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Article

Hessian Estimates for Dirichlet and Neumann Eigenfunctions of Laplacian

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By methods of stochastic analysis on Riemannian manifolds, we develop an approach to determine an explicit constant $c(D)$ for an n-dimensional compact manifold D with smooth boundary such that $\frac{\lambda}{n}\|\phi\|_\infty \le \|\mathrm{Hess}\,\phi\|_\infty \le c(D)\lambda\|\phi\|_\infty$ holds for any Dirichlet eigenfunction ϕ of $-\Delta$ on D with eigenvalue λ. Our results provide the sharp Hessian estimate $\|\mathrm{Hess}\,\phi\|_\infty \lesssim \lambda^{\frac{n+3}{4}}\|\phi\|_{L^2}$. Corresponding Hessian estimates for Neumann eigenfunctions are derived in the second part of the paper.

1 Introduction

Let D be an n-dimensional compact Riemannian manifold with smooth boundary ∂D. We write $(\phi,\lambda)\in \mathrm{Eig}(\Delta)$ if ϕ is a Dirichlet eigenfunction of $-\Delta$ on D with eigenvalue $\lambda>0$, that is, $-\Delta\phi=\lambda\phi$. We always assume eigenfunctions ϕ to be normalized in $L^2(D)$ such that $\|\phi\|_{L^2}=1$. According to [16], there exist two positive constants $c_1(D)$ and $c_2(D)$ such that

$$c_1(D)\sqrt{\lambda}\,\|\phi\|_\infty \leqslant \|\nabla\phi\|_\infty \leqslant c_2(D)\sqrt{\lambda}\,\|\phi\|_\infty,\quad (\phi,\lambda)\in \mathrm{Eig}(\Delta), \tag{1.1}$$

where we write $\|\nabla\phi\|_\infty := \|\,|\nabla\phi|\,\|_\infty$ for simplicity. An analogous statement for Neumann eigenfunctions has been derived by Hu, Shi, and Xu [9]. Subsequently, by methods of stochastic analysis on Riemannian manifolds, Arnaudon, Thalmaier, and Wang [2] determined explicit constants $c_1(D)$ and $c_2(D)$ in (1.1) for Dirichlet and Neumann eigenfunctions. From this, together with the uniform estimate of ϕ (see [7, 8, 12]),

$$\|\phi\|_\infty \le c_D\lambda^{\frac{n-1}{4}}$$

for some positive constant c_D, the optimal uniform bound of the gradient writes as

$$\|\nabla\phi\|_\infty \lesssim \lambda^{\frac{n+1}{4}}.$$

Results of this type have been used to study gradient estimates for unit spectral projection operators and to give a new proof of Hörmander's multiplier theorem; see [23–25].

Concerning higher order estimates of eigenfunctions, not much is known. Very recently, Steinerberger [17] studied Laplacian eigenfunctions of $-\Delta$ with Dirichlet boundary conditions on bounded domains

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$\Omega \subset \mathbb{R}^n$ with smooth boundary and proved a sharp Hessian estimate for the eigenfunctions which reads as

$$\|\operatorname{Hess}\phi\|_\infty \lesssim \lambda^{\frac{n+3}{4}},$$

where

$$\|\operatorname{Hess}\phi\|_\infty := \sup\left\{|\operatorname{Hess}\phi(v,v)|(x) : x \in \mathbb{R}^n,\ v \in \mathbb{R}^n,\ |v| = 1\right\}.$$

To the best of our knowledge, higher order estimates of eigenfunctions for Euclidean domains first appeared in [6] (see Lemma C.1 in the Appendix there which is easily adapted to cover the Hessian estimate in the Euclidean case).

It is natural to ask under which geometric assumptions such estimates extend to compact manifolds (with boundary). Following the lines of [2], for the Hessian of an eigenfunction ϕ, one may consider the question how to derive explicit numerical constants $C_1(D)$ and $C_2(D)$ such that

$$C_1(D)\lambda\,\|\phi\|_\infty \leqslant \|\operatorname{Hess}\phi\|_\infty \leqslant C_2(D)\lambda\,\|\phi\|_\infty, \quad (\phi,\lambda) \in \operatorname{Eig}(\Delta). \tag{1.2}$$

Note that for eigenfunctions of the Laplacian, one trivially has

$$|\operatorname{Hess}\phi| \geq \frac{1}{n}\,|\Delta\phi| = \frac{\lambda}{n}\,|\phi|,$$

and thus there is always the obvious lower bound

$$\frac{\|\operatorname{Hess}\phi\|_\infty}{\|\phi\|_\infty} \geq \frac{\lambda}{n}.$$

For this reason, we may concentrate in the sequel on upper bounds for $\|\operatorname{Hess}\phi\|_\infty/\|\phi\|_\infty$.

In [2] a derivative formula for Dirichlet eigenfunctions has been given from where an upper bound for the gradient of the eigenfunction could be derived directly. Let us briefly describe this method. Assume that X_t is a Brownian motion on $D\setminus\partial D$ with generator $\frac{1}{2}\Delta$, and write $X_t(x)$ to indicate the starting point $X_0 = x$. Then $X_\cdot(x)$ is defined up to the first hitting time $\tau_D = \inf\{t > 0 \colon X_t(x) \in \partial D\}$ of the boundary. For $x \in \partial D$ we use the convention that $X_\cdot(x)$ is defined with lifetime $\tau_D \equiv 0$; in this case the subsequent statements usually hold automatically.

Suppose that $Q_t \colon T_xD \to T_{X_t(x)}D$ is defined by

$$DQ_t = -\frac{1}{2}\operatorname{Ric}^\sharp(Q_t)\,dt, \quad Q_0 = \mathbf{id},$$

where $D := /\!/_t\, d\, /\!/_t^{-1}$ with $/\!/_t := /\!/_{0,t} \colon T_xD \to T_{X_t(x)}D$ parallel transport along $X(x)$ and $\operatorname{Ric}^\sharp(v)(w) = \operatorname{Ric}(v,w)$ for $v,w \in TD$. Suppose that $(\phi,\lambda) \in \operatorname{Eig}(\Delta)$. Then, for $v \in T_xD$ and any $k \in C_b^1([0,\infty);\mathbb{R})$, that is, k bounded with bounded derivative, the process

$$e^{\lambda t/2}\left(k(t)\,\langle\nabla\phi(X_t), Q_t(v)\rangle - \phi(X_t)\int_0^t \langle\dot{k}(s)Q_s(v), /\!/_s dB_s\rangle\right), \quad t \leq \tau_D$$

is a martingale. From this, by taking expectation, a formula involving $\nabla\phi$ can be obtained which allows to derive an upper bound for $|\nabla\phi|$ on D by estimating $|\nabla\phi|$ on the boundary ∂D and carefully choosing the function k. Along this circle of ideas, our aim is to establish a similar strategy for the Hessian of an eigenfunction ϕ.

In view of the fact that $P_t\phi = e^{-\lambda t/2}\,\phi$ where P_t is the semigroup generated by $\frac{1}{2}\Delta$, we focus first on martingales which are appropriate for attaining uniform Hessian estimates of eigenfunctions. Let us start with some background on Bismut-type formulas for second-order derivatives of heat semigroups. A second-order differential formula for the heat semigroup P_t was first obtained by Elworthy and Li [5, 13] for a non-compact manifold, however, with restrictions on the curvature of the manifold.

An intrinsic formula for $\mathbf{Hess}\, P_t f$ has been given by Stroock [18] for a compact Riemannian manifold, and a localized version of such a formula was obtained in [1, 3] adopting martingale arguments. For the Hessian of the Feynman–Kac semigroup of an operator $\Delta + V$ with a potential function V on manifolds, we refer the reader to [14, 15, 19].

For a complete Riemannian manifold M without boundary, an appropriate version of a Bismut-type Hessian formula gives the following estimate (see [3], Corollary 4.3, together with Lemma 2.2 below)

$$\|\mathbf{Hess}\, P_t f\|_\infty \le \left(K_1\sqrt{t} + \frac{K_2 t}{2} + \frac{2}{t}\right) e^{K_0 t}\ \|f\|_\infty,$$

where

$$\begin{aligned} K_0 &:= \sup\left\{-\mathbf{Ric}(v,v) : y \in M,\ v \in T_yM,\ |v| = 1\right\}; \\ K_1 &:= \sup\left\{|R|(y) : y \in M\right\}; \\ K_2 &:= \sup\left\{|(\mathbf{d}^*R + \nabla\mathbf{Ric})^\sharp(v,w)|(y) : y \in M,\ v,w \in T_yM,\ |v| = |w| = 1\right\} \end{aligned} \tag{1.3}$$

and

$$|R|(y) := \sup\left\{\sqrt{\sum_{i,j=1}^n R(e_i, v, w, e_j)^2(y)} : |v| \le 1, |w| \le 1\right\}$$

for an orthonormal base $\{e_i\}_{i=1}^n$ of T_yM.

Thus, if $f = \phi$ and $(\phi, \lambda) \in \mathbf{Eig}(\Delta)$, then

$$\|\mathbf{Hess}\, \phi\|_\infty \le \left(K_1\sqrt{t} + \frac{K_2 t}{2} + \frac{2}{t}\right) e^{(K_0+\lambda/2)t}\ \|\phi\|_\infty$$

for any $t > 0$. Letting $t = \frac{1}{K_0+\lambda/2}$ then yields the estimate

$$\frac{\|\mathbf{Hess}\, \phi\|_\infty}{\|\phi\|_\infty} \le \left(K_1\sqrt{\frac{2}{2K_0+\lambda}} + \frac{K_2}{2K_0+\lambda} + 2K_0 + \lambda\right) e\,.$$

To carry over such results to (compact) manifolds D with boundary, the influence of the boundary has to be studied. In this paper, we shall adopt a martingale approach to the Hessian of Dirichlet eigenfunctions. This approach is based on the construction of a suitable martingale which builds a relation between $\mathbf{Hess}\, \phi$ and $\mathbf{d}\phi$ and then to estimate $C_2(D)$ in (1.2) by searching for explicit constants C_1, C_2 and C_3 such that

$$\|\mathbf{Hess}\, \phi\|_\infty \le C_1 \|\mathbf{Hess}\, \phi\|_{\partial D,\infty} + C_2 \|\nabla \phi\|_{\partial D,\infty} + C_3 \|\nabla \phi\|_\infty, \tag{1.4}$$

where $\|\mathbf{Hess}\, \phi\|_{\partial D,\infty} := \sup_{x\in\partial D} |\mathbf{Hess}\, \phi|(x)$ and $\|\nabla \phi\|_{\partial D,\infty} := \sup_{x\in\partial D} |\nabla \phi|(x)$. The final estimate for $|\mathbf{Hess}\, \phi|$ is then received by combining the last inequality with estimate (1.1) in [2].

Let us start with the general principle behind the construction of the relevant martingale. Let $k \in C_b^1([0,\infty);\mathbb{R})$ and define an operator-valued process $W_t^k : T_xD \otimes T_xD \to T_{X_t(x)}D$ as solution to the following covariant Itô equation

$$DW_t^k(v,w) = R(//_t dB_t, Q_t(k(t)v))Q_t(w) - \frac{1}{2}(\mathbf{d}^*R + \nabla\mathbf{Ric})^\sharp(Q_t(k(t)v), Q_t(w))\,dt - \frac{1}{2}\mathbf{Ric}^\sharp(W_t^k(v,w))\,dt,$$

with initial condition $W_0^k(v,w)=0$; see Section 2 in [4]. In explicit terms this gives

$$W_t^k(v,w) = Q_t\int_0^t Q_s^{-1}R(//_s dB_s, Q_s(k(s)v))Q_s(w) - \frac{1}{2}Q_t\int_0^t Q_s^{-1}(\mathbf{d}^*R+\nabla\mathbf{Ric})^\sharp(Q_s(k(s)v),Q_s(w))\,ds. \tag{1.5}$$

Here the operator $\mathbf{d}^*R$ is defined by $\mathbf{d}^*R(v_1,v_2) := -\operatorname{tr}\nabla_{\boldsymbol{\cdot}}R(\boldsymbol{\cdot},v_1)v_2$ and thus satisfies

$$\langle \mathbf{d}^*R(v_1,v_2),v_3\rangle = \langle(\nabla_{v_3}\mathbf{Ric}^\sharp)(v_1),v_2\rangle - \langle(\nabla_{v_2}\mathbf{Ric}^\sharp)(v_3),v_1\rangle$$

for all $v_1,v_2,v_3\in T_xD$ and $x\in D$. Then the process

$$M_t := e^{\lambda t/2}\,\mathrm{Hess}\,\phi\big(Q_t(k(t)v),Q_t(v)\big) + e^{\lambda t/2}\,\mathbf{d}\phi(W_t^k(v,v)) - e^{\lambda t/2}\,\mathbf{d}\phi(Q_t(v))\int_0^t\langle Q_s(\dot k(s)v), //_s dB_s\rangle \tag{1.6}$$

is a martingale on $[0,\tau_D]$ in the sense that $(M_{t\wedge\tau_D})_{t\geq 0}$ is a globally defined martingale where $\tau_D=\inf\{t>0: X_t(x)\in\partial D\}$ denotes the first hitting time of $X_{\boldsymbol{\cdot}}(x)$ of the boundary ∂D. The martingale property of (1.6) now allows to establish an inequality of the type (1.4) by equating the expectations at time 0 and at time $t\wedge\tau_D$. This approach then requires to estimate the boundary values of $|\mathbf{d}\phi|$ and $|\,\mathrm{Hess}\,\phi|$, in order to obtain the wanted upper bound for $\|\,\mathrm{Hess}\,\phi\|_\infty$. To this end, we establish the required estimates in Lemmas 2.4–2.5 by using information on the second fundamental form II and the second derivative of N, where for $X,Y\in T_x\partial D$ and $x\in\partial D$, the second fundamental form is defined by

$$\mathrm{II}(X,Y) = -\langle\nabla_X N,\ Y\rangle.$$

Finally, let

$$\ell(t) := \ell_{k,\sigma}(t) := \begin{cases} \cos\sqrt{k}t - \frac{\sigma}{\sqrt{k}}\sin\sqrt{k}t, & k>0,\\ 1-\sigma t, & k=0,\\ \cosh\sqrt{-k}t - \frac{\sigma}{\sqrt{-k}}\sinh\sqrt{-k}t, & k<0. \end{cases} \tag{1.7}$$

We state now the first main result of the paper. To this end we denote by $\rho_{\partial D}$ the distance function to the boundary ∂D which is smooth in an open neighborhood of ∂D if the boundary of D is smooth.

