

ISOTROPIC RANDOM SPIN WEIGHTED FUNCTIONS ON S^2 VS ISOTROPIC RANDOM FIELDS ON S^3

MICHELE STECCONI

ABSTRACT. We show that an isotropic random field on $SU(2)$ is not necessarily isotropic as a random field on S^3 , although the two spaces can be identified. The ambiguity is due to the fact that the notion of isotropy on a group and on a sphere are different, the latter being much stronger. We show that any isotropic random field on S^3 is necessarily a superposition of uncorrelated random harmonic homogeneous polynomials, such that the one of degree d is necessarily a superposition of uncorrelated random spin weighted functions of every possible spin weight in the range $\{-\frac{d}{2}, \dots, \frac{d}{2}\}$, each of which is isotropic in the sense of $SU(2)$. Moreover, for a random field of fixed degree, each spin weight appears with the same magnitude, in a sense to be specified.

In addition we will give an overview of the theory of spin weighted functions and Wigner D -matrices, with the purpose of gathering together many different points of view and adding ours. As a byproduct of this survey we will prove some new properties of the Wigner matrices and a formula relating the operators $\bar{\partial}\bar{\partial}$ and the horizontal Laplacian of the Hopf fibration $S^3 \rightarrow S^2$, in the sense of [4].

1. INTRODUCTION

In this paper we compare the theory of random spin weighted functions on the sphere S^2 , with that of random fields on the hypersphere S^3 . We will see the two theories in the same light, but we will clarify the distinction between the corresponding notions of isotropy.

A function with spin weight $s \in \frac{1}{2}\mathbb{Z}$ on S^2 is a section of the spin s bundle (see [8, 18]). In the convention¹ of this paper this is defined to be the complex line bundle of degree $2s$ over the sphere S^2 and denoted by $\mathcal{T}^{\otimes s} \rightarrow S^2$, see Section 2.4. These objects have received a lot of attention in the last years (see [1, 2, 15]), due to their application in the statistical analysis of cosmological and astrophysical data (see [1]), in particular related to the Cosmic Microwave Background (see [15]).

A convenient way to treat such objects is the so-called “pull-back approach” (this is the point of view adopted in [2, 8]), which consists in the identification of the vector space of (smooth, continuous, square integrable, etc..) sections of $\mathcal{T}^{\otimes s}$ with a subspace of complex valued functions on S^3 , or on $SO(3)$ if $s \in \mathbb{Z}$, see Theorem 2.22 (Pull-back Correspondence). The reason why this is possible is that under the natural maps $S^3 \simeq$

2020 *Mathematics Subject Classification.* Primary 20C35, 60G60; Secondary 33C55, 53C20, 60B20.

Key words and phrases. Isotropic random fields, Riemannian geometry, spherical harmonics, random waves, Wigner D -matrices.

¹There is some ambiguity in the literature, regarding the sign of s . We will justify our choice in Section 2.4.

² $SU(2) \rightarrow SO(3) \rightarrow S^2$, the pull-back of $\mathcal{T}^{\otimes s}$ becomes a trivial bundle $\underline{\mathbb{C}} := \mathbb{C} \times SU(2)$ on $SU(2)$. If $s \in \mathbb{Z}$, then the pull-back bundle is already trivial on $SO(3)$ ³:

$$\begin{array}{ccc} \underline{\mathbb{C}} & \longrightarrow & S^3 \\ & & \downarrow \\ \mathcal{T}^{\otimes \frac{1}{2}} \text{ or } \underline{\mathbb{C}} & \longrightarrow & SO(3) \\ & & \downarrow \\ \mathcal{T}^{\otimes s}, \text{ with } s \in \frac{1}{2}\mathbb{Z} & \longrightarrow & S^2 \end{array}$$

Under this point of view, a random spin s function $\sigma_X: S^2 \rightarrow \mathcal{T}^{\otimes s}$ is thought as a complex random field $X: S^3 \rightarrow \mathbb{C}$ on the hypersphere $S^3 \simeq SU(2) \subset \mathbb{C}^2$, with a prescribed behavior under multiplication by a phase:

$$X(z \cdot e^{it}) = X(z)e^{-ist}.$$

In this case, we say that X has *right spin* $= -s$. The minus sign is explained by the fact that the function X represents the collection of all the coordinate expressions for the section σ_X , thus it has to be interpreted as a dual object, see Remark 2.17.

As in most models, we don't want the sphere S^2 to have special points or directions. Consequently, the random fields that we care about are only those that reflect such *isotropy*. In more rigorous terms, this means that we will study the random spin weighted functions that are invariant under the automorphisms of the bundle $\mathcal{T}^{\otimes s} \rightarrow S^2$ induced by orientation preserving rotations, i.e. elements of the group $SO(3)$. We will explain in Section 2.4 how this notion of change of variables, from the point of view of random fields on $SU(2) \simeq S^3$, translates to invariance in law under the composition with left multiplication by any element. This condition is usually called *isotropy* in the context of random fields on groups (compare with [16]). On the other hand, a random field $X: S^3 \rightarrow \mathbb{C}$ on a sphere is said to be *isotropic* if it is invariant in law under composition with any orientation-preserving isometry of S^3 , i.e. any element of $SO(4)$. This latter notion is clearly stronger than the previous, indeed the round metric on S^3 is in fact a bi-invariant metric on the group $SU(2)$ and any orientation-preserving isometry of S^3 is a composition of a left and a right multiplication by two elements of the group (see [9, 19]).

One of the purposes of this paper is to compare the two above notions of isotropy, which we will call *left-invariance* and *bi-invariance* (we will give the precise definition in Section 5). In addition we will consider also *right-invariant* random fields, so that X is bi-invariant if and only if it is both left and right invariant. The significance of such comparison is that the study of bi-invariant (isotropic for S^3) random fields is strictly related to the study of random waves on S^3 . Given a compact Riemannian manifold M , we will say that a random field $X: M \rightarrow \mathbb{C}$ is a *monochromatic random wave* of frequency $\lambda \in \mathbb{R}$ if X satisfies, almost surely, the Helmholtz equation for the eigenvalue $-\lambda^2$:

$$\Delta_M X = -\lambda^2 X,$$

²With this symbol we denote diffeomorphisms.

³If $s \in \mathbb{Z}$, then $\mathcal{T}^{\otimes s}$ is a true tensor power of the tangent bundle $TS^2 = \mathcal{T}^{\otimes 1}$ and $SO(3)$ is isomorphic to the frame orthonormal bundle of S^2 . When $s \in \frac{1}{2}\mathbb{Z}$ is not an integer then $\mathcal{T}^{\otimes \frac{1}{2}}$ is just a notation for the bundle of degree 1, justified by the fact that $(\mathcal{T}^{\otimes \frac{1}{2}})^{\otimes 2} \cong \mathcal{T}^{\otimes 1}$.

with Δ_M being the Laplace-Beltrami operator.⁴ For reasons that we will explain later (see Section 2) in this paper we will take on S^3 the round metric of a sphere of radius 2, so that for each $\ell \in \frac{1}{2}\mathbb{N}$, the eigenfunctions are all those complex valued functions whose real and imaginary parts are the restriction of real homogeneous harmonic polynomials on \mathbb{R}^4 , where the ones of degree 2ℓ are relative to the eigenvalue $-\ell(\ell+1)$, for all $\ell \in \frac{1}{2}\mathbb{N}$ ⁵.

It is well known that any square integrable random field $X: S^3 \rightarrow \mathbb{C}$ admits a *spectral representation* as a sum

$$(1.1) \quad X = \sum_{\ell, i} a_i^\ell \phi_i^\ell,$$

for some complex random variables a_i^ℓ , where ϕ_i^ℓ is an orthonormal basis of eigenfunctions of degree 2ℓ . Then, X is bi-invariant (i.e. isotropic on S^3) if and only if the fields X^ℓ are jointly bi-invariant, meaning that the joint law of the whole family $(X^\ell)_\ell$ is invariant under the transformations of the form: $(X^\ell)_\ell \sim (\theta^* X^\ell)_\ell$ for all $\theta \in SO(4)$, and in this case they are automatically pairwise uncorrelated. From the point of view of the group $SU(2)$, a similar statement is known under the name of Stochastic Peter-Weyl theorem (for which we refer to [16, Proposition 5.4] and [16, Theorem 5.5]. See also [16, Section 2.5] or [12, Section 4.7] for the standard Peter-Weyl theorem.). It says that there is a decomposition

$$(1.2) \quad X = \sum_{\ell, m, s} b_{m, s}^\ell D_{m, s}^\ell,$$

for a suitable collection of complex random variables $b_{m, s}^\ell$, where $D_{m, s}^\ell: SU(2) \rightarrow \mathbb{C}$ are the coefficients (indexed as in equation (1.3), below) of the ℓ^{th} Wigner matrix $D^\ell: SU(2) \rightarrow U(2\ell+1)$. Again, the field X is left-invariant if and only if the collection of fields $X^\ell = \sum_{m, s} b_{m, s}^\ell D_{m, s}^\ell$ are jointly left-invariant, meaning that the joint law of the whole family $(X^\ell)_\ell$ is invariant under the transformations of the form: $(X^\ell)_\ell \sim (L_g^* X^\ell)_\ell$ for all $g \in SU(2)$, and again in this case they are automatically uncorrelated.

A key observation is that the two decompositions above are essentially the same, due to the fact that, when seen on $S^3 \simeq SU(2)$, the functions

$$(1.3) \quad \phi_{m, s}^\ell = \frac{\sqrt{2\ell+1}}{4\pi} D_{m, s}^\ell, \quad \forall \ell \in \frac{1}{2}\mathbb{N} \text{ and } m, s \in \{-\ell, -\ell+1, \dots, \ell\},$$

are the hyperspherical harmonics of degree 2ℓ , thus they form an orthonormal⁶ basis of eigenfunctions of the Laplacian on S^3 . Although this is a well known fact (see [13], for instance), we will report a simple proof for completeness, see Proposition 3.5. An important feature of such basis is that $\phi_{m, s}^\ell$ is a function with pure left spin $-m$ and pure right spin $-s$, see Definition 2.26.

Remark 1.1. With a different normalization, as L^2 sections of $\mathcal{T}^{\otimes s}$, and via the pull-back correspondence, one defines the so-called *spin weighted spherical harmonics* $Y_{m, s}^\ell: S^2 \rightarrow \mathcal{T}^{\otimes s}$, see Remark 3.6. In particular the functions $Y_{m, 0}^\ell: S^2 \rightarrow \mathbb{C}$ are the standard spherical harmonics⁷.

⁴Here, we are using the term *monochromatic random wave* in a broad sense, whereas in the context of Riemannian geometry (see [5, 21]), the same terminology is often used to indicate that the random field is of the form $X = \sum_i a_i \phi_i$, for some family of i.i.d. complex Gaussian random variables $a_i \in \mathbb{C}$, and with ϕ_i being an orthonormal basis of the eigenspace relative to the eigenvalue $-\lambda^2$.

⁵We have $\Delta_{S^3} = 4\Delta_{2S^3}$.

⁶They are orthonormal with respect to the L^2 product of $2S^3$.

⁷The convention on the index m might differ from the usual one.

The decompositions (1.1) and (1.2) imply that any “isotropic” random field is a sum of uncorrelated “isotropic” random waves of frequency $\sqrt{\ell(\ell+1)}$:

$$X = \sum_{\ell \in \frac{1}{2}\mathbb{N}} X^\ell, \quad X^\ell = \sum_{m,s=-\ell}^{\ell} a_{m,s}^\ell \phi_{m,s}^\ell,$$

for all $\ell \in \frac{1}{2}\mathbb{N}$. Here, the $a_{m,s}^\ell := a_i^\ell$ are the complex random variables defined by Equation (1.1), with respect to the basis given in (1.3) indexed by $i = (m, s)$. This is true for both notions of isotropy: in the sense of S^3 (bi-invariance) and in the sense of $SU(2)$ (left-invariance).

Theorem 1.2. *The field X is left, right or bi invariant if and only if the fields X^ℓ are jointly left, right or bi invariant, respectively. Moreover, in this case the fields X^ℓ are pairwise uncorrelated.*

Proof. For left and right invariance, this is a direct application of [16, Proposition 5.4]. The statement for bi-invariance then follows from the fact that being bi-invariant is equivalent to being at the same time left and right invariant, see Section 5. \square

To see the true difference between these notions of invariance we have to consider a further decomposition, into the spaces of functions spanned by the coefficients of each column of the Wigner matrices.

Theorem 1.3. *If $X: S^3 \rightarrow \mathbb{C}$ is an almost surely square integrable random field, then it can be decomposed as a sum of random fields $X_{\bullet,s}^\ell$, for all $\ell \in \frac{1}{2}\mathbb{N}$ and $s \in \{-\ell, -\ell+1, \dots, \ell\}$, such that the series*

$$X = \sum_{\ell \in \frac{1}{2}\mathbb{N}} \sum_{s=-\ell}^{\ell} X_{\bullet,s}^\ell, \quad X_{\bullet,s}^\ell = \sum_{m=-\ell}^{\ell} a_{m,s}^\ell \phi_{m,s}^\ell,$$

converges almost surely in $L^2(S^3)$. Each of the fields $X_{\bullet,s}^\ell$ is a random harmonic polynomial of degree 2ℓ , i.e. an eigenfunction of Δ_{2S^3} with eigenvalue $\ell(\ell+1)$ and, at the same time, the pull-back of a section of $\mathcal{T}^{\otimes s}$, i.e. a random spin weighted function on S^2 , with spin weight s .

Proof. This is actually a deterministic statement about $L^2(SU(2))$. The first part about the decomposition into $X_{\bullet,s}^\ell$ follows from Proposition 3.2. The fact that each $X_{\bullet,s}^\ell$ is harmonic is proved by Theorem 3.5. Finally, the last sentence is due to the Pull-back Correspondence: Theorem 2.22. \square

To measure the relative magnitude of the component with spin s , in the decomposition we introduce a probability $\mathbb{E}RS[X]$ on $\frac{1}{2}\mathbb{Z}$, see Section 5:

$$\mathbb{E}RS[X](\{s\}) := \sum_{\ell \in \frac{1}{2}\mathbb{N}} \mathbb{E} \left\{ \frac{\|X_{\bullet,s}^\ell\|^2}{\|X\|^2} \right\},^8$$

where $\|\cdot\|$ is the Hilbert norm of $L^2(S^3)$. In particular, given $s \in \frac{1}{2}\mathbb{Z}$, such probability charges the singleton $\{s\}$ if and only if $\sum_{\ell} X_{\bullet,s}^\ell \neq 0$. We stress the fact that $\mathbb{E}RS[X]$ does not depend only on the marginal probabilities of the variables $a_{m,s}^\ell$, but takes into account their joint probability due to the normalization factor $\|X\|^{-2}$.

The first main result of this paper states that the random fields of type $X_{\bullet,s}^\ell$ are the true building blocks of left-invariant random fields.

⁸The notation $\mathbb{E}RS[X]$ stands for the expectation of the right spin measure $RS[X]$, defined by Equation (5.7).

Theorem 1.4. *The field X is left-invariant if and only if the fields $X_{\bullet,s}^\ell$ are jointly left-invariant. The fields $X_{\bullet,s}^\ell$ and $X_{\bullet,s'}^{\ell'}$ are uncorrelated whenever $\ell \neq \ell'$ or $s' \notin \{s, -s\}$ and their correlation structure has to satisfy certain relations, stated in Theorem 5.7. Moreover, for any probability μ on $\{-\ell, \dots, \ell\}$, there exists a left-invariant random field $X = X^\ell$ of degree 2ℓ such that $\mu = \mathbb{E}RS[X]$.*

Proof. The first part of the theorem and the correlation structure is the content of Theorem 5.7. The last statement on the possibility of realizing any probability μ is proved by Theorem 5.13. \square

Remark 1.5. The field X realizing the probability μ in the theorem above is not at all unique. In fact, it can always be constructed in a such a way that $\|X\|^2 = C$ almost surely, see the proof of Theorem 5.13. Non-uniqueness then follows from the fact that there exist left-invariant random fields Y of degree 2ℓ , so that the latter construction produces a field $X \neq Y$ (in law) with $\mathbb{E}RS[X] = \mu = \mathbb{E}RS[Y]$.

On the other hand, the condition of being bi-invariant is stronger and requires an equal presence of all spin weights.

Theorem 1.6. *Any bi-invariant square integrable random field $X = X^\ell: S^3 \rightarrow \mathbb{C}$ of degree $2\ell \in \mathbb{N}$ is a superposition of random spin weighted functions $X_{\bullet,s}^\ell$ of every spin weight $s \in \{-\ell, -\ell+1, \dots, \ell\}$, whose correlation structure has to satisfy the relations of Theorem 5.9. Moreover, each spin weight appears with the same magnitude, meaning that $\mathbb{E}RS[X]$ has to be uniform.*

Proof. Theorem 5.9 gives the first part and 5.15 shows that $\mathbb{E}RS[X]$ is uniform. \square

Remark 1.7. In both the situations of Theorems 1.3 and 1.6 it is possible to have correlation of the components with opposite spin weight $X_{\bullet,s}^\ell$ and $X_{\bullet,-s}^\ell$. For instance, it happens when X^ℓ is as in Example 5.5, because of Theorem 1.11. This reflects the possible correlation of the real and imaginary part of X . In the bi-invariant case, such correlation it is determined by the number

$$(1.4) \quad \mathbb{E} \left\{ \langle \overline{X}^\ell, X^\ell \rangle_{L^2(SU(2))} \right\} \in \mathbb{C},$$

see Theorem 5.9. For instance, if X is circularly symmetric, then the number (1.4) vanishes and thus all of the random spin weighted functions in the decomposition of X given by Theorem 1.6 are uncorrelated.

In particular, a bi-invariant random field of fixed degree X^ℓ cannot be further decomposed into simpler bi-invariant fields, contrary to what happens in the left-invariant case. The explanation of this phenomenon is to be found in the decomposition of the space $L^2(SU(2))$ into irreducible components for the action of $SO(4)$ in comparison to that of $SU(2)$, see Proposition 3.2.

