On clear choices with ordinal valued binary relations

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Abstract: Good or bad choices based on a crisp outranking relation correspond to dominant and absorbent kernels in the corresponding digraph. In this paper we generalize both concepts to an ordinal valued outranking relation. An efficient algorithm for computing these choices is proposed. Furthermore, we show the formal correspondence between ordinal valued choices and kernels in ordinal valued digraphs.

Keywords: Decision Making, Outranking methods, Fuzzy Choices, Kernels, Ordinal valued digraphs

1. Introduction

In the so-called outranking methods (see Roy and Bouyssou, 1993), the preference relation on a set of alternatives is constructed as a pairwise comparison evaluating the level of credibility of the fact that one alternative is at least as good as the other. In the case of a decision problem, where possibly a unique alternative has to be selected as the best one, stable and dominant subsets have been proposed as good choices (Roy, 1968).

In this paper we shall thoroughly investigate this approach and first, generalize it in the crisp case to clearly good, bad and ambiguous choices. The formal equivalence of good and bad choices with dominant and absorbent kernels in the corresponding outranking digraph is established.

In a second part we generalize clearly good, bad and ambiguous choices to ordinal valued outranking relations and show the formal link between crisp and valued choices. This result allows us to implement an efficient algorithm for computing ordinal valued choices in a practically relevant outranking digraph.

In a last part dominant and absorbent kernels are extended to ordinal valued outranking digraphs. The main result of the paper establishes the formal equivalence between valued choices and corresponding valued kernels.

2. Choices from crisp binary relations

2.1. On good and bad choices

Let us consider a finite, non empty set of alternatives X and a binary relation R. If $(a, b) \in R$, $a, b \in X$, we consider that "*a is at least as good as b*". X and R define a directed graph (digraph for short) $G = (X, R)$ where X represents the set of vertices and (a, b) is an arc if and only if $(a, b) \in R$.

A *choice* in G is a non empty subset Y of X. Singletons $\{x\}$, $\forall x \in X$ are called *single choices* whereas $Y = X$ is called the *greedy* choice.

A *stable choice* in G is either a single choice or a choice $Y \subseteq X$ such that $\forall a \neq b \in Y$, $(a, b) \notin R$.

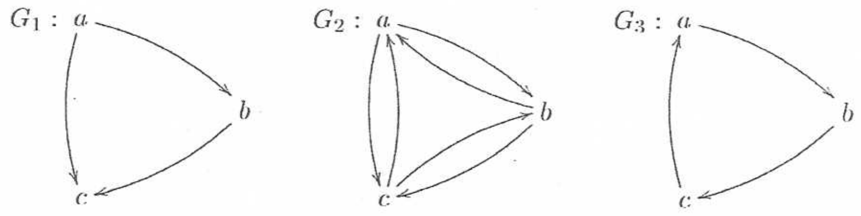
A *dominant choice* in G is either the greedy choice or a choice $Y \subset X$ such that $\forall a \notin Y$, $\exists b \in Y$, $(b, a) \in R$.

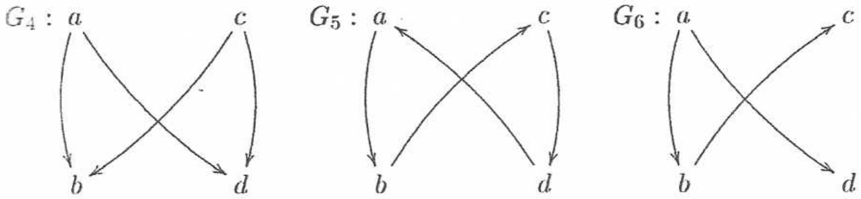
An *absorbent choice* in G is either the greedy choice or a choice $Y \subset X$ such that $\forall a \notin Y$, $\exists b \in Y$, $(b, a) \in R^t$, i.e. the transpose of relation R.

Definition 1 (Good, bad and ambiguous choices).
Let G be a digraph. A *good* (*bad*, respectively) choice is a stable and dominant (absorbent, respectively) choice in G. An *ambiguous* choice is a stable and both, dominant and absorbent choice in G. A *clear* choice is either a good or a bad choice which is non ambiguous.

We denote $\mathcal{C}^{\text{good}}(G)$ $(\mathcal{C}^{bad}(G))$ the possibly empty set of good (bad) choices in G.

Consider the following examples of digraphs:





In G_1, single choices $\{a\}$ and $\{c\}$ are clearly good, respectively bad, whereas in G_2 all single choices are both good and bad together, thus ambiguous. In fact they are equivalent and any choice order is ambiguous. There is no clear nor ambiguous choice in G_3.

In G_4, choices $\{a, c\}$ and $\{b, d\}$ are clearly good, respectively bad, whereas in G_5, the same are ambiguous and there appears no clear choice. Finally, in G_6, $\{a, c\}$ is a clear good choice whereas $\{c, d\}$ is a clear bad choice.

2.2. Dominant and absorbent kernels

Any choice $Y \subseteq X$ can be defined with the use of a subset characteristic (row) vector $Y(\cdot) = \big(Y(a),\, Y(b), \dots\big)$ where

$$Y(a) = \begin{cases} 1 & \text{if } a \in Y \\ 0 & \text{otherwise.} \end{cases}, \ \forall a \in X. \tag{1}$$

Definition 2 (Dominant and absorbent kernels).

We call *dominant kernel* a solution Y (if any) of the Boolean system of equations:

$$(Y \circ R)(a) = \bigvee_{b \neq a} (Y(b) \wedge R(b, a)) = \overline{Y}(a) = 1 - Y(a), \quad \text{for all } a \in X, \tag{2}$$

where $\overline{Y}$ represents the complement of Y, R represents the Boolean matrix associated with relation R, $\vee$ and $\wedge$ represent respectively "disjunction" and "conjunction" for the 2-element Boolean lattice $\mathcal{B} = \{0, 1\}$, and $\circ$ represents the standard relational composition operator.

We call *absorbent kernel* a solution Y (if any) of the Boolean system of equations:

$$(R \circ Y^t)(a) = \bigvee_{b \neq a} (R(a, b) \wedge Y^t(b)) = \overline{Y}^t(a) = 1 - Y^t(a), \quad \text{for all } a \in X. \tag{3}$$

where Y^t represents a transposed (column) characteristic vector.

We denote $\mathcal{K}^{\text{dom}}(G)$ ($\mathcal{K}^{\text{abs}}(G)$) the possibly empty set of dominant (absorbent) kernels in G, i.e. solutions of Equation System (2) (3, respectively).

Proposition 1.