Theorem 1.1. Let D be a compact Riemannian manifold with smooth boundary ∂D. Let K_0,K_1,K_2,σ be non-negative constants such that $\mathbf{Ric}\geq -K_0$, $|R|\leq K_1$ and $|\mathbf{d}^*R+\nabla\mathbf{Ric}|\leq K_2$ on D, and that $|\mathrm{II}|\leq\sigma$. Assume that the distance function $\rho_{\partial D}=\mathrm{dist}(x,\partial D)$ is smooth on the tubular neighborhood $\partial_{r_1}D := \{x\in D: \rho_{\partial D}(x)\leq r_1\}$ of ∂D. Let k,β,γ be constants such that $|\mathbf{Sect}|\leq k$ on $\partial_{r_1}D$, and that

$$|\nabla(\Delta\rho_{\partial D})|\leq\beta,\quad |\Delta^2\rho_{\partial D}|\leq\gamma\quad \text{on } \partial_{r_0}D, \tag{1.8}$$

where $r_0=r_1\wedge\ell^{-1}(1/2)$. Then for any non-trivial $(\phi,\lambda)\in\mathbf{Eig}_N(\Delta)$,

$$\frac{\|\,\mathrm{Hess}\,\phi\|_\infty}{\|\phi\|_\infty}\leq A(D)+C(D)\lambda \tag{1.9}$$

where

$$A(D) = 2(n-1)\sigma\,\mathrm{e}\left(\alpha + \frac{1}{\pi\varepsilon}\right) + 2c_0\,\mathrm{e}\left(K_1 - \frac{9\alpha}{r_0} - \frac{6}{r_0^2} - 3\beta\right)$$
$$+ \frac{(K_1+3\beta)\alpha\sqrt{\mathrm{e}} + \frac{1}{2}K_2c_0\sqrt{\mathrm{e}} + \gamma + \frac{3(\alpha^2+3\sqrt{\mathrm{e}}\alpha^2+2\beta)}{r_0} + \frac{3(1+2\sqrt{\mathrm{e}})\alpha}{r_0^2}}{\frac{6}{r_0}+2\alpha} + \frac{1}{4}\frac{K_2\alpha}{(6/r_0+2\alpha)^2}$$
$$+ 2c_0\,\mathrm{e}(\mathrm{e}^{nr_0\sigma/2}+1)\left(\frac{9\alpha+\sigma n}{r_0} + \frac{6}{r_0^2} + 3\beta + 2K_0 + 2\sigma^2 + 4\,\mathrm{e}^{nr_0\sigma+1}\left(\frac{6}{r_0}+2\alpha\right)^2\right)$$
$$+ \alpha\,\mathrm{e}\left(2\,\mathrm{e}^{nr_0\sigma/2}+1\right)\left(\left(2K_0+2\sigma^2+\frac{\sigma n}{r_0}\right)\varepsilon + \varepsilon^{-1} + 4\,\mathrm{e}^{(nr_0\sigma+1)/2}\left(\frac{6}{r_0}+2\alpha\right)\right) \tag{1.10}$$

and

$$C(D) = \left((n-1)\sigma + \alpha(2\,\mathrm{e}^{nr_0\sigma/2}+1)\right)\mathrm{e}\,\varepsilon + \frac{\alpha(\sqrt{\mathrm{e}}+1)}{6/r_0+2\alpha} + 2\left(2\,\mathrm{e}^{nr_0\sigma/2}+1\right)c_0\,\mathrm{e}, \tag{1.11}$$

for any $\varepsilon > 0$, $\alpha = 2(n-1)\max\{\sigma, k\}$ and $c_0 = \sqrt{\frac{2}{\pi}} + \frac{1}{4}\sqrt{\frac{\pi}{2}}$.

Remark 1.2. (1) Let λ_1 be the first Dirichlet eigenvalue of $-\Delta$. From inequality (1.9), we see that

$$\frac{\|\,\mathbf{Hess}\,\phi\|_\infty}{\|\phi\|_\infty} \le \left(\frac{A(D)}{\lambda_1} + C(D)\right)\lambda.$$

(2) If the manifold has constant sectional curvature and mean curvature on $\partial_{r_0}D$, that is, $H = \theta$, $\mathbf{Sect} = k$ on $\partial_{r_0}D$, then for $\rho_{\partial D}(x) \le \ell^{-1}(0) \wedge r_0$,

$$\Delta\rho_{\partial D} = \frac{\ell'_{\theta,(n-1)k}}{\ell_{\theta,(n-1)k}}(\rho_{\partial D}).$$

As a consequence, the upper bound of $|\nabla(\Delta\rho_{\partial D})|$ and $|\Delta^2\rho_{\partial D}|$ can be calculated explicitly, as

$$|\nabla(\Delta\rho_{\partial D})|(x) \le 4((n-1)k+\sigma^2),\quad |\Delta^2\rho_{\partial D}|(x) \le 8\max\left\{\sigma, \sqrt{(n-1)k}\right\}((n-1)k+\sigma^2),$$

for $\rho_{\partial D}(x) \le i_0 \wedge \ell^{-1}(1/2)$. For the general case, from the second variation formula of $\rho_{\partial D}$ (see (2.10) below) we see that further information about $|\nabla\mathbf{II}|$, $|\nabla^2\mathbf{II}|$, $|R|$, $|\nabla R|$, and $|\nabla^2 R|$ on $\partial_{r_0}D$ is needed to derive upper bounds for $|\nabla(\Delta\rho_{\partial D})|$ and $|\Delta^2\rho_{\partial D}|$.

Turning now to Hessian estimates for Neumann eigenfunctions, let us denote by $\mathbf{Eig}_N(\Delta)$ the set of non-trivial (ϕ,λ) for the Neumann eigenproblem, that is, ϕ is non-constant, $\Delta\phi = -\lambda\phi$ and $N\phi|_{\partial D} = 0$ for the unit inward normal vector field N of ∂D. Proceeding along the previous ideas, the main difference is that we can no longer consider the process only up to the first hitting the boundary ∂D. When constructing the suitable martingales, the boundary behaviour of the process must be included. We shall use reflecting Brownian motion as base process to deal with this problem. Due to recent work on Bismut-type Hessian formula for the Neumann semigroup [4], we have the following formula linking $\mathbf{Hess}\,P_tf$ and $\mathbf{d}f$ intrinsically:

$$\mathbf{Hess}\,P_tf(v,v) = \mathbb{E}\left[-\mathbf{d}f(\tilde{Q}_t(v))\int_0^t \langle \tilde{Q}_s(\dot{k}(s)v), //_s dB_s\rangle + \mathbf{d}f(\tilde{W}_t^k(v,v))\right],$$

where $\tilde{Q}$ and $\tilde{W}^k$ are defined in (3.1) and (3.2) in Section 3 below. By taking into account that $P_t\phi = \mathrm{e}^{-\frac{1}{2}\lambda t}\phi$ and estimating $\tilde{Q}_\cdot$ and $\tilde{W}_\cdot$ carefully under suitable curvature conditions, we obtain the following theorem which gives an upper estimate for $\mathbf{Hess}\,\phi$ of the type (1.2) with an explicit constant $C_2(D)$.

Theorem 1.3. Let D be an n-dimensional compact Riemannian manifold with boundary ∂D. Let K_0, K_1, K_2 be non-negative constants such that $\mathbf{Ric} \geq -K_0$, $|R| \leq K_1$ and $|\mathbf{d}^*R + \nabla\mathbf{Ric}| \leq K_2$ on D,and let $\sigma_1, \sigma_2, \sigma$ be non-negative constants such that $-\sigma_1 \leq \mathbf{II} \leq \sigma$ and $|\nabla^2 N - R(N)| \leq \sigma_2$ on the boundary ∂D. Assume the distance function $\rho_{\partial D}$ to the boundary ∂D is smooth on $\partial_{r_1} D := \{x \in D : \rho_{\partial D}(x) \leq r_1\}$ and let k be constant such that $\mathbf{Sect} \leq k$ on $\partial_{r_1} D$. Then for any non-trivial $(\phi, \lambda) \in \mathbf{Eig}_N(\Delta)$,

$$\frac{\|\,\mathbf{Hess}\;\phi\|_\infty}{\|\phi\|_\infty} \leq A(N) + C(N)\lambda,$$

where

$$A(N) := \left\{K_1 + 2K_0 + \frac{2\sigma_1 n}{r_0} + 4\sigma_1^2 + \frac{K_2 + 2\sigma_2(\frac{n}{r_0} + 2\sigma_1)}{2\sqrt{2K_0 + 2\sigma_1(\frac{n}{r_0} + 2\sigma_1)}}\right.$$
$$\left. + \frac{\sigma_2 n r_0}{2}\left(\varepsilon\left(K_0 + \frac{n\sigma_1}{r_0} + 2\sigma_1^2\right) + \frac{1}{\varepsilon}\right)\right\} \mathrm{e}^{\frac{3}{2} n r_0 \sigma_1 + 1}$$

and

$$C(N) := \left(1 + \frac{\varepsilon\sigma_2 n r_0}{2}\right) \mathrm{e}^{\frac{3}{2} n r_0 \sigma_1 + 1}$$

for $r_0 = r_1 \wedge \ell^{-1}(0)$ and any $\varepsilon > 0$.

Remark 1.4. Let λ_1 be the first Neumann eigenvalue of $-\Delta$. Then Theorem 1.3 gives

$$\frac{\|\,\mathbf{Hess}\;\phi\|_\infty}{\|\phi\|_\infty} \leq \left(\frac{A(N)}{\lambda_1} + C(N)\right)\lambda.$$

The remainder of the paper is organized as follows. In Section 2 we first show for Dirichlet eigenfunctions

$$\|\,\mathbf{Hess}\,\phi\|_\infty / \|\phi\|_\infty \leq C(D)\lambda \tag{1.12}$$

by verifying that the process (1.6) is a martingale, in combination with boundary estimates for $|\,\mathbf{Hess}\,\phi|$. Section 3 then deals with Neumann eigenfunctions where we give a proof of Theorem 1.3 by using Bismut-type Hessian formulae for the Neumann semigroup along with an estimate of the local time.

2 Hessian Estimates of Dirichlet Eigenfunctions

This section is dedicated to the approach described in the Introduction. The proof of Theorem 1.1 is divided into two steps by first showing Theorem 2.11 with some auxiliary function h, which will be constructed in Section 2.3.

2.1 Preliminary

We start by defining the fundamental martingale which will serve as basis for our method.

Theorem 2.1. On a compact Riemannian manifold D with boundary ∂D, let $X_\cdot(x)$ be a Brownian motion starting from $x \in D$ and denote by $\tau_D = \inf\{t \geq 0 : X_t(x) \in \partial D\}$ its first hitting time of ∂D. Define Q_t and W_t^k as above where $k \in C_b^1([0,\infty); \mathbb{R})$. Then, for $(\phi, \lambda) \in \mathbf{Eig}_N(\Delta)$ and $v \in T_x D$, the process

$$e^{\lambda t/2}\Big(\mathrm{Hess}\,\phi\big(Q_t(k(t)v),Q_t(v)\big)+\mathbf{d}\phi(W_t^k(v,v))-\mathbf{d}\phi(Q_t(v))\int_0^t\langle Q_s(\dot k(s)v),//_s dB_s\rangle\Big) \tag{2.1}$$

is a martingale on $[0,\tau_D]$.

