $A = F^{-1}(W)$ and $\tilde{A} = \tilde{F}^{-1}(W)$. Let $E \subset M$ be a closed tubular neighborhood (it exists because A is closed), meaning that $E = \operatorname{int}(E) \cup \partial E$ is diffeomorphic

[9]Meaning: in Whitney strong topology. In particular if $C \subset M$ is closed and $U \subset J$ is open, then the set $\{f \in \mathcal{C}^0(M, J)\colon f(C) \subset U\}$ is open, see [15].

to the unit ball of a metric vector bundle over A (via a diffeomorphism that preserves A). Denote by $\pi\colon E \to A$ the retraction map. Since $\tilde{F}$ is $\mathcal{C}^0-$close to F we can assume that there is a homotopy F_t connecting $F = F_0$ and $\tilde{F} = F_1$ such that $F_t(\partial E) \subset J\backslash W$. Define analogously $\tilde{\pi} : \tilde{E} \to \tilde{A}$ in such a way that $\tilde{E} \subset \mathrm{int}(E)$. It follows that there is an inclusion of pairs $u : (E, \partial E) \to (E, E\backslash\tilde{E})$. By construction, the function F_t induces a well defined mapping of pairs $F_t\colon (E, \partial E) \to (J, J\backslash W)$ for every $t \in [0, 1]$, in particular there is a homotopy between F_0 and F_1 (meant as maps of pairs). Moreover with $t = 1$, this map is the composition of u and the map $F_1\colon (E, E\backslash\tilde{E}) \to (J, J\backslash W)$.

The fact that W is closed and cooriented guarantees the existence of a Thom class $\phi \in H^r(J, J\backslash W)$, where r is the codimension of W. By transversality we have that also A and $\tilde{A}$ are cooriented with Thom classes $F_0^*\phi = \phi_E \in H^r(E, \partial E) \cong H^r(E, E\backslash A)$ and $F_1^*\phi = \phi_{\tilde{E}} \in H^r(\tilde{E}, \partial\tilde{E}) \cong H^r(\tilde{E}, \tilde{E}\backslash\tilde{A})$. We now claim the commutativity of the diagram below.

$$\begin{array}{ccccc}
 & & H^{*+r}(J, J\backslash W) & & \\
 & \swarrow^{F_1^*} & & \searrow^{F_1^*=F_0^*} & \\
H^{*+r}(\tilde{E}, \partial\tilde{E}) & \xrightarrow{\eta^{-1}} & H^{*+r}(E, E\backslash\tilde{E}) & \xrightarrow{u^*} & H^{*+r}(E, \partial E) \\
\uparrow{\scriptstyle \tilde{\pi}^*(\cdot)\cup\phi_{\tilde{E}}} & & & & \uparrow{\scriptstyle \pi^*(\cdot)\cup\phi_E} \\
H^*(\tilde{A}) & & \xleftarrow[\pi^*]{\hspace{6em}} & & H^*(A)
\end{array}$$

(where η is the excision isomorphism). For what regards the upper triangular diagram, the commutativity simply follows from the fact that all the maps F_t are homotopic and that the excision homomorphism is the inverse of that induced by the inclusion $(E, E\backslash\tilde{E}) \subset (\tilde{E}, \partial\tilde{E})$. To show that the lower rectangle commutes, observe that since $\tilde{\pi}$ is homotopic to the identity of $\tilde{E}$ we have that $\pi \circ \tilde{\pi}$ is homotopic to $\pi|_{\tilde{E}}$. Thus the commutativity follows from the property of the cup product, saying that for all $\varphi \in H^*(A)$ we have

$$\begin{aligned} u^* \circ \eta^{-1} \circ \left(\tilde{\pi}^* \left(\pi|_{\tilde{A}}\right)^* \varphi\right) \cup \phi_{\tilde{E}} &= \left(u^* \circ \eta^{-1} \circ \left(\pi|_{\tilde{E}}\right)^* \varphi\right) \cup \left(u^* \circ \eta^{-1} \circ F_1^*\phi\right) \\ &= \pi^*\varphi \cup \phi_E, \end{aligned}$$

where in the last equality we used the identity $u^* \circ \eta^{-1} \circ F_1^* = F_0^*$ implied by the commutativity of the upper triangle. Since the vertical maps are (Thom) isomorphisms, there exists a homomorphism $U\colon H^*(\tilde{A}) \to H^*(A)$ such that $U \circ \pi^* =$id. □

Remark 19. The above proof also provides a way to determine how small should the perturbation be. In fact we showed that if $F_t\colon M \to J$ is a homotopy such that $F_1 \pitchfork W$ and $F_t(\partial E) \subset J\backslash W$ for all $t \in [0, 1]$, where E is a closed tubular neighborhood of $F^{-1}(W)$, then the map $\tilde{F} = F_1$ satisfies (2.7). Notice that to have such property it is enough that $\tilde{F} \pitchfork W$ and $\tilde{F}|_{\partial E}$ is $\mathcal{C}^0-$close to $F|_{\partial E}$. This implies that the size of the $\mathcal{C}^0$ neighborhood of F in which the identity (2.7) holds depends only on the restriction of F to a codimension 1 submanifold.

Corollary 20. *Let M be a compact manifold of dimension m. Let $W \subset J^r(M, \mathbb{R}^k)$ be a Whitney stratified submanifold of codimension $1 \le l \le m$ being transverse to the fibers of the canonical projection $\pi \colon J^r(M, \mathbb{R}^k) \to M$. Then for any number $n \in \mathbb{N}$ there exists a smooth function $\psi \in \mathcal{C}^\infty(M, \mathbb{R}^k)$ such that $j^r\psi \pitchfork W$ and*

$$b_i\left((j^r\psi)^{-1}(W)\right) \ge n, \quad \forall i = 0, \dots, m-l.$$

Proof. Let $B \subset J^r(M, \mathbb{R}^k)$ be a small neighbourhood of a regular point $j^r_p f$ of W so that $(B, B \cap W) \cong (\mathbb{R}^{N+l}, \mathbb{R}^N \times \{0\})$. Moreover we can assume that there is a neighbourhood $U \cong \mathbb{R}^m$ of $p \in M$ and a commutative diagram of smooth maps

$$\begin{array}{ccccc} & \mathbb{R}^m \times \mathbb{R}^k \times \{0\} & \hookrightarrow & \mathbb{R}^m \times \mathbb{R}^k \times \mathbb{R}^l & \\ \nearrow^{\cong} & & \nearrow^{\cong} & \downarrow & \\ B \cap W & \hookrightarrow & B & & \mathbb{R}^m \\ & & \downarrow^{\pi} & \nearrow^{\cong} & \\ & & U & & \end{array} \tag{2.8}$$

This follows from the fact that $\pi|_W$ is a submersion, because of the transversality assumption. For any $0 \le i \le m-l$ consider the smooth map

$$\varphi_i \colon \mathbb{R}^m \to \mathbb{R}^l, \quad u \mapsto \left(\sum_{\ell=1}^{i+1}(u_\ell)^2 - 1, \sum_{\ell=i+2}^{m}(u_\ell)^2 - 1, u_{m-l+3}, \dots, u_m\right)$$

