Étale Coverings in Homotopical $\mathcal{D}$-Geometry

Alisa Govzmann, Damjan Pištalo, and Norbert Poncin

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Abstract

In recent years, the foundations of homotopical algebraic geometry over the ring $\mathcal{D}$ of differential operators have been extended, and the conjecture that this geometry is an appropriate framework for studying the solution space of a system of partial differential equations modulo symmetries has gained traction. In the present work we define and describe étale coverings in homotopical $\mathcal{D}$-geometry. In particular, the full characterization of finitely presented morphisms of $\mathcal{D}$-algebras turns out to be quite interesting. We are not aiming for a standard text that gives definitions, states results and proves them more or less rigorously, but all too often offers little insight for the uninformed reader. Instead, we opted for a smooth derivation of the used abstract definitions from more basic ones, thereby emphasizing the reasons for the various choices and facilitating a deeper understanding.

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Contents

1 Introduction

Building on ideas of Beilinson, Costello, Drinfeld, Gwilliam, Schreiber, Paugam, Toën, Vezzosi, and Vinogradov [3, 7, 30, 31, 37, 38, 42], Di Brino and two of the authors of this paper have introduced homotopical algebraic geometry over the ring $\mathcal{D}$ of differential operators of an underlying affine scheme, as a suitable framework for investigating the solution space of a system of partial differential equations up to symmetries [8, 9, 32]. The implementation of the associated research program requires in particular that the tuple

$$(\mathtt{DG}\mathcal{D}\mathtt{M}, \mathtt{DG}\mathcal{D}\mathtt{M}, \mathtt{DG}\mathcal{D}\mathtt{A}, \tau, \mathbf{P})$$

be a homotopical algebraic geometric context (HAGC) in the sense of [38], where $\mathtt{DG}\mathcal{D}\mathtt{M}$ is the symmetric monoidal model category of differential graded $\mathcal{D}$-modules, the subcategory $\mathtt{DG}\mathcal{D}\mathtt{A}$

is the model category of differential graded $\mathcal{D}$-algebras, τ is an appropriate model pre-topology on the opposite category of $\mathtt{DG\mathcal{D}A}$ and $\mathbf{P}$ is a compatible class of morphisms.

The (challenging) proof of this 'HAGC theorem' is based on a new simplified perspective on the concept of homotopy fiber sequence [34] and a generalization of the long exact sequence of Puppe, which are themselves based on the notion of model of a homotopy pullback, model square or homotopy fiber square in any model category [16], [15]. In contrast with Quillen's definition, this novel approach to model categorical homotopy fiber sequences [17] does not rely on the additional structure of an action and is much easier to apply [18].

The proof of the 'HAGC theorem' involves proving in our homotopical $\mathcal{D}$-geometric environment that flat morphisms are the same as strongly flat ones [18]. The latter not only requires a handy concept of homotopy fiber sequence, but in addition Quillen's Tor spectral sequence – which connects the graded Tor functors in homology with the homology of the derived tensor product of two differential graded $\mathcal{D}$-modules over a differential graded $\mathcal{D}$-algebra – to be valid in the derived $\mathcal{D}$-geometric world.

In the present work we define and explicitly describe étale coverings in homotopical $\mathcal{D}$-geometry. Especially the complete characterization of finitely presented morphisms of $\mathcal{D}$-algebras turns out to be quite interesting. We are not aiming for a standard text which gives definitions, states results and proves them more or less rigorously, but all too often offers little insight to the uninformed reader. Instead, we opted for a smooth derivation of the used abstract definitions from more basic ones, thereby emphasizing the reasons for the various choices and facilitating a deeper understanding. We hope that the text will therefore also be useful for doctoral students and researchers who want to acquire knowledge in homotopical algebraic geometry.

We are convinced that we can combine all the above results to complete the proof of the 'HAGC theorem', to show that solid concepts of derived stack and geometric derived stack do exist in homotopical $\mathcal{D}$-geometry, and thus to make an important step towards the full implementation of the mentioned 'Partial Differential Equations and Symmetries Program'.

Conventions and notations. We assume that the reader is familiar with model categories. We adopt the definition of a model category that is used in [21]. More precisely, a model category is a category $\mathtt{M}$ that is equipped with three classes of morphisms called weak equivalences, fibrations and cofibrations. The category $\mathtt{M}$ has all small limits and colimits and the 2-out-of-3 axiom, the retract axiom and the lifting axiom are satisfied. Moreover $\mathtt{M}$ comes equipped with a fixed functorial cofibration - trivial fibration factorization system (Cof - TrivFib factorization) and a fixed functorial trivial cofibration - fibration factorization system (TrivCof - Fib factorization). Finally, we use the Quillen homotopy category $\mathtt{Ho}(\mathtt{M})$ of $\mathtt{M}$ which is the 'on the nose' localization of $\mathtt{M}$ at its class W of weak equivalences. We denote the localization functor $\mathtt{M} \to \mathtt{Ho}(\mathtt{M})$ by $L_{\mathtt{M}}$.

2 Zooming in on spectra

2.1 Derived $\mathcal{D}$-schemes

Let $(X, \mathcal{O}_X)$ be a smooth scheme, denote by $\mathcal{D}_X$ the sheaf of rings of differential operators over X, and set $\mathcal{O} := \mathcal{O}_X(X)$ and $\mathcal{D} := \mathcal{D}_X(X)$.

We refer to the opposite of the category $\mathtt{DG}_+\mathtt{qcCAlg}(\mathcal{D}_X)$ (resp., the category $\mathtt{DG}\mathcal{D}\mathtt{A}$) of sheaves of differential non-negatively graded $\mathcal{O}_X$-quasi-coherent commutative $\mathcal{D}_X$-algebras (resp., differential non-negatively graded $\mathcal{D}$-algebras), as the category $\mathtt{D\_Aff}(\mathcal{D}_X)$ (resp., the category $\mathtt{DAff}(\mathcal{D})$) of **derived affine X-$\mathcal{D}_X$-schemes** (resp., of **derived affine $\mathcal{D}$-schemes**). If X is a smooth affine algebraic variety, there is an equivalence of categories $\mathtt{D\_Aff}(\mathcal{D}_X) \cong \mathtt{DAff}(\mathcal{D})$.

In the following we assume that X is a smooth scheme and investigate the categories $\mathtt{DG}\mathcal{D}\mathtt{A}$ and $\mathtt{DAff}(\mathcal{D})$. If an object $\mathcal{A} \in \mathtt{DG}\mathcal{D}\mathtt{A}$ is viewed as object in the opposite category, we denote it by $\operatorname{Spec}(\mathcal{A}) \in \mathtt{DAff}(\mathcal{D})$. Moreover, we will consider the category $\mathtt{DG}\mathcal{D}\mathtt{M}$ of differential (non-negatively) graded $\mathcal{D}$-modules and the category $\mathtt{Aff}(\mathcal{D}) := (\mathcal{D}\mathtt{A})^{\mathrm{op}}$ of **affine $\mathcal{D}$-schemes**.

Let us recall that a **differential graded $\mathcal{D}$-algebra** [8] is a differential graded-commutative unital $\mathcal{O}$-algebra, as well as a differential graded $\mathcal{D}$-module (a (non-negatively graded) chain complex in $\mathcal{D}$-modules), such that vector fields act as derivations. Further, the morphisms of $\mathtt{DG}\mathcal{D}\mathtt{A}$ are the morphisms of $\mathtt{DG}\mathcal{D}\mathtt{M}$ that respect the multiplications and units. The category $\mathtt{Aff}(\mathcal{D})$ is a full subcategory of $\mathtt{DAff}(\mathcal{D})$.

By definition of (derived) affine $\mathcal{D}$-schemes, a morphism $f : \operatorname{Spec}(\mathcal{A}) \to \operatorname{Spec}(\mathcal{B})$ in $(\mathtt{D})\mathtt{Aff}(\mathcal{D})$ is exactly a morphism $\tilde{f} : \mathcal{B} \to \mathcal{A}$ in $(\mathtt{DG})\mathcal{D}\mathtt{A}$. We omit the symbol 'tilde', whenever no confusion is possible.

We mention some useful properties of (derived) affine $\mathcal{D}$-schemes.

Since a coproduct is a product in the opposite category, a coproduct $\coprod_\alpha \operatorname{Spec}(\mathcal{A}_\alpha)$ in

$(\mathtt{D})\mathtt{Aff}(\mathcal{D})$ is a product $\prod_\alpha \mathcal{A}_\alpha$ in $(\mathtt{DG})\mathcal{D}\mathtt{A}$, i.e.,

$$\coprod_\alpha \operatorname{Spec}(\mathcal{A}_\alpha) = \operatorname{Spec}(\prod_\alpha \mathcal{A}_\alpha) \ ,$$

and a coproduct of maps

$$\coprod_\alpha f_\alpha : \coprod_\alpha \operatorname{Spec}(\mathcal{A}_\alpha) \to \operatorname{Spec}(\mathcal{A})$$

in $(\mathtt{D})\mathtt{Aff}(\mathcal{D})$ is a product of maps

$$\langle f_\alpha \rangle_\alpha : \mathcal{A} \to \prod_\alpha \mathcal{A}_\alpha$$

in $(\mathtt{DG})\mathcal{D}\mathtt{A}$. Since $\mathtt{DG}\mathcal{D}\mathtt{A}$ is a model category, it has all small limits and colimits, and so does $(\mathtt{D})\mathtt{Aff}(\mathcal{D})$.

Note further that a standard affine scheme $\operatorname{Spec}(R) \in \mathtt{Aff}$ is a locally ringed space $(|\operatorname{Spec}(R)|, \mathcal{O}_{|\operatorname{Spec(R)}|})$, where $|\operatorname{Spec}(R)|$ is the prime spectrum of a commutative ring $R \in \mathtt{CR}$ and where $\mathcal{O}_{|\operatorname{Spec}(R)|}$ is a specific structure sheaf over $|\operatorname{Spec}(R)|$ constructed from R. Usually $\operatorname{Spec}(R)$ denotes both, the locally ringed space and the underlying topological space. There is an equivalence of categories $\mathtt{Aff} \cong \mathtt{CR}^{\mathrm{op}}$.

In our above setting, we do not define $\operatorname{Spec}(\mathcal{A}) \in (\mathtt{D})\mathtt{Aff}(\mathcal{D})$ as a space, but we treat $(\mathtt{D})\mathtt{Aff}(\mathcal{D})$ as the category $(\mathtt{DG})\mathcal{D}\mathtt{A}^{\mathrm{op}}$.

2.2 Derived spectrum functor

Let $\mathtt{M}$ be a model category, let $c^\bullet : \mathtt{M} \to \mathtt{Fun}(\Delta, \mathtt{M})$ be the constant cosimplicial object functor from $\mathtt{M}$ to the category $\mathtt{Fun}(\Delta, \mathtt{M})$ of $\mathtt{M}$-valued functors out of the simplicial category Δ, and let $Q^\bullet$ be the cofibrant replacement functor of the Reedy model structure on $\mathtt{Fun}(\Delta, \mathtt{M})$ (recall that we systematically fix functorial factorizations that provide a cofibration - trivial fibration and a trivial cofibration - fibration decomposition). This resolution functor comes with a natural transformation $\iota : Q^\bullet \to \mathrm{id}$ that is objectwise a trivial fibration [21]. The wiskering $\iota\, c^\bullet$ is a natural transformation from $Q^\bullet \circ c^\bullet$ to $c^\bullet$, which is defined at $m \in \mathtt{M}$ by $(\iota\, c^\bullet)_m = \iota_{c^\bullet(m)}$. This implies that $Q^\bullet \circ c^\bullet$ together with $\iota\, c^\bullet$ is a cofibrant resolution functor (Γ, i) in the sense of [37, Paragraph 4.2].

Following the same reference, we thus define the functor

$$\underline{h} : \mathtt{M} \ni m \mapsto \underline{h}_m := \underline{\operatorname{Hom}}(-, m) := \operatorname{Hom}(Q^\bullet(c^\bullet(-)), m) \in \mathtt{Fun}(\mathtt{M}^{\mathrm{op}}, \mathtt{SSet})\check{\ }, \tag{1}$$

where superscript $\vee$ means that the model structure on the target category is the left Bousfield localization with respect to the weak equivalences in $\mathtt{M}$ of the objectwise model structure induced by the model structure of simplicial sets. Since $\underline{h}$ preserves fibrant objects and weak equivalences between them, it can be right derived. Writing $\mathtt{M}^\vee$ instead of $\mathtt{Fun}(\mathtt{M}^{\mathrm{op}}, \mathtt{SSet})^\vee$, we thus get the functor

$$\mathbb{R}\underline{h} : \mathtt{Ho}(\mathtt{M}) \ni m \mapsto \underline{h}_{R(m)} = \underline{\mathrm{Hom}}(-, R(m)) = \mathrm{Hom}(Q^\bullet(c^\bullet(-)), R(m)) \in \mathtt{Ho}(\mathtt{M}^\vee) ,$$

where R is the fibrant replacement functor in $\mathtt{M}$.

Since the standard Yoneda functor, viewed as functor

$$h : \mathtt{M} \ni m \mapsto h_m := \mathrm{Hom}(-, m) \in \mathtt{Fun}(\mathtt{M}^{\mathrm{op}}, \mathtt{SSet})^\vee,$$

preserves weak equivalences by definition of the model structure $\vee$, it induces a functor

$$\mathtt{Ho}(h) : \mathtt{Ho}(\mathtt{M}) \to \mathtt{Ho}(\mathtt{M}^\vee) ,$$

which is the factorization of $L_{\mathtt{M}^\vee} \circ h : \mathtt{M} \to \mathtt{Ho}(\mathtt{M}^\vee)$. Hence

$$\mathtt{Ho}(h) \circ L_{\mathtt{M}} = L_{\mathtt{M}^\vee} \circ h ,$$

so that

$$\mathtt{Ho}(h) : \mathtt{Ho}(\mathtt{M}) \ni m \mapsto h_m = \mathrm{Hom}(-, m) \in \mathtt{Ho}(\mathtt{M}^\vee) .$$

The two functors $\mathbb{R}\underline{h}$ and $\mathtt{Ho}(h)$ are known to be isomorphic [37, Lemma 4.2.2] (i.e., for every $m \in \mathtt{M}$ their values at m are isomorphic) and fully faithful [37, Theorem 4.2.3].