Proof. Due to the compactness of D it is sufficient to check that (2.1) is a local martingale on $[0,\tau_D)$. Fixing a time $T>0$, for $v\in T_xD$, we let

$$N_t(v,v)=\mathrm{Hess}\,P_{T-t}\phi(Q_t(v),Q_t(v))+(\mathbf{d}P_{T-t}\phi)(W_t(v,v)),\quad t\le T\wedge\tau_D,$$

where

$$W_t(v,v)=Q_t\int_0^t Q_r^{-1}R\big(//_r dB_r,Q_r(v)\big)Q_r(v)-\frac{1}{2}Q_t\int_0^t Q_r^{-1}(\mathbf{d}^*R+\nabla\mathbf{Ric})^\sharp\big(Q_r(v),Q_r(v)\big)\,dr.$$

Then $N_t(v,v)$ is a local martingale; see for instance the proof of [19, Lemma 2.7] in case that potential $V\equiv 0$. Since $(\phi,\lambda)\in\mathrm{Eig}(\Delta)$, we know that $P_{T-t}\phi(X_t)=e^{-\lambda(T-t)/2}\,\phi(X_t)$ and thus

$$e^{\lambda t/2}\left(\mathrm{Hess}\,\phi(Q_t(v),Q_t(v))+(\mathbf{d}\phi)(W_t(v,v))\right)$$

is also a local martingale. Furthermore, consider

$$N_t^k(v,v):=e^{\lambda t/2}\,\mathrm{Hess}\,\phi(Q_t(k(t)v),Q_t(v))+(e^{\lambda t/2}\,\mathbf{d}\phi)(W_t^k(v,v)).$$

According to the definition of $W_t^k(v,v)$, resp. $W_t(v,v)$, and in view of the fact that $N_t(v,v)$ is a local martingale, it is easy to see that

$$e^{\lambda t/2}\,\mathrm{Hess}\,\phi(Q_t(k(t)v),Q_t(v))+(e^{\lambda t/2}\,\mathbf{d}\phi)(W_t^k(v,v))-\int_0^t e^{\lambda s/2}\,\mathrm{Hess}\,\phi(Q_s(\dot k(s)v),Q_s(v))\,ds$$

is a local martingale as well. From the formula

$$e^{\lambda t/2}\,\mathbf{d}\phi(Q_t(v))=\mathbf{d}\phi(v)+\int_0^t e^{\lambda s/2}(\mathrm{Hess}\,\phi)(//_s dB_s,Q_s(v))$$

it follows that

$$\int_0^t e^{\lambda s/2}(\mathrm{Hess}\,\phi)(Q_s(\dot k(s)v),Q_s(v))\,ds-e^{\lambda t/2}\,\mathbf{d}\phi(Q_t(v))\int_0^t\langle Q_s(\dot k(s)v),//_s dB_s\rangle \tag{2.2}$$

is a local martingale. We conclude that

$$(e^{\lambda t/2}\,\mathrm{Hess}\,\phi)(Q_t(k(t)v),Q_t(v))+(e^{\lambda t/2}\,\mathbf{d}\phi)(W_t^k(v,v))-e^{\lambda t/2}\,\mathbf{d}\phi(Q_t(v))\int_0^t\langle Q_s(\dot k(s)v),//_s dB_s\rangle$$

is a local martingale. ■

We shall use the following estimates to proceed with the Hessian formula for ϕ.

Lemma 2.2. Assume that $\mathbf{Ric} \geq -K_0$, $|R| \leq K_1$ and $|\mathbf{d}^*R+\nabla\mathbf{Ric}| \leq K_2$ on D for non-negative constants K_0, K_1 and K_2. Let $k \in C_b^1([0,\infty);\mathbb{R})$. For $t \geq 0$ and $\delta > 0$, it holds

$$|Q_t| \leq \mathrm{e}^{K_0 t/2} \quad \text{and} \tag{2.3}$$

$$\mathbb{E}\left[\left|W_t^k(v, \dot{k}(t)v)\right| 1_{\{t\leq\tau_D\}}\right] \leq \left(K_1\left(\int_0^t k(s)^2\,ds\right)^{1/2} + \frac{K_2}{2}\int_0^t |k(s)|\,ds\right)\mathrm{e}^{K_0 t}\,|\dot{k}(t)|, \tag{2.4}$$

where K_0, K_1 and K_2 are defined as in (1.3).

Proof. The first inequality follows from the lower Ricci curvature bound condition and the definition of Q_t. For $0 \leq s \leq t$, the damped parallel transport $Q_{s,t} = Q_t Q_s^{-1}\colon T_{X_s}D \to T_{X_t}D$ satisfies

$$DQ_{t,s} = -\frac{1}{2}\mathbf{Ric}^\sharp(Q_{t,s})\,dt, \quad Q_{s,s} = \mathbf{id}.$$

Thus, the lower bound of Ricci curvature $-K_0$ yields

$$|Q_{s,t}| \leq \mathrm{e}^{K_0(t-s)/2}.$$

According to the definition of W_t^k (see (1.5)), we have

$$\begin{aligned}
\mathbb{E}\left(|W_t^k(v,v)| 1_{\{t\leq\tau_D\}}\right) &\leq \mathbb{E}\left[1_{\{t\leq\tau_D\}}\left|Q_t\int_0^t Q_s^{-1}R(/\!/_s dB_s, Q_s(k(s)v))Q_s(v)\right|\right]\\
&\quad + \frac{1}{2}\mathbb{E}\left[1_{\{t\leq\tau_D\}}\left|Q_t\int_0^t Q_s^{-1}(\mathbf{d}^*R+\nabla\mathbf{Ric})(Q_s(k(s)v), Q_s(v))\,ds\right|\right]\\
&\leq \mathrm{e}^{\frac{K_0 t}{2}}\,\mathbb{E}\left[1_{\{t\leq\tau_D\}}\left|\mathrm{e}^{-\frac{K_0 t}{2}}Q_t\int_0^t Q_s^{-1}R(/\!/_s dB_s, Q_s(k(s)v))Q_s(v)\right|^2\right]^{1/2}\\
&\quad + \frac{K_2}{2}\mathbb{E}\left[1_{\{t\leq\tau_D\}}\left|\mathrm{e}^{\frac{1}{2}K_0 t}\int_0^t \mathrm{e}^{\frac{1}{2}K_0 s}|k(s)|\,ds\right|\right].
\end{aligned} \tag{2.5}$$

Moreover,

$$\begin{aligned}
&d\left|\mathrm{e}^{-\frac{1}{2}K_0 t}Q_t\int_0^t Q_s^{-1}R(/\!/_s dB_s, Q_s(k(s)v))Q_s(v)\right|^2\\
&= 2\,\mathrm{e}^{-K_0 t}\left\langle R(/\!/_t dB_t, Q_t(k(t)v))Q_t(v), Q_t\int_0^t Q_s^{-1}R(/\!/_s dB_s, Q_s(k(s)v))Q_s(v)\right\rangle\\
&\quad + \mathrm{e}^{-K_0 t}\left|R^\sharp(Q_t(k(t)v), Q_t(v))\right|_{\mathrm{HS}}^2\,dt\\
&\quad - \mathrm{e}^{-K_0 t}\,\mathbf{Ric}\left(Q_t\int_0^t Q_s^{-1}R(/\!/_s dB_s, Q_s(k(s)v))Q_s(v), Q_t\int_0^t Q_s^{-1}R(/\!/_s dB_s, Q_s(k(s)v))Q_s(v)\right)dt\\
&\quad - K_0\,\mathrm{e}^{-K_0 t}\left|Q_t\int_0^t Q_s^{-1}R(/\!/_s dB_s, Q_s(k(s)v))Q_s(v)\right|^2 dt\\
&\overset{m}{\leq} \mathrm{e}^{-K_0 t}\left|R^\sharp(Q_t(k(t)v), Q_t(v))\right|_{\mathrm{HS}}^2\,dt \leq K_1^2\,\mathrm{e}^{-K_0 t}|Q_t|^4 k(t)^2\,dt \leq K_1^2\,\mathrm{e}^{K_0 t}k(t)^2\,dt, \quad t \leq \tau_D.
\end{aligned}$$

Combining this with (2.5), we have

$$\mathbb{E}\left(|W_t^k(v,v)| 1_{\{t\leq\tau_D\}}\right) \leq K_1\,\mathrm{e}^{\frac{1}{2}K_0 t}\left(\int_0^t \mathrm{e}^{K_0 s}k(s)^2\,ds\right)^{1/2} + \frac{K_2}{2}\,\mathrm{e}^{K_0 t}\int_0^t |k(s)|\,ds.$$

This completes the proof. ∎

By the results above, the following Hessian formula for eigenfunctions ϕ is obtained.

Theorem 2.3. Let D be a compact Riemannian manifold with boundary ∂D. Let $X_\bullet(x)$ be a Brownian motion starting from $x \in D$ and τ_D be its first hitting time of ∂D. Suppose that k is a non-negative function in $C_b^1([0,\infty);\mathbb{R})$ such that $k(0)=1$. Then for $(\phi,\lambda) \in \mathbf{Eig}(\Delta)$, $t \geq 0$ and $v \in T_xD$,

$$
\begin{aligned}
(\mathbf{Hess}\,\phi)(v,v) &= \mathbb{E}^x\left[\mathrm{e}^{(t\wedge\tau_D)\lambda/2}(\mathbf{Hess}\,\phi)(Q_{t\wedge\tau_D}(k(t\wedge\tau_D)v), Q_{t\wedge\tau_D}(v)) + \mathrm{e}^{(t\wedge\tau_D)\lambda/2}(\mathbf{d}\phi)(W^k_{t\wedge\tau_D}(v,v))\right] \\
&\quad - \mathbb{E}^x\left[\mathrm{e}^{(t\wedge\tau_D)\lambda/2}\,\mathbf{d}\phi(Q_{t\wedge\tau_D}(v))\int_0^{t\wedge\tau_D}\langle Q_s(\dot{k}(s)v), //_s dB_s\rangle\right]. \qquad (2.6)
\end{aligned}
$$

Proof. The claim follows by taking expectation of the martingale (2.1) at time 0 and $t\wedge\tau_D$. Recall that $|Q_t| \leq \mathrm{e}^{K_0 t/2}$. For $x \in \partial D$ formula (2.6) is obviously tautological since $\tau_D \equiv 0$. ■

To derive Hessian estimates of ϕ from Theorem 2.3 requires estimates of $\mathbf{Hess}\,\phi$ on the boundary ∂D. To this end, we first note the following observation. Since $\phi = 0$ on the boundary ∂D, we have $\nabla\phi = N(\phi)N$ on ∂D. We extend the normal vector field N to a tubular neighborhood of ∂D as $N = \nabla\rho_{\partial D}$ where $\rho_{\partial D}(x) = \mathrm{dist}(x,\partial D)$ denotes the smooth distance function close to the boundary (see Remark 2.5 below for the details).

Lemma 2.4. For $x \in \partial D$ let $H(x)$ be the mean curvature of the boundary. Then,

$$N^2(\phi)(x) = -H(x)N(\phi)(x), \quad x \in \partial D.$$

Remark 2.5. Assuming that the boundary ∂D is smooth, let N be the unit inward normal vector field N on ∂D. Furthermore, let

$$\Phi\colon [0,r_0[\,\times\partial D \to D, \quad (r,x)\mapsto \exp_x(rN), \qquad (2.7)$$

be the geodesic from $x \in \partial D$ orthogonal to ∂D and parametrized by its arc length r. As the differential of Φ at any point $(0,x)$ has full rank, we find $\varepsilon_0 > 0$ such that Φ is a diffeomorphism from $[0,\varepsilon_0[\times\partial D$ onto the open neighborhood $\{x\in D\colon \rho_{\partial D}(x) < \varepsilon_0\}$ of ∂D in D. This allows to extend N to a tubular (collar) neighborhood of ∂D as $\Phi_*\frac{\partial}{\partial r}$. By construction then $\nabla_N N = 0$. If X is a vector field on ∂D tangential to ∂D, we extend it to the neighborhood of ∂D as being independent of the real variable in the product $[0,\varepsilon_0[\,\times\partial D$. By construction, close to the boundary, the distance function $\rho_{\partial D}(x) = \mathrm{dist}(x,\partial D)$ is smooth and satisfies $N = \nabla\rho_{\partial D}$.

Proof of Lemma 2.4. On the boundary ∂D we have

$$
\begin{aligned}
0 = \lambda\phi = \Delta\phi &= \sum_{i=1}^n \langle\nabla_{X_i}\nabla\phi, X_i\rangle = \langle\nabla_N\nabla\phi, N\rangle + \sum_{i=2}^n\langle\nabla_{X_i}\nabla\phi, X_i\rangle \\
&= \langle\nabla_N\nabla\phi, N\rangle + \sum_{i=2}^n\langle\nabla_{X_i}(N(\phi)N), X_i\rangle \\
&= N\langle\nabla\phi, N\rangle + \sum_{i=2}^n X_iN(\phi)\langle N, X_i\rangle + N(\phi)\sum_{i=2}^n\langle\nabla_{X_i}N, X_i\rangle \\
&= N^2(\phi) + N(\phi)\sum_{i=2}^n \Pi(X_i,X_i) = N^2(\phi) + N(\phi)\,\mathbf{tr}\,\Pi,
\end{aligned}
$$

where for $x \in \partial D$, $\{X_i\}_{1\leq i\leq n}$ denotes an orthonormal basis of T_xD with $X_1 = N$. As $(\mathbf{tr}\,\Pi)(x) = H(x)$, $x \in \partial D$, the proof is completed. ■

The following lemma is taken from [2, Lemma 2.4 and Proposition 2.5] and allows to estimate the values of $|\nabla\phi|$ on the boundary. Here we use $\alpha^+ + \sqrt{\frac{2}{\pi t}}$ as upper bound for the right-hand-side in [2, Eq. (2.29)].

Lemma 2.6. Let $\alpha_0 \in \mathbb{R}$ such that

$$\Delta\rho_{\partial D} \leq \alpha_0 \tag{2.8}$$

outside $\mathbf{Cut}(\partial D)$. Then for any $t > 0$,

$$\|\nabla\phi\|_{\partial D,\infty} = \|N(\phi)\|_{\partial D,\infty} \leq \|\phi\|_\infty\, e^{\lambda t/2}\left(\alpha_0^+ + \frac{\sqrt{2}}{\sqrt{\pi t}}\right).$$

In particular,

$$\|\nabla\phi\|_{\partial D,\infty} \leq \|\phi\|_\infty\, e^{1/2}\left(\alpha_0^+ + \frac{\sqrt{2\lambda}}{\sqrt{\pi}}\right). \tag{2.9}$$

Remark 2.7. With constants $K_0, \theta > 0$ such that $\mathbf{Ric} \geq -K_0$ on D and $H \geq -\theta$ on the boundary ∂D, where $H(x)$ is the mean curvature of D at $x \in D$, let

$$\alpha_0 = \max\left\{\theta, \sqrt{(n-1)K_0}\right\}.$$

Then estimate (2.8) holds true for this α_0.

