Thermal drag effect in quantum Hall circuits
Supplemental material

Edvin G. Idrisov
Department of Physics and Materials Science, University of Luxembourg, Luxembourg

Ivan P. Levkivskyi
Dropbox Ireland, One Park Place, Hatch Street Upper, Dublin, Ireland

Eugene V. Sukhorukov
Département de Physique Théorique, Université de Genève, CH-1211 Genève 4, Switzerland
(Dated: July 21, 2022)

A: DERIVATION OF EQS. (10) AND (11)

To find temperatures $T_{\rm c,u}$ and $T_{\rm c,d}$ in Eq. (8) in main text, we solve self-consistently the energy balance equations at each Ohmic contact, namely $J_{\rm c,u} = J^{\rm c}_{\rm out,u}$ and $J_{\rm c,d} = J^{\rm c}_{\rm out,d}$. According to Eqs. (8) and (9) from main text, the heat currents on the left hand side of balance equations are given by quantum heat fluxes

$$J_{\rm c,u} = \frac{\pi T^2_{\rm c,u}}{12}, \quad J_{\rm c,d} = \frac{\pi T^2_{\rm c,d}}{12}. \tag{S-1}$$

Now, using the scattering matrix, $\hat{\mathcal{U}}(\omega)$, from Eq. (6) and Eqs. (8), (9) of main text, the outgoing fluxes are given by

$$\begin{aligned} J^{\rm c}_{\rm out,u} = {} & \frac{1}{2}\int\frac{d\omega}{2\pi}|\mathcal{C}(\omega)|^2\left[S_{\rm in,u}(\omega)-S_0(\omega)\right] + \frac{1}{2}\int\frac{d\omega}{2\pi}|\mathcal{B}(\omega)|^2\left[S_{\rm in,d}(\omega)-S_0(\omega)\right] \\ & + \frac{1}{2}\int\frac{d\omega}{2\pi}|\mathcal{A}(\omega)|^2\left[S_{\rm c,u}(\omega)-S_0(\omega)\right] + \frac{1}{2}\int\frac{d\omega}{2\pi}|\mathcal{B}(\omega)|^2\left[S_{\rm c,d}(\omega)-S_0(\omega)\right], \end{aligned} \tag{S-2}$$

and

$$\begin{aligned} J^{\rm c}_{\rm out,d} = {} & \frac{1}{2}\int\frac{d\omega}{2\pi}|\mathcal{B}(\omega)|^2\left[S_{\rm in,u}(\omega)-S_0(\omega)\right] + \frac{1}{2}\int\frac{d\omega}{2\pi}|\mathcal{C}(\omega)|^2\left[S_{\rm in,d}(\omega)-S_0(\omega)\right] \\ & + \frac{1}{2}\int\frac{d\omega}{2\pi}|\mathcal{B}(\omega)|^2\left[S_{\rm c,u}(\omega)-S_0(\omega)\right] + \frac{1}{2}\int\frac{d\omega}{2\pi}|\mathcal{A}(\omega)|^2\left[S_{\rm c,d}(\omega)-S_0(\omega)\right], \end{aligned} \tag{S-3}$$

where $S_{l,\alpha}(\omega) = \omega/(1-e^{-\omega/T_{l,\alpha}})$ is given by equilibrium spectral density in Eq. (8) from main text and $S_0(\omega) = \omega\theta(\omega)$. Next, using the property of scattering matrix, $|\mathcal{A}(\omega)|^2+2|\mathcal{B}(\omega)|^2+|\mathcal{C}(\omega)|^2 = 1$ the above expressions take the following forms

$$\begin{aligned} J^{\rm c}_{\rm out,u} = {} & \frac{\pi T^2_{\rm in,u}}{12} + \frac{1}{2}\int\frac{d\omega}{2\pi}|\mathcal{B}(\omega)|^2\left[S_{\rm in,d}(\omega)-S_{\rm in,u}(\omega)\right] + \\ & \frac{1}{2}\int\frac{d\omega}{2\pi}|\mathcal{A}(\omega)|^2\left[S_{\rm c,u}(\omega)-S_{\rm in,u}(\omega)\right] + \frac{1}{2}\int\frac{d\omega}{2\pi}|\mathcal{B}(\omega)|^2\left[S_{\rm c,d}(\omega)-S_{\rm in,u}(\omega)\right], \end{aligned} \tag{S-4}$$

and similarly

$$\begin{aligned} J^{\rm c}_{\rm out,d} = {} & \frac{\pi T^2_{\rm in,d}}{12} + \frac{1}{2}\int\frac{d\omega}{2\pi}|\mathcal{B}(\omega)|^2\left[S_{\rm in,u}(\omega)-S_{\rm in,d}(\omega)\right] + \\ & \frac{1}{2}\int\frac{d\omega}{2\pi}|\mathcal{A}(\omega)|^2\left[S_{\rm c,d}(\omega)-S_{\rm in,d}(\omega)\right] + \frac{1}{2}\int\frac{d\omega}{2\pi}|\mathcal{B}(\omega)|^2\left[S_{\rm c,u}(\omega)-S_{\rm in,d}(\omega)\right]. \end{aligned} \tag{S-5}$$

Finally, using the balance equations, $J_{\rm c,u} = J^{\rm c}_{\rm out,u}$ and $J_{\rm c,d} = J^{\rm c}_{\rm out,d}$ mentioned above and introducing the dimensionless variable of integration, $z = \omega\tau_{\rm c}$ we arrive to systems of coupled equations given in Eq. (10) of main text,

$$\begin{aligned} \frac{\pi T^2_{\rm c,u}}{12} &= \frac{\pi T^2_{\rm in,u}}{12} + J^{\mathcal{B}}_{\substack{\rm in,d\\ \rm in,u}}(\lambda) + J^{\mathcal{A}}_{\substack{\rm c,u\\ \rm in,u}}(\lambda) + J^{\mathcal{B}}_{\substack{\rm c,d\\ \rm in,u}}(\lambda), \\ \frac{\pi T^2_{\rm c,d}}{12} &= \frac{\pi T^2_{\rm in,d}}{12} + J^{\mathcal{B}}_{\substack{\rm in,u\\ \rm in,d}}(\lambda) + J^{\mathcal{A}}_{\substack{\rm c,d\\ \rm in,d}}(\lambda) + J^{\mathcal{B}}_{\substack{\rm c,u\\ \rm in,d}}(\lambda). \end{aligned} \tag{S-6}$$

The functions on the right hand side of above equations are given by

$$\begin{aligned} J^{i}_{\substack{l,\alpha\\ k,\beta}}(\lambda) &= \pi[T^2_{l,\alpha}\cdot I_i(\lambda,\tau_{\rm c}T_{l,\alpha}) - T^2_{k,\beta}\cdot I_i(\lambda,\tau_{\rm c}T_{k,\beta})]/12, \\ I_i(\lambda,a) &= \frac{6}{(\pi a)^2}\int_0^\infty \frac{dzzg_i(z,\lambda)}{\exp(z/a)-1}, \quad i=\mathcal{A},\mathcal{B}, \end{aligned} \tag{S-7}$$

where $g_{\mathcal{A}}(z,\lambda) = |iz+\lambda^2-1|^2/|(z+i)^2+\lambda^2|^2$, $g_{\mathcal{B}}(z,\lambda) = \lambda^2 z^2/|(z+i)^2+\lambda^2|^2$. This is the Eq. (11) from main text.