Clearly 0 is a regular value for φ_i, with preimage[10] $\varphi_i^{-1}(0) \cong S^i \times S^{m-l-i}$ and it is contained in the unit ball of radius 2. Let $C \subset \mathbb{R}^m$ be a set of $n(m-l+1)$ points such that $|c - c'| \ge 5$ for all pair of distinct elements $c, c' \in C$. Now choose a partition $C = C_0 \sqcup C_1 \sqcup \dots C_{m-l}$ in sets of cardinality n and define a smooth map $\varphi \colon \mathbb{R}^m \to \mathbb{R}^l$ such that $\varphi(x) = \varphi_i(x - c)$ whenever $\operatorname{dist}(x, C_i) \le 2$. We may also assume that 0 is a regular value for φ. Notice that $\varphi^{-1}(0)$ has a connected component

$$S \cong \{1, \dots, n\} \times \left(S^0 \times S^{m-l} \sqcup S^1 \times S^{m-l-1} \sqcup \dots S^{m-l} \times S^0\right).$$

Construct a smooth (non necessarily holonomic) section $F \colon U \to J^r(U, \mathbb{R}^k)$ such that $F \pitchfork W$ and such that $F = (u, 0, \varphi)$ on a neighbourhood of S, so that $F^{-1}(W)$ still contains S as a connected component, hence $b_i(F^{-1}(W)) \ge n$ for all $i = 0, \dots, m-l$.

Let $E \subset U$ be a closed tubular neighborhood of $F^{-1}(W)$. To conclude we use the holonomic approximation theorem [8, p. 22], applied to $F \colon U \to J^r(U, \mathbb{R}^k) \cong U \times \mathbb{R}^{k+l}$ near the codimension 1 submanifold $\partial E \subset U$. Such theorem ensures that for any $\varepsilon > 0$ there exists a diffeomorphism $h \colon U \to U$, an open neighborhood $O_{\partial E} \subset U$ of ∂E and a smooth function $\psi \colon U \to \mathbb{R}^k$ such that

$$\operatorname{dist}_{\mathcal{C}^0}\left((j^r f)|_{h(O_{\partial E})}, F|_{h(O_{\partial E})}\right) < \varepsilon, \quad \text{and} \quad \operatorname{dist}_{\mathcal{C}^0}(h, \mathrm{id}) < \varepsilon.$$

[10] Except for the case $l = 1$. Here one should adjust the definition of φ_i in order to have $b_i(\varphi_i^{-1}(0)) > 0$.

Moreover, we can assume that $j^r\psi \pitchfork W$, by Thom transversality Theorem (see [15] or [8]). In particular, it follows that

$$\mathrm{dist}_{\mathcal{C}^0}\left((j^r f)\circ h|_{\partial E}, F|_{\partial E}\right) < (1 + C(F))\cdot\varepsilon,$$

where $C(F)$ is the lipshitz constant of $F|_U$, which can be assumed to be finite (if not, replace $U \cong \mathbb{R}^m$ with an open ball that still contains $F^{-1}(W)$). Consider the smooth manifold $J = J^r(U, \mathbb{R}^k)$. By the diagram (2.8) it follows that $W \subset J$ is a closed and cooriented smooth submanifold, so that by Theorem 18 and Remark 19 we know that if $\varepsilon > 0$ is small enough, then the map $\tilde{F} = (j^r f)\circ h$ satisfies the identity (2.7). Therefore for each $i = 0, \dots, m-l$, we have

$$\begin{aligned} b_i\left((j^r f)^{-1}(W)\right) &= b_i\left(((j^r f)\circ h)^{-1}(W)\right) \\ &\geq b_i\left((F\circ h)^{-1}(W)\right) \\ &= b_i\left(F^{-1}(W)\right) \\ &\geq n. \end{aligned}$$

□

3. Random Algebraic Geometry

3.1. **Kostlan maps.** In this section we give the definition of a random Kostlan polynomial map $P : \mathbb{R}^{m+1} \to \mathbb{R}^k$, which is a Gaussian Random Field (GRF) that generalizes the notion of Kostlan polynomial.

Definition 21 (Kostlan polynomial maps)**.** Let $d, m, k \in \mathbb{N}$. We define the degree d homogeneous Kostlan random map as the measure on $\mathbb{R}[x]^k_{(d)} = \mathbb{R}[x_0, \dots, x_m]^k_{(d)}$ induced by the gaussian random polynomial:

$$P_d^{m,k}(x) = \sum_{\alpha\in\mathbb{N}^{m+1},\ |\alpha|=d} \xi_\alpha x^\alpha,$$

where $x^\alpha = x_0^{\alpha_0}\dots x_m^{\alpha_m}$ and $\{\xi_\alpha\}$ is a family of independent gaussian random vectors in $\mathbb{R}^k$ with covariance matrix

$$K_{\xi_\alpha} = \binom{d}{\alpha}\mathbb{1}_k = \left(\frac{d!}{\alpha_0!\dots\alpha_m!}\right)\mathbb{1}_k.$$

We will call $P_d^{m,k}$ the *Kostlan polynomial* of type (d, m, k) (we will simply write $P_d = P_d^{m,k}$ when the dimensions are understood).

(In other words, a Kostlan polynomial map $P_d^{m,k}$ is given by a list of k independent Kostlan polynomials of degree d in $m+1$ homogeneous variables.)

There is a non-homogeneous version of the Kostlan polynomial, which we denote as

$$p_d(u) = P_d(1, u) = \sum_{\beta\in\mathbb{N}^m,\ |\beta|\leq d} \xi_\beta u^\beta \in \mathcal{G}^\infty(\mathbb{R}^m, \mathbb{R}^k), \tag{3.1}$$

where $u=(u_1,\dots,u_m)\in\mathbb{R}^m$ and $\xi_\beta\sim N\left(0,\binom{d}{\beta}\mathbb{1}_k\right)$ are independent. Here we use the notation of [20], where $\mathcal{G}^\infty(\mathbb{R}^m,\mathbb{R}^k)$ denotes the space of gaussian random field on $\mathbb{R}^m$ with values in $\mathbb{R}^k$ which are $\mathcal{C}^\infty$. Next Proposition collects some well known facts on the Kostlan measure.

Proposition 22. *Let P_d be the Kostlan polynomial of type (d,m,k) and p_d be its dehomogenized version, as defined in (3.1).*

(1) For every $x,y\in\mathbb{R}^{m+1}$:

$$K_{P_d}(x,y)=\left(x^Ty\right)^d\mathbb{1}_k.$$

Moreover, given $R\in O(m+1)$ and $S\in O(k)$ and defined the polynomial $\tilde{P}_d(x)=SP_d(Rx)$, then P_d and $\tilde{P}_d$ are equivalent[11].

(2) For every $u,v\in\mathbb{R}^n$

$$K_{p_d}(u,v)=(1+u^Tv)^d\mathbb{1}_k.$$

Moreover, if $R\in O(m)$ and $S\in O(k)$ and defined the polynomial $\tilde{p}_d(x)=Sp_d(Rx)$, then p_d and $\tilde{p}_d$ are equivalent.