Next, we introduce some results on local time estimate of reflecting Brownian motion, which is also a tool in the boundary estimate of $|\mathbf{Hess}\,\phi|$. Let us recall some basic notations on it. The reflecting Brownian motion on D with generator $\frac{1}{2}\Delta$ satisfies the SDE

$$dX_t = //_t \circ dB_t^x + \frac{1}{2}N(X_t)\,dl_t, \quad X_0 = x,$$

where B_t^x is a standard Brownian motion on the Euclidean space $T_xD \cong \mathbb{R}^n$ and l_t is the local time supported on ∂D (see [22] for details). Now we turn to the problem of estimating $\mathbb{E}[e^{\alpha l_t/2}]$ for $\alpha > 0$ by exploiting a specific class of functions h.

Lemma 2.8. Suppose that $h \in C^\infty(D)$ such that $h \geq 1$ and $N \log h \geq 1$. For $\alpha > 0$ let

$$K_{h,\alpha} = \sup\{-\Delta \log h + \alpha|\nabla \log h|^2\}.$$

Then,

$$\mathbb{E}[e^{\alpha l_t/2}] \leq \|h\|_\infty^\alpha \exp\left(\frac{\alpha}{2}K_{h,\alpha}t\right).$$

Proof. By Itô's formula we have

$$\begin{aligned} dh^{-\alpha}(X_t) &= \langle\nabla h^{-\alpha}(X_t), //_t\, dB_t\rangle + \frac{1}{2}\Delta h^{-\alpha}(X_t)\,dt + \frac{1}{2}Nh^{-\alpha}(X_t)\,dl_t \\ &\leq \langle\nabla h^{-\alpha}(X_t), //_t\, dB_t\rangle - \alpha h^{-\alpha}(X_t)\left(-\frac{1}{2}K_{h,\alpha}\,dt + \frac{1}{2}N\log h(X_t)\,dl_t\right). \end{aligned}$$

Hence,

$$M_t := h^{-\alpha}(X_t)\exp\left(-\frac{\alpha}{2}K_{h,\alpha}t + \frac{\alpha}{2}\int_0^t N\log h(X_s)\,dl_s\right)$$

is a local supermartingale. Therefore, by Fatou's lemma and taking into account that $h \geq 1$, we get

$$\mathbb{E}\left[h^{-\alpha}(X_t)\exp\left(-\frac{\alpha}{2}K_{h,\alpha}t + \frac{\alpha}{2}\int_0^t N\log h(X_s)\,dl_s\right)\right] \leq h^{-\alpha}(x) \leq 1.$$

Since $N\log h(x) \geq 1$ we conclude that

$$\mathbb{E}\left[\exp\left(\frac{\alpha}{2}l_t\right)\right] \leq \mathbb{E}\left[\exp\left(\frac{\alpha}{2}\int_0^t N\log h(X_s)\,dl_s\right)\right] \leq \|h\|_\infty^\alpha \exp\left(\frac{\alpha}{2}K_{h,\alpha}t\right).$$

■

At the end of this subsection, we collect some Hessian comparison results for $\rho_{\partial D}$. Let p be the orthogonal projection of x on ∂D, and let $\gamma(s) = \exp_p(sN), s \in [0, \rho_{\partial D}(x)]$ be the geodesic from p to x. Let $\{J(s)\}_{s\in[0,\rho_{\partial D}(x)]}$ be the Jacobi field along γ such that $J(\rho_{\partial D}(x)) = v$ for $v \in T_xD$, and $\dot{J}(0) = -\mathrm{II}^\sharp(J(0)) \in T_p\partial D$, where $\langle \mathrm{II}^\sharp(J(0)), w\rangle = \mathrm{II}(J(0), w)$ for $w \in T_p\partial D$. From the variation formula of $\rho_{\partial D}$, we know that

$$\mathrm{Hess}\,\rho_{\partial D}(v,v) = -\mathrm{II}(J(0),J(0)) + \int_0^{\rho_{\partial D}(x)} \left(|\dot{J}(s)|^2 - \langle R(\dot\gamma(s),J(s))\dot\gamma(s),J(s)\rangle\right) ds. \tag{2.10}$$

The following result is essentially due to Kasue [10, 11] (see also Theorem A.1 in [20]).

Lemma 2.9 (Hessian Comparison). Let σ and k be non-negative constants such that $|\mathrm{II}| \leq \sigma$ and $|\mathrm{Sect}| \leq k$ on $\partial_{r_0}D$, where $\rho_{\partial D}$ is smooth $\partial_{r_0}D$. Then,

$$\frac{\ell'_{\sigma,k}}{\ell_{\sigma,k}}(\rho_{\partial D}(x)) \leq \mathrm{Hess}\,\rho_{\partial D}(v,v) \leq \frac{\ell'_{-\sigma,-k}}{\ell_{-\sigma,-k}}(\rho_{\partial D}(x)), \quad \rho_{\partial D} \leq r_0 \wedge \ell_{\sigma,k}^{-1}(0).$$

Moreover, for $\rho_{\partial D}(x) \leq r_0 \wedge \ell_{\sigma,k}^{-1}(\frac{1}{2})$,

$$|\,\mathrm{Hess}\,\rho_{\partial D}| \leq 2\max\{\sigma, \sqrt{k}\}.$$

Proof. The proof of the first inequality can be found in [22, Theorem 1.2.2]. Based on this, we have for $k, \sigma \geq 0$,

$$\mathrm{Hess}\,\rho_{\partial D}(v,v) \leq \max\{\sigma, \sqrt{k}\}.$$

On the other hand, for $\rho_{\partial D}(x) \leq r_0 \wedge \ell_{k,\sigma}^{-1}(\frac{1}{2})$,

$$\mathrm{Hess}\,\rho_{\partial D}(v,v) \geq \frac{\ell'_{k,\sigma}(\rho_{\partial D}(x))}{\ell_{k,\sigma}(\rho_{\partial D}(x))} \geq 2\ell_{k,\sigma}(\rho_{\partial D}(x)) \geq -2\max\{\sigma, \sqrt{k}\}.$$

This completes the proof of the second inequality. ■

2.2 Hessian estimate of Dirichlet eigenfunctions

Lemmas 2.4, 2.6, and 2.8 allow to derive an estimate of $|\,\mathrm{Hess}\,\phi|$ on the boundary ∂D.

Lemma 2.10. Let K_0, σ be non-negative constants such that $\mathrm{Ric} \geq -K_0$, $|\mathrm{II}| \leq \sigma$. Suppose that the distance function $\rho_{\partial D}$ is smooth on $\partial_{r_0}D := \{x : \rho_{\partial D}(x) \leq r_0\}$ for some constant $r_0 > 0$. Then for $x \in \partial D$,

$$\begin{aligned}\left\|\,\mathrm{Hess}(\phi)\right\|_{\partial D,\infty} \leq{}& (n-1)\sigma\,\|N(\phi)\|_{\partial D,\infty} \\ &+ \|h\|_\infty^\sigma\, e^{\frac{1}{2}(K_0+\sigma K_{h,\sigma})t}\left(C_1\frac{1}{\sqrt{t}} + (C_2 + C_1\lambda)\sqrt{t}\right)\|\phi\|_\infty \\ &+ \|h\|_\infty^\sigma\, e^{\frac{1}{2}(K_0+\sigma K_{h,\sigma})t}\left(\frac{1}{\sqrt{t}} + (C_3+\lambda)\sqrt{t}\right)\|\nabla\phi\|_\infty \\ &+ \|h\|_\infty^\sigma\, e^{\frac{1}{2}(K_0+\sigma K_{h,\sigma})t}\sqrt{t}C_4\|\,\mathrm{Hess}\,\phi\|_\infty,\end{aligned}$$

where $h \in C^\infty(D)$ such that $h \geq 1$ and $N \log h \geq 1$ and

$$K_{h,\sigma} = \sup\{-\Delta \log h + \sigma |\nabla \log h|^2\},$$

and the constants C_1, C_2, C_3, C_4 are defined as

$$C_1 = \left\| \Delta \rho_{\partial D} \right\|_{\partial_{r_0} D},$$

$$C_2 = \left\| \Delta(\psi(\rho_{\partial D}))\Delta\rho_{\partial D} + 2|\psi'(\rho_{\partial D})| \cdot |\nabla(\Delta\rho_{\partial D})| + |\Delta^2 \rho_{\partial D}| \right\|_{\partial_{r_0} D},$$

$$C_3 = \left\| \, |\psi''(\rho_{\partial D})| + 3|\psi'(\rho_{\partial D})\Delta\rho_{\partial D}| + 3|\nabla(\Delta\rho_{\partial D})| \, \right\|_{\partial_{r_0} D},$$

$$C_4 = \left\| 2|\psi'(\rho_{\partial D})| + 2(n-1)|\,\mathbf{Hess}\,\rho_{\partial D}| \, \right\|_{\partial_{r_0} D},$$

where $\psi \in C^2(\mathbb{R}^+, [0,1])$ satisfies $\psi(0) = 1$, $\psi'(0) = 0$ and $\psi(r) = 0$ for $r > r_0$.

Proof. Given $x \in \partial D$, let $\{X_i\}_{1 \leq i \leq n}$ be an orthonormal basis of $T_x D$ with $X_1 = N$. Then,

$$\begin{aligned} |\,\mathbf{Hess}(\phi)(X_i, X_j)| &= |\nabla \mathbf{d}\phi(X_i, X_j)| = |\langle \nabla_{X_i} \nabla \phi, X_j\rangle| \\ &= |X_i \langle \nabla \phi, X_j\rangle - \langle \nabla \phi, \nabla_{X_i} X_j\rangle|. \end{aligned}$$

By assumption we have $|\mathbf{II}| \leq \sigma$. If $X_i,\ X_j \in T_x \partial D$, that is $i, j \neq 1$, then $\langle \nabla\phi, X_j\rangle|_{\partial D} = 0$ and

$$|\,\mathbf{Hess}(\phi)(X_i, X_j)| = |-N(\phi)\langle N, \nabla_{X_i} X_j\rangle| \leq \sigma |N(\phi)|. \tag{2.11}$$

If $X_i = X_j = N$, that is, $i = j = 1$, then $\nabla_N N|_{\partial D} = 0$ and

$$|\,\mathbf{Hess}(\phi)(N, N)| = |N^2(\phi)| \leq |HN(\phi)| \leq (n-1)\sigma |N(\phi)|. \tag{2.12}$$

If $X_j \in T_x \partial D$ and $X_i = N$ (i.e., $j \neq 1$ and $i = 1$), then

$$|\,\mathbf{Hess}(\phi)(X_j, N)|(x) = |N X_j(\phi)|(x). \tag{2.13}$$

In order to get control on (2.13), we shall use a probabilistic argument based on the Brownian motion on D reflected at the boundary. Before going into the details, we recall our conventions on the extension of vector fields from ∂D to a tubular neighborhood of the boundary; see Remark 2.5.

Let N be the extension of the normal vector field to a tubular neighborhood $\partial_{r_0} D := \{x : \rho_{\partial D}(x) \leq r_0\}$ of ∂D and define

$$\varphi(x) = \psi(\rho_{\partial D}(x))\mathbf{div}(\phi N), \quad x \in \partial_{r_0} D, \tag{2.14}$$

where $\psi \in C^2(\mathbb{R}^+, [0,1])$ satisfies $\psi(0) = 1$, $\psi'(0) = 0$ and $\psi(r) = 0$ for $r > r_0$. Using the formula $\mathbf{div}(\phi N) = N(\phi) + \phi\, \mathbf{div}(N)$, along with Lemma 2.4, we observe for $x \in \partial D$,

$$N(\varphi)(x) = \psi'(0)\mathbf{div}(\phi N) + N(\mathbf{div}(\phi N)) = 0.$$

Thus, φ satisfies the Neumann boundary conditions on D.

Let now X_t be the reflecting Brownian motion on D and $P_t^N f(x) = \mathbb{E}^x[f(X_t)]$ for $f \in \mathcal{B}_b(D)$ the corresponding Neumann semigroup. According to the Kolmogorov equation, we have

$$\varphi(x) = P_t^N(\varphi)(x) - \frac{1}{2}\int_0^t P_s^N(\Delta\varphi)(x)\, ds.$$

Taking derivative on both sides of the equation yields

$$X_i(\varphi)(x) = X_i(P_t^N \varphi)(x) - \frac{1}{2}\int_0^t X_i(P_s^N \Delta\varphi)(x)\, ds,$$

where X_i is tangential to ∂D. We first observe that for $x \in \partial D$,

$$\begin{aligned} X_i(\varphi)(x) &= X_i(\psi(\rho_{\partial D}))(x)\mathbf{div}(\phi N)(x) + \psi(\rho_{\partial D}(x))X_i(\mathbf{div}(\phi N))(x) = X_i(\mathbf{div}(\phi N))(x) \\ &= X_i N(\phi)(x) + X_i(\phi)(x)\mathbf{div}(N)(x) + \phi(x)X_i(\mathbf{div}(N))(x) \\ &= X_i N(\phi)(x). \end{aligned}$$

To deal with the upper bound, we use the Bismut formula established in [22, Theorem 3.2.1] for the compact manifold D, which gives

$$|\nabla P_t^N f| \le \frac{1}{\sqrt{t}} \mathrm{e}^{\frac{1}{2}K_0 t}\, \mathbb{E}^x[\mathrm{e}^{\sigma l_t}]^{\frac{1}{2}} \|f\|_\infty,$$

where l_t is the local time supported on ∂D. By Lemma 2.8 of the previous subsection, we have

$$\mathbb{E}^x[\mathrm{e}^{\sigma l_t}] \le \|h\|_\infty^{2\sigma} \exp\left(\sigma K_{h,2\sigma} t\right),$$

where $h \in C^\infty(D)$ such that $h \ge 1$ and $N \log h \ge 1$ and

$$K_{h,2\sigma} = \sup\{-\Delta \log h + 2\sigma |\nabla \log h|^2\}.$$

We then conclude that

$$|X_i N(\phi)|(x) \le \|h\|_\infty^\sigma \mathrm{e}^{\frac{1}{2}(K_0 + \sigma K_{h,2\sigma})t} \left[\frac{1}{\sqrt{t}} \|\varphi\|_{B(x,r_0)} + \sqrt{t}\, \|\Delta\varphi\|_{B(x,r_0)}\right]. \tag{2.15}$$