Remark 1.8. From this we see that, if not constantly equal to the zero section, an isotropic random spin function $S^2 \rightarrow \mathcal{T}^{\otimes s}$ is *never* isotropic as a random field in the sense of S^3 , because it has a fixed spin.

We will actually prove stronger statements than the above, see Theorem 5.11 and Theorem 5.14. In particular, the latter implies that any bi-invariant monochromatic random wave $X = X^\ell$ of frequency $\sqrt{\ell(\ell+1)}$ is necessarily supported on the whole eigenspace of Δ_{2S^3} relative to the eigenvalue $\ell(\ell+1)$. More precisely, X^ℓ is a superposition of monochromatic random waves with pure left and right spin, such that any two of them are uncorrelated unless both their right and left spin are opposite. Moreover, each pair

of right and left spin $(m, s) \in \{-\ell, -\ell + 1 \dots \ell\}^2$ appears with the same magnitude, see Theorem 5.14.

~ • ~

A second purpose of this paper is to give an overview of the theory of spin weighted functions and of the special properties of the Wigner functions, trying to gather together various different points of view. This is the content of Sections 2 and 3.

Traditionally, the Wigner functions $D_{m,s}^\ell: S^3 \rightarrow \mathbb{C}$ (see Section 3), are indexed by $(\ell, m, s) \in \Lambda$, where

$$(1.5) \quad \Lambda := \left\{ (\ell, m, s) \in \frac{1}{2}\mathbb{Z}^3 : \ell \pm m, \ell \pm s \in \mathbb{N} \right\}.$$

As mentioned above, the Wigner functions are an orthogonal basis of $L^2(S^3)$ that enjoys many special properties. The most important for us is that for any fixed ℓ they are, at the same time, a basis of an eigenspace of $\Delta_{SU(2)}$ (see equation (1.3)), and the coefficients of an irreducible unitary matrix representation $D^\ell = (D_{m,s}^\ell)_{m,s}$ of $SU(2)$, see [16, Theorem 3.14]. At the same time, $D_{m,s}^\ell$ is a function with pure left and right spin (see Definition 2.8) and an eigenfunctions of the spin Laplacian, i.e. the operator $\bar{\partial}\bar{\partial}$ constructed from the spin raising and spin lowering operators (see [8]), thus by normalizing them in the space $L^2(S^2, \mathcal{T}^{\otimes s})$ one gets the so-called *spin weighted spherical harmonics* $Y_{m,s}^\ell$. In fact, we will show that the following formula holds when $SU(2)$ is endowed with the metric of the sphere of radius 2:

$$(1.6) \quad \Delta_{SU(2)} = \bar{\partial}\bar{\partial} + \frac{d^2}{d\psi^2} - i \frac{d}{d\psi}.$$

By Theorem 1.3, the law of a square integrable random field $X: S^3 \rightarrow \mathbb{C}$ is characterized by the family of random variables $a_{m,s}^\ell \in \mathbb{C}$, called *spectral* or *Fourier coefficients*. In Section 5 we will study the correlation structure of these random variables in the cases of left, right and bi invariance, resulting in Theorems 5.7, 5.8 and 5.9. Theorems 1.3, 1.4 and 1.6 are proved as a consequence of this study. Moreover, we show the following by combining the latter theorems with the results of [1].

Corollary 1.9. *Assume that the coefficients $a_{m,s}^\ell$ of a square integrable random field $X: S^3 \rightarrow \mathbb{C}$ are independent and circularly symmetric. Then, X is bi-invariant if and only if it is complex Gaussian and the variance of $a_{m,s}^\ell$ depends only on ℓ .*

Proof. See Section 5.2. □

The interest of Corollary 1.9 is in the fact that it characterizes the fields X that are a superposition of independent isotropic Gaussian monochromatic random waves in terms of conditions that, apparently, have nothing to do with Gaussianity. Notice that the theorem implies the independence of the real and imaginary parts of $a_{m,s}^\ell$.

Many authors already devoted their attention to the study of isotropic random fields (i.e. left-invariant in the language of the present paper) in terms of the collection of spectral coefficients, see for instance [1–3, 16] and proved equivalent statements to Theorem 5.7. The analogous result, Theorem 5.8, for right-invariance is obtained by the repeating the same arguments changing the roles of s and m and Theorem 5.9 for the case of bi-invariance is obtained by combining the previous two. We will nevertheless show how to obtain such characterization by studying the problem from the more abstract point of view of collections of random vectors $V_i \in \mathbb{C}^{2\ell_i+1}$ that are jointly invariant in law under the action of the collection of matrices D^ℓ . We call such notion *D-invariance* and study it in details in Section 4. This is convenient in that it allows, essentially, to study left

and right-invariance at the same time, thanks to Lemma 5.4. In this context, we observe the following.

Theorem 1.10. *A collection $\mathcal{V} = (V_i)_{i \in I}$ of random vectors $V_i \in \mathbb{C}^{2\ell_i+1}$ is (strongly) D -invariant if and only if*

$$(V_i)_{i \in I} \stackrel{\text{law}}{=} (D^\ell(\gamma)V_i)_{i \in I},$$

where $\gamma \in SU(2)$ is a uniformly distributed random element, independent of \mathcal{V} .

Proof. See Theorem 4.5 in Section 4. \square

The above result holds also for a weaker notion of D -invariance, that will be specified in Section 4. This allows to obtain a characterization of the correlation structure of any D -invariant random vector, see Theorem 4.7. From this, we deduce Theorems 5.7, 5.8 and 5.9. In fact, thanks to Theorem 1.10, the problem is reduced to the computation of the correlation structure of the collection of random matrices $D^\ell(\gamma)$.

Theorem 1.11. *The correlation structure of the collection of random matrices $(D^\ell(\gamma))_{\ell \in \frac{1}{2}\mathbb{N}}$ is described by the following identities.*

$$(1.7) \quad \mathbb{E} \left\{ D_{m,s}^\ell(\gamma) \overline{D_{m',s'}^{\ell'}(\gamma)} \right\} = \frac{\delta_{\ell,\ell'} \delta_{s,s'} \delta_{m,m'}}{2\ell+1};$$

$$(1.8) \quad \mathbb{E} \left\{ D_{m,s}^\ell(\gamma) D_{m',s'}^{\ell'}(\gamma) \right\} = \frac{\delta_{\ell,\ell'} \delta_{-s,s'} \delta_{-m,m'} (-1)^{2\ell-m+s}}{2\ell+1}.$$

Proof. See Section 3.3. \square

We point out that the first set of identities (1.7) is a reformulation of Schur's orthogonality relations and these identities are valid for any irreducible unitary matrix representation D^ℓ , while the second set (1.8) derives from special properties of the Wigner functions, thus they are due also to the choice of basis in which the representation is written.

In the Section 2.4, we review the theory of spin weighted functions, starting from the definition of the spin weight given by Newman and Penrose in their seminal paper [18]. We put a special focus on the Riemannian aspects of the pull-back correspondence, showing that under the hypothesis that the $SU(2) \cong 2S^3$ is considered as a sphere of radius 2, there are Riemannian coverings

$$SU(2) \rightarrow \frac{1}{|s|} S(\mathcal{T}^{\otimes s}) = \left\{ v \in \mathcal{T}^{\otimes s} : \|v\| = \frac{1}{s} \right\},$$

for every $s \in \frac{1}{2}\mathbb{Z} \setminus \{0\}$. Thanks to this, we can compare the spin Laplacian $\bar{\partial}\bar{\partial}$ with the decomposition of the Laplace-Beltrami operator into a vertical and a horizontal parts described in [4], finding that the horizontal Laplacian relative to the Riemannian submersion $2S^3 \cong SU(2) \rightarrow S^2$ is the operator

$$(1.9) \quad \Delta_h = \bar{\partial}\bar{\partial} - i \frac{d}{d\psi};$$

Notice that Δ_h preserves the spaces of spin weighted functions, indeed it was proved by Kuwabara in [13] that the operator $\Delta_h = \Delta_{SU(2)} - \frac{d^2}{d\psi^2}$ is in fact the Bochner Laplacian acting on smooth sections of $\mathcal{T}^{\otimes s}$ and relative to the Chern connection.

Remark 1.12. In particular, $\bar{\partial}\bar{\partial} \neq \Delta_h$ is not the Bochner Laplacian of the Chern connection.

With Proposition 2.3, we will show that taking the sphere of radius 2 is not really a choice, in that the factor 2 is built in the structure of the Hopf fibration $S^3 \rightarrow S^2$, which is a Riemannian submersion if and only if the radius of the first sphere is the double of that of the second. This is not a novelty, but it explains why in the decomposition of the Laplacians, given in (1.6), there are no factors, whereas in the literature, similar equations are written with a factor $\frac{1}{4}$ in front of $\Delta_{S^3} = 4\Delta_{2S^3}$, see [13, equation (5.4)].

In Section 3 we turn our attention to Wigner functions, giving an overview of their properties. In preparing this survey, we noticed a few results in this context, that we weren't able to find in the literature. One is Theorem 1.11 above and the other is a description of the maps

$$(1.10) \quad D_{\bullet,s}^\ell: SU(2) \rightarrow S^{4\ell+1}$$

given by the columns of the Wigner matrices (analogous things can be said about the rows). Let $\ell \in \frac{1}{2}\mathbb{N}$ and let us identify $\mathbb{C}^{2\ell+1}$ with the complex vector space generated by vectors e_s indexed by $s \in \{-\ell, -\ell+1, \dots, \ell\}$. Then the map $D_{\bullet,s}^\ell$ parametrizes the orbit of the element e_s of the canonical basis of $\mathbb{C}^{2\ell+1}$, under the unitary action $D^\ell: SU(2) \rightarrow U(2\ell+1)$. By studying these maps we deduce properties of all the orbits $O^\ell(v) = \{D^\ell(g)v: g \in SU(2)\}$ of the action. We prove the following facts.

Theorem 1.13. *The following holds.*

- (1) *The map (1.10) induces an embedding $\frac{1}{|s|}S(\mathcal{T}^{\otimes s}) \simeq O^\ell(e_s) \subset S^{4\ell+1}$ for all $s \in \{-\ell, \dots, \ell\} \setminus \{0\}$.*
- (2) *For $s = 0$ there are two cases: if $\ell \in 2\mathbb{N}$, then $O^\ell(e_0) \simeq S^1$; while $O^\ell(e_0) \simeq S^2$ otherwise.*
- (3) *For almost every $v \in \mathbb{C}^{2\ell+1}$, the orbit $O^\ell(v)$ is diffeomorphic to $SU(2)$ when $\ell \notin \mathbb{N}$ and to $SO(3)$ when $\ell \in \mathbb{N} \setminus \{0\}$.*
- (4) *Let $v \in \mathbb{C}^{2\ell+1}$ be any vector and let n be the dimension of its orbit $O^\ell(v)$. If $A \subset O^\ell(v)$ is a measurable subset of positive n -dimensional volume, then $\text{span}(A) = \mathbb{C}^{2\ell+1}$.*

Proof. The first three statements are the content of Theorem 3.13, while the fourth and last one is Theorem 3.12. \square

1.1. Acknowledgements. This paper originated from a discussion with Antonio Lerario, Domenico Marinucci and Maurizia Rossi. The author would like to thank them for many valuable suggestions, corrections and remarks. Moreover, the author wishes to thank both anonymous reviewers for their careful reading and their helpful and constructive comments.

2. SPIN BUNDLES

2.1. Preliminary definitions and notations. We will make use of some basic notions in Riemannian Geometry, for which we refer to the book [14]. Given two Riemannian manifolds, we will use the symbol \simeq for diffeomorphisms and the symbol \cong for isometries.

The space of quaternions \mathbb{H} will be represented as the following space of matrices:

$$\mathbb{H} = \left\{ h(\alpha, \beta) := \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \mid \alpha, \beta \in \mathbb{C} \right\} \simeq \mathbb{C}^2 \simeq \{x^0 + y^0 \underline{i} + x^1 \underline{j} + y^1 \underline{k} \mid x^i, y^i \in \mathbb{R}\} \simeq \mathbb{R}^4,$$

so that its algebraic generators $\underline{i}, \underline{j}, \underline{k}$ are defined by the following identity

$$x^0 + y^0 \underline{i} + x^1 \underline{j} + y^1 \underline{k} = (x^0 + y^0 \underline{i}) + (x^1 + y^1 \underline{i}) \underline{j} = \alpha + \beta \underline{j}.$$

For reasons that will become clear later (see Remark 2.1), we will need to put the enlarged metric $\langle \cdot, \cdot \rangle_{\mathbb{H}} = 4\langle \cdot, \cdot \rangle_{\mathbb{C}^2}$ on \mathbb{H} , i.e. the metric defined by the scalar product:

$$\langle h_1, h_2 \rangle_{\mathbb{H}} := 2\Re \operatorname{tr} \left(\overline{h_1}^T \cdot h_2 \right) = \operatorname{tr} \left(\overline{h_1}^T \cdot h_2 + \overline{h_2}^T \cdot h_1 \right).$$

With such representation, we obtain $SU(2) = \{g \in \mathbb{C}^{2 \times 2} : \overline{g}^T g = \mathbb{1}, \det(g) = 1\}$ as the subgroup of length 2 quaternions, i.e. the sphere of radius 2 in $(\mathbb{H}, \langle \cdot, \cdot \rangle_{\mathbb{H}})$:

$$\begin{aligned} SU(2) &= \left\{ h(\alpha, \beta) := \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \mid \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\} \\ &= \{h \in \mathbb{H} : \|h\|_{\mathbb{C}^2} = 1\} = \{h \in \mathbb{H} : \|h\|_{\mathbb{H}} = 2\} \\ &\cong 2S^3. \end{aligned}$$

Remark 2.1. The reason for taking the sphere of radius 2 is that this is the only way to make the Hopf fibration a Riemannian submersion onto the standard round sphere S^2 , see Proposition 2.3.

As it is well known, the Lie group $SU(2)$ is isomorphic to $\operatorname{Spin}(3)$, the universal covering Lie group of $SO(3)$:

$$\begin{aligned} SO(3) &= \{R \in \mathbb{R}^{3 \times 3} : R^T R = \mathbb{1}\} = \{R = (u, v, p) \text{ positive orthonormal basis of } \mathbb{R}^3\} \simeq \\ &\simeq \{(v, p) \in TS^2 : |v| = 1\} \simeq \mathbb{RP}^3. \end{aligned}$$

Of course, $SO(3)$ is also identified with the group of all Riemannian isometries of the round sphere $S^2 \cong \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. Points in the Riemann sphere \mathbb{CP}^1 will be denoted as

$$[z_0 : z_1] = \frac{z_0}{z_1} = \zeta = \frac{1}{\eta} \in \mathbb{CP}^1,$$

where $(z_0, z_1) \in \mathbb{C}^2 \setminus \{0\}$, $\zeta, \eta \in \mathbb{C} \cup \{\infty\}$. In particular,

$$[1 : 0] = \frac{1}{0} = \infty \quad \text{and} \quad [0 : 1] = \frac{0}{1} = 0.$$

As a Riemannian manifold, the Riemann sphere \mathbb{CP}^1 , with the Fubini-Study metric is isometric⁹ to the standard round sphere $S^2 = \{p \in \mathbb{R}^3 : |p| = 1\}$ via the bijection $\Phi : S^2 \rightarrow \mathbb{CP}^1$ defined by the stereographic projections from the north ($e_3 \mapsto \infty$) pole, to the equatorial plane:

(2.1)

$$S^2 \ni p(\varphi, \theta) = \begin{pmatrix} x \\ y \\ t \end{pmatrix} \stackrel{10}{=} \begin{pmatrix} \cos(\varphi) \sin(\theta) \\ \sin(\varphi) \sin(\theta) \\ \cos(\theta) \end{pmatrix} \mapsto \begin{aligned} \zeta &= \frac{x + iy}{1 - t} = \cot\left(\frac{\theta}{2}\right) e^{i\varphi} \\ \eta &= \frac{x - iy}{1 + t} = \tan\left(\frac{\theta}{2}\right) e^{-i\varphi} \end{aligned} \in \mathbb{CP}^1.$$

Notice that

$$\Phi(e_1) = 1; \quad \Phi(e_2) = i; \quad \Phi(e_3) = \infty; \quad \Phi(-e_3) = 0.$$

Remark 2.2. As usual, the orientation of S^2 is defined by considering e_1, e_2 to be a positive frame in $T_{e_3} S^2$, i.e. taking the point of view of a *polar bear* who's looking down at the north pole below him. The stereographic projection defined in (2.1) defines the

⁹The Fubini-Study metric on \mathbb{CP}^1 appears in the literature with various different normalizations, which differ by a constant factor. We take the normalization such that the volume of \mathbb{CP}^1 is 4π .

¹⁰We choose the letter t instead of z for the vertical coordinate in order to stress the fact that $t \in \mathbb{R}$ and not in \mathbb{C} .

opposite orientation, the one for which the same frame $\{e_1, e_2\}$ is a positive basis of $T_{-e_3}S^2$. Indeed, for points $\Phi(x, y, t) = \zeta$ near $\Phi(-e_3) = 0 \in \mathbb{C}$ we have

$$\zeta = x + iy + O(|\zeta|^2).$$

In other words, S^2 is oriented from the point of view of a *hamster* who's looking down while running (or not) inside a hollow sphere. The hamster and the polar bear measure the same angle between a given pair of tangent vectors, but with opposite sign. In this paper we want S^2 to have the *polar bear* orientation and a coherent complex structure, meaning that for any non-zero tangent vector $v \in T_p S^2$, the basis (v, iv) should be positively oriented. At the same time we choose to use the stereographic projection in the form defined in (2.1), because this is the most frequent form in the literature, see [8]. For this reason, we regard Φ as an antiholomorphic map. In other words, we have a biholomorphism:

$$\Phi: S^2 \rightarrow \overline{\mathbb{CP}^1},$$

where $\overline{\mathbb{CP}^1}$ is the complex manifold $(\mathbb{CP}^1, -J)$, where J denotes the standard complex structure on \mathbb{CP}^1 .