The simplicial Hom functor $\underline{h}$ has a dual variant, the simplicial spectrum functor $\underline{\mathrm{Spec}}$, which is used when $\mathtt{M}^{\mathrm{op}}$ is a category of commutative algebras and $\mathtt{M}$ the corresponding category of affine schemes. If we denote by $c_\bullet : \mathtt{M}^{\mathrm{op}} \to \mathtt{Fun}(\Delta^{\mathrm{op}}, \mathtt{M}^{\mathrm{op}})$ the constant simplicial object functor over $\mathtt{M}^{\mathrm{op}}$, this spectrum functor is the functor

$$\underline{\mathrm{Spec}} : (\mathtt{M}^{\mathrm{op}})^{\mathrm{op}} \ni a \mapsto \underline{\mathrm{Hom}}(a, -) := \mathrm{Hom}(a, R_\bullet(c_\bullet(-))) \in \mathtt{Fun}(\mathtt{M}^{\mathrm{op}}, \mathtt{SSet})^\vee,$$

where $R_\bullet$ is the fibrant replacement functor of the Reedy model structure on $\mathtt{Fun}(\Delta^{\mathrm{op}}, \mathtt{M}^{\mathrm{op}})$ (see (1)). Since $\underline{\mathrm{Spec}}$ preserves cofibrant objects and weak equivalences between them, it can be right derived:

$$\mathbb{R}\underline{\mathrm{Spec}} : (\mathtt{Ho}(\mathtt{M}^{\mathrm{op}}))^{\mathrm{op}} \ni a \mapsto \underline{\mathrm{Hom}}(Q(a), -) = \mathrm{Hom}(Q(a), R_\bullet(c_\bullet(-))) \in \mathtt{Ho}(\mathtt{M}^\vee) ,$$

where Q is the cofibrant replacement functor in $\mathtt{M}^{\mathrm{op}}$. If $m = \mathrm{Spec}(a)$, the value of the **derived spectrum functor** $\mathbb{R}\underline{\mathrm{Spec}}$ at a is isomorphic to the values of $\mathbb{R}\underline{h}$ and $\mathtt{Ho}(h)$ at m [40, Section 5.1].

The **mapping space** of morphisms between two objects m and m' of a model category $\mathtt{M}$ is denoted by $\mathrm{Map}_{\mathtt{M}}(m, m')$ or just $\mathrm{Map}(m, m')$ [21, Chapter 5]. This simplicial set depends on the choice of cofibrant and fibrant resolution functors (if they have not been fixed), but is well defined as an object in $\mathtt{Ho}(\mathtt{SSet})$ [38]. For any $a, a' \in \mathtt{M}^{\mathrm{op}}$, there is an equivalence of simplicial sets

$$\mathrm{Map}_{\mathtt{M}^{\mathrm{op}}}(a, a') \simeq \mathbb{R}\underline{\mathrm{Spec}}(a)(a') = \underline{\mathrm{Hom}}(Q(a), a') = \mathrm{Hom}(Q(a), R_\bullet(c_\bullet(a'))) \tag{2}$$

[41].

3 Definitions of étale coverings in $\mathcal{D}$-geometry

In this section - as in the rest of the work - our aim is not only to give definitions but also to explain them. Indeed, there are often several possible generalizations of standard algebraic concepts to a homotopical algebraic context (see below). All can be of interest for specific problems.

3.1 Étale morphisms in a homotopical algebraic context

Remark 1. A **homotopical algebraic context** (HAC for short) is a triplet $(\mathtt{C}, \mathtt{C}_0, \mathtt{A})$ consisting of a symmetric monoidal model category $\mathtt{C}$, a full subcategory $\mathtt{C}_0 \subset \mathtt{C}$ and a full subcategory $\mathtt{A} \subset \mathtt{CMon}(\mathtt{C})$ of the category of commutative monoids in $\mathtt{C}$. This data has to fulfill minimal assumptions that allow to do reasonable linear and commutative algebra in the considered HAC [38, Section 1.2]. In particular, it is possible to define the notion **étale morphisms** in $\mathtt{CMon}(\mathtt{C})$ [38, Definition 1.2.6.7].

Recall from the introduction that $(\mathtt{DG}\mathcal{D}\mathtt{M}, \mathtt{DG}\mathcal{D}\mathtt{M}, \mathtt{DG}\mathcal{D}\mathtt{A})$ is a HAC [9]. Hence, we can consider étale morphisms in $\mathtt{DG}\mathcal{D}\mathtt{A}$.

In this paper we also consider the trivial HAC $(\mathcal{D}\mathtt{M}, \mathcal{D}\mathtt{M}, \mathcal{D}\mathtt{A})$. This means that we consider the complete and cocomplete category $\mathcal{D}\mathtt{M}$ of $\mathcal{D}$-modules as equipped with the trivial model structure, i.e., the weak equivalences are the isomorphisms and all morphisms are fibrations and cofibrations. The category $\mathcal{D}\mathtt{M}$ is moreover closed symmetric monoidal for the Hom and tensor product functors over functions $\mathcal{O}$ [8]. This symmetric monoidal model category obviously fulfills the HAC requirements in the sense of [38]. The category $\mathcal{D}\mathtt{A} := \mathtt{CMon}(\mathcal{D}\mathtt{M})$ of $\mathcal{D}$-algebras and the category $\mathtt{Mod}(\mathcal{A}) := \mathtt{Mod}_{\mathcal{D}\mathtt{M}}(\mathcal{A})$ of modules in $\mathcal{D}\mathtt{M}$ over $\mathcal{A} \in \mathcal{D}\mathtt{A}$ are also endowed with the trivial model structure. Furthermore, in view of what we said in the first paragraph, we can consider étale morphisms in $\mathcal{D}\mathtt{A}$.

3.2 Étale coverings of affine k-schemes

The category of schemes will be denoted $\mathtt{Sch}$. An $\mathtt{Sch}$-morphism f is just a morphism $f = (|f|, f^*)$ of locally ringed spaces.

Definition 1. *A family $(f_i : U_i \to U)_{i\in I}$ of $\mathtt{Sch}$-morphisms is an* **étale covering of the scheme** U*, if and only if*

1. $f_i : U_i \to U$ is an étale morphism of schemes (in the standard sense), for all $i \in I$, and

2. $\cup_{i\in I} |f_i|(|U_i|) = |U|$.

We denote $\mathtt{Aff} \subset \mathtt{Sch}$ the full subcategory of affine schemes and $\mathtt{CR}^{\mathrm{op}}$ the equivalent opposite category of the category of commutative rings. In the case $U_i, U \in \mathtt{Aff}$, i.e., $U_i = \mathrm{Spec}(R_i)$ and $U = \mathrm{Spec}(R)$, the previous definition can be reformulated in terms of commutative rings.

Definition 2. *A family $(f_i : \mathrm{Spec}(R_i) \to \mathrm{Spec}(R))_{i\in I}$ of $\mathtt{Aff}$-morphisms is an* **étale covering of the affine scheme** $\mathrm{Spec}(R)$*, if and only if*

1' $f_i : R \to R_i$ is an étale morphism of commutative rings (in the standard sense), for all $i \in I$, and

2' there exists a finite subset $J \subset I$, such that the canonical arrow $R \to \prod_j R_j$ is a faithfully flat morphism of commutative rings.

Flat modules over a commutative ring R generalize projective and thus free modules over R. A morphism $f : R \to S$ of commutative rings is **flat** if S is a flat module over R, or equivalently, if the base change functor $S \otimes_R - : \mathtt{Mod}(R) \to \mathtt{Mod}(S)$ is exact. If $S \otimes_R -$ is exact and faithful, we say that the underlying ring morphism $f : R \to S$ is **faithfully flat**. It is well-known that a flat morphism of commutative rings $f : R \to S$ is faithfully flat, if and only if the morphism $f : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ of affine schemes is surjective on the underlying topological spaces. This shows that faithful flatness is a covering property.

Proposition 1. *A family* $(f_i : \mathrm{Spec}(R_i) \to \mathrm{Spec}(R))_{i\in I}$ *of* `Aff`*-morphisms is an étale covering of the affine scheme* $\mathrm{Spec}(R)$ (*Definition* 2) *if and only if it is an étale covering of the scheme* $\mathrm{Spec}(R)$ (*Definition* 1).

Proof. The conditions 1 and 1' are obviously equivalent. Hence it suffices to show that if 1 holds, then 2 and 2' are equivalent. Since an étale `Sch`-map is open, the subsets $|f_i|(|U_i|) \subset |U|$ are Zariski open. As affine schemes are quasi-compact, it follows that

$$\cup_{i\in I}\ |f_i|(|\mathrm{Spec}(R_i)|) = |\mathrm{Spec}(R)| \tag{3}$$

if and only if there exists a finite subset $J \subset I$, such that

$$\cup_{j\in J}\ |f_j|(|\mathrm{Spec}(R_j)|) = |\mathrm{Spec}(R)|\ , \tag{4}$$

so that it now suffices to prove that (4) is equivalent to 2'. Because a coproduct $\coprod_\alpha f_\alpha : \coprod_\alpha U_\alpha \to U$ of `Sch`-maps $f_\alpha : U_\alpha \to U$ is étale, if the f_α are, and because a finite coproduct of affine schemes is the affine scheme

$$\coprod_{j\in J} \mathrm{Spec}(R_j) = \mathrm{Spec}(\prod_{j\in J} R_j)\ ,$$

1 implies that

$$\coprod_{j\in J} f_j : \mathrm{Spec}(\prod_{j\in J} R_j) \to \mathrm{Spec}(R) \quad \text{and} \quad \prod_{j\in J} f_j : R \to \prod_{j\in J} R_j$$

are étale and therefore flat. Hence $\prod_{j\in J} f_j$ is satisfies 2', if and only if

$$|\coprod_{j\in J} f_j| : |\coprod_{j\in J} \mathrm{Spec}(R_j)| \to |\mathrm{Spec}(R)|$$

is surjective, i.e., if and only if

$$(\coprod_{j\in J} |f_j|)(\coprod_{j\in J} |\mathrm{Spec}(R_j)|) = \cup_{j\in J}\ |f_j|(|\mathrm{Spec}(R_j)|) = |\mathrm{Spec}(R)|\ ,$$

i.e., if and only if (4) holds. □

Since an affine k-scheme is an affine scheme $\mathrm{Spec}(A) \to \mathrm{Spec}(k)$, i.e., a ring morphism $k \to A$, its coordinate ring gets promoted from a commutative ring $A \in$ `CR` to a k-algebra $A \in k$`A`: the category `Aff(k)` of affine k-schemes is equivalent to the category opposite to the category k`A` of k-algebras. In particular the category `Aff`(k) of standard affine k-schemes coincides with the category $\mathtt{Aff}_{k\mathtt{M}} := k\mathtt{A}^{\mathrm{op}} = \mathtt{CMon}(k\mathtt{M})^{\mathrm{op}}$ of affine k-schemes in the trivial HAC $(k\mathtt{M}, k\mathtt{M}, k\mathtt{A})$. Furthermore, from the previous proof it follows that 1' implies that $R \to \prod_j R_j$ is étale. This motivates

Definition 3 ([38], Section 2.1.1). *A family* $(f_i : \mathrm{Spec}(A_i) \to \mathrm{Spec}(A))_{i\in I}$ *of* $\mathtt{Aff}(k)$*-morphisms is an* **étale covering of the affine k-scheme** $\mathrm{Spec}(A)$*, if and only if there exists a finite subset* $J \subset I$*, such that the canonical arrow* $A \to \prod_j A_j$ *is a faithfully flat morphism of k-algebras and is étale as morphism of commutative rings.*

Definition 3 is a variant that ignores the morphisms f_i with $i \in I \setminus J$ and is more general than Definition 2. The reason for this generalization is that it is better suited to the objectives of [38, Chapter 2.1].

3.3 Étale coverings of affine $\mathcal{D}$-schemes

We are now ready to define étale coverings of an affine $\mathcal{D}$-scheme $\mathrm{Spec}(\mathcal{A}) \in \mathtt{Aff}(\mathcal{D}) = \mathcal{D}\mathtt{A}^{\mathrm{op}}$ (as mentioned above, we do not define $\mathrm{Spec}(\mathcal{A})$ as a space but just treat it as an object $\mathcal{A} \in \mathcal{D}\mathtt{A}$ viewed in the opposite category).

Definition 4. *A family* $(f_i : \mathrm{Spec}(\mathcal{A}_i) \to \mathrm{Spec}(\mathcal{A}))_{i\in I}$ *of* $\mathtt{Aff}(\mathcal{D})$*-morphisms is an* **étale covering of the affine $\mathcal{D}$-scheme** $\mathrm{Spec}(\mathcal{A})$*, if and only if there exists a finite subset* $J \subset I$*, such that the canonical arrow* $\prod_j f_j : \mathcal{A} \to \mathcal{P} := \prod_j \mathcal{A}_j$ *is faithfully flat and étale. This means that the universal* $\mathcal{D}\mathtt{A}$*-morphism* $\prod_j f_j$ *is étale in the sense of Subsection 3.1 and that the base change functor*

$$\mathcal{P} \otimes_{\mathcal{A}} - : \mathtt{Mod}_{\mathcal{D}\mathtt{M}}(\mathcal{A}) \to \mathtt{Mod}_{\mathcal{D}\mathtt{M}}(\mathcal{P})$$

is faithful and exact.

The tensor product over $\mathcal{A}$ was described in [9] and will be further explained in Subsection 4.1. Definition 4 makes sense as the categories $\mathtt{Mod}_{\mathcal{D}\mathtt{M}}(\mathcal{B})$ ($\mathcal{B} \in \mathcal{D}\mathtt{A}$) are abelian.

3.4 Étale coverings of derived affine k-schemes

The following notion of formal covering is valid in any HAC $(\mathtt{C}, \mathtt{C}_0, \mathtt{A})$ and is a homotopical and categorical version of the joint covering property encoded in the faithful flatness in Definition 2, 3 and 4.

Definition 5 ([38], Definition 1.2.5.1). *A family* $(f_i : A \to A_i)_{i\in I}$ *of* $\mathtt{CMon}(\mathtt{C})$*-morphisms is a* **formal covering of the monoid** A*, if and only if the family of derived base change functors*

$$(A_i \otimes^{\mathbb{L}}_A - : \mathtt{Ho}(\mathtt{Mod}_{\mathtt{C}}(A)) \to \mathtt{Ho}(\mathtt{Mod}_{\mathtt{C}}(A_i)))_{i\in I}$$

is conservative in the sense that a morphism ϕ of the source homotopy category is an isomorphism if *and only if all the* $A_i \otimes^{\mathbb{L}}_A \phi$ *are isomorphisms of the target homotopy category.*

If k is a commutative ring, we can consider the HAC $(\mathtt{DG}k\mathtt{M}, \mathtt{DG}k\mathtt{M}, \mathtt{DG}k\mathtt{A})$ (as elsewhere in this paper, we work with non-negatively graded chain complexes of k-modules and the corresponding commutative algebras) and define the category of derived affine k-schemes by setting $\mathtt{DAff}(k) := \mathtt{DG}k\mathtt{A}^{\mathrm{op}}$. Recall that in view of Remark 1 we can speak of étale $\mathtt{DG}k\mathtt{A}$-morphisms.

Definition 6 ([38], Definition 2.2.2.12)**.** *A family* $(f_i : \mathrm{Spec}(\mathcal{A}_i) \to \mathrm{Spec}(\mathcal{A}))_{i\in I}$ *of* $\mathtt{DAff}(k)$-*morphisms is an* **étale covering of the derived affine k-scheme** $\mathrm{Spec}(\mathcal{A})$*, if and only if*

1. $f_i : \mathcal{A} \to \mathcal{A}_i$ *is an étale* $\mathtt{DG}k\mathtt{A}$-*morphism, for all* $i \in I$*, and*
2. *there exists a finite subset* $J \subset I$*, such that the family* $(f_j : \mathcal{A} \to \mathcal{A}_j)_{j\in J}$ *of* $\mathtt{DG}k\mathtt{A}$-*morphisms is a formal covering of* $\mathcal{A}$ *.*

3.5 Étale coverings of derived affine $\mathcal{D}$-schemes

We are now prepared to define étale coverings of a derived affine $\mathcal{D}$-scheme $\mathrm{Spec}(\mathcal{A}) \in \mathtt{DAff}(\mathcal{D}) = \mathtt{DG}\mathcal{D}\mathtt{A}^{\mathrm{op}}$.