To each good (bad) choice $K \in \mathcal{C}^{\text{good}}(G)$ ($\mathcal{C}^{\text{bad}}(G)$) corresponds a unique dominant (absorbent) kernel characteristic vector $Y \in \mathcal{K}^{\text{dom}}(G)$ ($\mathcal{K}^{\text{abs}}(G)$).

Proof. To show the equivalence between a solution Y of Equation System (2) and a good choice K in $G = (X, R)$, we may rearrange the elements of the set X in such a way that Y is split into two disjoint parts: a 1 valued part Y_K and a 0–valued part $Y_{\overline{K}}$. Rearranging in the same way rows and columns of matrix R, we obtain the following matrix representation of Equation System (2):

$$[Y_K \; Y_{\overline{K}}] \circ \begin{bmatrix} R_{KK} & R_{K\overline{K}} \\ R_{\overline{K}K} & R_{\overline{K}\overline{K}} \end{bmatrix} = [\overline{Y}_K \; \overline{Y}_{\overline{K}}]. \tag{4}$$

It is easily seen that Y exactly characterizes a good, i.e. stable and dominant, choice K when and only when, on the one hand, R_{KK}, is 0-valued, i.e. $Y_K \circ R_{KK} \vee Y_{\overline{K}} \circ R_{\overline{K}K} = \overline{Y}_K$ is 0-valued. And, on the other hand, $R_{K\overline{K}}$ is such that $Y_K \circ R_{K\overline{K}} \vee Y_{\overline{K}} \circ R_{\overline{K}\overline{K}} = \overline{Y}_{\overline{K}}$ is 1-valued. A same argument applies to the equivalence between bad choices and absorbent kernels. □

Stable and absorbent choices were originally introduced by J. von Neumann and O. Morgenstern under the name "game solution" in the context of game theory (v. Neumann and Morgenstern, 1944). J. Riguet introduced the name "noyau (kernel)" for the von Neumann "game solution" (Riguet, 1948). This kind of choices was studied by C. Berge in the context of the Nim game modelling (Berge, 1958, 1970). More results on (absorbent) kernels, concerning solutions of different games, have been reported by G. Schmidt and T. Ströhlein (Schmidt and Ströhlein, 1985, 1989). Recently, J. Bang-Jensen and G. Gutin reviewed the link between kernel-solvability and perfect graphs (Bang-Jensen and Gutin, 2001).

Stable and dominant choices (dominant kernels) were introduced by B. Roy in the context of the multicriteria Electre decision aid methods (Roy, 1968, 1985; Roy and Bouyssou, 1993). Ambiguous choices, i.e. both dominant and absorbent choices at the same time, were proposed by R. Bisdorff as potential cluster candidates in the context of multicriteria clustering (Bisdorff, 2002a).

The absorbent version of the kernel equations (Equation System 3) was first introduced by Schmidt and Ströhlein (1985, 1989) in the context of their thorough exploration of relational algebra. The dominant version (Equation System 2) was introduced by Kitaïnik (1993) and subsequently used by Bisdorff and Roubens (1996a, 1996b, 1997).

2.3. Computing good and bad choices from a crisp binary relation

Finding the sets (if any) of good and bad choices in a digraph G is an enumerationally difficult problem (Chvátal, 1973). However, if G is acyclic, its unique clearly bad choice may be computed in polynomial time by a dual fix-point algorithm attributed to von Neumann (see Schmidt and Ströhlein, 1989). In the context of the Electre decision aid methods, Roy has developed a similar algorithm for computing the unique good choice in G (Roy, 1969).

Unfortunately, multicriteria aggregation procedures rarely deliver an acyclic digraph. In this case, finite domain solvers such as clp(FD) (Codognet and Diaz, 1996) and GNU-Prolog (Diaz, 2001) may nevertheless provide an efficient tool for extracting kernels from random digraphs of not too large order, i.e. number of vertices. The key idea is to solve the kernel defining Equation Systems (2) (resp. (3)) in the associated $\mathcal{B} = \{0, 1\}$ computation domain with the help of a constraint enumeration of all possible characteristic vector solutions. Efficient dynamic propagation techniques, based on the specific kernel defining equations, help keep the set of effectively to be inspected instantiations rather limited.

In Table 1 we report summary statistics concerning resolution times of the dominant kernel equation system for samples of 500 random digraphs with an average of 25%, 50% and 75% arcs defined on 10 to 60 vertices. Under Linux RH 7.2 (kernel 2.4.19) and GNU-Prolog version 1.2.13, we may compute, on a 2.0 Ghz Pentium 4 system, the set of good choices in a digraph of order 50 in less than 3 seconds. That the problem is NP-complete is noticeable on the nearly exponential growth of the resolution times in relation with the number of vertices of the digraph. Kernels in sparse digraphs appear to be more difficult to compute for graphs of increasing order.

Table 1: Computation times for dominant kernels (in milliseconds) in $\{0, 1\}$-valued digraphs

number of vertices	10	20	30	40	50	60
25% of arcs on average						
min	50	108	215	472	1123	3111
mean	52	112	248	659	1912	5357
max	56	119	297	889	2851	8902
50% of arcs on average						
min	51	111	248	542	1213	2675
mean	53	116	267	634	1477	3206
max	57	120	293	846	1918	3866
75% of arcs on average						
min	50	114	256	527	1020	1868
mean	53	117	266	563	1113	2057
max	55	120	280	725	1374	2313

As a set of potential decision alternatives is generally of limited dimension (less than 50) in the domain of multicriteria decision aid, the operational performance achieved with the help of the GNU-Prolog FD-solver appears quite satisfactory. Outranking relations resulting from a multicriteria preference aggregation procedure are, however, rarely crisp in nature. They generally appear associated with credibility levels (see Bisdorff, 2002b; Fodor and Roubens, 1994).

This issue will be investigated in the next Section.

3. Basic credibility calculus for good, bad and ambiguous choices

3.1 L-valued binary relations

We now consider a finite set of alternatives X and a binary relation R whose credibility is evaluated as follows:

For all $a, b \in X$ and m positive integer, $R(a, b)$ belongs to the finite set $L : \{c_0, \ldots, c_m, \ldots, c_{2m}\}$ that constitutes a $(2m + 1)$-element chain $c_0 < \ldots < c_{2m}$. $R(a, b)$ may be understood as the *level of credibility* that "*a is at least as good as b*". We call such a relation an L-valued binary relation (or shortly L-vbr).

We denote $L^{\geq m} : \{c_m, \ldots, c_{2m}\}$, $L^{>m} : \{c_{m+1}, \ldots, c_{2m}\}$, $L^{<m} : \{c_0, \ldots, c_{m-1}\}$ and $L^{\leq m} : \{c_0, \ldots, c_m\}$.