B: HEATING EFFECTS

In the case of $\lambda = 1$, from Eq. (11) of main text, we have $g_{\mathcal{A}}(z,\lambda=1) = g_{\mathcal{B}}(z,\lambda=1) = 1/(z^2+4)$ and the asymptotics of the integral in Eq. (11) of main text have the simple form

$$I_i(\lambda=1,a) = \frac{6}{(\pi a)^2}\int_0^\infty \frac{dzz}{e^{z/a}-1}\frac{1}{z^2+4} \simeq \begin{cases} 1/4-(\pi a)^2/40, & a\ll 1, \\ 3/2\pi a, & a\gg 1. \end{cases} \tag{S-8}$$

Substituting these asymptotics into Eq. (10) from the main text, we further obtain the final results for the temperatures $T_{\rm c,u}$ and $T_{\rm c,d}$ in Eq. (12) of the main text.

In the case of weak coupling, $\lambda \ll 1$, one can substitute the expansion of amplitudes of scattering matrix $g_{\mathcal{A}}(z,\lambda\ll 1) \approx [1-4\lambda^2z^2/(1+z^2)^2]/(1+z^2)$ and $g_{\mathcal{B}}(z,\lambda\ll 1)\approx \lambda^2z^2/(1+z^2)^2$ into Eq. (11) from main text and get

$$I_{\mathcal{A}}(\lambda\ll 1,a) \simeq I_{\mathcal{A}}(\lambda=0,a) - 4\lambda^2\cdot\delta I_{\mathcal{A}}(\lambda=0,a), \quad \text{and} \quad I_{\mathcal{B}}(\lambda\ll 1,a)\simeq \lambda^2\cdot\delta I_{\mathcal{B}}(\lambda=0,a), \tag{S-9}$$

where the asymptotics of integrals on the right hand side are given by

$$I_{\mathcal{A}}(\lambda=0,a) = \frac{6}{(\pi a)^2}\int_0^\infty \frac{dzz}{e^{z/a}-1}\frac{1}{1+z^2} \simeq \begin{cases} 1-2(\pi a)^2/5, & a\ll 1, \\ 3/\pi a, & a\gg 1, \end{cases} \tag{S-10}$$

$$\delta I_{\mathcal{A}}(\lambda=0,a) = \frac{6}{(\pi a)^2}\int_0^\infty \frac{dzz}{e^{z/a}-1}\frac{z^2}{(1+z^2)^3} \simeq \begin{cases} 2(\pi a)^2/5, & a\ll 1, \\ 3/2\pi a, & a\gg 1, \end{cases} \tag{S-11}$$

$$\delta I_{\mathcal{B}}(\lambda=0,a) = \frac{6}{(\pi a)^2}\int_0^\infty \frac{dzz}{e^{z/a}-1}\frac{z^2}{(1+z^2)^2} \simeq \begin{cases} 2(\pi a)^2/5, & a\ll 1, \\ 3/8\pi a, & a\gg 1. \end{cases} \tag{S-12}$$

Substituting the above asymptotics in the case of $\max\{\tau_{\rm c}T_{l,\alpha}\}\ll 1$ ($a\ll 1$) into Eq. (10) from the main text, one obtains the following system of equations

$$\begin{cases}\dfrac{\pi T_{\mathrm{c,u}}^2(\lambda)}{12}-\dfrac{\pi T_{\mathrm{in,u}}^2}{12}=\lambda^2\left[\dfrac{\pi T_{\mathrm{in,d}}^2}{12}\dfrac{2(\pi\tau_{\mathrm{c}}T_{\mathrm{in,d}})^2}{5}-\dfrac{\pi T_{\mathrm{in,u}}^2}{12}\dfrac{2(\pi\tau_{\mathrm{c}}T_{\mathrm{in,u}})^2}{5}\right]\\ +\dfrac{\pi T_{\mathrm{c,u}}^2(\lambda)}{12}\left[1-\dfrac{2(\pi\tau_{\mathrm{c}}T_{\mathrm{c,u}}(\lambda))^2}{5}-4\lambda^2\dfrac{2(\pi\tau_{\mathrm{c}}T_{\mathrm{c,u}}(\lambda))^2}{5}\right]\\ -\dfrac{\pi T_{\mathrm{in,u}}^2}{12}\left[1-\dfrac{2(\pi\tau_{\mathrm{c}}T_{\mathrm{in,u}})^2}{5}-4\lambda^2\dfrac{2(\pi\tau_{\mathrm{c}}T_{\mathrm{in,u}})^2}{5}\right]\\ +\lambda^2\left[\dfrac{\pi T_{\mathrm{c,d}}^2(\lambda)}{12}\dfrac{2(\pi\tau_{\mathrm{c}}T_{\mathrm{c,d}}(\lambda))^2}{5}-\dfrac{\pi T_{\mathrm{in,u}}^2}{12}\dfrac{2(\pi\tau_{\mathrm{c}}T_{\mathrm{in,u}})^2}{5}\right],\\ \\ \dfrac{\pi T_{\mathrm{c,d}}^2(\lambda)}{12}-\dfrac{\pi T_{\mathrm{in,d}}^2}{12}=\lambda^2\left[\dfrac{\pi T_{\mathrm{in,u}}^2}{12}\dfrac{2(\pi\tau_{\mathrm{c}}T_{\mathrm{in,u}})^2}{5}-\dfrac{\pi T_{\mathrm{in,d}}^2}{12}\dfrac{2(\pi\tau_{\mathrm{c}}T_{\mathrm{in,d}})^2}{5}\right]\\ +\dfrac{\pi T_{\mathrm{c,d}}^2(\lambda)}{12}\left[1-\dfrac{2(\pi\tau_{\mathrm{c}}T_{\mathrm{c,d}}(\lambda))^2}{5}-4\lambda^2\dfrac{2(\pi\tau_{\mathrm{c}}T_{\mathrm{c,d}}(\lambda))^2}{5}\right]\\ -\dfrac{\pi T_{\mathrm{in,d}}^2}{12}\left[1-\dfrac{2(\pi\tau_{\mathrm{c}}T_{\mathrm{in,d}})^2}{5}-4\lambda^2\dfrac{2(\pi\tau_{\mathrm{c}}T_{\mathrm{in,d}})^2}{5}\right]\\ +\lambda^2\left[\dfrac{\pi T_{\mathrm{c,u}}^2(\lambda)}{12}\dfrac{2(\pi\tau_{\mathrm{c}}T_{\mathrm{c,u}}(\lambda))^2}{5}-\dfrac{\pi T_{\mathrm{in,d}}^2}{12}\dfrac{2(\pi\tau_{\mathrm{c}}T_{\mathrm{in,d}})^2}{5}\right],\end{cases}\tag{S-13}$$