Definition 7. *A family* $(f_i : \mathrm{Spec}(\mathcal{A}_i) \to \mathrm{Spec}(\mathcal{A}))_{i\in I}$ *of* $\mathtt{DAff}(\mathcal{D})$-*morphisms is an* **étale covering of the derived affine $\mathcal{D}$-scheme** $\mathrm{Spec}(\mathcal{A})$*, if and only if*

1. *each* $\mathtt{DG}\mathcal{D}\mathtt{A}$-*morphism* $f_i : \mathcal{A} \to \mathcal{A}_i$ *is étale (in the sense of Subsection 3.1), and*
2. *there is a finite subset* $J \subset I$*, such that the family* $(f_j : \mathcal{A} \to \mathcal{A}_j)_{j\in J}$ *of* $\mathtt{DG}\mathcal{D}\mathtt{A}$-*morphisms is a formal covering of* $\mathcal{A}$ *(in the sense of Definition 5).*

Remark 2. If $(\mathtt{C}, \mathtt{C}_0, \mathtt{A})$ is a HAC, the formal covering families in $\mathtt{CMon(C)}$ are stable under equivalences, homotopy pushouts and compositions. They thus define a model topology on $(\mathtt{CMon(C)})^{\mathrm{op}}$ (see [38]). In particular, if the HAC is $(\mathtt{DG}\mathcal{D}\mathtt{M}, \mathtt{DG}\mathcal{D}\mathtt{M}, \mathtt{DG}\mathcal{D}\mathtt{A})$, the formal coverings used in Definition 7 define a model topology on $\mathtt{DAff}(\mathcal{D})$. This fact will be crucial in the proof that étale coverings of derived affine $\mathcal{D}$-schemes form a model pre-topology τ on $\mathtt{DAff}(\mathcal{D}) = \mathtt{DG}\mathcal{D}\mathtt{A}^{\mathrm{op}}$ (see Section 1). Moreover, we have to show that they satisfy the conditions [38, Assumption 1.3.2.2] needed to apply the general theory of stacks [38, Part 1] in our homotopical $\mathcal{D}$-geometric setting – which is an important step on our way to establish this context as a suitable framework for a coordinate-independent study of solutions of partial differential equations modulo symmetries and in particular the study of the Batalin-Vilkovisky complex.

4 Étale maps in $\mathcal{D}$-geometry

4.1 Étale and strongly étale morphisms

However, this step requires a full understanding of Definition 4 and Definition 7, so that we now describe étale morphisms in $\mathcal{D}\mathtt{A}$ and in $\mathtt{DG}\mathcal{D}\mathtt{A}$, which are particular cases of étale morphisms in $\mathtt{CMon(C)}$, where $(\mathtt{C}, \mathtt{C_0}, \mathtt{A})$ $(\mathtt{C_0} \subset \mathtt{C}, \mathtt{A} \subset \mathtt{CMon(C)})$ is a HAC.

The notion of étale morphism of schemes is the algebraic analogue of the notion of local diffeomorphism of smooth manifolds. A morphism of schemes is étale if and only if it is locally of finite presentation and formally étale. The definition of étale morphisms in $\mathtt{CMon(C)}$ is similar:

Definition 8. *[38, Definition 1.2.6.7] A* $\mathtt{CMon(C)}$*-morphism is* **étale***, if and only if it is*

1. *finitely presented (see Definition 11), and*
2. *formally étale (see Definition 17).*

As said before, Definition 8 is valid in particular in $\mathcal{D}\mathtt{A}$ and $\mathtt{DG}\mathcal{D}\mathtt{A}$.

Definition 9. *A* $\mathtt{DG}\mathcal{D}\mathtt{A}$*-morphism* $f : \mathcal{A} \to \mathcal{B}$ *is* **strongly étale***, if and only if*

1. f *is strong (see Definition 10), and*
2. $H_0(f) : H_0(\mathcal{A}) \to H_0(\mathcal{B})$ *is an étale* $\mathcal{D}\mathtt{A}$*-morphism (see Definition 8).*

Moreover:

Definition 10. 1. *Let* $\mathcal{A} \in \mathtt{DG}\mathcal{D}\mathtt{A}$*. A* module $M \in \mathtt{Mod}_{\mathtt{DG}\mathcal{D}\mathtt{M}}(\mathcal{A})$ *is* **strong***, if the canonical* $\mathtt{Mod}_{\mathtt{G}\mathcal{D}\mathtt{M}}(H_0(\mathcal{A}))$*-morphism*

$$\phi_{\bullet,\mathcal{A},M} : H_\bullet(\mathcal{A}) \otimes_{H_0(\mathcal{A})} H_0(M) \to H_\bullet(M) \tag{5}$$

is an isomorphism, i.e., if all $\mathtt{Mod}_{\mathcal{D}\mathtt{M}}(H_0(\mathcal{A}))$*-morphisms* $\phi_{k,\mathcal{A},M}$ *are isomorphisms.*

2. *A* $\mathtt{DG}\mathcal{D}\mathtt{A}$-morphism $f : \mathcal{A} \to \mathcal{B}$ *is* **strong***, if* $\mathcal{B} \in \mathtt{Mod}_{\mathtt{DG}\mathcal{D}\mathtt{M}}(\mathcal{A})$ *is strong.*

Remark 3. Since we consider bounded-below differential graded $\mathcal{D}$-algebras, we expect that étale $\mathtt{DG}\mathcal{D}\mathtt{A}$-morphisms are exactly the strongly étale $\mathtt{DG}\mathcal{D}\mathtt{A}$-morphisms [38, Theorem 2.2.2.6]. In this case it suffices - if we want to understand étale $\mathtt{DG}\mathcal{D}\mathtt{A}$-morphisms - to describe finitely presented and formally étale $\mathcal{D}\mathtt{A}$-morphisms. This will be done in the following subsections.

So the conjecture is that a $\mathtt{DG\mathcal{D}A}$-morphism $f : \mathcal{A} \to \mathcal{B}$ is étale exactly if its induced map in degree zero homology is étale and the homology of $\mathcal{B}$ can be fully reconstructed from its zero degree part by acting on it with the homology of $\mathcal{A}$.

However, we must first properly understand the tensor product in the previous definition and the morphism defined on it. We start with the following general definitions.

Let $(\mathtt{C}, \otimes, I, \mathsf{Hom})$ be a closed symmetric monoidal category with all small limits and colimits. Consider a commutative algebra in $\mathtt{C}$, i.e., a commutative monoid $(\mathcal{A}, \mu, \eta)$. The corresponding algebra morphisms are defined naturally and the category of commutative algebras in $\mathtt{C}$ is denoted by $\mathtt{Alg}_{\mathtt{C}}$. A (left) *$\mathcal{A}$-module in* $\mathtt{C}$ is a $\mathtt{C}$-object M together with a $\mathtt{C}$-morphism $\nu : \mathcal{A} \otimes M \to M$, such that the usual associativity and unitality diagrams commute. *Morphisms of $\mathcal{A}$-modules in* $\mathtt{C}$ are also defined in the obvious way and the category of $\mathcal{A}$-modules in $\mathtt{C}$ is denoted by $\mathtt{Mod}_{\mathtt{C}}(\mathcal{A})$. The category of right $\mathcal{A}$-modules in $\mathtt{C}$ is defined analogously. Since $\mathcal{A}$ is commutative, the categories of left and right modules are equivalent (one passes from one type of action to the other by precomposing with the braiding 'com'). The tensor product $\otimes_{\mathcal{A}}$ of two modules $M', M'' \in \mathtt{Mod}_{\mathtt{C}}(\mathcal{A})$ is defined as the coequalizer in $\mathtt{C}$ of the maps

$$\psi' := (\nu_{M'} \otimes \mathrm{id}_{M''}) \circ (\mathrm{com} \otimes \mathrm{id}_{M''}), \psi'' := \mathrm{id}_{M'} \otimes \nu_{M''} :$$
$$(M' \otimes \mathcal{A}) \otimes M'' \cong M' \otimes (\mathcal{A} \otimes M'') \rightrightarrows M' \otimes M'' \xrightarrow{\kappa} M' \otimes_{\mathcal{A}} M'' \ .$$

Since $\mathcal{A} \in \mathtt{Alg}_{\mathtt{C}}$ is commutative, the $\mathtt{C}$-object $M' \otimes_{\mathcal{A}} M''$ inherits an $\mathcal{A}$-module structure from those of M' and M''.

The categories $\mathtt{C} = \mathcal{D}\mathtt{M}$ and $\mathtt{C} = \mathtt{DG}\mathcal{D}\mathtt{M}$ satisfy the conditions above on $\mathtt{C}$ with $\otimes = \otimes_{\mathcal{O}}$. In these cases, we have $\mathtt{Alg}_{\mathtt{C}} = \mathcal{D}\mathtt{A}$ and $\mathtt{Alg}_{\mathtt{C}} = \mathtt{DG}\mathcal{D}\mathtt{A}$, respectively. In order to define a $\mathtt{Mod}_{\mathcal{D}\mathtt{M}}(\mathsf{A})$-morphism (an A- and $\mathcal{D}$-linear map) $\phi : M' \otimes_{\mathsf{A}} M'' \to M$ ($\mathsf{A} \in \mathcal{D}\mathtt{A}$), one starts defining an $\mathcal{O}$-bilinear map $\varphi : M' \times M'' \to M$, hence, an $\mathcal{O}$-linear map $\varphi : M' \otimes_{\mathcal{O}} M'' \to M$. One then checks that φ is a $\mathcal{D}\mathtt{M}$-morphism, i.e., is $\mathcal{D}$-linear, or, equivalently, is linear for the action ∇_θ of vector fields θ. Recall that, by definition,

$$\nabla_\theta(m' \otimes m'') = (\nabla_\theta m') \otimes m'' + m' \otimes (\nabla_\theta m'') \ .$$

If, for $a \in \mathsf{A}$,

$$\varphi((a \cdot m') \otimes m'') = \varphi(m' \otimes (a \cdot m'')) \ ,$$

it follows from the universal property of a coequalizer, that there is a unique $\mathcal{D}$-linear map $\phi : M' \otimes_{\mathsf{A}} M'' \to M$, such that $\phi \circ \kappa = \varphi$. Finally, if φ is A-bilinear on $M' \times M''$, then ϕ is

A-linear on $M' \otimes_{\mathsf{A}} M''$. Indeed,

$$\varphi(m', m'') = \varphi(m' \otimes m'') = \phi(\kappa(m' \otimes m'')) = \phi(m' \otimes_{\mathsf{A}} m'') .$$

We now come back to Definition 10. The tensor product in (5) makes sense since $H_0(\mathcal{A}) \in \mathcal{D}\mathtt{A}$ and $H_k(\mathcal{A}), H_0(M) \in \mathtt{Mod}_{\mathcal{D}\mathtt{M}}(H_0(\mathcal{A}))$. In order to define the morphism $\phi_{k,\mathcal{A},M}$ (we will write ϕ), we apply the just detailed method to the preceding $\mathcal{D}$-algebra and modules in $\mathcal{D}\mathtt{M}$ over that algebra. Let φ be the map

$$\varphi : H_k(\mathcal{A}) \times H_0(M) \ni ([a_k], [m_0]) \mapsto [\nu(a_k \otimes m_0)] \in H_k(M) ,$$

where the $\mathtt{DG}\mathcal{D}\mathtt{M}$-morphism $\nu : \mathcal{A} \otimes_{\mathcal{O}} M \to M$ is the action of $\mathcal{A}$ on M. It is easy to check that φ is well-defined on homology classes. In view of the $\mathcal{D}\mathtt{A}$-morphism

$$\mathcal{O} \ni f \mapsto f \cdot [1_{\mathcal{A}}] = [f \cdot 1_{\mathcal{A}}] \in H_0(\mathcal{A}) ,$$

the function algebra $\mathcal{O}$ can be interpreted as sub-$\mathcal{D}$-algebra of $H_0(\mathcal{A})$. Hence, if φ is $H_0(\mathcal{A})$-bilinear, it is in particular $\mathcal{O}$-bilinear. As for $H_0(\mathcal{A})$-bilinearity, taking into account that the action $*$ of $H_0(\mathcal{A})$ on $H_k(\mathcal{A})$ is induced by the multiplication $*$ in $\mathcal{A}$, and that the action ν of $H_0(\mathcal{A})$ on $H_k(M)$ is induced by the action ν of $\mathcal{A}$ on M, we get

$$\varphi([a_0] * [a_k], [m_0]) = \varphi([a_0 * a_k], [m_0]) = [\nu((a_0 * a_k) \otimes m_0)] ,$$

$$\nu([a_0] \otimes \varphi([a_k], [m_0])) = \nu([a_0] \otimes [\nu(a_k \otimes m_0)]) = [\nu(a_0 \otimes \nu(a_k \otimes m_0))] = [\nu((a_0 * a_k) \otimes m_0)] ,$$

and

$$\varphi([a_k], \nu([a_0] \otimes [m_0])) = \varphi([a_k], [\nu(a_0 \otimes m_0)]) = [\nu(a_k \otimes \nu(a_0 \otimes m_0))] = [\nu((a_k * a_0) \otimes m_0)] ,$$

so that φ is actually $H_0(\mathcal{A})$-bilinear and thus $\mathcal{O}$-bilinear. We now check linearity of

$$\varphi : H_k(\mathcal{A}) \otimes_{\mathcal{O}} H_0(M) \to H_k(M)$$

with respect to the action ∇_θ by vector fields θ. As the $\mathcal{D}$-action in homology is induced by the $\mathcal{D}$-action on the underlying complex, we have

$$\varphi(\nabla_\theta([a_k] \otimes [m_0])) = \varphi((\nabla_\theta[a_k]) \otimes [m_0]) + [a_k] \otimes (\nabla_\theta[m_0])) =$$

$$\varphi([\nabla_\theta a_k] \otimes [m_0]) + \varphi([a_k] \otimes [\nabla_\theta m_0]) = [\nu((\nabla_\theta a_k) \otimes m_0)] + [\nu(a_k \otimes (\nabla_\theta m_0))] =$$

$$[\nu(\nabla_\theta(a_k \otimes m_0))] = \nabla_\theta[\nu(a_k \otimes m_0)] = \nabla_\theta \, \varphi([a_k] \otimes [m_0]) .$$

In view of what has been said above, it follows that φ induces a unique $\mathtt{Mod}_{\mathcal{D}\mathtt{M}}(H_0(\mathcal{A}))$-morphism $\phi : H_k(\mathcal{A}) \otimes_{H_0(\mathcal{A})} H_0(M) \to H_k(M)$.

4.2 Finitely presented morphisms

In view of Definition 8, we now take an interest in finitely presented morphisms in $\mathtt{DG}\mathcal{D}\mathtt{A}$ and $\mathcal{D}\mathtt{A}$. This means that we consider morphisms between commutative algebras over the non-commutative ring of differential operators and their derived counterpart.

