

A chain of adjunctions between BA and the variety generated by a semi-primal bounded lattice expansion.

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Semi-primal bounded lattice expansion

Definition

A finite algebra \mathbf{L} is *semi-primal* if every map $f : L^n \rightarrow L$ which preserves subalgebras is term-definable.

Proposition

Let \mathbf{L} be a finite algebra with bounded lattice reduct. Then TFAE:

- \mathbf{L} is semi-primal.
- For every $a \in L$ the map $T_a : L \rightarrow L$

$$T_a(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{if } x \neq a \end{cases}$$

is term-definable.

Throughout the presentation \mathbf{L} will denote a semi-primal algebra with bounded lattice reduct and (smallest) subalgebra $\mathbf{2}$.

Examples

- Chain-based: the $(n + 1)$ -element Łukasiewicz-chain $\mathbf{L}_n = (\{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}, \wedge, \vee, \oplus, \neg, 0, 1)$ where

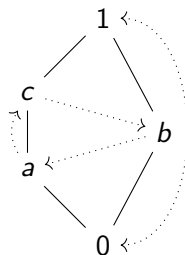
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$$x \oplus y = \min(x + y, 1) \text{ and } \neg x = 1 - x.$$

- Not chain-based: $\mathbf{L} = (N_5, \wedge, \vee, f, 0, 1)$:



(Davey, Schumann, Werner 1991)

Stone duality and semi-primal natural duality

For $BA = \mathbf{ISP}(\mathbf{2})$ we have the famous **Stone duality**:

$$\text{Stone} \begin{array}{c} \xrightarrow{\Pi} \\ \xleftarrow{\Sigma} \end{array} BA$$

$\Sigma(B) = BA(B, \mathbf{2})$ and $\Pi(X) = \text{Stone}(X, \mathbf{2})$.

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For $\mathcal{A} := \mathbf{ISP}(\mathbf{L})$ we have the similar **natural duality**

$$\text{Stone}_{\mathbf{L}} \begin{array}{c} \xrightarrow{P} \\ \xleftarrow{S} \end{array} \mathcal{A}$$

“ $S(\mathbf{A}) = \mathcal{A}(\mathbf{A}, \mathbf{L})$ and $P(X) = \text{Stone}_{\mathbf{L}}(X, \mathbf{L})$ ”

The category Stone_L

The category Stone_L has objects (X, ν) where

- $X \in \text{Stone}$,
- $\nu : X \rightarrow \mathbb{S}(L)$,
- $\nu^{-1}(\mathbf{S}\downarrow)$ is closed for every subalgebra $\mathbf{S} \leq L$.

A morphism $f : (X, \nu) \rightarrow (Y, \nu')$ is a continuous map with

$$\nu'(f(x)) \subseteq \nu(x)$$

for every $x \in X$.

Semi-primal natural duality (2)

$$\text{Stone}_{\mathbf{L}} \begin{array}{c} \xrightarrow{P} \\ \xleftarrow{S} \end{array} \mathcal{A}$$

$(L, v_L) \in \text{Stone}_{\mathbf{L}}$ with $v_L(a) = \langle a \rangle$. Then

$$P(X, v) = \text{Stone}_{\mathbf{L}}((X, v), (L, v_L))$$

$$f \in P(X, v) \Leftrightarrow f \in \text{Stone}(X, L) \text{ and } f(x) \in v(x)$$

and

$$S(\mathbf{A}) = (\mathcal{A}(\mathbf{A}, \mathbf{L}), im)$$

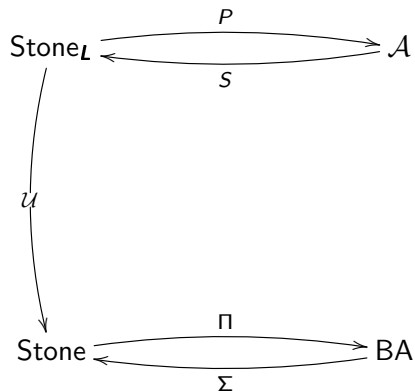
$$im(h) = h(A)$$

The adjoint functors on the dual side

$$\text{Stone}_L \begin{array}{c} \xrightarrow{P} \\ \xleftarrow{S} \end{array} \mathcal{A}$$