2.2. Riemannian submersions. To have a clear view of the double covering map $SU(2) \rightarrow SO(3)$, let us fix an isometric action of $SU(2)$ on \mathbb{CP}^1 . Since \mathbb{CP}^1 is the space of all complex lines $\ell \subset \mathbb{C}^2$ and \mathbb{H} acts on \mathbb{C}^2 via \mathbb{C} -linear automorphisms, there is an obvious action of $h \in \mathbb{H}$ on \mathbb{CP}^1 , defined by $\ell \mapsto h(\ell)$. In terms of our chosen coordinates, the action of $h = h(\alpha, \beta)$ is expressed by a Möbius transformation:

$$M(h) = M(\alpha, \beta) : \frac{z_0}{z_1} \mapsto \frac{\alpha z_0 - \bar{\beta} z_1}{\beta z_0 + \bar{\alpha} z_1}.$$

A standard exercise (left to the reader) is to prove that such diffeomorphism is an isometry precisely when $|\alpha|^2 + |\beta|^2 = 1$. Therefore, the above expression determines uniquely a homomorphism $M: SU(2) \rightarrow SO(3)$. Since its kernel is $\{\pm \mathbb{1}_2\}$ and $SU(2) \cong S^3$ is simply connected, the map M is the universal cover of $SO(3)$, hence it is equivalent to $\text{Spin}(3) \rightarrow SO(3)$.

The action of $SO(3)$ on the sphere is transitive, so that, choosing to view the sphere as the orbit of the point $\infty \in \mathbb{CP}^1$, we get two compatible principal circle bundles over S^2 :

$$(2.2) \quad \begin{array}{ccccc} 2S^1 & \hookrightarrow & SU(2) & \xrightarrow{\cdot \infty} & \overline{\mathbb{CP}^1} \\ \downarrow :2 & & \downarrow M & & \downarrow \Phi^{-1} \\ S^1 & \hookrightarrow & SO(3) & \xrightarrow{\cdot e_3} & S^2 \end{array}$$

where $SU(2) \rightarrow S^2$ is the map $g \mapsto M(g)\infty$, corresponding, via Φ , to the Hopf fibration $S^3 = \{(\alpha, \beta) \in \mathbb{C}^2 : |\alpha|^2 + |\beta|^2 = 1\} \rightarrow \mathbb{CP}^1$:

$$(2.3) \quad h(\alpha, \beta) \mapsto \zeta = \frac{\alpha}{\beta},$$

while $SO(3) \rightarrow S^2$ is given by the action on the north pole $e_3 = \Phi^{-1}(\infty)$, i.e. the map $R \mapsto Re_3$.

Let the space \mathbb{R}^{N^2} be identified with the space $\mathbb{R}^{N \times N}$ of square matrices of order N . Then it is easy to see that the Euclidean metric can be written as

$$g_{\mathbb{R}^{N^2}} \langle A, B \rangle := \text{tr}(A^T B).$$

This metric is invariant under left and right multiplication by any matrix $g = g^{-T} \in SO(N)$, in that the trace is invariant under conjugation:

$$\mathrm{tr}((g_1 A g_2)^T (g_1 B g_2)) = \mathrm{tr}(g_2^{-1} A^T B g_2) = \mathrm{tr}(A^T B).$$

From this, it follows that its restriction to any subgroup $G \subset SO(N)$ is a bi-invariant metric. In particular, the metric induced by the inclusions $SO(3) \subset \mathbb{R}^9$ and $SU(2) \subset SO(4) \subset \mathbb{R}^{16}$ are bi-invariant. We also get an inclusion $\mathbb{R}^8 = \mathbb{C}^{2 \times 2} \subset \mathbb{R}^{4 \times 4}$ by identifying \mathbb{C} -linear endomorphisms of \mathbb{C}^2 with the set of \mathbb{R} -linear endomorphisms of \mathbb{R}^4 that satisfy the Cauchy-Riemann equations. Since this inclusion has a diagonal form, the induced metric on \mathbb{R}^8 is doubled: $g_{\mathbb{R}^{16}} = 2g_{\mathbb{R}^8}$. The same happens with the inclusion $\mathbb{R}^4 \simeq \mathbb{H} \subset \mathbb{C}^{2 \times 2}$, so that

$$g_{\mathbb{R}^{16}} = 2g_{\mathbb{R}^8} = 4g_{\mathbb{R}^4}.$$

From this we see that the metric induced on $SU(2)$ by the inclusion in $SO(4) \subset \mathbb{R}^{16}$ corresponds to the round metric of a sphere of radius 2. It turns out that such metric is the only one for which the Hopf fibration $\cdot \infty: SU(2) \rightarrow S^2$ is a Riemannian submersion.

Proposition 2.3. *Assume that \mathbb{CP}^1 is given the round metric of radius r , namely $\mathbb{CP}^1 \cong rS^2$. There is a unique choice of bi-invariant Riemannian metrics on $SU(2)$ and $SO(3)$ such that all the labeled maps $(:2, M, \cdot \infty, \cdot e_3, \Phi^{-1})$ in the diagram (2.2) are Riemannian submersions. Such choice is $2rS^3 \cong SU(2) \subset (\mathbb{C}^2, 4r^2 g_{\mathbb{R}^4})$ and $SO(3) \subset (\mathbb{R}^{3 \times 3}, \frac{r^2}{2} g_{\mathbb{R}^9})$. In particular the lengths of the fibers are $4\pi r$ and $2\pi r$.*

Proof. The metric on $SO(3)$ defined by the inclusion $SO(3) \subset \mathbb{R}^{3 \times 3}$ (with its standard metric) is bi-invariant. Choosing an orthonormal basis of $T_1 SO(3)$ we get an identification of the Lie algebra $T_1 SO(3) \cong \mathbb{R}^3$, such that the adjoint action of $SO(3)$ on $T_1 SO(3)$ is isometric and, moreover, it is given by the identity: $SO(3) \rightarrow SO(\mathbb{R}^3)$. Any bi-invariant metric on $SO(3)$ is thus defined by a metric on \mathbb{R}^3 which is invariant by the action of $SO(3)$, but there is only one such metric, up to a constant factor. The map $SU(2) \rightarrow SO(3)$ is a Riemannian submersion if and only if it is a local isometry. This implies that the metric on $SO(3)$ defines uniquely the metric on $SU(2)$ and vice versa. Moreover, the lifted metric on $SU(2)$ is bi-invariant and the same argument implies that it is the only bi-invariant metric on $SU(2)$, up to a constant factor. Since the round metric, obtained from the inclusion $SU(2) \subset \mathbb{R}^4$ is bi-invariant, it follows that there is a unique choice, corresponding to $SU(2) \cong aS^3$ for some a that can be determined by a computation of the volumes. Indeed, since the map $SU(2) \rightarrow \mathbb{CP}^1$ is a Riemannian submersion, by using the coarea formula we deduce that

$$2\pi^2 a^3 = \mathrm{vol}(aS^3) = (2\pi a) \cdot (4\pi r^2),$$

from which we obtain $a = 2r$. Here, we are also using the fact that the fibers of the Hopf fibration $SU(2) \cong aS^3 \rightarrow \mathbb{CP}^1$ are geodesic circles and thus have length $2\pi a$. Finally, we get the normalization of the metric on $SO(3)$ by observing that the fibers of the map $\cdot e_3: SO(3) \rightarrow S^2$ must have length $2r\pi$, given that the leftmost vertical map in the diagram (2.2) is a Riemannian submersion. Thus the following tangent vector v must have length r , being a generator with “period” 2π of the fiber over $e_3 \in S^2$:

$$v = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T_1 SO(3) \subset \mathbb{R}^9.$$

□

A similar reasoning can be applied to define the orientation of $SU(2)$ and $SO(3)$, by observing that taking a point $p \in S^2$, the fiber of over p of both projections can be canonically embedded as a small loop around p and thus defines an orientation for S^2 .

Proposition 2.4. *There is a unique choice of orientations on $SU(2)$ and $SO(3)$ such that all the vertical maps are, locally, orientation preserving and such that the horizontal sequences define the standard polar bear orientation on S^2 .*

In the following, we will always consider $SU(2)$, $SO(3)$ and S^2 with the orientation provided by the above proposition, which we will keep referring to as the *polar bear orientation*.

2.3. Euler angles. All the matrices $h(\alpha, \beta) \in SU(2)$ can be written in terms of Euler angles, via the following surjective parametrization:

$$\alpha = \cos\left(\frac{\theta}{2}\right) e^{i\frac{\varphi}{2}} e^{i\frac{\psi}{2}}; \quad \beta = \sin\left(\frac{\theta}{2}\right) e^{-i\frac{\varphi}{2}} e^{i\frac{\psi}{2}},$$

with $\varphi \in [0, 2\pi]$, $\theta \in [0, \pi]$ and $\psi \in [-2\pi, 2\pi]$. Notice that then φ, θ are precisely the polar coordinates of the point $p(\varphi, \theta) = g(\varphi, \theta, \psi)\infty = R(\varphi, \theta, \psi)e_3$ (see the identities (2.1)).

$$(2.4) \quad g(\varphi, \theta, \psi) := \begin{pmatrix} e^{i\frac{\varphi}{2}} & 0 \\ 0 & e^{-i\frac{\varphi}{2}} \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix} \begin{pmatrix} e^{i\frac{\psi}{2}} & 0 \\ 0 & e^{-i\frac{\psi}{2}} \end{pmatrix} \\ =: g_3(\varphi)g_2(\theta)g_3(\psi).$$

Remark 2.5. Such notation for the Euler angle in $SU(2)$ is different to that of [16, Sec 3.2]. More precisely, one is obtained from the other after the transformation

$$\beta \mapsto -\bar{\beta} \quad \text{i.e.} \quad \theta \mapsto -\theta.$$

We made this change in order to have an elegant formula (2.3) for the Hopf fibration $SU(2) \rightarrow \mathbb{CP}^1$. Nevertheless, the convention Euler angles φ, θ, ψ for $SO(3)$ is the same as in the book [16], namely the so-called *zyz* convention.

The image of $g_3(\psi)$ and $g_2(\theta)$ under the quotient $SU(2) \rightarrow SO(3)$ are the matrices of the standard rotations around e_3 and e_2 :

Proposition 2.6.

$$g_3(\psi) \mapsto R_3(\psi) = \begin{pmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad g_2(\theta) \mapsto R_2(\theta) = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}.$$

Therefore

$$g(\varphi, \theta, \psi) \mapsto R(\varphi, \theta, \psi) := R_3(\varphi)R_2(\theta)R_3(\psi).$$

Proof. It is sufficient to check that $g_2(\frac{\pi}{2})\infty = 1$ and $g_2(\theta)i = i$. Moreover, $g_3(\psi)\infty = \infty$ and $g_3(\frac{\pi}{2})1 = i$. \square

To get a surjective parametrization of $SO(3)$, it is sufficient to take all matrices of the form $R(\varphi, \theta, \psi)$, with $\varphi \in [0, 2\pi]$, $\theta \in [0, \pi]$, $\psi \in [0, 2\pi]$. Indeed $R(\varphi, \theta, \psi) = R(\varphi, \theta, \psi + 2\pi)$, while $g(\varphi, \theta, \psi) = -g(\varphi, \theta, \psi + 2\pi)$.

Remark 2.7. The principal bundle structure in (2.2) is the one defined by the right multiplication by matrices of the form $g_3(\psi)$ and $R_3(\psi)$. From this point of view, the leftmost vertical map in the diagram corresponds to the double covering

$$\mathbb{R}/4\pi\mathbb{Z} \rightarrow \mathbb{R}/2\pi\mathbb{Z}.$$

2.4. Definition of the Spin-weighted functions. Newman and Penrose define the spin weight as follows [18]: *a quantity u defined on S^2 has spin weight s if, whenever a tangent vector ρ at any point x on the sphere transforms under coordinate change by $\rho' = e^{i\psi}\rho$, then the quantity at this point x transforms by $u' = e^{is\psi}u$.*

In [8] the authors introduce the mathematical model for spin weighted functions, viewing them as sections of complex line bundles on S^2 . Similar approaches have been taken in [2, 15]. From the statement of Newman and Penrose, it is immediately clear that a function on S^2 with spin weight equal to 1 should be a section of the bundle $\mathcal{T}^{\otimes 1} := TS^2$ endowed with its standard complex structure. Moreover, it is also clear that a spin s function is a section of $\mathcal{T}^{\otimes s} = \mathcal{T}^{\otimes 1} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathcal{T}^{\otimes 1}$ (s times), because the transition functions of the latter bundle are, by definition, the s^{th} power of those of $\mathcal{T}^{\otimes 1}$. This is true for all $s \geq 0$. When $s \leq 0$ we can argue in the same way, after noticing that the transition functions for $\mathcal{T}^{\otimes -1}$ are the inverse of those of $\mathcal{T}^{\otimes 1}$, thus $\mathcal{T}^{\otimes -1} = (TS^2)^*$ is the dual (i.e. the inverse in the group of all line bundles) of TS^2 .

Definition 2.8. Let $s \geq 0$, we define the bundles:

$$\mathcal{T}^{\otimes s} := (TS^2)^{\otimes s} = TS^2 \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} TS^2, \quad s \text{ times};$$

$$\mathcal{T}^{\otimes -s} := (TS^2)^{\otimes -s} = ((TS^2)^*)^{\otimes s} = (TS^2)^* \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} (TS^2)^*, \quad s \text{ times}.$$

Remark 2.9. Complex line bundles on the sphere are classified by their Chern class c_1 , that is an element of the second cohomology group: $c_1 \in H^2(S^2, \mathbb{Z}) \cong \mathbb{Z}$, or equivalently by their Euler characteristic χ , when thought as real oriented rank 2 vector bundles. They are related by the cap product $c_1 \frown [S^2] = \chi$, see [10, 11]. In fact, it is well known that in general the set of isomorphism classes of smooth complex line bundles form an abelian group in which the opposite element of L is L^* . In the case of the sphere (or any compact oriented surface), the group is isomorphic to \mathbb{Z} and the isomorphism is given exactly by the Euler characteristic.

From Remark 2.9 we see that Definition 2.8 can be extended to all (and not more) $s \in \frac{1}{2}\mathbb{Z} = \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \dots$

Definition 2.10. For any $s \in \frac{1}{2}\mathbb{Z}$, we define the *spin s bundle* $\mathcal{T}^{\otimes s}$ to be the complex line bundle on S^2 having Euler characteristic

$$\chi(\mathcal{T}^{\otimes s}) = 2s.$$

In terms of the Riemann sphere \mathbb{CP}^1 , we have that the holomorphic line bundle $\mathcal{O}(2s) \rightarrow \mathbb{CP}^1$ has Euler characteristic $\chi(\mathcal{O}(2s)) = 2s$. Therefore the spin s bundle $\mathcal{T}^{\otimes s}$ must be isomorphic to the smooth complex line bundle underlying $\Phi^*\mathcal{O}(-2s)$ because Φ reverses the orientation, or, equivalently, to $\Phi^*\overline{\mathcal{O}(2s)}$. The latter carries a canonical structure of holomorphic¹¹ line bundle over the complex manifold S^2 . Moreover, since there are no other holomorphic vector bundles over \mathbb{CP}^1 , the Euler characteristic uniquely determines also a holomorphic structure on $\mathcal{T}^{\otimes s}$.

Technically speaking, Definition 2.10 defines the spin s line bundle only up to isomorphisms, both in the category of smooth complex line bundle and in that of the holomorphic ones. In the rest of the paper we will keep the same notation $\mathcal{T}^{\otimes s}$ to denote a very precise holomorphic line bundle.

Definition 2.11. We define the holomorphic line bundle $\mathcal{T}^{\otimes s} \rightarrow S^2$ to the pull-back via the stereographic projection $\Phi: S^2 \rightarrow \mathbb{CP}^1$ of the holomorphic line bundle $\overline{\mathcal{O}(2s)} \rightarrow \mathbb{CP}^1$:

$$\mathcal{T}^{\otimes s} := \Phi^*\overline{\mathcal{O}(2s)}, \quad \forall s \in \frac{1}{2}\mathbb{Z}.$$

¹¹A complex vector bundle $E \rightarrow S$ over a complex manifold S is holomorphic if and only if the total space E is a complex manifold and the projection map $E \rightarrow S$ is holomorphic, see [11].

Proposition 2.12. *Let $s \in \frac{1}{2}\mathbb{N}$, then the spin s bundle is the $2s$ tensor power of the spin $\frac{1}{2}$ bundle.*

$$\begin{aligned}\mathcal{T}^{\otimes s} &= \left(\mathcal{T}^{\otimes \frac{1}{2}}\right)^{\otimes 2s}, \\ \mathcal{T}^{\otimes -s} &= \left(\mathcal{T}^{\otimes \frac{1}{2}}\right)^{\otimes (-2s)} = \left(\left(\mathcal{T}^{\otimes \frac{1}{2}}\right)^*\right)^{\otimes 2s} = \left(\mathcal{T}^{\otimes -\frac{1}{2}}\right)^{\otimes 2s}.\end{aligned}$$

2.5. $SU(2)$ is the radius 2 sphere bundle of $\mathcal{T}^{\otimes \frac{1}{2}}$.

Definition 2.13. Let $L \rightarrow M$ be a complex line bundle, endowed with an Hermitian norm $\|\cdot\|$. Then its radius $r > 0$ sphere bundle is the circle bundle $rS(L) \rightarrow M$

$$rS(L) := \{v \in L : \|v\| = r\}.$$

Remark 2.14. If L is a holomorphic line bundle over a Kähler Riemann surface, then the total space $rS(L)$ inherits a Riemannian metric, via the Chern connection of L , which makes $S(L) \rightarrow M$ a Riemannian submersion.