4.2.1 Finitely presented morphisms of differential graded $\mathcal{D}$-algebras

The definition of finitely presented morphisms of commutative rings via a finite number of generators and a finite number of relations, naturally leads to locally finitely presented morphisms of schemes. The latter can be characterized as those $\mathtt{Sch}$-morphisms $X \to S$ that satisfy

$$\mathrm{Colim}_i \, \mathrm{Hom}_{\mathtt{Sch}\downarrow S}(T_i, X) = \mathrm{Hom}_{\mathtt{Sch}\downarrow S}(\mathrm{Lim}_i \, T_i, X) \, , \tag{6}$$

for any direct system of affine schemes $(T_i)_{i\in I}$ over S [36, Proposition 32.6.1] (the category $\mathtt{Sch} \downarrow S$ is the slice category of $\mathtt{Sch}$ over S). Finitely presented morphisms in the proper model category $\mathtt{DG}\mathcal{D}\mathtt{A}$ [9, Theorem 3.2.3] are a particular case of the following 'categorification', 'dualization' and 'homotopification' of (6):

Definition 11. *[38, Definition 1.2.3.1] A morphism $m \to n$ in a proper model category $\mathtt{M}$ is* **finitely presented***, if and only if, for any filtered diagram of objects $(p_i)_{i\in\mathtt{I}}$ in $m \downarrow \mathtt{M}$, the canonical morphism*

$$\mathrm{HoColim}_{i\in\mathtt{I}} \, \mathrm{Map}_{m\downarrow\mathtt{M}}(n, p_i) \to \mathrm{Map}_{m\downarrow\mathtt{M}}(n, \mathrm{HoColim}_{i\in\mathtt{I}} \, p_i) \tag{7}$$

is an isomorphism in $\mathtt{Ho}(\mathtt{SSet})$.

Indeed, since filtered categories categorize directed sets, filtered diagrams or functors out of a filtered index category (the corresponding colimit is called a filtered colimit) categorize the notion of direct system over a directed set (the corresponding colimit is referred to as direct or inductive limit). Since we have the case $\mathtt{M} = \mathtt{DG}\mathcal{D}\mathtt{A} = \mathtt{DAff}(\mathcal{D})^{\mathrm{op}}$ in mind, it is natural to work dually with respect to the $\mathtt{Sch}$-situation considered in (6). We use the mapping space and the homotopy colimit, thereby incorporating the model structure of $\mathtt{M}$ and working in the context of homotopical algebraic geometry. Also remember that the mapping space (resp., the homotopy colimit of a functor) is only well-defined in the homotopy category of simplicial sets (resp., the homotopy category of the functor's target category).

Recall further that for every small category $\mathtt{S}$ and every model category $\mathtt{C}$ the functor category $\mathtt{Fun}(\mathtt{S}, \mathtt{C})$ can be equipped with a projective model structure, if $\mathtt{C}$ is cofibrantly generated. If the projective model structure exists, the diagonal functor $\Delta : \mathtt{C} \to \mathtt{Fun}(\mathtt{S}, \mathtt{C})$ – which sends each object to the corresponding constant diagram and each morphism to the corresponding constant natural transformation – preserves fibrations and trivial fibrations and the adjunction

$$\mathrm{Colim} : \mathtt{Fun}(\mathtt{S}, \mathtt{C}) \rightleftarrows \mathtt{C} : \Delta$$

is thus a Quillen adjunction. From Brown's Lemma [21, Lemma 1.1.12] it follows that a left Quillen functor sends weak equivalences between cofibrant objects to weak equivalences, so that its (total) left derived functor exists. In the case of the colimit functor, the left derived functor $\mathbb{L}_{\mathrm{proj}}\,\mathrm{Colim}$ is referred to as the homotopy colimit functor and denoted

$$\mathrm{HoColim} : \mathtt{Ho}(\mathtt{Fun}_{\mathrm{proj}}(\mathtt{S}, \mathtt{C})) \to \mathtt{Ho}(\mathtt{C}) \ .$$

It is computed using a cofibrant replacement functor Q_{proj} of the projective model structure on $\mathtt{Fun}(\mathtt{S}, \mathtt{C})$. The case of the homotopy limit functor is dual, so that it can be computed with a fibrant replacement functor R_{inj} of the injective model structure on $\mathtt{Fun}(\mathtt{S}, \mathtt{C})$, which exists when $\mathtt{C}$ is combinatorial. So, for every $\mathtt{S}$-diagram F in $\mathtt{C}$ or functor $F \in \mathtt{Fun}(\mathtt{S}, \mathtt{C})$, we have

$$\mathrm{HoColim}(F) = \mathrm{Colim}(Q_{\mathrm{proj}}(F)) \quad (\text{resp.,} \quad \mathrm{HoLim}(F) = \mathrm{Lim}(R_{\mathrm{inj}}(F))) \ . \tag{8}$$

4.2.2 Finitely presented morphisms of algebras over differential operators

As already mentioned, Definition 11 applies in particular to morphisms of the proper model categories $\mathtt{DG}\mathcal{D}\mathtt{A}$ and $\mathcal{D}\mathtt{A}$. Indeed, every trivial model category is left and right proper, because pushouts and pullbacks preserve isomorphisms. With regard to Remark 3, we focus on the case $\mathtt{M} = \mathcal{D}\mathtt{A} = \mathtt{Aff}(\mathcal{D})^{\mathrm{op}}$. It is to be expected that Definition 11 reduces in this case to the characterization:

Proposition 2. *An object $\mathcal{B}$ in $\mathcal{A} \downarrow \mathcal{D}\mathtt{A}$ is* **finitely presented***, if and only if, for any filtered diagram $(\mathcal{C}_i)_{i\in\mathtt{I}}$ of objects in $\mathcal{A} \downarrow \mathcal{D}\mathtt{A}$, the canonical morphism*

$$\mathrm{Colim}_{i\in\mathtt{I}}(\mathrm{Hom}_{\mathcal{A}\downarrow\mathcal{D}\mathtt{A}}(\mathcal{B}, \mathcal{C}_i)) \to \mathrm{Hom}_{\mathcal{A}\downarrow\mathcal{D}\mathtt{A}}(\mathcal{B}, \mathrm{Colim}_{i\in\mathtt{I}}\, \mathcal{C}_i) \tag{9}$$

is a bijection.

Homotopical geometric definitions are designed to reduce to the corresponding standard definitions for $\mathbb{Z}$-modules and commutative rings. Since modules and algebras over differential operators have a richer and very special structure, it makes sense to carefully prove the previous proposition.

Proof. As $\mathcal{D}\mathtt{A}$ is endowed with the trivial model structure, the same is true for the under-category $\mathcal{A} \downarrow \mathcal{D}\mathtt{A}$ (since the three distinguished classes of morphisms of $\mathcal{A} \downarrow \mathcal{D}\mathtt{A}$ are defined using the forgetful functor). It follows that the Reedy model structure on simplicial objects $\mathtt{Fun}(\Delta^{\mathrm{op}}, \mathcal{A} \downarrow \mathcal{D}\mathtt{A})$ in $\mathcal{A} \downarrow \mathcal{D}\mathtt{A}$ is also trivial. Indeed, the Reedy weak equivalences are the natural transformations that are object-wise weak equivalences in $\mathcal{A} \downarrow \mathcal{D}\mathtt{A}$, i.e., they are the natural isomorphisms. Since the Reedy fibrations (resp., cofibrations) are the natural transformations such that certain object-wise induced universal morphisms are fibrations (resp., cofibrations) in $\mathcal{A} \downarrow \mathcal{D}\mathtt{A}$, all natural transformations are Reedy fibrations (resp., Reedy cofibrations). In addition, the projective model structure on $\mathtt{Fun}(\mathtt{I}, \mathcal{A} \downarrow \mathcal{D}\mathtt{A})$ is trivial for every index category $\mathtt{I}$. Finally, it follows from Equation (2) that, for every $\mathcal{B}, \mathcal{D} \in \mathcal{A} \downarrow \mathcal{D}\mathtt{A}$, there is an equivalence of simplicial sets

$$\mathrm{Map}_{\mathcal{A}\downarrow\mathcal{D}\mathtt{A}}(\mathcal{B}, \mathcal{D}) \simeq \mathrm{Hom}_{\mathcal{A}\downarrow\mathcal{D}\mathtt{A}}(Q(\mathcal{B}), R_\bullet(c_\bullet(\mathcal{D}))) \;, \tag{10}$$

where Q is a cofibrant replacement functor in $\mathcal{A} \downarrow \mathcal{D}\mathtt{A}$, where $R_\bullet$ a fibrant replacement functor in the Reedy model category $\mathtt{Fun}(\Delta^{\mathrm{op}}, \mathcal{A} \downarrow \mathcal{D}\mathtt{A})$ and where $c_\bullet$ is the constant simplicial object functor. Since we can take $Q = \mathrm{id}$ and $R_\bullet = \mathrm{id}$, we get

$$\mathrm{Map}_{\mathcal{A}\downarrow\mathcal{D}\mathtt{A}}(\mathcal{B}, \mathcal{D}) \simeq \mathrm{Hom}_{\mathcal{A}\downarrow\mathcal{D}\mathtt{A}}(\mathcal{B}, c_\bullet(\mathcal{D})) = c_\bullet(\mathrm{Hom}_{\mathcal{A}\downarrow\mathcal{D}\mathtt{A}}(\mathcal{B}, \mathcal{D})) \;. \tag{11}$$

With all that said, we return to applying Definition 11 to $\mathcal{D}\mathtt{A}$. From Equation (11) it follows that a $\mathcal{D}\mathtt{A}$-morphism $\mathcal{A} \to \mathcal{B}$ is finitely presented if and only if, for every filtered diagram $\mathcal{C}_\star \in \mathtt{Fun}(\mathtt{I}, \mathcal{A} \downarrow \mathcal{D}\mathtt{A})$, the canonical morphism

$$\mathrm{HoColim}_{i\in\mathtt{I}}\, c_\bullet(\mathrm{Hom}_{\mathcal{A}\downarrow\mathcal{D}\mathtt{A}}(\mathcal{B}, \mathcal{C}_i)) \to c_\bullet(\mathrm{Hom}_{\mathcal{A}\downarrow\mathcal{D}\mathtt{A}}(\mathcal{B}, \mathrm{HoColim}_{i\in\mathtt{I}}\, \mathcal{C}_i)) \tag{12}$$

is an isomorphism in $\mathtt{Ho}(\mathtt{SSet})$. Moreover, the homotopy colimit of $\mathcal{C}_\star$ is

$$\mathrm{HoColim}(\mathcal{C}_\star) = \mathrm{Colim}\,(\mathfrak{Q}_{\mathrm{proj}}(\mathcal{C}_\star))_\star = \mathrm{Colim}_{i\in\mathtt{I}}\, \mathcal{C}_i \;,$$

since the projective model structure on $\mathtt{Fun}(\mathtt{I}, \mathcal{A} \downarrow \mathcal{D}\mathtt{A})$ is trivial. Hence the RHS of (12) is given by

$$c_\bullet(\mathrm{Hom}_{\mathcal{A}\downarrow\mathcal{D}\mathtt{A}}(\mathcal{B}, \mathrm{Colim}_{i\in\mathtt{I}}\, \mathcal{C}_i)) \;. \tag{13}$$

On the other hand, the LHS homotopy colimit in (12) is equal to

$$\mathrm{Colim}(Q_{\mathrm{proj}}(c_\bullet(\mathrm{Hom}_{\mathcal{A}\downarrow\mathcal{D}\mathtt{A}}(\mathcal{B}, \mathcal{C}_\star))))_{\bullet,\star} \;, \tag{14}$$

where Q_{proj} denotes a cofibrant replacement functor of the projective model structure of $\texttt{Fun}(\texttt{I}, \texttt{SSet})$. In view of (12), (13) and (14), an object $\mathcal{B} \in \mathcal{A} \downarrow \mathcal{D}\texttt{A}$ is finitely presented if and only if, for every filtered diagram $\mathcal{C}_\star \in \texttt{Fun}(\texttt{I}, \mathcal{A} \downarrow \mathcal{D}\texttt{A})$, the canonical morphism

$$\text{Colim}_{i\in\texttt{I}}(Q_{\text{proj}}(c_\bullet(\text{Hom}_{\mathcal{A}\downarrow\mathcal{D}\texttt{A}}(\mathcal{B}, \mathcal{C}_\star))))_{\bullet,i} \to c_\bullet(\text{Hom}_{\mathcal{A}\downarrow\mathcal{D}\texttt{A}}(\mathcal{B}, \text{Colim}_{i\in\texttt{I}}\, \mathcal{C}_i)) \tag{15}$$

is an isomorphism in $\texttt{Ho}(\texttt{SSet})$.

Lemma 1. *The universal map*

$$\text{Colim}_{i\in\texttt{I}}(Q_{\text{proj}}(c_\bullet(\text{Hom}_{\mathcal{A}\downarrow\mathcal{D}\texttt{A}}(\mathcal{B}, \mathcal{C}_\star))))_{\bullet,i} \xrightarrow{\sim}\!\!\!\!\!\to c_\bullet(\text{Colim}_{i\in\texttt{I}}(\text{Hom}_{\mathcal{A}\downarrow\mathcal{D}\texttt{A}}(\mathcal{B}, \mathcal{C}_i))) \;. \tag{16}$$

is a trivial fibration of simplicial sets.

Proof of Lemma 1. In the previous definition and paragraph, the symbols $\star$ and $\bullet$ are used to denote the entries of the considered functors of $\texttt{Fun}(\texttt{I}, \texttt{SSet}) = \texttt{Fun}(\texttt{I}, \texttt{Fun}(\Delta^{\text{op}}, \texttt{Set}))$. The natural transformation $\eta : Q_{\text{proj}} \Rightarrow \text{id}$ between endofunctors of $\texttt{Fun}(\texttt{I}, \texttt{SSet})$ assigns to every functor $F_{\bullet,\star} \in \texttt{Fun}(\texttt{I}, \texttt{SSet})$ a natural trivial fibration

$$\eta_F : (Q_{\text{proj}}(F_{\bullet,\star}))_{\bullet,\star} \xrightarrow{\sim}\!\!\!\!\!\to F_{\bullet,\star}$$

of the projective model structure of $\texttt{Fun}(\texttt{I}, \texttt{SSet})$, whose components

$$\eta_{F,i} : (Q_{\text{proj}}(F_{\bullet,\star}))_{\bullet,i} \xrightarrow{\sim}\!\!\!\!\!\to F_{\bullet,i}$$

are therefore trivial fibrations of $\texttt{SSet}$. This holds in particular for

$$F_{\bullet,\star} = c_\bullet(\text{Hom}_{\mathcal{A}\downarrow\mathcal{D}\texttt{A}}(\mathcal{B}, \mathcal{C}_\star)) \;,$$

i.e., the components

$$\eta_i : (Q_{\text{proj}}(c_\bullet(\text{Hom}_{\mathcal{A}\downarrow\mathcal{D}\texttt{A}}(\mathcal{B}, \mathcal{C}_\star))))_{\bullet,i} \xrightarrow{\sim}\!\!\!\!\!\to c_\bullet(\text{Hom}_{\mathcal{A}\downarrow\mathcal{D}\texttt{A}}(\mathcal{B}, \mathcal{C}_i))$$

are trivial fibrations in $\texttt{SSet}$, i.e., are trivial Kan fibrations. Since a filtered colimit of trivial Kan fibrations is a trivial Kan fibration [36, Lemma 14.30.7], we get that

$$\text{Colim}_{i\in\texttt{I}}\, \eta_i : \text{Colim}_{i\in\texttt{I}}(Q_{\text{proj}}(c_\bullet(\text{Hom}_{\mathcal{A}\downarrow\mathcal{D}\texttt{A}}(\mathcal{B}, \mathcal{C}_\star))))_{\bullet,i} \xrightarrow{\sim}\!\!\!\!\!\to \text{Colim}_{i\in\texttt{I}}(c_\bullet(\text{Hom}_{\mathcal{A}\downarrow\mathcal{D}\texttt{A}}(\mathcal{B}, \mathcal{C}_i)))$$

is a trivial Kan fibration. The announced result now follows from the fact that colimits in $\texttt{SSet}$ are computed level-wise. □

Proof of Proposition 2 (Continuation). The canonical arrow (15) is the composite of the universal arrow (16) and the universal arrow

$$c_\bullet(\mathrm{Colim}_{i\in\mathtt{I}}(\mathrm{Hom}_{\mathcal{A}\downarrow\mathcal{D}\mathtt{A}}(\mathcal{B},\mathcal{C}_i))) \to c_\bullet(\mathrm{Hom}_{\mathcal{A}\downarrow\mathcal{D}\mathtt{A}}(\mathcal{B},\mathrm{Colim}_{i\in\mathtt{I}}\,\mathcal{C}_i))\ . \tag{17}$$

Lemma 1 implies that (15) is an isomorphism in $\mathtt{Ho}(\mathtt{SSet})$ if and only if 17 is an isomorphism in $\mathtt{Ho}(\mathtt{SSet})$. Therefore it suffices to prove that if $\varphi : S \to T$ is a morphism in $\mathtt{Set}$, then $c_\bullet(\varphi) : c_\bullet(S) \to c_\bullet(T)$ is an isomorphism in $\mathtt{Ho}(\mathtt{SSet})$ if and only if $\varphi : S \to T$ is a bijection.

