

The Minor Order for Homomorphisms via Natural Dualities

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Joint work with Bruno Teheux

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AAA 100 - February 5-7, 2021

The Minor Order

Let $f: A^3 \rightarrow B$ be a function. Some **minors** of f :

- $g(x_1, x_2, x_3) = f(x_2, x_1, x_3)$
- $g(x_1, x_2, x_3, x_4) = f(x_1, x_2, x_3)$
- $g(x_1, x_2) = f(x_1, x_2, x_1)$

We write $g \preceq f$.

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Let $f: A^3 \rightarrow B$ be a function. Some **minors** of f :

- $g(x_1, x_2, x_3) = f(x_2, x_1, x_3)$ permuting arguments
- $g(x_1, x_2, x_3, x_4) = f(x_1, x_2, x_3)$ adding/deleting inessential arguments
- $g(x_1, x_2) = f(x_1, x_2, x_1)$ identifying arguments

We write $g \preceq f$.

⇒ preorder \preceq on $\mathcal{F}_{AB} := \bigcup_{n \geq 1} \text{Set}(A^n, B)$

⇒ equivalence \equiv on \mathcal{F}_{AB}

⇒ partial order \leq on \mathcal{F}_{AB}/\equiv

Restriction to Homomorphisms

We replace \mathcal{F}_{AB} by $\mathcal{A}_{AB} := \bigcup_{n \geq 1} \mathcal{A}(A^n, B)$.

Here, \mathcal{A} is a category of algebras with a **natural duality**:

$\mathcal{A} \longleftrightarrow$ category of structured top. spaces \mathcal{X}

$$A \in \mathcal{A} \longleftrightarrow A^* \in \mathcal{X}$$

$$f \in \mathcal{A}(A, B) \longleftrightarrow f^* \in \mathcal{X}(B^*, A^*)$$

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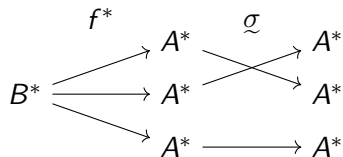
In particular, we assume that this duality is **logarithmic**:

$$A \times B \rightleftarrows A^* \oplus B^*$$

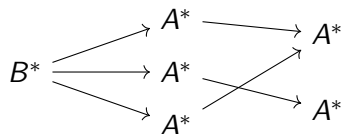
$$A^n \rightleftarrows nA^*$$

Dual Minors

Let $f: A^3 \rightarrow B$ be a homomorphism with dual morphism $f^*: B^* \rightarrow 3A^*$.
Some **dual minors** of f^* :



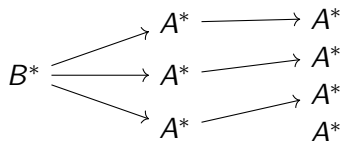
permuting arguments



identifying arguments

Proposition (Kerkhoff 2013)

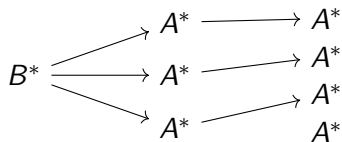
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adding/deleting inessential arguments

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adding/deleting inessential arguments

\Rightarrow preorder \preceq_d on $\mathcal{X}_{B^*A^*} := \bigcup_{n \geq 1} \mathcal{X}(B^*, nA^*)$

$\Rightarrow g \preceq f \Leftrightarrow g^* \preceq_d f^*$

$\Rightarrow \mathcal{A}_{AB}/\equiv$ isomorphic to $\mathcal{X}_{B^*A^*}/\equiv_d$

Finite Boolean Algebras

Let $2^j, 2^k$ be two finite Boolean algebras. We describe $\mathcal{BA}_{2^j, 2^k} / \cong$.