The complex line bundle $\mathcal{O}(-1) \rightarrow \mathbb{CP}^1$ is defined as

$$\mathcal{O}(-1) = \{(\ell, h) \in \mathbb{CP}^1 \times \mathbb{C}^2 : h \in \ell\} \xrightarrow{\pi_1} \mathbb{CP}^1.$$

Since the fiber over $\ell \in \mathbb{CP}^1$ is $\ell \subset \mathbb{C}^2$, the bundle $\mathcal{O}(-1)$ is called the *tautological* bundle. Notice also that by restricting to the complement of the zero section $\mathbb{CP}^1 \times \{0\} \subset \mathcal{O}(-1)$, we get a tautological diffeomorphism

$$\begin{aligned}(2.5) \quad \tau : \mathbb{H} \setminus \{0\} &\xrightarrow{\cong} \mathcal{O}(-1) \setminus \mathbb{CP}^1 \times \{0\} \\ h = h(\alpha, \beta) &\mapsto \tau_h := \left(\frac{\alpha}{\beta}, \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right).\end{aligned}$$

In particular, this map transports the metric $g_{\mathbb{H}}$ into an Hermitian bundle metric on $\mathcal{O}(-1)$, such that the restriction of τ to $SU(2) = 2S^3 \subset \mathbb{H}$, is an isomorphism of principal $2S^1$ bundles. Thus, we deduce that according to the *polar bear* orientation on $SU(2)$, the map τ descends to an orientation preserving isometry of the total spaces:

$$\tau : SU(2) \rightarrow 2S(\mathcal{O}(-1)) = \{(\ell, h) \in \mathcal{O}(-1) : |h| = 2\}.$$

Forgetting about the holomorphic structure, we have that $\overline{\mathcal{O}(-1)} \cong \mathcal{O}(1)$ as smooth complex hermitian line bundles over \mathbb{CP}^1 , so that $SU(2) = 2S(\mathcal{O}(-1))$ and $2S(\overline{\mathcal{O}(1)})$ are isomorphic as oriented Riemannian manifolds, and the same for $2S(\overline{\mathcal{O}(1)})$ and $2S(\phi^*\mathcal{O}(-1))$. Remembering Definition 2.11, we proved the following fact.

Proposition 2.15. *There is an isomorphism of oriented Riemannian circle bundles*

$$SU(2) \cong 2S\left(\mathcal{T}^{\otimes \frac{1}{2}}\right) \rightarrow S^2,$$

where the projection map $SU(2) \rightarrow S^2$ is given by the action on the north pole $e_3 = \Phi^{-1}(\infty) \in S^2$, as in the diagram (2.2).

Remark 2.16. The total space of $\mathcal{O}(-1)$ is, by definition, the Kähler manifold obtained as the blow-up of \mathbb{C}^2 at the point 0. It can be seen that then, in the category of smooth manifolds, $\mathcal{O}(-1)$ is diffeomorphic to $\mathbb{CP}^2 \setminus \{pt\}$.

2.6. Spin weighted functions. Let $p \in \mathbb{CP}^1$ and $v \in \mathcal{T}_p^{\otimes \frac{1}{2}} \setminus \{0\}$, then the fiber over p of $\mathcal{T}^{\otimes s}$ is

$$\mathcal{T}_p^{\otimes s} = \left\{ \sum_i v_1^i \otimes \cdots \otimes v_{2s}^i, \text{ s.t. } v_j^i \in \mathcal{T}_p^{\otimes \frac{1}{2}} \right\} = \{z \cdot v \otimes \cdots \otimes v : z \in \mathbb{C}\}.$$

When v changes: $v' = wv$, the vector $v^{\otimes 2s} = v \otimes \cdots \otimes v$ changes accordingly to:

$$(v')^{\otimes 2s} = w^{2s} v^{\otimes 2s}.$$

When $s < 0$, the above description still makes sense, if $v^{\otimes (-2s)} \in \mathcal{T}^{\otimes -\frac{1}{2}}$ is defined as the linear form $\mathcal{T}^{\otimes s} \rightarrow \mathbb{C}$ such that $\langle v^{\otimes (-1)}, v \rangle = 1$, with respect to the duality pairing $\mathcal{T}^{\otimes -s} = \mathcal{T}^{\otimes s*}$.

Remark 2.17. Notice that the coordinates of an element $\tau = zv^{\otimes 2s} = z'(v')^{\otimes 2s} \in \mathcal{T}^{\otimes s}$ have spin weight $= -s$:

$$z' = w^{-2s} z,$$

indeed “the coordinates of vectors are covectors, hence they belong to the dual bundle”.

Remark 2.18. In the book [16, p. 287] we see that the transition functions for the bundle $\mathcal{T}^{\otimes s}$ are

$$(2.6) \quad f_{R_2}(x) = \exp(is\psi_{R_2 R_1}) f_{R_1}(x),$$

where $\psi_{R_1 R_2}$ is the angle between $\frac{\partial}{\partial \varphi_{R_1}}$ and $\frac{\partial}{\partial \varphi_{R_2}}$, measured in the usual way, from the outside of the sphere, see Remark 2.2.

$$\frac{\partial}{\partial \varphi_{R_2}} = e^{-i\psi_{R_2 R_1}} \frac{\partial}{\partial \varphi_{R_1}}.$$

Therefore the rule (2.6) is equivalent to the transition rule for $(TS^2)^{\otimes s} = \mathcal{T}^{\otimes s}$, for any $s \in \mathbb{N}$:

$$f_{R_2}(x) \left(\frac{\partial}{\partial \varphi_{R_2}} \right)^{\otimes s} = f_{R_1}(x) \left(\frac{\partial}{\partial \varphi_{R_1}} \right)^{\otimes s}.$$

Definition 2.19. We define a Hermitian bundle metric on $\mathcal{T}^{\otimes s}$ such that for any $g \in SU(2)$ we have

$$\|(\tau_g)^{\otimes 2s}\| = \frac{1}{|s|};$$

for any $s \in \frac{1}{2}\mathbb{Z} \setminus \{0\}$, where $\tau: SU(2) \rightarrow \mathcal{T}^{\otimes \frac{1}{2}}$ is defined as in (2.5).

In this way, the map $\tau^{\otimes 2s}$ is a $2|s|$ -fold covering of the radius $\frac{1}{|s|}$ circle bundle of $\mathcal{T}^{\otimes s}$.

$$(2.7) \quad \tau^{\otimes 2s}: SU(2) \rightarrow \frac{1}{|s|} S(\mathcal{T}^{\otimes s}) = \left\{ v \in \mathcal{T}^{\otimes s} : \|v\| = \frac{1}{|s|} \right\}.$$

The above map is a principal bundle with respect to the right action of the cyclic subgroup of order $2|s|$ generated by the element $g_3\left(\frac{2\pi}{s}\right) \in SU(2)$, which is the finite group

$$\begin{aligned} \sqrt[2s]{1} &:= \{h \in \mathbb{H} : h^{2s} = 1\} \\ &= \left\{ g_3\left(\frac{2\pi}{s}\right), g_3\left(\frac{2\pi}{s}2\right), \dots, g_3\left(\frac{2\pi}{s}(2|s|-1)\right), g_3\left(\frac{2\pi}{s}2|s|\right) \right\} \cong \mathbb{Z}_{2|s|.} \end{aligned}$$

By declaring the map (2.7) to be a Riemannian covering (i.e. a covering that is also a Riemannian submersion), we can define a metric on the total space of $\frac{1}{|s|}S(\mathcal{T}^{\otimes s})$. We define the resulting Riemannian manifold as

$$SU(2)^{\otimes 2s} := SU(2)/\sqrt[2s]{1} \cong \frac{1}{|s|}S(\mathcal{T}^{\otimes s}).$$

Remark 2.20. As Riemannian manifolds, there is no difference between $SU(2)^{\otimes 2s}$ and $SU(2)^{\otimes -2s}$, but they have opposite orientations, since they are the total spaces of a pair of circle bundles over S^2 that are dual to each other.

Remark 2.21. By definition, $SU(2)^{\otimes 2s} \rightarrow \mathbb{CP}^1$ is a Riemannian circle bundle having fibers of length $\frac{2\pi}{|s|}$. In particular $SU(2)^{\otimes 2} \cong SO(3)$.

We can in fact extend the commutative diagram in (2.2) to every $s \in \frac{1}{2}\mathbb{Z} \setminus \{0\}$:

$$\begin{array}{ccccc} 2S^1 & \hookrightarrow & SU(2) & \xrightarrow{\cdot\infty} & \overline{\mathbb{CP}^1} \\ \downarrow \cdot s & & \downarrow \cdot 2s & & \downarrow \cong \\ \frac{1}{s}S^1 & \hookrightarrow & SU(2)^{\otimes 2s} & \longrightarrow & S^2 \end{array}$$

Let $\sigma: S^2 \rightarrow \mathcal{T}^{\otimes s}$ be a section. Then, obviously, for any point $p \in S^2$ and $v \in \mathcal{T}^{\otimes \frac{1}{2}}$, we have

$$\sigma(p) = z_\sigma(p, v)v^{\otimes 2s},$$

for some $z_\sigma(p, v) \in \mathbb{C}$. A convenient way to understand this $z_\sigma(p, v)$ is to observe that for any such p, v there exists a unique $g \in SU(2)$ such that $g \cdot \infty = p$ and $\tau_g = v$. It follows that the section $\sigma: S^2 \rightarrow \mathcal{T}^{\otimes s}$ is uniquely determined by a function $F_\sigma: SU(2) \rightarrow \mathbb{C}$ such that

$$(2.8) \quad \sigma(g \cdot \infty) = F_\sigma(g) (\tau_g)^{\otimes 2s}.$$

It is easy to see that a function $F: SU(2) \rightarrow \mathbb{C}$ is associated with a section σ of $\mathcal{T}^{\otimes s}$ if and only if

$$(2.9) \quad F(g \cdot g_3(\psi)) = F(g)e^{-is\psi},$$

for any $\psi \in \mathbb{R}$. Thus, we have the following well known characterization of spin weighted functions, see [2, 7].

Theorem 2.22 (Pull-back Correspondence). *Sections of $\mathcal{T}^{\otimes s}$ are in bijections with functions $F: SU(2) \rightarrow \mathbb{C}$ that satisfy the rule (2.9), via the identity (2.8).*

Definition 2.23. We say that F_σ is the *pull-back* of σ (see [2]) and that F has *right spin* $= -s$.

Remark 2.24. This change of sign in the spin weight is explained by the fact that $F_\sigma(g)$ is actually a function that expresses the *coordinates* (see Remark 2.17) of σ in the trivialization of the bundle $\mathcal{T}^{\otimes s}$ determined by g .

Corollary 2.25. *Sections of $\mathcal{T}^{\otimes s}$ are (particular) functions on $SU(2)^{\otimes 2s}$.*

Definition 2.26. We denote the set of all smooth functions on $SU(2)$ with *right spin* $= s \in \frac{1}{2}\mathbb{Z}$ as $\mathcal{R}(s)$.

$$\mathcal{R}(s) := \{F \in \mathcal{C}^\infty(SU(2)) : F(g \cdot g_3(\psi)) = F(g)e^{is\psi}\}.$$

Similarly, the set of functions with *left spin* $= m \in \frac{1}{2}\mathbb{Z}$ is

$$\mathcal{L}(m) := \{F \in \mathcal{C}^\infty(SU(2)) : F(g_3(\psi) \cdot g) = e^{im\psi}F(g)\}.$$

We say that a function $F: SU(2) \rightarrow \mathbb{C}$ has *pure left spin* or *pure right spin* if it belongs to some of the spaces $\mathcal{L}(m)$ or $\mathcal{R}(s)$, for some m, s . We say that F has *pure spin* if it has both pure left spin and pure right spin.

Theorem 2.22 says that

$$\mathcal{R}(-s) = \mathcal{C}^\infty \left(S^2 \Big| \mathcal{T}^{\otimes s} \right).$$

Remark 2.27. In terms of the coordinates α, β , the left and right multiplication by $g_3(\psi)$ are given by the following identities:

$$h(\alpha, \beta) \cdot g_3(\psi) = h(\alpha e^{i\frac{\psi}{2}}, \beta e^{i\frac{\psi}{2}}); \quad g_3(\psi) \cdot h(\alpha, \beta) = h(\alpha e^{i\frac{\psi}{2}}, \beta e^{-i\frac{\psi}{2}}).$$

3. WIGNER FUNCTIONS

For any $\ell \in \frac{1}{2}\mathbb{N}$, consider the vector space $\mathcal{H}_\ell = \mathbb{C}[z_0, z_1]_{(2\ell)}$ as a subspace of $L^2(SU(2))$ of complex dimension $2\ell + 1$. The resulting Hilbert product on \mathcal{H}_ℓ corresponds (up to a constant factor) with the Bombieri-Weyl product, for which an orthonormal basis is given by the rescaled monomials: for any $z = h(z_0, z_1) \in SU(2)$,

$$\psi_m^\ell(z) := \binom{2\ell}{\ell+m}^{\frac{1}{2}} z_0^{\ell+m} z_1^{\ell-m}, \quad m = -\ell, \ell+1, \dots, \ell \in \frac{1}{2}\mathbb{Z}.$$

Then we define the extended Wigner function $\hat{D}_{m,s}^\ell: \mathbb{H} \rightarrow \mathbb{C}$ on any matrix $h = h(\alpha, \beta) \in \mathbb{H}$ by

$$(3.1) \quad \sum_{m=-\ell}^{\ell} \psi_m^\ell(z) \hat{D}_{m,s}^\ell(h) = \psi_s^\ell \left(h^{-1} \begin{pmatrix} z_0 & -\bar{z}_1 \\ z_1 & \bar{z}_0 \end{pmatrix} \right) = (\bar{\alpha}z_0 + \bar{\beta}z_1)^{\ell+s} (-\beta z_0 + \alpha z_1)^{\ell-s} \binom{2\ell}{\ell-s}^{\frac{1}{2}}.$$

The (standard) Wigner function is the restriction to $SU(2)$:

$$\hat{D}_{m,s}^\ell|_{SU(2)} = D_{m,s}^\ell: SU(2) \rightarrow \mathbb{C},$$

$$D^\ell: SU(2) \rightarrow U(2\ell+1); \quad D^\ell(g) = (D_{m,s}^\ell(g))_{-\ell, \leq m, s \leq \ell}.$$

We will also be interested in the columns and rows of D^ℓ , which we will denote as follows.

$$D_{\bullet,s}^\ell, D_{m,\bullet}^\ell: SU(2) \rightarrow S^{2\ell} \subset \mathbb{C}^{2\ell+1}.$$

Moreover, we define the spaces

$$\mathcal{D}_{\bullet,s}^\ell := \text{span}_{\mathbb{C}} \{ D_{m,s}^\ell: m = -\ell, \dots, \ell \} \subset \mathcal{C}^\infty(SU(2), \mathbb{C});$$

$$\mathcal{D}_{m,\bullet}^\ell := \text{span}_{\mathbb{C}} \{ D_{m,s}^\ell: s = -\ell, \dots, \ell \} \subset \mathcal{C}^\infty(SU(2), \mathbb{C});$$

$$\mathcal{D}^\ell := \mathcal{D}_{\bullet,-\ell}^\ell + \dots + \mathcal{D}_{\bullet,\ell}^\ell \subset \mathcal{C}^\infty(SU(2), \mathbb{C}).$$

3.1. Irreducible representations. Let us consider the *pull-back action* of $\Theta \in SO(4)$ on $L^2(SU(2))$, defined by: $F \mapsto \Theta^{-*}F := (\Theta^{-1})^*F = F \circ (\Theta^{-1})$. Since $SO(4) = \{L_{g_1} \circ R_{g_2}: g_1, g_2 \in SU(2)\}$, we can lift it to an action of $SU(2) \times SU(2)$ ¹². We will use the convention of left actions: the pair $(g_1, g_2) \in SU(2) \times SU(2)$ acts on a function $F \in L^2(SU(2))$ as $\Theta = L_{g_1} \circ R_{g_2}^{-1}$, that is:

$$(g_1, g_2) \cdot F = L_{g_1}^{-*} \circ R_{g_2}^* F(z) = F(g_1^{-1} z g_2).$$

¹²The group $SO(4)$ is not isomorphic to $SU(2) \times SU(2)$, but this will not be important for our purpose.

In particular, we have the two different actions of $SU(2) \times 1$ and $1 \times SU(2)$ on $L^2(SU(2))$ corresponding to the pull-back of left and right multiplications. We will refer to them as *left* and *right pull-back actions*. Moreover, by identifying the subgroup $K := \{g_3(\psi) : \psi \in [0, 4\pi]\} \subset SU(2)$ with $U(1)$, we see that the pull-back actions restrict to a unitary action of $U(1) \times U(1)$. Clearly, the left and right spin are related to this action, indeed $\mathcal{L}(m) \cap \mathcal{R}(s)$ are invariant subspaces.

The space $\mathcal{H}_\ell \subset L^2(SU(2))$ is $SU(2) \times 1$ invariant¹³. By definition, the Wigner matrices are the matrices that correspond to such unitary representation of $SU(2)$.