The solution of these equations for $\max\{\tau_{\mathrm{c}}T_{l,\alpha}\}\ll 1$ is given by Eq. (13) from the main text.

In the opposite regime, $\min\{\tau_{\mathrm{c}}T_{l,\alpha}\}\gg 1$, one can obtain the result which is non-perturbative in λ. Namely, by calculating the integrals in this regime ($a\gg 1$) we get the following asymptotic results

$$\begin{aligned}I_{\mathcal{A}}(\lambda,a)&=a\int_0^\infty dz\frac{z/a}{e^{z/a}-1}\frac{z^2+(\lambda^2-1)^2}{z^4+2(\lambda^2+1)z^2+(\lambda^2-1)^2}=[a\to\infty]\\ &\simeq a\int_0^\infty dz\frac{z^2+(\lambda^2-1)^2}{z^4+2(\lambda^2+1)z^2+(\lambda^2-1)^2}=a\cdot\mathcal{F}_{\mathcal{A}}(\lambda),\quad \mathcal{F}_{\mathcal{A}}(\lambda)=\frac{\pi}{4}(2-\lambda^2),\end{aligned}\tag{S-14}$$

and similarly

$$\begin{aligned}I_{\mathcal{B}}(\lambda,a)&=a\int_0^\infty dz\frac{z/a}{e^{z/a}-1}\frac{z^2\lambda^2}{z^4+2(\lambda^2+1)z^2+(\lambda^2-1)^2}=[a\to\infty]\\ &\simeq a\int_0^\infty dz\frac{z^2\lambda^2}{z^4+2(\lambda^2+1)z^2+(\lambda^2-1)^2}=a\cdot\mathcal{F}_{\mathcal{B}}(\lambda),\quad \mathcal{F}_{\mathcal{B}}(\lambda)=\frac{\pi\lambda^2}{4}.\end{aligned}\tag{S-15}$$

After the substitution of Eq. (S-14) and (S-15) into Eq. (10) from the main text, the system of equations to find the temperatures of Ohmic contacts takes the form

$$\begin{cases}\frac{\pi T_{\mathrm{c,u}}^2(\lambda)}{12}=\frac{\pi T_{\mathrm{in,u}}^2}{12}+\left[\frac{\pi T_{\mathrm{in,d}}^2}{12}\frac{6}{\pi\tau_{\mathrm{c}}T_{\mathrm{in,d}}}-\frac{\pi T_{\mathrm{in,u}}^2}{12}\frac{6}{\pi\tau_{\mathrm{c}}T_{\mathrm{in,u}}}\right]\cdot\mathcal{F}_{\mathcal{B}}(\lambda)\\ +\left[\frac{\pi T_{\mathrm{c,u}}^2}{12}\frac{6}{\pi\tau_{\mathrm{c}}T_{\mathrm{c,u}}}-\frac{\pi T_{\mathrm{in,u}}^2}{12}\frac{6}{\pi\tau_{\mathrm{c}}T_{\mathrm{in,u}}}\right]\cdot\mathcal{F}_{\mathcal{A}}(\lambda)+\left[\frac{\pi T_{\mathrm{c,d}}^2}{12}\frac{6}{\pi\tau_{\mathrm{c}}T_{\mathrm{c,d}}}-\frac{\pi T_{\mathrm{in,u}}^2}{12}\frac{6}{\pi\tau_{\mathrm{c}}T_{\mathrm{in,u}}}\right]\cdot\mathcal{F}_{\mathcal{B}}(\lambda),\\ \\ \frac{\pi T_{\mathrm{c,d}}^2(\lambda)}{12}=\frac{\pi T_{\mathrm{in,d}}^2}{12}+\left[\frac{\pi T_{\mathrm{in,u}}^2}{12}\frac{6}{\pi\tau_{\mathrm{c}}T_{\mathrm{in,u}}}-\frac{\pi T_{\mathrm{in,d}}^2}{12}\frac{6}{\pi\tau_{\mathrm{c}}T_{\mathrm{in,d}}}\right]\cdot\mathcal{F}_{\mathcal{B}}(\lambda)\\ +\left[\frac{\pi T_{\mathrm{c,d}}^2}{12}\frac{6}{\pi\tau_{\mathrm{c}}T_{\mathrm{c,d}}}-\frac{\pi T_{\mathrm{in,d}}^2}{12}\frac{6}{\pi\tau_{\mathrm{c}}T_{\mathrm{in,d}}}\right]\cdot\mathcal{F}_{\mathcal{A}}(\lambda)+\left[\frac{\pi T_{\mathrm{c,u}}^2}{12}\frac{6}{\pi\tau_{\mathrm{c}}T_{\mathrm{c,u}}}-\frac{\pi T_{\mathrm{in,d}}^2}{12}\frac{6}{\pi\tau_{\mathrm{c}}T_{\mathrm{in,d}}}\right]\cdot\mathcal{F}_{\mathcal{B}}(\lambda).\end{cases}\tag{S-16}$$

As a first approximation, we choose the solution at $\lambda=0$, namely, we substitute $T_{\mathrm{c,u}}=T_{\mathrm{in,u}}$ and $T_{\mathrm{c,d}}=T_{\mathrm{in,d}}$ into the right hand side. Therefore, only the first term on the right hand side survives, and we get the final solution at

$\min\{\tau_c T_{l,\alpha}\} \gg 1$

$$\begin{aligned}\frac{\pi T_{\rm c,u}^2(\lambda)}{12} &= \frac{\pi T_{\rm in,u}^2}{12} + \frac{3\lambda^2}{2}\left[\frac{\pi T_{\rm in,d}^2}{12}\frac{1}{\pi\tau_c T_{\rm in,d}} - \frac{\pi T_{\rm in,u}^2}{12}\frac{1}{\pi\tau_c T_{\rm in,u}}\right],\\ \frac{\pi T_{\rm c,d}^2(\lambda)}{12} &= \frac{\pi T_{\rm in,d}^2}{12} + \frac{3\lambda^2}{2}\left[\frac{\pi T_{\rm in,u}^2}{12}\frac{1}{\pi\tau_c T_{\rm in,u}} - \frac{\pi T_{\rm in,d}^2}{12}\frac{1}{\pi\tau_c T_{\rm in,d}}\right].\end{aligned} \tag{S-17}$$

This result is provided in Eq. (14) of main text.

