Derived categories of (nested) Hilbert schemes

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Abstract

In this paper we provide several results regarding the structure of derived categories of (nested) Hilbert schemes of points. We show that the criteria of Krug–Sosna and Addington for the universal ideal sheaf functor to be fully faithful resp. a \( \mathbb{P} \)-functor are sharp. Then we show how to embed multiple copies of the derived category of the surface using these fully faithful functors. We also give a semiorthogonal decomposition for the nested Hilbert scheme of points on a surface, and finally we give an elementary proof of a semiorthogonal decomposition due to Toda for the symmetric product of a curve.

1 Introduction

Hilbert schemes of points on surfaces are a classical object of study, providing very explicit moduli spaces of sheaves with interesting links to resolution of singularities and representation theory.

In the past two decades their derived categories have been studied thoroughly, especially in the context of the derived McKay correspondence [10, 16]. They provide an important source of interesting behaviour for derived categories of smooth projective varieties, such as the construction of non-standard autoequivalences [1, 20, 22] or interesting fully faithful functors [24, 5]. A good understanding of their derived categories, and functors relating them, has e.g. led to a more abstract interpretation of their deformation theory [5].

In this paper we give several different results regarding the derived categories of Hilbert schemes of points on surfaces, and closely related varieties, namely nested Hilbert schemes and symmetric products of curves.

Converses First we will give two results which provide converses to important criteria in the literature, which are closely related to understanding the deformation theory of Hilbert schemes via fully faithful functors, and the construction of new auto-equivalences. The first result we discuss provides a converse to the fully faithfulness criterion [24, theorem 1.2], where \( S \) is a smooth projective surface and \( S^{[n]} \) the associated Hilbert scheme of \( n \) points. Throughout the article we will denote \( F \) for the Fourier–Mukai functor \( \Phi_J \), where \( J \) is the ideal sheaf for the universal subscheme on \( S \times S^{[n]} \).
**Theorem A.** Let $S$ be a smooth projective surface, and $n \geq 2$. Then

\[(1) \quad F: \mathcal{D}^b(S) \to \mathcal{D}^b(S[n])\]

is fully faithful if and only if $H^1(S, \mathcal{O}_S) = H^2(S, \mathcal{O}_S) = 0$.

This strengthens [24, theorem 1.2] from an if to an if and only if. From now on we will also say that $\mathcal{O}_S$ is an *exceptional* object, if the vanishing $H^1(S, \mathcal{O}_S) = H^2(S, \mathcal{O}_S) = 0$ holds.

The second result is a converse to the criterion of Addington [1, theorem 3.1]. This result motivated the definition of a $\mathbb{P}^n$-functor, which is an interesting way of obtaining auto-equivalences of a variety.

**Theorem B.** Let $S$ be a smooth projective surface, and $n \geq 2$. Then

\[(2) \quad F: \mathcal{D}^b(S) \to \mathcal{D}^b(S[n])\]

is either a $\mathbb{P}^m$-functor for some $m \geq 1$ or a spherical functor if and only if $S$ is a K3 surface (in which case it is a $\mathbb{P}^m$-functor for $m = n - 1$).

This strengthens [1, theorem 3.1], which shows $F$ is a $\mathbb{P}^{n-1}$-functor if $S$ is a K3 surface, from an if to an if and only if. Remark that for any surface there are $\mathbb{P}^{n-1}$-functors $\mathcal{D}^b(S) \to \mathcal{D}^b(S[n])$, but they are constructed differently, see remark 13 for more information.

**Multiple copies** The third result gives an extension of the fully faithfulness result of theorem A. Given a line bundle $L \in \text{Pic}(S)$, there is a natural way to get an associated line bundle $D_L \in \text{Pic}(S[n])$; see [14] or section 2.1 for details.

**Theorem C.** Let $n \geq 3$. Let $S$ be a smooth projective surface, such that $\mathcal{O}_S$ is exceptional. Let

\[(3) \quad \mathcal{D}^b(S) = \langle L_1, \ldots, L_m, A \rangle\]

be a semiorthogonal decomposition, with $L_i$ line bundles and $A$ some (possibly empty) complement to the exceptional collection. Then we have an induced semiorthogonal decomposition

\[(4) \quad \mathcal{D}^b(S[n]) = \langle F(\mathcal{D}^b(S)) \otimes D_{L_1}, \ldots, F(\mathcal{D}^b(S)) \otimes D_{L_m}, \mathcal{B} \rangle\]

where $\mathcal{B}$ is defined as the complement.

**Symmetric powers of curves** Next we give an elementary proof of a recent result of Toda which describes the derived category of the symmetric powers $C^{(n)}$ of a curve $C$ [34, corollary 5.11]. Here we rely on the classical geometry of the Abel–Jacobi map and Schwarzenberger’s description of the symmetric powers, together with a recent general result in homological projective geometry due to Jiang–Leung [18] generalising Orlov’s projective bundle formula to not necessarily locally free sheaves.\(^1\)

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\(^1\)Jiang–Leung have independently included a proof of theorem D as an application of their projective bundle formula in the second version of their preprint. It is now the content of [18, corollary 3.8].
**Theorem D** (Toda). For \( n = g, \ldots, 2g-2 \), there exists a semiorthogonal decomposition

\[
D^b(C^{(n)}) = \langle D^b(\text{Jac } C), \ldots, D^b(\text{Jac } C), D^b(C^{(2g-2-n)}) \rangle.
\]

Here \( n = g, \ldots, 2g-2 \) constitutes the non-trivial range. For \( n \geq 2g-1 \) the symmetric power is the projectivisation of a locally free sheaf. Hence there is a semiorthogonal decomposition of \( D^b(C^{(n)}) \) with all components equivalent to \( D^b(\text{Jac } C) \), and no contribution from lower-dimensional symmetric powers of the curve. For \( n \leq g-1 \) the derived category is expected to be indecomposable. So (5) would be a decomposition into indecomposable pieces. For the proof of theorem D and more context, see section 6.

**Nested Hilbert schemes** Finally we describe the derived categories of *nested* Hilbert schemes of points. These are usually considered as tools to study the geometry of Hilbert schemes of points, but it turns out that their derived categories also exhibit an interesting phenomenon.

The proof of this decomposition follows in a straightforward way from the above-mentioned recent result of Jiang–Leung, and we have included it mostly to exhibit a large family of situations in which the Jiang–Leung result can be applied. For more details and an explicit description of the functors, see section 7.

**Theorem E.** Let \( S \) be a smooth projective surface. Then we have a semiorthogonal decomposition

\[
D^b(S^{[n-1,n]}) = \langle D^b(S \times S^{[n-1]}), D^b(S^{[n-2,n-1]}) \rangle.
\]

Hence by induction one can obtain the following corollary.

**Corollary F.** Let \( S \) be a smooth projective surface. Then for \( n \geq 2 \) we have a semiorthogonal decomposition

\[
D^b(S^{[n-1,n]}) = \langle D^b(S \times S^{[n-1]}), D^b(S \times S^{[n-2]}), \ldots, D^b(S) \rangle.
\]

There are no cohomological conditions on the surface, and e.g. when \( S \) is a K3 surface or an abelian surface, the decomposition (7) is in terms of indecomposable derived categories.

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2 Preliminaries

In this section we briefly recall some of the notation and constructions we will use throughout this article.

We denote \( k \) for the field \( \mathbb{C} \) of complex numbers. Some of the results in the literature that we are referring to are only proven using complex geometric methods, but we expect that all results in this article are valid for an algebraically closed field of characteristic 0.