Proposition 3.1. *Any unitary irreducible representation of $SU(2)$, is equivalent to one and only one of the Wigner matrices $D^\ell : SU(2) \rightarrow U(2\ell + 1)$ for some $\ell \in \frac{1}{2}\mathbb{N}$. In the sense of representation theory this means that the dual of $SU(2)$ is the set*

$$\hat{SU}(2) = \{(D^\ell, \mathbb{C}^{2\ell+1})\}_{\ell \in \frac{1}{2}\mathbb{N}}.^{14}$$

Proof. See [16, Theorem 3.14]. □

By the Peter-Weyl theorem [16], the space $L^2(SU(2))$ splits as an orthogonal sum of the spaces of matrix coefficients \mathcal{D}^ℓ , which are irreducible spaces for the whole pull-back action, i.e. the action of $SO(4)$. The Peter-Weyl theorem asserts that, in fact, each of the spaces \mathcal{D}^ℓ splits again into the spaces of columns coefficients $\mathcal{D}_{\bullet,s}^\ell$, which are irreducible for the left pull-back action of $SU(2) \times 1$ and give equivalent unitary representations. Finally, the space $\mathcal{D}_{\bullet,s}^\ell$ is $U(1) \times 1$ invariant and thus it splits again into irreducible subspaces for such circle action. Since $U(1)$ is abelian, its irreducible representations are forced to be 1-dimensional, thus we conclude that there exists an orthonormal basis of $\mathcal{D}_{\bullet,s}^\ell$ consisting of functions with pure left spin. Notice that the monomials are special as a basis of \mathcal{H}_ℓ , in that $\psi_m^\ell \in \mathcal{L}(m)$. The consequence of this choice of basis is that the coefficients of the corresponding matrix representation have pure right spin, and precisely $D_{m,s}^\ell \in \mathcal{R}(-s)$:

$$\sum_m \psi_m^\ell(z) D_{m,s}^\ell(g \cdot g_3(\psi)) = \psi_s^\ell(g_3(\psi)^{-1} g^{-1} z) = e^{-is\psi} \psi_s^\ell(g^{-1} z) = \sum_m \psi_m^\ell(z) D_{m,s}^\ell(g) e^{-is\psi}.$$

Then, from unitarity $D(g)_{m,s}^\ell = \overline{D_{s,m}^\ell(g^{-1})}$, we deduce that $D_{m,s}^\ell \in \mathcal{L}(-m) \cap \mathcal{R}(-s)$, hence the orthonormal basis of pure spin functions for $\mathcal{D}_{\bullet,s}^\ell$ is indeed given by the Wigner functions $D_{m,s}^\ell$. Moreover, by Schur's orthogonality relations (see [16]), we deduce that, with respect to the metric $SU(2) = 2S^3$, we have

$$\|D_{m,s}^\ell\|_{L^2(SU(2))}^2 = \frac{\text{vol}(SU(2))}{2\ell + 1} = \frac{16\pi^2}{2\ell + 1}.$$

The first part of Theorem 1.3 follows from the next proposition.

Proposition 3.2. *We have the following orthogonal decompositions of $L^2(SU(2))$.*

$$L^2(SU(2)) = \bigoplus_{\ell \in \frac{1}{2}\mathbb{N}} \mathcal{D}^\ell = \bigoplus_{\ell \in \frac{1}{2}\mathbb{N}} \bigoplus_{s=-\ell, \dots, +\ell} \mathcal{D}_{\bullet,s}^\ell = \bigoplus_{\ell \in \frac{1}{2}\mathbb{N}} \bigoplus_{s=-\ell, \dots, +\ell} \bigoplus_{m=-\ell, \dots, +\ell} \mathbb{C} D_{m,s}^\ell,$$

- (1) *The space \mathcal{D}^ℓ is irreducible for $SO(4)$;*
- (2) *The space $\mathcal{D}_{\bullet,s}^\ell$ is irreducible for the action of $SU(2) \times U(1)$;*
- (3) *The space $\mathcal{D}_{m,\bullet}^\ell$ is irreducible for the action of $U(1) \times SU(2)$;*

¹³Notice that \mathcal{H}_ℓ is not invariant for the action of $1 \times SU(2)$, because multiplication on the right would mix the variables z_i and \bar{z}_i .

¹⁴If G is a compact group, the notation \hat{G} stands for its dual, i.e. the collection of all isomorphism classes of irreducible unitary representations of G .

(4) $D_{m,s}^\ell \in \mathcal{L}(-m) \cap \mathcal{R}(-s)$ has left spin $= -m$ and right spin $= -s$.

Remark 3.3. In the literature, one might find different conventions for the definitions of $D_{m,s}^\ell$. Anyway, the above decomposition characterizes the functions $D_{m,s}^\ell$ up to multiplication by a phase. The additional requirement that $D_{m,s}^\ell(g_2(\theta)) \in \mathbb{R}$, which is true in the case of this paper, determines them uniquely up to a sign.

We report an additional property of the functions $D_{m,s}^\ell$. The Proposition below is not new, see [20, Equation (2)].

Proposition 3.4. *For any m, s, ℓ and $g \in SU(2)$, we have*

$$\overline{D_{m,s}^\ell(g)} = D_{m,s}^\ell(\bar{g}) = D_{-m,-s}^\ell(g)(-1)^{2\ell+m-s}.$$

Proof. For the first identity, observe that $\overline{\psi_s^\ell(z)} = \psi_s^\ell(\bar{z})$, so that by taking the conjugate in (3.1) we have

$$\sum_m \psi_m^\ell(\bar{z}) \overline{D_{m,s}^\ell(g)} = \psi_s^\ell(\bar{g}^{-1}\bar{z}) = \sum_m \psi_m^\ell(\bar{z}) D_{m,s}^\ell(\bar{g}) e^{-is\psi}.$$

To prove the second identity, we use the fact that for all $z = h(z_0, z_1) \in \mathbb{H}$, we have

$$\psi_s^\ell(g_2(\pi)^{-1}z) = (z_1)^{\ell+s} (-z_0)^{\ell-s} \binom{2\ell}{\ell+s}^{\frac{1}{2}} = (-1)^{\ell-s} \psi_{-s}^\ell(z).$$

Thus, $D_{m,s}^\ell(g_2(\pi)) = \delta_{-m,s}(-1)^{\ell+m}$. From this, by using the representation of $g \in SU(2)$ via Euler angles, as in (2.4), $g = g_3(\phi)g_2(\theta)g_3(\psi)$ and the fact that $D_{m,s}^\ell$ have pure left and right spin, we obtain

$$\begin{aligned} D_{m,s}^\ell(\bar{g}) &= e^{im\phi} D_{m,s}^\ell(g_2(\pi+\theta-\pi)) e^{is\psi} = e^{im\phi} \sum_{a,b} D_{m,a}^\ell(g_2(\pi)) D_{a,b}^\ell(g_2(\theta)) D_{b,s}^\ell(g_2(-\pi)) e^{is\psi} = \dots \\ &\dots = e^{im\phi} \sum_{a,b} \delta_{-m,a}(-1)^{\ell+m} D_{a,b}^\ell(g_2(\theta)) \delta_{s,-b}(-1)^{(\ell-s)} e^{is\psi} = \dots \\ &\dots = (-1)^{2\ell+m-s} e^{im\phi} D_{-m,-s}^\ell(g_2(\theta)) e^{is\psi} = (-1)^{2\ell+m-s} D_{-m,-s}^\ell(g). \end{aligned}$$

□

3.2. Laplacians. The above decomposition of $L^2(SU(2))$ can be also seen as a consequence of the decomposition of the Laplacian into a vertical and a horizontal part, in the sense of [4]. Indeed the map

$$\cdot_\infty: SU(2) \rightarrow S^2$$

is a Riemannian submersion with totally geodesic fibers (its fibers are big circles in $2S^3$) and therefore the Laplace-Beltrami operator can be written as a sum of two commuting self-adjoint operators:

$$(3.2) \quad \Delta_{SU(2)} = \Delta_h + \Delta_v.$$

Following [4], the vertical Laplacian Δ_v is defined as the Laplace-Beltrami operator of the fibers of \cdot_∞ . In our conventions, it corresponds to second derivative with respect to the Euler angle ψ :

$$\Delta_v F(g) := \frac{d^2}{d\psi^2} \Big|_{\psi=0} F(g \cdot g_3(\psi));$$

while Δ_h is defined by the identity (3.2). Clearly, functions with pure right spin are eigenfunctions of Δ_v , indeed

$$\ker(\Delta_v + s^2) = \mathcal{R}(s) + \mathcal{R}(-s).$$

What's most remarkable about the decomposition (3.2) is that Δ_h and Δ_v commute (this is proved in [4, Theorem 1.5]). This implies that there exists an orthogonal decomposition of $L^2(SU(2))$ into common eigenspaces of $\Delta_{SU(2)}$, Δ_v and Δ_h (it is sufficient to find a Hilbert basis of common eigenfunctions for $\Delta_{SU(2)}$ and Δ_v , that is [4, Proposition 1.3]).

By the following observation, we conclude that such common eigenspaces are exactly the spaces of column coefficients of the Wigner matrices.

Proposition 3.5. *Both components of the function $\hat{D}_{m,s}^\ell: \mathbb{H} = \mathbb{R}^4 \rightarrow \mathbb{R}^2 = \mathbb{C}$ are homogeneous real harmonic polynomials¹⁵ of degree 2ℓ . Therefore, under the normalization*

$$\phi_{m,s}^\ell := \frac{\sqrt{2\ell+1}}{4\pi} D_{m,s}^\ell,$$

the collection $\phi_{m,s}^\ell$, for all $\ell, m, s \in \frac{1}{2}\mathbb{N}$ with $-\ell \leq m, s \leq \ell$, form an orthonormal basis of $L^2(SU(2))$ of spherical harmonics on $SU(2) = 2S^3$, i.e. eigenfunctions of $\Delta_{SU(2)}$, with eigenvalue $-\frac{1}{2}\ell(2\ell+2)$.

This, combined with Proposition 3.2, completes the proof of Theorem 1.3.

Remark 3.6. To obtain the so-called *spin weighted spherical harmonics* $Y_{m,s}^\ell$ one has to normalize $D_{m,s}^\ell$ as a section of $\mathcal{T}^{\otimes s}$, in which case the L^2 norm is given by integrating on the sphere S^2 . Thus they are obtained by multiplying $\phi_{m,s}^\ell$ by the square root of the length of the fiber of $SU(2) \rightarrow S^2$, which is 4π :

$$Y_{m,s}^\ell = \sqrt{\frac{2\ell+1}{4\pi}} D_{m,s}^\ell.$$

Here, $D_{m,s}^\ell$ is interpreted as a spin s function on S^2 , i.e. a section of $\mathcal{T}^{\otimes s} \rightarrow S^2$ via the Pull-back Correspondence, see Theorem 2.22.

Proof. It is clear by the definitions (3.1) that the Wigner function $D_{m,s}^\ell$ is a homogeneous polynomials with complex coefficients of degree 2ℓ in the variables $\alpha, \bar{\alpha}, \beta, \bar{\beta}$. It follows that its real and imaginary parts are real homogeneous polynomials of degree 2ℓ in the real coordinates of $\mathbb{H} = \mathbb{R}^4$. The Laplacian of $\mathbb{H} = \mathbb{C}^2$, with the metric $\langle \cdot, \cdot \rangle_{\mathbb{H}} = 4\langle \cdot, \cdot \rangle_{\mathbb{R}^4}$ is the operator

$$\Delta_{\mathbb{H}} = \frac{1}{4} \Delta_{\mathbb{R}^4} = \frac{1}{4} \frac{\partial}{\partial \alpha} \frac{\partial}{\partial \bar{\alpha}} + \frac{1}{4} \frac{\partial}{\partial \beta} \frac{\partial}{\partial \bar{\beta}}: \mathcal{C}^\infty(\mathbb{H}, \mathbb{C}) \rightarrow \mathcal{C}^\infty(\mathbb{H}, \mathbb{C}).¹⁶$$

By noticing that $\psi_m^\ell(z)$ is holomorphic, i.e. it can be written as a polynomial in z_0 and z_1 , where $z = h(z_1, z_2)$, we can conclude by proving the following Lemma.

Lemma 3.7. *If $F(z) = f(z_0, z_1)$ is holomorphic, then the function $\phi_z: \mathbb{H} \rightarrow \mathbb{C}$, such that $(\alpha, \beta) \mapsto F(h(\alpha, \beta)^{-1}z) \in \mathbb{C}$, is harmonic for every $z \in \mathbb{H}$, that is $\Delta_{\mathbb{H}}\phi_z = 0$.*

Let $z = h(z_0, z_1)$ and $h = h(\alpha, \beta)$, so that

$$\begin{aligned} \phi_z(\alpha, \beta) &= F(h(\alpha, \beta)^{-1}h(z_0, z_1)) = \\ &= F\left(\begin{pmatrix} \bar{\alpha} & -\beta \\ \bar{\beta} & \alpha \end{pmatrix} \begin{pmatrix} z_0 & -\bar{z}_1 \\ z_1 & \bar{z}_0 \end{pmatrix}\right) = f(\bar{\alpha}z_0 - \beta z_1, \bar{\beta}z_0 + \alpha z_1). \end{aligned}$$

¹⁵Meaning that they are polynomials $p(x_\alpha, y_\alpha, x_\beta, y_\beta)$ in the 4 real coordinates of $\mathbb{H} = \mathbb{R}^4$, satisfying the equation $\Delta_{\mathbb{R}^4}p = 0$.

¹⁶Here the smooth structure is that of $\mathbb{H} \simeq \mathbb{R}^4$, so $\mathcal{C}^\infty(\mathbb{H}, \mathbb{C}) = \mathcal{C}^\infty(\mathbb{R}^4, \mathbb{R}^2)$.

Then

$$\begin{aligned}
4\Delta_{\mathbb{H}}\phi_z(\alpha, \beta) &= \\
&= \frac{\partial}{\partial\alpha} \frac{\partial}{\partial\bar{\alpha}} + \frac{\partial}{\partial\beta} \frac{\partial}{\partial\bar{\beta}} f(\bar{\alpha}z_0 - \beta z_1, \bar{\beta}z_0 + \alpha z_1) = \\
&= \frac{\partial}{\partial\alpha} \left(z_0 \left(\frac{\partial f}{\partial z_0} \right) (\bar{\alpha}z_0 - \beta z_1, \bar{\beta}z_0 + \alpha z_1) \right) + \frac{\partial}{\partial\beta} \left(z_0 \left(\frac{\partial f}{\partial z_1} \right) (\bar{\alpha}z_0 - \beta z_1, \bar{\beta}z_0 + \alpha z_1) \right) \\
&= z_0 z_1 \left(\frac{\partial^2 f}{\partial z_0 \partial z_1} \right) (\bar{\alpha}z_0 - \beta z_1, \bar{\beta}z_0 + \alpha z_1) - z_0 z_1 \left(\frac{\partial^2 f}{\partial z_1 \partial z_0} \right) (\bar{\alpha}z_0 - \beta z_1, \bar{\beta}z_0 + \alpha z_1) = 0.
\end{aligned}$$

□

Corollary 3.8. *The spaces $\mathcal{D}_{\bullet,s}^\ell$ are common eigenspaces of $\Delta_{SU(2)}$, Δ_v and Δ_h with eigenvalue:*

$$\begin{aligned}
\Delta_{SU(2)}|_{\mathcal{D}_{\bullet,s}^\ell} &= -\frac{1}{2}\ell(2\ell+2), \\
\Delta_v|_{\mathcal{D}_{\bullet,s}^\ell} &= -s^2, \\
\Delta_h|_{\mathcal{D}_{\bullet,s}^\ell} &= -\frac{1}{2}\ell(2\ell+2) + s^2 = -(\ell-s)(\ell+s+1) - s.
\end{aligned}$$

In terms of the Laplacian defined in terms of the spin raising and spin lowering operators, it was proved in by Newman and Penrose [18] (see also [7, 8, 16]) that the Wigner functions are eigenfunctions of the self-adjoint operator $\bar{\partial}\partial: \mathcal{R}(-s) \rightarrow \mathcal{R}(-s)$:

$$\bar{\partial}\partial D_{m,s}^\ell = -(\ell-s)(\ell+s+1)D_{m,s}^\ell.$$

Notice also that $\mathcal{R}(-s) = \ker(\frac{d}{d\psi} + is)$. Thus, we obtain the formulas (1.6) and (1.9):

$$\Delta_{SU(2)} = \bar{\partial}\partial + \frac{d^2}{d\psi^2} - i\frac{d}{d\psi}; \quad \Delta_h = \bar{\partial}\partial - i\frac{d}{d\psi}$$

3.3. Random Wigner matrices. In this section we will think the group $SU(2)$ as a probability space, with the volume measure μ of $2S^3$ normalized to 1. Using the notation of random elements, we will write γ for a random element of $SU(2)$ such that for every measurable subset $A \subset SU(2)$ and measurable function $F: SU(2) \rightarrow \mathbb{R}$,

$$\mathbb{P}\{\gamma \in A\} = \frac{\text{vol}(A)}{\text{vol}(SU(2))}; \quad \mathbb{E}\{F(\gamma)\} = \frac{1}{\text{vol}(SU(2))} \int_{SU(2)} F(g) d\mu(g).$$

In other words, γ is a random variable with values in $SU(2)$ whose law is the Haar probability measure of $SU(2)$. This is just a convenient notation.

Let $\langle v, w \rangle = \bar{v}^T w$ be the Hermitian product of $\mathbb{C}^{2\ell+1}$, then the real part $\Re\langle w, v \rangle$ is the standard Euclidean product, under the identification $\mathbb{C}^{2\ell+1} = \mathbb{R}^{2(2\ell+1)}$. The unitarity of the representation D^ℓ means that the matrices $D^\ell(g)$ act on $\mathbb{C}^{2\ell+1}$ as unitary operators. Thus, in particular, they preserves all spheres $rS^{4\ell+1}$.

By means of the homomorphism map $D^\ell: SU(2) \rightarrow U(2\ell+1)$, we get a collection of random matrices $D^\ell(\gamma)$, for all $\ell \in \frac{1}{2}\mathbb{N}$ such that

$$(D^\ell(\gamma))_{\ell \in \frac{1}{2}\mathbb{N}} \stackrel{\text{law}}{=} (D^\ell(g)D^\ell(\gamma))_{\ell \in \frac{1}{2}\mathbb{N}},$$

and for each $v \in S^{4\ell+1}$, there is a collection of random vectors in $S(\mathbb{C}^{2\ell+1}) = S^{4\ell+1}$, for all $\ell \in \frac{1}{2}\mathbb{N}$.

Definition 3.9. $\gamma_v^\ell := D^\ell(\gamma)v$ for every $v \in S^{4\ell+1}$.

