One-Dimensional Quantum Systems with Ground State of Jastrow Form Are Integrable

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(Received 25 January 2022; accepted 7 September 2022; published 3 October 2022)

Exchange operator formalism describes many-body integrable systems using phase-space variables involving an exchange operator that acts on any pair of particles. We establish an equivalence between models described by exchange operator formalism and the complete infinite family of parent Hamiltonians describing quantum many-body models with ground states of Jastrow form. This makes it possible to identify the invariants of motion for any model in the family and establish its integrability, even in the presence of an external potential. Using this construction we establish the integrability of the long-range Lieb-Liniger model, describing bosons in a harmonic trap and subject to contact and Coulomb interactions in one dimension. We further identify a variety of models exemplifying the integrability of Hamiltonians in this family.

DOI: 10.1103/PhysRevLett.129.150601

Integrability in both classical and quantum many-body systems is associated with the existence of conserved quantities. At the quantum level, the latter correspond to operators that commute with the system Hamiltonian and govern the nonequilibrium dynamics and thermalization of a system in isolation [1,2]. Several integrable models have been realized in the laboratory, prompting their use as a test bed for quantum many-body physics, statistical mechanics, and nonequilibrium phenomena [3,4].

The integrability of a system may be proven by finding the set of conserved quantities. In one spatial dimension, this is possible in systems that are exactly solved using Bethe ansatz, which posits that the wave function of any quantum eigenstate admits an expansion in terms of plane waves with suitable coefficients and quasimomenta. The latter set the integrals of motion, are also known as the Bethe roots or rapidities, and serve as "good" quantum numbers [5,6]. An alternative framework is the exchange operator formalism (EOF) [7,8], in which the Hamiltonian of the quantum system admits a decoupled form in terms of generalized momenta, which readily allows for the identification of integrals of motion. This approach can be applied to the study of excited states, as demonstrated in systems with inverse-square interactions [9,10]. An encompassing notion of quantum integrability relies on scattering without diffraction, encoded in the Yang-Baxter equation [11–13], when collisions between particles can be described exclusively as a sequence of two-body scattering events. The system is then solvable by algebraic Bethe ansatz, i.e., using the quantum inverse scattering method. Integrals of motion can be derived from the transfer matrix [13] or invoking the asymptotic Bethe ansatz [14,15]. While a definite notion of quantum integrability remains under debate, many of these approaches are closely interrelated [6,16]. In particular, EOF is related to the Yang-Baxter equation and asymptotic Bethe ansatz [17,18].

An important class of quantum systems is characterized by a ground state of (Bijl-Dingle-) Jastrow form, in which the wave function is simply the pairwise product of a pair function [19–21]. This facilitates the computation of correlation functions in these systems [14]. The family of parent Hamiltonians with Jastrow wave functions (PHJ, for short) can be determined by solving an inverse problem: by acting with the kinetic energy operator in the ground-state wave function, one can recast the resulting terms in the form of a many-body Schrödinger equation, thus identifying the parent Hamiltonian. This approach has its roots in the early works by Calogero and Sutherland [22–24]. It has been extended in a number of ways [25,26] and by now, for identical particles without internal degrees of freedom, the complete family of PHJ is known both in one and higher spatial dimensions, provided that the ground-state wave function includes at most the product of one-particle and two-particle functions [27,28]. The corresponding Hamiltonians generally contain two-body and three-body interactions. It was shown by Kane *et al.* [29] that the three-body contribution does not affect the low-energy physics. Further, the conditions for the three-body term to vanish or reduce to a constant have been long established in the homogeneous case, in the absence of an external potential [14,30,31].

Paradigmatic instances of PHJ are integrable. Hard-core bosons in the Tonks-Girardeau regime, realized in the laboratory with ultracold gases [32,33], have ground state of Jastrow form [34–36] and are integrable, being related to noninteracting fermions via the Bose-Fermi duality [3,34,37]. The Calogero-Sutherland model with a Jastrow ground state has a harmonic spectrum, it can be mapped to a set of independent harmonic oscillators [38–40], and satisfies the asymptotic Bethe ansatz [14,15]. Similarly, the attractive Lieb-Liniger (LL) model of bosons subject to contact interactions, used to describe ultracold gases in tight waveguides [41,42], has a bright quantum soliton as Jastrow ground state [43]. This system is solvable by coordinate Bethe ansatz, which yields the Bethe roots as integrals of motion [5,6,44,45].

