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Modular Forms of Degree 2 and Curves of Genus 2 in Characteristic 2

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We describe the ring of modular forms of degree 2 in characteristic 2 using its relation with curves of genus 2.

1 Introduction

In the 1960s, Igusa determined the ring of Siegel modular forms of degree 2 using the close connection between the moduli of principally polarized abelian surfaces and the moduli of curves of genus 2, see [9]. Since curves of genus 2 in characteristic different from 2 can be described by binary sextics, he could use the invariant theory of binary sextics and establish the link with modular forms using Thomae's formulas that express theta constants in terms of cross ratios of zeros of binary sextics. Recently, in a joint work with Carel Faber, we exhibited in [2] a more direct version of this without the detour of Thomae's formulas and used it for describing vector-valued Siegel modular forms of degree 2.

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In positive characteristic, modular forms of degree 2 are described as sections of powers of the determinant bundle L of the Hodge bundle $\mathbb{E}$ on the moduli space $\mathcal{A}_2 \otimes \mathbb{F}_p$ of principally polarized abelian surfaces in characteristic p. It turns out that for $p \geq 5$ the situation is very close to the case of characteristic 0. For $p \geq 5$, the graded ring

$$\mathcal{R}_2(\mathbb{F}_p) = \oplus_k H^0(\mathcal{A}_2 \otimes \mathbb{F}_p, L^k)$$

has generators of weight $4, 6, 10, 12$, and 35 and there is a relation of degree 70 expressing the square of the odd weight generator in terms of the even weight generators, just as in characteristic 0. In fact, Igusa determined the ring $\mathcal{R}_2(\mathbb{Z})$ of integral modular forms and the reduction map $\mathcal{R}_2(\mathbb{Z}) \to \mathcal{R}_2(\mathbb{F}_p)$ is surjective for $p \geq 5$. We refer to [1, 10, 11, 15, 16].

The situation in characteristics 2 and 3 is different. The ring $\mathcal{R}_2(\mathbb{F}_3)$ was determined in [7] using the relation with binary sextics.

In characteristic 2, curves of genus 2 can no longer be described by binary sextics. Nevertheless, as Igusa showed in [9], there is still a close relationship with the invariant theory of binary sextics. In this paper we use invariants of curves of genus 2 in characteristic 2 to determine the ring of modular forms of degree 2. The result is as follows.

Theorem 1.1. The ring $\mathcal{R}_2(\mathbb{F}_2)$ is generated by modular forms of weights 1, 10, 12, 13, and 48 with one relation of weight 52:

$$\mathcal{R}_2(\mathbb{F}_2) = \mathbb{F}_2[\psi_1, \chi_{10}, \psi_{12}, \chi_{13}, \chi_{48}]/(R)$$

with $R = \chi_{13}^4 + \psi_1^3 \chi_{10} \chi_{13}^3 + \psi_1^4 \chi_{48} + \chi_{10}^4 \psi_{12}$. The ideal of cusp forms is generated by χ_{10}, χ_{13} and χ_{48}.

The modular form ψ_1 is the Hasse invariant vanishing on the locus of non-ordinary abelian surfaces; the form χ_{10} vanishes doubly on the locus of abelian surfaces that are products of elliptic curves. All the forms $\psi_1, \chi_{10}, \psi_{12}, \chi_{13}$ and χ_{48} are constructed using invariant theory, see Section 4.

We point out that we find non-zero (regular) vector-valued modular forms of weights not allowed in characteristic 0, like $(3, -1)$ or $(2, 0)$.

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2 Modular Forms

We denote by $\mathcal{A}_g$ the moduli stack of principally polarized abelian varieties of dimension g and by $\mathbb{E}_g$ the Hodge bundle on $\mathcal{A}_g$. It extends to a Faltings–Chai-type toroidal compactification $\tilde{\mathcal{A}}_g$. We write $L = \det(\mathbb{E}_g)$. For $g > 1$ sections of L^k over $\mathcal{A}_g$ extend automatically to $\tilde{\mathcal{A}}_g$, a property known as the Koecher principle, see [5, Prop. 1.5, p. 140]. We write $M_k(\Gamma_g) = H^0(\tilde{\mathcal{A}}_g \otimes \mathbb{F}_2, L^k)$ for the space of scalar-valued modular forms of weight k. Moreover, we set

$$\mathcal{R}_g(\mathbb{F}_2) = \oplus_k M_k(\Gamma_g)\,,$$

the ring of scalar-valued modular forms of degree g in characteristic 2. It is a finitely generated $\mathbb{F}_2$-algebra. It is well-known that $H^0(\tilde{\mathcal{A}}_g, L^k) = (0)$ for $k < 0$. For $g = 1$ we know by Deligne [3] that $\mathcal{R}_1(\mathbb{F}_2) = \mathbb{F}_2[c_1, \Delta]$ with c_1 of weight 1 and Δ a cusp form of weight 12.

We are interested in the case $g = 2$. In the following we shall simply write $\mathbb{E}$ for the Hodge bundle $\mathbb{E}_2$.