We will abbreviate the bounded derived category of coherent sheaves \( \mathcal{D}^b(\text{coh} X) \) on a smooth projective variety to \( \mathcal{D}^b(X) \). If \( X \) and \( Y \) are smooth projective varieties, and \( \mathcal{P} \in \mathcal{D}^b(X \times Y) \), then the associated Fourier–Mukai transform is defined as

\[
\Phi_{\mathcal{P}} = \Phi^X \rightarrow Y : \mathcal{D}^b(X) \to \mathcal{D}^b(Y) : \mathcal{E} \mapsto Rq_* (p^* \mathcal{E} \otimes L \mathcal{P})
\]

where \( p \) and \( q \) denote the projections

\[
\begin{array}{c}
X \\
\leftarrow \hspace{2cm} \leftarrow \hspace{2cm} \leftarrow \\
p \hspace{2cm} \hspace{2cm} \hspace{2cm} q \\
X \times Y \\
Y
\end{array}
\]

The left (resp. right) adjoint of a Fourier–Mukai transform \( \Phi_{\mathcal{P}} \) is again a Fourier–Mukai transform, with kernel

\[
\begin{align*}
\mathcal{P}^L &:= \mathcal{P}^\vee \otimes^L q^* \omega_Y [\dim Y] \\
\mathcal{P}^R &:= \mathcal{P}^\vee \otimes^L p^* \omega_X [\dim X].
\end{align*}
\]

2.1 Induced sheaves and objects on Hilbert schemes

Throughout the article \( S \) will be a smooth projective surface, and \( S [n] \) denotes the Hilbert scheme of \( n \) points on \( S \), which is a smooth projective variety of dimension \( 2n \); see [13, theorem 2.4].

Let \( Z_n \subset S \times S [n] \) be the universal family of length \( n \) subschemes. Then we have a short exact sequence

\[
0 \to J_{Z_n} \to \mathcal{O}_{S \times S [n]} \to \mathcal{O}_{Z_n} \to 0
\]

on \( S \times S [n] \). We will denote the induced exact triangle of Fourier–Mukai transforms by

\[
F \to F' \to F'' \to F[1],
\]

where

\[
\begin{align*}
F &= \Phi_{J_{Z_n}}; \\
F' &= \Phi_{\mathcal{O}_{S \times S [n]}} \cong H^*(S, -) \otimes k \mathcal{O}_{S [n]};
\end{align*}
\]
We have $F'' = \Phi_{Cl^2}$. In the literature, the objects in the image of $F''$ are called tautological objects and denoted by $B[1] := F''(B)$ for $B \in D^b(S)$.

Let $S^{(n)} := S^n/\mathbb{Z}_n$ denote the symmetric product, let $\pi : S^n \to S^{(n)}$ be the quotient morphism, and let $\mu : S^{(n)} \to S^{(n)}$ be the Hilbert–Chow morphism. By [14, proposition 4.6], there is an injective group homomorphism

\[(14) \quad \text{Pic}(S) \to \text{Pic}(S^{(n)}), \quad L \mapsto \mathcal{D}_L := \mu'((\pi_*L^{\mathbb{B}n})^{\mathbb{Z}_n}).\]

### 2.2 Formulae for Hom-spaces

The results summarised in this subsection will be used in section 3 and section 5.

By [19], we have the following formulae for the graded Hom-spaces between the above objects on the Hilbert scheme:

\[(15) \quad \text{Hom}^*(F'A \otimes \mathcal{D}_L, F'B \otimes \mathcal{D}_M) \cong H^*(S, A)^{\vee} \otimes H^*(S, B) \otimes \text{Sym}^n \text{Hom}^*(L, M),\]

\[(16) \quad \text{Hom}^*(F'A \otimes \mathcal{D}_L, F''B \otimes \mathcal{D}_M) \cong H^*(S, A)^{\vee} \otimes \text{Hom}^*(L, B \otimes M) \otimes \text{Sym}^{n-1} \text{Hom}^*(L, M),\]

\[(17) \quad \text{Hom}^*(F''A \otimes \mathcal{D}_L, F'B \otimes \mathcal{D}_M) \cong \text{Hom}^*(A \otimes L, M) \otimes H^*(S, B) \otimes \text{Sym}^{n-1} \text{Hom}^*(L, M),\]

\[(18) \quad \text{Hom}^*(F''A \otimes \mathcal{D}_L, F''B \otimes \mathcal{D}_M) \cong (\text{Hom}^*(A \otimes L, B \otimes M) \otimes \text{Sym}^{n-1} \text{Hom}^*(L, M)) \]

\[\oplus (\text{Hom}^*(A \otimes L, M) \otimes \text{Hom}^*(L, B \otimes M) \otimes \text{Sym}^{n-2} \text{Hom}^*(L, M)).\]

For (17) and (18), see [19, theorem 3.17]. For (15), see [19, remark 3.21]. For (16), see [19, remark 3.20], which is a straight-forward generalisation of [31, corollary 35]. Here, $A^{\vee}$ stands for the derived dual and $H^*(S, A)^{\vee}$ stands for the graded dual of the graded vector space $H^*(S, A) := \bigoplus_{i \in \mathbb{Z}} H^i(S, A)[-i]$. This means that if, for example, $H^*(S, A)$ is concentrated in degrees 0, 1, and 2, the dual $H^*(S, A)^{\vee}$ is concentrated in degrees $-2$, $-1$, and 0. Furthermore, $\text{Sym}^n H^*(S, \mathcal{O}_S)$ denotes the symmetric power in the graded sense, which means that

\[(19) \quad \text{Sym}^n H^*(S, \mathcal{O}_S) = \bigoplus_{i = j \geq n} \text{Sym}^i H^\text{even}(S, \mathcal{O}_S) \otimes \bigwedge^j H^\text{odd}(S, \mathcal{O}_S).\]

We have $\mathcal{D}_{S^{(n)}} \cong \mathcal{O}_{S^{(n)}}$. Hence by setting $L = \mathcal{O}_S = M$, the formulae (15)–(18) specialise to the following, as summarised in [25, theorem 6.1]:

\[(20) \quad \text{Hom}^*(F'A, F'B) \cong H^*(S, A)^{\vee} \otimes H^*(S, B) \otimes \text{Sym}^n H^*(S, \mathcal{O}_S),\]

\[(21) \quad \text{Hom}^*(F'A, F''B) \cong H^*(S, A)^{\vee} \otimes H^*(S, B) \otimes \text{Sym}^{n-1} H^*(S, \mathcal{O}_S),\]

\[(22) \quad \text{Hom}^*(F''A, F'B) \cong H^*(S, A^{\vee}) \otimes H^*(S, B) \otimes \text{Sym}^{n-1} H^*(S, \mathcal{O}_S),\]

\[(23) \quad \text{Hom}^*(F''A, F''B) \cong (\text{Hom}^*(A, B) \otimes \text{Sym}^{n-2} H^*(S, \mathcal{O}_S)) \]

\[\oplus (H^*(S, A^{\vee}) \otimes H^*(S, B) \otimes \text{Sym}^{n-2} H^*(S, \mathcal{O}_S)).\]

### 2.3 Equivariant sheaves and the Bridgeland–King–Reid–Haiman equivalence

The results summarised in this subsection will be used in section 4.
Let $G$ be a finite group acting on a smooth projective variety $X$. In our applications, the group $G$ will be the symmetric group $\Sigma_n$. We will recall a few facts about (the category of) $G$-equivariant sheaves on $X$, and its derived category. For further details, we refer to [10, section 4], [21, section 2.2]. A $G$-equivariant sheaf on $X$ is a pair $(E, \lambda)$ where $E$ is a coherent sheaf on $X$ and $(\lambda_g : E \to g^*E)_{g \in G}$ is a $G$-linearisation, i.e. a collection of isomorphisms such that for every $h, g \in G$ the following diagram commutes:

\[
\begin{array}{ccc}
E & \xrightarrow{\lambda_g} & g^*E \\
\downarrow{\lambda_{hg}} & & \downarrow{(h \circ g)^*}
\end{array}
\]

We denote the abelian category of $G$-equivariant sheaves on $X$ and its bounded derived category by $D^b_G(X) := D^b(\text{coh}_G X)$. Equivariant sheaves can be canonically identified with coherent sheaves on the corresponding quotient stack $[X/G]$, i.e. we have equivalences $\text{coh}_G X \cong \text{coh}[X/G]$ and $D^b_G(X) \cong D^b([X/G])$.

If $H \leq G$ is a subgroup we have the functor

\[
(25) \quad \text{Re}^H_G : \text{coh}_G X \to \text{coh}_H X
\]

restricting $G$-linearisations to $H$-linearisations.