One may thus wonder the extent to which the ground-state correlations can determine the complete integrability of the system, and what are the required conditions for this to be the case. In this Letter, we show that the complete family of one-dimensional many-body quantum models with ground state of Jastrow form is integrable. To this end, we first establish the equivalence between this family and models described by EOF. In doing so, we identify explicitly the integrals of motion. Our construction holds in the presence of an external potential, which allows us to show the integrability of the long-range Lieb-Liniger model, describing bosons confined in a harmonic trap and subject to both contact and Coulomb interactions in one spatial dimension [27,46].

Systems described by EOF.—Consider the family of one-dimensional systems of identical particles without internal degrees of freedom. It will prove useful to consider those models subject to pairwise interactions that are possibly supplemented with three-body interactions. In this context, EOF is a powerful framework due to Polychronakos that explicitly exhibits the integrability of a many-body quantum system in one spatial dimension [7,8]. Its application has been particularly fruitful in Calogero-Sutherland-Moser systems involving two-body inverse-square interactions [14,22,23,25,47,48], as discussed in [7,8].

Let M_{ij} denote the exchange operator, which exchanges the positions of two particles labeled by i and j, respectively. This operator is Hermitian, idempotent $M_{ij}^2 = \mathbb{I}$ and symmetric with respect to the indices, i.e., $M_{ij} = M_{ji}$. For any one-body operator $A_j \equiv A(x_j)$, it obeys the relations $M_{ij}A_j = A_i M_{ij}$ and $M_{ij}A_k = A_k M_{ij}$ for distinct i, j, k [7,8,49]. Note that when for spinless identical particles, M_{ij} can be identified with the permutation of two particles. In terms of the canonical position and momentum coordinates, x_i and $p_j = -\mathrm{i}\hbar\partial/\partial x_j$, one can introduce the generalized momenta

$$\pi_i = p_i + \mathrm{i}\sum_{j\neq i} V_{ij} M_{ij}, \tag{1}$$

for particles $j = 1, \ldots, N$. The so-called prepotential function $V_{ij} = V(x_i - x_j)$ should be antisymmetric (i.e., $V_{ij} = -V_{ji}$) to guarantee the Hermiticity of the generalized momenta. Using the latter, one can construct permutation-invariant quantities $I_n \equiv \sum_i \pi_i^n$. In particular, I_2 is quadratic in p_i's, and resembles the Hamiltonian of many-body systems. To describe states of N particles, consider the tensor product of the single-particle Hilbert space $\mathcal{H}$, i.e., $\mathcal{H}^{\otimes N}$. For indistinguishable particles, states are restricted to the bosonic or fermionic subspaces of $\mathcal{H}^{\otimes N}$, denoted as $\mathcal{H}_\zeta$ with $\zeta = +1$ for spinless bosons and $\zeta = -1$ for spinless fermions. We define the projector $\mathcal{P}_\zeta$ onto $\mathcal{H}_\zeta$ as [51] $\mathcal{P}_\zeta \psi(x_1, x_2, \ldots, x_N) = (1/N!)\sum_\sigma \zeta^\sigma \psi(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_N})$, where σ denotes a permutation of the tuple $(1, 2, \ldots, N)$. Projecting I_2 onto the subspace $\mathcal{H}_\zeta$ and using

$$M_{ij}\mathcal{P}_\zeta = \mathcal{P}_\zeta M_{ij} = \zeta \mathcal{P}_\zeta, \tag{2}$$

we obtain $\mathcal{P}_\zeta I_2 \mathcal{P}_\zeta/(2m) = \mathcal{P}_\zeta H_0 \mathcal{P}_\zeta$, where H_0 is the translation-invariant quantum many-body Hamiltonian defined as follows

$$H_0 = \sum_i \frac{p_i^2}{2m} + \frac{1}{m}\left[\sum_{i<j}(\zeta\hbar V'_{ij} + V_{ij}^2) - \sum_{i<j<k} V_{ijk}\right], \tag{3}$$