If $R(a, b) \in L^{>m}$ we say that the proposition "$(a, b) \in R$" is *L-true*.

If, however, $R(a,b) \in L^{<m}$ we say that the proposition "$(a,b) \in R$" is *L-false*. If $R(a,b) = c_m$, i.e. the median level, we say that the proposition "$(a,b) \in R$" is *L-undetermined*. *L*-undeterminedness may be assimilated to a kind of *missing value* status (see Bisdorff, 2002a).

We define on L an antitone unary *contradiction* operator $\neg$ such that $\neg c_i = c_{(2m-i)}$ for $i = 0, \ldots 2m$. If $R(a,b) = c_{m+k}$, i.e. the proposition "*a* is at least as good as *b*" is more or less true, then its negation, $\neg R(a,b)$, i.e. the proposition "*a* is *not* at least as good as *b*" is more or less false. $L^{\leq m}$ thus represents the order reversed mirror of $L^{\geq m}$ and the *median* credibility level c_m appears as the fix point of the $\neg$-operator.

In order to respect the ordinal character of the credibility calculus, we consider that the credibility level of a *conjunction* (a *disjunction*, respectively) of *L*-valued propositions is given by the "min" (the "max", respectively) operator defined on L.

In this setting the classic Boolean lattice $\mathcal{B} = \{0, 1\}$ appears as a degenerated limit case with no median level defined. We shall designate L_3 the three-valued limit case ($m = 1$), which corresponds to a natural bipolarization of *L*-valuedness preserving the median, logically undetermined level c_m (see Bisdorff, 1999).

3.1 Qualification of good and bad choices

We denote $G^L = (X, R)$ a digraph with vertices set X and *L*-valued binary relation R. Consider a choice Y in G^L. In accordance with the crisp case, we define following *L*-valued qualifications of Y.

The level of *stability qualification* of Y is defined as

$$\Delta^{\text{sta}}(Y) = \begin{cases} c_{2m} & \text{if } Y \text{ is a single choice,} \\ \min_{\substack{b \neq a \\ b \in Y}} \min_{\substack{a \neq b \\ a \in Y}} \{\neg R(a,b)\} & \text{otherwise.} \end{cases} \quad (5)$$

Y is considered to be *L-stable* if $\Delta^{\text{sta}}(Y) \in L^{>m}$.

The level of *dominance qualification* of $Y \neq X$ corresponds to

$$\Delta^{\text{dom}}(Y) = \begin{cases} c_{2m} & \text{if } Y \text{ is a greedy choice,} \\ \min_{a \notin Y} \max_{b \in Y} \{R(b,a)\} & \text{otherwise.} \end{cases} \quad (6)$$

Y is considered to be *L-dominant* if $\Delta^{\text{dom}}(Y) \in L^{>m}$.

The level of *absorbance qualification* of $Y \neq X$ is equal to

$$\Delta^{\text{abs}}(Y) = \begin{cases} c_{2m} & \text{if } Y \text{ is a greedy choice.} \\ \min_{a \notin Y} \max_{b \in Y} \{R(a,b)\} & \text{otherwise.} \end{cases} \tag{7}$$

Y is considered to be *L-absorbent* if $\Delta^{\text{abs}}(Y) \in L^{>m}$.

The qualification of Y being a *good choice*, i.e. stable and dominant, corresponds to

$$Q^{\text{good}}(Y) = \min(\Delta^{\text{sta}}(Y), \Delta^{\text{dom}}(Y)). \tag{8}$$

The qualification of Y being a *bad choice*, i.e. stable and absorbent, corresponds to

$$Q^{\text{bad}}(Y) = \min\left(\Delta^{\text{sta}}(Y), \Delta^{\text{abs}}(Y)\right) \tag{9}$$

The qualification of Y being an *ambiguous choice*, i.e. stable, dominant and absorbent, is equal to

$$Q^{\text{amb}}(Y) = \min\{\Delta^{\text{sta}}(Y), \Delta^{\text{abs}}(Y), \Delta^{\text{dom}}(Y)\} \tag{10}$$

Definition 3 (Good, bad and ambiguous L-valued choices).
If $Q^{\text{good}}(Y) \in L^{>m}$, i.e. Y is L-stable and L-dominant, we accept Y as an *L-good choice*. If, however, $Q^{\text{bad}}(Y) \in L^{>m}$, i.e. Y is L-stable and L-absorbent, we consider Y as an *L-bad choice*.

If $Q^{\text{amb}}(Y) \in L^{>m}$, Y is considered to be *L-ambiguous*. In addition, if $Q^{\text{good}}(Y) = Q^{\text{bad}}(Y)$, we call Y a *clearly L-ambiguous* choice. If, however, $Q^{\text{good}} > Q^{\text{bad}} > c_m$, we call Y a *more good than bad* choice. Inversely, if $Q^{\text{bad}}(Y) > Q^{\text{good}}(Y) > c_m$, we call Y a *more bad than good* choice.

Finally, a *clearly L-good* (*L-bad*) choice is an L-stable and L-dominant (L-absorbent) choice which is not L-ambiguous.

We summarize all possible qualifications for choices in a given G^L in Figure 1.

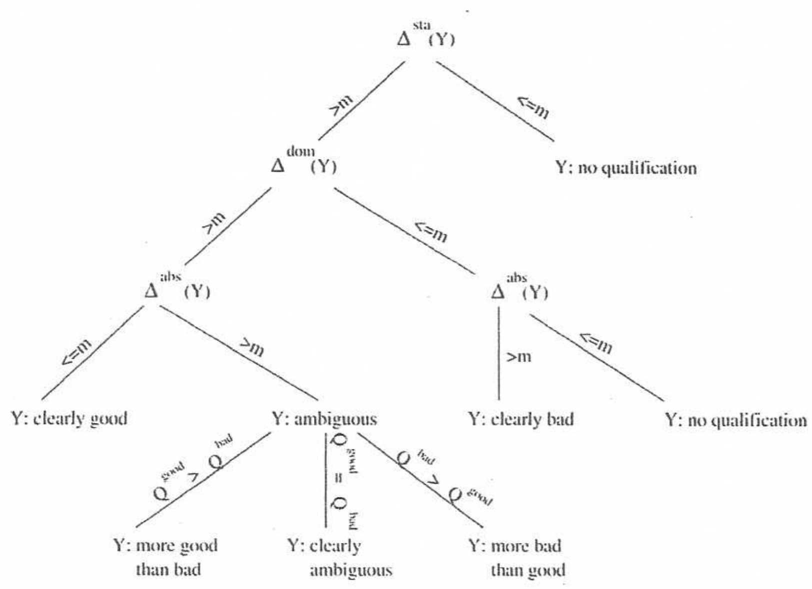


Figure 1: Qualification of choices

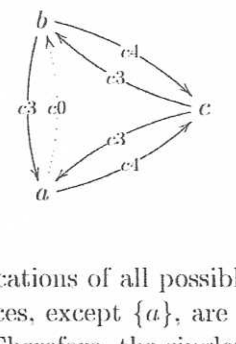
Furthermore, we denote $\mathcal{C}^{\text{good}}(G^L)$ ($\mathcal{C}^{\text{abs}}(G^L)$) the possibly empty set of L-good (L-bad) choices in G^L.

