Congruence Theorems in the past, present, and future

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Congruence Theorems for triangles are, since the time of Euclid and nowadays worldwide, an important topic of school mathematics. Ensuring that two triangles are congruent is useful to solve practical problems and to prove results in geometry. Moreover, Congruence Theorems are a nice example of the historical development of mathematics. Finally, generalizing such results is an accessible research exercise. We present some reflections made from a mathematician's viewpoint and invite historians and didacticians to further investigate the topic of Congruence Theorems.

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Teaching Congruence Theorems for triangles

While studying subsets of the plane, it makes sense to identify those that are equal up to an isometry[1], and the notion of *congruent* precisely does that. This is a concrete example of *equivalence relation,* which is a very handy concept to have in mind: both in mathematics and in real life one should decide what the important features are and focus on them.

After having introduced the notion of congruence, there comes the problem of telling whether two given sets are congruent. After having explicitly constructed isometries, pupils may appreciate the Congruence Theorems as a valuable shortcut to prove that an isometry exists. These results are like new gadgets in the *mathematical toolbox* at the pupils' disposal. The key point becomes to decide whether some Congruence Theorem applies, which amounts to check whether all assumptions hold.

Knowing that two triangles are congruent (more generally, similar) is useful for applications. There are several exercises stemming from real-life situations that show how Congruence Theorems for triangles have been applied since two thousand years to solve practical problems. To be honest, the Congruence Theorems for triangles are completed by the results of trigonometry that let one compute all side lengths and all angle measures, a set of procedures known under the name of *solving the triangle.* One advantage of the Congruence Theorems is that they can be understood (even by younger pupils) without making use of trigonometry.

Another plus point for the Congruence Theorems for triangles is that they are a collection of results that fit nicely together, so they constitute a small *mathematical theory.* Moreover, they provide a playground for pupils to *distinguish truth from falsehood*: some sets of assumptions give a true theorem, others are not sufficient and lead to counterexamples. As it often happens for mathematical statements, in the classical Congruence Theorems none of the assumptions can be removed without invalidating the result.

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[1] Occasionally (e.g., for placing tiles that cannot be flipped) it makes sense to allow only a composition of rotations and translations. For questions concerning only the shape, scaling should be allowed.

Congruence Theorems for triangles around the world

One could define a “Congruence Theorem for triangles” as a mathematical result listing a set of conditions that ensure that a triangle is determined up to congruence, possibly restricting to those conditions involving only the length of sides and the measure of interior angles. With this definition there are at least two further Congruence Theorems with respect to the classical ones:

- (acuteSSA) Two acute triangles are congruent if two pairs of corresponding sides are equal in length, and one pair of corresponding angles are equal in measurement.
- (perimeterAA) Two triangles are congruent if they have the same perimeter and the same angles.

So let us restrict to more classical Congruence Theorems, and let’s collect their current formulations:

- (SSS) Two triangles are congruent if three pairs of corresponding sides are equal in length.
- (ASA) Two triangles are congruent if two pairs of corresponding angles are equal in measurement, and the included sides are equal in length.
- (AAS) Two triangles are congruent if two pairs of corresponding angles are equal in measurement, and one pair of corresponding sides, different from the included ones, are equal in length.
- (AAcorrS) Two triangles are congruent if two pairs of corresponding angles are equal in measurement, and one pair of corresponding sides are equal in length.
- (SAS) Two triangles are congruent if two pairs of corresponding sides are equal in length, and the included angles are equal in measurement.
- (SsA) Two triangles are congruent if two pairs of corresponding sides are equal in length, and the pair of angles opposite to the bigger of these sides are equal in measurement (if the two sides are equal, then we can use any of the two sides).
- (HL) Two right-angled triangles are congruent if their hypotenuses and one pair of corresponding legs are equal in length.

Historians may explain how come the mathematical results having the status of “Congruence Theorem for triangles” vary with the language. We content ourselves to supporting this claim by considering the set of Congruence Theorems that one finds on [Wikipedia] in various languages[2]: in the table below a checkmark stands for a theorem, and a checkmark in parenthesis stands for a result which is mentioned but not with the status of a Congruence Theorem. Looking at the table, keep in mind the following facts: AAS and AAcorrS are variants of ASA because knowing two angles amounts to knowing three angles; maybe SsA is neglected because knowing two pairs of corresponding sides and one pair of corresponding angles is not sufficient to ensure congruence; HL is a special case of SsA.

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[2] There are reasons to believe that the Wikipedia sample is representative of what one finds in school texts.