where $V_{ijk} = V_{ij}V_{jk} + V_{jk}V_{ki} + V_{ki}V_{ij}$ is fully symmetric and a prime denotes the spatial derivative. The form of H_0 will play an important role in proving the integrability of the family of Hamiltonians generated by EOF and PHJ. Specific choices of the prepotential function $V(x)$ gives rise to well-known models. For $V(x) = \lambda/x$, V_{ijk} vanishes by permutation symmetry and one recovers the Hamiltonian of identical particles with inverse-square interactions [14,22]. For $V(x) = \lambda\cot(ax)$, V_{ijk} is a constant and H_0 involves the inverse sine square potentials. The case $V(x) = c\,\mathrm{sgn}(x)$, corresponding V_{ijk} being a negative constant, gives rise to the celebrated Lieb-Liniger (LL) model [44,45] describing ultracold gases in tight waveguides [3,41]. For all these cases where V_{ijk} vanishes or is a constant, I_n commute with each other. As the system Hamiltonian coincides with I_2 on the bosonic or fermionic sector, the set of I_n can be identified as invariants of motion, i.e., $[I_n, I_m] = 0$. We note that all the models that have been shown to be integrable by the EOF in Ref. [7] happen to have a ground-state wave function of Jastrow form, which we discuss next.

Parent Hamiltonians with Jastrow ground state.—Consider a homogenous one-dimensional many-body quantum system described by a ground state of Jastrow form [19–21],

$$\Phi_0(x_1, \ldots, x_N) = \prod_{i<j} f_{ij}, \tag{4}$$

this is, the pairwise product of the pair function $f_{ij} = f(x_i - x_j)$ [14]. In the "beautiful models" [14] that concern us here, quantum statistics is encoded in the symmetry of $f(x)$ which is an even function for bosons and odd for fermions, i.e., without resorting to the use of permanents or determinants. The case of one dimensional anyons can similarly be taken into account by including a phase factor

θ, i.e., $f(x) = e^{-i\theta \mathrm{sgn}(x)} f(-x)$ [52–54]. The complete family of PHJ of ground state of the Jastrow form (4) has been identified in one spatial dimension [27] and includes paradigmatic models such as the LL gas with contact interactions [44,45] and the rational Calogero-Sutherland model with inverse-square interactions [22,23], as well as the recently introduced long-range LL model [46]. For a given choice of f, the parent Hamiltonian H_0 takes the form

$$H_0 = \sum_i \frac{p_i^2}{2m} + \frac{\hbar^2}{m}\left[\sum_{i<j}\frac{f''_{ij}}{f_{ij}} + \sum_{i<j<k}\left(\frac{f'_{ij}f'_{ik}}{f_{ij}f_{ik}} - \frac{f'_{ij}f'_{jk}}{f_{ij}f_{jk}} + \frac{f'_{ik}f'_{jk}}{f_{ik}f_{jk}}\right)\right]. \quad (5)$$

Here, f' and f'' denote the first and second spatial derivatives of f, respectively. The explicit expressions for this Hamiltonian directly follow from evaluating the Laplacian on the Jastrow wave function (4) and recasting all resulting terms in the form of a Schrödinger equation.

Equivalence of EOF and PHJ for spinless indistinguishable particles.—We now establish the correspondence between EOF and PHJ for spinless identical particles. Comparing the EOF Hamiltonian and the PHJ in Eqs. (3) and (5), the two-body terms are equal if $\hbar^2 f''(x_{ij})/f(x_{ij}) = \zeta\hbar V'_{ij} + V_{ij}^2$. Thus, the prepotential reads

$$V_{ij} = \zeta\hbar \frac{d}{dx_{ij}} \log(f_{ij}) = \zeta\hbar \frac{f'_{ij}}{f_{ij}}. \quad (6)$$

Independently of whether the pair function is symmetric or antisymmetric, its logarithmic derivative is guaranteed to be odd $f'_{ij}/f_{ij} = -f'_{ji}/f_{ji}$. Thus, this property holds for spinless bosons and fermions. The antisymmetry of the prepotential in Eq. (6) guarantees the Hermiticity condition of the associated generalized momenta in the EOF,

$$\pi_i = p_i + i\zeta\hbar \sum_{j\neq i} \frac{f'_{ij}}{f_{ij}} M_{ij}. \quad (7)$$

The prepotential in Eq. (6) further ensures the equivalence of the three-body interaction in the EOF and the PHJ. Thus, any spinless system described by EOF, as in Eq. (3), has a ground state of Jastrow form with a pair function $f_{ij} = \exp[\int^{x_{ij}} dy\, V(y)/(\zeta\hbar)]$. Conversely, the complete infinite family of PHJ can be recast in the EOF provided (6) is satisfied. This makes it possible to identify the class of PHJ that is integrable as we shall see later.