In the Chow group with rational coefficients of codimension 1 cycles of $\tilde{\mathcal{A}}_2$ the class of the locus $\mathcal{A}_{1,1}$ of products of elliptic curves satisfies the relation

$$2[\overline{\mathcal{A}}_{1,1}] + [D] = 10\,\lambda_1\,, \tag{1}$$

where λ_1 is the 1st Chern class of $\mathbb{E}$ and D the divisor added to compactify $\mathcal{A}_2$, see [14, (8.4)]. Therefore, there exists a non-zero modular form χ_{10} of weight 10 over $\mathbb{F}_2$ vanishing doubly on the divisor $\overline{\mathcal{A}}_{1,1} \otimes \mathbb{F}_2$. As the cycle relation shows χ_{10} is a cusp form. The form is determined up to a non-zero multiplicative constant and we shall normalize χ_{10} later.

Since $\mathcal{A}_{1,1}$ is the image of a degree 2 morphism $\mathcal{A}_1 \times \mathcal{A}_1 \to \mathcal{A}_{1,1}$ by restriction to the locus $\mathcal{A}_{1,1} \otimes \mathbb{F}_2$ we find an exact sequence

$$0 \to M_{k-10}(\Gamma_2) \to M_k(\Gamma_2) \to \mathrm{Sym}^2(M_k(\Gamma_1))\,. \tag{2}$$

Another divisor that we will use is the locus V_1 of principally polarized abelian surfaces of 2-rank ≤ 1. The cycle class of this locus is known; it is the vanishing locus of the map $L \to L^{\otimes 2}$ induced by Verschiebung on the universal abelian surface and we thus find (cf. [6])

$$[V_1] = \lambda_1\,.$$

Hence, there exists a modular form ψ_1 of weight 1, determined up to a non-zero multiplicative constant, with divisor V_1. It is called the Hasse invariant. This is not a cusp form since it restricts to the Hasse invariant c_1 on the boundary component $\mathcal{A}_1^* \otimes \mathbb{F}_2$ in the Satake compactification $\mathcal{A}_2^* \otimes \mathbb{F}_2$. We shall normalize ψ_1 later.

The existence of ψ_1 implies $\dim M_k(\Gamma_2) \leq \dim M_{k+1}(\Gamma_2)$. The exact sequence (2) and the structure of $\mathcal{R}_1(\mathbb{F}_2)$ imply that

$$\dim M_k(\Gamma_2) = 1 \quad \text{for } k = 1, \ldots, 9 \quad \text{and } \dim M_{10}(\Gamma_2) = 2. \tag{3}$$

In order to estimate $\dim M_k(\Gamma_2)$ for larger k we can apply semi-continuity (cf. [8, Thm. 12.8]) to L defined over $\tilde{\mathcal{A}}_2/\mathbb{Z}$ and deduce $\dim H^0(\tilde{\mathcal{A}}_2 \otimes \mathbb{C}, L^k) \leq \dim H^0(\tilde{\mathcal{A}}_2 \otimes \mathbb{F}_2, L^k)$. Using this and the exact sequence (2) we can deduce

$$\dim M_{11}(\Gamma_2) = 2, \quad \text{and } 3 \leq \dim M_k(\Gamma_2) \leq 4 \text{ for } 12 \leq k \leq 19, \tag{4}$$

but we will deduce this in another way later, see (7).

Similarly, we can consider vector-valued modular forms. For each irreducible representation ρ of $\mathrm{GL}(2, \mathbb{Q})$ there is a corresponding bundle $\mathbb{E}_\rho$. Since we are dealing with $g = 2$ the ρ in question correspond to pairs (j, k) of integers with $j \geq 0$ and $\mathbb{E}_\rho = \mathrm{Sym}^j(\mathbb{E}) \otimes \det(\mathbb{E})^{\otimes k}$. We write $M_{j,k}(\Gamma_2)$ for $H^0(\mathcal{A}_2 \otimes \mathbb{F}_2, \mathrm{Sym}^j(\mathbb{E}) \otimes \det(\mathbb{E})^{\otimes k})$ and $S_{j,k}(\Gamma_2)$ for the space of cusp forms. Again, these sections extend over $\tilde{\mathcal{A}}_2 \otimes \mathbb{F}_2$ by the Koecher principle. But note that $\rho \otimes \mathbb{F}_2$ might be reducible.

Lemma 2.1. We have $\dim S_{6,8}(\Gamma_2) \geq 1$ and every element of $S_{6,8}(\Gamma_2)$ vanishes on $\mathcal{A}_{1,1} \otimes \mathbb{F}_2$. If $\dim S_{6,8}(\Gamma_2) \geq 2$ then $M_{6,-2}(\Gamma_2) \neq 0$.