Example 1.

Consider the following graph:

G^L_7 is such that $X = \{a, b, c\}$, $L = \{c_0, c_1, c_2 = c_m, c_3, c_4\}$ and $R : \{R(b, a) = R(c, a) = R(c, b) = c_3,\ R(b, c) = R(a, c) = c_4,\ R(a, b) = c_0\}$.

G^L_7 :



In Table 2 we show qualifications of all possible choices in G^L_7. All singletons are L-stable. All choices, except $\{a\}$, are L-dominant. All choices, except $\{b\}$ are L-absorbent. Therefore, the singleton $\{b\}$ gives a clearly L-good single choice whereas $\{a\}$ gives a clearly L-bad single choice. We may

Table 2: Qualification of good and bad choices

choice	Δ^{sta}	Δ^{dom}	Δ^{abs}	Q^{good}	Q^{bad}	Q^{amb}
$\{a\}$	c_4	c_0	c_3	c_0	c_3	c_0
$\{b\}$	c_4	c_3	c_0	c_3	c_0	c_0
$\{c\}$	c_4	c_3	c_4	c_3	c_4	c_3
$\{a,b\}$	c_1	c_4	c_3	c_1	c_1	c_1
$\{a,c\}$	c_0	c_3	c_3	c_0	c_0	c_0
$\{b,c\}$	c_0	c_3	c_3	c_0	c_0	c_0
$\{a,b,c\}$	c_0	c_4	c_4	c_0	c_0	c_0

notice that $\{c\}$ gives an L-ambiguous, but more bad than good single choice. Possible pair choices, as well as the greedy choice, are neither L-good nor L-bad, as they all fail the required L-stability condition. Single choice $\{b\}$ clearly appears as the *best* choice recommendation one can deduce from this valued outranking relation. Deciding, whether $\{a\}$ or $\{c\}$ gives the worst choice, is not immediate and we shall come back later to an argument for eventually discriminating between these these two choices. .

3.2. Computing good and bad L-valued choices

Having to inspect, in a given digraph G^L with n vertices, all 2^n possible choices, in order to uncover the sets $\mathcal{C}^{\text{good}}(G^L)$ and $\mathcal{C}^{\text{abs}}(G^L)$ is not a satisfactory operational perspective. Fortunately the following proposition, relating crisp and L-valued choices, comes to our rescue.

Let R be an L-vbr. The *median-level cut relation* $R^{\geq m}$ associated with R is a crisp relation such that $(a,b) \in R^{\geq m}$ if and only if $R(a,b) \in L^{\geq m}$.

The corresponding *strict median-level cut relation* $R^{>m}$ is a crisp relation associated with R such that $(a,b) \in R^{>m}$ if and only if $R(a,b) \in L^{>m}$.

Proposition 2 (L. Kitaïnik, 1993).
Let $G^L = (X,R)$ be an L-valued digraph and let $G^{>m} = (X, R^{>m})$ represent the associated crisp, strict median-level cut digraph.

$$\mathcal{C}^{x}(G^L) \subseteq \mathcal{C}^{x}(G^{>m}) \textit{ with } x \in \{\text{good}, \text{bad}\}.$$

Proof. Consider $Y \neq \emptyset$ such that $\Delta^{\text{sta}}(Y) \in L^{>m}$.

$$\begin{aligned}\Delta^{\text{sta}}(Y) \in L^{>m} &\Leftrightarrow \min_{\substack{b \neq a \\ b \in Y}} \min_{\substack{a \neq b \\ a \in Y}} (\neg R(a,b)) \in L^{>m} \\ &\Leftrightarrow \forall a \neq b \in Y : R(a,b) \in L^{<m} \; (R(a,b) \notin L^{\geq m}) \\ &\Leftrightarrow \forall a \neq b \in Y : (a,b) \notin R^{\geq m} \\ &\Leftrightarrow Y \text{ is a stable set in } G^{\geq m} = (X, R^{\geq m}).\end{aligned}$$

Finally, $\Delta^{\text{sta}}(Y) \in L^{>m} \Rightarrow Y$ is a stable set in $G^{>m}$ since $R^{\geq m} \supset R^{>m}$ and any stable set in $G^{\geq m}$ is contained in the set of all stable sets in $G^{>m}$.

Consider now $Y \neq \emptyset$ such that $\Delta^{\text{dom}}(Y) \in L^{>m}$.

$$\begin{aligned}\Delta^{\text{dom}}(Y) \in L^{>m} &\Leftrightarrow \min_{a \notin Y} \max_{b \in Y} R(b,a) \in L^{>m} \\ &\Leftrightarrow \forall a \notin Y, \exists b \in Y : R(b,a) \in L^{>m} \\ &\Leftrightarrow \forall a \notin Y, \exists b \in Y : (b,a) \in R^{>m} \\ &\Leftrightarrow Y \text{ is a dominant set in } G^{>m}.\end{aligned}$$

Using the same arguments,

$$\Delta^{\text{abs}}(Y) \in L^{>m} \Leftrightarrow Y \text{ is an absorbent set in } G^{>m}.$$

The proposition is an immediate consequence of the previous results. □

This important result, from an operational point of view, may be strengthened when considering L-valued relations without median-valued, i.e. L-undetermined or missing, arcs.

Corollary 1.

If $R(a,b) \neq c_m$ for all $a, b \in X$, there exists a bijection between L-good and/or L-bad choices in $G^L = (X, R)$ and good and/or bad choices in $G^{>m} = (X, R^{>m})$.