Embedding in an external potential.—The embedding of a system described by EOF in an external potential is known in the case of a harmonic trap [7]. For the embedding of a homogenous system in an arbitrary trapping potential, we draw inspiration from supersymmetric quantum mechanics [55] and introduce the one-body superpotential $W_i \equiv W(x_i)$ in terms of which the external trapping potential U_i will be identified. We define the operators

$$a_i = \frac{\pi_i}{\sqrt{2m}} - iW_i, \qquad a_i^\dagger = \frac{\pi_i}{\sqrt{2m}} + iW_i, \quad (8)$$

and the permutation-invariant quantities $\tilde{I}_n \equiv \sum_i h_i^n$, where $h_i \equiv a_i^\dagger a_i$. Projecting $\tilde{I}_1$ onto $\mathcal{H}_\zeta$, we find $\mathcal{P}_\zeta \tilde{I}_1 \mathcal{P}_\zeta = \mathcal{P}_\zeta H \mathcal{P}_\zeta$, where H is the Hamiltonian of the system in the presence of the trap, i.e.,

$$H = H_0 + \sum_i U_i - \zeta\sqrt{\frac{2}{m}} \sum_{i<j} V_{ij}(W_i - W_j), \quad (9)$$

with the external potential U_i being determined by the Riccati equation

$$U_i = W_i^2 - \frac{\hbar}{\sqrt{2m}} W'_i. \quad (10)$$

As a familiar example, when H_0 is the homogeneous Calogero model with inverse-square interactions [22] and $W_i = \sqrt{m/2}\,\omega x_i$, Eq. (9) reduces to the rational Calogero-Sutherland model [23,47] including a harmonic trap.

In PHJ, the ground-state wave functions is not limited to the homogeneous form (4), but also includes more general ground states

$$\Psi_0 = \prod_{i<j} f_{ij} \prod_i \exp(v_i) = \Phi_0 \prod_i \exp(v_i), \quad (11)$$

where the one-body function $v_i = v(x_i)$ accounts for the role of the external potential $U_i = U(x_i)$ that breaks translational invariance [27]. Specifically, if H_0 is the parent Hamiltonian of Φ_0 in Eq. (4), then Ψ_0 has the parent Hamiltonian

$$H = H_0 + \sum_i U_i + \frac{\hbar^2}{m} \sum_{i<j} (v'_i - v'_j) \frac{f'_{ij}}{f_{ij}}, \quad (12)$$

with the one-body local external potential U_i given in terms of the function v_i by

$$U_i = \frac{\hbar^2}{2m} [(v'_i)^2 + v''_i]. \quad (13)$$

As a result, the Hamiltonian H includes the external potential U_i and an additional pairwise (two-body) potential which is generally of long-range character.

The equivalence between EOF and PHJ require that the one-body and potential and the additional long-range term are equal in both representations. Comparing Eqs. (10) and (13), the superpotential W_i and the function v_i entering the one-body function of the Jastrow form are related by

$$W_i = -\frac{\hbar}{\sqrt{2m}} v_i'. \tag{14}$$

Upon substituting Eq. (14) into Eqs. (9) and (12), we find that the additional long-range potentials coincide, given the correspondence Eq. (6) is identified. The ground state of the Hamiltonian with the external potential in terms of the prepotential and the superpotential is

$$\Psi_0 = \exp\left(-\frac{\sqrt{2m}}{\hbar}\sum_i \int^{x_i} W(y)dy\right)\prod_{i<j}\exp\left[\frac{\int^{x_{ij}} dyV(y)}{\zeta\hbar}\right]. \tag{15}$$

This establishes the equivalence between EOF and PHJ in the presence of external potential.