Proof. The proof of this proposition simply follows by computing explicitly the covariance functions and observing that they are invariant under orthogonal change of coordinates in the target and the source. For example, in the case of P_d we have:

$$\begin{aligned}K_{P_d}(x,y)&=\mathbb{E}\{P_d(x)P_d(y)^T\}=\\&=\sum_{|\alpha|,|\alpha'|=d}\mathbb{E}\left\{\xi_\alpha\xi_{\alpha'}^T\right\}x^\alpha y^{\alpha'}=\\&=\sum_{|\alpha|=d}\binom{d}{\alpha}(x_0y_0)^{\alpha_0}\dots(x_my_m)^{\alpha_m}\mathbb{1}_k=\\&=(x_0y_0+\dots+x_my_m)^d\mathbb{1}_k,\end{aligned}$$

from which the orthogonal invariance is clear. The case of p_d follows from the identity:

$$K_{p_d}(u,v)=K_{P_d}\left((1,u),(1,v)\right).$$

. ∎

3.2. **Properties of the rescaled Kostlan.** The main feature here is the fact that the local model of a Kostlan polynomial has a rescaling limit. The orthogonal invariance is used to prove that the limit does not depend on the point where we center the local model, hence it is enough to work around the point $(1,0,\dots,0)\in S^m$. These considerations lead to introduce the Gaussian Random Field $X_d:\mathbb{R}^m\to\mathbb{R}^k$ (we call it the *rescaled Kostlan*) defined by:

$$X_d(u)=P_d^{m,k}\left(1,\frac{u_1}{\sqrt{d}},\dots,\frac{u_m}{\sqrt{d}}\right). \tag{3.2}$$

[11]Two random fields are said to be *equivalent* if they induce the same probability measure on $\mathcal{C}^\infty(\mathbb{R}^m,\mathbb{R}^k)$.

Next result gives a description of the properties of the rescaled Kostlan polynomial, in particular its convergence in law as a random element of the space of smooth functions, space which, from now, on we will always assume to be endowed with the weak Whitney's topology as in [20].

Theorem 23 (Properties of the rescaled Kostlan)**.** *Let $X_d : \mathbb{R}^m \to \mathbb{R}^k$ be the Gaussian random field defined in (3.2).*

1. (The limit) *Given a family of independent gaussian random vectors $\xi_\beta \sim N\left(0, \frac{1}{\beta!}\mathbb{1}_k\right)$, the series*
$$X_\infty(u) = \sum_{\beta \in \mathbb{N}^m} \xi_\beta u^\beta,$$
is almost surely convergent in $\mathcal{C}^\infty(\mathbb{R}^m, \mathbb{R}^k)$ to the Gaussian Random Field[12] *$X_\infty \in \mathcal{G}^\infty(\mathbb{R}^m, \mathbb{R}^k)$.*
2. (Convergence) *$X_d \Rightarrow X_\infty$ in $\mathcal{G}^\infty(\mathbb{R}^m, \mathbb{R}^k)$, that is:*
$$\lim_{d \to +\infty} \mathbb{E}\{F(X_d)\} = \mathbb{E}\{F(X_\infty)\}$$
for any bounded and continuous function $F\colon \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{R}^k) \to \mathbb{R}$. Equivalently, we have
$$\mathbb{P}\{X_\infty \in \operatorname{int}(A)\} \le \liminf_{d \to +\infty} \mathbb{P}\{X_d \in A\} \le \limsup_{d \to +\infty} \mathbb{P}\{X_d \in A\} \le \mathbb{P}\{X_\infty \in \overline{A}\} \tag{3.3}$$
for any Borel subset $A \subset \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{R}^k)$.
3. (Nondegeneracy of the limit) *The support of X_∞ is the whole $\mathcal{C}^\infty(\mathbb{R}^m, \mathbb{R}^k)$. In other words, for any non empty open set $U \subset \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{R}^k)$ we have that $\mathbb{P}\{X_\infty \in U\} > 0$.*
4. (Probabilistic Transversality) *For $d \ge r$ and $d = \infty$, we have $\operatorname{supp}(j^r_p X_d) = J^r_p(\mathbb{R}^m, \mathbb{R}^k)$ for every $p \in \mathbb{R}^m$ and consequently for every submanifold $W \subset J^r(\mathbb{R}^m, \mathbb{R}^k)$, we have*
$$\mathbb{P}\{j^r X_d \pitchfork W\} = 1.$$
5. (Existence of limit probability) *Let $V \subset J^r(\mathbb{R}^m, \mathbb{R}^k)$ be an open set whose boundary is a (possibly stratified) submanifold*[13]*. Then*
$$\lim_{d \to +\infty} \mathbb{P}\{j^r_p X_d \in V,\ \forall p \in \mathbb{R}^m\} = \mathbb{P}\{j^r_p X_\infty(\mathbb{R}^m) \in V,\ \forall p \in \mathbb{R}^m\}.$$
In other words, we have equality in (3.3) for sets of the form $U = \{f\colon j^r f \in V\}$.
6. (Kac-Rice densities) *Let $W \subset J^r(\mathbb{R}^m, \mathbb{R}^k)$ be a semialgebraic subset of codimension m, such that*[14] *$W \pitchfork J^r_p(\mathbb{R}^m, \mathbb{R}^k)$ for all $p \in M$ (i.e. W is transverse to fibers of the projection of the jet space). Then for all $d \ge r$ and for $d = +\infty$ there exists a locally bounded function $\rho^W_d \in L^\infty_{loc}(\mathbb{R}^m)$ such that*[15]
$$\mathbb{E}\#\{u \in A\colon j^r_u X_d \in W\} = \int_A \rho^W_d,$$

[12] X_∞ is indeed a random analytic function, commonly known as the Bargmann-Fock ensemble.
[13] For example V could be a semialgebraic set
[14] In this paper the symbol $\pitchfork$ stands for "it is transverse to".
[15] A formula for ρ^W_d is presented in [20], as a generalization of the classical Kac-Rice formula.

for any Borel subset $A \subset \mathbb{R}^m$. Moreover $\rho_d^W \to \rho_\infty^W$ in L^∞_{loc}.

7. (Limit of Betti numbers) *Let $W \subset J^r(\mathbb{R}^m, \mathbb{R}^k)$ be any closed semialgebraic subset transverse to fibers. Then:*

$$\lim_{d\to+\infty} \mathbb{E}\left\{b_i\left((j^r X_d)^{-1}(W) \cap \mathbb{D}^m\right)\right\} = \mathbb{E}\left\{b_i\left((j^r X_\infty)^{-1}(W) \cap \mathbb{D}^m\right)\right\}, \tag{3.4}$$

where $b_i(Z) = \dim H_i(Z, \mathbb{R})$. Moreover, if the codimension of W is $l \geq 1$, then the r.h.s. in equation (3.4) is strictly positive for all $i = 0, \dots, m-l$.

Proof. The proof uses a combination of results from [20].

(1) Let $S_d = \sum_{|\beta|\leq d} \xi_\beta u^\beta \in \mathcal{G}^\infty(M, \mathbb{R}^k)$. The covariance function of S_d converges in Whitney's weak topology:
$$K_{S_d}(u,v) = \sum_{|\beta|\leq d} \frac{u^\beta v^\beta}{\beta!} \mathbb{1}_k \xrightarrow{\mathcal{C}^\infty} \exp(u^T v)\mathbb{1}_k.$$
It follows by [20, Theorem 3] that S_d converges in $\mathcal{G}^\infty(M, \mathbb{R}^k)$, moreover since all the terms in the series are independent we can conclude with the Ito-Nisio [16] Theorem [16] that indeed the convergence holds almost surely.

(2) By [20, Theorem 3] it follows from convergence of the covariance functions:
$$K_{X_d}(u,v) = \left(1 + \frac{u^T v}{d}\right)^d \mathbb{1}_k \quad \xrightarrow{\mathcal{C}^\infty} \quad K_{X_\infty}(u,v) = \exp(u^T v)\mathbb{1}_k$$

(3) The support of X_∞ contains the set of polynomial functions $\mathbb{R}[u]^k$, which is dense in $\mathcal{C}^\infty(\mathbb{R}^m, \mathbb{R}^k)$, hence the thesis follows from [20, Theorem 6].