Language	SSS	ASA	AAS	AAcorrS	SAS	SsA	HL
English	✓	✓	✓	(✓)	✓		✓
Chinese	✓	✓	✓		✓		✓
Italian	✓	✓		✓	✓		(✓)
French	✓	✓	✓		✓	✓	
Spanish	✓	✓	✓		✓	✓	
Japanese	✓	✓		✓	✓		✓
Portuguese	✓	✓	✓		✓		✓
German	✓	✓			✓	✓	
Russian	✓	✓	✓		✓	(✓)	
Arabic	✓	✓			✓		
Hindi	✓	✓	✓		✓	(✓)	

This analysis shows that, although mathematical results are immutable truths, what is "important" is a matter of choice. However, one may foresee some sort of uniformization because the English Wikipedia is an inspiration to Wikipedia in other languages. We do hope that the numeration of the Congruence Theorems from [Hilbert] is generally replaced by the more convenient abbreviations like SSS (e.g., in Spanish LLL because of the word 'Lado')[3]. We also hope that SsA gets a better status worldwide because it is a useful theorem, while HL should get a lower status because it is an immediate consequence of Pythagoras' theorem (and it should be presented together with the analogous "LL" statement, namely the special case of SAS where we know both legs of the right triangle).

A mathematical subtlety in the Congruence Theorems for triangles

Provided that we have agreed upon what the set of Congruence Theorems is, let us see whether we also agree on the formulation of these results. First, notice that presenting congruent figures as those "having the same shape and size" is didactically meaningful, although this is not mathematically exact because of the vague meaning of the terms 'shape' and 'size'.

In the Congruence Theorems for triangles there are six important quantities, namely the measure of the three interior angles and the length of the three sides. Given two triangles, we must ensure that some of these quantities match, and for this reason one speaks of "corresponding" sides or angles. This notion of correspondence is crucial for some of the statements because we must specify the relative position of the known sides and angles.

[3] Languages without an alphabet (or with the unfortunate coincidence that the words for angle and side have the same initial) can still use nicknames like Side-Angle-Side. In the Chinese, Japanese, and Hindi Wikipedia the English abbreviations are used, however this choice may be didactically not optimal.

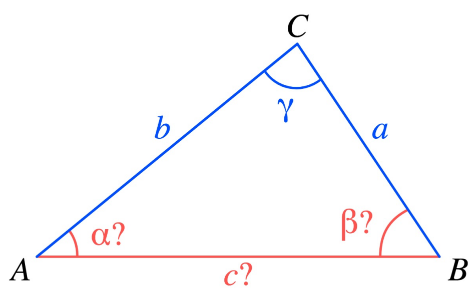
Indeed, consider for example the triangles that are similar to the right triangle with side lengths (3,4,5). If we want such a triangle with one side length equal to 1, then it is not clear whether this side is the short leg, the long leg, or the hypothenuse, so we cannot determine the triangle up to congruence even if we know three angles and one side. Even worse, the existence of *5-con triangles* means that if we are not careful with matching (i.e., we do not specify the relative position of the known objects), then there are non-congruent triangles of which we know three angles and two sides. To see how neglected the notion of correspondence is, simply (as the author did) test some pupils or teachers with the following question:

> Suppose you know the measure of three angles and the length of two sides of a triangle. Is the triangle determined up to congruence?

In the future, all prospective teachers should reflect on the 'correspondence issue' and should know about 5-con triangles, and schoolbooks could speak about both. Notice that there are 5-con triangles with very nice side lengths, for example (8, 12, 18) and (12, 18, 27).

Let us propose a didactical solution for the above-mentioned issue. Declare that the word 'corresponding' is not mathematically precise unless the correspondence is explicitly given. There are six possible bijections from the set of vertices of a triangle to the set of vertices of another triangle. What is implicitly understood in the Congruence Theorems (as we have formulated them) is that we have given names to the vertices of the two triangles (e.g., ABC and A'B'C') and we fix 'corresponding vertices' (say, A and A') and then 'corresponding angles' are simply angles at corresponding vertices, while 'corresponding sides' are simply sides between corresponding vertices. Notice that there is no mathematical rule that says that vertices described using the same letter (e.g., A and A') must match: although this is usually the case, it is not necessary the case, especially if the triangles are part of a larger geometrical constructions. Once teachers and pupils have understood the hidden convention and its dangers, then there is no problem in using it (which avoids defining a notation in the statements of the Congruence Theorems).

A mathematically more satisfying solution to the "corresponding issue" in the Congruence Theorems would be speaking of just one triangle. Indeed, another way to view those theorems is in terms of constructions. The question shifts from comparing two triangles to asking whether a single triangle can be built whenever some of its parts are given. More precisely, we know some angles and some sides (possibly with some specifications as how they are related to each other) and ask whether we can construct only one triangle (unique up to isometry) which has these data. Instead of drawing two triangles, we just draw one. This is done in some countries, for example the above picture about the SAS Congruence Theorem stems from the Russian [Wikipedia].