Let $\chi$ be a one-dimensional representation of $G$, which we identify with a character $\chi : G \to k^\times$. Then there is an induced autoequivalence

\[
(26) \quad M_\chi : \text{coh}_G X \to \text{coh}_G X
\]

sending an equivariant sheaf $(E, \lambda)$ to $(E, \lambda^\chi)$, where $\lambda^\chi_g = \chi(g) \cdot \lambda_g$. We will later consider the special case $M_{\alpha_2}$, where $\alpha_2$ is the unique non-trivial character of $\Sigma_2$.

If the group $G$ acts trivially on $X$, a $G$-linearisation of a sheaf $E$ is the same as a $G$-action on the sheaf. Hence in this case every $G$-equivariant sheaf $(E, \lambda)$ has a well-defined subsheaf of $G$-invariants $E^G \subset E$. This defines a functor

\[
(27) \quad (-)^G : \text{coh}_G X \to \text{coh} X.
\]

All three of the functors $\text{Res}$, $M_\chi$, and $(-)^G$ are exact, hence they induce functors between the equivariant derived categories.

Let now $G$ act on two smooth projective varieties $X$ and $Y$. Then every $G$-equivariant morphism $f : X \to Y$ induces a pullback $f^* : \text{coh}_G Y \to \text{coh}_G X$, as well as a derived version $Lf^* : D^b_G(Y) \to D^b_G(X)$. A special case that we will consider later is the following. Let $S$ be a smooth projective surface. We consider $\Sigma_n$ acting trivially on $S$ and by permutation of the factors on $S^n$. Then, the embedding of the small diagonal

\[
(28) \quad \delta : S \hookrightarrow S^n : x \mapsto (x, \ldots, x)
\]

is $\Sigma_n$-equivariant. Hence we get a pullback $L\delta^* : D^b_{\Sigma_n}(S^n) \to D^b(S)$.

To define the derived McKay correspondence for Hilbert schemes of points on surface, we consider the diagram

\[
\begin{array}{ccc}
P^S & \xrightarrow{\pi} & S^n \\
\downarrow{q} & & \downarrow{\pi} \\
S[n] & \xrightarrow{\mu} & S^{(n)}
\end{array}
\]
where \( I^S \coloneqq (S^{[n]} \times_{S^{[n]}} S^n)_{\text{red}} \) is the reduced fibre product, also called the \emph{isospectral Hilbert scheme}, and \( p \) and \( q \) are the projections. With this notation,

\begin{align}
\Psi \coloneqq (-)_{\leq n} \circ R_q \circ L p^* \colon D^b_{\text{Sh}}(S^n) & \to D^b(S^{[n]}) \tag{30} \\
\end{align}

is an equivalence, called \emph{derived McKay correspondence}; see [10], [16], [21, proposition 2.8].

### 3 A converse to the fully faithfulness criterion of Krug–Sosna

In this section we prove theorem A. As pointed out in the introduction, one direction has been proven in [24, theorem 1.2], so it suffices to show that if \( F \) is fully faithful, then \( H^i(S, \mathcal{O}_S) = H^i(S, \mathcal{O}_S) = 0 \).

We will denote \( q \coloneqq h^1(S, \mathcal{O}_S) \) and \( p_g \coloneqq h^2(S, \mathcal{O}_S) \), so that we have to show that \( p_g = q = 0 \).

This is done by exhibiting objects \( A, B \) in \( D^b(S) \) for which we can compute \( \text{Ext}^i_{S^{[n]}}(FA, FB) \) explicitly and compare it to \( \text{Ext}^i_S(A, B) \), to obtain a contradiction unless \( p_g = q = 0 \).

We do this in the following lemmas.

If \( V^* \) is a graded vector space, then we will denote

\begin{align}
\text{topdeg}(V^*) & \coloneqq \max \{ i \in \mathbb{Z} \mid V^i \neq 0 \}. \tag{31}
\end{align}

**Lemma 1.** Assume that \( F \) is fully faithful. Then \( p_g = 0 \).

**Proof.** Assume on the contrary that \( p_g \neq 0 \). This implies that \( \text{topdeg} \text{Sym}^i H^*(S, \mathcal{O}_S) = 2i \) for all \( i \in \mathbb{N} \).

We set \( A \) to be a line bundle on \( S \) such that \( H^*(S, A^\vee) \) is concentrated in degree 0 (we can take \( A^\vee \) to be a sufficiently high power of an ample line bundle), and \( B \coloneqq k(x) \) to be the skyscraper sheaf of some point \( x \in S \). Then we have

\begin{align}
\begin{bmatrix}
\text{topdeg} H^*(S, B) \\
\text{topdeg} H^*(S, A^\vee) \\
\text{topdeg} H^*(S, A^\vee) \\
\text{topdeg} \text{Hom}^*(A, B)
\end{bmatrix} & = 0 \tag{32} \\
\end{align}

Using (20)–(23), this implies

\begin{align}
\begin{bmatrix}
\text{topdeg} \text{Hom}^*(F'A, F'B) \\
\text{topdeg} \text{Hom}^*(F'A, F''B) \\
\text{topdeg} \text{Hom}^*(F''A, F'B) \\
\text{topdeg} \text{Hom}^*(F''A, F''B)
\end{bmatrix} & = 2(n - 1). \tag{33}
\end{align}

By the exact sequence

\begin{align}
0 & = \text{Hom}^{2n}(F''A, F'B) \to \text{Hom}^{2n}(F'A, F'B) \to \text{Hom}^{2n}(FA, F'B) \tag{34}
\end{align}

we get that \( \text{Hom}^{2n}(FA, F'B) \neq 0 \). On the other hand, the exact sequence

\begin{align}
0 & = \text{Hom}^{2n}(F'A, F''B) \to \text{Hom}^{2n}(FA, F''B) \to \text{Hom}^{2n+1}(F''A, F''B) = 0 \tag{35}
\end{align}

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shows that \( \text{Hom}^n(FA, F'B) = 0 \). Hence the exact sequence

\[
\text{(36)} \quad \text{Hom}^n(FA, FB) \to \text{Hom}^n(FA, F'B) \to \text{Hom}^n(FA, F''B) = 0
\]
gives \( \text{Hom}^n(FA, FB) \neq 0 \). But if \( F \) were fully faithful, \( \text{Hom}^*(FA, FB) \cong \text{Hom}^*(A, B) \) would be concentrated in degree 0. This shows that \( p_g = 0 \). \( \square \)

We still need to show that \( q = 0 \).

**Lemma 2.** Assume that \( F \) is fully faithful. Then \( q \leq n - 1 \).

**Proof.** Assume on the contrary that \( q \geq n \). By the previous lemma we have that \( p_g = 0 \), hence \( \text{topdeg Sym}^i H^*(S, \mathcal{O}_S) = i \) for all \( i \leq n \). Setting as above \( A \) to be a line bundle with \( H^*(S, A^i) \) being concentrated in degree 0 and \( B = k(x) \), we get

\[
\text{topdeg Hom}^*(F'A, F'B) = n
\]

\[
\text{topdeg Hom}^*(F''A, F''B)
\]

By the exact sequence

\[
\text{(37)} \quad \begin{aligned}
\text{topdeg Hom}^*(F'A, F'B) \\
\text{topdeg Hom}^*(F''A, F''B)
\end{aligned}
\]

\[
= n - 1.
\]

we get that \( \text{Hom}^n(FA, F'B) \neq 0 \). On the other hand, the exact sequence

\[
\text{(38)} \quad 0 = \text{Hom}^n(F''A, F'B) \to \text{Hom}^n(F'A, F'B) \to \text{Hom}^n(FA, F'B)
\]

shows that \( \text{Hom}^n(FA, F'B) = 0 \). Hence the exact sequence

\[
\text{(39)} \quad 0 = \text{Hom}^n(FA, F''B) \to \text{Hom}^n(FA, F'B) \to \text{Hom}^n(F''A, F''B) = 0
\]
gives \( \text{Hom}^n(FA, FB) \neq 0 \). But if \( F \) were fully faithful, \( \text{Hom}^*(FA, FB) \cong \text{Hom}^*(A, B) \) would be concentrated in degree 0. This shows that \( q \leq n - 1 \). \( \square \)

We can rule out another case by similar methods as follows.