Proof. We consider only the case of a clearly L-good choice Y (the other cases can be treated in the same manner) : $\Delta^{\text{sta}}(Y) \in L^{>m}$, $\Delta^{\text{dom}}(Y) \in L^{>m}$, $\Delta^{\text{abs}}(Y) \in L^{\leq m}$:

$$\Delta^{\text{sta}}(Y) \in L^{>m} \quad \Leftrightarrow \quad Y \text{ is a stable set in } G^{\geq m} \equiv G^{>m},$$

$$\Delta^{\text{dom}}(Y) \in L^{>m} \quad \Leftrightarrow \quad Y \text{ is a dominant in } G^{>m},$$

$$\Delta^{\text{abs}}(Y) \in L^{\leq m} \quad \Leftrightarrow \quad Y \text{ is not an absorbent set in } G^{>m}.$$

Thus, there is a bijection between clearly L-good choices in G^L and clearly good choices in the associated crisp digraph $G^{>m}$. □

Proposition (2) allows us to implement Algorithm 1 for computing L-valued choices.

Algorithm 1 (Computing L-valued choices).

computeValuedChoices($G^L = (X, R)$)
  step 1: *strict median-level cut*
    $R^{>m} \leftarrow$ *cut R above median level c_m*
  step 2: *enumerating kernels using the FD solver in GNU-Prolog*
    $\mathcal{C}^{\text{good}} \leftarrow$ *good choices from* $Y \circ R^{>m} = \overline{Y}$
    $\mathcal{C}^{\text{bad}} \leftarrow$ *bad choices from* $R^{>m} \circ Y^t = \overline{Y}^t$
  step 4: *computing the qualification of the associated choices*
    for $Y_i \in \mathcal{C}^{\text{good}} \cup \mathcal{C}^{\text{bad}}$ **:**
      $\Delta^{\text{sta}}(Y_i) \leftarrow \min_{a \in Y_i}[\min_{b \neq a \in Y_i} \neg R(a, b)]$
      $\Delta^{\text{dom}}(Y_i) \leftarrow \min_{a \notin Y_i}[\max_{b \neq a \in Y_i} R(b, a)]$
      $\Delta^{\text{abs}}(Y_i) \leftarrow \min_{a \notin Y_i}[\max_{b \neq a \in Y_i} R(a, b)]$
      $Q^{\text{good}}(Y_i) \leftarrow \min[\Delta^{\text{sta}}(Y_i), \Delta^{\text{dom}}(Y_i)]$
      $Q^{\text{bad}}(Y_i) \leftarrow \min[\Delta^{\text{sta}}(Y_i), \Delta^{\text{abs}}(Y_i)]$
      $Q^{\text{amb}}(Y_i) \leftarrow \min[Q^{\text{good}}(Y_i), Q^{\text{bad}}(Y_i)]$
      output $(Y_i, Q^{\text{good}}(Y_i), Q^{\text{bad}}(Y_i), Q^{\text{amb}}(Y_i))$
    endfor
  endcomputeValuedChoices

In Step **1** we compute from R the strict median level cut relation $R^{>m}$. In Step **2** we enumerate, with the help of the FD-Solver in GNU-Prolog, all dominant and absorbent kernels in $G^{>m} = (X, R^{>m})$ and compute the corresponding sets of good and bad choices. As the L-good and L-bad choices in G^L are necessarily part of these sets, we may reduce, in Step **3**,

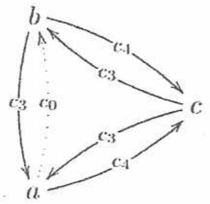
Table 3: Outranking relation R_8 observed on the set of potential car selections

R	$vwgc$	$r9gt$	$gsax$	$p305$	$tahg$	$audi$	$r18g$	$alfa$
$vwgc$	-	c_{75}	c_{70}	c_{62}	c_0	c_0	c_0	c_0
$r9gt$	c_{76}	-	c_{90}	c_{100}	c_{82}	c_{82}	c_{82}	c_{80}
$gsax$	c_{70}	c_{86}	-	c_{100}	c_{100}	c_{46}	c_{80}	c_{91}
$p305$	c_{64}	c_{65}	c_{94}	-	c_{88}	c_{22}	c_{94}	c_{74}
$tahg$	c_{33}	c_{57}	c_{93}	c_{100}	-	c_0	c_{80}	c_{86}
$audi$	c_0	c_{73}	c_{64}	c_{92}	c_{76}	-	c_{96}	c_{80}
$r18g$	c_0	c_{63}	c_{73}	c_{85}	c_{82}	c_{70}	-	c_{81}
$alfa$	c_0	c_{60}	c_{64}	c_{60}	c_{77}	c_0	c_0	-

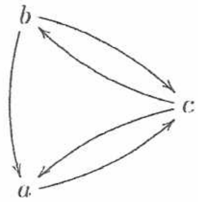
the computation of the L-valued qualifications to the union of the good or bad choices in $G^{>m}$.

In order to illustrate Algorithm 1, let us reconsider Example 1 :

G^L_7 :



$G^{>c_2}_7$:



In $G^{>c_2}_7$ one obtains, as in G^L_7, two good choices, $\{b\}$ and $\{c\}$, and two bad choices, $\{a\}$ and $\{c\}$. $\{a\}$ is a clearly bad choice, $\{b\}$ is a clearly good choice and $\{c\}$ is an ambiguous choice. Thus we recover here the results obtained by inspecting the whole set of possible choices (see Table 2).

Example 2 (Perny, 1992).
Eight cars are proposed as decision alternatives: $X = \{vwgc, r9gt, gsax, p305, tahg, audi, r18g, alfa\}$. A previous multicriteria decision analysis has generated the L-valued outranking relation R_8 carrying a meaning of "*to be at least as good as*" shown in Table 3, where $L = \{c_i \mid i = 0..100\}$ and $c_m = c_{50}$.

Step **1** computes from relation R_8 the corresponding crisp strict median level cut relation $R_8^{>50}$. Given the crisp digraph $G^{>50} = (X, R_8^{>50})$, Step **2**

Table 4: Qualification of L-good and L-bad choices in $G^L = (X, R_8)$

choices	Δ^{sta}	Δ^{dom}	Δ^{abs}	Q^{good}	Q^{bad}	Q^{amb}
L-good						
$\{r9gt\}$	c_{100}	c_{76}	c_{57}	c_{76}	c_{57}	c_{57}
$\{vwgc, r18g\}$	c_{100}	c_{70}	c_0	c_{70}	c_0	c_0
$\{vwgc, audi\}$	c_{100}	c_{70}	c_0	c_{70}	c_0	c_0
L-bad						
$\{vwgc, tahg\}$	c_{67}	c_0	c_{76}	c_0	c_{67}	c_0
$\{vwgc, alfa\}$	c_{76}	c_0	c_{74}	c_0	c_{74}	c_0
$\{gsax\}$	c_{100}	c_{46}	c_{64}	c_{46}	c_{64}	c_{46}
$\{p305\}$	c_{100}	c_{22}	c_{60}	c_{22}	c_{60}	c_{22}

delivers following sets of good and bad choices:

$$\begin{aligned} \mathcal{C}^{good}(G^{>50}) &= \{\{r9gt\}, \{vwgc, r18g\}, \{vwgc, audi\}\}, \\ \mathcal{C}^{bad}(G^{>50}) &= \{\{p305\}, \{gsax\}, \{vwgc, tahg\}, \{vwgc, alfa\}\}. \end{aligned}$$

Following Propositions 1 and 2, as well as Corollary 1 (relation R_8 doesn't contain any L-undetermined arc), we may compute (see Table 4), in Step **3**, qualifications of all L-good and L-bad choices in the original digraph $G^L = (X, R_8)$.