Stone Duality:

$$2^k \longleftrightarrow \{1, \dots, k\} =: [k]$$

$$f: (2^j)^n \rightarrow 2^k \longleftrightarrow f^*: [k] \rightarrow n[j]$$

Every map $[k] \rightarrow n[j]$ corresponds to some map $[k] \rightarrow [j]$, and modulo equivalence this is one-to-one.

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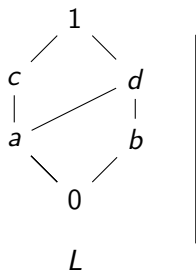
$$\mathcal{BA}_{2^i, 2^j} / \cong \simeq \bigsqcup_{[f^*] \text{ maximal}} [f^*] \downarrow \simeq \bigsqcup_{j^k} \Pi_k^\partial$$

The minor homomorphism poset is a **disjoint union of (order-reversed) partition lattices**.

Finite Distributive Lattices: Example

For (L, \wedge, \vee) , we want to determine \mathcal{DL}_L/\equiv .

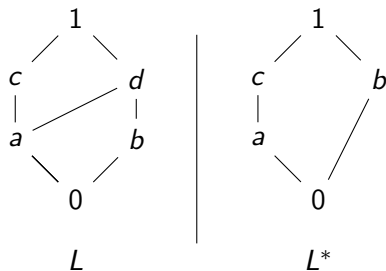
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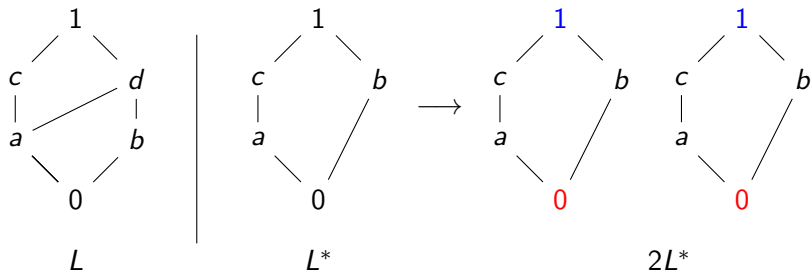
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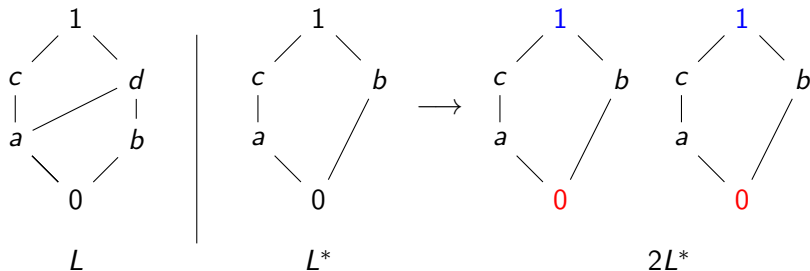
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$$\mathcal{DL}_L/\equiv \simeq \bigsqcup_{30} \pi_2^\partial \uplus \bigsqcup_{29} \pi_1^\partial \uplus \bigsqcup_6 \pi_0^\partial$$

Finite Distributive Lattices: General Outline

Let $L \in \mathcal{DL}_{fin}$.

$$(L, \wedge, \vee) \longleftrightarrow (L^*, \leq, 0, 1)$$

- \Rightarrow Look at the Hasse diagram of $L^* \setminus \{0, 1\}$
- \Rightarrow Let C_1, \dots, C_n be its connected components
- $\Rightarrow f^*: L^* \rightarrow L^*$ determines a maximal element $[F^*]$
- \Rightarrow Essential arity of $F^*: L^* \rightarrow nL^*$ depends on $f^*(C_i)$