According to the definition of φ in (2.14), we have

$$\|\varphi\|_\infty \le \|\nabla\phi\|_\infty + \|\mathbf{div}(N)\|_{\partial_{r_0} D} \|\phi\|_\infty$$

By commutation rules, we calculate

$$\begin{aligned} \Delta\varphi &= \Delta((\psi(\rho_{\partial D}))\mathbf{div}(\phi N)) \\ &= \Delta(\psi(\rho_{\partial D}))\mathbf{div}(\phi N) + 2\psi'(\rho_{\partial D})N(\mathbf{div}(\phi N)) + \psi(\rho_{\partial D}))\Delta(\mathbf{div}(\phi N)) \\ &= \Delta(\psi(\rho_{\partial D}))(\phi\mathbf{div}(N) + N(\phi)) + 2\psi'(\rho_{\partial D})\Big(\phi N(\mathbf{div}(N)) + N(\phi)\mathbf{div}(N) + N^2(\phi)\Big) \\ &\quad + \psi(\rho_{\partial D}))\Delta(\mathbf{div}(\phi N)) \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} \Delta(\mathbf{div}(\phi N)) &= \mathbf{div}((\Box - \mathbf{Ric}^\sharp)(\phi N)) \\ &= \mathbf{div}(\Delta(\phi)N) + \mathbf{div}(\phi\Box N) + 2\mathbf{div}(\nabla_{\nabla\phi}N) - \phi\mathbf{div}(\mathbf{Ric}^\sharp(N)) - \mathbf{Ric}(N, \nabla\phi) \\ &= -\lambda\mathbf{div}(\phi N) + \phi\mathbf{div}((\Box - \mathbf{Ric}^\sharp)N) + \langle \Box N, \nabla\phi\rangle + 2\mathbf{div}(\nabla_{\nabla\phi}N) - \mathbf{Ric}(N, \nabla\phi), \end{aligned} \tag{2.17}$$

where $\Box = \mathbf{tr}\,\nabla^2$ and $\mathbf{Ric}^\sharp : TD \to TD$ such that $\langle \mathbf{Ric}^\sharp(v), w\rangle = \mathbf{Ric}(v, w)$ for $v, w \in T_x D$, $x \in D$. Let $\{e_i\}_{1\le i\le n}$ be orthonormal basis of TD about x satisfying $\nabla e_i(x) = 0$. We then have

$$\nabla_{\nabla\phi} N = \sum_{i=1}^n (e_i(\phi))\nabla_{e_i} N,$$

and as a consequence

$$\begin{aligned} \mathbf{div}(\nabla_{\nabla\phi}N) &= \sum_{i=1}^n \left(\langle \nabla e_i(\phi), \nabla_{e_i} N\rangle + e_i(\phi)\mathbf{div}(\nabla_{e_i} N)\right) \\ &= \langle \mathbf{Hess}(\phi), \nabla N\rangle + \langle \nabla\phi, \sum_{i=1}^n \mathbf{div}(\nabla_{e_i} N)e_i\rangle \\ &= \langle \mathbf{Hess}(\phi), \nabla N\rangle + \langle \nabla\phi, \nabla(\mathbf{div}(N))\rangle. \end{aligned}$$

Combining this with (2.17) yields

$$\begin{aligned} \Delta(\mathbf{div}(\phi N)) &= \phi(-\lambda\mathbf{div}(N) + \Delta(\mathbf{div}(N))) - \lambda N(\phi) + 2\langle \mathbf{Hess}(\phi), \nabla N\rangle + 2\langle \nabla\phi, \nabla(\mathbf{div}(N))\rangle \\ &\quad + \langle \Box N, \nabla\phi\rangle - \mathbf{Ric}(N, \nabla\phi). \end{aligned}$$

From the fact that $N = \nabla\rho_{\partial D}$ and the Weitzenböck formula, we observe that

$$\mathbf{div}(N) = \Delta\rho_{\partial D}, \quad \langle\nabla_{\cdot}N, \cdot\rangle = \mathbf{Hess}\,\rho_{\partial D} \quad \text{and} \quad \langle\Box N, \nabla\phi\rangle - \mathbf{Ric}(N, \nabla\phi) = \langle\nabla\Delta\rho_{\partial D}, \nabla\phi\rangle. \tag{2.18}$$

Combining the equations (2.16), (2.17) and (2.18), we conclude

$$\begin{aligned}\Delta\varphi &= \Delta(\psi(\rho_{\partial D}))(\Delta\rho_{\partial D})\phi + \Delta(\psi(\rho_{\partial D}))N(\phi) + 2\psi'(\rho_{\partial D})\Big(\phi N(\Delta\rho_{\partial D}) + N(\phi)\Delta\rho_{\partial D} + N^2(\phi)\Big)\\ &\quad + \psi(\rho_{\partial D})\phi(-\lambda\Delta\rho_{\partial D} + \Delta^2\rho_{\partial D}) - \lambda\psi(\rho_{\partial D})N(\phi) + 2\psi(\rho_{\partial D})\langle\mathbf{Hess}(\phi), \nabla N\rangle + 3\psi(\rho_{\partial D})\langle\nabla\Delta\rho_{\partial D}, \nabla\phi\rangle\\ &= \big(\Delta(\psi(\rho_{\partial D}))(\Delta\rho_{\partial D}) + 2\psi'(\rho_{\partial D})N(\Delta\rho_{\partial D}) + \psi(\rho_{\partial D})(\Delta^2\rho_{\partial D} - \lambda\Delta\rho_{\partial D})\big)\,\phi\\ &\quad + \big(\Delta(\psi(\rho_{\partial D})) + 2\psi'(\rho_{\partial D})\Delta\rho_{\partial D} - \lambda\psi(\rho_{\partial D})\big)N(\phi) + 3\psi(\rho_{\partial D})\langle\nabla\Delta\rho_{\partial D}, \nabla\phi\rangle\\ &\quad + 2\psi'(\rho_{\partial D})N^2(\phi) + 2\psi(\rho_{\partial D})\langle\mathbf{Hess}(\phi), \nabla N\rangle,\end{aligned}$$

which together with (2.15) implies that

$$\begin{aligned}&|X_i N(\phi)|(x)\\ &\le \|h\|_\infty^\sigma\, e^{\frac{1}{2}(K_0+2\sigma K_{h,\sigma})t}\left(C_1\frac{1}{\sqrt{t}} + (C_2 + C_1\lambda)\sqrt{t}\right)\|\phi\|_\infty\\ &\quad + \|h\|_\infty^\sigma\, e^{\frac{1}{2}(K_0+\sigma K_{h,2\sigma})t}\left(\frac{1}{\sqrt{t}} + (C_3+\lambda)\sqrt{t}\right)\|\nabla\phi\|_\infty\\ &\quad + \|h\|_\infty^\sigma\, e^{\frac{1}{2}(K_0+\sigma K_{h,2\sigma})t}\sqrt{t}\,C_4\|\,\mathbf{Hess}\,\phi\|_\infty,\end{aligned}$$

where

$$\begin{aligned}C_1 &= \|\Delta\rho_{\partial D}\|_{\partial_{r_0}D},\\ C_2 &= \|\Delta(\psi(\rho_{\partial D}))\Delta\rho_{\partial D} + 2|\psi'(\rho_{\partial D})|\cdot|\nabla(\Delta\rho_{\partial D})| + |\Delta^2\rho_{\partial D}|\|_{\partial_{r_0}D},\\ C_3 &= \|\,|\psi''(\rho_{\partial D})| + 3|\psi'(\rho_{\partial D})\Delta\rho_{\partial D}| + 3|\nabla(\Delta\rho_{\partial D})|\,\|_{\partial_{r_0}D},\\ C_4 &= \|2|\psi'(\rho_{\partial D})| + 2(n-1)|\,\mathbf{Hess}\,\rho_{\partial D}|\,\|_{\partial_{r_0}D}.\end{aligned}$$

The proof is completed by combining the above estimate with (2.11) and (2.12). ■

Combining the estimates in Lemmas 2.6 and 2.10 with Theorem 2.3, we are now in a position to prove our main result.

Theorem 2.11. Let D be a compact Riemannian manifold with boundary ∂D. Let K_0, K_1, K_2 and σ be non-negative constants such that $\mathbf{Ric} \ge -K_0$, $|R| \le K_1$ and $|\mathbf{d}^*R + \nabla\mathbf{Ric}| \le K_2$ on D, and that $|\mathbf{II}| \le \sigma$ on the boundary ∂D. Assume that the distance function $\rho_{\partial D}$ is smooth on the tubular neighborhood $\partial_{r_0}D = \{x\colon \rho_{\partial D}(x) \le r_0\}$ of ∂D for some constant $r_0 > 0$, and let $\alpha, \beta, \gamma \in \mathbb{R}$ be such that

$$|\,\mathbf{Hess}\,\rho_{\partial D}| \le \frac{\alpha}{n-1}, \quad |\nabla(\Delta\rho_{\partial D})| \le \beta, \quad |\Delta^2\rho_{\partial D}| \le \gamma \quad \text{on } \partial_{r_0}D. \tag{2.19}$$

Letting $h \in C^\infty(D)$ with $\mathbf{min}_D\, h = 1$ and $N\,\mathbf{log}\,h|_{\partial D} \ge 1$, then

$$\begin{aligned}\frac{\|\,\mathbf{Hess}\,\phi\|}{\|\phi\|_\infty} &\le 2(n-1)\sigma e\left(\alpha + \sqrt{\frac{2\lambda}{\pi}}\right) + \frac{K_2\alpha}{16(\frac{3}{r_0}+\alpha)^2}\\ &\quad + \frac{2\left(K_1 + \frac{9\alpha}{r_0} + \frac{6}{r_0^2} + 3\beta\right)\alpha\sqrt{e} + K_2 c_0\sqrt{e} + \frac{6}{r_0}(\alpha^2 + 2\beta) + \frac{12\alpha}{r_0^2} + 2\gamma + 2\alpha(1+\sqrt{e})\lambda}{\frac{12}{r_0} + 4\alpha}\\ &\quad + 2\alpha\, e\,(2\|h\|_\infty^\sigma + 1)\,\mathbf{max}\left\{\sqrt{\lambda + 2K_0 + \sigma K_{h,2\sigma}},\ 4\sqrt{e}\|h\|_\infty^\sigma\left(\frac{3}{r_0} + \alpha\right)\right\}\\ &\quad + 2c_0 e\,(\|h\|_\infty^\sigma + 1)\,\mathbf{max}\left\{\lambda + 2K_0 + \sigma K_{h,2\sigma},\ 4\,e\,\|h\|_\infty^{2\sigma}\left(\frac{6}{r_0} + 2\alpha\right)^2\right\}\\ &\quad + 2c_0\, e\left(K_1 + \|h\|_\infty^\sigma\left(\frac{9\alpha}{r_0} + \frac{6}{r_0^2} + 3\beta + \lambda\right)\right),\end{aligned} \tag{2.20}$$

for $c_0 = \sqrt{\frac{2}{\pi}} + \frac{1}{4}\sqrt{\frac{\pi}{2}}$.

Proof. According to formula (2.6) we have

$$\begin{aligned}|\operatorname{Hess}\phi(v,v)| &= \mathbb{E}\Big[e^{\lambda(t\wedge\tau_D)/2}\operatorname{Hess}\phi\big(Q_{t\wedge\tau_D}(k(t\wedge\tau_D)v),Q_{t\wedge\tau_D}(v)\big)\Big]\\ &\quad+\mathbb{E}\Big[e^{\lambda(t\wedge\tau_D)/2}\,\mathbf{d}\phi(W^k_{t\wedge\tau_D}(v,v))\Big]\\ &\quad-\mathbb{E}\left[e^{\lambda(t\wedge\tau_D)/2}\,\mathbf{d}\phi(Q_{t\wedge\tau_D}(v))\int_0^{t\wedge\tau_D}\langle Q_s(\dot{k}(s)v),//_s dB_s\rangle\right].\end{aligned}$$

Taking $k(s)=(t-s)/t$ for $s\in[0,t]$ and $v\in T_xD$, $|v|=1$, in the equation yields

$$\begin{aligned}|\operatorname{Hess}\phi(v,v)| &\le \mathbb{E}\left[1_{\{\tau_D\le t\}}\,e^{(\frac{\lambda}{2}+K_0)\tau_D}\,\frac{t-\tau_D}{t}\,\|\operatorname{Hess}(\phi)\|_{\partial D,\infty}\right]\\ &\quad+\|\mathbf{d}\phi\|_\infty\left(K_1\sqrt{t}+\frac{K_2}{2}t\right)e^{\left(\frac12\lambda+K_0\right)t}\\ &\quad+\|\mathbf{d}\phi\|_\infty\frac{e^{\left(\frac12\lambda+K_0\right)t}}{\sqrt{t}}.\end{aligned}$$