**Lemma 3.** Assume that \( F \) is fully faithful. Then \( q \leq n - 2 \).

**Proof.** By the previous two lemmas we have that \( p_g = 0 \) and that \( q \leq n - 1 \). Assume that \( q = n - 1 \). Then we have

\[
\text{topdeg Sym}^n H^*(S, \mathcal{O}_S)
\]

\[
\text{topdeg Sym}^{n-1} H^*(S, \mathcal{O}_S)
\]

\[
\text{topdeg Sym}^{n-2} H^*(S, \mathcal{O}_S) = n - 2.
\]

We set \( A = k(x) \) and \( B = k(y) \) to be skyscraper sheaves of two different points \( x, y \) of \( S \). Then

\[
\text{topdeg H}^*(S, A^i) = 2
\]

\[
\text{topdeg H}^*(S, A^i)
\]

\[
\text{topdeg H}^*(S, B)
\]

\[
= 0
\]

\[
\square
\]
and

(43) \( \text{Hom}^* (A, B) = 0 \).

Hence by (20)–(23),

\[
\begin{align*}
\text{topdeg} \text{Hom}^* (F'' A, F' B) &= n + 1 \\
\text{topdeg} \text{Hom}^* (F' A, F'' B) &= n \\
\text{topdeg} \text{Hom}^* (F' A, F' B) &= n - 1
\end{align*}
\]

(44)

The exact sequence

(45) \( \text{Hom}^{n+1} (F' A, F B) \to \text{Hom}^{n+1} (F'' A, F' B) \to \text{Hom}^{n+1} (F'' A, F'' B) = 0 \)

shows that \( \text{Hom}^{n+1} (F'' A, F B) \neq 0 \). The exact sequence

(46) \( 0 = \text{Hom}^n (F' A, F' B) \to \text{Hom}^{n+1} (F' A, F B) \to \text{Hom}^{n+1} (F' A, F' B) = 0 \)

gives \( \text{Hom}^{n+1} (F' A, F B) = 0 \). Hence the exact sequence

(47) \( \text{Hom}^n (F A, F B) \to \text{Hom}^{n+1} (F'' A, F B) \to \text{Hom}^{n+1} (F' A, F B) = 0 \)

shows that \( \text{Hom}^n (F A, F B) \neq 0 \). But if \( F \) were fully faithful, we would have

(48) \( \text{Hom}^* (F A, F B) \equiv \text{Hom}^* (A, B) = 0 \).

This shows that \( q \leq n - 2 \). \( \square \)

**Lemma 4.** Assume that \( F \) is fully faithful. Then \( q = 0 \).

**Proof:** By the previous lemmas we have that \( p_q = 0 \) and \( q \leq n - 2 \). Assume on the contrary that \( 0 < q \leq n - 2 \) (and \( n \geq 3 \)). Then we have

\[
\begin{align*}
\text{topdeg} \text{Sym}^n \text{H}_{\bullet} (S, \mathcal{O}_S) \\
\text{topdeg} \text{Sym}^{n-1} \text{H}_{\bullet} (S, \mathcal{O}_S) \\
\text{topdeg} \text{Sym}^{n-2} \text{H}_{\bullet} (S, \mathcal{O}_S)
\end{align*}
\]

(49) \( \equiv q \)

such that

(50) \( \text{(Sym}^n \text{H}_{\bullet} (S, \mathcal{O}_S))^{\theta} \equiv (\text{Sym}^{n-1} \text{H}_{\bullet} (S, \mathcal{O}_S))^{\theta} \equiv (\text{Sym}^{n-2} \text{H}_{\bullet} (S, \mathcal{O}_S))^{\theta} \equiv k \).

We set \( A = B = k(x) \) to be the skyscraper sheaf of some point \( x \in S \) which gives

\[
\begin{align*}
\text{topdeg} \text{H}_{\bullet} (S, A^\vee) \\
\text{topdeg} \text{H}_{\bullet} (A, B)
\end{align*}
\]

(51) \( \equiv 2 \)

and

(52) \( \text{H}_{\bullet}^2 (S, A^\vee) \equiv \text{Hom}^2 (A, B) \equiv \text{H}_{\bullet}^0 (S, B) \equiv k \)
Plugging this into (22) and (23) gives $\text{Hom}^{q+2}(F''A, F'B) \cong k$ and $\text{Hom}^{q+2}(F''A, F''B) \cong k^2$. Hence the short exact sequence

\[(53) \quad \text{Hom}^{q+2}(F''A, F'B) \rightarrow \text{Hom}^{q+2}(F''A, F''B) \rightarrow \text{Hom}^{q+3}(F''A, FB)\]

gives $\text{Hom}^{q+3}(F''A, FB) \neq 0$. Furthermore, combining (51) with (20) and (21) gives

\[(54) \quad \text{topdeg} \text{Hom}^*(F'A, F'B) = \text{topdeg} \text{Hom}^*(F'A, F''B) = q.\]

Hence the short exact sequence

\[(55) \quad 0 = \text{Hom}^{q+2}(F'A, F''B) \rightarrow \text{Hom}^{q+3}(F'A, FB) \rightarrow \text{Hom}^{q+3}(F'A, F'B) = 0\]

shows $\text{Hom}^{q+3}(F'A, FB) = 0$. Using the exact sequence

\[(56) \quad \text{Hom}^{q+2}(F'A, FB) \rightarrow \text{Hom}^{q+3}(F''A, FB) \rightarrow \text{Hom}^{q+3}(F'A, FB) = 0\]

we now get $\text{Hom}^{q+2}(F'A, FB) \neq 0$. However, for $q > 0$ we have $\text{Hom}^{q+2}(A, B) = 0$, so $F$ is not fully faithful. This contradiction shows that $q = 0$. \[\square\]

Remark 5. For higher-dimensional varieties the Hilbert scheme becomes (very) singular (unless $n = 2, 3$), and it is better to consider symmetric quotient stacks. In this case there is no universal ideal sheaf, but one can define the truncated ideal sheaf \cite{24, §5} and it turns out that if $\mathcal{O}_X$ is exceptional then the associated Fourier–Mukai transform is fully faithful \cite{24, proposition 5.6}. It would be interesting to know whether this condition is also necessary.

When $n = 2$ it is shown in \cite{5, theorem A} that $F = \Phi_Y : D^b(X) \rightarrow D^b(X^{[2]})$ is fully faithful if $\mathcal{O}_X$ is exceptional, and in fact this follows from the fully faithfulness of $D^b(X) \rightarrow D^b([X^2/\mathbb{Z}_2])$, see also \cite{5, remark 11}. It would be interesting to know whether this condition is also necessary.

When $n = 3$ neither sufficiency nor necessity are known, and the question is closely related to the conjecture that $\Psi : D^b([X^3/\mathbb{Z}_3]) \rightarrow D^b(X^{[3]})$ is fully faithful.

4 A converse to the $\mathbb{P}^n$-functor criterion of Addington

In this section we prove theorem B. Let $X$ and $Y$ be smooth projective varieties. For a Fourier–Mukai transform $F = \Phi_Y : D^b(X) \rightarrow D^b(Y)$ we write $F^L = \Phi_{[y]} : D^b(X) \rightarrow D^b(Y)$ for its left resp. right adjoint functor; compare section 2. The unit $\eta : \text{id}_{D^b(X)} \rightarrow F^R \circ F$ of the adjunction $F \dashv F^R$ is induced by a morphism $\eta : \mathcal{O}_X \rightarrow \mathcal{P}^R \star \mathcal{P}$ where $\mathcal{P}^R \star \mathcal{P}$ is the convolution product of the Fourier–Mukai kernels; see \cite{2} or \cite{12}. Hence one can define

\[(57) \quad C := \text{cone}(\eta : \text{id}_{D^b(X)} \rightarrow F^R \circ F)\]

as the Fourier–Mukai transform along cone($\mathcal{O}_X \rightarrow \mathcal{P}^R \star \mathcal{P}$).