Proof. We know that over $\mathbb{C}$ we have $\dim S_{6,8}(\Gamma_2) = 1$. Hence, by semi-continuity of dimensions there exists a cusp form of weight $(6, 8)$ in characteristic 2. Note that the Hodge bundle $\mathbb{E}$ pulled back via

$$\mathcal{A}_1 \times \mathcal{A}_1 \to \mathcal{A}_{1,1} \hookrightarrow \mathcal{A}_2$$

can be written as $p_1^*(\mathbb{E}_1) \oplus p_2^*(\mathbb{E}_1)$ with $\mathbb{E}_1$ the Hodge bundle of $\mathcal{A}_1$ and p_1, p_2 the two projections. The pullback of $\chi_{6,8}$ to $(\mathcal{A}_1 \times \mathcal{A}_1) \otimes \mathbb{F}_2$ must vanish because it lands in

$$\bigoplus_{j=0}^{6} S_{14-j}(\Gamma_1) \otimes S_{8+j}(\Gamma_1)$$

and this is (0). If we develop a form $\varphi \in S_{6,8}(\Gamma_2)$ vanishing on $\mathcal{A}_{1,1} \otimes \mathbb{F}_2$ in the normal direction of the divisor $\mathcal{A}_{1,1} \otimes \mathbb{F}_2$ we see that the 1st ("non-constant") term in its Taylor expansion lands in

$$\bigoplus_{j=0}^{6} S_{15-j}(\Gamma_1) \otimes S_{9+j}(\Gamma_1)$$

and the only non-zero factor here is $S_{12}(\Gamma_1) \otimes S_{12}(\Gamma_1)$ and the image is symmetric under the interchange of factors. But $\dim S_{12}(\Gamma_1) = 1$. Therefore, if φ_1, φ_2 are two linearly independent cusp forms of weight $(6, 8)$ there exists a non-trivial linear combination that vanishes with order ≥ 2 along $\mathcal{A}_{1,1} \otimes \mathbb{F}_2$. This yields a modular form divisible by χ_{10}, hence a regular section of $\mathrm{Sym}^6(\mathbb{E}) \otimes \det(\mathbb{E})^{-2}$, not identically zero on $\mathcal{A}_{1,1} \otimes \mathbb{F}_2$. ■

We thus have at least one non-zero modular form $f \in S_{6,8}(\Gamma_2)$ that vanishes with multiplicity 1 on $\mathcal{A}_{1,1} \otimes \mathbb{F}_2$. Thus, we also find a rational modular form f/χ_{10} with possible poles along $\mathcal{A}_{1,1} \otimes \mathbb{F}_2$.

We can analyze the order of the poles of the coordinates of f/χ_{10} near a generic point of $\mathcal{A}_{1,1}$. We consider f near a generic point of $\mathcal{A}_{1,1} \otimes \mathbb{F}_2$ and write its Taylor expansion as

$$f = (\gamma_0, \ldots, \gamma_6) \quad \text{with } \gamma_i = \sum_{r \geq 0} \gamma_{i,r} t^r,$$

where t is a local normal coordinate. If for fixed i we have $\gamma_{i,r} = 0$ for $r < r_0$ then

$$\gamma_{i,r_0} \in QS_{14+r_0-i}(\Gamma_1) \otimes QS_{8+r_0+i}(\Gamma_1),$$

where QS denotes quasi-modular cusp forms, see [17, Sec. 5] for the notion of quasi-modular form. From the dimensions of $S_k(\Gamma_1)$ we see as in the proof of Lemma 2.1 that

$\mathrm{ord}_{\mathcal{A}_{1,1}}(\gamma_3) \geq -1$ and we can even get the estimate

$$\mathrm{ord}_{\mathcal{A}_{1,1}}(\gamma_0, \ldots, \gamma_6) \geq (2, 1, 0, -1, 0, 1, 2)\,,$$

but we will not use this.

3 Curves of Genus 2 in Characteristic 2

Let C be a smooth projective curve of genus 2 over a perfect field k of characteristic 2. Let K be the canonical divisor (class) of C. Then by Riemann–Roch $\dim H^0(C, O(nK)) = 2n-1$ for $n \geq 2$. If ξ_0, ξ_1 is a basis of $H^0(C, O(K))$, then there is an element $\eta \in H^0(C, O(3K))$ such that $\xi_0^3, \xi_0^2\xi_1, \xi_0\xi_1^2, \xi_1^3, \eta$ form a basis of $H^0(C, O(3K))$. Then

$$\xi_0^6, \xi_0^5\xi_1, \ldots, \xi_1^6,\ \xi_0^3\eta, \xi_0^2\xi_1\eta, \xi_0\xi_1^2\eta, \xi_1^3\eta$$

are linearly independent in $H^0(C, O(6K))$ and since $\eta^2 \in H^0(C, O(6K))$ there must be a linear relation

$$\eta^2 + c_3\eta + c_6 = 0$$

with c_3 (resp. c_6) homogeneous of degree 3 (resp. 6) in ξ_0, ξ_1. Setting $x = \xi_1/\xi_0$ and $y = \eta/\xi_0^3$ we can write the equation as

$$y^2 + a\,y = b \tag{5}$$

with $a \in k[x]$ non-zero of degree ≤ 3 and $b \in k[x]$ of degree ≤ 6. The hyperelliptic involution of the curve C is given by $y \mapsto y+a$. The fixed points are given by the equation $a = 0$ (or $c_3 = 0$ in projective coordinates); hence, there are 3, 2 or 1 ramification points. This corresponds to the 2-rank of $\mathrm{Jac}(C)$ being equal to 2, 1 or 0.