C: THERMAL DRAG CURRENT

In this section we provide the calculation of outgoing heat current in down passive circuit in case of $\lambda \ll 1$ and $T_{\rm in,d} = 0$ and $\tau_c T_{\rm in,u} \ll 1$. Using the definition of heat current Eq. (9) in main text one has

$$\begin{aligned}J_{\rm d} &= \frac{1}{2}\int\frac{d\omega}{2\pi}\left[|\mathcal{B}(\omega)|^2(S_{\rm in,u}(\omega) - S_0(\omega)) + |\mathcal{A}(\omega)|^2(S_{\rm in,d}(\omega) - S_0(\omega))\right]\\ &+ \frac{1}{2}\int\frac{d\omega}{2\pi}\left[|\mathcal{B}(\omega)|^2(S_{\rm c,u}(\omega) - S_0(\omega)) + |\mathcal{C}(\omega)|^2(S_{\rm c,d}(\omega) - S_0(\omega))\right],\end{aligned} \tag{S-18}$$

where $S_0(\omega) = \omega\theta(\omega)$ ground state contribution, $S_{\rm in,u}(\omega) = \omega/(1 - e^{-\omega/T_{\rm in,u}})$, $S_{\rm in,d} = S_0(\omega)$, $S_{\rm c,u}(\omega) = \omega/(1 - e^{-\omega/f_{\rm c,u}(\lambda)T_{\rm in,u}})$, $S_{\rm c,d}(\omega) = \omega/(1 - e^{-\omega/f_{\rm c,d}(\lambda)T_{\rm in,u}})$ and according to Eq. (13) from main text $f_{\rm c,u}(\lambda) = \sqrt[4]{1+3\lambda^2}$, $f_{\rm c,d}(\lambda) = \sqrt[4]{2\lambda^2}$. Introducing the dimensionless integration variable $y = \omega/T_{\rm in,u}$, we write the heat current as a sum of four terms

$$J = J_1 + J_2 + J_3 + J_4, \tag{S-19}$$

where in the leading order with respect to $\tau_c T_{\rm in,u} \ll 1$ they read

$$\begin{aligned}J_1 &= \frac{T_{\rm in,u}^2}{2}\int\frac{dy}{2\pi}\frac{(\lambda\tau_c T_{\rm in,u})^2 y^2}{\tau_c^4 T_{\rm in,u}^4 y^4 + 2\tau_c^2 T_{\rm in,u}^2 y^2(1+\lambda^2) + (1-\lambda^2)^2}\left[\frac{y}{1-e^{-y}} - y\theta(y)\right]\\ &\simeq \frac{T_{\rm in,u}^2}{2}(\lambda\tau_c T_{\rm in,u})^2\int\frac{dy}{2\pi}y^3\left[\frac{1}{1-e^{-y}} - \theta(y)\right],\end{aligned} \tag{S-20}$$

$$J_2 = 0, \tag{S-21}$$

$$\begin{aligned}J_3 &= \frac{T_{\rm in,u}^2}{2}\int\frac{dy}{2\pi}\frac{(\lambda\tau_c T_{\rm in,u})^2 y^2}{\tau_c^4 T_{in,u}^4 y^4 + 2\tau_c^2 T_{\rm in,u}^2 y^2(1+\lambda^2) + (1-\lambda^2)^2}\left[\frac{y}{1-e^{-y/f_{\rm c,u}(\lambda)}} - y\theta(y)\right]\\ &= (f_{\rm c,u}(\lambda)\to 1) \simeq \frac{T_{\rm in,u}^2}{2}(\lambda\tau_c T_{\rm in,u})^2\int\frac{dy}{2\pi}y^3\left[\frac{1}{1-e^{-y}} - \theta(y)\right],\end{aligned} \tag{S-22}$$

$$\begin{aligned}J_4 &= \frac{T_{\rm in,u}^2}{2}\int\frac{dy}{2\pi}\frac{\left[1 + (\tau_c T_{\rm in,u} y)^2\right](\tau_c T_{\rm in,u} y)^2}{\tau_c^4 T_{\rm in,u}^4 y^4 + 2\tau_c^2 T_{\rm in,u}^2 y^2(1+\lambda^2) + (1-\lambda^2)^2}\left[\frac{y}{1-e^{-y/f_{\rm c,d}(\lambda)}} - y\theta(y)\right]\\ &\simeq \frac{T_{\rm in,u}^2}{2}(\tau_c T_{\rm in,u})^2\left[f_{\rm c,d}(\lambda)\right]^4\int\frac{dy}{2\pi}y^3\left[\frac{1}{1-e^{-y}} - \theta(y)\right] = ([f_{\rm c,d}(\lambda)]^4 \simeq 2\lambda^2)\\ &\simeq T_{\rm in,u}^2(\lambda\tau_c T_{\rm in,u})^2\int\frac{dy}{2\pi}y^3\left[\frac{1}{1-e^{-y}} - \theta(y)\right].\end{aligned} \tag{S-23}$$

Thus the sum of four terms gives the result presented in Eq. (16) of the main text

$$J_{\rm d} = \frac{T_{\rm in,u}^2}{\pi}(\lambda\tau_c T_{\rm in,u})^2\underbrace{\int dy y^3\left[\frac{1}{1-e^{-y}} - \theta(y)\right]}_{2\pi^4/15} = \frac{8\lambda^2(\pi\tau_c T_{\rm in,u})^2}{5}J_{\rm Q}, \tag{S-24}$$

where $J_{\rm Q} = \pi T_{\rm in,u}^2/12$.