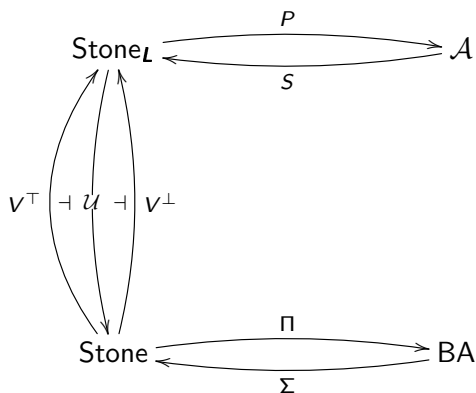
$$\text{Stone} \begin{array}{c} \xrightarrow{\Pi} \\ \xleftarrow{\Sigma} \end{array} \text{BA}$$

The adjoint functors on the dual side



Let \mathcal{U} be the forgetful functor.

The adjoint functors on the dual side



Let \mathcal{U} be the forgetful functor.

$V^\top(X) = (X, v^\top)$ where $v^\top(x) = L$ for all $x \in X$.

$V^\perp(X) = (X, v^\perp)$ where $v^\perp(x) = 2$ for all $x \in X$.

The adjoint functors on the dual side (2)

$V^{\top} \dashv \mathcal{U} \dashv V^{\perp}$:

$$f \in \text{Stone}_{\mathbf{L}}((X, v^{\top}), (Y, v')) \Leftrightarrow v'(f(x)) \subseteq v^{\top}(x) \Leftrightarrow f \in \text{Stone}(X, Y)$$

$$f \in \text{Stone}_{\mathbf{L}}((X, v), (Y, v^{\perp})) \Leftrightarrow v^{\perp}(f(x)) \subseteq v(x) \Leftrightarrow f \in \text{Stone}(X, Y)$$

The adjoint functors on the dual side (2)

$V^T \dashv \mathcal{U} \dashv V^\perp$:

$$f \in \text{Stone}_L((X, v^T), (Y, v')) \Leftrightarrow v'(f(x)) \subseteq v^T(x) \Leftrightarrow f \in \text{Stone}(X, Y)$$

$$f \in \text{Stone}_L((X, v), (Y, v^\perp)) \Leftrightarrow v^\perp(f(x)) \subseteq v(x) \Leftrightarrow f \in \text{Stone}(X, Y)$$

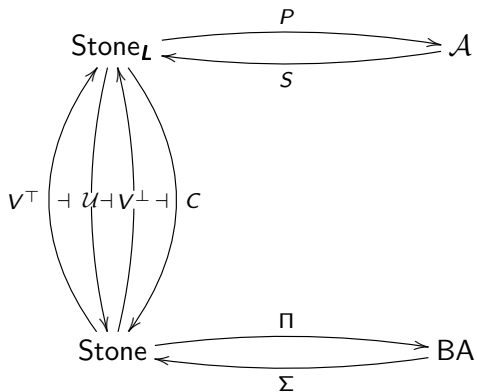
Consider the functor $C : \text{Stone}_L \rightarrow \text{Stone}$ given by

$$C(X, v) = \{x \in X \mid v(x) = 2\}$$

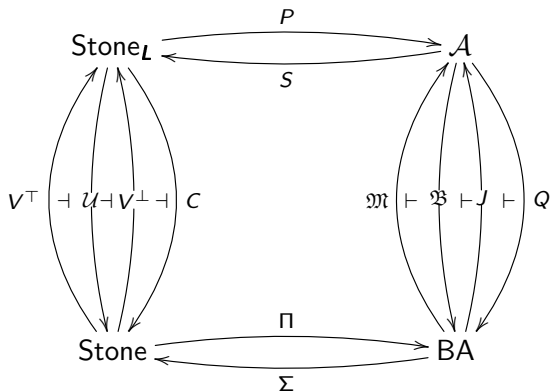
on objects and by $f \mapsto f \upharpoonright_{C(X, v)}$ on morphisms.

$$f \in \text{Stone}_L((X, v^\perp), (Y, v')) \Leftrightarrow v'(f(x)) \subseteq v^\perp \Leftrightarrow f \in \text{Stone}(X, C(Y, v'))$$

The big picture



The big picture



Next goal: Understand the adjunction on the algebraic side.