(4) Let $d \geq r$ or $d = +\infty$. We have that
$$\begin{aligned}\operatorname{supp}(j^r_u X_d) &= \{j^r_u f \colon f \in \mathbb{R}[u]^k \text{ of degree} \leq d\} = \\ &= \operatorname{span}\{j^r_u f \colon f(v) = (v-u)^\beta \text{ with } |\beta| \leq d\} = \\ &= \operatorname{span}\{j^r_u f \colon f(v) = (v-u)^\beta \text{ with } |\beta| \leq r\} = \\ &= J^r_u(\mathbb{R}^m, \mathbb{R}^k).\end{aligned}$$
The fact that $\mathbb{P}\{j^r X_d \pitchfork W\} = 1$ follows [20, Theorem 8].

(5) Let $A = \{f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{R}^k) \colon j^r f \in V\}$. If $f \in \partial A$, then $j^r f \in \overline{V}$ and there is a point $u \in \mathbb{R}^m$ such that $j^r_u f \in \partial V$. Let ∂V be stratified as $\partial V = \coprod Z_i$ with each Z_i a submanifold. If $j^r f \pitchfork \partial V$ then it means that $j^r f$ is transversal to all the Z_i and there exists one of them which contains $j^r_u f$ (i.e. the jet of f intersect ∂V). Therefore the intersection would be transversal *and nonempty*, and then there exists a small Whitney-neighborhood of f such that for every g in this neighborhood $j^r g$ still intersects ∂V. This means that there is a neighborhood of f consisting of maps that are not in A, which means f has a neighborhood contained in A^c. It follows

[16] It may not be trivial to apply the standard Ito-Nisio theorem, which actually regards convergence of series in a Banach space. See Theorem 36 of [20] for a statement that is directly applicabile to our situation

that $f \notin \overline{A}$ and consequently $f \notin \partial A$, which is a contradiction. Therefore we have that

$$\partial A \subset \{f \in \mathcal{C}^\infty(R^m, \mathbb{R}^k) \colon f \text{ is not transverse to } \partial V\}.$$

It follows by point (4) that $\mathbb{P}\{X \in \partial A\} = 0$, so that we can conclude by points (2) and (3).

(6) By previous points, we deduce that we can apply the results described in section 7 of [20].
(7) This proof is postponed to Section 3.3.

□

Given a $\mathcal{C}^\infty$ Gaussian Random Field $X : \mathbb{R}^m \to \mathbb{R}^k$, let us denote by $[X]$ the probability measure induced on $\mathcal{C}^\infty(\mathbb{R}^m, \mathbb{R}^k)$ and defined by:

$$[X](U) = \mathbb{P}(X \in U),$$

for every U belonging to the Borel $\sigma-$algebra relative to the weak Whitney topology, see [15] for details on this topology. Combining Theorem 23 with Skorohod Theorem [1, Theorem 6.7] one gets that it is possible to represent $[X_d]$ with equivalent fields $\tilde{X}_d$ such that $\tilde{X}_d \to \tilde{X}_\infty$ almost surely in $\mathcal{C}^\infty(\mathbb{R}^m, \mathbb{R}^k)$. This is in fact equivalent to point (2) of Theorem 23. In other words there is a (not unique) choice of the gaussian coefficients of the random polynomials in (3.1), for which the covariances $\mathbb{E}\{\tilde{X}_d \tilde{X}_{d'}^T\}$ are such that the sequence converges almost surely. We leave to the reader to check that a possible choice is the following. Let $\{\gamma_\beta\}_{\beta \in \mathbb{N}^m}$ be a family of i.i.d. gaussian random vectors $\sim N(0, \mathbb{1}_k)$ and define for all $d < \infty$

$$\tilde{X}_d = \sum_{|\beta| \le d} \binom{d}{\beta}^{\frac{1}{2}} \gamma_\beta \left(\frac{u}{\sqrt{d}}\right)^\beta$$

and

$$\tilde{X}_\infty = \sum_\beta \left(\frac{1}{\beta!}\right)^{\frac{1}{2}} \gamma_\beta u^\beta.$$

Proposition 24. *$\tilde{X}_d \to \tilde{X}_\infty$ in $\mathcal{C}^\infty(\mathbb{R}^m, \mathbb{R}^k)$ almost surely.*

However, we stress the fact that in most situations: when one is interested in the sequence of probability measures $[X_d]$, it is sufficient to know that such a sequence exists.

3.3. **Limit laws for Betti numbers and the generalized square-root law.** Let $W_0 \subset J^r(\mathbb{R}^m, \mathbb{R}^k)$ be a semialgebraic subset. Consider the random set

$$S_d = \{p \in \mathbb{D}^m \colon j_p^r X_d \in W_0\},$$

where $X_d \colon \mathbb{R}^m \to \mathbb{R}^k$ is the rescaled Kostlan polynomial from Theorem 23 (see Figure 2). We are now in the position of complete the proof of Theorem 23 by showing point (7). Let us start by proving the following Lemma.

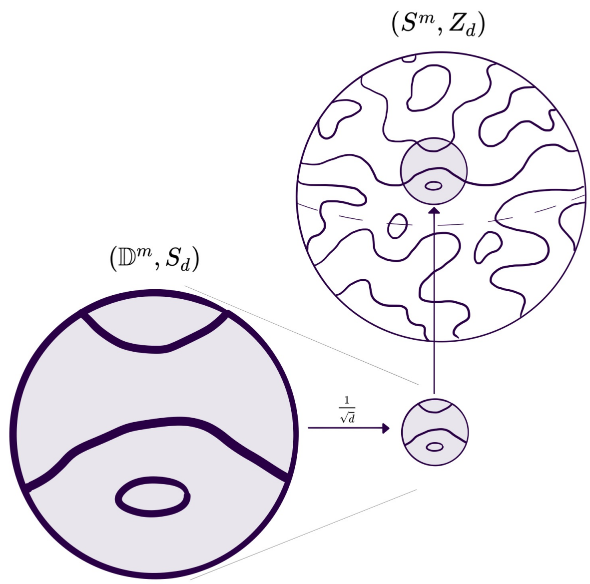

FIGURE 2. The random set $S_d = \{X_d = 0\} \subset \mathbb{D}^m$ is a rescaled version of $Z_d \cap D(p, d^{-1/2})$, where $Z_d = \{\psi_d = 0\}$.

Lemma 25. *Let r be the codimension of W_0 and suppose $0 \leq i \leq m - r \leq m - 1$. Then*

$$\mathbb{E}\{b_i(S_\infty)\} > 0.$$

Proof. From Corollary 20 we deduce that there exists a function $f \in \mathcal{C}^\infty(\mathbb{D}^m, \mathbb{R}^k)$ such that $j^r f \pitchfork W_0$ and $b_i\left((j^r f)^{-1}(W_0)\right) \neq 0$. Since the condition on f is open, there is an open neighbourhood O of f where $b_i((j^r g)^{-1}(W_0)) = c > 0$ for all $g \in O$. Thus $\mathbb{P}\{b_i(S_\infty) = c\} > 0$ because every open set has positive probability for X_∞, by 23.3. therefore $\mathbb{E}\{b_i(S_\infty)\} > 0$. □

We complete the proof of Theorem 23 with the next Proposition.

