MATHEMATICAL HOURGLASSES, JUST IN TIME

Mathematical hourglasses have infinite precision, and the beings who are supposed to use them can flip them in no time. There are usually finitely many hourglasses, and they can also be used simultaneously.
It is customary that the times the hourglasses display are positive multiples of some time unit or, in other words, that the times associated to any two of them have a rational ratio.

The empty hourglass could display zero, and an hourglass where the sand cannot descend could display infinity. Negative times can be seen as positive times before a future event: for example, if we are 2 time units left before an hourglass becomes empty, then we could say that we are at time -2 from this event. And, for example, if we have the 3-hourglass (measuring 3 time units), then we are at time -15 when we start the process of flipping the 3-hourglass 5 times.

One rule of mathematical hourglasses is that it is impossible, just with one hourglass, to estimate a time that is strictly between 0 and the time of the hourglass: in other words, with the *n*-hourglass we can build (or measure) precisely the time intervals that are multiples of *n*.

By using two hourglasses displaying n and m time units respectively, which times can we build (or measure)?

We can easily build all times which are the sum of a non-negative multiple of *n* and a non-negative multiple of *m.* Suppose that *n>m.* Then we can also build the difference *n-m* by starting simultaneously both hourglasses, and measuring the time between the two events “the *m*-hourglass becomes empty” and “the *n*-hourglass becomes empty”. More generally, we can build a positive time of the form *n-xm* where *x* is a positive integer. In particular, we can build the remainder of *n* after division by *m*.

Suppose that we could build new hourglasses, meaning that we could make the *x*-hourglass as soon as we can measure *x* time units. Then with a chain of remainders (following Euclid’s Algorithm) we could measure the greatest common divisor of *m* and *n*, and therefore all its multiples. This is all we can do with two hourglasses because summing a (possibly negative) multiple of *m* and a (possibly negative) multiple of *n* gives precisely all (possibly negative) multiples of the greatest common divisor of *m* and *n.*

Suppose that we can flip an hourglass at any intermediate time. Then, provided that we can measure *x* time units, we can build a sort of *x*-hourglass with two larger hourglasses. Namely, we can start one of them, and after *x* time units we flip it back and we start the second one. By repeating this procedure, we can measure tacts of *x* time units. By coordinating these and other procedures, we might achieve the measure of any time unit that is within reach from the hourglasses we start with (for finitely many hourglasses we are stuck with times that are multiples of the greatest common divisors of the individual times).
Thus we can produce many mathematical riddles about hourglasses.

Notice that, if we have two hourglasses (multiples of the time unit) but we don’t know their times, then at least we can find out the ratio of their times. Indeed, if we start them simultaneously and we flip them as soon as they are empty, then they become empty simultaneously for the first time at the least common multiple of their times.

Now imagine hourglasses with less and less sand. Imagine that a grain of sand can be arbitrarily small, so that one can make hourglasses displaying every non-negative multiple of the time unit (possibly not rational). Moreover, imagine that there are infinitely many hourglasses at our disposal. We could then rephrase some mathematical results in an original way, for example the fact that the sum of the harmonic sum is infinite while a geometric series is not: *If we flip in sequence hourglasses displaying 1/n time units, then this procedure will keep us occupied for eternity. However, if we do the same with hourglasses displaying 1/2^n time units, then we are finished in one time unit.*