Integrability via projection formalism.—For quantum systems with classical analog, as the PHJ, one can define quantum integrability by promoting the Poisson bracket into commutators in the definition of classical integrability. Polychronakos [7] pursued along this line and showed that $I_n \equiv \sum_i \pi_i^n$ become integrals of motion, i.e., $[I_n, I_m] = 0$, $\forall\ n, m$, in the restricted case in which V_{ijk} vanishes or is a constant. Having shown that any spinless model described by EOF is a PHJ with a Jastrow ground state, we next establish the integrability of the complete family of PHJ models, i.e., without restrictions on the three-body potential V_{ijk} or the external potential U_i.

Note that any physical observable $\mathcal{O}$ for spinless indistinguishable particles must be permutation invariant, i.e., $\mathcal{O}(x_1, x_2, \ldots, x_N) = \mathcal{O}(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_N})$, $\forall$ permutation σ. As a consequence,

$$[\mathcal{P}_\zeta, \mathcal{O}] = 0, \tag{16}$$

which can be easily checked by acting on any wave function in $\mathcal{H}^{\otimes N}$ [49]. Equation (16) implies a permutation-invariant observable is block diagonal on $\mathcal{H}_\zeta$ and its orthogonal complement. We define an observable is local if it only involves derivatives with respect to the coordinates up to a finite order. Then permutation invariance and locality implies that if a permutation-invariant and local observable $\mathcal{O}$ vanishes on $\mathcal{H}_\zeta$, then it also vanishes on the full Hilbert space $\mathcal{H}^{\otimes N}$. That is [56],

$$\mathcal{P}_\zeta\mathcal{O}\mathcal{P}_\zeta = \mathcal{P}_\zeta\mathcal{O} = \mathcal{O}\mathcal{P}_\zeta = 0 \Leftrightarrow \mathcal{O} = 0, \tag{17}$$

for a permutation-invariant and local observable $\mathcal{O}$.

Equations (16) and (17) lead to the following theorem regarding the commutators of two permutation-invariant observables, which is extremely useful in proving integrability.

Theorem 1. For two permutation-invariant and local observable $\mathcal{O}_n$ and $\mathcal{O}_m$, the following three conditions are equivalent to each other: (i) $[\mathcal{O}_n, \mathcal{O}_m] = 0$; (ii) $\mathcal{P}_\zeta[\mathcal{O}_n, \mathcal{O}_m]\mathcal{P}_\zeta = 0$; (iii) $[\mathcal{P}_\zeta\mathcal{O}_n\mathcal{P}_\zeta, \mathcal{P}_\zeta\mathcal{O}_m\mathcal{P}_\zeta] = 0$.

The equivalence between (i) and (ii) is a consequence of Eq. (17). The equivalence between (ii) and (iii) follows from

$$\begin{aligned}\mathcal{P}_\zeta[\mathcal{O}_n, \mathcal{O}_m]\mathcal{P}_\zeta &= \mathcal{P}_\zeta\mathcal{O}_n\mathcal{O}_m\mathcal{P}_\zeta^2 - \mathcal{P}_\zeta\mathcal{O}_m\mathcal{O}_n\mathcal{P}_\zeta^2\\ &= \mathcal{P}_\zeta\mathcal{O}_n\mathcal{P}_\zeta\mathcal{O}_m\mathcal{P}_\zeta - \mathcal{P}_\zeta\mathcal{O}_m\mathcal{P}_\zeta\mathcal{O}_n\mathcal{P}_\zeta\\ &= [\mathcal{P}_\zeta\mathcal{O}_n\mathcal{P}_\zeta, \mathcal{P}_\zeta\mathcal{O}_m\mathcal{P}_\zeta],\end{aligned}$$

where we have used Eq. (16).

Theorem 2. Both the quantum mechanical homogenous model (3) and the inhomogeneous model (9) generated in EOF are integrable, with the integral of motion being I_n for the homogenous model and $\tilde{I}_n$ for the inhomogeneous model.

To prove Theorem 2, let us first observe a very interesting property due to the projection operator $\mathcal{P}_\zeta$ and the exchange operator M_{ij}. Although the generalized momentum π_i involves N degrees of freedomdue to the prepotential term, when it is multiplied by M_{ij} from the left, it still satisfies the exchange rule for one-body operators [49]. As a consequence,

$$\mathcal{P}_\zeta\pi_i^n\mathcal{P}_\zeta = \mathcal{P}_\zeta M_{ij}^2\pi_i^n\mathcal{P}_\zeta = \mathcal{P}_\zeta M_{ij}\pi_i^n M_{ij}\mathcal{P}_\zeta = \mathcal{P}_\zeta\pi_j^n\mathcal{P}_\zeta. \tag{18}$$

A similar equation also holds for h_i. For a more general version of this identity, see Ref. [49].