We point out that a curve of genus 2 defined by an equation (5) comes with a basis dx/a, $x\,dx/a$ of $H^0(C, O(K))$.

The choices we made for arriving at equation (5) are a basis of $H^0(C, O(K))$ and an element of η in $H^0(C, O(3K))$. Clearly, another choice of η is of the form $\epsilon\eta + \theta$ with ϵ a unit in k and θ homogeneous of degree 3 in ξ_0, ξ_1, or $y \to uy + v$ for $u \in k^*$ and $v \in k[x]$ of degree ≤ 3.

The induced action on the pair (a,b) is by

$$(a,b) \mapsto (a/u, (b+v^2+av)/u^2) \quad \text{with } u \in k^*, v \in k[x] \text{ of degree } \leq 3\,.$$

Another choice of basis of $H^0(C, O(K))$ is given by an element of $\mathrm{GL}(V)$ with $V = \langle \xi_0, \xi_1 \rangle$. The action on x, y is by

$$x \mapsto (\alpha x + \beta)/(\gamma x + \delta), \quad y \mapsto y/(\gamma x + \delta)^3\,.$$

Let Y be the algebraic stack of triples (π, α, β) with $\pi : C \to S$ a curve of genus 2, $\alpha : \mathcal{O}_S^{\oplus 2} \xrightarrow{\sim} \pi_* \omega_\pi$ and $\beta \in \pi_*(\omega^{\otimes 3})$ nowhere in the image of $\mathrm{Sym}^3(\pi_*(\omega_\pi)) \to \pi_*(\omega_\pi^{\otimes 3})$. We may view it as the stack of curves of genus 2 with a framed Hodge bundle and a section of $\omega_\pi^{\otimes 3}$ satisfying the additional condition that it yields an equation as (5). There is an obvious action of $\mathrm{GL}(2)$ and an action of $\mathrm{Sym}^3(\mathcal{O}_S^{\oplus 2})$.

We thus consider the stack

$$[Y/\mathrm{GL}(2) \ltimes \mathrm{Sym}^3(\mathcal{O}_S^{\oplus 2})]$$

and we can identify it with the moduli space $\mathcal{M}_2 \otimes \mathbb{F}_2$ of curves of genus 2 in characteristic 2.

We now describe a concrete form of this stack. Let V be a 2-dimensional k-vector space generated by x_1, x_2. Consider the subspace of $\mathrm{Sym}^3(V) \times \mathrm{Sym}^6(V)$ of pairs (a,b) satisfying the condition that the pair defines a non-singular curve; in the affine version this amounts to a and $(a')^2 b + (b')^2$ (with a' and b' the derivative) having no root in common.

We write $V_{j,l}$ for $\mathrm{Sym}^j(V) \otimes \det(V)^{\otimes l}$. We let the semi-direct product $\mathrm{GL}(V) \ltimes V_{3,-1}$ act on $V_{3,-1} \times V_{6,-2}$ via twisting the two actions of $\mathrm{GL}(V)$ and $\mathrm{Sym}^3(V)$. Without twisting an element $M \in \mathrm{GL}(V)$ acts by

$$(a,b) \mapsto (a(\alpha x_1 + \beta x_2, \gamma x_1 + \delta x_2), b(\alpha x_1 + \beta x_2, \gamma x_1 + \delta x_2))$$

and $v \in \mathrm{Sym}^3(V)$ by

$$(a,b) \mapsto (a, b + v^2 + va)\,.$$

After twisting $c \cdot \mathrm{Id}_V$ acts via c on $V_{3,-1}$ and by c^2 on $V_{6,-2}$ and this is compatible with the equation $y^2 + ay = b$ if we let $c \cdot \mathrm{Id}_V$ act on y by $y \mapsto cy$. It is also compatible with $b \mapsto b + v^2 + av$ for $v \in V_{3,-1}$.

We thus consider

$$\mathcal{X}^0 \subset \mathcal{X} = V_{3,-1} \times V_{6,-2}\,,$$

where $\mathcal{X}^0$ is the open substack given by the condition that $y^2 + ay = b$ defines a smooth projective curve of genus 2.

The moduli space $\mathcal{M}_2 \otimes \mathbb{F}_2$ can be identified with the stack quotient

$$[\mathcal{X}^0/\mathrm{GL}(V) \ltimes V_{3,-1}].$$

Note that by our choice of twisting the stabilizer of a pair (a, b) contains (id_V, a) as it should, since the generic curve has an automorphism group of order 2. The pullback of the Hodge bundle $\mathbb{E}$ on $\mathcal{M}_2 \otimes \mathbb{F}_2$ is the equivariant bundle V.