The most qualified good choice is given by single choice $\{r9gt\}$. As the same alternative also appears to be a more or less qualified bad choice, it is, however, L-ambiguous, but more good than bad. Two clearly good, but less qualified, choices are given by the pairs $\{vwgc, r18g\}$ and $\{vwgc, audi\}$. The proposed outranking relation reveals furthermore fours different, clearly more or less L-bad choices: $\{vwgc, tahg\}$, $\{vwgc, alfa\}$, $\{gsax\}$ and $\{p305\}$.

Let us now extend the crisp kernel equation systems to the L-valued case.

4. Relating L-valued choices and kernels

4.1. L-valued dominant and absorbent kernels

The key to the equivalence between good and bad choices and the corresponding solutions of Equation Systems (2) and (3) lies in the characteristic vector representation of the choices (see Equation 1).

In case of an L-vbr R defined on a set X, we may extend the characteristic vector representation of a choice Y in the following sense: $\tilde{Y}(\cdot) = (\tilde{Y}(a_1), \ldots,)$ is a row vector such that $\forall a_i \in X : \tilde{Y}(a_i) \in L = \{c_0, \ldots, c_m, \ldots, c_{2m}\}$ gives the credibility level of the assertion that "*element a_i is part of the choice Y*". From the semantics of our credibility calculus follows the logical denotation below. $\forall a \in X$:

$$\tilde{Y}(a) \in \begin{cases} L^{>m} & \text{if } \textit{more or less } a \in Y, \\ L^{<m} & \text{if } \textit{more or less } a \notin Y, \\ c_m & \text{if } a \in Y \text{ is } \textit{undetermined}. \end{cases} \tag{11}$$

We may thus formulate following L-valued versions of the dominant, resp. absorbent kernel equation systems. Let $G^L = (X, R)$ be an L-valued digraph. $\forall a, b \in X$:

$$(\tilde{Y} \circ R)(a) = \max_{b \neq a} \left[\min\left(\tilde{Y}(b), R(b,a)\right)\right] = \neg\tilde{Y}(a) \tag{12}$$

$$(R \circ \tilde{Y}^t)(a) = \max_{b \neq a} \left[\min\left(R(a,b), \tilde{Y}^t(b)\right)\right] = \neg\tilde{Y}^t(a) \tag{13}$$

The L-valued composition operator '$\circ$' uses 'max' and 'min' as disjunction and conjunction operators. As in the crisp case, we ignore in our computation the diagonal terms of R. If $L = \mathcal{B} = \{c_0, c_1\}$, we recover the respective crisp Equation Systems.

We denote $\mathcal{Y}^{\text{dom}}(G^L)$ ($\mathcal{Y}^{\text{abs}}(G^L)$, respectively) the set of solutions from Equations Systems 12 (13, respectively).

Example 3.
$G^L_9 = (X, R)$ is such that $X = \{a, b, c\}$, $L = \{c_0, \ldots, c_{10}\}$ with $m = 5$ and $R = \{R(a,b) = c_2\}$, $R(a,c) = c_9$, $R(b,a) = c_6$, $R(b,c) = c_{10}$, $R(c,a) = c_7$, $R(c,b) = c_8\}$.

The corresponding dominant kernel equation system:

$$\left[\tilde{Y}(a)\ \tilde{Y}(b)\ \tilde{Y}(c)\right] \circ \begin{bmatrix} - & c_2 & c_9 \\ c_6 & - & c_{10} \\ c_7 & c_8 & - \end{bmatrix} = \left[\neg\tilde{Y}(a)\ \neg\tilde{Y}(b)\ \neg\tilde{Y}(c)\right], \tag{14}$$

admits the set of solutions shown in Table 5: Solution $\tilde{Y}_0$, completely L-undetermined, doesn't characterize any choice at all and we may ignore it. Solutions $\tilde{Y}_1$, $\tilde{Y}_2$ and $\tilde{Y}^*_3$ characterize the same single choice $\{b\}$, whereas solutions $\tilde{Y}_4$ and $\tilde{Y}^*_5$ characterize the same single choice $\{c\}$. It is worthwhile

Table 5: Solutions of Equation System 14

solution	$\tilde{Y}(a)$	$\tilde{Y}(b)$	$\tilde{Y}(c)$
$\tilde{Y}_0$	c_5	c_5	c_5
$\tilde{Y}_1$	c_4	$\mathbf{c_6}$	c_4
$\tilde{Y}_2$	c_4	$\mathbf{c_7}$	c_3
$\tilde{Y}_3^*$	c_4	$\mathbf{c_8}$	c_2
$\tilde{Y}_4$	c_4	c_4	$\mathbf{c_6}$
$\tilde{Y}_5^*$	c_3	c_3	$\mathbf{c_7}$

noticing that both subsets of solutions are organized as chains of more and more logically determined, we say *sharper*, solutions. In each chain we consider the maximal solutions, $\tilde{Y}_3^*$ and $\tilde{Y}_5^*$, with respect to this denotational sharpness, to be the actual L-valued dominant kernels we are looking for.

In general, let $\tilde{Y}, \tilde{Y}'$ be two admissible solutions of Equation Systems (12) and (13). We say that $\tilde{Y}$ is *sharper* than $\tilde{Y}'$, noted $\tilde{Y}' \preccurlyeq \tilde{Y}$ iff $\forall a \in X$: either $\tilde{Y}(a) \leq \tilde{Y}'(a) \leq c_m$ or $c_m \leq \tilde{Y}'(a) \leq \tilde{Y}(a)$.

As obvious from Example **3**, relation "$\preccurlyeq$" models in general a partial order on the sets $\mathcal{Y}^{\text{dom}}$ and $\mathcal{Y}^{\text{abs}}$ of solutions of Equation System 12, respectively 13.