Proposition 26.

$$\lim_{d \to \infty} \mathbb{E}\{b_i(S_d)\} = \mathbb{E}\{b_i(S_\infty)\}.$$

Proof. Let $b_i(S_d) = b_d$. Define a random field $Y_d = (X_d, x_d) \colon \mathbb{R}^m \to \mathbb{R}^k \times \mathbb{R}$ to be the rescaled Kostlan polynomial of type $(m, k+1)$. Consider the semialgebraic subset $W' = W \cap J^r(\mathbb{D}^m, \mathbb{R}^k)$ of the real algebraic smooth manifold $J^r(\mathbb{R}^m, \mathbb{R}^k)$ and observe that $S_d = (j^r X_d)^{-1}(W')$ is compact. Now Theorem 8, along with Remark 12, implies the existence of a semialgebraic submanifold $\hat{W}' \subset J^{r+1}(\mathbb{R}^m, \mathbb{R}^{k+1})$ of codimension m and a constant C, such that

$$b_d \leq C \# \left\{ \left(j^{r+1}(Y_d)\right)^{-1} (\hat{W}') \right\} =: N_d$$

whenever $j^r X_d \pitchfork W'$ and $j^{r+1}Y_d \pitchfork \hat{W}'$, hence almost surely, because of Theorem 23.4. Since $Y_d \Rightarrow Y_\infty$ by 23.2, we see that $[b_d, N_d] \Rightarrow [b_\infty, N_\infty]$ and it is not restrictive to assume that $(b_i, N_d) \to (b_i, N_\infty)$ almost surely, by Skorokhod's theorem (see [1, Theorem 6.7]). Moreover $\mathbb{E}\{N_d\} \to \mathbb{E}\{N_\infty\}$ by Theorem 23.6. Now we can conclude with Fatou's Lemma as follows

$$\begin{aligned} 2\mathbb{E}\{N_\infty\} &= \mathbb{E}\{\liminf_d N_d + N_\infty - |b_d - b_\infty|\} \leq \\ &\leq \liminf_d \mathbb{E}\{N_d + N_\infty - |b_d - b_\infty|\} = \\ &= 2\mathbb{E}\{N_\infty\} - \limsup_d \mathbb{E}\{|b_d - b_\infty|\}, \end{aligned}$$

so that

$$\limsup_d \mathbb{E}\{|b_d - b_\infty|\} \leq 0.$$

□

In the sequel, with the scope of keeping a light notation, for a given $W \subset J^r(S^m, \mathbb{R}^k)$ and $\psi : S^m \to \mathbb{R}^k$ we will denote by $Z_d \subseteq S^m$ the set

$$Z_d = j^r\psi^{-1}(W).$$

If W is of codimension m, then by Theorem 23, Z_d is almost surely a finite set of points and the expectation of this number is given by next result.

Theorem 27 (Generalized square-root law for cardinality)**.** *Let $W \subset J^r(S^m, \mathbb{R}^k)$ be a semialgebraic intrinsic subset of codimension m. Then there is a constant $C_W > 0$ such that:*

$$\mathbb{E}\{\#Z_d\} = C_W d^{\frac{m}{2}} + O(d^{\frac{m}{2}-1}).$$

Moreover, the value of C_W can be computed as follows. Let $Y_\infty = e^{-\frac{|u|^2}{2}} X_\infty \in \mathcal{G}^\infty(\mathbb{D}^m, \mathbb{R}^k)$ and let $W_0 \subset J^r(\mathbb{D}^m, \mathbb{R}^k)$ be the local model for W. Then

$$C_W = m\frac{\mathrm{vol}(S^m)}{\mathrm{vol}(S^{m-1})}\mathbb{E}\#\{u \in \mathbb{D}^m \colon j^r_u Y_\infty \in W_0\}.$$

In order to prove Theorem 27, we will need a preliminary Lemma, which ensures that we will be in the position of using the generalized Kac-Rice formula of point (6) from Theorem 23.

Lemma 28. *If $W \subset J^r(M, \mathbb{R}^k)$ is intrinsic, then W is transverse to fibers.*

Proof. Since the result is local it is sufficient to prove it in the case when $M = \mathbb{R}^m$. In this case we have a natural identification (see [15, Chapter 2, Section 4])

For any point $u \in \mathbb{R}^m$ we consider the embedding $i_u \colon \mathbb{D}^m \to \mathbb{R}^m$ obtained as the isometric inclusion in the disk with center u and let $\tau_u \colon \mathbb{R}^m \to \mathbb{R}^m$ be the translation map $x \mapsto u + x$. Let $u, v \in \mathbb{R}^m$ be two points with distance smaller than 1, he fact that the submanifold W is intrinsic implies that $j^r_v f \in W$ if and only if $(j^r i_u)^*(j^r_v f) \in W_0$,

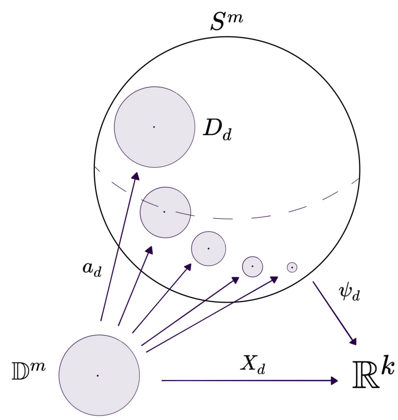

FIGURE 3. A family of shrinking embedding of the unit disk.

where $W_0 \subset J^r(\mathbb{D}^m, \mathbb{R}^k)$ is the model for W. From this we deduce that also the jet $j^r_u(f \circ \tau_{v-u})$ is in W, since:

$$\begin{aligned}(j^r i_v)^* (j^r_v f) &= j^r(\tau_{v-u} \circ i_u)^*(j^r_v f) \\ &= (j^r i_u)^* \left(j^r (\tau_{v-u})^* \left(j^r_{\tau_{v-u}(u)} f\right)\right) \\ &= (j^r i_u)^* \left(j^r_u (f \circ \tau_{v-u})\right).\end{aligned}$$

By interchanging the role of u and v, we conclude that $j^r_u(f \circ \tau_{v-u}) \in W$ if and only if $j^r_v f \in W$. Notice that such statement is thus true for any couple of points $u, v \in \mathbb{R}^m$, regardless of their distance.

We thus claim that $T(W)$ is of the form $\mathbb{R}^m \times \bar{W}$, under the natural identification (see [15, Sec. 2.4]):

$$T\colon J^r(\mathbb{R}^m, \mathbb{R}^k) \cong \mathbb{R}^m \times J^r_0(\mathbb{R}^m, \mathbb{R}^k), \qquad j^r_u f \mapsto (u, j^r_0(f \circ \tau_u)).$$

To see this, observe that if $(v, j^r_0 g) \in T(W)$, hence $(v, j^r_0 g) = T(j^r_v f)$ for a jet $j^r_v f \in W$ such that $g = f \circ \tau_v$, then $(u, j^r_0 g) = T\left(j^r_u(f \circ \tau_{v-u})\right) \in T(W)$. □

The reason why we consider intrinsic submanifold is to be able to easily pass to the rescaled Kostlan polynomial $X_d \in \mathcal{G}^\infty(\mathbb{D}^m, \mathbb{R}^k)$ by composing ψ_d with the embedding of the disk a^R_d defined by:

$$a^R_d \colon \mathbb{D}^m \hookrightarrow S^m, \quad u \mapsto \frac{R\begin{pmatrix} 1 \\ \frac{u}{\sqrt{d}} \end{pmatrix}}{\sqrt{\left(1 + \frac{|u|^2}{d}\right)}} \tag{3.5}$$

for any $R \in O(m+1)$ (see Figure 3).