The random vector γ_v is supported on the orbit of v under the action of $SU(2)$, that is the set

$$O^\ell(v) := \{D^\ell(g)v \in \mathbb{C}^{2d+1} : g \in SU(2)\},$$

and its law is invariant by the unitary transformations of the form $D^\ell(g)$, for every $g \in SU(2)$. By the theory of smooth group actions (see [12]), $O^\ell(v)$ is a smooth embedded submanifold of $S^{4\ell+1}$ and the canonical map

$$(3.3) \quad D^\ell(\cdot)v : SU(2) \rightarrow SU(2)/H^\ell(v) \cong O^\ell(v)$$

is a principal bundle, whose structure group $H^\ell(v) := \{g \in SU(2) : D^\ell(g)v = v\}$ is the isotropy subgroup of v . Notice that this implies that the law of γ_v is equivalent to that of $D^\ell(g)\gamma_v$ for any $g \in SU(2)$, therefore it coincides with the normalized Riemannian volume measure of $O^\ell(v) \subset S^{4\ell+1}$.

Proposition 3.10. *The random vectors γ_v are centered: $\mathbb{E}\{\gamma_v\} = 0$ for all $\ell \neq 0$, while $\gamma_v = v$ is constant if $\ell = 0$. Their correlation is characterized by the following identities. For every $v, w \in \mathbb{C}^{2\ell+1}$ and $v', w' \in \mathbb{C}^{2\ell'+1}$, we have*

$$(3.4) \quad \overline{w}^T K^\ell(v, v')w := \mathbb{E} \left\{ \overline{\langle \gamma_v, w \rangle} \langle \gamma_{v'}, w' \rangle \right\} = \delta_{\ell, \ell'} \frac{\overline{\langle v, v' \rangle} \langle w, w' \rangle}{2\ell + 1}.$$

Proof. Clearly, the vector $e = \mathbb{E}\{\gamma_v\}$ generates an invariant subspace of $\mathbb{C}^{2\ell+1}$, thus $e = 0$, because of the irreducibility of D^ℓ , except in the case $\ell = 0$, where $D^0(g) = 1$. The identities (3.4) are just a reformulation of the Schur's orthogonality relations, see [16]. We repeat the proof here, since this result will be of fundamental importance for the rest of the paper. Let us define $K(v, v') : \mathbb{C}^{2\ell'+1} \rightarrow \mathbb{C}^{2\ell+1}$ to be the (complex) linear operator

$$K(v, v')(w) = \mathbb{E}\{\gamma_v (\overline{\gamma_{v'}})^T\} w.$$

Then $K(v, v')$ intertwines the two unitary representations D^ℓ and $D^{\ell'}$:

$$\begin{aligned} K(v, v')(D^{\ell'}(g)w) &= \mathbb{E} \left\{ \gamma_v \left((\overline{\gamma_{v'}})^T D^{\ell'}(g)w \right) \right\} = \dots \\ &= \mathbb{E} \left\{ \gamma_v \left(w^T D^{\ell'}(g)^T \overline{\gamma_{v'}} \right) \right\} = \mathbb{E} \left\{ \gamma_v w^T \overline{D^{\ell'}(g^{-1}) \gamma_{v'}} \right\} = \dots \end{aligned}$$

and using the D -invariance of the collection γ_v , we have

$$\dots = \mathbb{E} \left\{ D^\ell(g) \gamma_v w^T \overline{D^{\ell'}(g^{-1}) D^{\ell'}(g) \gamma_{v'}} \right\} = D^\ell(g) \mathbb{E} \left\{ \gamma_v (\overline{\gamma_{v'}})^T \right\} w = D^\ell(g) K(v, v')w.$$

By Schur's lemma (see [16]), the operator $K(v, v')$ is certainly equal to 0 if $\ell \neq \ell'$, because it intertwines two non equivalent irreducible representations. If $\ell = \ell'$ and $i = 0$, then $K(v, v')$ is an endomorphism of $\mathbb{C}^{2\ell+1}$, thus it has an eigenvalue. By Schur's lemma (or rather, its proof) again, the relative eigenspace is invariant, hence it must be the whole space, so that we conclude that there exists $\lambda \in \mathbb{C}$ such that

$$K(v, v') = \lambda \mathbb{1}.$$

Now, λ can be computed, by taking the trace.

$$\lambda(2\ell + 1) = \text{tr}(K(v, v')) = \mathbb{E} \left\{ \text{tr}(\gamma_v (\overline{\gamma_{v'}})^T) \right\} = \mathbb{E} \left\{ \text{tr}((\overline{\gamma_{v'}})^T \gamma_v) \right\} = \mathbb{E} \left\{ \langle \gamma_{v'}, \gamma_v \rangle \right\} = \langle v', v \rangle.$$

□

We are now ready to give the proof of Theorem 1.11.

Proof of Theorem 1.11. The first identities (1.7) follows directly from Theorem 3.10, when $w = e_m, v = e_s \in \mathbb{C}^{2\ell+1}$ and $w' = e_{m'}, v' = e_{s'} \in \mathbb{C}^{2\ell'+1}$ are the vectors of the canonical basis. The second identities (1.8) are deduced from the first, via Proposition 3.4

$$\mathbb{E} \left\{ D_{m,s}^\ell(\gamma) D_{m',s'}^{\ell'}(\gamma) \right\} = \mathbb{E} \left\{ D_{m,s}^\ell(\gamma) \left(\overline{D_{-m',-s'}^{\ell'}(\gamma)} \right) \right\} (-1)^{2\ell+m'-s'}.$$

□

Remark 3.11. Notice that each of the random functions $D_{m,s}^\ell(\gamma)$ is circularly symmetric, for all $(m, s) \neq (0, 0)$, while $D_{0,0}^\ell(\gamma) \in \mathbb{R}$. However, this is not true for the law of the whole collection of random variables $(D_{m,s}^\ell(\gamma))_{\ell,m,s}$, otherwise the left hand side in the second equation (1.8) would be always = 0.

3.4. Orbits. The irreducibility of the representation $D^\ell: SU(2) \rightarrow U(2\ell+1)$, can be equivalently expressed by saying that for each $v \in S^{4\ell+1}$ the orbit O_v^ℓ spans the whole space (it is a consequence of Theorem 3.10):

$$\text{span}(O_v^\ell) = \mathbb{C}^{2\ell+1}.$$

In fact, a consequence of Proposition 3.5 is that the same property holds for any non negligible subset of O_v^ℓ .

Theorem 3.12. *Let $A \subset O^\ell(v)$ be a measurable subset such that $\text{vol}(A) \neq 0$, then $\text{span}(A) = \mathbb{C}^{2\ell+1}$.*

Proof. Let $w, v \in \mathbb{C}^{2\ell+1}$, we will show that if $\text{span}(A)$ is contained in w^\perp , then $\mathbb{P}\{\gamma_v \in A\} = 0$. Let $f: SU(2) \rightarrow \mathbb{R}$ be the function $f: g \mapsto \Re(\bar{w}^T D^\ell(g)v) \in \mathbb{R}$. By Theorem 3.5, f is an eigenfunction of $\Delta_{SU(2)}$, therefore its nodal set $f^{-1}(0)$ has Hausdorff dimension $n-1$ (see [6]). Thus we conclude:

$$\mathbb{P}\{\gamma_v \in A\} \leq \mathbb{P}\{\gamma_v \in w^\perp\} = \text{vol}(f^{-1}(0)) = 0.$$

□

Despite Theorem 3.12, it could be that the homeomorphism type and even the dimension of the orbits $O^\ell(v)$ are different for different choices of $v \in S^{4\ell+1}$. This is what happens for the columns of the matrix D^ℓ , which parametrize the orbits of the canonical basis, i.e. the vectors $e_s, s = -\ell, \dots, \ell$:

$$\gamma_{e_s} = D_{\bullet,s}^\ell(\gamma).$$

These orbits are special, in that they correspond to the orbits of the monomial basis, under the identification $\mathbb{C}^{2\ell+1} \cong \mathcal{H}_\ell$.

Theorem 3.13. *The following holds.*

- (1) *The map (3.3) induces an embedding $SU(2)^{\otimes 2s} \simeq O^\ell(e_s) \subset S^{4\ell+1}$ for all $s \in \{-\ell, \dots, \ell\} \setminus \{0\}$.*
- (2) *For $s = 0$ there are two cases: if $\ell \in 2\mathbb{N}$, then $S^1 \simeq O^\ell(e_0)$; while $S^2 \simeq O^\ell(e_0)$ otherwise.*
- (3) *The above map descends to a smooth map of the sphere S^2 in $\mathbb{CP}^{2\ell}$, which is an embedding, except in the case $\ell \in 2\mathbb{N}$ and $s = 0$.*
- (4) *For almost every $v \in S^{4\ell+1}$, the orbit $O^\ell(v)$ is diffeomorphic to $SU(2)$ when $\ell \notin \mathbb{N}$ and to $SO(3)$ when $\ell \in \mathbb{N} \setminus \{0\}$.*

Proof. To prove (1) and (2) it is sufficient to compute the isotropy group of e_s . Let us take $\mathbb{C}^{2\ell+1} = \mathcal{H}_\ell$, with e_s corresponding to the monomial ψ_s^ℓ . Then the isotropy group of e_s is the set of matrices $h(\alpha, \beta) \in SU(2)$ such that

$$(3.5) \quad (\bar{\alpha}z_0 + \bar{\beta}z_1)^{\ell+s}(-\beta z_0 + \alpha z_1)^{\ell-s} = z_0^{\ell+s}z_1^{\ell-s}.$$

By evaluating the above identity of polynomials at the points $(z_0, z_1) = (1, 0), (0, 1)$, we see that either $\alpha = 0$ or $\beta = 0$. Now, observe that with $\alpha = 0$ (and consequently, $|\beta| = 1$), the equation (3.5) becomes

$$(-1)^{\ell-s}(\beta)^{-2s}z_1^{\ell+s}z_0^{\ell-s} = z_0^{\ell+s}z_1^{\ell-s}.$$

This admits solutions only if $s = 0$ and $\ell \in 2\mathbb{N}$. In the case $\beta = 0$, the equation to solve is

$$\alpha^{-2s} = 1,$$

whose set of solution is, by definition, the subgroup $\sqrt[2s]{1} \subset SU(2)$. In synthesis we just proved that for all $s \in \{-\ell, \dots, \ell\} \setminus \{0\}$, the isotropy subgroup of e_s is

$$H^\ell(e_s) = \sqrt[2s]{1};$$

while, for $s = 0$, we have two cases:

$$H^\ell(e_0) = \left\{ h(e^{i\frac{\psi}{2}}, 0) : \psi \in [0, 4\pi] \right\}, \text{ if } \ell \notin 2\mathbb{N};$$

$$H^\ell(e_0) = \left\{ h(e^{i\frac{\varphi+\psi}{2}}, e^{i\frac{-\varphi+\psi}{2}}) : \varphi \in [0, 2\pi], \psi \in [0, 4\pi] \right\}, \text{ if } \ell \in 2\mathbb{N}.$$

Point (3) follows from the fact that $D_{\bullet,s}^\ell : SU(2) \rightarrow S^{4\ell+1}$ is a spin s function, hence it maps the fibers of the circle bundle $SU(2)^{\otimes s} \rightarrow S^2$ to fibers of the Hopf fibrations $S^{4\ell+1} \rightarrow \mathbb{CP}^{2\ell}$.

Point (4) is a consequence of the so-called *Principal Orbit Theorem*. In one of its stronger forms, proved in [17], it says that union of orbits that are not maximal, both in the senses maximal dimension and minimal isotropy group, form a subset of codimension 2. Since being an embedding is an open condition, for any point v close enough to $e_{\frac{1}{2}} \in S^{4\ell+1}$ the orbit is $O^\ell(v) \simeq SU(2)$, therefore it is has typical orbit type of the action. In the case $\ell \in \mathbb{N}$ then $s \in \mathbb{N}$ as well and thus every map $D_{\bullet,s}^\ell$ descends to a map $SU(2)^{\otimes 2} = SO(3) \rightarrow S^{4\ell+1}$, so that we can repeat the previous argument, but for e_1 . \square

4. D-INVARIANCE

In this section we study collection of vectors in $\mathbb{C}^{2\ell+1}$ that are invariant under the action of Wigner matrices, with the purpose of applying our results to the spectral coefficients $a_{m,s}^\ell$ of the decomposition as in Theorem 1.3. To this end, let us introduce some terminology.

Definition 4.1. We say that a collection of random vectors $(v_i)_{i \in I}$ with $v_i \in \mathbb{C}^{2\ell_i+1}$ is *strongly D-invariant* if their joint law is equivalent to that of the collection $(D^{\ell_i}(g)v_i)_{i \in I}$ for any $g \in SU(2)$.

Definition 4.2. Two random vectors $V, V' \in \mathbb{C}^N$ are said to be *2-weakly equivalent* if they have the same expectation: $\mathbb{E}\{V\} = \mathbb{E}\{V'\}$ and the same self-correlation matrices: $\mathbb{E}\{V\bar{V}^T\} = \mathbb{E}\{V'\bar{V}'^T\}$ and $\mathbb{E}\{VV^T\} = \mathbb{E}\{V'(V')^T\}$. In this case, we will write

$$V \sim_{2-w} V'.$$

Definition 4.3. We say that a collection $(V_i)_{i \in I}$ of random vectors $V_i \in \mathbb{C}^{2\ell_i+1}$ is 2-weakly D -invariant if for any i, j and $g \in SU(2)$, we have

$$(V_i, V_j) \sim_{2-w} (D^{\ell_i}(g)V_i, D^{\ell_j}(g)V_j).$$

Given any collection $\mathcal{V} = (V_i)_{i \in I}$ of random vectors $V_i \in \mathbb{C}^{2\ell_i+1}$, there is an easy way to construct a strongly D -invariant one, simply by multiplying it by an independent random matrix $D^\ell(\gamma)$.

Remark 4.4. The collection $\mathcal{V} = (D^\ell(\gamma)V_i)_{i \in I}$ obtained in such way is always strongly D -invariant.

In fact, it is almost tautological that any D -invariant collections is essentially of this form, since such operation can be seen as a projection on the space of strongly D -invariant probability measures. We will not enter into the details of this point of view, but we will give a concrete statement, to be precise.

Theorem 4.5. A collection $\mathcal{V} = (V_i)_{i \in I}$ of random vectors $V_i \in \mathbb{C}^{2\ell_i+1}$ is strongly or 2-weakly D -invariant, respectively, if and only if

$$(V_i)_{i \in I} \stackrel{\text{law}}{=} (D^\ell(\gamma)V_i)_{i \in I} \quad \text{or} \quad (V_i)_{i \in I} \sim_{2-w} (D^\ell(\gamma)V_i)_{i \in I},$$

where $\gamma \in SU(2)$ is a random element independent from \mathcal{V} , in the sense of Section 3.3.

Proof. The *if* part of the statement simply follows from Remark 4.4, hence it is sufficient to show the *only if* statement. Let us start from the case of strong D -invariance. Let $F_i: \mathbb{C}^{2\ell_i+1} \rightarrow \mathbb{R}$, for all $i \in I$, be any collection of measurable functions. Then, the strong D -invariance yields

$$(4.1) \quad \mathbb{E}\{F_i(V_i)\} = \mathbb{E}\{F_i(D^\ell(\gamma)V_i)\}.$$

Therefore the joint distribution of the two collections $(V_i)_{i \in I}$ and $(D^\ell(\gamma)V_i)_{i \in I}$ are the same. In the case of a 2-weak D -invariant collection, we observe that equation (4.1) holds for all functions F_i that are real polynomials of degree at most 2, which implies that the expectations and the correlation matrices of the two collections $(V_i)_{i \in I}$ and $(D^\ell(\gamma)V_i)_{i \in I}$ coincide. \square

Definition 4.6. For all $\ell \in \frac{1}{2}\mathbb{N}$, we define the matrix $\varepsilon(\ell) \in \mathbb{C}^{(2\ell+1) \times (2\ell+1)}$ as follows

$$\varepsilon(\ell)_{m,m'} = \delta_{-m,m'}(-1)^{\ell-m}, \quad \text{that is} \quad \varepsilon(\ell) = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & -1 & \\ 0 & 0 & 1 & 0 & \dots \\ 0 & -1 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \end{pmatrix},$$

where the coordinates are indexed by $m = -\ell, -\ell+1, \dots, \ell$, so that $\ell - m$ takes all integer values between 0 and $2\ell+1$.

Theorem 4.7. Let $\mathcal{V} = (V_i)_{i \in I}$ be a 2-weakly invariant collection of random vectors $V_i \in \mathbb{C}^{2\ell_i+1}$. Then $\mathbb{E}\{V_i\} = 0$, whenever $\ell_i \neq 0$ and the correlation structure satisfies the following identities, for all $i, j \in I$.

$$(4.2) \quad \begin{aligned} \mathbb{E}\{V_i(\overline{V_j})^T\} &= \frac{\delta_{\ell_i, \ell_j}}{2\ell_i+1} \mathbb{1}_{(2\ell_i+1)} \left(\mathbb{E}\{\langle \overline{V_i}, V_j \rangle\} \right); \\ \mathbb{E}\{V_i(V_j)^T\} &= \frac{\delta_{\ell_i, \ell_j}}{2\ell_i+1} \varepsilon(\ell_i) \left(\sum_{k=-\ell_i}^{\ell_i} (-1)^{\ell_i+k} \mathbb{E}\{\langle e_k, V_i \rangle \langle e_{-k}, V_j \rangle\} \right). \end{aligned}$$

In particular, V_i and V_j are uncorrelated if $\ell \neq \ell'$. Moreover, the m^{th} component of V_i is correlated only with the m^{th} and the $(-m)^{\text{th}}$ components of V_j and the correlation depends only on the parity of $\ell - m$. Finally, the variance of each components of V_i depends only on ℓ_i .