By Lemma 2.10, we have

$$\begin{aligned}|\operatorname{Hess}\phi(v,v)| &\le \mathbb{E}\Bigg[1_{\{\tau_D\le t\}}\,e^{(\frac{\lambda}{2}+K_0)\tau_D}\,\frac{t-\tau_D}{t}\Bigg((n-1)\sigma\,\|N\phi\|_{\partial D,\infty}\\ &\quad+\|h\|_\infty^\sigma\,e^{\frac12(K_0+\sigma K_{h,2\sigma})(t-\tau_D)}\left(C_1\frac{1}{\sqrt{t-\tau_D}}+(C_2+\lambda C_1)\sqrt{t-\tau_D}\right)\|\phi\|_\infty\\ &\quad+\|h\|_\infty^\sigma\,e^{\frac12(K_0+\sigma K_{h,2\sigma})(t-\tau_D)}\left(\frac{1}{\sqrt{t-\tau_D}}+(C_3+\lambda)\sqrt{t-\tau_D}\right)\|\mathbf{d}\phi\|_\infty\\ &\quad+\|h\|_\infty^\sigma\,e^{\frac12(K_0+\sigma K_{h,2\sigma})(t-\tau_D)}\sqrt{t-\tau_D}\,C_4\|\operatorname{Hess}\phi\|_\infty\Bigg\}\Bigg]\\ &\quad+\|\mathbf{d}\phi\|_\infty\left(K_1\sqrt{t}+\frac{K_2}{2}t\right)e^{\left(\frac12\lambda+K_0\right)t}\\ &\quad+\|\mathbf{d}\phi\|_\infty\frac{e^{\left(\frac12\lambda+K_0\right)t}}{\sqrt{t}},\end{aligned}\tag{2.21}$$

where C_1, C_2, C_3 and C_4 are defined as in Lemma 2.10. Combining this with the fact that

$$\frac{t-\tau_D}{t}\,\frac{1}{\sqrt{t-\tau_D}}=\frac{\sqrt{t-\tau_D}}{t}\le\frac{1}{\sqrt{t}}$$

and then substituting back into (2.21) and using (2.9) to estimate $\|N\phi\|_{\partial D,\infty}$, we obtain

$$\begin{aligned}|\operatorname{Hess}\phi(v,v)| &\le (n-1)\sigma\,e^{\left(\frac{\lambda}{2}+K_0\right)t}\sqrt{e}\left(\alpha+\sqrt{\frac{2\lambda}{\pi}}\right)\|\phi\|_\infty\\ &\quad+\|h\|_\infty^\sigma\,e^{\left(\frac{\lambda}{2}+K_0+\frac{\sigma K_{h,2\sigma}}{2}\right)t}\left(\frac{C_1}{\sqrt{t}}+(C_2+C_1\lambda)\sqrt{t}\right)\|\phi\|_\infty\\ &\quad+\|h\|_\infty^\sigma\,e^{\left(\frac{\lambda}{2}+K_0+\frac{\sigma K_{h,2\sigma}}{2}\right)t}\left(\frac{1}{\sqrt{t}}+(C_3+\lambda)\sqrt{t}\right)\|\mathbf{d}\phi\|_\infty\\ &\quad+C_4\|h\|_\infty^\sigma\,e^{\left(\frac{\lambda}{2}+K_0+\frac{\sigma K_{h,2\sigma}}{2}\right)t}\sqrt{t}\,\|\operatorname{Hess}\phi\|_\infty\\ &\quad+\left(\frac{1}{\sqrt{t}}+K_1\sqrt{t}+\frac{K_2}{2}t\right)e^{\left(\frac12\lambda+K_0\right)t}\|\mathbf{d}\phi\|_\infty.\end{aligned}\tag{2.22}$$

Now let

$$t=t_0:=\frac{1}{\max\left\{\lambda+2K_0+\sigma K_{h,2\sigma},\ 4\,e\,\|h\|_\infty^{2\sigma}C_4^2\right\}}.$$

Then

$$C_4\,\|h\|_\infty^\sigma\, \mathrm{e}^{(\frac{\lambda}{2}+K_0+\frac{\sigma K_{h,2\sigma}}{2})t_0}\sqrt{t_0}\,\|\,\mathrm{Hess}\,\phi\|_\infty \le \frac{1}{2}\,\|\,\mathrm{Hess}\,\phi\|_\infty$$

and inequality (2.22) implies

$$\begin{aligned}
\|\,\mathrm{Hess}\,\phi\|_\infty \le\ & 2(n-1)\sigma\,\mathrm{e}\left(\alpha+\sqrt{\frac{2\lambda}{\pi}}\right)\|\phi\|_\infty\\
&+2C_1\|h\|_\infty^\sigma\sqrt{\mathrm{e}}\max\left\{\sqrt{\lambda+2K_0+\sigma K_{h,2\sigma}},\ 2\sqrt{\mathrm{e}}\|h\|_\infty^\sigma C_4\right\}\|\phi\|_\infty+\frac{(C_2+C_1\lambda)}{C_4}\|\phi\|_\infty\\
&+2\sqrt{\mathrm{e}}(\|h\|_\infty^\sigma+1)\max\left\{\sqrt{\lambda+2K_0+\sigma K_{h,2\sigma}},\ 2\sqrt{\mathrm{e}}\|h\|_\infty^\sigma C_4\right\}\|\mathbf{d}\phi\|_\infty\\
&+\frac{2\big(K_1+\|h\|_\infty^\sigma(C_3+\lambda)\big)\sqrt{\mathrm{e}}}{\max\left\{\sqrt{\lambda+2K_0+\sigma K_{h,2\sigma}},\ 2\sqrt{\mathrm{e}}\|h\|_\infty^\sigma C_4\right\}}\|\mathbf{d}\phi\|_\infty\\
&+\frac{K_2\sqrt{\mathrm{e}}}{\max\left\{\lambda+2K_0+\sigma K_{h,2\sigma},\ 4\,\mathrm{e}\,\|h\|_\infty^{2\sigma}C_4^2\right\}}\|\mathbf{d}\phi\|_\infty. \qquad (2.23)
\end{aligned}$$

As $\Delta\rho_{\partial D}\le\alpha$ and $\mathbf{Ric}\ge -K_0$, the constant A in [2, Eq. (1.7) in Theorem 1.1] is bounded by $\alpha+\sqrt{\frac{2}{\pi}(\lambda+K_0)}$. Thus we conclude from [2, Eq. (1.7)] that

$$\begin{aligned}
\frac{\|\mathbf{d}\phi\|_\infty}{\|\phi\|_\infty} &\le \sqrt{\mathrm{e}}\left(\alpha+\sqrt{\frac{2}{\pi}(\lambda+K_0)}+\frac{\lambda+K_0}{4\left(\alpha+\sqrt{\frac{2}{\pi}(\lambda+K_0)}\right)}\right)\\
&\le \sqrt{\mathrm{e}}\left(\alpha+\left(\sqrt{\frac{2}{\pi}}+\frac{1}{4}\sqrt{\frac{\pi}{2}}\right)\sqrt{\lambda+K_0}\right).
\end{aligned}$$

Combining this with (2.23) implies that

$$\begin{aligned}
\|\,\mathrm{Hess}\,\phi\|_\infty \le\ & 2(n-1)\sigma\,\mathrm{e}\left(\alpha+\sqrt{\frac{2\lambda}{\pi}}\right)\|\phi\|_\infty\\
&+2C_1\|h\|_\infty^\sigma\sqrt{\mathrm{e}}\max\left\{\sqrt{\lambda+2K_0+\sigma K_{h,2\sigma}},\ 2\sqrt{\mathrm{e}}\|h\|_\infty^\sigma C_4\right\}\|\phi\|_\infty+\frac{(C_2+C_1\lambda)}{C_4}\|\phi\|_\infty\\
&+2\sqrt{\mathrm{e}}(\|h\|_\infty^\sigma+1)\max\left\{\sqrt{\lambda+2K_0+\sigma K_{h,2\sigma}},\ 2\sqrt{\mathrm{e}}\|h\|_\infty^\sigma C_4\right\}\alpha\sqrt{\mathrm{e}}\,\|\phi\|_\infty\\
&+\frac{2(K_1+\|h\|_\infty^\sigma(C_3+\lambda))\alpha\,\mathrm{e}}{\max\left\{\sqrt{\lambda+2K_0+\sigma K_{h,2\sigma}},\ 2\sqrt{\mathrm{e}}\|h\|_\infty^\sigma C_4\right\}}\|\phi\|_\infty\\
&+\frac{K_2\alpha\,\mathrm{e}}{\max\left\{\lambda+2K_0+\sigma K_{h,2\sigma},\ 4\,\mathrm{e}\,\|h\|_\infty^{2\sigma}C_4^2\right\}}\|\phi\|_\infty\\
&+2\mathrm{e}(\|h\|_\infty^\sigma+1)\max\left\{\sqrt{\lambda+2K_0+\sigma K_{h,2\sigma}},\ 2\sqrt{\mathrm{e}}\|h\|_\infty^\sigma C_4\right\}\left(\sqrt{\frac{2}{\pi}}+\frac{1}{4}\sqrt{\frac{\pi}{2}}\right)\sqrt{\lambda+K_0}\|\phi\|_\infty\\
&+\frac{2(K_1+\|h\|_\infty^\sigma(C_3+\lambda))\mathrm{e}}{\max\left\{\sqrt{\lambda+2K_0+\sigma K_{h,2\sigma}},\ 2\sqrt{\mathrm{e}}\|h\|_\infty^\sigma C_4\right\}}\left(\sqrt{\frac{2}{\pi}}+\frac{1}{4}\sqrt{\frac{\pi}{2}}\right)\sqrt{\lambda+K_0}\,\|\phi\|_\infty\\
&+\frac{K_2\mathrm{e}}{\max\left\{\lambda+2K_0+\sigma K_{h,2\sigma},\ 4\,\mathrm{e}\,\|h\|_\infty^{2\sigma}C_4^2\right\}}\left(\sqrt{\frac{2}{\pi}}+\frac{1}{4}\sqrt{\frac{\pi}{2}}\right)\sqrt{\lambda+K_0}\,\|\phi\|_\infty. \qquad (2.24)
\end{aligned}$$

Using condition (2.19), the constants C_1, C_2, C_3 and C_4 then become

$$\begin{aligned}
C_1 &= \alpha,\\
C_2 &= \|\psi'\|_\infty(\alpha^2+2\beta)+\|\psi''\|_\infty\alpha+\gamma,\\
C_3 &= 3\|\psi'\|_\infty\alpha+\|\psi''\|_\infty+3\beta,\\
C_4 &= 2\|\psi'\|_\infty+2\alpha.
\end{aligned}$$

To simplify the upper bounds, we observe that

$$\frac{2\big(K_1+\|h\|_\infty^\sigma(C_3+\lambda)\big)\alpha\,\mathrm{e}}{\max\big\{\sqrt{\lambda+2K_0+\sigma K_{h,2\sigma}},\,2\sqrt{\mathrm{e}}\|h\|_\infty^\sigma C_4\big\}}\le\frac{(K_1+C_3+\lambda)\alpha\sqrt{\mathrm{e}}}{C_4};$$

$$\frac{K_2\alpha\,\mathrm{e}}{\max\big\{\lambda+2K_0+\sigma K_{h,2\sigma},\,4\,\mathrm{e}\,\|h\|_\infty^{2\sigma}C_4^2\big\}}\le\frac{K_2\alpha}{4C_4^2};$$

$$\frac{2\big(K_1+\|h\|_\infty^\sigma(C_3+\lambda)\big)\mathrm{e}}{\max\big\{\sqrt{\lambda+2K_0+\sigma K_{h,2\sigma}},\,2\sqrt{\mathrm{e}}\|h\|_\infty^\sigma C_4\big\}}\sqrt{\lambda+K_0}\le 2\big(K_1+\|h\|_\infty^\sigma(C_3+\lambda)\big)\mathrm{e};$$

$$\frac{K_2\mathrm{e}}{\max\big\{\lambda+2K_0+\sigma K_{h,2\sigma},\,4\,\mathrm{e}\,\|h\|_\infty^{2\sigma}C_4^2\big\}}\sqrt{\lambda+K_0}\le\frac{K_2\mathrm{e}}{\max\big\{\sqrt{\lambda+2K_0+\sigma K_{h,2\sigma}},\,2\sqrt{\mathrm{e}}\|h\|_\infty^\sigma C_4\big\}}\le\frac{K_2\sqrt{\mathrm{e}}}{2C_4}.$$

Let $c_0=\sqrt{\frac{2}{\pi}}+\frac{1}{4}\sqrt{\frac{\pi}{2}}$. Then,

$$\begin{aligned}\frac{\|\operatorname{Hess}\phi\|_\infty}{\|\phi\|_\infty}&\le 2(n-1)\sigma\mathrm{e}\left(\alpha+\sqrt{\frac{2\lambda}{\pi}}\right)\\&\quad+2\alpha\left(\|h\|_\infty^\sigma(\sqrt{\mathrm{e}}+\mathrm{e})+\mathrm{e}\right)\max\left\{\sqrt{\lambda+2K_0+\sigma K_{h,2\sigma}},\,2\sqrt{\mathrm{e}}\|h\|_\infty^\sigma C_4\right\}\\&\quad+\frac{2(K_1+C_3+\lambda)\alpha\sqrt{\mathrm{e}}+K_2c_0\sqrt{\mathrm{e}}+2(C_2+\alpha\lambda)}{2C_4}+\frac{K_2\alpha}{4C_4^2}\\&\quad+2c_0\mathrm{e}(\|h\|_\infty^\sigma+1)\max\left\{\lambda+2K_0+\sigma K_{h,2\sigma},\,4\,\mathrm{e}\,\|h\|_\infty^{2\sigma}C_4^2\right\}\\&\quad+2c_0\,\mathrm{e}(K_1+\|h\|_\infty^\sigma(C_3+\lambda)).\end{aligned}\tag{2.25}$$

Now let

$$\psi(r)=\begin{cases}\left(\frac{r_0-r}{r_0}\right)^3, & 0\le r\le r_0;\\ 0, & r>r_0,\end{cases}.\tag{2.26}$$

Then $\psi'\le\frac{3}{r_0}$ and $\psi''\le\frac{6}{r_0^2}$. With these estimates, the constants C_1, C_2, C_3, and C_4 are explicit. ■

2.3 Proof of Theorem 1.1

In this subsection we describe Wang's construction of functions h satisfying the requirements of Lemma 2.8 (see [21,p. 1436] or [22, Theorem 3.2.9] for the details). His construction is performed under the following condition.