Definition 4 (L-valued dominant and absorbent kernels).
We call *L-dominant* (*L-absorbent*) kernel a solution (if any) $\tilde{Y}$ of Equation System (12) (resp. (13)) such that $\tilde{Y}(a) \neq c_m$ for all $a \in X$ and $\tilde{Y}$ is a maximal sharp solution in the set $\mathcal{Y}^{\text{dom}}(G^L)$ ($\mathcal{Y}^{\text{abs}}(G^L)$) of admissible solutions.

Taking into account only completely determined (no L-undetermined components) and efficient, non-dominated (maximal sharpest) solutions is coherent with the logical semantics of our credibility calculus, where L-valued disjunction, modelled with the help of the max–operator, verifies, as in the crisp case, some kind of L-valued idempotency property. If a same proposition "$a \in Y$" is true at credibility levels c_{m+k} or c_{m+r} with $0 < k, r \leq m$, then the proposition is true at the credibility level of their disjunction, i.e. $\max(c_{m+k}, c_{m+r})$.

We denote $\mathcal{F}^{\text{dom}}(G^L)$ ($\mathcal{F}^{\text{abs}}(G^L)$) the possibly empty set of L-dominant

Table 6: L-valued kernels in the digraph of Example 1

kernel	$\tilde{Y}(a)$	$\tilde{Y}(b)$	$\tilde{Y}(c)$
L-dominant			
$\tilde{Y}_1^*$	c_1	c_4	c_0
$\tilde{Y}_2^*$	c_1	c_1	c_3
L-absorbent			
$\tilde{Y}_3^*$	c_3	c_1	c_1
$\tilde{Y}_4^*$	c_0	c_0	c_4

(L-absorbent) kernels we may find in a given digraph G^L.

Reconsider Example **3**: Following Definition 4, solutions $\tilde{Y}_3^*$ and $\tilde{Y}_5^*$ (see Table 5) give the L-dominant kernels in G_9^L. Let us denote K_1 the dominant kernel characterized through $\tilde{Y}_3^*$. The L-dominant kernel tells us that, – first, "$a \notin K_1$" is true at credibility level $\neg c_4 = c_6$, –secondly, "$b \in K_1$" is true at credibility level c_8, and – finally, "$c \notin K_1$" is true at credibility level $\neg c_2 = c_8$. $\tilde{Y}_3^*$ therefore characterizes an L-valued single choice $\{b\}$. A similar computation, involving the corresponding absorbent kernel equation systems reveals solutions $[Y(a) = c_7, Y(b) = c_4, Y(c) = c_3]$ and $[Y(a) = c_1, Y(b) = c_1, Y(c) = c_9]$ as the L-absorbent kernels in G_9^L.

Let us also reconsider Example **1**: As shown in Table 6, the digraph G_7 admits the set $\mathcal{F}^{\text{good}}(G_7) = \{\tilde{Y}_1^*, \tilde{Y}_2^*\}$ of L-dominant kernels and the set $\mathcal{F}^{\text{bad}}(G_7) = \{\tilde{Y}_3^*, \tilde{Y}_4^*\}$ of L-dominant kernels. The L-dominant kernels confirm that single choice $\{b\}$ clearly appears as the *best* choice recommendation one can deduce from this valued outranking relation. Similarly, the L-absorbent kernels clearly denote $\{c\}$ as the *worst* choice compared with $\{a\}$ (see Table 6).

From this last example, we may notice that L-valued kernels and choices give similar, but not completely identical answers with respect to the qualification of good and bad choices in a given L-valued digraph. In the last part we show the formal correspondence between both.

4.2 Relating L-valued kernels and L-valued choices

Let $G^L = (X, R)$ be an L-valued digraph. In accordance with the logical denotation of the L-valued characteristic vector construction (see Equation 11), we may define the choice $K_{\tilde{Y}}$ associated with a given L-valued kernel

$\tilde{Y}$ as follows:

$$K_{\tilde{Y}} \subset X \ : \begin{cases} a \in K_{\tilde{Y}} & \text{if } \tilde{Y}(a) > c_m, \\ a \notin K_{\tilde{Y}} & \text{if } \neg\tilde{Y}(a) \geq c_m, \end{cases} \tag{15}$$

We denote $\mathcal{K}(\mathcal{F}^{\text{dom}}(G^L))$ ($\mathcal{K}(\mathcal{F}^{\text{abs}}(G^L))$, respectively) the set of good (bad, respectively) choices we may construct from the corresponding L-valued kernels.

Theorem 1.
With $(x, y) = \{(\text{good}, \text{dom}), (\text{bad}, \text{abs})\}$*:*

$$\mathcal{C}^x(G^L) = \mathcal{K}(\mathcal{F}^y(G^L)).$$

The proof of this theorem, requires the following lemmas.

We call π the *median polarization operator* for L-valued assertions defined as follows:

$$\pi P = \begin{cases} c_{2m} & \text{iff } P > c_m, \\ c_0 & \text{iff } P < c_m, \\ c_m & \text{otherwise.} \end{cases} \tag{16}$$

where $P \in L$ is a proposition evaluated in L.

Lemma 1. *Let $P, Q \in L$ be two L-valued propositions:*

$$\pi(\min(P, Q)) = \min(\pi(P), \pi(Q)) \tag{17}$$
$$\pi(\max(P, Q)) = \max(\pi(P), \pi(Q)) \tag{18}$$
$$\pi(\neg(P)) = \neg(\pi(P)) \tag{19}$$
$$P = Q \Rightarrow \pi(P) = \pi(Q) \tag{20}$$

Proof. The following L-valued truth table establishes Equivalence (17).

P	Q	$\min(P, Q)$	$\pi(P)$	$\pi(Q)$	$\pi(\min(P, Q))$	$\min(\pi(P), \pi(Q))$
L-true	L-true	L-true	c_{2m}	c_{2m}	c_{2m}	c_{2m}
L-true	c_m	c_m	c_{2m}	c_m	c_m	c_m
L-true	L-false	L-false	c_{2m}	c_0	c_0	c_0
c_m	L-true	c_m	c_m	c_0	c_0	c_0
c_m	c_m	c_m	c_m	c_m	c_m	c_m
c_m	L-false	L-false	c_m	c_0	c_0	c_0
L-false	L-true	L-false	c_0	c_{2m}	c_0	c_0
L-false	c_m	L-false	c_0	c_m	c_0	c_0
L-false	L-false	L-false	c_0	c_0	c_0	c_0

A similar table establishes Equivalences (18) and (19) whereas Implication (20) follows immediately from the definitions. □

Let $G^L = (X, R)$ be an L-valued digraph and let $\pi G^L = (X, \pi R)$ be the associated median polarized $L_3 = \{c_0, c_m, c_{2m}\}$-valued digraph.