Lemma 2. *(i) Two objects of a model category* $\mathtt{M}$ *are related by a zigzag of weak equivalences of* $\mathtt{M}$ *if and only if they are isomorphic as objects of* $\mathtt{Ho}(\mathtt{M})$.

(ii) If $\psi : X \to Y$ *is an* $\mathtt{M}$*-morphism, then* ψ *is a weak equivalence in* $\mathtt{M}$ *if and only if* $L_\mathtt{M}\psi : X \to Y$ *is an isomorphism in* $\mathtt{Ho}(\mathtt{M})$.

Proof of Lemma 2. The direct implications of the statements (i) and (ii) are obvious.

We prove the converse implication of (i). Let $X, Y \in \mathtt{M}$ and let

$$[\varphi] \in \mathrm{Hom}_{\mathtt{Ho}(\mathtt{M})}(X,Y) = \mathrm{Hom}_\mathtt{M}(RQX, RQY)/\approx$$

be an isomorphism with inverse $[\psi]$, where R is a fibrant replacement functor, Q a cofibrant replacement functor and $[\varphi]$ the homotopy class $[\varphi]_\approx$ of φ. Since $[\psi\circ\varphi] = [\mathrm{id}_{RQX}]$ and $[\varphi\circ\psi] = [\mathrm{id}_{RQY}]$, we conclude that φ and ψ are inverse homotopy equivalences between fibrant-cofibrant objects, i.e., that they are weak equivalences. Therefore we have a zigzag of weak equivalences

$$\begin{array}{ccccc} RQX & \xleftarrow[r_{QX}]{\sim} & QX & \xrightarrow[q_X]{\sim} & X \\ \sim\big\downarrow\varphi & & & & \\ RQY & \xleftarrow[r_{QY}]{\sim} & QY & \xrightarrow[q_Y]{\sim} & Y \end{array}$$

As for the converse implication of (ii), observe that if $\psi : X \to Y$ is a morphism in $\mathtt{M}$ such that the homotopy class $L_\mathtt{M}\psi = [RQ\psi]$ is an isomorphism in $\mathtt{Ho}(\mathtt{M})$, the previous diagram becomes the commutative diagram

$$\begin{array}{ccccc} RQX & \xleftarrow[r_{QX}]{\sim} & QX & \xrightarrow[q_X]{\sim} & X \\ \sim\big\downarrow RQ\psi & & \big\downarrow Q\psi & & \big\downarrow\psi \\ RQY & \xleftarrow[r_{QY}]{\sim} & QY & \xrightarrow[q_Y]{\sim} & Y \end{array}$$

The conclusion now follows from the 2-out-of-3 axiom. □

Proof of Proposition 2 (Continuation). From Lemma 2 it follows that $c_\bullet(\varphi)$ is an isomorphism in $\mathtt{Ho(SSet)}$, i.e.,

$$L_{\mathtt{SSet}}(c_\bullet(\varphi)) = [R_{\mathtt{SSet}}Q_{\mathtt{SSet}}c_\bullet(\varphi)] \in \mathrm{Hom}_{\mathtt{Ho(SSet)}}(c_\bullet(S), c_\bullet(T))$$

is an isomorphism, if and only if $c_\bullet(\varphi)$ is a weak homotopy equivalence between the simplicial sets $c_\bullet(S)$ and $c_\bullet(T)$, i.e., a $\mathtt{SSet}$-morphism $c_\bullet(\varphi) : c_\bullet(S) \to c_\bullet(T)$ whose geometric realization $|c_\bullet(\varphi)| : |c_\bullet(S)| \to |c_\bullet(T)|$ is a weak homotopy equivalence of topological spaces in the standard sense that $|c_\bullet(\varphi)|$ induces isomorphisms

$$\pi_n(|c_\bullet(\varphi)|, x) : \pi_n(|c_\bullet(S)|, x) \to \pi_n(|c_\bullet(T)|, |c_\bullet(\varphi)|(x)) \quad (x \in |c_\bullet(S)|, n \geq 1) \tag{18}$$

between all homotopy groups and a bijection

$$\pi_0(|c_\bullet(\varphi)|) : \pi_0(|c_\bullet(S)|) \to \pi_0(|c_\bullet(T)|) \tag{19}$$

between the sets of path components.

Because the geometric realization $|c_\bullet(S)|$ is homeomorphic to the set S equipped with the discrete topology τ_{dis} [28] and because a discrete space is locally path connected so that its path components coincide with its connected components i.e. with its singletons, we conclude that

$$\pi_0(|c_\bullet(S)|) \cong \pi_0(S, \tau_{\mathrm{dis}}) \cong S\,. \tag{20}$$

On the other hand, the homotopy groups

$$\pi_n(|c_\bullet(S)|, x) \cong \pi_n(S, \tau_{\mathrm{dis}}, x) = \{1\} \tag{21}$$

are all trivial. Indeed, every element is the homotopy class of an n-loop

$$\sigma \in C^0((S^n, s), (S, \tau_{\mathrm{dis}}, x))$$

at x. However, since the image $\sigma(S^n)$ is connected and $\sigma(s) = x$, the loop σ is necessarily the constant loop at x.

The equations (21) and (20) imply that the necessary and sufficient conditions (18) and (19) for $c_\bullet(\varphi)$ to be an isomorphism in $\mathtt{Ho(SSet)}$ reduce to the condition that $\varphi : S \to T$ is a bijection. □

Remark 4. We have just shown that the derived version (7) of finitely presented objects boils down to the classical version (6) even if we replace commutative rings with commutative algebras over differential operators. The characterization 2 can be used as a definition of finitely presented objects in any locally small category with filtered colimits [25].

4.2.3 Explicit description of finitely presented morphisms

We will now work towards an explicit description of finitely presented objects in $\mathcal{A} \downarrow \mathcal{D}\mathtt{A}$.

Let $\mathcal{A} \in \mathcal{D}\mathtt{A}$. We can exclude the case $\mathcal{A} = \{0_{\mathcal{A}}\}$ without loss of generality, so that $1_{\mathcal{A}} \neq 0_{\mathcal{A}}$. Since $\mathcal{O}$ is the initial object in $\mathcal{D}\mathtt{A}$, there exists a unique $\mathcal{D}\mathtt{A}$-morphism

$$\iota : \mathcal{O} \ni f \mapsto f \cdot_{\mathcal{A}} 1_{\mathcal{A}} \in \mathcal{A} \ , \tag{22}$$

where $\cdot_{\mathcal{A}}$ is the $\mathcal{D}$-action on $\mathcal{A}$. This morphism is injective and allows us to view $\mathcal{O}$ as a $\mathcal{D}$-subalgebra of $\mathcal{A}$.

Denote now by $\mathcal{A}[\mathcal{D}] := \mathcal{A} \otimes_{\mathcal{O}} \mathcal{D}$ the ring of linear differential operators with coefficients in $\mathcal{A}$ [32]. It is well-known that

$$\mathtt{Mod}_{\mathcal{D}\mathtt{M}}(\mathcal{A}) \cong \mathtt{Mod}(\mathcal{A}[\mathcal{D}]) =: \mathcal{A}[\mathcal{D}]\mathtt{M} \tag{23}$$

[3] and that

$$\mathcal{A}[\mathcal{D}]\mathtt{A} := \mathtt{CMon}(\mathcal{A}[\mathcal{D}]\mathtt{M}) \cong \mathtt{CMon}(\mathtt{Mod}_{\mathcal{D}\mathtt{M}}(\mathcal{A})) \cong \mathcal{A} \downarrow \mathcal{D}\mathtt{A} \ , \tag{24}$$

where the last isomorphism results roughly speaking from the 'equivalence' of the $\mathcal{A}$-action on the left side and the morphism out of $\mathcal{A}$ on the right side and was described in detail in [9]. In the following we work in $\mathcal{A}[\mathcal{D}]\mathtt{A}$ because this is the most economical setting for our purpose.

Proposition 3. *For any $\mathcal{D}$-algebra $\mathcal{A}$, an $\mathcal{A}[\mathcal{D}]$-algebra $\mathcal{F}$ is an $\mathcal{A}$-algebra and a $\mathcal{D}$-module, such that vector fields act as derivations on the $\mathcal{A}$-action $\triangleleft_{\mathcal{F}}$ and on the algebra multiplication $\star_{\mathcal{F}}$, and such that, for any $f \in \mathcal{O}$ and any $\varphi \in \mathcal{F}$, the actions $\cdot_{\mathcal{F}}$ and $\cdot_{\mathcal{A}}$ by differential operators on $\mathcal{F}$ and $\mathcal{A}$, respectively, and the action $\triangleleft_{\mathcal{F}}$ are compatible in the sense that*

$$f \cdot_{\mathcal{F}} \varphi = (f \cdot_{\mathcal{A}} 1_{\mathcal{A}}) \triangleleft_{\mathcal{F}} \varphi \ . \tag{25}$$

Indeed, in view of (22), there are two actions of $\mathcal{O}$ on $\mathcal{F}$, and the preceding proposition asks that they coincide.

Proof. Starting for instance from the definition of a commutative monoid in $\mathcal{A}$-modules in $\mathcal{D}\mathtt{M}$, one gets that an $\mathcal{A}[\mathcal{D}]$-algebra $\mathcal{F}$ is a $\mathcal{D}$-module with a $\mathcal{D}$-linear $\mathcal{A}$-action $\triangleleft_{\mathcal{F}}$ defined on $\mathcal{A} \otimes_{\mathcal{O}} \mathcal{F}$, and endowed with $\mathcal{A}$- and $\mathcal{D}$-linear multiplication and unit maps $\star_{\mathcal{F}}$ and $\eta_{\mathcal{F}}$ defined on $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{F}$ and $\mathcal{A}$, respectively. One checks that this means exactly that an $\mathcal{A}[\mathcal{D}]$-algebra $\mathcal{F}$ is a $\mathcal{D}$-module and an $\mathcal{A}$-algebra, whose $\mathcal{A}$-action $\triangleleft_{\mathcal{F}}$ and algebra multiplication $\star_{\mathcal{F}}$ are bilinear for the actions by degree zero differential operators $\mathcal{O}$, and are acted upon by vector fields according to the derivation-rule. It is straightforwardly seen that the bilinearity conditions are equivalent to the compatibility condition (25). $\square$

Corollary 1. *For any $\mathcal{D}$-algebra $\mathcal{A}$, an $\mathcal{A}[\mathcal{D}]$-module $\mathcal{M}$ (i.e., a module $\mathcal{M} \in \mathtt{Mod}_{\mathcal{D}\mathtt{M}}(\mathcal{A})$) is a $\mathcal{D}$-module and an $\mathcal{A}$-module, such that vector fields act as derivations on the $\mathcal{A}$-action $\triangleleft_{\mathcal{M}}$ and such that Property (25) is satisfied.*

This corollary can also be checked starting from the definition of an $\mathcal{A}$-module in $\mathcal{D}\mathtt{M}$.

An $\mathcal{A}[\mathcal{D}]$-algebra morphism is an $\mathcal{A}$- and $\mathcal{D}$-linear map that respects the multiplications and the units. The category of $\mathcal{A}[\mathcal{D}]$-algebras is an algebraic category in the sense of [1]. Due to [1, Theorem 3.12.], the definition 2 simplifies and reads:

Definition 12. *An object $\mathcal{B} \in \mathcal{A}[\mathcal{D}]\mathtt{A}$ is* **finitely presented** *if it is the coequalizer of a parallel pair $g, h : \mathcal{F}_1 \rightrightarrows \mathcal{F}_2$ of $\mathcal{A}[\mathcal{D}]\mathtt{A}$-morphisms between two free $\mathcal{A}[\mathcal{D}]$-algebras over two finite sets.*

Free objects over a set in a category $\mathtt{C}$ (or over an object of another category $\mathtt{D}$) generalize vector spaces V with their bases B. Since B is just a set, the inclusion $B \to V$ is just a set-theoretic map. Therefore, in the general situation, we need a forgetful or faithful functor $\Phi : \mathtt{C} \to \mathtt{Set}$, or, more generally, $\Phi : \mathtt{C} \to \mathtt{D}$. A free $\mathtt{C}$-object over a $\mathtt{D}$-object d is then defined in the natural way, i.e., is defined as a pair (c, i) where c is a $\mathtt{C}$-object and $i : d \to \Phi(c)$ is a $\mathtt{D}$-morphism such that for any $\mathtt{C}$-object c' and any $\mathtt{D}$-morphism $i' : d \to \Phi(c')$, there exists a unique $\mathtt{C}$-morphism $f : c \to c'$ such that $\Phi(f) \circ i = i'$. If a forgetful functor $\mathtt{D} \leftarrow \mathtt{C} : \text{For}$ has a left adjoint $\mathcal{F}$, then, for any $d \in \mathtt{D}$, the image $\mathcal{F}(d) \in \mathtt{C}$ is a (the) free $\mathtt{C}$-object over d. The left adjoint $\mathcal{F}$ is the **free functor**. Conversely, if a free $\mathtt{C}$-object exists over all $d \in \mathtt{D}$, the left adjoint $\mathcal{F}$ is defined in the obvious way [27].