Proof of Theorem 27. Let us consider the set function $\mu_d \colon \mathcal{B}(S^m) \mapsto \mathbb{R}$ such that $A \mapsto \mathbb{E}\{\#(j^r X_d)^{-1}(W) \cap A\}$. It is explained in [20] that μ_d is a Radon measure on S^m.

Because of the invariance under rotation of P_d, by Haar's theorem μ needs to be proportional to the volume measure. Therefore for any Borel subset $A \subset S^m$ we have $\mathbb{E}\{\#Z_d\} = \mu_d(S^m) = \mu_d(A)\mathrm{vol}(A)^{-1}\mathrm{vol}(S^m)$. Define $Y_d \in \mathcal{G}^\infty(\mathbb{D}^m, \mathbb{R}^k)$ as

$$Y_d = \left(1 + \frac{|u|^2}{d}\right)^{-\frac{d}{2}} X_d.$$

Observe that $Y_d \Rightarrow Y_\infty = \exp(-\frac{|u|^2}{2})X_\infty$ and that Y_d is equivalent to the GRF $\psi_d \circ a_d^R$ for any $R \in O(m+1)$.

Now let $W_0 \subset J^r(\mathbb{D}^m, \mathbb{R}^k)$ be the (semialgebraic) model of W. By the same proof of point (7) from Theorem 23, adapted to Y_d, there is a convergent sequence of functions $\rho_d \to \rho_{+\infty} \in L^1(\mathbb{D}^m)$ such that

$$\mathbb{E}\{\#(j^r Y_d)^{-1}(W_0)\} = \int_{\mathbb{D}^m} \rho_d \to \int_{\mathbb{D}^m} \rho_\infty = \mathbb{E}\{\#(j^r Y_\infty)^{-1}(W_0)\}.$$

In conclusion we have for $A = a_d^R(\mathbb{D}^m)$, as $d \to +\infty$

$$\begin{aligned}
\mathbb{E}\{\#Z_d\} &= \mu_d(A)\mathrm{vol}(A)^{-1}\mathrm{vol}(S^m) \\
&= \mathbb{E}\{\#(j^r Y_d)^{-1}(j^r\varphi^*(W))\}\mathrm{vol}(A)^{-1}\mathrm{vol}(S^m) \\
&= \mathbb{E}\{\#(j^r Y_d)^{-1}(W_0)\}\left(\frac{\int_0^\pi |\sin\theta|^{m-1}d\theta}{\int_0^{\arctan\left(d^{-\frac{1}{2}}\right)} |\sin\theta|^{m-1}d\theta}\right) \\
&= \mathbb{E}\{\#(j^r Y_\infty)^{-1}(W_0)\} m \frac{\mathrm{vol}(S^m)}{\mathrm{vol}(S^{m-1})} d^{\frac{m}{2}} + O(d^{\frac{m}{2}-1}).
\end{aligned}$$

□

Building on the previous results, we can now prove the general case for Betti numbers of a random singualrity.

Theorem 29 (Generalized square-root law for Betti numbers)**.** *Let $W \subset J^r(S^m, \mathbb{R}^k)$ be a closed semialgebraic intrinsic (as defined in Definition 11) of codimension $1 \le l \le m$. Then there are constants $b_W, B_W > 0$ depending only on W such that*

$$b_W d^{\frac{m}{2}} \le \mathbb{E}\{b_i(Z_d)\} \le B_W d^{\frac{m}{2}} \quad \forall i = 0, \dots, m-l$$

and $\mathbb{E}\{b_i(Z_d)\} = 0$ for all other i.

Proof. The proof is divided in two parts, first we prove the upper bound, using the square-root law from Theorem 27, then the we use Theorem 7 to deduce the lower bound. The globalization step for the lower bound is a generalization of the so-called "barrier method" from [11, 22].

1. Assume W is smooth with codimension s. Let us consider

$$P_d^{m,k+1}|_{S^m} = \Psi_d = (\psi_d, \psi_d^1) \in \mathcal{G}^\infty(S^m, \mathbb{R}^{k+1})$$

and Let $\hat{W} \subset J^{r+1}(S^m, \mathbb{R}^{k+1})$ be the intrinsic semialgebraic submanifold coming from Theorem 8 and Remark 12. Thus, using Theorems 8 and 27, we get

$$\mathbb{E}\{b_i(Z_d)\} \le N_W \mathbb{E}\#\{(j^{r+1}\Psi_d)^{-1}(\hat{W})\} \le N_W C_{\hat{W}} d^{\frac{m}{2}}.$$

2. Consider the embeddings of the m dimensional disk $a_d^R \colon \mathbb{D}^m \hookrightarrow S^m$ defined in (3.5). For any fixed $d \in \mathbb{N}$, choose a finite subset $F_d \subset O(m+1)$ such that the images of the corresponding embeddings $\{a_d^R(\mathbb{D}^m)\}_{R\in F_d}$ are disjoint. Denoting by Z_d^R the union of all connected components of Z_d that are entirely contained in $a_d^R(\mathbb{D}^m)$, we have

$$b_i(Z_d) \ge \sum_{R\in F_d} b_i(Z_d^R).$$

Let $W_0 \subset J^r(\mathbb{D}^m, \mathbb{R}^k)$ be the model of W as an intrinsic submanifold, it is closed and semialgebraic. By Definition 11, we have

$$(a_d^R)^{-1}\left((j^r\psi_d)^{-1}(W)\right) = \left(j^r(\psi_d \circ a_d^R)\right)^{-1}(W_0) \subset \mathbb{D}^m. \tag{3.6}$$

Recall that for any $R \in O(m+1)$, the GRF $\psi_d \circ a_d^R$ is equivalent to $Y_d \in \mathcal{G}^\infty(\mathbb{D}^m, \mathbb{R}^k)$ defined in 3.3, hence taking expectation in Equation (3.6) we find

$$\mathbb{E}\{b_i(Z_d)\} \ge \#(F_d)\mathbb{E}\{b_i(S_d)\},$$

where $S_d = (j^r(Y_d))^{-1}(W_0)$. is easy to see (repeating the same proof) that Theorem 23.7 holds also for the sequence $Y_d \Rightarrow Y_\infty$, so that $\mathbb{E}\{S_d\} \to \mathbb{E}\{S_\infty\}$. We can assume that $\mathbb{E}\{S_\infty\} > 0$, because of Lemma 25, thus for big enough d, the numbers $\mathbb{E}\{b_i(S_d)\}$ are bounded below by a constant $C > 0$. Now it remains to estimate the number $\#(F_d)$. Notice that $a_d^R(\mathbb{D}^m)$ is a ball in S^m of a certain radius ε_d, hence it is possible to choose F_d to have at least $N_m\varepsilon_d^{-1}$ elements, for some dimensional constant $N_m > 0$ depending only on m. We conclude by observing that