On the other hand, since the integrals of motions I_n and $\tilde{I}_n$ are permutation invariant and local, one can reduce their commutativity to condition (iii) in Theorem 1. Using Eq. (18) or the analogous equation for h_i, it follows that the condition (iii) in Theorem 1 is satisfied, with $\mathcal{O}_n = I_n$ or $\mathcal{O}_n = \tilde{I}_n$. This concludes the proof of the integrability of the Hamiltonians (3) and (9).

A few comments are in order. First, since we have proved the equivalence between EOF and PHJ, the Hamiltonians (5) and (12) are therefore also integrable. Second, it is possible to build integrals of motions for families of classical models generated by the EOF and PHJ according to the quantum-classical correspondence. One can expand the powers in I_n and $\tilde{I}_n$ and compute $\mathcal{P}_\zeta I_n\mathcal{P}_\zeta$ and $\mathcal{P}_\zeta\tilde{I}_n\mathcal{P}_\zeta$ explicitly with Eq. (2). Then one is left with the expressions $\mathcal{P}_\zeta K_n\mathcal{P}_\zeta$ and $\mathcal{P}_\zeta\tilde{K}_n\mathcal{P}_\zeta$, where K_n and $\tilde{K}_n$ contain only the phase space variables but no exchange operators. In particular, we note that $K_2 = H_0$ and $\tilde{K}_1 = H$; see Ref. [49], where one obtains H_0 by projecting I_2 onto $\mathcal{H}_\zeta$. According to Theorem 1, K_n's and $\tilde{K}_n$'s must commute on the whole Hilbert space $\mathcal{H}^{\otimes N}$, respectively. Transitioning to the classical model, where the commutator is demoted to Poisson brackets, the Poisson brackets of K_n's and $\tilde{K}_n$'s must vanish, respectively. Thus, we see that K_n's and $\tilde{K}_n$'s are also the integrals of motion for the classical model with Hamiltonians (3), (5) and (9), (12), respectively.

Discussion.— It is worth noting that the Jastrow wave functions Φ_0, Ψ_0 may not be the true ground state of the corresponding PHJ if they cannot be properly normalized. Nevertheless, the family of models generated by EOF and PHJ is always integrable, regardless of the normalization of the Jastrow wave function.

For example, if $f_{ij} = \exp(g|x_{ij}|)$, H_0 becomes the well-known LL model [27]. However, the Jastrow wave function is normalizable only when $g < 0$, which corresponds to the McGuire bright soliton [43]. Therefore, Φ_0 is no longer the ground state wave function of the repulsive LL model. However, as we have discussed previously, the integrability of the Hamiltonian is not affected, so our result reproduces the integrability of the LL model with the integral of motion being I_n or K_n. More interestingly, upon introducing the external harmonic potential, according to Eq. (11), Ψ_0 becomes normalizable even if Φ_0 is not and Eq. (5) corresponds to the Lieb-Liniger-Coulomb model introduced in Refs. [46], i.e.,

$$H = \sum_i \left[\frac{p_i^2}{2m} + \frac{1}{2} m\omega^2 x_i^2\right] + g\sum_{i<j}\left[\frac{2\hbar^2}{m}\delta(x_{ij}) - \frac{m\omega}{\hbar}|x_{ij}|\right], \tag{19}$$

with ground state $E_0 = (N\hbar\omega/2) - (g^2\hbar^2/m)[N(N^2-1)/6]$. This system describes harmonically confined bosons subject to contact and Coulomb interactions or gravitational attraction in one spatial dimension. Reference [46] characterized its EOF representation and ground state properties. Using Theorem 2, we conclude that this system is integrable, with the integrals of motion being $\tilde{I}_n \equiv \sum_i h_i^n$.

Further physical examples of integrable PHJ systems are provided in Supplemental Material [49], which includes Refs. [30,57]. The proof leading to the integrability of PHJ essentially takes advantage of the permutation invariance and EOF. As a result it can be applied to models defined on the real line as well as those embedded in an external potential. Likewise, it holds for systems with hard-wall confinement or a ring geometry, provided the pair function f_{ij} and the one body potential v_i or W_i fulfill the corresponding boundary conditions.