4 Invariants and Modular Forms

We write elements $a \in \mathrm{Sym}^3(V)$ and $b \in \mathrm{Sym}^6(V)$ as

$$a = \sum_{i=0}^{3} a_i\, x_1^{3-i} x_2^i, \quad b = \sum_{i=0}^{6} b_i\, x_1^{6-i} x_2^i\,.$$

Following standard usage, by an invariant for the action of $\mathrm{GL}(V) \ltimes \mathrm{Sym}^3(V)$ we mean a polynomial in the coordinates $a_0, \ldots, a_3$ and $b_0, \ldots, b_6$ of the pair (a, b) in $\mathrm{Sym}^3(V) \times \mathrm{Sym}^6(V)$ that is invariant under $\mathrm{SL}(V) \ltimes \mathrm{Sym}^3(V)$. We define $\mathcal{K}$ as the ring of $\mathrm{SL}(V) \ltimes \mathrm{Sym}^3(V)$-invariants under the action on $\mathcal{X}$.

The simplest example of such an invariant is the expression

$$K_1 = a_0 a_3 + a_1 a_2,$$

the square root of the discriminant of a.

Since the pullback to $\mathcal{X}^0$ of the Hodge bundle $\mathbb{E}$ under the Torelli map $\mathcal{M}_2 \to \mathcal{A}_2$ is the equivariant bundle V, pulling back of scalar-valued modular forms gives invariants. This provides us with a homomorphism

$$\mu : \mathcal{R}_2(\mathbb{F}_2) \to \mathcal{K}\,.$$

Since the image of the Torelli map $\mathcal{M}_2 \hookrightarrow \mathcal{A}_2$ is the open set that is the complement of $\mathcal{A}_{1,1}$, the zero locus of χ_{10}, we see that an invariant defines a rational modular form that becomes regular after multiplication by a power of χ_{10}. Therefore, we can extend the map μ to

$$\mathcal{R}_2(\mathbb{F}_2) \xrightarrow{\mu} \mathcal{K} \xrightarrow{\nu} \mathcal{R}_2(\mathbb{F}_2)_{\chi_{10}}$$

such that $\nu \cdot \mu = \mathrm{id}$; here, the index χ_{10} indicates localization at the multiplicative system generated by χ_{10}.

Since we know the existence of a modular form $\psi_1 \in M_1(\Gamma_2)$ we see that $\mu(\psi_1)$ must equal K_1 as K_1 is the only invariant of weighted degree 3. (Here the weight of a_i (resp. b_j) is i (resp. j).) This implies in particular that $\nu(K_1)$ is a regular modular form of weight 1.

Similarly for covariants. By a covariant under the action of $\mathrm{GL}(V) \ltimes \mathrm{Sym}^3(V)$, we mean a polynomial in the coefficients a_i and b_j and x_1, x_2, which is an invariant for the action of $\mathrm{SL}(V) \ltimes \mathrm{Sym}^3(V)$.

The simplest example of a covariant is given by the polynomial a. It defines an a priori rational modular form of weight $(3, -1)$.

Lemma 4.1. The form $\nu(a)$ is a regular modular form of weight $(3, -1)$.

Proof. The pullback of the $\mathrm{Sym}^3(\mathbb{E})$ to $(\mathcal{A}_1 \times \mathcal{A}_1) \otimes \mathbb{F}_2$ decomposes as

$$\bigoplus_{j=0}^{3} p_1^*(\mathbb{E}_1)^{3-j} \otimes p_2^*(\mathbb{E}_1)^j$$

and the coefficients of a correspond to the factors. In view of the symmetry we have $\mathrm{ord}_{\mathcal{A}_{1,1}}(a_0) = \mathrm{ord}_{\mathcal{A}_{1,1}}(a_3)$ and $\mathrm{ord}_{\mathcal{A}_{1,1}}(a_1) = \mathrm{ord}_{\mathcal{A}_{1,1}}(a_2)$. As we saw above the form $\nu(K_1) = \nu(a_0 a_3 + a_1 a_2)$ is regular. By coordinate changes we may assume that $a_0 = 0$ or $a_1 = 0$, so it easily follows that $\mathrm{ord}_{\mathcal{A}_{1,1}}(a_i) \geq 0$ and the form is regular. Moreover, since its pullback lands in $\oplus_{j=0}^3 M_{2-j}(\Gamma_1) \otimes M_{j-1}(\Gamma_1)$ and $M_2(\Gamma_1) \otimes M_{-1}(\Gamma_1) = (0)$ it follows that $\mathrm{ord}_{\mathcal{A}_{1,1}}(a_0) \geq 1$ and $\mathrm{ord}_{\mathcal{A}_{1,1}}(a_3) \geq 1$. ■

Corollary 4.2. We have $\mathrm{ord}_{\mathcal{A}_{1,1}}(a_i) = 0$ for $i = 1, 2$ and $\mathrm{ord}_{\mathcal{A}_{1,1}}(a_i) \geq 1$ for $i = 0, 3$.

Proof. We note that ψ_1 does not vanish on $\mathcal{A}_{1,1} \otimes \mathbb{F}_2$. ■

Remark 4.3. The restriction of the modular form $\nu(a)$ to $\mathcal{A}_{1,1} \otimes \mathbb{F}_2$ is the form $c_1 \otimes 1 + 1 \otimes c_1 \in M_1(\Gamma_1) \otimes M_0(\Gamma_1) \oplus M_0(\Gamma_1) \otimes M_1(\Gamma_1)$.