The free $\mathcal{D}$-module over $S \in \mathtt{Set}$ is the $\mathcal{D}$-module $\bigoplus_{s \in S} \mathcal{D}\, s$. If S is finite of cardinality k, we denote its elements by $x_1, \ldots, x_k$, so that the free $\mathcal{D}$-module is then $\bigoplus_{\ell=1}^{k} \mathcal{D}\, x_\ell$. We thus get a free-forgetful adjunction

$$\mathcal{F} : \mathtt{Set} \rightleftarrows \mathcal{D}\mathtt{M} : \text{For}_{\mathcal{D}} \ . \tag{26}$$

Moreover, for any $\mathcal{A} \in \mathcal{D}\mathtt{A}$, we have two other free-forgetful adjunctions [9],

$$\mathcal{A} \otimes_{\mathcal{O}} - : \mathcal{D}\mathtt{M} \rightleftarrows \mathtt{Mod}_{\mathcal{D}\mathtt{M}}(\mathcal{A}) : \text{For}_{\mathcal{A}} \tag{27}$$

as well as

$$\mathcal{S}_{\mathcal{A}} : \mathtt{Mod}_{\mathcal{D}\mathtt{M}}(\mathcal{A}) \rightleftarrows \mathcal{A}[\mathcal{D}]\mathtt{A} : \text{For}_{\mathtt{A}} \ , \tag{28}$$

where $\mathcal{S}_{\mathcal{A}}$ is the symmetric tensor product functor (tensor product over $\mathcal{A}$). The composition of adjunctions being an adjunction, we obtain a free-forgetful adjunction

$$\mathcal{S}_{\mathcal{A}} \circ (\mathcal{A} \otimes_{\mathcal{O}} -) \circ \mathcal{F} : \mathtt{Set} \rightleftarrows \mathcal{A}[\mathcal{D}]\mathtt{A} : \mathrm{For}_{\mathcal{D}} \circ \mathrm{For}_{\mathcal{A}} \circ \mathrm{For}_{\mathtt{A}} \ .$$

Hence, the free $\mathcal{A}[\mathcal{D}]$-algebra over a set S is

$$\mathcal{S}_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{O}} (\bigoplus_{s\in S} \mathcal{D}\, s)) \ .$$

Proposition 4. *If $\mathcal{A} \in \mathcal{D}\mathtt{A}$ and $S \in \mathtt{Set}$, the free $\mathcal{A}[\mathcal{D}]$-algebra $\mathcal{F}_S$ over S is*

$$\mathcal{F}_S \cong \mathcal{S}_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{O}} (\bigoplus_{s\in S} \mathcal{D}\, s)) \cong \mathcal{A} \otimes_{\mathcal{O}} \mathcal{S}_{\mathcal{O}}(\bigoplus_{s\in S} \mathcal{D}\, s) \ ,$$

where $\cong$ are isomorphisms in $\mathcal{A}[\mathcal{D}]\mathtt{A}$. In particular, the symmetric tensor algebra functor commutes with the base change functor.

Proof. There are two free-forgetful adjunctions

$$\mathcal{S}_{\mathcal{O}} : \mathcal{D}\mathtt{M} \rightleftarrows \mathcal{D}\mathtt{A} : \mathrm{For}_{\mathtt{A}}$$

and, for any $\mathcal{A} \in \mathcal{D}\mathtt{A}$,

$$\mathcal{A} \otimes_{\mathcal{O}} - : \mathcal{D}\mathtt{A} \rightleftarrows \mathcal{A}[\mathcal{D}]\mathtt{A} : \mathrm{For}_{\mathcal{A}} \ .$$

The second adjunction is just the free-forgetful adjunction $C \amalg - : \mathtt{C} \rightleftarrows C \downarrow \mathtt{C} : \mathrm{For}_C$ between any cocomplete category and any of its undercategories. Indeed, for $\mathtt{C} = \mathcal{D}\mathtt{A}$, the coproduct $\amalg$ is the tensor product $\otimes_{\mathcal{O}}$, so that $\mathcal{A} \amalg - = \mathcal{A} \otimes_{\mathcal{O}} -$. Hence, the free-forgetful adjunction

$$(\mathcal{A} \otimes_{\mathcal{O}} -) \circ \mathcal{S}_{\mathcal{O}} \circ \mathcal{F} : \mathtt{Set} \rightleftarrows \mathcal{A}[\mathcal{D}]\mathtt{A} : \mathrm{For}_{\mathcal{D}} \circ \mathrm{For}_{\mathtt{A}} \circ \mathrm{For}_{\mathcal{A}} \ ,$$

which implies that the free $\mathcal{A}[\mathcal{D}]$-algebra over a set S is

$$\mathcal{A} \otimes_{\mathcal{O}} \mathcal{S}_{\mathcal{O}}(\bigoplus_{s\in S} \mathcal{D}\, s) \ .$$

Since a free object is defined via a universal property, the announced $\mathcal{A}[\mathcal{D}]\mathtt{A}$-isomorphism exists and is unique. □

Let us mention that an $\mathcal{A}[\mathcal{D}]$-ideal $\mathcal{I}$ of an $\mathcal{A}[\mathcal{D}]$-algebra $\mathcal{F}$ is an $\mathcal{A}[\mathcal{D}]$-subalgebra that has the standard ideal-property. A subset $\mathcal{I} \subset \mathcal{F}$ is an $\mathcal{A}[\mathcal{D}]$-subalgebra (see Proposition 3), if it is an $\mathcal{A}$-subalgebra and a $\mathcal{D}$-submodule, i.e., if it is closed under the addition $+_{\mathcal{F}}$, $\mathcal{A}$-action $\triangleleft_{\mathcal{F}}$, multiplication $\star_{\mathcal{F}}$, and $\mathcal{D}$-action $\cdot_{\mathcal{F}}$.

Remark 5. The $\mathcal{A}[\mathcal{D}]$-ideal $\mathcal{I}_\Phi$ of an $\mathcal{A}[\mathcal{D}]$-algebra $\mathcal{F}$, generated by a family $\Phi = (\varphi_i)_{i\in I}$ of elements of $\mathcal{F}$, is given by

$$\mathcal{I}_\Phi = \bigoplus_{i\in I} \mathcal{F} \star_{\mathcal{F}} (\mathcal{D} \cdot_{\mathcal{F}} \varphi_i) ,$$

where the RHS denotes the subset of $\mathcal{F}$ made of the finite sums of elements of the type

$$\varphi \star_{\mathcal{F}} (D \cdot_{\mathcal{F}} \varphi_i) \;\; (\varphi \in \mathcal{F}, D \in \mathcal{D}) .$$

The subset $\mathcal{I}_\Phi$ is actually an $\mathcal{A}[\mathcal{D}]$-ideal. Indeed, obviously, it has the ideal property and is closed under $\star_{\mathcal{F}}$, $+_{\mathcal{F}}$, and $\triangleleft_{\mathcal{F}}$ (since $\star_{\mathcal{F}}$ is $\mathcal{A}$-bilinear). To check stability under the action $\cdot_{\mathcal{F}}$ by differential operators, it suffices to check it for the action by functions f and vector fields θ. Closure under f is clear:

$$f \cdot_{\mathcal{F}} \sum_{\text{fin}} \varphi \star_{\mathcal{F}} (D \cdot_{\mathcal{F}} \varphi_i) = (f \cdot_{\mathcal{A}} 1_{\mathcal{A}}) \triangleleft_{\mathcal{F}} \sum_{\text{fin}} \varphi \star_{\mathcal{F}} (D \cdot_{\mathcal{F}} \varphi_i) = \sum_{\text{fin}} ((f \cdot_{\mathcal{A}} 1_{\mathcal{A}}) \triangleleft_{\mathcal{F}} \varphi) \star_{\mathcal{F}} (D \cdot_{\mathcal{F}} \varphi_i) .$$

Similarly for the closure under θ:

$$\theta \cdot_{\mathcal{F}} \sum_{\text{fin}} \varphi \star_{\mathcal{F}} (D \cdot_{\mathcal{F}} \varphi_i) = \sum_{\text{fin}} (\theta \cdot_{\mathcal{F}} \varphi) \star_{\mathcal{F}} (D \cdot_{\mathcal{F}} \varphi_i) + \sum_{\text{fin}} \varphi \star_{\mathcal{F}} ((\theta \circ D) \cdot_{\mathcal{F}} \varphi_i) .$$

We are now prepared to provide a more explicit description of finitely presented morphisms $\mathcal{A} \to \mathcal{B}$ in $\mathcal{D}\mathtt{A}$. This characterization is part of the description of étale $\mathcal{D}\mathtt{A}$-morphisms (see Definition 8). As shown above, a $\mathcal{D}\mathtt{A}$-morphism $\mathcal{A} \to \mathcal{B}$ or alternatively an object $\mathcal{B} \in \mathcal{A}[\mathcal{D}]\mathtt{A}$ is finitely presented if it satisfies the requirement of Definition 12.

Theorem 1. *A $\mathcal{D}\mathtt{A}$-morphism $\mathcal{A} \to \mathcal{B}$ is finitely presented if and only if $\mathcal{B}$ is the coequalizer of two parallel arrows $g, h : \mathcal{F}_Y \rightrightarrows \mathcal{F}_X$ between two free $\mathcal{A}[\mathcal{D}]$-algebras over finite sets $Y = \{y_1, \ldots, y_m\}$ and $X = \{x_1, \ldots, x_n\}$ $(m, n \in \mathbb{N})$. In other words, the morphism $\mathcal{A} \to \mathcal{B}$ is finitely presented in $\mathcal{D}\mathtt{A}$ if and only if there are two finite sets $Y = \{y_1, \ldots, y_m\}$ and $X = \{x_1, \ldots, x_n\}$ and two $\mathcal{A}[\mathcal{D}]\mathtt{A}$ - arrows $g, h : \mathcal{F}_Y \rightrightarrows \mathcal{F}_X$ such that*

$$\mathcal{B} \cong {}^{\mathcal{F}_X}\!/_{\mathcal{I}_{\Phi_{g,h,Y}}} \tag{29}$$

as $\mathcal{A}[\mathcal{D}]$-algebra, where $\mathcal{I}_{\Phi_{g,h,Y}}$ is the $\mathcal{A}[\mathcal{D}]$-ideal of $\mathcal{F}_X$ that is finitely generated by the family

$$\Phi_{g,h,Y} = (\phi_{g,h,\ell})_{\ell\in\{1,\ldots,m\}} := (g(1_{\mathcal{A}} \otimes y_\ell) - h(1_{\mathcal{A}} \otimes y_\ell))_{\ell\in\{1,\ldots,m\}} .$$

Proof. It suffices to prove that the RHS of (29) is a coequalizer of g and h in $\mathcal{A}[\mathcal{D}]\mathtt{A}$. Recall that

$$\mathcal{F}_X = \mathcal{A} \otimes_{\mathcal{O}} \mathcal{S}_{\mathcal{O}}(\bigoplus_{k=1}^{n} \mathcal{D}\, x_\ell) \quad \text{and} \quad \mathcal{F}_Y = \mathcal{A} \otimes_{\mathcal{O}} \mathcal{S}_{\mathcal{O}}(\bigoplus_{\ell=1}^{m} \mathcal{D}\, y_\ell) ,$$

and that

$$\mathcal{I}_{\phi_{g,h,Y}} = \bigoplus_{\ell=1}^{m} \mathcal{F}_X \star_{\mathcal{F}_X} (\mathcal{D} \cdot_{\mathcal{F}_X} \phi_{g,h,\ell}) \; .$$

Let $\pi : \mathcal{F}_X \to \mathcal{F}_X / \mathcal{I}_\phi$ (we omit the subscripts of ϕ) be the canonical $\mathcal{A}[\mathcal{D}]$-algebra morphism $\varphi \mapsto [\varphi]$.

We first explain that $\pi \circ g = \pi \circ h$. An arbitrary element of the common source $\mathcal{F}_Y$ of these morphisms is a finite sum of decomposable tensors $T = a \otimes v_1 \odot \ldots \odot v_p$ ($a \in \mathcal{A}, v_i \in \mathcal{D}\, y_\ell, \ell \in \{1, \ldots, m\}, p \in \mathbb{N}$). Let q be any of the morphisms g, h. We have

$$q(T) = (a \otimes 1_{\mathcal{O}}) \star_{\mathcal{F}_X} q(1_{\mathcal{A}} \otimes v_1) \star_{\mathcal{F}_X} \ldots \star_{\mathcal{F}_X} q(1_{\mathcal{A}} \otimes v_p) \; ,$$

since $a \otimes 1_{\mathcal{O}} = a \triangleleft_{\mathcal{F}_Y} (1_{\mathcal{A}} \otimes 1_{\mathcal{O}})$, so that its image by q is $a \otimes 1_{\mathcal{O}}$. As

$$1_{\mathcal{A}} \otimes v_i = 1_{\mathcal{A}} \otimes D y_\ell = D \cdot_{\mathcal{F}_Y} (1_{\mathcal{A}} \otimes y_\ell) \; ,$$

with self-explanatory notations, we get $q(1_{\mathcal{A}} \otimes v_i) = D \cdot_{\mathcal{F}_X} q(1_{\mathcal{A}} \otimes y_\ell)$, so that $g(1_{\mathcal{A}} \otimes v_i) - h(1_{\mathcal{A}} \otimes v_i) \in \mathcal{I}_\phi$. However, if $\phi_j - \psi_j \in \mathcal{I}_\phi$ for a finite number of j-s, then $\star_j \phi_j - \star_j \psi_j \in \mathcal{I}_\phi$. Indeed,

$$\phi_1 \star \phi_2 - \psi_1 \star \psi_2 = \phi_1 \star (\phi_2 - \psi_2) + (\phi_1 - \psi_1) \star \psi_2 \in \mathcal{I}_\phi \; .$$

It follows that $g(T) - h(T) \in \mathcal{I}_\phi$, so that $\pi(g(T)) = \pi(h(T))$.

It remains to show that the universal property holds. Hence, consider a second $\mathcal{A}[\mathcal{D}]$-algebra morphism $\rho : \mathcal{F}_X \to \mathcal{C}$ such that $\rho \circ g = \rho \circ h$, and prove that there is a unique $\mathcal{A}[\mathcal{D}]$-algebra map $\sigma : \mathcal{F}_X / \mathcal{I}_\phi \to \mathcal{C}$ such that $\sigma \circ \pi = \rho$. If σ exists, it is necessarily defined by $\sigma[\varphi] = \rho(\varphi)$, which ensures uniqueness. Define now σ by the latter equality. That map is well-defined, as ρ sends $\mathcal{I}_\phi$ to 0. Since π and ρ are $\mathcal{A}[\mathcal{D}]\mathtt{A}$-maps, it is straightforwardly checked that σ is an $\mathcal{A}[\mathcal{D}]\mathtt{A}$-map, i.e., that it is $\mathcal{A}$- and $\mathcal{D}$-linear and respects the multiplications and unities. This completes the proof. □

4.3 Formally étale $\mathcal{D}\mathtt{A}$-morphisms

To complete the description of étale $\mathcal{D}\mathtt{A}$-morphisms (and thus of étale $\mathtt{DG}\mathcal{D}\mathtt{A}$-morphisms), we must still describe formally étale maps in $\mathcal{D}\mathtt{A}$. However, this task is not as straightforward as it may seem at first. Namely, the abstract notion of formally étale maps in an HA context (Definition 17) uses the notion of cotangent complex. As detailed below, in the trivial HA context ($\mathcal{D}\mathtt{M}, \mathcal{D}\mathtt{M}, \mathcal{D}\mathtt{A}$) the latter is merely a $\mathcal{D}$-variant of the module of Kähler differentials. As for algebras over a commutative ring, this is not sufficient to correctly characterize formal étaleness. Instead, one should first embed $\mathcal{D}$-algebras into $\mathtt{DG}\mathcal{D}\mathtt{A}$ as complexes concentrated in degree zero, and then compute the cotangent complex in the non-trivial HA context $(\mathtt{DG}\mathcal{D}\mathtt{M}, \mathtt{DG}\mathcal{D}\mathtt{M}, \mathtt{DG}\mathcal{D}\mathtt{A})$.