Condition (A) There exist a non-negative constant σ such that $\mathrm{II}\le\sigma$ and a positive constant r_1 such that the distance function $\rho_{\partial D}$ to the boundary ∂D is smooth on $\partial_{r_1}D:=\{x\in D:\rho_{\partial D}(x)\le r_1\}$. Moreover, **Sect** $\le k$ on $\partial_{r_1}D$ for some positive constant k.

Under Condition (A), based on the Hessian comparison theorem, one then constructs a function h satisfying the necessary properties of Lemma 2.8 (see [21,p. 1436] or [22, Theorem 3.2.9] for the notation and the precise result), along with explicit upper bounds for $\|h\|_\infty$ and the constant $K_{h,\alpha}$. Modifying his construction one may take

$$\log h(x)=\frac{1}{\Lambda_0}\int_0^{\rho_{\partial D}(x)}(\ell(s)-\ell(r_0))^{1-n}\,ds\int_{s\wedge r_0}^{r_0}(\ell(u)-\ell(r_0))^{n-1}\,du,\tag{2.27}$$

where $\ell=\ell_{\sigma,\ell}$ is defined in (1.7), $r_0:=r_1\wedge\ell^{-1}(0)$ and

$$\Lambda_0:=(1-\ell(r_0))^{1-n}\int_0^{r_0}(\ell(s)-\ell(r_0))^{n-1}\,ds.$$

Then from the proof of [20, Theorem 1.1], we get:

$$K_{h,\alpha}\le K_\alpha:=\frac{n}{r_0}+\alpha\quad\text{and}\quad\|h\|_\infty\le\mathrm{e}^{\frac{1}{2}nr_0}.\tag{2.28}$$

By means of h, as constructed above, we are now in position to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. By estimate (2.25), we know that

$$\begin{aligned}
\frac{\|\operatorname{Hess}\phi\|_\infty}{\|\phi\|_\infty} &\le 2(n-1)\sigma\,\mathrm{e}\left(\alpha+\sqrt{\frac{2\lambda}{\pi}}\right)\\
&\quad+2\alpha\,\mathrm{e}\left(2\|h\|_\infty^\sigma+1\right)\max\left\{\sqrt{\lambda+2K_0+\sigma K_{h,2\sigma}},\ 2\sqrt{\mathrm{e}}\|h\|_\infty^\sigma C_4\right\}\\
&\quad+\frac{2(K_1+C_3+\lambda)\alpha\sqrt{\mathrm{e}}+K_2c_0\sqrt{\mathrm{e}}+2(C_2+\alpha\lambda)}{2C_4}+\frac{K_2\alpha}{4C_4^2}\\
&\quad+2c_0\,\mathrm{e}(\|h\|_\infty^\sigma+1)\max\left\{\lambda+2K_0+\sigma K_{h,2\sigma},\ 4\,\mathrm{e}\,\|h\|_\infty^{2\sigma}C_4^2\right\}\\
&\quad+2c_0\,\mathrm{e}\left(K_1+\|h\|_\infty^\sigma(C_3+\lambda)\right)\\
&\le 2(n-1)\sigma\,\mathrm{e}\,\alpha+4(\sqrt{\mathrm{e}})^3C_4\alpha\left(2\|h\|_\infty^{2\sigma}+\|h\|_\infty^\sigma\right)\\
&\quad+\frac{2(K_1+C_3)\alpha\sqrt{\mathrm{e}}+K_2c_0\sqrt{\mathrm{e}}+2C_2}{2C_4}+\frac{K_2\alpha}{4C_4^2}\\
&\quad+2c_0\,\mathrm{e}(\|h\|_\infty^\sigma+1)(2K_0+\sigma K_{h,2\sigma}+4\,\mathrm{e}\,\|h\|_\infty^{2\sigma}C_4^2)\\
&\quad+2c_0\,\mathrm{e}\left(K_1+\|h\|_\infty^\sigma C_3\right)\\
&\quad+2(n-1)\sigma\,\mathrm{e}\sqrt{\frac{2\lambda}{\pi}}+2\alpha\,\mathrm{e}\left(2\|h\|_\infty^\sigma+1\right)\sqrt{\lambda+2K_0+\sigma K_{h,2\sigma}}\\
&\quad+\left(\frac{\alpha\sqrt{\mathrm{e}}}{C_4}+\frac{\alpha}{C_4}+2c_0\,\mathrm{e}(2\|h\|_\infty^\sigma+1)\right)\lambda,
\end{aligned}$$

where C_2, C_3 and C_4 are defined as in Lemma 2.10. By means of the Cauchy–Schwarz inequality, we further obtain

$$\begin{aligned}
\frac{\|\operatorname{Hess}\phi\|_\infty}{\|\phi\|_\infty} &\le 2(n-1)\sigma\,\mathrm{e}\,\alpha+4(\sqrt{\mathrm{e}})^3C_4\alpha\left(2\|h\|_\infty^{2\sigma}+\|h\|_\infty^\sigma\right)\\
&\quad+\frac{2(K_1+C_3)\alpha\sqrt{\mathrm{e}}+K_2c_0\sqrt{\mathrm{e}}+2C_2}{2C_4}+\frac{K_2\alpha}{4C_4^2}\\
&\quad+2c_0\,\mathrm{e}(\|h\|_\infty^\sigma+1)(2K_0+\sigma K_{h,2\sigma}+4\,\mathrm{e}\,\|h\|_\infty^{2\sigma}C_4^2)\\
&\quad+2c_0\,\mathrm{e}\left(K_1+\|h\|_\infty^\sigma C_3\right)+\alpha\,\mathrm{e}\left(2\|h\|_\infty^\sigma+1\right)\frac{1}{\varepsilon}\\
&\quad+(n-1)\sigma\,\mathrm{e}\,\varepsilon\lambda+(n-1)\sigma\,\mathrm{e}\,\frac{2}{\pi\varepsilon}\\
&\quad+\alpha\,\mathrm{e}\left(2\|h\|_\infty^\sigma+1\right)\varepsilon(\lambda+2K_0+\sigma K_{h,2\sigma})\\
&\quad+\left(\frac{\alpha\sqrt{\mathrm{e}}}{C_4}+\frac{\alpha}{C_4}+2c_0\,\mathrm{e}(2\|h\|_\infty^\sigma+1)\right)\lambda,
\end{aligned}$$

for any $\varepsilon>0$. Using the explicit estimates of C_2, C_3 and C_4 obtained in the proof of Theorem 2.11 and letting

$$\begin{aligned}
A_h(D) &= 2(n-1)\sigma\,\mathrm{e}(\alpha+\frac{1}{\pi\varepsilon})+2c_0\,\mathrm{e}\left(K_1-\frac{9\alpha}{r_0}-\frac{6}{r_0^2}-3\beta\right)\\
&\quad+\frac{(K_1+3\beta)\alpha\sqrt{\mathrm{e}}+\frac{1}{2}K_2c_0\sqrt{\mathrm{e}}+\gamma+\frac{3(\alpha^2+3\sqrt{\mathrm{e}}\alpha^2+2\beta)}{r_0}+\frac{(3+6\sqrt{\mathrm{e}})\alpha}{r_0^2}}{\frac{6}{r_0}+2\alpha}+\frac{K_2\alpha}{4(\frac{6}{r_0}+2\alpha)^2}\\
&\quad+2c_0\,\mathrm{e}(\|h\|_\infty^\sigma+1)\left(\frac{9\alpha}{r_0}+\frac{6}{r_0^2}+3\beta+2K_0+\sigma K_{h,2\sigma}+4\,\mathrm{e}\,\|h\|_\infty^{2\sigma}(\frac{6}{r_0}+2\alpha)^2\right)\\
&\quad+\alpha\,\mathrm{e}\left(2\|h\|_\infty^\sigma+1\right)\left(\varepsilon(2K_0+\sigma K_{h,2\sigma})+\varepsilon^{-1}+4\sqrt{\mathrm{e}}\|h\|_\infty^\sigma\left(\frac{6}{r_0}+2\alpha\right)\right)
\end{aligned}$$

and

$$C_h(D) = (n-1)\sigma\, \mathrm{e}\, \varepsilon + \frac{\alpha(\sqrt{\mathrm{e}}+1)}{\frac{6}{r_0}+2\alpha} + (2c_0+\alpha\varepsilon)\, \mathrm{e}(2\|h\|_\infty^\sigma + 1),$$

we then obtain

$$\frac{\|\operatorname{Hess}\phi\|_\infty}{\|\phi\|_\infty} \le A_h(D) + C_h(D)\lambda.$$

Using h defined in (2.27) and substituting the estimates (2.28), we replace $K_{h,2\sigma}$ and $\|h\|_\infty$ by

$$\frac{n}{r_0} + 2\sigma \text{ and } \mathrm{e}^{nr_0/2},$$

respectively. Finally, by Lemma 2.9, the upper bound α in (2.19) can be chosen as $2(n-1)\max\{\sigma, \sqrt{k}\}$. This completes the proof of Theorem 1.1. ■

3 Hessian Estimates on Neumann Eigenfunctions of Laplacian

We use a stochastic approach as well to prove Theorem 1.3. Let us first recall the Hessian formulas for the Neumann semigroups, established recently in [4]. The reflecting Brownian motion on D with generator $\frac{1}{2}\Delta$ satisfies the SDE

$$dX_t = //_t \circ dB_t^x + \frac{1}{2}N(X_t)\, dl_t, \quad X_0 = x,$$

where B_t^x is a standard Brownian motion on the Euclidean space $T_xD \cong \mathbb{R}^n$. We write again $X_t = X_t(x)$ to indicate the starting point $x \in D$ (which may be on the boundary ∂D). Here $//_t : T_xD \to T_{X_t(x)}D$ denotes the ∇-parallel transport along $X_t(x)$ and l_t the local time of $X_t(x)$ supported on ∂D; see [22]. Note that the reflecting Brownian motion $X_t(x)$ is defined for all $t \ge 0$.

Suppose that $\tilde{Q}_t : T_xD \to T_{X_t(x)}D$ satisfies

$$D\tilde{Q}_t = -\frac{1}{2}\mathbf{Ric}^\sharp(\tilde{Q}_t)\, dt + \frac{1}{2}(\nabla N)^\sharp(\tilde{Q}_t)\, dl_t, \quad \tilde{Q}_0 = \mathbf{id}. \tag{3.1}$$

For $k \in C_b^1([0,\infty);\mathbb{R})$ define an operator-valued process $\tilde{W}_t^k : T_xD \otimes T_xD \to T_{X_t(x)}D$ as solution to the following covariant Itô equation:

$$\begin{aligned} D\tilde{W}_t^k(v,w) &= R(//_t\, dB_t, \tilde{Q}_t(k(t)v))\tilde{Q}_t(w) \\ &\quad - \frac{1}{2}(\mathbf{d}^*R + \nabla\mathbf{Ric})^\sharp(\tilde{Q}_t(k(t)v), \tilde{Q}_t(w))\, dt \\ &\quad - \frac{1}{2}(\nabla^2 N - R(N))^\sharp(\tilde{Q}_t(k(t)v), \tilde{Q}_t(w))\, dl_t \\ &\quad - \frac{1}{2}\mathbf{Ric}^\sharp(\tilde{W}_t^k(v,w))\, dt + \frac{1}{2}(\nabla N)^\sharp(\tilde{W}_t^k(v,w))\, dl_t, \end{aligned} \tag{3.2}$$

with initial condition $\tilde{W}_0^k(v,w) = 0$. Actually, $\tilde{W}_t^k(v,w)$ can be written in explicit form as

$$\begin{aligned} \tilde{W}_t^k(v,w) &= \tilde{Q}_t \int_0^t \tilde{Q}_s^{-1} R(//_s\, dB_s, \tilde{Q}_s(k(s)v))\tilde{Q}_s(w) \\ &\quad - \frac{1}{2}\tilde{Q}_t \int_0^t \tilde{Q}_s^{-1}(\mathbf{d}^*R + \nabla\mathbf{Ric})^\sharp(\tilde{Q}_s(k(s)v), \tilde{Q}_s(w))\, ds \\ &\quad - \frac{1}{2}\tilde{Q}_t \int_0^t \tilde{Q}_s^{-1}(\nabla^2 N - R(N))^\sharp(\tilde{Q}_s(k(s)v), \tilde{Q}_s(w))\, dl_s. \end{aligned}$$

Theorem 3.1 ([4]). Let D be a compact Riemannian manifold with boundary ∂D. Let $X(x)$ be the reflecting Brownian motion on D with starting point x (possibly on the boundary) and denote by $P_t f(x) = \mathbb{E}[f(X_t(x))]$ the corresponding Neumann semigroup acting on $f \in \mathcal{B}_b(D)$. Then, for $v \in T_x D$, $t \geq 0$ and $k \in C_b^1([0,\infty);\mathbb{R})$,

$$\mathrm{Hess}\, P_t f(v,v) = \mathbb{E}\left[-\mathbf{d}f(\tilde{Q}_t(v)) \int_0^t \langle \tilde{Q}_s(\dot{k}(s)v), //_s dB_s\rangle + \mathbf{d}f(\tilde{W}_t^k(v,v))\right].$$

By estimating $\tilde{W}^k$ and $\tilde{Q}$ in explicit terms, pointwise bounds for the Hessian of Neumann eigenfunctions can be obtained.