We will need the following two notions.
Definition 6. We say that a Fourier–Mukai functor \( F : \mathcal{D}^b(X) \to \mathcal{D}^b(Y) \) is spherical if the associated cotwist \( C := \text{cone}(\eta : \text{id}_{\mathcal{D}^b(X)} \to F^R \circ F) \) is an auto-equivalence such that there is an isomorphism \( F^R \cong C \circ F^L \).

Definition 7. We say that a Fourier–Mukai functor \( F : \mathcal{D}^b(X) \to \mathcal{D}^b(Y) \) is a \( \mathbb{P}^n \)-functor if there exists an auto-equivalence \( H \) of \( \mathcal{D}^b(X) \), called the \( \mathbb{P} \)-cotwist of \( F \), such that

1. \( F^R \circ F \cong \text{id}_{\mathcal{D}^b(X)} \oplus H \oplus \ldots \oplus H^n \);

2. the composition \( H \circ F^R \circ F \xrightarrow{\iota \circ F^R_0 \circ F} F^R \circ F \circ R \circ F \xrightarrow{F^R \circ \delta \circ F} R \circ F \), where \( i : H \to F^R \circ F \) is the embedding of the direct summand under the above isomorphism and \( \iota : F \circ F^R \to \text{id}_{\mathcal{D}^b(Y)} \) is the counit of adjunction, is of the form

\[
\begin{pmatrix}
* & * & \ldots & * \\
\text{id}_H & * & * & * \\
0 & \text{id}_H & * & * \\
\vdots & \vdots & \ddots & * \\
0 & 0 & \ldots & \text{id}_H & *
\end{pmatrix}
\]

(58) when written in terms of the decomposition

\[
H \oplus H^2 \oplus \ldots \oplus H^{n+1} \cong \text{id}_{\mathcal{D}^b(X)} \oplus H \oplus \ldots \oplus H^n;
\]

3. \( F^R \cong H^n \circ F^L \).

One important reason for the interest in spherical and \( \mathbb{P} \)-functors is that they induce autoequivalences, called twists, of their target categories; see [30, 3, 1].

Note that our assumption that \( X \) and \( Y \) are smooth and projective imply that \( \mathcal{D}^b(X) \) and \( \mathcal{D}^b(Y) \) possess Serre functors \( S_X = (-) \otimes \omega_X[\dim X] \) and \( S_Y = (-) \otimes \omega_Y[\dim Y] \). The condition \( F^R \cong C \circ F^L \) for a spherical functor is equivalent to \( S_Y \circ F \circ C \cong F \circ S_X \), see [1, page 225], and the condition \( F^R \cong H^n \circ F^L \) for a \( \mathbb{P}^n \)-functor is equivalent to \( S_Y \circ F \circ H^n \cong F \circ S_X \); see [1, page 246]. Hence we have the following property common to spherical functors and \( \mathbb{P} \)-functors on which our proof of theorem B relies.

Lemma 8. Let \( F : \mathcal{D}^b(X) \to \mathcal{D}^b(Y) \) be a Fourier–Mukai transform which is a spherical or a \( \mathbb{P} \)-functor. Then there exists an autoequivalence \( D \in \text{Aut}(\mathcal{D}^b(X)) \) such that

\[
(60) \quad S_Y \circ F \circ D \cong F \circ S_X.
\]

Here \( D \) is

- the cotwist \( C \) of \( F \), if \( F \) is a spherical functor;
- the \( n \)th power of the \( \mathbb{P} \)-cotwist \( H \) of \( F \), if \( F \) is a \( \mathbb{P}^n \)-functor.

The second ingredient is the observation that for a smooth projective surface \( S \) the functor \( F = \Phi_{\mathcal{C}_S} : \mathcal{D}^b(S) \to \mathcal{D}^b(S[n]) \) has a left inverse \( I : \mathcal{D}^b(S[n]) \to \mathcal{D}^b(S) \) satisfying a compatibility with the Serre functors. This left-inverse is given by the composition

\[
(61) \quad 1 : \mathcal{D}^b(S[n]) \xrightarrow{\Phi^{-1}} \mathcal{D}^b_{\mathcal{C}_S}(S^n) \xrightarrow{L^V} \mathcal{D}_{\mathcal{C}_S}^b(S) \xrightarrow{\text{Res}_{S/2}^b} \mathcal{D}^b_{\mathcal{C}_S}^b(S) \xrightarrow{\text{Res}_{S/2}^b} \mathcal{D}^b_{\mathcal{C}_S}(S) \xrightarrow{(-)^{\frac{2g}{2}}_{\mathcal{C}_S}} \mathcal{D}^b(S)
\]

where all functors are defined in section 2.3.
Lemma 9. The functor

\[(62) \quad I := (-)^{\mathbb{Z}_2} \circ M_{a_2} \circ \text{Res}_{\mathbb{Z}_2}^{\mathbb{Z}_2} \circ L \delta^* \circ \Psi^{-1} : \mathcal{D}^b(S[n]) \to \mathcal{D}^b(S) \]

satisfies \(I \circ F \equiv \text{id}_{\mathcal{D}^b(S)}\) and \(I \circ S_{[n]} \equiv S^n \circ I\).

**Proof.** The statement \(I \circ F \equiv \text{id}\) is proved in [23, theorem 3.6].

For the compatibility with the Serre functors, first note that \(\Psi^{-1}\), being an equivalence, commutes with the Serre functors, i.e.

\[(63) \quad \Psi^{-1} \circ S_{[n]} \equiv S_{[\text{Sym}^n S]} \circ \Psi^{-1}\]

where \(S_{[\text{Sym}^n S]}\) denotes the Serre functor of \(D^b_{S_n} (S^n)\). It is given by \(S_{[\text{Sym}^n S]} = -\otimes \omega_{S^n}[2n]\) where \(\omega_{S^n}\) is equipped with the natural \(\mathbb{Z}_n\)-linearisation. This linearisation restricts to the trivial action on the pullback \(\delta^* \omega_{S^n} \equiv \omega_{S^n}\). Hence for \(D^b_{S_n} (S^n)\) we have natural isomorphisms

\[(64) \quad \begin{align*}
(a_2 \otimes \text{Res}_{\mathbb{Z}_2}^{\mathbb{Z}_2} \delta^*(E \otimes \omega_{S^n}))^{\mathbb{Z}_2} & \equiv (a_2 \otimes \text{Res}_{\mathbb{Z}_2}^{\mathbb{Z}_2} \delta^*(E) \otimes \omega_{S^n}^{\mathbb{Z}_2})^{\mathbb{Z}_2} \\
& \equiv (a_2 \otimes \text{Res}_{\mathbb{Z}_2}^{\mathbb{Z}_2} \delta^*(E))^{\mathbb{Z}_2} \otimes \omega_{S^n}^{\mathbb{Z}_2}.
\end{align*}\]

This gives

\[(65) \quad (-)^{\mathbb{Z}_2} \circ M_{a_2} \circ \text{Res}_{\mathbb{Z}_2}^{\mathbb{Z}_2} \circ \delta^* \circ S_{[\text{Sym}^n S]} \equiv S^n \circ (-)^{\mathbb{Z}_2} \circ M_{a_2} \circ \text{Res}_{\mathbb{Z}_2}^{\mathbb{Z}_2} \circ \delta^* .\]

□

**Corollary 10.** Assume that \(S_{[n]} \circ F \circ D \equiv F \circ S_S\) for some \(D \in \text{Aut}(D^b(S))\). Then we have \(D \equiv S_{S}^{-(n-1)}\).

**Proof.** We postcompose both sides of the given isomorphism of functors by the left inverse \(I\). By lemma 9, we get on the left-hand side

\[(66) \quad I \circ S_{[n]} \circ F \circ D \equiv S^n \circ D ,\]

while the right-hand side is \(I \circ F \circ S_S \equiv S_S\). Now, postcomposing both sides with \(S_S^{-n}\) gives the assertion \(D \equiv S_{S}^{-(n-1)}\).

□

For \(x \in S\), we consider the subset \(U_x := \{ \xi \in S^n \mid x \neq \xi \} \subseteq S^n\). It is an open subset whose complement is of codimension 2.