The Boolean skeleton

Recall: For every $\ell \in L$ the map $T_\ell : L \rightarrow L$ given by

$$T_\ell(x) = \begin{cases} 1 & \text{if } x = \ell \\ 0 & \text{if } x \neq \ell \end{cases} \text{ is term-definable in } \mathbf{L}.$$

Definition (Maruyama 2011)

Given any $\mathbf{A} \in \mathcal{A}$ we define $\mathfrak{B}(\mathbf{A}) = \{a \in A \mid T_1(a) = a\}$. The *Boolean skeleton* of \mathbf{A} is the Boolean algebra

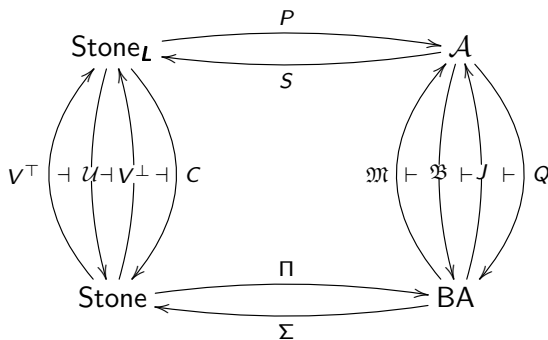
$$\mathfrak{B}(\mathbf{A}) = (\mathfrak{B}(\mathbf{A}), \wedge, \vee, T_0, 0, 1).$$

The Boolean skeleton (2)

Theorem

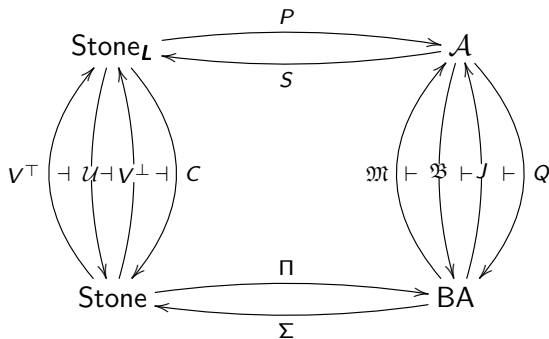
Let $\mathbf{A} \in \mathcal{A}$. Then $h \mapsto h \upharpoonright_{\mathfrak{B}(\mathbf{A})}$ is a homeomorphism

$$\mathcal{A}(\mathbf{A}, \mathbf{L}) \simeq \text{BA}(\mathfrak{B}(\mathbf{A}), \mathbf{2}).$$



$$\Pi U S(\mathbf{A}) = \Pi(\mathcal{A}(\mathbf{A}, \mathbf{L})) \simeq \Pi(\text{BA}(\mathfrak{B}(\mathbf{A}), \mathbf{2})) = \Pi \Sigma(\mathfrak{B}(\mathbf{A})) \simeq \mathfrak{B}(\mathbf{A}).$$

The functor J

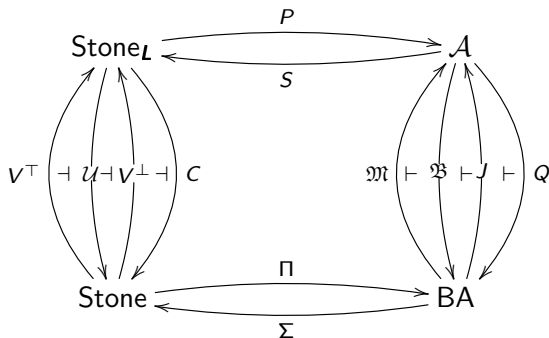


$$J(\mathbf{B}) = PV^\perp\Sigma(\mathbf{B}) = P(\mathbf{BA}(\mathbf{B}, \mathbf{2}), v^\perp)$$

$$J(\mathbf{B}) = \text{Stone}(\mathbf{BA}(\mathbf{B}, \mathbf{2}), \mathbf{2}) = \Pi\Sigma(\mathbf{B}) \simeq \mathbf{B}$$

$\Rightarrow J$ can be seen as **inclusion**.