Since the model structure of $\mathcal{D}\mathtt{A}$ is trivial, the same holds for the model structure of

$$\mathcal{A}[\mathcal{D}]\mathtt{A} = \mathtt{CMon}(\mathcal{A}[\mathcal{D}]\mathtt{M}) \cong \mathtt{CMon}(\mathtt{Mod}_{\mathcal{D}\mathtt{M}}(\mathcal{A})) \cong \mathcal{A} \downarrow \mathcal{D}\mathtt{A} \quad (\mathcal{A} \in \mathcal{D}\mathtt{A})$$

(see Equation (24)) and the model structure of

$$\mathcal{A}[\mathcal{D}]\mathtt{A} \downarrow \mathcal{B} \cong (\mathcal{A} \downarrow \mathcal{D}\mathtt{A}) \downarrow \mathcal{B} \quad (\mathcal{B} \in \mathcal{A} \downarrow \mathcal{D}\mathtt{A})$$

(for the definition of the induced model structure of a slice or coslice category, see [20]). Further, since the model structure of $\mathcal{D}\mathtt{M}$ is trivial, the same holds for the model structure of

$$\mathcal{A}[\mathcal{D}]\mathtt{M} \cong \mathtt{Mod}_{\mathcal{D}\mathtt{M}}(\mathcal{A}) \quad (\mathcal{A} \in \mathcal{D}\mathtt{A})$$

(see Equation (23); for the definition of this induced model structure, see [9, Theorem 2.2.3]).

Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a $\mathcal{D}\mathtt{A}$-map and let $\mathcal{M} \in \mathcal{B}[\mathcal{D}]\mathtt{M}$. To understand the derived derivations $\mathbb{D}\mathrm{er}_{\mathcal{A}}(\mathcal{B}, \mathcal{M})$ (see Definition 15), remark first that $\mathcal{B} \oplus \mathcal{M}$ is an object in $\mathcal{A}[\mathcal{D}]\mathtt{A} \downarrow \mathcal{B}$ for the following data:

1. $(b, m) + (b', m') := (b + b', m + m')\,,$
2. $D \cdot (b, m) := (D \cdot b, D \cdot m)\,,$
3. $a \triangleleft (b, m) := (\varphi(a) \star b, \varphi(a) \triangleleft m)\,,$
4. $(b, m) \star (b', m') := (b \star b', b \triangleleft m' + b' \triangleleft m)$, and
5. $\pi(b, m) := b\,,$

where $b, b' \in \mathcal{B}$, $m, m' \in \mathcal{M}$, $D \in \mathcal{D}$, $a \in \mathcal{A}$ and where the meaning of the right side operations is obvious.

To prove that the previous claim is true, it suffices to apply Proposition 3 and to show that π is an $\mathcal{A}$- and $\mathcal{D}$-linear map that respects the multiplications and the units. We will not elaborate this proof in detail.

Since, as said above, the model structure of $\mathcal{A}[\mathcal{D}]\mathtt{A} \downarrow \mathcal{B}$ is trivial, the RHS of Equation (43) in Definition 15 is just

$$\mathrm{Map}_{\mathtt{CMon}(\mathtt{Mod}_{\mathcal{D}\mathtt{M}}(\mathcal{A}))\downarrow \mathcal{B}}(\mathcal{B}, \mathcal{B} \oplus \mathcal{M}) = \mathrm{Hom}_{\mathcal{A}[\mathcal{D}]\mathtt{A}\downarrow \mathcal{B}}(\mathcal{B}, \mathcal{B} \oplus \mathcal{M}) \in \mathtt{Set}\;,$$

and we expect the derived derivations $\mathbb{D}\mathrm{er}_{\mathcal{A}}(\mathcal{B},\mathcal{M})$, i.e., the elements of the preceding set, to be just the standard derivations $\mathrm{Der}_{\mathcal{A}[\mathcal{D}]}(\mathcal{B},\mathcal{M})$.

Proposition 5. *For any $\mathcal{D}\mathtt{A}$-morphism $\varphi:\mathcal{A}\to\mathcal{B}$ and any $\mathcal{M}\in\mathcal{B}[\mathcal{D}]\mathtt{M}$, we have*

$$\mathbb{D}er_{\mathcal{A}}(\mathcal{B},\mathcal{M})=\mathrm{Der}_{\mathcal{A}[\mathcal{D}]}(\mathcal{B},\mathcal{M})\ ,$$

where the right side is the set of $\mathcal{A}$- and $\mathcal{D}$-linear maps from $\mathcal{B}$ to $\mathcal{M}$ that satisfy the standard derivation property.

Proof. A morphism $\psi\in\mathrm{Hom}_{\mathcal{A}[\mathcal{D}]\mathtt{A}\downarrow\mathcal{B}}(\mathcal{B},\mathcal{B}\oplus\mathcal{M})$ of the slice category $\mathcal{A}[\mathcal{D}]\mathtt{A}\downarrow\mathcal{B}$ is an $\mathcal{A}$- and $\mathcal{D}$-linear map $\psi:\mathcal{B}\to\mathcal{B}\oplus\mathcal{M}$ that respects the multiplications and the units, and satisfies $\pi\circ\psi=\mathrm{id}_{\mathcal{B}}$. A map $\psi:\mathcal{B}\to\mathcal{B}\oplus\mathcal{M}$ is the same as two maps $\gamma:\mathcal{B}\to\mathcal{B}$ and $\delta:\mathcal{B}\to\mathcal{M}$. The condition $\pi\circ\psi=\mathrm{id}_{\mathcal{B}}$ means that $\gamma=\mathrm{id}_{\mathcal{B}}$. The map ψ is $\mathcal{A}$-linear (resp., $\mathcal{D}$-linear) if and only if the map δ is $\mathcal{A}$-linear (resp., $\mathcal{D}$-linear). Similarly, the map ψ respects the multiplications if and only if δ satisfies the Leibniz rule. Units do not lead to a new condition. This completes the proof. □

In view of Proposition 8 and the triviality of the involved model structure (see above), for $\varphi:\mathcal{A}\to\mathcal{B}$ as usual, the cotangent complex $\mathbb{L}_{\mathcal{B}|\mathcal{A}}\in\mathcal{B}[\mathcal{D}]\mathtt{M}$ and the corresponding derivation $d\in\mathrm{Der}_{\mathcal{A}[\mathcal{D}]}(\mathcal{B},\mathbb{L}_{\mathcal{B}|\mathcal{A}})$ (since here the simplicial set of derived derivations is just a set, the 0-th homotopy group $\pi_0(-)$ can be omitted) are characterized by the fact that, for any $\mathcal{M}\in\mathcal{B}[\mathcal{D}]\mathtt{M}$, the map

$$\mathrm{Hom}_{\mathcal{B}[\mathcal{D}]\mathtt{M}}(\mathbb{L}_{\mathcal{B}|\mathcal{A}},\mathcal{M})\ni\psi\mapsto\psi\circ d\in\mathrm{Der}_{\mathcal{A}[\mathcal{D}]}(\mathcal{B},\mathcal{M}) \tag{30}$$

is a $1:1$ correspondence.

Since, due to Equation (39), the cotangent complex $\mathcal{L}_{\varphi}$ of a $\mathtt{CR}$-map $\varphi:R\to S$ coincides, in the absence of a model structure, with the module $\Omega_{S|R}\in\mathtt{Mod}(S)$ of Kähler differentials, we expect the cotangent complex $\mathbb{L}_{\mathcal{B}|\mathcal{A}}\in\mathcal{B}[\mathcal{D}]\mathtt{M}$ to be the module $\Omega_{\mathcal{B}|\mathcal{A}}\in\mathcal{B}[\mathcal{D}]\mathtt{M}$ of Kähler differentials, and we expect that $\Omega_{\mathcal{B}|\mathcal{A}}$ can be constructed in a similar way as $\Omega_{S|R}$. Below we clarify this idea.

First, remember that

$$(\mathcal{B}\otimes_{\mathcal{O}}-)\circ\mathcal{F}:\mathtt{Set}\rightleftarrows\mathcal{B}[\mathcal{D}]\mathtt{M}:\mathrm{For}_{\mathcal{D}}\circ\mathrm{For}_{\mathcal{B}}$$

is an adjunction (see Equations (26) and (27)), so that the left functor is the free $\mathcal{B}[\mathcal{D}]$-module functor.

Definition 13. *For any* $\mathcal{D}\mathtt{A}$*-morphism* $\varphi : \mathcal{A} \to \mathcal{B}$*, we denote by* $\Omega_{\mathcal{B}|\mathcal{A}} \in \mathcal{B}[\mathcal{D}]\mathtt{M}$ *the free* $\mathcal{B}[\mathcal{D}]$*-module* $\mathcal{B} \otimes_{\mathcal{O}} \left(\bigoplus_{b\in\mathcal{B}} \mathcal{D}\, db\right)$ *over the generating set* $\{db : b \in \mathcal{B}\}$*, modulo the relations*

1. $d(b + b') = db + db'\,,$
2. $d(b \star b') = b \otimes db' + b' \otimes db\,,$
3. $d(a \triangleleft 1_{\mathcal{B}}) = 0\,,$
4. $d(\theta \cdot b) = \theta \cdot db\,,$

where $b, b' \in \mathcal{B}$*,* $a \in \mathcal{A}$ *and* $\theta \in \Theta$*, where* $\Theta \subset \mathcal{D}$ *is the module of vector fields.*

We refer to $\Omega_{\mathcal{B}|\mathcal{A}}$ *as the* $\mathcal{B}[\mathcal{D}]$**-module of Kähler differentials** *of* φ*.*

The chosen presentation makes

$$d : \mathcal{B} \ni b \mapsto db \in \Omega_{\mathcal{B}|\mathcal{A}}$$

an $\mathcal{A}$- and $\mathcal{D}$-linear derivation, i.e., a map $d \in \mathrm{Der}_{\mathcal{A}[\mathcal{D}]}(\mathcal{B}, \Omega_{\mathcal{B}|\mathcal{A}})$ (the second and third relations above are equivalent to $\mathcal{A}$-linearity and the derivation property; the $\mathcal{O}$-linearity follows from the $\mathcal{A}$-linearity due to Equation (25)).

Proposition 6. *The module* $\Omega_{\mathcal{B}|\mathcal{A}}$ *and the derivation* $d : \mathcal{B} \to \Omega_{\mathcal{B}|\mathcal{A}}$ *satisfy the characterizing property* (30)*, i.e.,* $\Omega_{\mathcal{B}|\mathcal{A}}$ *is the cotangent complex of the underlying* $\mathcal{D}\mathtt{A}$*-map* $\varphi : \mathcal{A} \to \mathcal{B}$ *:*

$$\mathbb{L}_{\mathcal{B}|\mathcal{A}} = \Omega_{\mathcal{B}|\mathcal{A}}\ . \tag{31}$$

Proof. Let $\mathcal{M} \in \mathcal{B}[\mathcal{D}]\mathtt{M}$. To show that the map

$$\mathrm{Hom}_{\mathcal{B}[\mathcal{D}]\mathtt{M}}(\Omega_{\mathcal{B}|\mathcal{A}}, \mathcal{M}) \ni \psi \mapsto \psi \circ d \in \mathrm{Der}_{\mathcal{A}[\mathcal{D}]}(\mathcal{B}, \mathcal{M})$$

is bijective, we must prove that, for any $\delta \in \mathrm{Der}_{\mathcal{A}[\mathcal{D}]}(\mathcal{B}, \mathcal{M})$, there is a unique

$$\psi \in \mathrm{Hom}_{\mathcal{B}[\mathcal{D}]\mathtt{M}}(\Omega_{\mathcal{B}|\mathcal{A}}, \mathcal{M})$$

such that

$$\psi \circ d = \delta\ . \tag{32}$$

In order to define a $\mathcal{B}[\mathcal{D}]\mathtt{M}$-map ψ on $\Omega_{\mathcal{B}|\mathcal{A}}$, it suffices to define it on the generators db of the free $\mathcal{B}[\mathcal{D}]$-module used in the construction of $\Omega_{\mathcal{B}|\mathcal{A}}$, and then check that the morphism defined that way descends to the quotient. From this and Condition (32), it follows that if ψ does exist, it is unique. Define now the $\mathcal{B}[\mathcal{D}]\mathtt{M}$-morphism $\psi : \mathcal{B} \otimes_{\mathcal{O}} \left(\bigoplus_{b\in\mathcal{B}} \mathcal{D}\, db\right) \to \mathcal{M}$ by setting $\psi(db) = \delta(b)$. The fact that this morphism ψ descends to the quotient $\Omega_{\mathcal{B}|\mathcal{A}}$ is a direct consequence of the properties of δ. Hence, we defined a map $\psi \in \mathrm{Hom}_{\mathcal{B}[\mathcal{D}]\mathtt{M}}(\Omega_{\mathcal{B}|\mathcal{A}}, \mathcal{M})$ such that $\psi \circ d = \delta$. □

5 Appendix

The appendix renders natural some extensions of standard concepts of linear and commutative algebra to a homotopical algebraic context. In the main text, we adapt these extensions to our setting.

5.1 Cotangent complex of a morphism of commutative rings

Let $\texttt{CR}$ be the category of commutative rings and let $\varphi : R \to S$ be a $\texttt{CR}$-map. The **Kähler differentials** or Kähler differential 1-forms of φ are defined as the universal pair $(\Omega_\varphi, d_\varphi)$ or $(\Omega_{S|R}, d)$, where $\Omega_{S|R}$ is an S-module and where $d : S \to \Omega_{S|R}$ is an R-linear derivation. The universal property reads as follows. For any S-module N and any R-linear derivation $\delta \in \mathrm{Der}_R(S,N)$, there is a unique S-module morphism $s \in \mathrm{Hom}_S(\Omega_{S|R}, N)$ such that $\delta = s \circ d$. In other words, we have an isomorphism of S-modules

$$\mathrm{Der}_R(S,N) \cong \mathrm{Hom}_S(\Omega_{S|R}, N)\ . \tag{33}$$

Hence the functor $\mathrm{Der}_R(S,-)$ is corepresentable with corepresenting object Ω_φ.

Let

$$Q \xrightarrow{\psi} R \xrightarrow{\varphi} S$$

be $\texttt{CR}$-morphisms. In view of the universal property, there is a sequence of S-modules,

$$\Omega_\psi \otimes_R S \to \Omega_{\varphi\psi} \to \Omega_\varphi \to 0\ , \tag{34}$$

where the base change is needed to get an S-module. It is easily seen that the sequence is exact. We refer to it as the **Jacobi-Zariski exact sequence** associated to φ and ψ.