**Lemma 11.** For \(x \in S\), we have \(F(k(x))|_{U_x} \equiv \mathcal{O}_{U_x}\).

**Proof.** We use the exact triangle of functors (13). By definition of \(F''\) as the Fourier–Mukai transform along the structure sheaf of the universal family, we have \(F''(k(x))|_{U_x} \equiv 0\). Furthermore, as \(H^*(S, k(x)) \equiv k[0]\), we have \(F'(k(x)) \equiv \mathcal{O}_{S([n]}\). The assertion follows from the exact triangle \(F \to F' \to F'' \to F[1]\).

□

**Proposition 12.** Let \(S_{[n]} \circ F \circ D \equiv F \circ S_S\) for some \(D \in \text{Aut}(D^b(S))\). Then \(\omega_S \equiv \mathcal{O}_S\).
Proof. We apply both sides of the given isomorphism to the skyscraper sheaf \( k(x) \) of some point \( x \in S \). By corollary 10, we have

\[
(67) \quad D(k(x)) \cong S - (n-1)(k(x)) \cong k(x)[-2(n-1)].
\]

Hence the given isomorphism yields an isomorphism \( F(k(x)) \otimes \omega_{S[\omega]} \cong F(k(x)) \). Now, lemma 11 gives \( \omega_{S[\omega]} \otimes \omega_{U_{\omega}} \cong \omega_{U_{\omega}} \). Since \( S^{[\omega]} \) is normal and the complement of \( U_{\omega} \) is of codimension 2, this implies \( \omega_{S[\omega]} \cong \omega_{U_{\omega}} \). Recall from section 2.1 that there is an injective group homomorphism \( \text{Pic}(S) \hookrightarrow \text{Pic}(S^{[\omega]}) \), \( L \mapsto D_L \) satisfying \( D_{O_S} \cong O_{S[\omega]} \) and \( D_{O_S} \cong O_{S[\omega]} \); for the latter isomorphism, see e.g. [27, proposition 1.6]. Hence \( O_{S[\omega]} \cong O_{\omega_S} \) implies \( O_S \cong O_{\omega_S} \).

\( \square \)

Proof of theorem B. Assume that \( F: D^b(S) \rightarrow D^b(S^{[\omega]}) \) is a spherical functor or a \( \mathbb{P}^m \)-functor for some \( m \geq 1 \). We want to show that \( S \) is a K3 surface (in which case it is known by [1, theorem 3.1] that \( F \) is a \( \mathbb{P}^{n-1} \)-functor). By lemma 8 together with proposition 12, we see that \( \omega_{S[\omega]} \) is trivial. Hence \( S \) is a K3 or an abelian surface. But for an abelian surface, we have \( F \circ F = \text{id}_{D^b(S^{[\omega]})} \otimes [-1]^{\otimes 2} \otimes \cdots \otimes [-2(n-1)]^{\otimes 2} \otimes [-2n] \) as computed in [25, page 1199]. Hence if \( S \) is an abelian surface, \( F \) is neither a spherical nor a \( \mathbb{P} \)-functor.

\( \square \)

Remark 13. When \( S \) is a K3 surface the functor \( F \) was the first instance of a \( \mathbb{P} \)-functor [1, theorem 3.1] from \( D^b(S) \) to \( D^b(S^{[\omega]}) \). We now know that this is the only surface for which \( F \) gives such a functor. But there exist other constructions of \( \mathbb{P} \)-functors \( D^b(S) \) to \( D^b(S^{[\omega]}) \) which work for arbitrary surfaces; see [20, 22].

5 Embedding multiple copies

In this section we prove theorem C. The motivation for these results comes from a result for moduli of vector bundles on curves. In this context, the analogue of theorem A is the fully faithfulness of the Fourier–Mukai functor

\[
(68) \quad \Phi_E : D^b(C) \rightarrow D^b(MC(r, \mathcal{L})),
\]

where \( MC(r, \mathcal{L}) \) is the moduli space of rank \( r \) bundles with determinant \( \mathcal{L} \) and \( \mathcal{E} \) the universal vector bundle on \( C \times MC(r, \mathcal{L}) \). Here \( \gcd(r, \deg \mathcal{L}) \equiv 1 \) so that the moduli space is smooth and projective of dimension \( (r^2 - 1)/(g - 1) \). This is shown for \( r = 2 \) in [15, 26] and \( r \geq 2 \) in [7] (under suitable conditions on the genus and with \( \deg \mathcal{L} = 1 \) and more generally in [6]).

Now in [7] it was observed that \( \Phi_E \) can be twisted by \( \omega_{MC(r, \mathcal{L})}(-1) \), so that \( \Phi_E(D^b(C)) \) and \( \Phi_E(D^b(C)) \otimes \omega_{MC(r, \mathcal{L})}(-1) \) are semiorthogonal. This follows from \( MC(r, \mathcal{L}) \) being a smooth projective Fano variety of index 2.

Hilbert schemes of points on surfaces are never Fano\(^2\), but nevertheless a similar method of embedding multiple copies exists, as will be shown in this section.

**Proposition 14.** Let \( S \) be a smooth projective surface, and let \( L, M \in \text{Pic}(S) \) with \( \text{Hom}^0(L, M) = 0 \). Then, for \( n \geq 3 \), the subcategories (im \( F \) \( \otimes \) \( L \) \( \subset \) \( D^b(S^{[\omega]}) \) and (im \( F \) \( \otimes \) \( M \) \( \subset \) \( D^b(S^{[\omega]}) \) are semiorthogonal: for every \( A, B \in D^b(S) \), we have

\[
(69) \quad \text{Hom}^0(F(A \otimes D_L, FB \otimes D_M) = 0.
\]

---

\(^2\)As the Hilbert–Chow morphism is a crepant resolution, the anticanonical bundle is never positive on the exceptional divisor.
Proof. As \( n \geq 3 \), the vanishing \( \text{Hom}^\bullet(L, M) = 0 \) implies

(70) \[ \text{Sym}^n \text{Hom}^\bullet(L, M) = \text{Sym}^{n-1} \text{Hom}^\bullet(L, M) = \text{Sym}^{n-2} \text{Hom}^\bullet(L, M) = 0 . \]

This means that all the Hom-spaces (15)–(18) vanish. Then, by the exact triangle (13), also \( \text{Hom}^\bullet(FA \otimes D_L, F \otimes D_M) = 0 \).

\[ \square \]

Proof of theorem C. Let \( O_S \) be exceptional. By theorem A, the functor \( F : D^b(S) \to D^b(S[n]) \) is fully faithful. Let now \( L_1, \ldots, L_m \) be an exceptional collection of line bundles. Since tensor product by the associated line bundle \( D_Li \) is an autoequivalence of \( D^b(S[n]) \), the functor \( F(\cdot) \otimes D_Li : D^b(S) \to D^b(S[n]) \) is again fully faithful for every \( i = 1, \ldots, n \).

This means that the subcategories \( F(D^b(S)) \otimes D_Li \) of \( D^b(S[n]) \) are admissible. The semiorthogonality of these subcategories is provided by proposition 14.

\[ \square \]

Remark 15. Even when \( A \) is trivial, the category \( B \) in (4) is not. To see this it suffices to observe that the number of copies of \( D^b(S) \) in (4) does not grow with \( n \).

6 Semiorthogonal decompositions for symmetric products of curves

In this section we prove theorem D. The Hilbert scheme of points on a smooth projective curve is nothing but its symmetric power: \( C^{[n]} \cong C^{(n)} \). And \( C^{(n)} \) has a description in terms of a projectivisation of a coherent but not necessary locally free sheaf. This is the content of [32, theorem 4], which describes the Abel–Jacobi map

(71) \[ C^{(n)} \to J := \text{Jac} C \]

as a projectivisation of a coherent sheaf on \( J \).

For \( i \geq 2g-1 \) the Abel–Jacobi map has the structure of a projective bundle for a locally free sheaf, so one can just apply Orlov’s projective bundle formula [29, theorem 2.6] to describe \( D^b(C^{(n)}) \).

For \( n \leq g - 1 \) it is expected that \( D^b(C^{(n)}) \) is indecomposable. This is proven for \( n = 1 \) in [28] and for \( n \leq \left\lfloor \frac{g}{2} \right\rfloor \) in [9, 8].