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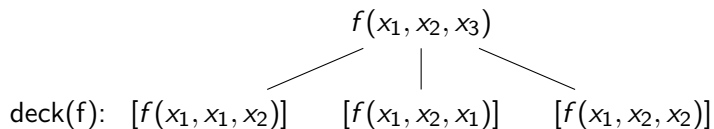
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$$\mathcal{DL}_L / \cong \simeq \bigsqcup_{f^*: L^* \rightarrow L^*} \Pi_{n-c_{f^*}}^{\partial}$$

where $c_{f^*} = \#\{i \mid f^*(C_i) \subseteq \{0, 1\}\}$

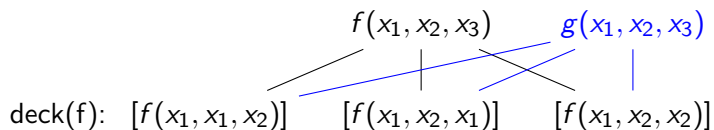
Weak Reconstructibility

A reconstruction problem:



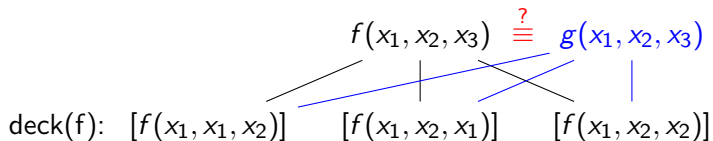
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\Rightarrow Let $f: A^n \rightarrow B$ be a homomorphism with $\text{ess}(f) > 2$

\Rightarrow Let $g: A^n \rightarrow B$ be a homomorphism with $\text{deck}(g) = \text{deck}(f)$

\Rightarrow Then $g \equiv f$.

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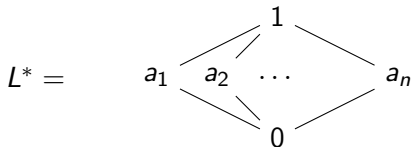
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- \Rightarrow Let $g: A^n \rightarrow B$ be a homomorphism with $\text{deck}(g) = \text{deck}(f)$
- \Rightarrow Then $g \equiv f$.
- $\Rightarrow \mathcal{A}_{AB}^{>2}$ is weakly reconstructible.

Finite Distributive Lattices: Boolean Reducts

Let $(L, \wedge, \vee) \in \mathcal{DL}_{\text{fin}}$. Is L the reduct of a Boolean algebra?

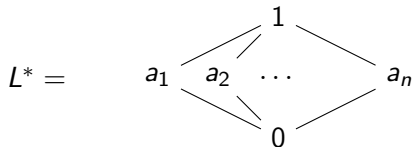
Suppose $L = 2^n$:



Finite Distributive Lattices: Boolean Reducts

Let $(L, \wedge, \vee) \in \mathcal{DL}_{\text{fin}}$. Is L the reduct of a Boolean algebra?

Suppose $L = 2^n$:



$$(nx + 2)^n = d_n x^n + \dots + d_1 x + d_0$$

$$\mathcal{DL}_L / \cong \simeq \bigsqcup_{d_n} \Pi_n^\partial \uplus \dots \uplus \bigsqcup_{d_1} \Pi_1^\partial \uplus \bigsqcup_{d_0} \Pi_0^\partial$$

Proposition

L is a Boolean reduct if and only if \mathcal{DL}_L / \cong is of the above form for some $n \in \mathbb{N}$.

Infinite Algebras

In the infinite case **topology** comes into play.

Consider the Boolean algebra $B = \{X \subseteq \mathbb{N} \text{ finite or co-finite}\}$

$B \xleftrightarrow{\cong} \mathbb{N} \cup \{\infty\}$ one-point compactification

$f: B^n \rightarrow B$ homomorphism $\xleftrightarrow{\cong} f^*: B^* \rightarrow nB^*$ continuous

$\Rightarrow \mathcal{BA}_B / \cong$ has uncountable antichains.

$\Rightarrow \mathcal{BA}_B / \cong$ has countable chains.

Thanks for your attention!

W.Poiger, B.Teheux, *The Minor Order for Homomorphisms via Natural Duality* (2021)
<https://arxiv.org/abs/2101.05545>