Proof. Let us denote $V_i = V$ and $V_j = V'$. By Theorem 4.5 we have

$$\mathbb{E} \{V_i(\overline{V_j})^T\} = \mathbb{E} \{D^\ell(\gamma)V_i(\overline{D^\ell(\gamma)V_j})^T\} = \mathbb{E} \{\gamma_V(\overline{\gamma_{V'}})^T\} = \dots$$

The last expectation is with respect to the independent pair of random variables γ and (V, V') . By taking first the one in γ (i.e. using Fubini's theorem), Proposition 3.10 yields

$$\dots = \delta_{\ell, \ell'} \mathbb{E} \{K^\ell(V, V')^T\} = \delta_{\ell, \ell'} \mathbb{E} \left\{ \frac{\langle \overline{V}, V' \rangle}{2\ell + 1} \mathbb{1}_{(2\ell+1)} \right\}.$$

This proves the first of the identities (4.2). To prove the second identity, we use again Theorem 4.5:

$$\mathbb{E} \{ \langle e_m, V_i \rangle \langle e_{-m'}, V_j \rangle \} = \mathbb{E} \{ (D_{m, \bullet}^\ell(\gamma))^T V (D_{m', \bullet}^{\ell'}(\gamma))^T V' \} = \mathbb{E} \{ V^T D_{m, \bullet}^\ell(\gamma) (D_{m', \bullet}^{\ell'}(\gamma))^T V' \} = \dots$$

Now, we use the same trick as before and apply the second formula of Theorem 1.11:

$$\begin{aligned} \dots &= \mathbb{E} \left\{ V^T \left(D_{m, \bullet}^\ell(\gamma) (D_{m', \bullet}^{\ell'}(\gamma))^T \right) V' \right\} = \sum_{k, k'} \mathbb{E} \left\{ \langle e_k, V \rangle \left(D_{m, k}^\ell(\gamma) D_{m', k'}^{\ell'}(\gamma) \right) \langle e_{k'}, V' \rangle \right\} = \dots \\ &= \sum_{k, k'} \mathbb{E} \left\{ \langle e_k, V \rangle \left(\frac{\delta_{\ell, \ell'} \delta_{-m, m'} \delta_{-k, k'} (-1)^{2\ell - m + k}}{2\ell + 1} \right) \langle e_{k'}, V' \rangle \right\} = \dots \\ &= \sum_k \mathbb{E} \left\{ \langle e_k, V \rangle \left(\frac{\delta_{\ell, \ell'} \delta_{-m, m'} (-1)^{\ell - m} (-1)^{\ell + k}}{2\ell + 1} \right) \langle e_{-k}, V' \rangle \right\}. \end{aligned}$$

□

5. RANDOM SPIN WEIGHTED FUNCTIONS

A random function between two topological space M, N is a measurable function

$$\Omega \times M \rightarrow N,$$

where $(\Omega, \mathcal{S}, \mathbb{P})$ is a probability space and M, N are endowed with their Borel σ -algebras. If X is a random function, it is always possible to identify Ω with the set N^M of all functions $\omega: M \rightarrow N$, so that $X(\omega, g) = \omega(g)$, and \mathcal{S} with a σ -algebra on N^M having the property that the evaluation map $N^M \times M \ni (\omega, g) \mapsto \omega(g) \in N$ is measurable $\mathcal{S} \otimes \mathcal{B}(M) \rightarrow \mathcal{B}(N)$. This enables us to use a shortened notation $X: M \rightarrow N$. Given a subset $A \subset N^M$, we say that $X \in A$ *almost surely*, if there exists a measurable set $S \subset N^M$ such that $A \subset S$ and $\mathbb{P}\{X \in S\} = 1$.

A *random spin weighted function* with *spin weight* s , here called also *random spin s function* for short, X_s is a random section of the $\mathcal{T}^{\otimes s}$ bundle, i.e. a random function

$$X_s: S^2 \rightarrow \mathcal{T}^{\otimes s},$$

that is almost surely a section of the bundle $\mathcal{T}^{\otimes s} \rightarrow S^2$. By Theorem 2.22, X_s can be equivalently defined as a (complex) random field, i.e. a random function with values in \mathbb{C}

$$X_s: SU(2) \rightarrow \mathbb{C},$$

such that $X_s \in \mathcal{R}(-s)$ almost surely.

We will focus on random fields that are isotropic. This word can be misleading in the case of $SU(2)$, because it can have two different meanings in the common language:

- A random field $X: G \rightarrow \mathbb{C}$ on a group G is said to be isotropic when it is invariant in law under the left pull-back action:

$$(5.1) \quad X(g^{-1}(\cdot)) \stackrel{\text{law}}{=} X(\cdot), \quad \forall g \in G.$$

- A random function on the sphere $X: S^3 \rightarrow \mathbb{C}$ is said to be isotropic if it is invariant in law under the pull-back action by elements of the group $SO(4)$ of (orientation preserving) isometries of the sphere: $X(\phi(\cdot)) \sim X(\cdot)$. In the case of $SU(2) \cong 2S^3$, this is equivalent to invariance under both the left and the right pull-back actions:

$$(5.2) \quad X(g^{-1}(\cdot)g') \stackrel{\text{law}}{=} X(\cdot), \quad \forall g, g' \in SU(2).$$

We see that since $SU(2)$ is both a group and a sphere, the word *isotropic* can be misleading, therefore we will not use it.

Definition 5.1. Let $X: SU(2) \rightarrow \mathbb{C}$ be any random function. We say that X is *left invariant* if it satisfies condition (5.1) and *right-invariant* if it satisfies the analogous condition for the right pull-back action. We say that X is *bi-invariant* if it satisfies condition (5.2), i.e. if it is both left and right-invariant.

The analogous notions of n -weak invariance and invariance for a collection of random fields are considered here as stated in [16, Definition 5.2]. In particular, we will be interested in the weakest among those notions of invariance, which takes into account just the correlation of pairs of variables. We recall it here, for the reader's convenience.

Definition 5.2. A collection of random fields $X^\ell: SU(2) \rightarrow \mathbb{C}$, for $\ell \in L$ is said to be 2-weakly (left, right or bi)-invariant if $\mathbb{E}|X^\ell(g)|^2 < \infty$ for every $\ell \in L$ and $g \in SU(2)$ and if the fields X^ℓ and $X^\ell \circ \phi$ have the same joint moments of order up to 2:

$$\mathbb{E} \{X^\ell \circ \phi(g)\} = \mathbb{E} \{X^\ell(g)\} \quad \text{and}$$

$$\mathbb{E} \{c^i(X^{\ell_1} \circ \phi(g_1)) c^j(X^{\ell_2} \circ \phi(g_2))\} = \mathbb{E} \{c^i(X^{\ell_1}(g_1)) c^j(X^{\ell_2}(g_2))\},$$

for every $\ell_1, \ell_2 \in L$, every $i, j \in \{0, 1\}$, every $g_1, g_2 \in SU(2)$ and for every $\phi: SU(2) \rightarrow SU(2)$ isometry of type L_g , R_g or $L_g \circ R_{g'}$. Here $c: \mathbb{C} \rightarrow \mathbb{C}$ is the complex conjugation $c(z) = \bar{z}$.

By the Stochastic Peter-Weyl theorem [16, Theorem 5.5] and [16, Proposition 5.4], any 2-weakly left-invariant field is automatically in L^2 almost surely:

$$\exists \Omega_0 \subset \Omega \text{ s.t. } \mathbb{P}\{\Omega_0\} = 1 \text{ and } X(\omega, \cdot) \in L^2 \text{ for every } \omega \in \Omega_0.$$

In particular, it can be written in terms of the Hilbert basis (1.3) for some collection of random variables $a_{m,s}^\ell \in \mathbb{C}$:

$$(5.3) \quad X(g) = \sum_{\ell, m, s} a_{m,s}^\ell \phi_{m,s}^\ell(g).$$

By Theorem 1.3, the series (5.3) converges almost surely in $L^2(SU(2))$ and almost surely pointwise in \mathbb{C} , for every $g \in SU(2)$. Let

$$X^\ell := \sum_{m,s} a_{m,s}^\ell \phi_{m,s}^\ell: SU(2) \rightarrow \mathbb{C}^{(2\ell+1) \times (2\ell+1)}$$

be the projections of X onto the subspaces \mathcal{D}^ℓ of matrix coefficients. It follows that if the field X is (2-weakly) left, right or bi invariant, then the collection of fields X^ℓ , for $s \in \frac{1}{2}\mathbb{Z}$ is (2-weakly) left, right or bi invariant as well and viceversa, see [16, Proposition 5.4].

Remarkably, again by [16, Proposition 5.4], the fields X^ℓ are uncorrelated for different $\ell \in \frac{1}{2}\mathbb{N}$, therefore Theorem 1.2. A further decomposition gives

$$X_{\bullet,s}^\ell := \sum_m a_{m,s}^\ell \phi_{m,s}^\ell; \quad X_{m,\bullet}^\ell := \sum_s a_{m,s}^\ell \phi_{m,s}^\ell.$$

The fields $X_{\bullet,s}^\ell, X_{m,\bullet}^\ell: SU(2) \rightarrow \mathbb{C}^{2\ell+1}$ are, respectively, the projections of X onto the subspaces $\mathcal{D}_{\bullet,s}^\ell$ and $\mathcal{D}_{m,\bullet}^\ell$. It follows that if the field X is (2-weakly) left-invariant, then the collection of fields $X_{\bullet,s}^\ell$, for $s \in \frac{1}{2}\mathbb{Z}$ is (2-weakly) left-invariant as well. However, the fields $X_{\bullet,s}^\ell, X_{\bullet,s'}^\ell$ need not to be uncorrelated.

Example 5.3. If $X = \sum_m a_{m,s}^\ell \phi_{m,s}^\ell$ is left-invariant, then $X' = \sum_m a_{m,s}^\ell (\phi_{m,s}^\ell + \phi_{m,s'}^\ell)$ is again left-invariant. In the second case, the corresponding fields $X_{\bullet,s}^\ell$ and $X_{\bullet,s'}^\ell$ are clearly correlated.

The same discussion can be repeated for right-invariance. Notice that the law of X can be thought as the law of the collections of random vectors

$$a_{\bullet,s}^\ell = \begin{pmatrix} a_{-\ell,s}^\ell \\ \vdots \\ a_{\ell,s}^\ell \end{pmatrix} \in \mathbb{C}^{2\ell+1} \quad \text{or} \quad a_{m,\bullet}^\ell = \begin{pmatrix} a_{m,-\ell}^\ell \\ \vdots \\ a_{m,\ell}^\ell \end{pmatrix} \in \mathbb{C}^{2\ell+1},$$

with $\ell \in \frac{1}{2}\mathbb{N}$, $m, s = -\ell, \dots, \ell$, corresponding respectively to the laws of $X_{\bullet,s}^\ell$ and $X_{m,\bullet}^\ell$. This observation allows us to adopt the point of view of the Section 4 above. The following theorem is similar to [16, Lemma 6.3].

Lemma 5.4. *Let $\gamma \in SU(2)$ be a random element distributed with the Haar measure. A square integrable random field X is, respectively, 2-weakly or strongly left-invariant, if and only if the collection of random vectors $\overline{a_{\bullet,s}^\ell}$, with $\ell \in \frac{1}{2}$ and $s \in \{-\ell, \dots, \ell\}$ is 2-weakly or strongly D -invariant:*

$$\left(\overline{a_{\bullet,s}^\ell}\right)_{\ell,s} \sim_{2-w} \text{ or } \stackrel{\text{law}}{=} \left(D^\ell(\gamma) \overline{a_{\bullet,s}^\ell}\right)_{\ell,s}.$$

Similarly, if X is 2-weakly or strongly right-invariant, then the collection of random vectors $(a_{m,\bullet}^\ell)_{\ell,m}$ is 2-weakly or strongly D -invariant:

$$(a_{m,\bullet}^\ell)_{\ell,m} \sim_{2-w} \text{ or } \stackrel{\text{law}}{=} (D^\ell(\gamma) a_{m,\bullet}^\ell)_{\ell,m}.$$

Proof. We have for any $g, z \in SU(2)$:

$$X_{\bullet,s}^\ell(g^{-1}z) = (a_{\bullet,s}^\ell)^T \phi_{\bullet,s}^\ell(g^{-1}z) = (a_{\bullet,s}^\ell)^T D^\ell(g^{-1}) \phi_{\bullet,s}^\ell(z) = \left(\overline{D^\ell(g) a_{\bullet,s}^\ell}\right)^T \phi_{\bullet,s}^\ell.$$

Similarly, from the point of view of the right pull back action:

$$X_{m,\bullet}^\ell(zg) = (\phi_{m,\bullet}^\ell(z))^T a_{m,\bullet}^\ell(zg) = (\phi_{m,\bullet}^\ell(z))^T D^\ell(g) a_{m,\bullet}^\ell(z),$$

because $\phi_{m,\bullet}^\ell$ is the m^{th} row of the matrix ϕ^ℓ . We conclude the proof by an application of Theorem 4.5. \square

Example 5.5. We leave to the reader to check that, in particular, if $a_{m,s}^\ell = D_{s,m}^\ell(\gamma) = \overline{D_{m,s}^\ell(\gamma^{-1})}$, then the field $X_{\bullet,s}^\ell$ is strongly left-invariant, $X_{m,\bullet}^\ell$ is strongly right-invariant and X^ℓ is strongly bi-invariant.

We immediately get the following Corollary.

Corollary 5.6. *Let $\gamma \in SU(2)$ be a random element distributed with the Haar measure. A square integrable random field X is, respectively, 2-weakly or strongly left-invariant, if and only if*

$$X(\cdot) \sim_{2-w} \text{ or } \stackrel{\text{law}}{=} X(\gamma(\cdot)).$$

It is, respectively, 2-weakly or strongly right-invariant, if and only if

$$X(\cdot) \sim_{2-w} \text{ or } \stackrel{\text{law}}{=} X((\cdot)\gamma).$$

Moreover, this proves the first part of Theorem 1.4.

Theorem 5.7. *Let $X: SU(2) \rightarrow \mathbb{C}$ be a 2-weakly left-invariant random field. Then $\mathbb{E}\{a_{m,s}^\ell\} = 0$ whenever $\ell \neq 0$. Moreover, the random variables $a_{m,s}^\ell$ have the following correlation structure:*

$$\mathbb{E}\left\{\overline{(a_{m,s}^\ell)} a_{m',s'}^{\ell'}\right\} = \delta_{m,m'} \frac{\delta_{\ell,\ell'}}{2\ell+1} \mathbb{E}\left\{\langle X_{\bullet,s}^\ell, X_{\bullet,s'}^\ell \rangle_{L^2(SU(2))}\right\}.$$

In particular, the variance of $a_{m,s}^\ell$ does not depend on m . Furthermore:

$$(5.4) \quad \mathbb{E}\left\{a_{m,s}^\ell(a_{m',s'}^{\ell'})\right\} = \delta_{-m,m'} (-1)^{\ell-m} \frac{\delta_{\ell,\ell'}}{2\ell+1} \left(\sum_{k=-\ell}^{\ell} (-1)^{\ell+k} \mathbb{E}\left\{a_{k,s}^\ell(a_{-k,s'}^\ell)\right\} \right).$$

Proof. A combination of Lemma 5.4 and Theorem 4.7 gives the thesis. \square

By repeating the same arguments, we obtain an analogous statement for right-invariant random fields.

Theorem 5.8. *Let $X: SU(2) \rightarrow \mathbb{C}$ be a 2-weakly right-invariant random field. Then $\mathbb{E}\{a_{m,s}^\ell\} = 0$ whenever $\ell \neq 0$. Moreover, the random variables $a_{m,s}^\ell$ have the following correlation structure:*

$$\mathbb{E}\left\{\overline{(a_{m,s}^\ell)} a_{m',s'}^{\ell'}\right\} = \delta_{s,s'} \frac{\delta_{\ell,\ell'}}{2\ell+1} \mathbb{E}\left\{\langle X_{m,\bullet}^\ell, X_{m',\bullet}^\ell \rangle_{L^2(SU(2))}\right\}.$$

In particular, the variance of $a_{m,s}^\ell$ does not depend on s . Furthermore:

$$(5.5) \quad \mathbb{E}\left\{a_{m,s}^\ell(a_{m',s'}^{\ell'})\right\} = \delta_{-s,s'} (-1)^{\ell-s} \frac{\delta_{\ell,\ell'}}{2\ell+1} \left(\sum_{k=-\ell}^{\ell} (-1)^{\ell+k} \mathbb{E}\left\{a_{m,k}^\ell(a_{m',-k}^\ell)\right\} \right).$$

Theorem 5.9. *Let $X: SU(2) \rightarrow \mathbb{C}$ be a 2-weakly bi-invariant random field. Then $\mathbb{E}\{a_{m,s}^\ell\} = 0$ whenever $\ell \neq 0$. Moreover, the random variables $a_{m,s}^\ell$ have the following correlation structure:*

$$(5.6) \quad \mathbb{E}\left\{\overline{(a_{m,s}^\ell)} a_{m',s'}^{\ell'}\right\} = \delta_{m,m'} \frac{\delta_{\ell,\ell'}}{(2\ell+1)^2} \delta_{s,s'} \mathbb{E}\left\{\|X^\ell\|_{L^2(SU(2))}^2\right\}.$$

In particular, the variance of $a_{m,s}^\ell$ depends only ℓ . Furthermore:

$$\mathbb{E}\left\{a_{m,s}^\ell(a_{m',s'}^{\ell'})\right\} = \delta_{-m,m'} \frac{\delta_{\ell,\ell'}}{(2\ell+1)^2} \delta_{-s,s'} (-1)^{\ell-m} (-1)^{\ell-s} \mathbb{E}\left\{\langle \overline{X^\ell}, X^\ell \rangle_{L^2(SU(2))}\right\} i^{2\ell}.$$

Proof. Equation (5.6) follows by the fact that X is both left and right invariant, thus both Theorem 5.7 and Theorem 5.8 hold. To prove the second equation, let us observe that, because of (5.4) and (5.5), we have that there exists a constant $C \in \mathbb{C}$ such that:

$$\mathbb{E}\left\{a_{m,s}^\ell a_{-m,-s}^\ell\right\} = C (-1)^{\ell-m} (-1)^{\ell-s}.$$

We find the value of C with the following computation.

$$\mathbb{E} \left\{ \left\langle \overline{X^\ell}, X^\ell \right\rangle \right\} = \mathbb{E} \left\{ \left\langle \sum_{m,s} \overline{a_{m,s}^\ell \phi_{m,s}^\ell}, \sum_{m',s'} \overline{a_{m',s'}^\ell \phi_{m',s'}^\ell} \right\rangle \right\} = \dots$$

Using Proposition 3.4 this is

$$\begin{aligned} \dots &= \sum_{m,m',s,s'} \mathbb{E} \left\{ \left\langle \overline{a_{m,s}^\ell \phi_{-m,-s}^\ell} (-1)^{2\ell+m-s}, a_{m',s'}^\ell \phi_{m',s'}^\ell \right\rangle \right\} = \dots \\ &= \sum_{m,s} (-1)^{2\ell+m-s} \mathbb{E} \{ a_{m,s}^\ell a_{-m,-s}^\ell \} = \dots \\ &= \sum_{m,s} (-1)^{2\ell+m-s} C (-1)^{\ell-m} (-1)^{\ell-s} = C (2\ell+1)^2 i^{2\ell}. \end{aligned}$$

□

5.1. Spectral probability. We use the Hilbert basis formed by the normalized Wigner functions, i.e. the spherical harmonics $\phi_{m,s}^\ell$, to define a notion of *spin* for every $f \in L^2(SU(2))$.