Research on the Congruence Theorems

Although Congruence Theorems for triangles have been known since almost two thousand years, because of [Euclid], there are open questions on this topic. It is a plus point for the Congruence Theorems that they provide open research questions whose statements can be understood by pupils. Moreover, since there are many research directions, the pupils can understand that a mathematical result can be generalized in many ways, as it is common in mathematical research.

First, although we already have the Congruence Theorems for triangles, we may not know all their possible proofs. To support this claim, consider that in 2014 [Dokai] published a new proof for the SAS Congruence Theorem, so most likely this is not the end of the story.

With some guidance, pupils can state more theorems that present conditions on a triangle that determine it up to congruence. For example, although there is no "SSA Congruence Theorem", an acute triangle is determined up to congruence if, in short, we know two sides and one angle. Similarly, we have already mentioned that knowing the three angles and the perimeter is also sufficient. As soon as one involves other objects than sides and angles, one may conjecture that there are infinitely many possible theorems. To support this claim, consider that there are thousands of points that are defined for a given triangle, see for example *The Encyclopedia of Triangle Centers* by [Kimberling and others]. As a simple example, a triangle is determined up to congruence if we know the angles and the radius of the circumscribed circle.

We could also move beyond triangles. By applying the Congruence Theorems for triangles, it is easy to prove some Congruence Theorems for convex quadrilaterals, for example the SASAS Congruence Theorem where we know three consecutive sides and the two angles that they form. For results on quadrilaterals, see [Vance], [Laudano and Vicenzi] and [Perucca and Torti]. The first reference states that exploring these results is an enriching activity for pupils:

> The theory of congruence for quadrilaterals, can provide some rich applications of congruence of triangles and other concepts taught in high school geometry, and opportunities for valuable practice in constructing counterexamples.

In the third reference, Perucca proved Congruence Theorems for convex n-gons for arbitrary n (e.g., for n>6 it suffices to know the lengths of enough diagonals). In fact, the situation would be much worse without the assumption of convexity: for example, knowing the exact position of n-1 vertices and all sides and all but two angles is not sufficient to determine an n-gon up to congruence because the two sides at the missing vertex could point outwards or inwards.

Mathematicians can investigate analogous theorems for more complicated shapes in higher dimension. For pupils, we also recommend considering various easy shapes in the plane. For example, it is an instructive mathematical exercise to prove Congruence Theorems for circles (it suffices to know the radius or the diameter or the length of the circumference or the area or…). Similarly, one could consider for example rhombi, trapezoids, T-shaped figures and so on. The teacher can, in principle, take any agreed-upon symbol and analyze which conditions guarantee that the shape and size cannot be changed, thus the pupils may get original exercises.

While trying to generalize the Congruence Theorems, pupils can prove easy results and they can find easy counterexamples to other statements. In fact, one could also play the game of *mathematical bets*, gaining points for each correct guess about whether a statement is true or false.

The most brilliant pupils, assisted for example by a mathematician from a nearby University, can also attack some of the open research questions and prove new Congruence Theorems for polygons. Finally, notice that instead of Congruence Theorems one could investigate Similarity Theorems in the same spirit. Let's rejoice in the fact that there are plenty of open questions that are accessible research exercises.

The author is a mathematician who discovered new Congruence Theorems and collected original reflections. She invites historians and didacticians to research the topic of Congruence Theorems.

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References

[Dokai] Dokai, M.C. (2014) *A Proof of Euclid's SAS (side angle side) Theorem of Congruence of Triangles via the Cross Section of a Double Cone*. IOSR Journal of Mathematics, Volume 10, Issue 1 Ver. III pp. 26–33.

[Euclid] Euclid (c. 300 BC) *Elements.* The Clay Mathematical Institute dedicated page http://www.claymath.org/euclid/index.

[Hilbert] Hilbert, D. (1899) *The Foundations of Geometry*. 2nd ed. Chicago, Open Court.

[Kimberling] Kimberling, C. and others (1998-2022) *Encyclopedia of Triangle Centers.* https://faculty.evansville.edu/ck6/encyclopedia/etc.html.

[Laudano and Vicenzi] Laudano, F. and Vicenzi, G. (2017) *Congruence Theorems for Quadrilaterals*. Journal for Geometry and Graphics, vol. 21 No. 1, 45–59.

[Perucca and Torti] Perucca, A. and Torti T. (2023) *Congruence Theorems for convex polygons involving sides, angles, and diagonals*. To appear in International Journal of Geometry.

[Vance] Vance, I.E. (1982) *Minimum Conditions for Congruence of Quadrilaterals*. School Science and Mathematics vol. 82 No. 5, 403–415.

[Wikipedia] Wikipedia (2022) The English Wikipedia page *Congruence (geometry)*. https://en.wikipedia.org/wiki/Congruence_(geometry) and the corresponding pages in Arabic, Chinese, Hindi, Italian, French, Japanese, Portuguese, Russian, Spanish.