D: NOISE OF THERMAL CURRENT FOR BALLISTIC CHANNEL

Using the definition (9) of the heat flux in the main text, we get the following expression for the noise power of thermal current

$$\mathcal{S}_{\mathrm{Q}}(\omega)=(R_{\mathrm{q}}/2)^2\int dte^{i\omega t}\left[\langle j^2(t)j^2(0)\rangle-\langle j^2(t)\rangle\langle j^2(0)\rangle\right]. \tag{S-25}$$

According to Wick's theorem $\langle j^2(t)j^2(0)\rangle=\langle j^2(t)\rangle\langle j^2(0)\rangle+2\langle j(t)j(0)\rangle\langle j(t)j(0)\rangle$, since each current operator is the combination of bosonic creation and annihilation operators. Subtracting the second term we get

$$\mathcal{S}_{\mathrm{Q}}(\omega)=(R_{\mathrm{q}}^2/2)\int dte^{i\omega t}\langle j(t)j(0)\rangle^2. \tag{S-26}$$

Next, using the FDT in Eq. (8) of the main text, we calculate the above expression explicitly

$$\begin{aligned}\mathcal{S}_{\mathrm{Q}}(\omega)&=(R_{\mathrm{q}}^2/2)\prod_{i=1}^{4}\int\frac{d\omega_i}{(2\pi)^4}\int dte^{i(\omega-\omega_1-\omega_3)t}\langle j(\omega_1)j(\omega_2)\rangle\langle j(\omega_3)j(\omega_4)\rangle=\frac{1}{2}\int\frac{d\omega_1}{2\pi}S(\omega-\omega_1)\cdot S(\omega_1)\\&=\frac{1}{2}\int\frac{d\omega_1}{2\pi}\frac{\omega-\omega_1}{1-e^{-(\omega-\omega_1)/T}}\cdot\frac{\omega_1}{1-e^{-\omega_1/T}}=\frac{T^3}{2}\int\frac{dy}{2\pi}\frac{\frac{\omega}{T}-y}{1-e^{-\frac{\omega}{T}+y}}\cdot\frac{y}{1-e^{-y}}.\end{aligned} \tag{S-27}$$

By introducing the dimensionless variable $x=y-\omega/2T$ and performing the integral with respect to x we obtain

$$\mathcal{S}_{\mathrm{Q}}(\omega)=\frac{T^3}{4\pi}\int dx\frac{\frac{\omega}{2T}-x}{1-e^{x-\frac{\omega}{2T}}}\cdot\frac{x+\frac{\omega}{2T}}{1-e^{-\left(x+\frac{\omega}{2T}\right)}}=\frac{\omega}{48\pi}[(2\pi T)^2+\omega^2][1+\coth(\omega/2T)]. \tag{S-28}$$

It is worth mentioning that the non-symmetrized noise is given by $\mathcal{S}_{\mathrm{Q}}(\omega)/2+\mathcal{S}_{\mathrm{Q}}(-\omega)/2$. Because of the quadratic term ω^2 in square brackets the finite frequency noise at zero temperature $T=0$ does not vanish: $\mathcal{S}_{\mathrm{Q}}(\omega)=\omega^3\mathrm{sgn}(\omega)/24\pi$. Finally, taking the limit $\omega\to 0$ in Eq. (S-28) we arrive to Eq. (18) from the main text.

E: NOISE OF THERMAL DRAG CURRENT

Similar calculations as in Sec. B can be done for the noise of heat current in the case of $\lambda\to 1$ and $\lambda\ll 1$ at $T_{\mathrm{in,d}}=0$ and $\tau_{\mathrm{c}}T_{\mathrm{in,u}}\ll 1$. Let demonstrate this for strong and weak coupling parameter λ. In both cases the result consists of 16 terms according to Eq. (19) and due to the fact that the incoming vector $\mathbf{J}_{\mathrm{in}}(\omega)$ in Eq. (6) of main text consists of 4 elements.

In the case of $\lambda\to 1$, the spectral functions are given by $S_{\mathrm{in,u}}(\omega)=\omega/(1-e^{-\omega/T_{\mathrm{in,u}}})$, $S_{\mathrm{in,d}}(\omega)=\omega\theta(\omega)$ and $S_{\mathrm{c,u}}(\omega)=S_{\mathrm{c,d}}(\omega)=\omega/(1-e^{-\sqrt{2}\omega/T_{\mathrm{in,u}}})$, since $T_{\mathrm{c,d}}=T_{\mathrm{c,u}}=T_{\mathrm{in,u}}/\sqrt{2}$. Now we calculate 16 terms at $\tau_{\mathrm{c}}T_{\mathrm{in,u}}\ll 1$ regime. They are given by

$$11)\quad\frac{1}{2}\int\frac{d\omega}{2\pi}\left[-\omega\theta(-\omega)\right]\omega\theta(\omega)\frac{1}{\left[4+(\omega\tau_{\mathrm{c}})^2\right]^2}=0, \tag{S-29}$$

$$\begin{aligned}&12)\quad\frac{1}{2}\int\frac{d\omega}{2\pi}\left[-\omega\theta(-\omega)\right]\frac{\omega}{1-e^{-\omega/T_{\mathrm{in,u}}}}\frac{1}{\left[4+(\omega\tau_{\mathrm{c}})^2\right]^2}=(y=\omega/T_{\mathrm{in,u}})=\\&\frac{T_{\mathrm{in,u}}^3}{2}\int\frac{dy}{2\pi}\frac{-y^2\theta(-y)}{1-e^{-y}}\underbrace{\frac{1}{[4+(\tau_{\mathrm{c}}T_{\mathrm{in,u}})^2\cdot y^2]^2}}_{\simeq 1/4^2}\simeq\frac{T_{\mathrm{in,u}}^3}{2}\frac{1}{2\pi}\frac{1}{4^2}\int dy\frac{y^2\theta(y)}{e^y-1},\end{aligned} \tag{S-30}$$

$$\begin{aligned}&13)\quad\frac{1}{2}\int\frac{d\omega}{2\pi}\left[-\omega\theta(-\omega)\right]\frac{\omega}{1-e^{-\omega/T_{\mathrm{c,d}}}}\frac{1}{4+(\omega\tau_{\mathrm{c}})^2}\left[1-\frac{3}{4+(\omega\tau_{\mathrm{c}})^2}\right]=(y=\omega/T_{\mathrm{in,u}},\quad T_{\mathrm{c,d}}=T_{\mathrm{in,u}}/\sqrt{2})=\\&\frac{T_{\mathrm{in,u}}^3}{2}\int\frac{dy}{2\pi}\frac{-y^2\theta(-y)}{1-e^{-\sqrt{2}y}}\underbrace{\frac{1}{4+(\tau_{\mathrm{c}}T_{\mathrm{in,u}})^2\cdot y^2}\left[1-\frac{3}{4+(\tau_{\mathrm{c}}T_{\mathrm{in,u}})^2\cdot y^2}\right]}_{\simeq 1/4^2}\simeq\frac{T_{\mathrm{in,u}}^3}{2}\frac{1}{2\pi}\frac{1}{4^2}\int dy\frac{y^2\theta(y)}{e^{\sqrt{2}y}-1},\end{aligned} \tag{S-31}$$