$$\varepsilon_d \approx d^{-\frac{m}{2}}.$$

□

Appendix 1: Examples of applications of Theorem 23

Example 30 (Zero sets of random polynomials)**.** Consider the zero set $Z_d \subset \mathbb{RP}^m$ of a random Kostlan polynomial $P_d = P_d^{m+1,1}$. Recently Gayet and Welschinger [11] have proved that given a compact hypersurface $Y \subset \mathbb{R}^m$ there exists a positive constant $c = c(\mathbb{R}^m, Y) > 0$ and $d_0 = d_0(\mathbb{R}^m, Y) \in \mathbb{N}$ such that for every point $x \in \mathbb{RP}^m$ and every large enough degree $d \ge d_0$, denoting by B_d any open ball of radius $d^{-1/2}$ in $\mathbb{RP}^m$, we have:

$$(B_d, B_d \cap Z_d) \cong (\mathbb{R}^m, Y) \tag{3.7}$$

(i.e. the two pairs are diffeomorphic) with probability larger than c. This result follows from Theorem 23 as follows. Let $\mathbb{D}^m \subset \mathbb{R}^m$ be the unit disk, and let $U \subset \mathcal{C}^\infty(\mathbb{D}^m, \mathbb{R})$ be the open set consisting of functions $g : \mathbb{D}^m \to \mathbb{R}$ whose zero set is regular (an open $\mathcal{C}^1$ condition satisfied almost surely by X_d, because of point (4)), entirely contained in the interior of $\mathbb{D}^m$ (an open $\mathcal{C}^0$ condition) and such that, denoting by $\mathbb{B} \subset \mathbb{R}^m$ the standard unit open ball, the first two conditions hold and $(\mathbb{B}, \mathbb{B} \cap \{g = 0\})$ is diffeomorphic to $(\mathbb{R}^m, Y)$ (this is another open $\mathcal{C}^1$ condition). Observe that, using the notation above:

$$(B_d, B_d \cap Z_d) \sim (\mathbb{B}, \mathbb{B} \cap \{X_d = 0\})$$

(this is simply because $X_d(u) = P_d(1, ud^{-1/2})$). Consequently point (5) of Theorem 23 implies that:

$$\begin{aligned}\lim_{d\to+\infty} \mathbb{P}\{(3.7)\} &= \lim_{d\to\infty} \mathbb{P}\left\{(\mathbb{B}, \mathbb{B} \cap \{X_d = 0\}) \sim (\mathbb{R}^m, Y)\right\} \\ &= \lim_{d\to\infty} \mathbb{P}\left\{X_d \in U\right\} \\ &= \mathbb{P}\left\{X_\infty \in U\right\} > 0.\end{aligned}$$

We stress that, as an extra consequence of Theorem 23, compared to [11] what we get is the existence of the limit of the probability of seeing a given diffeomorphism type.

Example 31 (Discrete properties of random maps)**.** Let $[X_d] \Rightarrow [X_\infty]$ be a converging family of gaussian random fields. In this example we introduce a useful tool for studying the asymptotic probability induced by X_d on discrete sets as $d \to \infty$. The key example that we have in mind is the case when we consider a codimension-one "discriminant" $\Sigma \subset \mathcal{C}^\infty(S^m, \mathbb{R}^k)$ which partitions the set of functions into many connected open sets. For instance Σ could be the set of maps for which zero is not a regular value: the complement of Σ consists of countably many open connected sets, each one of which corresponds to a rigid isotopy class of embedding of a smooth codimension-k submanifold $Z \subset S^m$. The following Lemma gives a simple technical tool for dealing with these situations.

Lemma 32. *Let E be a metric space and let $[X_d], [X_\infty]$ be a random fields such that $[X_d] \Rightarrow [X_\infty]$. Let also Z be a discrete space and $\nu\colon U \subset E \to Z$ be a continuous function defined on an open subset $U \subset E$ such that*[17] $\mathbb{P}\{X_\infty \in U\} = 1$*. Then, for any $A \subset Z$ we have:*

$$\exists \lim_{d\to\infty} \mathbb{P}\left\{X_d \in U,\ \nu(X_d) \in A\right\} = \mathbb{P}\left\{\nu(X_\infty) \in A\right\}.$$

Proof. Since $\nu^{-1}(A)$ is closed and open by continuity of ν, it follows that $\partial\nu^{-1}(A) \subset E \backslash U$. Therefore $\mathbb{P}\{X_\infty \in \partial\nu^{-1}(A)\} = 0$ and by Portmanteau's Theorem [1, Theorem 2.1], we conclude that

$$\mathbb{P}\{X_d \in \nu^{-1}(A)\} \xrightarrow[d\to\infty]{} \mathbb{P}\{X_\infty \in \nu^{-1}(A)\}, \quad \forall\ A \subset Z. \tag{3.8}$$

□

Equation (3.8), in the case of a discrete topological space such as Z, is equivalent to narrow convergence $\nu(X_d) \Rightarrow \nu(X)$, by Portmanteau's Theorem, because $\partial A = \emptyset$ for all

[17] Of course, $E \smallsetminus U = \Sigma$ is what we called "discriminant" in the previous discussion. Note that we do not require that $\mathbb{P}\{X_d \in U\} = 1$, however it will follow that $\lim_d \mathbb{P}\{X_d \in U\} = 1$.

subsets $A \subset Z$. Note also that to prove narrow convergence of a sequence of measures on Z, it is sufficient to show (3.8) for all $A = \{z\}$, indeed in that case the inequality

$$\liminf_{d\to\infty} \mathbb{P}\{\nu_d \in A\} = \liminf_{d\to\infty} \sum_{z\in A} \mathbb{P}\{\nu_d = z\} \geq \sum_{z\in A} \mathbb{P}\{\nu = z\} = \mathbb{P}\{\nu \in A\}$$

follows automatically from Fatou's lemma.

Following Sarnak and Wigman [24], let us consider one simple application of this Lemma. Let H_{m-1} be the set of diffeomorphism classes of smooth closed connected hypersurfaces of $\mathbb{R}^m$. Consider $U = \{f \in \mathcal{C}^\infty(\mathbb{D}^m, \mathbb{R}) : f \pitchfork 0\}$ and let $\nu(f)$ be the number of connected components of $f^{-1}(0)$ entirely contained in the interior of $\mathbb{D}^m$. For $h \in H_{m-1}$ let $\nu_h(f)$ be the number of those components which are diffeomorphic to $h \subset \mathbb{R}^m$. In the spirit of [24], we define the probability measure $\mu(f) \in \mathscr{P}(H_{m-1})$ as

$$\mu(f) = \frac{1}{\nu(f)} \sum_{h\in H_{m-1}} \nu_h(f)\delta_h.$$

Let us consider now the rescaled Kostlan polynomial $X_d : \mathbb{D}^m \to \mathbb{R}$ as in Theorem 23. The diffeomorphism type of each internal component of $f^{-1}(0)$ remains the same after small perturbations of f inside U, hence $\mu \colon U \to \mathscr{P}(H_{m-1})$ is a locally constant map, therefore by Lemma 32 we obtain that for any subset $A \subset \mathscr{P}(H_{m-1})$,

$$\exists \lim_{d\to\infty} \mathbb{P}\{X_d \in U \text{ and } \mu(X_d) \in A\} = \mathbb{P}\{\mu(X_\infty) \in A\}.$$