In the interesting range \( n = g, \ldots, 2g-2 \), where the Abel–Jacobi map is surjective but not a bundle, a semiorthogonal decomposition for \( D^b(C^{(n)}) \) is known, and forms the content of theorem D. In [34], this is shown using wall-crossing methods. We will give a more elementary proof using the description of \( C^{(n)} \) as a projectivisation of a coherent sheaf on the Jacobian \( J = \text{Jac} C \), together with the description of the derived category of such projectivisations [18, theorem 3.4]. Let us recall the statement of loc. cit.

For a coherent sheaf \( \mathcal{S} \) of rank \( r \) on \( X \) we will denote

(72) \[ X^{>i}(\mathcal{S}) := \{ x \in X \mid \text{rk} \mathcal{S}(x) > i \} \]

such that the singular locus \( \text{Sing}(\mathcal{S}) \) of \( \mathcal{S} \) is \( X^{>r}(\mathcal{S}) \), and the smooth part \( \text{Sing}(\mathcal{S})^{\text{sm}} \) of the singular locus is \( X^{>r}(\mathcal{S}) \setminus X^{>r+1}(\mathcal{S}) \).
**Theorem 16** (Jiang–Leung’s generalised projective bundle formula). Let $X$ be a smooth projective variety. Let $\mathcal{G}$ be a coherent sheaf of rank $r$, which locally admits a 2-step locally free resolution. Assume that

1. $\mathbb{P}(\mathcal{G})$ is irreducible of expected dimension $\dim X + r - 1$;
2. $\mathbb{P}(\mathcal{E}xt^1(\mathcal{G}, \mathcal{O}_X))$ is irreducible of expected dimension $\dim X - r - 1$;
3. the smooth part of the singular locus $\text{Sing}(\mathcal{G})^{\text{sm}}$ is non-empty of expected codimension $r + 1$ in $X$.

Then

1. $\mathbb{P}(\mathcal{E}xt^1(\mathcal{G}, \mathcal{O}_X)) \to \text{Sing}(\mathcal{G})$ is a resolution of singularities,
2. there exists a semiorthogonal decomposition

\[
\mathcal{D}b(\mathbb{P}(\mathcal{G})) = \langle \mathcal{D}b(\mathbb{P}(\mathcal{E}xt^1(\mathcal{G}, \mathcal{O}_X))), \mathcal{D}b(X), \ldots, \mathcal{D}b(X) \rangle.
\]

The fully faithful functors in (73) are

\[
\mathcal{R}q_1^*, \circ \mathcal{L}q_2^*: \mathcal{D}b(\mathbb{P}(\mathcal{E}xt^1(\mathcal{G}, \mathcal{O}_X))) \to \mathcal{D}b(\mathbb{P}(\mathcal{G}))
\]

where

\[
\begin{array}{ccc}
\mathbb{P}(\mathcal{G}) \times_X \mathbb{P}(\mathcal{E}xt^1(\mathcal{G}, \mathcal{O}_X)) & \xrightarrow{q_1} & \mathbb{P}(\mathcal{G}) \\
\mathbb{P}(\mathcal{E}xt^1(\mathcal{G}, \mathcal{O}_X)) & \xrightarrow{q_2} & \mathbb{P}(\mathcal{G})
\end{array}
\]

are the projections and

\[
\mathcal{L}\pi^*(-) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{G})}(i): \mathcal{D}b(X) \to \mathcal{D}b(\mathbb{P}(\mathcal{G}))
\]

for $i = 1, \ldots, r$, where $\pi: \mathcal{D}b(\mathbb{P}(\mathcal{G})) \to \mathcal{D}b(X)$ is the natural morphism.

Later, we will also use a special case of theorem 16 for the blowup in a singular center; see theorem 22.

**Remark 17.** Under the assumption that $\mathcal{G}$ locally admits a 2-step locally free resolution, we have

\[
\text{im}(\mathbb{P}(\mathcal{E}xt^1(\mathcal{G}, \mathcal{O}_X)) \to X) = \text{Sing}(\mathcal{G});
\]

see [18, remark 3.5]. We can replace condition 3 of theorem 16 by the (under the presence of conditions 1 and 2) equivalent condition that $\text{im}(\pi) = \text{Sing}(\mathcal{G})$ has the expected dimension $\dim X - r + 1$ in $X$. Indeed, if $\mathbb{P}(\mathcal{E}xt^1(\mathcal{G}, \mathcal{O}_X))$ and $\text{im}(\pi)$ are of the same dimension, the morphism $\pi$ must be generically finite. But the fibers of $\pi$ are projective spaces, hence, in particular, connected. That means that $\pi$ is generically an isomorphism. The locus over which $\pi$ is an isomorphism is exactly $\text{Sing}(\mathcal{G})^{\text{sm}}$; see again [18, remark 3.5].
Let us now recall some definitions and results from [32]. Fix a base point \( x_0 \in C \) and, for \( n \in \mathbb{N} \), write \( \mathcal{O}_{C}(n) := \mathcal{O}_C(n \cdot x_0) \). Let \( \mathcal{P} \in \text{Pic}(C \times J) \) be the Poincaré bundle, and let

\[
\begin{align*}
C \times J & \xrightarrow{p} J \quad \text{and} \quad J \xrightarrow{q} C \\
\end{align*}
\]

be the projections. There are for every \( n \in \mathbb{N} \) the Picard sheaves

\[
\begin{align*}
\mathcal{E}_n & := p_* (\mathcal{P} \otimes q^* \mathcal{O}_C(n)) \\
\mathcal{F}_n & := R^1 p_* (\mathcal{P} \otimes q^* \mathcal{O}_C(n)) .
\end{align*}
\]

We can write the canonical bundle of \( C \) in the form \( \omega_C \cong \mathcal{O}_C(2g - 2) \otimes \mathcal{K} \) for some degree zero line bundle \( \mathcal{K} \in \text{Pic}^0(C) \). We consider the automorphism \( \theta \) of \( J = \text{Pic}^0(C) \) given by \( \theta(E) := \mathcal{K} \otimes \mathcal{L}_E^\vee \). Then, for every \( n \geq 1 \), there is an isomorphism

\[
C^{(n)} \cong \mathbb{P}(\theta^* \mathcal{F}_{2g-2-n})
\]

of varieties over \( J \); see [32, theorem 4]. Furthermore, we have isomorphisms

\[
\begin{align*}
\theta^* \mathcal{E}_n & \cong p_* (\mathcal{P} \otimes q^* (\mathcal{O}_C(n) \otimes \mathcal{K})) \\
\theta^* \mathcal{F}_n & \cong R^1 p_* (\mathcal{P} \otimes q^* (\mathcal{O}_C(n) \otimes \mathcal{K})) ;
\end{align*}
\]

see [32, lemma 1]. We will also use the fact that

\[
\mathcal{E}_i = 0 \quad \text{for } i < g ;
\]

see [32, corollary 2].

**Lemma 18.** For \( n = g, \ldots, 2g - 2 \) we have \( \text{Ext}^1 (\theta^* \mathcal{F}_{2g-2-n}, \mathcal{O}_J) \cong \mathcal{F}_n \).