Conclusion.—We have established the equivalence between the families of one-dimensional many-body quantum systems generated by the exchange operator formalism and parent Hamiltonians with a ground-state wave function of Jastrow form, describing indistinguishable particles with no internal degrees of freedom. Making use of the projection operator onto the spinless bosonic or fermionic subspace, we have proved the integrability of all these systems by constructing explicitly the corresponding integrals of motion. Embedding these translation-invariant models in an external potential preserves the integrability, in the presence of long-range interactions, as we have illustrated in the long-range Lieb-Liniger model and related systems.

These findings advance the study of many-body physics by uncovering the implications of ground-state correlations on integrability. They should lead to manifold applications in the study of quantum solitons, quantum quenches, and the thermalization of isolated integrable systems (governed by integrals of motion), and strongly correlated regimes, generalizing the super-Tonks-Girardeau gas [58], among others. Our results bear also implications on numerical methods for strongly correlated systems such as variational methods and quantum Monte Carlo algorithms, in which the ubiquitous use of Jastrow trial wave functions may impose integrability on systems lacking it. It may be possible to extend our results to higher spatial dimensions [28], higher-order correlations [59], the inclusion of spin degrees of freedom [8], mixtures of different species [36], and distinguishable particles [60,61].

It is a pleasure to acknowledge discussions with Pablo Martinez Azcona and Aritra Kundu.

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[1] J. Dziarmaga, Adv. Phys. **59**, 1063 (2010).
[2] T. Mori, T. N. Ikeda, E. Kaminishi, and M. Ueda, J. Phys. B **51**, 112001 (2018).
[3] M. A. Cazalilla, R. Citro, T. Giamarchi, E. Orignac, and M. Rigol, Rev. Mod. Phys. **83**, 1405 (2011).
[4] X.-W. Guan, M. T. Batchelor, and C. Lee, Rev. Mod. Phys. **85**, 1633 (2013).
[5] M. Takahashi, *Thermodynamics of One-Dimensional Solvable Models* (Cambridge University Press, Cambridge, England, 1999).
[6] M. Gaudin, *The Bethe Wavefunction*, edited by J.-S. Caux (Cambridge University Press, Cambridge, England, 2014).
[7] A. P. Polychronakos, Phys. Rev. Lett. **69**, 703 (1992).
[8] A. P. Polychronakos, J. Phys. A **39**, 12793 (2006).
[9] L. Lapointe and L. Vinet, Commun. Math. Phys. **178**, 425 (1996).
[10] H. Ujino and M. Wadati, J. Phys. Soc. Jpn. **65**, 653 (1996).
[11] C. N. Yang, Phys. Rev. Lett. **19**, 1312 (1967).
[12] R. J. Baxter, Phil. Trans. R. Soc. A **289**, 315 (1978).
[13] V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge University Press, Cambridge, England, 1997).
[14] B. Sutherland, *Beautiful Models* (World Scientific, Singapore, 2004).
[15] B. Sutherland, Phys. Rev. Lett. **75**, 1248 (1995).
[16] J.-S. Caux and J. Mossel, J. Stat. Mech. (2011) P02023.
[17] H. Ujino and M. Wadati, J. Phys. Soc. Jpn. **64**, 4121 (1995).
[18] A. P. Polychronakos, Nucl. Phys. **B961**, 115243 (2020).
[19] A. Bijl, Physica (Utrecht) **7**, 869 (1940).