Moreover since in this case $X_d \in U$ with $\mathbb{P} = 1$, for all $d \in \mathbb{N}$ and the support of X_∞ is the whole $\mathcal{C}^\infty(\mathbb{D}^m, \mathbb{R})$, we have

$$\exists \lim_{d\to\infty} \mathbb{P}\{\mu(X_d) \in A\} = \mathbb{P}\{\mu(X_\infty) \in A\} > 0.$$

Example 33 (Random rational maps)**.** The Kostlan polynomial $P_d^{m,k+1}$ can be used to define random rational maps. In fact, writing $P_d^{m,k+1} = (p_0, \dots, p_k)$, then one can consider the map $\varphi_d^{m,k} : \mathbb{R}\mathrm{P}^m \dashrightarrow \mathbb{R}\mathrm{P}^k$ defined by:

$$\varphi_d^{m,k}([x_0, \dots, x_m]) = [p_0(x), \dots, p_m(x)].$$

(When $m > k$, with positive probability, this map might not be defined on the whole $\mathbb{R}\mathrm{P}^m$; when $m \leq k$ with probability one we have that the list $(p_0, \dots, p_k)$ has no common zeroes, and we get a well defined map $\varphi_d^{m,k} : \mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^k$.) Given a point $x \in \mathbb{R}\mathrm{P}^m$ and a small disk $D_d = D(x, d^{-1/2})$ centered at this point, the behavior of $\varphi_d^{m,k}|_{D_d}$ is captured by the random field X_d defined in (3.2): one can therefore apply Theorem 23 and deduce, asymptotic local properties of this map.

For example, when $m \leq k$ for any given embedding of the unit disk $q : \mathbb{D}^m \hookrightarrow \mathbb{R}\mathrm{P}^k$ and for every neighborhood U of $q(\partial\mathbb{D}^m)$ there exists a positive constant $c = c(q) > 0$ such that for big enough degree d and with probability larger than c the map

$$X_d = \varphi_d^{m,k} \circ a_d : \mathbb{D}^m \to \mathbb{R}\mathrm{P}^k$$

(defined by composing φ with the rescaling diffeomorphism $a_d : \mathbb{D}^m \to D_d$) is isotopic to q thorugh an isotopy $\{q_t : \mathbb{D}^m \to \mathbb{R}\mathrm{P}^k\}_{t\in I}$ such that $q_t(\partial\mathbb{D}^m) \subset U$ for all $t \in I$.

The random map $\varphi_d^{m,k}$ is strictly related to the random map $\psi_d^{m,k}\colon S^m \to \mathbb{R}^k$:

$$\psi_d^{m,k}(x) = P_d^{m,k}(x),$$

which is an easier object to work with. For example the random algebraic variety $\{\varphi_d = 0\}$ is the quotient of $\{\psi_d = 0\}$ modulo the antipodal map. If we denote by D_d any sequence of disks of radius $d^{-\frac{1}{2}}$ in the sphere, then $\psi_d|_{D_d} \approx X_d$, so that we can understand the local asymptotic behaviour of ψ_d using Theorem 23 (see Figure 2). For instance, from point (7) it follows that

$$\mathbb{E}\left\{b_i\left(\{\psi_d = 0\} \cap D_d\right)\right\} \to \mathbb{E}\left\{b_i\left(\{X_\infty = 0\} \cap \mathbb{D}^m\right)\right\}.$$

Example 34 (Random knots)**.** Kostlan polynomials offer different possible ways to define a "random knot". The first is by considering a random rational map:

$$\varphi_d^{1,3} : \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^3,$$

to which the discussion from Example 33 applies. (Observe that this discussion has to do with the *local* structure of the knot.)

Another interesting example of random knots, with a more global flavour, can be obtained as follows. Take the random Kostlan map $X_d : \mathbb{R}^2 \to \mathbb{R}^3$ (as in (3.2) with $m = 2$ and $k = 3$) and restrict it to $S^1 = \partial\mathbb{D}^m$ to define a random knot:

$$k_d = X_d|_{\partial\mathbb{D}^m} : S^1 \to \mathbb{R}^3.$$

The difference between this model and the previous one is that this is global, in the sense that as $d \to \infty$ we get a limit global model $k_\infty = X_\infty|_{\partial D} : S^1 \to \mathbb{R}^3$. What is interesting for this model is that the Delbruck–Frisch–Wasserman conjecture [5, 9], that a typical random knot is non-trivial, does not hold: in fact k_∞ charges every knot (included the unknot) with positive probability.

Proposition 35. *The random map:*

$$k_d = X_d|_{\partial\mathbb{D}^2} : S^1 \to \mathbb{R}^3.$$

is almost surely a topological embedding (i.e. a knot). Similarly, the random rational map $\varphi_d^{1,3} : \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^3$ *is almost surely an embedding.*

Proof. We prove the statement for k_d, the case of $\varphi_d^{1,3}$ is similar. Since S^1 is compact, it is enough to prove that k_d is injective with probability one.

Let $F_d = \mathbb{R}[x_0, x_1, x_2]^3_{(d)}$ be the space of triples of homogeeous polynomials of degree d in 3 variables. Recall that $k_d = X_d|_{\partial\mathbb{D}^2}$, where, if $P \in F_d$, we have set:

$$X_d(u) = P\left(1, \frac{u}{\sqrt{d}}\right), \quad u = (u_1, u_2) \in \mathbb{R}^2.$$

Let now $S^1 = \partial\mathbb{D}^2 \subset \mathbb{R}^2$ and $\phi : \left((S^1 \times S^1)\backslash\Delta\right) \times F_d \to \mathbb{R}^3$ be the map defined by

$$\phi(x, y, P) = P\left(1, \frac{x}{\sqrt{d}}\right) - P\left(1, \frac{y}{\sqrt{d}}\right).$$

Observe that $\phi \pitchfork \{0\}$. By the parametric transversality theorem we conclude that ϕ is almost surely transversal to $W = \{0\}$. This imples that, with probability one, the set

$$\{x \neq y \in S^1 \times S^1 \,|\, k_d(x) = k_d(y)\}$$

is a codimension-three submanifold of $S^1 \times S^1$ hence it is empty, so that k_d is injective. □

Theorem 23 implies now that the random variable $k_d \in C^\infty(S^1, \mathbb{R}^3)$ converges narrowly to $k_\infty \in C^\infty(S^1, \mathbb{R}^3)$, which is the restriction to $S^1 = \partial\mathbb{D}^2$ of X_∞. Note that, since the support of X_∞ is all $C^\infty(\mathbb{D}^2, \mathbb{R}^3)$, it follows that the support of k_∞ is all $C^\infty(S^1, \mathbb{R}^3)$ and in particular every knot (i.e. isotopy class of topological embeddings $S^1 \to \mathbb{R}^3$, a set with nonempty interior in the C^∞ topology) has positive probability by Theorem 23.3. Moreover, denoting by $\gamma_1 \sim \gamma_2$ two isotopic knots, we have that

$$\mathbb{P}\left(\partial\{k_\infty \sim \gamma\}\right) \leq \mathbb{P}\{k_\infty \text{ is not an immersion}\} = 0$$

by Theorem 23.4, because the condition of being an immersion is equivalent to that of being transverse to the zero section of $J^1(S^1, \mathbb{R}^3) \to S^1 \times \mathbb{R}^3$. Theorem 23.2, thus implies that for every knot $\gamma : S^1 \to \mathbb{R}^3$ we have:

$$\lim_{d\to\infty} \mathbb{P}\{k_d \sim \gamma\} = \mathbb{P}\{k_\infty \sim \gamma\} > 0.$$

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