The functor Q



$$\mathbf{A} \in \mathcal{A}_\omega \Rightarrow \mathbf{A} = \prod \mathbf{S}_i = \prod \{h(\mathbf{A}) \mid h \in \mathcal{A}(\mathbf{A}, L)\}$$

$$Q(\mathbf{A}) = 2^{\mathcal{A}(\mathbf{A}, 2)}$$

$\Rightarrow Q$ can be seen as **quotient**.

The functor \mathfrak{M} - Example

$\mathfrak{M}(\mathbf{B})$ is the largest algebra in \mathcal{A} which has \mathbf{B} as Boolean skeleton. For finite algebras this is $\mathfrak{M}(\mathbf{B}) = \mathbf{L}^{\text{BA}(\mathbf{B}, \mathbf{2})}$.

For $\text{MV}_n = \text{ISP}(\mathbf{L}_n)$ (Di Nola, Lettieri 2000):

$$\mathfrak{M}(B) = \{(b_1, \dots, b_n) \in B^n \mid b_1 \geq \dots \geq b_n\}$$

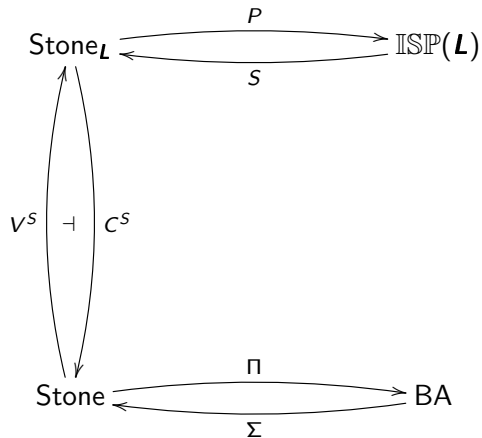
turn into MV_n -algebra

$$\neg(b_1, \dots, b_n) = (\neg b_n, \dots, \neg b_1)$$

$$(b \oplus c)_i = \bigvee_{j+k=i} b_j \wedge c_k$$

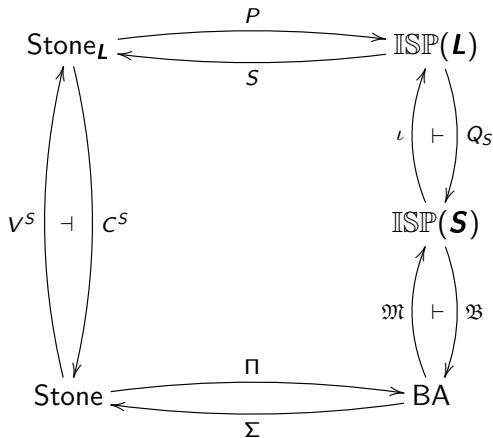
Another adjunction

For an arbitrary subalgebra $\mathbf{S} \leq \mathbf{L}$:

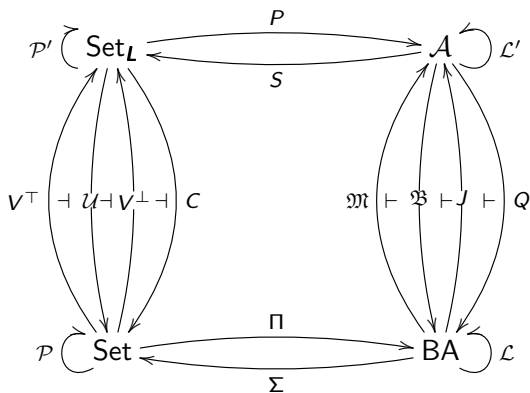


Another adjunction

For an arbitrary subalgebra $\mathbf{S} \leq \mathbf{L}$:



Many-valued modal logic



\mathcal{P} -coalgebras: (Kripke) frames

\mathcal{L} -algebras: Classical modal logic

\mathcal{P}' -coalgebras: L -frames

\mathcal{L}' -algebras: Modal logic over L

Thanks for your attention!