$$14)\quad \text{the result is the same as in case of 13), namely}\quad \simeq \frac{T_{\rm in,u}^3}{2}\frac{1}{2\pi}\frac{1}{4^2}\int dy\frac{y^2\theta(y)}{e^{\sqrt{2}y}-1}, \tag{S-32}$$

$$21) = 12), \tag{S-33}$$

$$22)\quad \frac{1}{2}\int\frac{d\omega}{2\pi}\frac{\omega}{e^{\omega/T_{\rm in,u}}-1}\frac{\omega}{1-e^{-\omega/T_{\rm in,u}}}\frac{1}{[4+(\omega\tau_{\rm c})^2]^2} = (y=\omega/T_{\rm in,u}) =$$
$$\frac{T_{\rm in,u}^3}{2}\int\frac{dy}{2\pi}\frac{y^2}{(e^y-1)(1-e^{-y})}\underbrace{\frac{1}{[4+(\tau_{\rm c}T_{\rm in,u})^2\cdot y^2]^2}}_{\simeq 1/4^2} \simeq \frac{T_{\rm in,u}^3}{2}\frac{1}{2\pi}\frac{1}{4^2}\int dy\frac{y^2}{(e^y-1)(1-e^{-y})}, \tag{S-34}$$

$$23)\quad \frac{1}{2}\int\frac{d\omega}{2\pi}\frac{\omega}{e^{\omega/T_{\rm in,u}}-1}\frac{\omega}{1-e^{-\sqrt{2}\omega/T_{\rm in,u}}}\frac{1}{4+(\omega\tau_{\rm c})^2}\left[1-\frac{3}{4+(\omega\tau_{\rm c})^2}\right] = (y=\omega/T_{\rm in,u}) =$$
$$\frac{T_{\rm in,u}^3}{2}\int\frac{dy}{2\pi}\frac{y^2}{(e^y-1)(1-e^{-\sqrt{2}y})}\underbrace{\frac{1}{4+(\tau_{\rm c}T_{\rm in,u})^2\cdot y^2}\left[1-\frac{3}{4+(\tau_{\rm c}T_{\rm in,u})^2y^2}\right]}_{\simeq 1/4^2} \simeq \frac{T_{\rm in,u}^3}{2}\frac{1}{2\pi}\frac{1}{4^2}\int dy\frac{y^2}{(e^y-1)(1-e^{-\sqrt{2}y})}, \tag{S-35}$$

$$24)\quad \text{the result is the same as in case of 23), namely}\quad \simeq \frac{T_{in,u}^3}{2}\frac{1}{2\pi}\frac{1}{4^2}\int dy\frac{y^2}{(e^y-1)(1-e^{-\sqrt{2}y})}, \tag{S-36}$$

$$31) = 13), \tag{S-37}$$

$$32) = 23), \tag{S-38}$$

$$33)\quad \frac{1}{2}\int\frac{d\omega}{2\pi}\frac{\omega}{e^{\sqrt{2}\omega/T_{\rm in,u}}-1}\frac{\omega}{1-e^{-\sqrt{2}\omega/T_{\rm in,u}}}\left[1-\frac{3}{4+(\omega\tau_{\rm c})^2}\right]^2 = (y=\omega/T_{\rm in,u}) =$$
$$\frac{T_{\rm in,u}^3}{2}\frac{1}{2\pi}\int dy\frac{y^2}{(e^{\sqrt{2}y}-1)(1-e^{-\sqrt{2}y})}\underbrace{\left[1-\frac{3}{4+(\tau_{\rm c}T_{\rm in,u})^2\cdot y^2}\right]^2}_{\simeq 1/4^2} \simeq \frac{T_{\rm in,u}^3}{2}\frac{1}{2\pi}\frac{1}{4^2}\int dy\frac{y^2}{(e^{\sqrt{2}y}-1)(1-e^{-\sqrt{2}y})}, \tag{S-39}$$

$$34) = 33), \tag{S-40}$$

$$41) = 14), \tag{S-41}$$

$$42) = 24), \tag{S-42}$$

$$43) = 34), \tag{S-43}$$

$$44) = 33). \tag{S-44}$$

Summing up all above 16 terms we get the result for zero frequency noise given in Eq. (20) of mian text

$$\mathcal{S}_{\rm out,d}(0) = (3\mathcal{I}/2\pi^2)\mathcal{S}_{\rm in,u}(0), \tag{S-45}$$

where $\mathcal{S}_{\rm in,u}(0) = S_Q$ and the dimensionless prefactor is given by

$$\mathcal{I} = \int \frac{dy}{16} P(y)P(-y) \approx 2.5782, \tag{S-46}$$

where $P(y) = y/(1-e^{-y}) + y\theta(y) + 2y/(1-e^{-\sqrt{2}y})$ and $\theta(y)$ is the Heaviside step function. To remind, $\sqrt{2}$ in the integrand of $\mathcal{I}$ originates from Ohmic contact temperatures $T_{\rm c,u} = T_{\rm c,d} = T_{\rm in,u}/\sqrt{2}$.