Another example is the expression

$$(a_0a_2 + a_1^2)x_1^2 + (a_0a_3 + a_1a_2)x_1x_2 + (a_1a_3 + a_2^2)x_2^2,$$

which then defines a regular modular form of weight $(2, 0)$. In fact, this lives in the 2nd factor of the decomposition $\mathrm{Sym}^2(V_{3,-1}) = V_{6,-2} \oplus V_{2,0}$. Note that the map $V_{2,0} \to V_{0,0}$ given by $\alpha x_1^2 + \beta x_1x_2 + \gamma x_2^2 \mapsto \beta$ defines an invariant in characteristic 2.

5 Modular Forms from Invariants

Igusa used in [9] invariants of binary sextics to construct modular forms and to construct a coarse moduli space of curves of genus 2. The $\mathbb{Z}$-algebra of even degree invariants of binary sextics describes a coarse moduli space

$$\mathcal{Y} = \mathrm{Proj}\left(\mathbb{Z}[J_2, J_4, J_6, J_8, J_{10}]/(J_4^2 - J_2J_6 + 4J_8)\right)$$

for curves of genus 2 over $\mathbb{Z}$.

To get characteristic 2 invariants (or covariants) one lifts the curve given by $y^2 + ay = b$ to the Witt ring, say defined by $(\tilde{a}, \tilde{b})$, and takes an invariant (or covariant) of the binary sextic defined by $\tilde{a}^2 + 4\tilde{b}$, divides these by an appropriate power of 2 and reduces these modulo 2. This defines invariants (or covariants).

If $f = \sum_{i=0}^{6} c_i x^{6-i}$ is the (universal) binary sextic then the invariant

$$J_2 = 2^{-2}(-120\, c_0c_6 + 20\, c_1c_5 - 8\, c_2c_4 + 3\, c_3^2)$$

gives by the substitution

$$c_i = 4\, b_i + \sum_{0 \le r,s \le 3, r+s=i} a_r a_s \tag{6}$$

the invariant $K_2 = (a_0a_3 + a_1a_2)^2$, that is, $K_2 = K_1^2$. For the formula for J_2 and the formulas of the invariants J_{2i} ($i = 2, 3, 5$) used below we refer to [13, pp. 139–140] and [12, p. 204].

Similarly, the invariant J_4 given by

$$2^{-7}(2640\, c_0^2c_6^2 - 880\, c_0c_1c_5c_6 + \ldots)$$

yields via $I_4 = J_2^2 - 24\,J_4$ an invariant K_4 in characteristic 2 that turns out to be reducible and divisible by K_1. We thus set $K_3 = K_4/K_1$ and get an invariant

$$\begin{aligned} K_3 = (a_0a_3 + a_1a_2)b_3^2 + (a_0^2a_3^2 + a_0a_2^3 + a_1^3a_3 + a_1^2a_2^2)b_3 + \\ (a_0a_3 + a_1a_2)a_1^2b_4 + (a_0a_3 + a_1a_2)a_2^2b_2 + \\ (a_0^2a_1a_3 + a_0^2a_2^2 + a_0a_1^2a_2 + a_1^4)b_5 + \\ (a_0a_2a_3^2 + a_3^2a_1^2 + a_1a_2^2a_3 + a_2^4)b_1 + \\ (a_0a_3 + a_1a_2)a_0^2b_6 + (a_0a_3 + a_1a_2)a_3^2b_0\,. \end{aligned}$$

Similarly, J_6 yields the invariant K_3^2 in characteristic 2. The invariant J_8 yields the invariant

$$\begin{aligned} K_8 = b_3^8 + (a_0a_3 + a_1a_2)^2b_3^6 + \\ (a_0a_3 + a_1a_2)(a_0^2a_3^2 + a_0a_2^3 + a_1^3a_3 + a_1^2a_2^2)b_3^5 + \ldots\,, \end{aligned}$$

and similarly from J_{10} we obtain

$$\begin{aligned} K_{10} = b_3^6(a_0a_3)^4 + b_3^5(a_0^5a_3^5 + a_0^4a_1a_2a_3^4 + a_0^4a_2^3a_3^3 + a_0^3a_1^3a_3^4) + \\ b_3^4\big(a_0^6a_3^6 + a_0^5a_1a_2a_3^5 + a_0^4a_1^2a_2^2a_3^4 + a_0^3a_1^3a_2^3a_3^3 + \\ a_0^6a_3^4b_6 + a_0^5a_1a_3^4b_5 + a_0^5a_2^2a_3^3b_5 + a_0^4a_1^2a_3^4b_4 + \\ a_0^4a_1a_2^3a_3^2b_5 + a_0^4a_2^6b_6 + a_0^4a_2^5a_3b_5 + a_0^4a_2^4a_3^2b_4 + \\ a_0^4a_2^2a_3^4b_2 + a_0^4a_2a_3^5b_1 + a_0^4a_3^6b_0 + a_0^3a_1^2a_3^5b_1 + \\ a_0^2a_1^4a_3^4b_2 + a_0^2a_1^3a_2a_3^4b_1 + a_0a_1^5a_3^4b_1 + a_1^6a_3^4b_0 + \\ a_0^4a_2^4b_5^2 + a_1^4a_3^4b_1^2\big) + \ldots\,. \end{aligned}$$

Finally, the invariant I_{15} defines an invariant K_{15}, but K_{15} can be expressed in terms of the invariants already obtained:

$$K_{15} = K_1^3K_3^4 + K_1^5K_{10} + K_1^4K_3K_8 + K_3^5.$$

Lemma 5.1. We have $\mathrm{ord}_{\mathcal{A}_{1,1}}(b_3) = -1$ and $\mathrm{ord}_{\mathcal{A}_{1,1}}(b_i) \geq 0$ if $i \neq 3$.