**Proof.** Note that \( 2g - 2 - n < g \). Hence by (82) we have \( 0 = \mathcal{E}_{2g-2-n} \). Combining this vanishing with (81) gives

\[
0 = \theta^* \mathcal{E}_{2g-2-n} \cong p_* (\mathcal{P} \otimes q^* (\mathcal{O}_C(2g - 2 - n) \otimes \mathcal{K})) .
\]

Hence the derived pushforward \( R^1 p_* (\mathcal{P} \otimes q^* (\mathcal{O}_C(2g - 2 - n) \otimes \mathcal{K})) \) is concentrated in degree 1, which means that

\[
R^1 p_* (\mathcal{P} \otimes q^* (\mathcal{O}_C(2g - 2 - n) \otimes \mathcal{K})) \cong \theta^* \mathcal{F}_{2g-2-n}[-1] .
\]

This implies

\[
\text{Ext}^1 (\theta^* \mathcal{F}_{2g-2-n}, \mathcal{O}_J) \cong \mathcal{H}^0 \left( \mathcal{R} \text{Hom}_J (R^1 p_* (\mathcal{P} \otimes q^* (\mathcal{O}_C(2g - 2 - n) \otimes \mathcal{K})), \mathcal{O}_J) \right)
\]

By Grothendieck–Verdier duality, we have

\[
\mathcal{R} \text{Hom}_J (R^1 p_* (\mathcal{P} \otimes q^* (\mathcal{O}_C(2g - 2 - n) \otimes \mathcal{K})), \mathcal{O}_J) \cong R^1 \text{Hom}_{\text{Pic}(C \times J)} (\mathcal{P} \otimes q^* (\mathcal{O}_C(2g - 2 - n) \otimes \mathcal{K}), \omega_p)[1]
\]
Note that $\omega_p \equiv q^*\omega_C \equiv q^*(O_C(2g-2) \otimes \mathcal{K})$. Hence

$$R^j\hom_{C,j}(\mathcal{P}^n \otimes q^*(O_C(2g-2-n) \otimes \mathcal{K})), o_p) \cong \mathcal{P} \otimes q^*O_C(n)[0].$$

Plugging (87) into (86) gives

$$\mathcal{H}^0 \left(R^j\hom_{j}(\mathcal{P}^n \otimes q^*(O_C(2g-2-n) \otimes \mathcal{K})), o_j) \cong R^jG^*(\mathcal{P} \otimes q^*O_C(n)) = \mathcal{F}_n \right.$$  

Combining (88) with (85) gives the assertion. \hfill \Box

**Proof of theorem D.** Set $\mathcal{S} := \mathcal{P}2g-2-n$. By (80) together with lemma 18, we have

$$P(\mathcal{Ext}^1(\mathcal{S}, \mathcal{O}_j)) \cong P(\mathcal{F}_n) \cong C^{(2g-2-n)}.$$  

Hence the semiorthogonal decomposition (73) from theorem 16 gives rise to (5), once we have checked that $\mathcal{S}$ satisfies the assumptions. Note that the isomorphism $P(\mathcal{Ext}^1(\mathcal{S}, \mathcal{O}_j)) \cong C^{(2g-2-n)}$ identifies the canonical map $\pi : P(\mathcal{Ext}^1(\mathcal{S}, \mathcal{O}_j)) \to J$ with $\theta \circ A\theta$, where $A\theta : C^{(2g-2-n)} \to J$ is the Abel–Jacobi map. In particular, $\text{im}(\pi) = \theta(\text{im}(A\theta))$.

By [32, proposition 4 and the following remark], we know that $\mathcal{S}$ has a two-term resolution by locally free sheaves. Note that $r := \text{rk}(\mathcal{S}) = \dim C(n) - \dim J + 1 = n - g + 1$. Hence $P(\mathcal{Ext}^1(\mathcal{S}, \mathcal{O}_j)) \cong C^{(2g-2-n)}$ has the expected dimension $2g - 2 - n = g - r - 1$.

Since $2g - 2 - n < g$, we have $\text{dim}(\text{im}(A\theta)) = \dim C^{(2g-2-n)} = 2g - 2 - n$; see [4, page 25]. Since, as noted above we have $\text{im}(\pi) = \theta(\text{im}(A\theta))$, we also have $\text{dim}(\text{im}(\pi)) = 2g - 2 - n$.

By remark 17, this shows that also the third condition of theorem 16 is fulfilled. \hfill \Box

**Remark 19.** The same technique has been used independently by the authors of [18] in the second version of their preprint.

### 7  Semiorthogonal decompositions for nested Hilbert schemes

In this section we prove theorem E. Nested Hilbert schemes are usually seen as a means to set up correspondences between Hilbert schemes of points, e.g. in the construction of actions of Heisenberg algebras on the cohomology of Hilbert schemes. But one can also study them for their own sake, as we do in this section.

**Definition 20.** Let $S$ be a smooth projective surface. Let $n \geq 2$. The nested Hilbert scheme is

$$s^{(n-1,n)} := \{ (\xi, \xi') \in s^{(n-1)} \times s^{(n)} \mid \xi \subset \xi' \} \hookrightarrow s^{(n-1)} \times s^{(n)}.$$  

By [11, theorem 3.0.1] the nested Hilbert scheme $s^{(n-1,n)}$ is a smooth projective variety of dimension $2n$, which is of Kodaira dimension $n\kappa(S)$ by the geometric description (91), together with [17, theorem 11.1.2].

**Remark 21.** In the literature, also nested Hilbert schemes of the form $s^{(m,n)} \subset s^{(m)} \times s^{(n)}$ for $m < n$ are considered. However, for $m \neq n - 1$ they are singular; see [11, theorem 3.0.1]. Hence we will only consider the case $m = n - 1$.  

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As summarised in [1, §3.1] the nested Hilbert scheme $S^{[n-1,n]}$ comes with two projection morphisms which we will denote as

$$\begin{align*}
S^{[n-1,n]} & \xrightarrow{f_n} S^{[n-1,n]} \xrightarrow{g_n} S^{[n,n]}.
\end{align*}$$

These send the pair $(\zeta, \xi)$ to $\zeta$ (resp. $\xi$). Moreover, we have the morphism

$$q_n : S^{[n-1,n]} \to S$$

which sends the pair $(\zeta, \xi)$ to the difference $\xi \setminus \zeta$. The morphism

$$\phi_n := q_n \times g_n : S^{[n-1,n]} \to S \times S^{[n]}$$

has as its image the universal subscheme $Z_n$, and is a resolution of singularities of $Z_n$. Recall that $Z_n$ is singular, as soon as $n \geq 3$. However, it is always Cohen–Macaulay as it is flat and finite over the smooth base $S^{[n]}$. For $n = 1$ it is nothing but the diagonal, whilst for $n = 2$ it is $\text{Bl}_\Delta S \times S$.

The morphism

$$y_n = q_n \times f_n : S^{[n-1,n]} \to S \times S^{[n-1]}$$

on the other hand is the blowup of $S \times S^{[n-1]}$ in the universal subscheme $Z_{n-1}$.

The proof of theorem E relies on a generalisation of Orlov’s blowup formula [29, theorem 4.3], which in this case cannot be applied as the center of the blowup in (94) is singular. This can be remedied by using a more general version of the blowup formula which is an instance of a projective bundle formula for not necessarily locally free sheaves in homological projective geometry [18, theorem 3.4], see theorem 16. The special case we will apply is described in [18, §3.1.2].

**Theorem 22** (Jiang–Leung’s generalised blowup formula). Let $X$ be a smooth projective variety. Let $Z \hookrightarrow X$ be a Cohen–Macaulay subscheme of codimension 2 which is cut out by the ideal sheaf $\mathcal{I}_Z$. This sheaf admits a locally free resolution

$$0 \to \mathcal{F} \xrightarrow{\sigma} \mathcal{E} \to \mathcal{I}_Z \to 0$$

where $\text{rk} \mathcal{E} = \text{rk} \mathcal{F} + 1$.

Then there exists a semiorthogonal decomposition

$$D^b(\text{Bl}_Z X) = \left\{ D^b(\tilde{Z}), D^b(X) \right\},$$

where $\tilde{Z} \to Z$ is the (Springer-type) resolution of singularities given by

$$\tilde{Z} := \mathbb{P}(\mathcal{E}^2 \iota^!(\mathcal{I}_Z, \mathcal{O}_X)) = \{(x, [H_x]) | \text{ im } \sigma^\vee(x) \subseteq H_x \} \subseteq \mathbb{P}_X(\mathcal{F}^\vee)$$

With this result available to us, the proof of theorem E is very short.

**Proof of theorem E.** By the proof of [14, lemma 4.7] we have that $Z_{n-1}$ is Cohen–Macaulay, and it is of codimension 2. So it suffices to check that the resolution $\mathbb{P}(\mathcal{E}^2 \iota^!(\mathcal{I}_{Z_{n-1}}, \mathcal{O}_{S \times S^{[n]}}))$ of $Z_{n-1}$ is isomorphic to $S^{[n-2,n-1]}$. But this is the content of [33, theorem 2]. \[\square\]
References