Roughly speaking, the sequence (34) evokes a sequence resulting from the application of some right exact functor, say $\Omega_{-|R} \otimes_- S$. Hence the question whether there is a left derived functor of the Kähler differentials functor $\Omega_{-|R} \otimes_- S$. The target category of $\Omega_{-|R} \otimes_- S$ is the category $\texttt{Mod}(S)$ of S-modules and the source category is the category $\texttt{Alg}(R) \downarrow S$ of commutative associative unital R-algebras over S. However, since the source category is non-abelian, the concepts of right exact functor $\Omega_{-|R} \otimes_- S$ and left derived functor of $\Omega_{-|R} \otimes_- S$ are meaningless.

To compute a left derived functor of a right exact covariant functor, in particular $L_i(\Omega_{-|R} \otimes_- S)(S)$ $(i \geq 0)$ – in case it were possible, we would use a projective resolution $P_\bullet$ of S. Setting

$$\mathcal{L}_\varphi := \Omega_{P_\bullet|R} \otimes_{P_\bullet} S\ , \tag{35}$$

we would get

$$L_i(\Omega_{-|R} \otimes_{-} S)(S) = H_i(\mathcal{L}_\varphi) \;, \tag{36}$$

in particular,

$$H_0(\mathcal{L}_\varphi) = \Omega_{S|R} \;. \tag{37}$$

What has just been said can be done rigorously. Indeed, derived functors can be computed, not only in abelian categories, but also in model categories. The model categorical counterpart of projective resolutions are cofibrant replacements. It turns out that the closed simplicial model category $\texttt{SAlg}(R)$ of simplicial R-algebras and the overcategory $\texttt{SAlg}(R) \downarrow S$ are well-suited for this purpose.

The underlying $\texttt{CR}$-map $\varphi : R \to S$ can be interpreted as an $\texttt{SAlg}(R)$-map, and a 'cofibration – trivial fibration' decomposition

$$R \rightarrowtail Q_\bullet(S) \stackrel{\sim}{\twoheadrightarrow} S$$

can be chosen, so that $Q_\bullet(S) \in \texttt{SAlg}(R)$ is a cofibrant replacement of S (we can actually choose a replacement $Q_\bullet(S) = R[X]$ that is free over R). When applying the Kähler differentials functor, we obtain the cotangent complex:

Definition 14. *Let $\varphi : R \to S$ be a $\texttt{CR}$-map and let $Q_\bullet := Q_\bullet(S)$ be a cofibrant replacement of S in $\texttt{SAlg}(R)$. The* **cotangent complex** *of φ is the simplicial S-module*

$$\mathcal{L}_\varphi = \Omega_{Q_\bullet|R} \otimes_{Q_\bullet} S \in \texttt{SMod}(S) \simeq \texttt{DG}_+\texttt{Mod}(S) \;. \tag{38}$$

Equation (38) is the mathematically correct version of Equation (35).

If we view the Kähler differentials functor as the functor

$$\Omega_{-|R} \otimes_{-} S : \texttt{SAlg}(R) \downarrow S \to \texttt{DG}_+\texttt{Mod}(S)$$

between model categories, we can consider its derived functor

$$\mathbb{L}(\Omega_{-|R} \otimes_{-} S) : \texttt{Ho}(\texttt{SAlg}(R) \downarrow S) \to \texttt{Ho}(\texttt{DG}_+\texttt{Mod}(S)) \;,$$

which we refer to as the **cotangent complex functor** of $R \to S$. Indeed, its value at S can be computed via a cofibrant replacement $Q_\bullet := Q_\bullet(S)$ in $\texttt{SAlg}(R)$ (hence, in $\texttt{SAlg}(R) \downarrow S$):

$$\mathbb{L}(\Omega_{-|R} \otimes_{-} S)(S) = \Omega_{Q_\bullet|R} \otimes_{Q_\bullet} S = \mathcal{L}_\varphi \in \texttt{Ho}(\texttt{DG}_+\texttt{Mod}(S)) \;. \tag{39}$$

Equation (39) is the correct variant of Equation (36). From here it follows that the cotangent complex does not depend on the choice of the cofibrant replacement, if we view it as an object of the homotopy category of the model category $\mathtt{DG_+Mod}(S)$.

Recall that, for any $M \in \mathtt{Mod}(S)$, the S-modules

$$D_n(S|R, M) := H_n(\mathcal{L}_\varphi \otimes_S M) \quad (n \in \mathbb{N})$$

(in particular, the S-modules

$$D_n(S|R, S) := H_n(\mathcal{L}_\varphi) \quad (n \in \mathbb{N})\ )$$

are the **André-Quillen homology modules** of $\varphi : R \to S$ with coefficients in M (resp., in S). It can be checked that, as expected,

$$H_0(\mathcal{L}_\varphi) = H_0(\mathbb{L}(\Omega_{-|R} \otimes_- S)(S)) = \Omega_{S|R} = \Omega_\varphi \tag{40}$$

(see Equation (37)).

Remark 6. Equation (40) shows that the concept of cotangent complex is the derived counterpart of the notion of Kähler differentials.

5.2 Cotangent complex in a Homotopical Algebraic Context

To extend the content of Subsection 5.1 to any homotopical algebraic context (HAC) $(\mathtt{C}, \mathtt{C_0}, \mathtt{A})$ (see above), we rely on an additional observation.

Let $\varphi : R \to S$ be a $\mathtt{CR}$-map and let $N \in \mathtt{Mod}(S)$. Equation (33) characterizes derivations $\mathrm{Der}_R(S, N)$ as morphisms in $\mathtt{Mod}(S)$. The next proposition characterizes these derivations as morphisms in $\mathtt{Alg}(R) \downarrow S$.

Proposition 7. *Let $\varphi : R \to S$ and N be as above. We denote by $S \oplus N$ the obvious R-module endowed with the multiplication*

$$(s, n) \star (s', n') = (ss', s \triangleleft n' + s' \triangleleft n)\ , \tag{41}$$

where juxtaposition (resp., $\triangleleft$) denotes the multiplication in S (resp., the S-action in N). The multiplication $\star$ allows one to encode the derivation property as algebra-morphism property: there is a 1:1 correspondence

$$\mathrm{Der}_R(S, N) \cong \mathrm{Hom}_{\mathtt{Alg}(R)\downarrow S}(S, S \oplus N)\ . \tag{42}$$

The multiplication $\star$ is very natural from various perspectives. One of them consists in thinking about the proof of Equation (42).

Proof. Notice first that a morphism $\alpha : S \to S \oplus N$ in the slice category $\mathtt{Alg}(R) \downarrow S$ is a morphism of the type $\alpha : s \mapsto (s, \delta(s))$, with $\delta : S \to N$. Since $\alpha(ss') = (ss', \delta(ss'))$ and $\alpha(s) \star \alpha(s') = (s, \delta(s)) \star (s', \delta(s'))$, the algebra-morphism property of α and the derivation property of δ are equivalent if and only if $\star$ is defined by Equation (41). □

The definition of $S \oplus N$ has a categorical–homotopical counterpart in an arbitrary HAC $(\mathtt{C}, \mathtt{C}_0, \mathtt{A})$ (see [38, Section 1.2.1]). Proposition 7 serves then as definition for derivations (see [38, Definition 1.2.1.1]):

Definition 15. *Let $\varphi : A \to B$ be a morphism in $\mathtt{CMon}(\mathtt{C})$ and let $N \in \mathtt{Mod}_{\mathtt{C}}(B)$.* **Derived A-derivations** $\mathbb{D}\mathrm{er}_A(B, N)$ **from** B **to** N *is the simplicial set*

$$\mathbb{D}\mathrm{er}_A(B, N) := \mathrm{Map}_{\mathtt{CMon}(\mathtt{Mod}_{\mathtt{C}}(A))\downarrow B}(B, B \oplus N) \in \mathtt{Ho}(\mathtt{SSet}) \;, \tag{43}$$

where Map *denotes the mapping space in the model category* $\mathtt{CMon}(\mathtt{Mod}_{\mathtt{C}}(A)) \downarrow B$ *.*

We thus get a functor $\mathbb{D}\mathrm{er}_A(B, -) : \mathtt{Ho}(\mathtt{Mod}_{\mathtt{C}}(B)) \to \mathtt{Ho}(\mathtt{SSet})$ that can be lifted to a functor

$$\mathbb{D}\mathrm{er}_A(B, -) \in \mathtt{Fun}(\mathtt{Mod}_{\mathtt{C}}(B), \mathtt{SSet}) \;. \tag{44}$$

The symbol $\check{}$ means that we consider the functor category, which is naturally equipped with the object-wise model structure induced by the model structure of $\mathtt{SSet}$, as endowed with the left Bousfield localization of the object-wise model structure by the weak equivalences of the model structure of $\mathtt{Mod}_{\mathtt{C}}(B)^{\mathrm{op}}$. Of course, we view these weak equivalences as morphisms in $\mathtt{Fun}(\mathtt{Mod}_{\mathtt{C}}(B), \mathtt{SSet})$, using the simplicial Yoneda embedding

$$\underline{\mathrm{Spec}} : \mathtt{Mod}_{\mathtt{C}}(B)^{\mathrm{op}} \ni M \mapsto \underline{\mathrm{Hom}}(M, -) \in \mathtt{Fun}(\mathtt{Mod}_{\mathtt{C}}(B), \mathtt{SSet})\check{} \;.$$

Here

$$\underline{\mathrm{Hom}}(M, -) = \mathrm{Hom}(M, R_\bullet(c_\bullet(-))) \;,$$

where $c_\bullet$ is the constant simplicial object functor from $\mathtt{Mod}_{\mathtt{C}}(B)$ to $\mathtt{Fun}(\Delta^{\mathrm{op}}, \mathtt{Mod}_{\mathtt{C}}(B))$ and where $R_\bullet$ is a fibrant replacement functor in the Reedy model category $\mathtt{Fun}(\Delta^{\mathrm{op}}, \mathtt{Mod}_{\mathtt{C}}(B))$ of simplicial objects in $\mathtt{Mod}_{\mathtt{C}}(B)$ (see [37, Section 4.2]. Since

$$\mathbb{R}\underline{\mathrm{Spec}} : \mathtt{Ho}(\mathtt{Mod}_{\mathtt{C}}(B))^{\mathrm{op}} \to \mathtt{Ho}(\mathtt{Fun}(\mathtt{Mod}_{\mathtt{C}}(B), \mathtt{SSet})\check{}) \;,$$

we find in view of Equation (2) that

$$\mathbb{R}\underline{\mathrm{Spec}}(M)(-) = \mathrm{Map}_{\mathtt{Mod}_{\mathtt{C}}(B)}(M, -) \in \mathtt{Fun}(\mathtt{Mod}_{\mathtt{C}}(B), \mathtt{SSet}) \;, \tag{45}$$

for every $M \in \mathtt{Mod}_{\mathtt{C}}(B)$.

The next result generalizes Equation (33) (see Equations (44) and (45) and see [38, Proposition 1.2.1.2]).

Proposition 8. *Let $\varphi : A \to B$ be a morphism in $\mathtt{CMon}(\mathtt{C})$. There is an object $\mathbb{L}_{B|A} \in \mathtt{Mod}_{\mathtt{C}}(B)$ and an element $d \in \pi_0(\mathbb{D}\mathrm{er}_A(B, \mathbb{L}_{B|A}))$, such that the induced natural transformation*

$$- \circ d : \mathrm{Map}_{\mathtt{Mod}_{\mathtt{C}}(B)}(\mathbb{L}_{B|A}, -) \to \mathbb{D}\mathrm{er}_A(B, -)$$

has components

$$(- \circ d)_N : \mathrm{Map}_{\mathtt{Mod}_{\mathtt{C}}(B)}(\mathbb{L}_{B|A}, N) \to \mathbb{D}\mathrm{er}_A(B, N) \quad (N \in \mathtt{Mod}_{\mathtt{C}}(B))$$

that are isomorphisms in $\mathtt{Ho}(\mathtt{SSet})$.

In view of Equation (33) and Remark 6, it is natural to give the following

Definition 16. *Let $\varphi : A \to B$ be a morphism in $\mathtt{CMon}(\mathtt{C})$. We refer to the corepresenting object $\mathbb{L}_{B|A} \in \mathtt{Ho}(\mathtt{Mod}_{\mathtt{C}}(B))$ of the derivations functor $\mathbb{D}\mathrm{er}_A(B, -)$ as the* **cotangent complex** *of φ (or of B over A). In the case $A = \mathbf{1}$, where $\mathbf{1}$ is the unit of the symmetric monoidal category $\mathtt{C}$, we set $\mathbb{L}_B := \mathbb{L}_{B|\mathbf{1}}$ and refer to this module as the cotangent complex of B.*

As the unit $\mathbf{1}$ of $\mathtt{C}$ is the initial object of $\mathtt{CMon}(\mathtt{C})$, Proposition 1.2.1.6 (1) in [38] implies that, for any $\mathtt{CMon}(\mathtt{C})$-morphism $A \to B$, the sequence

$$\mathbb{L}_A \otimes^{\mathbb{L}}_A B \to \mathbb{L}_B \to \mathbb{L}_{B|A} \tag{46}$$

is a homotopy cokernel sequence in $\mathtt{Mod}_{\mathtt{C}}(B)$ (this generalizes the Jacobi-Zariski right exact sequence).

5.3 Formally étale morphisms in a Homotopical Algebraic Context

The notion of étale morphism of schemes is the algebraic analogue of the notion of local diffeomorphism of smooth manifolds. A morphism of schemes is étale if and only if it is locally of finite presentation and formally étale. The definition of a formally étale morphism of schemes is dual to that of a formally étale morphism of *commutative rings* and consists of a lifting property of a local diffeomorphism.

Let k be a commutative $\mathbb{Q}$-algebra. A morphism $\varphi : R \to S$ of discrete *commutative k-algebras* is formally étale, if $\tau_{\leq 1}\mathcal{L}_{S|R} \cong 0$, where τ stands for truncation. It can be shown that this definition coincides with the usual definition via the lifting property mentioned in the previous paragraph. That motivates the definition saying that a $\mathtt{DG}k\mathtt{A}$-morphism $\varphi : A \to B$ is (derived) formally étale, if $\mathbb{L}_{B|A} \cong 0$ [29]. In view of the homotopy cokernel sequence

$$\mathbb{L}_A \otimes_A^{\mathbb{L}} B \to \mathbb{L}_B \to \mathbb{L}_{B|A}$$

(see Equation (46)), it is now natural to give the

Definition 17. *In a homotopical algebraic context* $(\mathtt{C}, \mathtt{C}_0, \mathtt{A})$ *, a* $\mathtt{CMon}(\mathtt{C})$*-morphism* $\varphi : A \to B$ *is* **formally étale***, if the morphism*

$$\mathbb{L}_A \otimes_A^{\mathbb{L}} B \to \mathbb{L}_B$$

is an isomorphism in $\mathtt{Ho}(\mathtt{Mod}_{\mathtt{C}}(B))$*.*

This definition holds in particular in $\mathtt{DG}\mathcal{D}\mathtt{A}$, i.e., for $\mathtt{C} = \mathtt{DG}\mathcal{D}\mathtt{M}$, and in $\mathcal{D}\mathtt{A}$, i.e., for $\mathtt{C} = \mathcal{D}\mathtt{M}$.

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email: alisa.govzmann@uni.lu; *email:* damjan.pistalo@uni.lu; *email:* norbert.poncin@uni.lu.