Proof. In characteristic 0 we have $\dim S_{6,8}(\Gamma_2) = 1$. The covariant $\tilde{f}$, the universal binary sextic, defines a rational modular form $\nu(\tilde{f})$ such that $\chi_{10}\nu(\tilde{f}) \in S_{6,8}(\Gamma_2)$. A non-zero element χ of $S_{6,8}(\Gamma_2)$ vanishes on $\mathcal{A}_{1,1}$ and the coordinates of χ/χ_{10} thus have order ≥ -1 along $\mathcal{A}_{1,1}$. Since the a_i are regular near $\mathcal{A}_{1,1} \otimes \mathbb{F}_2$ we see using the formulas (6) that $\mathrm{ord}_{\mathcal{A}_{1,1}}(b_i) \geq -1$. By an argument as in Lemma 2.1 if the vanishing order equals -1, the only non-zero term in the 1st ("non-constant") term of the Taylor development sits in the middle coordinate and lands in $S_{12}(\Gamma_1) \otimes S_{12}(\Gamma_1)$ and thus we see that $\mathrm{ord}_{\mathcal{A}_{1,1}}(b_i) \geq 0$ for $i \neq 3$. The invariant K_3 defines a rational modular form $\nu(K_3)$ of weight 3. Since it is not a multiple of ψ_1^3 and $\dim M_3(\Gamma_2) = 1$ it cannot be regular. Therefore, $\mathrm{ord}(b_3) = -1$. ■

Using this lemma we obtain modular forms $\psi_1 = \nu(K_1) \in M_1(\Gamma_2)$ and

$$\chi_{13} = \nu(K_3)\chi_{10} \in S_{13}(\Gamma_2)$$

as the form of K_3 shows that $\nu(K_3)$ has order -2 along $\mathcal{A}_{1,1}$ since K_1 does not vanish on $\mathcal{A}_{1,1}$. We see that χ_{13} is not divisible by ψ_1, hence $\dim M_{12}(\Gamma_2) < \dim M_{13}(\Gamma_2)$. In view of the estimates on the dimensions of $M_k(\Gamma_2)$ implied by (2) we conclude that

$$\dim M_{12}(\Gamma_2) = 3 \quad \text{and} \quad \dim M_{13}(\Gamma_2) = 4\,. \tag{7}$$

By inspecting the expression of the invariant K_{10} in terms of the a_i and b_j we see that $\nu(K_{10})$ is a modular form of weight 10 that vanishes with multiplicity 2 along $\mathcal{A}_{1,1}$. Therefore, by the cycle class of $\mathcal{A}_{1,1}$ given in (1) it is a multiple of χ_{10} and we normalize χ_{10} by setting

$$\chi_{10} = \nu(K_{10})\,.$$

Consider $K_{12} = K_8K_1^4 + K_3^4 + K_1^3K_3^3$. It starts as follows

$$(a_0^2a_3^2 + a_0a_1a_2a_3 + a_0a_2^3 + a_1^3a_3)^4\, b_3^4 + \ldots$$

and one can check that it has order 0 along $\mathcal{A}_{1,1}$. This defines a modular form

$$\psi_{12} = \nu(K_{12})\,.$$

Remark 5.2. From the fact that χ_{10} (resp. ψ_{12}) vanishes with multiplicity 2 (resp. 0) along $\mathcal{A}_{1,1}$ we see that the order of a_0 and a_3 along $\mathcal{A}_{1,1}$ is 1.

Similarly, $\nu(K_8)$ has order -8 along $\mathcal{A}_{1,1}$, hence we find

$$\chi_{48} = \nu(K_8)\chi_{10}^4 \in S_{48}(\Gamma_2)\,.$$

In this way we obtain modular forms $\psi_1, \chi_{10}, \psi_{12}, \chi_{13}, \chi_{48}$.