In the case of $\lambda \ll 1$, the spectral functions are given by $S_{\rm in,u}(\omega) = \omega/(1-e^{-\omega/T_{\rm in,u}})$, $S_{\rm in,d}(\omega) = \omega\theta(\omega)$ and $S_{\rm c,d}(\omega) = \omega/(1-e^{-\omega/f_{\rm c,d}(\lambda)T_{\rm in,u}})$ and $S_{\rm c,u}(\omega) = \omega/(1-e^{-\omega/f_{\rm c,u}(\lambda)T_{\rm in,u}})$. Here the pre-factors are obtained from Eq. (13) of main text at $T_{\rm in,d} = 0$

$$f_{\rm c,d}(\lambda) = \sqrt[4]{2\lambda^2}, \quad f_{\rm c,u}(\lambda) = \sqrt[4]{1-2\lambda^2}. \tag{S-47}$$

Introducing the dimensionless variable of integration $y = \omega/T_{\rm in,u}$ in Eq. (19) of main text we calculate all 16 terms step by step and keep only the leading terms with respect to $\lambda \ll 1$ and $\tau_{\rm c}T_{\rm in,u} \ll 1$. Namely,

$$11) \quad \frac{1}{2}\int \frac{d\omega}{2\pi}[-\omega\theta(-\omega)]\omega\theta(\omega)[...] = 0, \tag{S-48}$$

$$12) \quad \frac{T^3_{\rm in,u}}{2}\frac{1}{2\pi}\int \frac{dy y^2\theta(y)}{e^y-1}\frac{(\lambda\tau_{\rm c}T_{\rm in,u}y)^2\left[(\tau_{\rm c}T_{\rm in,u}y)^2+(1-\lambda^2)^2\right]}{\left[(\tau_{\rm c}T_{\rm in,u}y)^4+2(1+\lambda^2)(\tau_{\rm c}T_{\rm in,u}y)^2+(1-\lambda^2)^2\right]^2} \simeq \frac{T^3_{\rm in,u}}{2}\frac{1}{2\pi}(\lambda\tau_{\rm c}T_{\rm in,u})^2\int\frac{dy y^4\theta(y)}{e^y-1}, \tag{S-49}$$

$$\begin{aligned}13) \quad &\frac{T^3_{\rm in,u}}{2}\frac{1}{2\pi}\int \frac{dy y^2\theta(y)}{e^{y/f_{\rm c,d}(\lambda)}-1}\frac{(\tau_{\rm c}T_{\rm in,u}y)^2\left[1+(\tau_{\rm c}T_{\rm in,u}y)^2\right]\left[(\tau_{\rm c}T_{\rm in,u}y)^2+(1-\lambda^2)^2\right]}{\left[(\tau_{\rm c}T_{\rm in,u}y)^4+2(1+\lambda^2)(\tau_{\rm c}T_{\rm in,u}y)^2+(1-\lambda^2)^2\right]^2}\\ &\simeq \frac{T^3_{\rm in,u}}{2}\frac{1}{2\pi}(\tau_{\rm c}T_{\rm in,u})^2\left[f_{\rm c,d}(\lambda)\right]^5\int\frac{dy y^4\theta(y)}{e^y-1} = (\left[f_{\rm c,d}(\lambda)\right]^5 \simeq 2\sqrt[4]{2}\lambda^2\sqrt{\lambda}) \simeq \frac{T^3_{\rm in,u}}{2}\frac{1}{2\pi}(\tau_{\rm c}T_{\rm in,u})^2 2\sqrt[4]{2}\lambda^2\sqrt{\lambda}\int\frac{dy y^4\theta(y)}{e^y-1}\\ &\propto \lambda^2\sqrt{\lambda}, \quad \text{(this is sub-leading correction, thus we ignore this term),}\end{aligned} \tag{S-50}$$

$$\begin{aligned}14) \quad &\frac{T^3_{\rm in,u}}{2}\frac{1}{2\pi}\int \frac{dy y^2\theta(y)}{e^{y/f_{\rm c,u}(\lambda)}-1}\frac{(4\lambda_{\rm c}T_{\rm in,u}y)^2\left[(\tau_{\rm c}T_{\rm in,u}y)^2+(1-\lambda^2)^2\right]}{\left[(\tau_{\rm c}T_{\rm in,u}y)^4+2(1+\lambda^2)(\tau_{\rm c}T_{\rm in,u}y)^2+(1-\lambda^2)^2\right]^2}\\ &\simeq \frac{T^3_{\rm in,u}}{2}\frac{1}{2\pi}(\lambda\tau_{\rm c}T_{\rm in,u})^2\left[f_{\rm c,u}(\lambda)\right]^5\int\frac{dy y^4\theta(y)}{e^y-1} = (\left[f_{\rm c,u}(\lambda)\right]^5 \to 1) \simeq \frac{T^3_{\rm in,u}}{2}\frac{1}{2\pi}(\lambda\tau_{\rm c}T_{\rm in,u})^2\int\frac{dy y^4\theta(y)}{e^y-1},\end{aligned} \tag{S-51}$$

$$21) = 12), \tag{S-52}$$

$$\begin{aligned}22) \quad &\frac{T^3_{\rm in,u}}{2}\frac{1}{2\pi}\int \frac{dy y^2}{(e^y-1)(1-e^{-y})}\frac{(\lambda\tau_{\rm c}T_{\rm in,u}y)^4}{\left[(\tau_{\rm c}T_{\rm in,u}y)^4+2(1+\lambda^2)(\tau_{\rm c}T_{\rm in,u}y)^2+(1-\lambda^2)^2\right]^2}\\ &\simeq \frac{T^3_{\rm in,u}}{2}\frac{1}{2\pi}(\lambda\tau_{\rm c}T_{\rm in,u})^4\int\frac{dy y^6}{(e^y-1)(1-e^{-y})} \propto \lambda^4, \quad \text{(this is the sub-leading correction, thus we ignore this term),}\end{aligned} \tag{S-53}$$

$$23)\quad \frac{T_{\rm in,u}^3}{2}\frac{1}{2\pi}\int\frac{dyy^2}{(e^y-1)(1-e^{-y/f_{\rm c,d}(\lambda)})}\frac{(\lambda\tau_{\rm c}T_{\rm in,u}y)^2\left[1+(\tau_{\rm c}T_{\rm in,u}y)^2\right](\tau_{\rm c}T_{\rm in,u}y)^2}{[(\tau_{\rm c}T_{\rm in,u}y)^4+2(1+\lambda^2)(\tau_{\rm c}T_{\rm in,u}y)^2+(1-\lambda^2)^2]^2}$$
$$\simeq\frac{T_{\rm in,u}^3}{2}\frac{1}{2\pi}(\lambda\tau_{\rm c}T_{\rm in,u})^2(\tau_{\rm c}T_{\rm in,u})^2\int\frac{dyy^6}{(e^y-1)(1-e^{-y/f_{\rm c,d}(\lambda)})}=(1/(1-e^{-y/f_{\rm c,d}(\lambda)})\to\theta(y))$$
$$\simeq\frac{T_{\rm in,u}^3}{2}\frac{1}{2\pi}(\lambda\tau_{\rm c}T_{\rm in,u})^2(\tau_{\rm c}T_{\rm in,u})^2\int\frac{dyy^6\theta(y)}{(e^y-1)}$$
$$\propto\lambda^2(\tau_{\rm c}T_{\rm in,u})^4,\quad\text{(this is the sub-leading correction, thus we ignore this term)},\tag{S-54}$$