Corollary 3.2. We keep the assumptions of Theorem 3.1. Let K_0, K_1, K_2 and σ_1, σ_2 be non-negative constants such that $\mathbf{Ric} \geq -K_0$, $|R| \leq K_1$ and $|\mathbf{d}^* R + \nabla \mathbf{Ric}| \leq K_2$ on D, and $\mathbf{II} \geq -\sigma_1$, $|\nabla^2 N + R(N)| < \sigma_2$ on the boundary ∂D. Then, for $(\phi, \lambda) \in \mathbf{Eig}_N(D)$,

$$\begin{aligned}|\mathrm{Hess}\,\phi|(x) \leq& \,\mathrm{e}^{(\frac{1}{2}\lambda + K_0)t}\, \mathbb{E}[\mathrm{e}^{\sigma_1 l_t}] \left(\frac{1}{\sqrt{t}} + K_1\sqrt{t} + \frac{K_2}{2}t\right) \|\mathbf{d}\phi\|_\infty \\ &+ \frac{\sigma_2}{2}\,\mathrm{e}^{(K_0 + \frac{\lambda}{2})t}\, \mathbb{E}\left[\mathrm{e}^{\frac{1}{2}\sigma_1 l_t} \int_0^t \mathrm{e}^{\frac{1}{2}\sigma_1 l_s}\, dl_s\right] \|\mathbf{d}\phi\|_\infty.\end{aligned}$$

Proof. By [4, Theorem 4.1] the Hessian of the semigroup can be estimated as

$$\begin{aligned}|\mathrm{Hess}\, P_t f| \leq& \left(K_1\sqrt{t} + \frac{K_2}{2}t + \frac{1}{\sqrt{t}}\right) \mathbb{E}\left[\mathrm{e}^{\sigma_1 l_t}\right] \mathrm{e}^{K_0 t}\, \|\nabla f\|_\infty \\ &+ \frac{\sigma_2}{2}\mathbb{E}\left[\mathrm{e}^{\frac{1}{2}\sigma_1 l_t} \int_0^t \mathrm{e}^{\frac{1}{2}\sigma_1 l_s}\, dl_s\right] \mathrm{e}^{K_0 t}\, \|\nabla f\|_\infty.\end{aligned}$$

We complete the proof by observing that $P_t\phi = \mathrm{e}^{-\lambda t/2}\,\phi$. ■

Combining Theorem 3.2 and Lemma 2.8, we are now in a position to prove Theorem 1.3.

Theorem 3.3. Let D be an n-dimensional compact Riemannian manifold with boundary ∂D. Let $K_0, K_1, K_2, \sigma_1, \sigma_2$ be non-negative constants such that $\mathbf{Ric} \geq -K_0$, $|R| \leq K_1$ and $|\mathbf{d}^* R + \nabla \mathbf{Ric}| \leq K_2$ on D, and that $\mathbf{II} \geq -\sigma_1$ and $|\nabla^2 N - R(N)| \leq \sigma_2$ on the boundary ∂D. For $h \in C^\infty(D)$ with $\min_D h = 1$ and $N \log h|_{\partial D} \geq 1$, let $K_{h,\alpha} := \sup_D\{-\Delta \log h + \alpha|\nabla \log h|^2\}$ with α a non-negative constant. Then for any non-trivial $(\phi, \lambda) \in \mathbf{Eig}_N(\Delta)$,

$$\frac{\|\mathrm{Hess}\,\phi\|_\infty}{\|\phi\|_\infty} \leq A_h(N) + C_h(N)\lambda,$$

where

$$\begin{aligned}A_h(N) :=& \Bigg(K_1 + 2K_0 + 2\sigma_1 K_{h,2\sigma_1} + \frac{K_2 + 2\sigma_2 K_{h,\sigma_1}}{2\sqrt{2K_0 + 2\sigma_1 K_{h,2\sigma_1}}} \\ &+ \sigma_2 \ln\|h\|_\infty \left[\varepsilon(K_0 + \sigma_1 K_{h,2\sigma_1}) + \frac{1}{\varepsilon}\right]\Bigg) \|h\|_\infty^{3\sigma_1}\,\mathrm{e}; \\ C_h(N) :=& (1 + \varepsilon\sigma_2 \ln\|h\|_\infty)\|h\|_\infty^{3\sigma_1}\,\mathrm{e}\end{aligned}$$

for any $\varepsilon > 0$.

Proof. By Lemma 2.8, we have

$$\mathbb{E}[\mathrm{e}^{\sigma_1 l_t}] \leq \mathbb{E}[\mathrm{e}^{\sigma_1 l_t}] \leq \|h\|_\infty^{2\sigma_1} \exp\left(\sigma_1 K_{h,2\sigma_1} t\right),$$

and

$$\mathbb{E}[\mathrm{e}^{\sigma_1 l_t}] \leq \|h\|_\infty^{2\sigma_1} \exp\left(\sigma_1 K_{h,2\sigma_1} t\right).$$

Moreover, we observe that

$$\begin{aligned}\mathbb{E}\left[e^{\frac{1}{2}\sigma_1 l_t}\int_0^t e^{\frac{1}{2}\sigma_1 l_s}\,dl_s\right] &\le \frac{2(\mathbb{E}[e^{(\sigma_1+\varepsilon)l_t}]-1)}{\sigma_1+\varepsilon}\\ &\le \frac{2}{\sigma_1+\varepsilon}\left(\|h\|_\infty^{2(\sigma_1+\varepsilon)}\exp\left((\sigma_1+\varepsilon)K_{h,2(\sigma_1+\varepsilon)}t\right)-1\right)\\ &\le \frac{2}{\sigma_1+\varepsilon}\left(\|h\|_\infty^{2(\sigma_1+\varepsilon)}\exp\left((\sigma_1+\varepsilon)K_{h,(\sigma_1+\varepsilon)}t\right)-1\right)\\ &\le \frac{2}{\sigma_1+\varepsilon}\left(\|h\|_\infty^{2(\sigma_1+\varepsilon)}-1\right)+\frac{2}{\sigma_1+\varepsilon}\|h\|_\infty^{2(\sigma_1+\varepsilon)}\left[\exp\left((\sigma_1+\varepsilon)K_{h,2(\sigma_1+\varepsilon)}t\right)-1\right]\\ &\le 4\|h\|_\infty^{2(\sigma_1+\varepsilon)}\ln\|h\|_\infty+2\|h\|_\infty^{2(\sigma_1+\varepsilon)}\exp\left((\sigma_1+\varepsilon)K_{h,2(\sigma_1+\varepsilon)}t\right)K_{h,2(\sigma_1+\varepsilon)}t.\end{aligned}$$

Letting ε tend to 0, we arrive at

$$\mathbb{E}\left[e^{\frac{1}{2}\sigma_1 l_t}\int_0^t e^{\frac{1}{2}\sigma_1 l_s}\,dl_s\right]\le 4\|h\|_\infty^{2\sigma_1}\ln\|h\|_\infty+2\|h\|_\infty^{2\sigma_1}\exp\left(\sigma_1 K_{h,2\sigma_1}t\right)K_{h,2\sigma_1}t.$$

Therefore, combining this with Theorem 3.2, we obtain

$$\begin{aligned}\frac{\|\operatorname{Hess}\phi\|_\infty}{\|\mathbf{d}\phi\|_\infty} &\le e^{(\frac{1}{2}\lambda+K_0)t}\left(\frac{1}{\sqrt{t}}+K_1\sqrt{t}+\frac{K_2}{2}t\right)\|h\|_\infty^{2\sigma_1}\exp\left(\sigma_1K_{h,2\sigma_1}t\right)\\ &\quad+\sigma_2\, e^{(\frac{1}{2}\lambda+K_0)t}\left[2\ln\|h\|_\infty+K_{h,\sigma_1}t\right]\|h\|_\infty^{2\sigma_1}\exp\left(\sigma_1K_{h,2\sigma_1}t\right)\\ &\le e^{(\frac{1}{2}\lambda+K_0)t}\left(\frac{1}{\sqrt{t}}+K_1\sqrt{t}+\frac{K_2}{2}t\right)\|h\|_\infty^{2\sigma_1}\exp\left(\sigma_1K_{h,2\sigma_1}t\right)\\ &\quad+\sigma_2\, e^{(\frac{1}{2}\lambda+K_0)t}\left[2\ln\|h\|_\infty+K_{h,\sigma_1}t\right]\|h\|_\infty^{2\sigma_1}\exp\left(\sigma_1K_{h,2\sigma_1}t\right).\end{aligned}$$

Letting $t=\left(\lambda+2K_0+2\sigma_1K_{h,2\sigma_1}\right)^{-1}$, we get

$$\begin{aligned}\frac{\|\operatorname{Hess}\phi\|_\infty}{\|\mathbf{d}\phi\|_\infty} &\le \left(\frac{K_1}{\sqrt{\lambda+2K_0+2\sigma_1K_{h,2\sigma_1}}}+\sqrt{\lambda+2K_0+2\sigma_1K_{h,2\sigma_1}}\right.\\ &\quad\left.+\frac{K_2+2\sigma_2K_{h,\sigma_1}}{2(\lambda+2K_0+2\sigma_1K_{h,2\sigma_1})}+2\sigma_2\ln\|h\|_\infty\right)\|h\|_\infty^{2\sigma_1}\sqrt{e}.\end{aligned}$$

On the other hand, it is already shown in [2] that

$$\frac{\|\mathbf{d}\phi\|_\infty}{\|\phi\|_\infty}\le\frac{1}{\sqrt{t}}\mathbb{E}[e^{\sigma_1 l_t}]^{1/2}e^{\frac{1}{2}(K_0+\lambda)t}\le\frac{1}{\sqrt{t}}\|h\|_\infty^{\sigma_1}\exp\left(\frac{1}{2}(\lambda+\sigma_1K_{h,2\sigma_1}+K_0)t\right).$$

Let $t=\left(\lambda+K_0+\sigma_1K_{h,2\sigma_1}\right)^{-1}$. Then we get

$$\frac{\|\mathbf{d}\phi\|_\infty}{\|\phi\|_\infty}\le\sqrt{\lambda+K_0+\sigma_1K_{h,2\sigma_1}}\,\|h\|_\infty^{\sigma_1}\sqrt{e}.$$

We then conclude that

$$\begin{aligned}\frac{\|\operatorname{Hess}\phi\|_\infty}{\|\phi\|_\infty} &\le \left(\lambda+K_1+2K_0+2\sigma_1K_{h,2\sigma_1}+\frac{K_2+2\sigma_2K_{h,\sigma_1}}{2\sqrt{\lambda+2K_0+2\sigma_1K_{h,2\sigma_1}}}\right.\\ &\quad\left.+2\sigma_2\ln\|h\|_\infty\sqrt{\lambda+K_0+\sigma_1K_{h,2\sigma_1}}\right)\|h\|_\infty^{3\sigma_1}e\\ &\le \left(\lambda+K_1+2K_0+2\sigma_1K_{h,2\sigma_1}+\frac{K_2+2\sigma_2K_{h,\sigma_1}}{2\sqrt{2K_0+2\sigma_1K_{h,2\sigma_1}}}\right.\\ &\quad\left.+2\sigma_2\ln\|h\|_\infty\sqrt{\lambda+K_0+\sigma_1K_{h,2\sigma_1}}\right)\|h\|_\infty^{3\sigma_1}e.\end{aligned}$$

Using the Cauchy–Schwarz inequality, we further obtain

$$\frac{\|\operatorname{Hess}\phi\|_\infty}{\|\phi\|_\infty} \le \Bigg(\lambda + K_1 + 2K_0 + 2\sigma_1 K_{h,2\sigma_1} + \frac{K_2 + 2\sigma_2 K_{h,\sigma_1}}{2\sqrt{2K_0 + 2\sigma_1 K_{h,2\sigma_1}}} + \varepsilon\sigma_2 \ln\|h\|_\infty(\lambda + K_0 + \sigma_1 K_{h,2\sigma_1}) + \frac{1}{\varepsilon}\sigma_2 \ln\|h\|_\infty\Bigg)\|h\|_\infty^{3\sigma_1}\,\mathrm{e},$$

for any $\varepsilon > 0$. Letting

$$A_h(N) := \Bigg(K_1 + 2K_0 + 2\sigma_1 K_{h,2\sigma_1} + \frac{K_2 + 2\sigma_2 K_{h,\sigma_1}}{2\sqrt{2K_0 + 2\sigma_1 K_{h,2\sigma_1}}} + \sigma_2 \ln\|h\|_\infty \left(\varepsilon(K_0 + \sigma_1 K_{h,2\sigma_1}) + \frac{1}{\varepsilon}\right)\Bigg)\|h\|_\infty^{3\sigma_1}\,\mathrm{e}$$

and

$$C_h(N) := (1 + \varepsilon\sigma_2 \ln\|h\|_\infty)\|h\|_\infty^{3\sigma_1}\,\mathrm{e},$$

we then obtain

$$\frac{\|\operatorname{Hess}\phi\|_\infty}{\|\phi\|_\infty} \le A_h(N) + C_h(N)\lambda.$$

■

Proof of Theorem 1.3. From the conditions we see that Condition (**A**) is satisfied. Then, the Hessian estimate of Neumann eigenfunctions in Theorem 3.3 remain valid by substituting the h defined in (2.27). Then under replacing

$$K_{h,\alpha} \text{ and } \|h\|_\infty$$

by

$$K_\alpha := \frac{n}{r_0} + \alpha \text{ and } \mathrm{e}^{nr_0/2}$$

respectively, the conclusion is just listed in Theorem 1.3. ■

Conflict of Interest and Ethics Statements

On behalf of all authors, the corresponding author Li-Juan Chen declares that there is no conflict of interest. Data sharing is not applicable to the article as no datasets were created or analysed in this study.

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