The relation $K_{12} = K_8K_1^4 + K_3^4 + K_1^3K_3^3$ implies that

$$\chi_{10}^4\psi_{12} = \chi_{48}\psi_1^4 + \chi_{13}^4 + \chi_{13}^3\psi_1^3\chi_{10}\,. \tag{8}$$

Inside the ring $\mathcal{R}_2(\mathbb{F}_2)$ we thus found generators

$$\psi_1, \chi_{10}, \psi_{12}, \chi_{13}, \chi_{48}$$

with a relation (8) of weight 52. These generate a subring $R = \oplus_k R_k$ of $\mathcal{R}_2(\mathbb{F}_2)$ with generating function for the dimensions

$$G = \frac{1-t^{52}}{(1-t)(1-t^{10})(1-t^{12})(1-t^{13})(1-t^{48})}$$

and $\dim R_k = k^3/1080 + O(k^2)$. To see that $R = \mathcal{R}_2(\mathbb{F}_2)$ one can use the degree of $\mathrm{Proj}(R)$

$$2\ \deg(\lambda_1^3) = \frac{1}{1440} = \frac{52}{1\cdot 10\cdot 12\cdot 13\cdot 48}\,,$$

(see [6]) or argue with the dimensions as follows. Let $d(k) = \dim M_k(\Gamma_2)$ and $r(k) = \dim R_k$.

Proposition 5.3. We have $d(k) = r(k)$ for $k \geq 0$.

Proof. We have $d(k) \geq r(k)$ and we observed equality for $k = 0, \ldots, 13$ using the upper bound provided by (2), see (3) and (7). We assume by induction that we have $d(k) = r(k)$ for $k \leq m$. Then by (2) we get an upper bound $d(k) \leq r(k-10) + c(k)(c(k)+1)/2$ for $k \leq m+10$ with $c(k) = \dim M_k(\Gamma_1) = \lfloor k/12 \rfloor + 1$. But we have $r(k) - r(k-10) = c(k)(c(k)+1)/2$ for $k \not\equiv 0,\ 1,\ 2 \pmod{12}$. Indeed,

$$G - t^{10}G - \frac{1}{(1-t)(1-t^{12})^2} = -t^{12}\,\frac{t^{26}+t^{25}+t^{24}+t^{13}+t^{12}+1}{t^{60}-t^{48}-t^{12}+1}\,.$$

Hence, $d(k) = r(k)$ for $k \leq m + 10$ and $k \not\equiv 0, 1, 2 \pmod{12}$. But using $d(k+1) - d(k) \geq r(k+1) - r(k)$, to be proved in the next lemma, we get $d(k) = r(k)$ for all $k \leq m + 10$. Induction finishes the proof. ∎

Lemma 5.4. We have $d(k+1) - d(k) \geq r(k+1) - r(k)$ for $k \geq 0$.

Proof. Because of (8) we have $R_{k+1} = \psi_1 R_k \oplus N_{k+1}$ with N_{k+1} the subspace with basis the forms $\chi_{10}^a \psi_{12}^b \chi_{13}^c \chi_{48}^d$ with $10\,a + 12\,b + 13\,c + 48\,d = k+1$ with $a, b, c, d \geq 0$ and $c \leq 3$. We thus have $r(k+1) - r(k) = \dim N_{k+1}$. We claim that $N_{k+1} \cap \psi_1 M_k(\Gamma_2) = (0)$, and this implies $d(k+1) - d(k) \geq \dim N_{k+1}$. Suppose we have an $f \in M_k(\Gamma_2)$ with $f \notin R_k$ and $\psi_1 f \in N_{k+1}$. We can write $\psi_1 f = P$ with P a polynomial in $\chi_{10}, \psi_{12}, \chi_{13}, \chi_{48}$, where no power of χ_{13} is > 3. Then $P = \nu(Q)$ with Q a polynomial in

$$K_{10},\ K_8 K_1^4 + K_3^4 + K_3^3 K_1^3,\ K_3 K_{10},\ K_8 K_{10}^4$$

that is divisible by K_1 by assumption. In order that a non-zero Q is divisible by K_1 we need at least one monomial of Q involving a power ≥ 4 of $K_3 K_{10}$ since K_1, K_3, K_8, and K_{10} are algebraically independent. But this is excluded. ∎

Remark 5.5. In characteristic different from 2 the invariant E of degree 15 gives rise to a generator, a modular form χ_{35} of weight 35. But in our case, we get

$$K_{15} = K_3 K_{12} + K_{10} K_1^5$$

and $\nu(K_{15})$ has order -2 along $\mathcal{A}_{1,1} \otimes \mathbb{F}_2$, so if we set $\chi_{25} = \chi_{10}\, \nu(K_{15}) \in S_{25}(\Gamma_2)$ we have

$$\chi_{25} = \chi_{13} \psi_{12} + \chi_{10}^2 \psi_1^5$$

and do not get a new generator.

Remark 5.6. The zero locus of ψ_1 is the closure of the locus V_1 of abelian surfaces of 2-rank ≤ 1. The zero locus of the ideal (ψ_1, χ_{13}) consists of two components: the locus V_0 of 2-rank 0, that is, the supersingular locus, and the closure of the intersection of $\mathcal{A}_{1,1} \cap V_1$, both with multiplicity 2. This fits the cycles classes:

$$13\lambda_1^2 = 3\lambda_1^2 + 10\lambda_1^2\,.$$

The class of V_0 is $3\lambda_2 = (3/2)\lambda_1^2$, with λ_2 the 2nd Chern class of $\mathbb{E}$, see [6, Thm. 2.4] and [4, Thm. 12.4].

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