$$24)\quad \frac{T_{\rm in,u}^3}{2}\frac{1}{2\pi}\int\frac{dyy^2}{(e^y-1)(1-e^{-y/f_{\rm c,u}(\lambda)})}\frac{(\lambda\tau_{\rm c}T_{\rm in,u}y)^4}{[(\tau_{\rm c}T_{\rm in,u}y)^4+2(1+\lambda^2)(\tau_{\rm c}T_{\rm in,u}y)^2+(1-\lambda^2)^2]^2}$$
$$\simeq\frac{T_{\rm in,u}^3}{2}\frac{1}{2\pi}(\lambda\tau_{\rm c}T_{\rm in,u})^4\int\frac{dyy^6}{(e^y-1)(1-e^{-y/f_{\rm c,u}(\lambda)})}=(1/(1-e^{-y/f_{\rm c,u}(\lambda)})\to1/(1-e^{-y}))$$
$$\simeq\frac{T_{\rm in,u}^3}{2}\frac{1}{2\pi}(\lambda\tau_{\rm c}T_{\rm in,u})^4\int\frac{dyy^6}{(e^y-1)(1-e^{-y})}\propto\lambda^4,\quad\text{(this is the sub-leading correction, thus we ignore this term)},\tag{S-55}$$

$$31)=13),\quad\text{(this is the sub-leading correction, thus we ignore this term)},\tag{S-56}$$

$$32)=23)\quad\text{(this is the sub-leading correction, thus we ignore this term)},\tag{S-57}$$

$$33)\quad \frac{T_{\rm in,u}^3}{2}\frac{1}{2\pi}\int\frac{dyy^2}{(e^{y/f_{\rm c,d}(\lambda)}-1)(1-e^{-y/f_{\rm c,d}(\lambda)})}\frac{(\tau_{\rm c}T_{\rm in,u}y)^4[1+(\tau_{\rm c}T_{\rm in,u}y)^2]^2}{[(\tau_{\rm c}T_{\rm in,u}y)^4+2(1+\lambda^2)(\tau_{\rm c}T_{\rm in,u}y)^2+(1-\lambda^2)^2]^2}$$
$$\simeq\frac{T_{\rm in,u}^3}{2}\frac{1}{2\pi}(\tau_{\rm c}T_{\rm in,u})^4[f_{\rm c,d}(\lambda)]^7\int\frac{dyy^6}{(e^y-1)(1-e^{-y})}=([f_{\rm c,d}(\lambda)]^7\simeq2\sqrt[4]{8}\lambda^3\sqrt{\lambda})$$
$$\simeq\frac{T_{\rm in,u}^3}{2}\frac{1}{2\pi}(\tau_{\rm c}T_{\rm in,u})^4 2\sqrt[4]{8}\lambda^3\sqrt{\lambda}\int\frac{dyy^6}{(e^y-1)(1-e^{-y})}$$
$$\propto\lambda^3\sqrt{\lambda},\quad\text{(this is the sub-leading correction, thus we ignore this term)},\tag{S-58}$$

$$34)\quad \frac{T_{\rm in,u}^3}{2}\frac{1}{2\pi}\int\frac{dyy^2}{(e^{y/f_{\rm c,d}(\lambda)}-1)(1-e^{-y/f_{\rm c,u}(\lambda)})}\frac{(\lambda\tau_{\rm c}T_{\rm in,u}y)^2[1+(\tau_{\rm c}T_{\rm in,u}y)^2](\tau_{\rm c}T_{\rm in,u}y)^2}{[(\tau_{\rm c}T_{\rm in,u}y)^4+2(1+\lambda^2)(\tau_{\rm c}T_{\rm in,u}y)^2+(1-\lambda^2)^2]^2}$$
$$\simeq\frac{T_{\rm in,u}^3}{2}\frac{1}{2\pi}(\lambda\tau_{\rm c}T_{\rm in,u})^2(\tau_{\rm c}T_{\rm in,u})^2\int\frac{dyy^6\theta(y)}{e^y-1}$$
$$\propto\lambda^2(\tau_{\rm c}T_{\rm in,u})^4,\quad\text{(this is the sub-leading correction, thus we ignore this term)},\tag{S-59}$$

$$41)=14),\tag{S-60}$$

$$42)=24),\quad\text{this is the sub-leading correction, thus we ignore this term},\tag{S-61}$$

$$43)=34),\quad\text{this is the sub-leading correction, thus we ignore this term},\tag{S-62}$$

$$
\begin{aligned}
44)\quad & \frac{T_{\rm in,u}^3}{2}\frac{1}{2\pi}\int \frac{dyy^2}{(e^{y/f_{\rm c,u}(\lambda)}-1)(1-e^{-y/f_{\rm c,u}(\lambda)})}\frac{(\lambda\tau_{\rm c}T_{\rm in,u}y)^4}{\left[(\tau_{\rm c}T_{\rm in,u}y)^4+2(1+\lambda^2)(\tau_{\rm c}T_{\rm in,u}y)^2+(1-\lambda^2)^2\right]^2} \\
&= (f_{\rm c,u}(\lambda)\simeq 1) \simeq \frac{T_{\rm in,u}^3}{2}\frac{1}{2\pi}(\lambda\tau_{\rm c}T_{\rm in,u})^4\int\frac{dyy^6\theta(y)}{(e^y-1)(1-e^{-y})} \\
&\propto \lambda^4(\tau_{\rm c}T_{\rm in,u})^4, \quad \text{(this is the sub-leading correction, thus we ignore this term).}
\end{aligned}
\tag{S-63}
$$

Summing up all the leading 4 terms we get the following equation

$$
\mathcal{S}_{\rm out,d}(0) = 4\frac{T_{\rm in,u}^3}{2}\frac{1}{2\pi}(\lambda\tau_{\rm c}T_{\rm in,u})^2\underbrace{\int_0^\infty\frac{dyy^4}{e^y-1}}_{24\zeta(5)} = 4\frac{T_{\rm in,u}^3}{2}\frac{1}{2\pi}(\lambda\tau_{\rm c}T_{\rm in,u})^2 24\zeta(5). \tag{S-64}
$$

Now approximating $\zeta(5)\approx 1.0369$ and taking into account that $\mathcal{S}_{\rm in,u}(0)=\pi T^3/6$ we get Eq. (22) from main text, namely $\mathcal{S}_{\rm out,d}(0)=(3\lambda^2\mathcal{K}/2)(\tau_{\rm c}T_{\rm in,u})^2\mathcal{S}_{\rm in,u}(0)$.