Definition 5.10. Let $F \in L^2(SU(2))$, and let $\|F\|$ denote its $L^2(SU(2))$ norm. There exist coefficients $a_{m,s}^\ell \in \mathbb{C}$ such that

$$F = \sum_{\ell,m,s} a_{m,s}^\ell \phi_{m,s}^\ell.$$

We define the *spectral probability* of F to be the probability measure $\Sigma[F]$ on the space $\Lambda = \{(\ell, m, s)\}$ (defined in (1.5)) such that

$$\Sigma[F](\{(\ell, m, s)\}) := \frac{|a_{m,s}^\ell|^2}{\|F\|^2}.$$

Similarly, we define the *left spin* and the *right spin* of F as the probability measures on $\frac{1}{2}\mathbb{Z}$ such that for every singleton $m, s \in \frac{1}{2}\mathbb{Z}$, we have

$$(5.7) \quad LS[F](\{m\}) := \frac{\sum_{\ell,s} |a_{m,s}^\ell|^2}{\|F\|^2} \quad \text{and} \quad RS[F](\{s\}) := \frac{\sum_{\ell,m} |a_{m,s}^\ell|^2}{\|F\|^2}.$$

Moreover, we call *bi spin* of F , the probability on $\frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z}$ defined for every singleton (m, s) as

$$BS[F](\{(m, s)\}) := \sum_{\ell \in \frac{1}{2}\mathbb{N}} \frac{|a_{m,s}^\ell|^2}{\|F\|^2}.$$

In particular, a function F has pure right spin $= -s$, and thus it is section of $\mathcal{T}^{\otimes s}$, if and only if $RS[F]$ is the delta measure on s (and similarly in the case of pure left spin).

Let us consider a square integrable random field, i.e. a random function $X: SU(2) \rightarrow \mathbb{C}$, such that $X \in L^2(SU(2))$ almost surely, so that

$$X = \sum_{\ell,m,s} a_{m,s}^\ell \phi_{m,s}^\ell.$$

Then we define another associated spectral probability on Λ :

$$\Sigma\mathbb{E}[X](\{\ell, m, s\}) = \frac{\mathbb{E}|a_{m,s}^\ell|^2}{\mathbb{E}\|X\|^2}.$$

This has to be compared with the expectation of the random probability $\Sigma[X]$, that is

$$\mathbb{E}\Sigma[X](\{\ell, m, s\}) = \mathbb{E} \left(\frac{|a_{m,s}^\ell|^2}{\|X\|^2} \right).$$

Similarly, we define $LS\mathbb{E}[X]$, $RS\mathbb{E}[X]$, and $BS\mathbb{E}[X]$. Thus, we have 2 probability measures on Λ , that are associated to the random field X and that give a sense of the distribution of the left and right spin of X and of its homogeneous components, i.e of the relative magnitude of the random variables $a_{m,s}^\ell$.

In general, the probabilities $\Sigma\mathbb{E}[X]$ and $\mathbb{E}\Sigma[X]$ might be different, in that the first takes into account only the correlation structure of the variables $a_{m,s}^\ell$, i.e. it depends on the field X up to 2-weak equivalence. In fact, even less, it just depends on the marginal distributions of the coefficients. For this reason, we will call $\Sigma\mathbb{E}[X]$, the *weak spectral probability* of X . On the other hand, the expected spectral probability $\mathbb{E}\Sigma[X]$ depends on higher moments and on the joint distribution, thus it should be considered as a more descriptive data, thus we call it *strong spectral probability* of X .

Theorem 5.11. *Let X be 2-weakly left-invariant random field. Then, the weak spectral measure $\Sigma\mathbb{E}[X]$ is uniform on the sets*

$$(5.8) \quad \{\ell\} \times \{-\ell, \dots, \ell\} \times \{s\},$$

for all $\ell \in \frac{1}{2}\mathbb{N}$. If X is strongly left-invariant, then the same is true for the strong spectral measure $\mathbb{E}\Sigma[X]$. The analogous statement is true when X is right-invariant, in which case the measure is uniform on the sets of the form

$$\{\ell\} \times \{m\} \times \{-\ell, \dots, \ell\}.$$

Proof. From Theorem 5.7 we see that the variance of $a_{m,s}^\ell$ does not depend on m , when X is 2-weakly left-invariant, which means exactly that $\Sigma\mathbb{E}[X](\ell, m, s)$ is uniform on sets of the form (5.8). In case X is strongly left-invariant, we have that the field

$$Y = \frac{1}{\|X\|} X$$

is again strongly left-invariant. Therefore we can apply the first part of the theorem to it, but in this case

$$LS\mathbb{E}[Y] = \mathbb{E}LS[Y] = \mathbb{E}LS[X].$$

□

Corollary 5.12. *Let $X = \sum_{\ell,s} X_{\bullet,s}^\ell$ be 2-weakly left-invariant random field. Then for all $\ell \in \frac{1}{2}\mathbb{N}$ and $s \in \{-\ell, \dots, \ell\}$, the measure $LS\mathbb{E}[X_{\bullet,s}^\ell]$ is the uniform probability on $\{-\ell, \dots, \ell\}$. If X is strongly left-invariant, then also $\mathbb{E}LS[X_{\bullet,s}^\ell] = LS\mathbb{E}[X_{\bullet,s}^\ell]$ is uniform. The analogous statement is true when X right-invariant.*

Theorem 5.13. *Any probability measure μ on $\{-\ell, \dots, \ell\}$ can be realized as $\mu = \mathbb{E}RS[X^\ell]$, with X^ℓ being a strongly left-invariant random field.*

Proof. Let $\gamma_{-\ell}, \dots, \gamma_\ell \in SU(2)$ be independent uniform random elements of $SU(2)$. Define

$$X(g) := \sum_s \mu(\{s\})^{\frac{1}{2}} \phi_{s,s}^\ell(\gamma \cdot g) = \sum_s \mu(\{s\})^{\frac{1}{2}} (D_{s,\bullet}^\ell(\gamma))^T \phi_{\bullet,s}^\ell(g).$$

X is strongly left-invariant by construction. Moreover, the coefficients are $a_{m,s}^\ell = \mu(\{s\})^{\frac{1}{2}} D_{s,m}^\ell(\gamma)$, so that the random vectors $a_{\bullet,s}^\ell$ are orthogonal almost surely and have constant length, due to the unitarity of $D^\ell(g)$, for any $g \in SU(2)$. It follows that

$\|X\|_{L^2(SU(2))}^2 = C$ is constant almost surely. Therefore, the strong and weak right spin measures are equal to μ :

$$\mathbb{E}RS[X](\{s\}) = \sum_m \mathbb{E}(\mu(\{s\})|D_{s,m}^\ell(\gamma)|^2) = \mu(\{s\}).$$

□

This concludes the proof of Theorem 1.4. The following result implies Theorem 1.6.

Theorem 5.14. *Let X be a 2-weakly bi-invariant random field. Then the weak spectral probability $\Sigma\mathbb{E}[X]$ is uniform on all sets of the form*

$$\{\ell\} \times \{-\ell, \dots, \ell\} \times \{-\ell, \dots, \ell\}.$$

If X is strongly left-invariant, then the same is true for the strong spectral probability $\mathbb{E}\Sigma[X]$.

Proof. Follows from Theorem 5.9 and the same argument used in the proof of the previous theorem. □

Corollary 5.15. *Let $X = \sum_\ell X^\ell$ be a 2-weakly bi-invariant random field. Then for all $\ell \in \frac{1}{2}\mathbb{N}$ and $m, s \in \{-\ell, \dots, \ell\}^2$, the measure $BS\mathbb{E}[X^\ell]$ is the uniform probability on $\{-\ell, \dots, \ell\}$. If X is strongly left-invariant, then $\mathbb{E}BS[X_{\bullet,s}^\ell] = BS\mathbb{E}[X_{\bullet,s}^\ell]$.*

5.2. The Gaussian case. Let us turn our focus to everyone's favorite random fields, the Gaussian ones. We say that $V \in \mathbb{R}^N$ is a real Gaussian random vector if all linear combinations of its components are Gaussian random variables. For simplicity, we will only consider the case of centered Gaussian. We say that a complex random variable $a \in \mathbb{C}$ is complex Gaussian if and only if, it is a circularly symmetric Gaussian random vector in \mathbb{R}^2 , i.e. there exists $\sigma \in \mathbb{R}$ and $\xi_1, \xi_2 \sim N(0, \sigma^2)$ independent such that

$$a = \sigma \left(\frac{1}{\sqrt{2}}\xi_1 + i\frac{1}{\sqrt{2}}\xi_2 \right),$$

in this case we write $a \sim N_{\mathbb{C}}(0, \sigma)$. We say that $V \in \mathbb{C}^N$ is a complex Gaussian random vector if all \mathbb{C} -linear combinations of its components are complex Gaussian. The distribution of a real Gaussian random vector $V \in \mathbb{C}^N = \mathbb{R}^{2N}$ is determined by the correlation matrices

$$K := \mathbb{E}\{V\bar{V}^T\} \quad \text{and} \quad C := \mathbb{E}\{VV^T\}.$$

A real Gaussian random vector is complex Gaussian if and only if $C = 0$ and we write $V \sim N_{\mathbb{C}}(0, K)$.

Definition 5.16. A random field $X: SU(2) \rightarrow \mathbb{C}$ is Gaussian (complex or real) if for every finite set of points $g_i \in SU(2)$, the random vector is Gaussian (complex or real)

$$(X(g_1), \dots, X(g_N)) \in \mathbb{C}^N.$$

It is straightforward to see that an almost surely square integrable random field $X: SU(2) \rightarrow \mathbb{C}$ is Gaussian (complex or real) if and only if the coefficients $a_{m,s}^\ell$ of the decomposition

$$X = \sum_{\ell, m, s} a_{m,s}^\ell \phi_{m,s}^\ell$$

are a family of jointly Gaussian (complex or real) random variables¹⁷, i.e. any finite subset of them form a (complex or real) Gaussian random vector. In particular, if X is complex Gaussian, then

$$\mathbb{E}\{a_{m,s}^\ell a_{m',s'}^{\ell'}\} = 0,$$

for every $m, m', s, s', \ell, \ell' \in \frac{1}{2}\mathbb{Z}$.

Theorem 5.17. *A complex Gaussian random field $X: SU(2) \rightarrow \mathbb{C}$ is 2-weakly (left, right or bi)-invariant if and only if it is strongly (left, right or bi)-invariant. In particular, it is left-invariant if and only if the fields X^ℓ are independent, for all $\ell \neq 0$ we have*

$$(5.9) \quad a_{\bullet,s}^\ell \sim N(0, \sigma(\ell, s)^2 \mathbb{1}_{(2\ell+1)})^{18}$$

for some $\sigma(\ell, s) \in \mathbb{R}$ and there are constants $K(s, s') \in \mathbb{C}$ such that

$$\mathbb{E}\{a_{\bullet,s}^\ell \overline{(a_{\bullet,s'}^\ell)^T}\} = K(s, s') \mathbb{1}_{(2\ell+1)}.$$

The analogous statement holds if X is right-invariant. Moreover, X is bi-invariant if and only if all the variables $a_{m,s}^\ell$ form an independent family and condition (5.9) holds with $\sigma(\ell, s) = \sigma(\ell)$.

Proof. The first statement follows from Lemma 5.4 and the fact that the distribution of a Gaussian vector is uniquely determined by the correlation structure of the variables $a_{m,s}^\ell$. By a further examination of the characterization given in Theorems 5.7 and Theorem 5.9 we complete the proof. \square

Moreover, in [1, 3] it is proved that the only left-invariant random fields for which the variables $a_{m,s}^\ell$ are independent are the Gaussian ones. This, combined with Theorem 5.17 and Theorem 1.3, proves Corollary 1.9.

BIBLIOGRAPHY

1. Paolo Baldi, Domenico Marinucci, and Veeravalli Varadarajan, *On the characterization of isotropic Gaussian fields on homogeneous spaces of compact groups*, Electronic Communications in Probability **12** (2007), no. none, 291 – 302.
2. Paolo Baldi and Maurizia Rossi, *Representation of Gaussian isotropic spin random fields*, Stochastic Process. Appl. **124** (2014), no. 5, 1910–1941. MR 3170229
3. Paolo Baldi and Stefano Trapani, *Fourier coefficients of invariant random fields on homogeneous spaces of compact Lie groups*, Annales de l'Institut Henri Poincaré, Probabilités et Statistiques **51** (2015), no. 2, 648 – 671.
4. Lionel Bérard Bergery and Jean-Pierre Bourguignon, *Laplacians and Riemannian submersions with totally geodesic fibres*, Illinois Journal of Mathematics **26** (1982), no. 2, 181 – 200.
5. Yaiza Canzani and Boris Hanin, *Local universality for zeros and critical points of monochromatic random waves*, 2020.
6. I. Chavel, B. Randol, and J. Dodziuk, *Eigenvalues in riemannian geometry*, ISSN, Elsevier Science, 1984.
7. Michael Eastwood and Paul Tod, *Edth-a differential operator on the sphere*, Mathematical Proceedings of the Cambridge Philosophical Society **92** (1982), no. 2, 317.
8. Daryl Geller and Domenico Marinucci, *Spin wavelets on the sphere*, J. Fourier Anal. Appl. **16** (2010), no. 6, 840–884. MR 2737761
9. Allen Hatcher, *Algebraic topology*, Cambridge Univ. Press, Cambridge, 2000.

¹⁷Clearly if $a_{m,s}^\ell$ are jointly Gaussian, then X is a Gaussian random field. The converse, follows from the fact that

$$a_{m,s}^\ell = \int_{SU(2)} X(g) \overline{\phi_{m,s}^\ell(g)} d\mu(g),$$

hence $a_{m,s}^\ell$ can be expressed as the almost sure limit of a sequence of linear combination of random variables of the form $X(g)$, therefore it is Gaussian.

¹⁸Here, $\mathbb{1}_{(2\ell+1)}$ stands for the identity matrix of size $(2\ell+1) \times (2\ell+1)$.

10. Dale Husemoller, *Fibre bundles*, third ed., Graduate Texts in Mathematics, vol. 20, Springer-Verlag, New York, 1994. MR 1249482
11. D. Huybrechts and Springer-Verlag (Berlin), *Complex geometry: An introduction*, Universitext (Berlin. Print), Springer, 2005.
12. Alexander Kirillov, Jr, *An introduction to Lie groups and Lie algebras*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2008.
13. Ruishi Kuwabara, *On spectra of the Laplacian on vector bundles*, J. Math., Tokushima Univ. **16** (1982), 1–23 (English).
14. J.M. Lee, *Introduction to riemannian manifolds*, Graduate Texts in Mathematics, Springer International Publishing, 2019.
15. Anatoliy Malyarenko, *Invariant random fields in vector bundles and application to cosmology*, Ann. Inst. Henri Poincaré Probab. Stat. **47** (2011), no. 4, 1068–1095. MR 2884225
16. Domenico Marinucci and Giovanni Peccati, *Random fields on the sphere: Representation, limit theorems and cosmological applications*, London Mathematical Society Lecture Note Series, Cambridge University Press, 2011.
17. D. Montgomery, H. Samelson, and C. T. Yang, *Exceptional orbits of highest dimension*, Annals of Mathematics **64** (1956), no. 1, 131–141.
18. E. T. Newman and R. Penrose, *Note on the Bondi-Metzner-Sachs group*, J. Mathematical Phys. **7** (1966), 863–870. MR 194172
19. Takahashi Tsunero Ochiai, Takushiro, *The group of isometries of a left invariant riemannian metric on a Lie group.*, Mathematische Annalen **223** (1976), 91–96.
20. D. A. Varshalovich, A. N. Moskalev, and V. K. Khersonskii, *Quantum theory of angular momentum*, World Scientific, Singapore, 1988.
21. Steve Zelditch, *real and complex zeros of riemannian random waves*, IJBF Online, K. G. Saur, 2009.

LABORATOIRE DE MATHÉMATIQUE JEAN LERAY, NANTES UNIVERSITY, NANTES, FRANCE

Current address: Laboratoire de Mathématique Jean Leray, 2, rue de la Houssinière BP 92208

Email address: michele.stecconi@univ-nantes.fr

Received 30/JUL/2021