[20] R. B. Dingle, Lond. Edinb. Dublin Philos. Mag. J. Sci. **40**, 573 (1949).
[21] R. Jastrow, Phys. Rev. **98**, 1479 (1955).
[22] F. Calogero, J. Math. Phys. (N.Y.) **12**, 419 (1971).
[23] B. Sutherland, J. Math. Phys. (N.Y.) **12**, 246 (1971).
[24] B. Sutherland, Phys. Rev. A **4**, 2019 (1971).
[25] F. Calogero and C. Marchioro, J. Math. Phys. (N.Y.) **14**, 182 (1973).
[26] P. J. Gambardella, J. Math. Phys. (N.Y.) **16**, 1172 (1975).
[27] A. del Campo, Phys. Rev. Res. **2**, 043114 (2020).
[28] M. Beau and A. del Campo, SciPost Phys. Core **4**, 30 (2021).
[29] C. L. Kane, S. Kivelson, D. H. Lee, and S. C. Zhang, Phys. Rev. B **43**, 3255 (1991).
[30] F. Calogero, Lett. Nuovo Cimento **13**, 411 (1975).
[31] L. Šamaj and Z. Bajnok, *Introduction to the Statistical Physics of Integrable Many-Body Systems* (Cambridge University Press, Cambridge, England, 2013).
[32] T. Kinoshita, T. Wenger, and D. S. Weiss, Science **305**, 1125 (2004).
[33] B. Paredes, A. Widera, V. Murg, O. Mandel, S. Fölling, I. Cirac, G. V. Shlyapnikov, T. W. Hänsch, and I. Bloch, Nature (London) **429**, 277 (2004).
[34] M. Girardeau, J. Math. Phys. (N.Y.) **1**, 516 (1960).
[35] M. D. Girardeau, E. M. Wright, and J. M. Triscari, Phys. Rev. A **63**, 033601 (2001).
[36] M. D. Girardeau and A. Minguzzi, Phys. Rev. Lett. **99**, 230402 (2007).
[37] M. Girardeau, H. Nguyen, and M. Olshanii, Opt. Commun. **243**, 3 (2004).
[38] N. Kawakami, J. Phys. Soc. Jpn. **62**, 4163 (1993).
[39] K. Vacek, A. Okiji, and N. Kawakami, J. Phys. A **27**, L201 (1994).
[40] N. Gurappa and P. K. Panigrahi, Phys. Rev. B **59**, R2490 (1999).
[41] M. Olshanii, Phys. Rev. Lett. **81**, 938 (1998).
[42] E. H. Lieb, R. Seiringer, and J. Yngvason, Phys. Rev. Lett. **91**, 150401 (2003).
[43] J. B. McGuire, J. Math. Phys. (N.Y.) **5**, 622 (1964).
[44] E. H. Lieb and W. Liniger, Phys. Rev. **130**, 1605 (1963).
[45] E. H. Lieb, Phys. Rev. **130**, 1616 (1963).
[46] M. Beau, S. M. Pittman, G. E. Astrakharchik, and A. del Campo, Phys. Rev. Lett. **125**, 220602 (2020).
[47] B. Sutherland, J. Math. Phys. (N.Y.) **12**, 251 (1971).
[48] J. Moser, Adv. Math. **16**, 197 (1975).
[49] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.129.150601 for details about the properties of the exchange operators, the projection operator to the bosonic and fermionic subspaces, derivations of Eq. (3), proof of integrability, and a user guide to construct one-dimensional integrable models as well as various examples, which includes Ref. [50] that does not appear in the reference section of the main text.
[50] A. Erdelyi, *Higher Transcendental Functions* (Dover Publications, Mineola, NY, 2007), Vol. 3.
[51] J. W. Negele and H. Orland, *Quantum Many-Particle Systems* (Perseus Books, Reading, MA, 1998).
[52] A. Kundu, Phys. Rev. Lett. **83**, 1275 (1999).
[53] M. D. Girardeau, Phys. Rev. Lett. **97**, 100402 (2006).
[54] M. T. Batchelor, X.-W. Guan, and N. Oelkers, Phys. Rev. Lett. **96**, 210402 (2006).
[55] F. Cooper, A. Khare, and U. Sukhatme, Phys. Rep. **251**, 267 (1995).
[56] This observation was put forward first by Polychronakos [7], using a heuristic locality argument. However, we emphasize the argument misses an important ingredient, the permutation invariance. In [49], we give a formal proof inspired by the coordinate Bethe ansatz [6,31].
[57] M. Beau (unpublished).
[58] G. E. Astrakharchik, J. Boronat, J. Casulleras, and S. Giorgini, Phys. Rev. Lett. **95**, 190407 (2005).
[59] G. Carleo, L. Cevolani, L. Sanchez-Palencia, and M. Holzmann, Phys. Rev. X **7**, 031026 (2017).
[60] S. R. Jain and A. Khare, Phys. Lett. A **262**, 35 (1999).
[61] S. M. Pittman, M. Beau, M. Olshanii, and A. del Campo, Phys. Rev. B **95**, 205135 (2017).