Lemma 2.

$$\mathcal{C}^x(G^L) = \mathcal{C}^x(\pi G^L), \quad x \in \{\text{good}, \text{bad}\}.$$

Proof. The definition of L-good (L-bad) choices only involves the L-valued logical operators: min, max, $\neg$ and equality $=$. Lemma 1 assures that operator π is a natural transformation for these operators. □

Let $\pi\mathcal{F}^x(G^L)$ with $x \in \{\text{dom}, \text{abs}\}$ represent the set of median polarized L-dominant or L-absorbent kernels.

Lemma 3.

$$\pi\mathcal{F}^x(G^L) = \mathcal{F}^x(\pi G^L), \quad x \in \{\text{dom}, \text{abs}\}.$$

Proof. Same arguments as in the proof of Lemma 2. □

Proof of Theorem 1.
Consider, first, L-dominant kernels and L-good choices. An immediate consequence of Lemmas 1, 2 and 3 is that, on the one hand, there exists a bijection between $\mathcal{C}^{\text{good}}(G^L)$ and $\mathcal{C}^{\text{good}}(\pi G^L)$. And, on the other hand, there exists a bijection between $\pi\mathcal{F}^{\text{dom}}(G^L)$ and $\mathcal{F}^{\text{dom}}(\pi G^L)$.

We just need now to prove that there also exists a bijection between $\mathcal{C}^{\text{good}}(\pi G^L)$ and $\mathcal{F}^{\text{dom}}(\pi G^L)$.

Let $\tilde{Y} \in \mathcal{F}^{\text{dom}}(\pi G^L)$. From Definition 4, we know that $\tilde{Y}$ must be completely L-determined, i.e cannot contain any c_m valued element. Being solution of Equation System 12, therefore, implies that, on the one hand, the median polarized stability qualification of the choice Y characterized by $\tilde{Y}$ must be c_{2m} valued and, on the other hand, the median polarized dominance qualification of the choice Y modelled by $\tilde{Y}$ must also be c_{2m} valued. Thus $\tilde{Y}$ necessarily characterizes an L_3-good choice.

Inversely, let $K \in \mathcal{C}^{\text{good}}(\pi G^L)$. If we characterize this choice with the help of an L_3-valued characteristic vector $\tilde{Y}_K$ such that $\tilde{Y}_K(a) = c_{2m}$ if $a \in K$ and $\tilde{Y}_K(a) = c_0$ if $a \notin K$, we may easily check, that $\tilde{Y}_K$ gives a solution of the dominant kernel defining Equation System. As $\tilde{Y}_K$ contains by construction no c_m-valued element and such a solution is always maximal sharp with respect to L_3, $\tilde{Y}_K$ gives necessarily an L_3-dominant kernel in πG^L.

Same arguments apply to L-absorbent kernels and L-bad choices. □

Finally, let us show the equivalence between the subset-wise qualification of choices as introduced in Section **2** and the corresponding element-wise qualification provided by the L-valued kernels.

4.3. Element-wise and subset-wise qualification

Let $\tilde{Y}$ be an L-dominant or L-absorbent kernel in a given L-valued digraph G^L.

Definition 5 (Element-wise qualification of kernels).
The element-wise qualification Q_e of $\tilde{Y}$ to be a good or bad choice characterization is defined as follows:

$$Q_e(\tilde{Y}) = \min\big[\min_{\tilde{Y}(a)>c_m} \tilde{Y}(a), \min_{\tilde{Y}(a)<c_m} \neg\tilde{Y}(a)\big]$$

Let $K_{\tilde{Y}}$ denote the choice characterized by the vector $\tilde{Y}$.

Theorem 2.
With $(x, y) = \{(\text{dom}, \text{good}), (\text{abs}, \text{bad})\}$*:*

$$\forall \tilde{Y} \in \mathcal{F}^{x}(G^L): \quad Q_e(\tilde{Y}) = Q^{y}(K_{\tilde{Y}}).$$

Proof. We give the proof for L-dominant kernels and L-good choices. The absorbent/bad case takes similar arguments.

Let G^L be an L-valued digraph supporting a non empty set of L-dominant kernels and let $\tilde{Y} \in \mathcal{F}^{\text{dom}}(G^L)$ represent such a kernel. Following Theorem 1 we may suppose that the associated choice $K_{\tilde{Y}} \in \mathcal{C}^{\text{good}}(G^L)$ is such that $Q^{\text{good}}(K_{\tilde{Y}}) = c_{m+k}$ with $k > 0$.

Let us rewrite the L-dominant kernel definig Equation System 12 using a similar matrix decomposition as in the proof of Proposition 1:

$$[\tilde{Y}_K \; \tilde{Y}_{\overline{K}}] \circ \begin{bmatrix} R_{KK} & R_{K\overline{K}} \\ R_{\overline{K}K} & R_{\overline{K}\overline{K}} \end{bmatrix} = [\neg\tilde{Y}_K \; \neg\tilde{Y}_{\overline{K}}]. \quad (21)$$

$\tilde{Y}$ being solution of Equation System 21 and $Q^{\text{good}}(K_{\tilde{Y}}) = c_{m+k}$ imply that

$$\forall a \in K : c_m < \tilde{Y}(a) \leq c_{m+k} \quad \text{and} \quad \forall a \notin Y : c_m > \tilde{Y}(a) \geq c_{m-k}.$$

Hence,

$$\forall \tilde{Y} \in \mathcal{F}^{\text{dom}}(G^L) : \quad Q_e(\tilde{Y}) \leq Q^{\text{good}}(K_{\tilde{Y}}).$$

Let us assume that $\exists \tilde{Y} \in \mathcal{F}^{\text{dom}}(G^L)$ such that $Q_e(\tilde{Y}) < Q^{\text{good}}(K_{\tilde{Y}}) = c_{m+k}$.

Following Theorem 1 this implies that $\exists r$ such that $0 < r < k$ and $\tilde{Y}(a) \geq c_{m+r} : a \in K_{\tilde{Y}}$ and $\tilde{Y}(a) \leq c_{m-r} : a \notin K_{\tilde{Y}}$.

Now, let $\tilde{Y}^*$ be such that $\tilde{Y}^*(a) = \max\left(\tilde{Y}(a), c_{m+k}\right)$ when $a \in K_{\tilde{Y}}$ and $\tilde{Y}^*(a) = \min\left(\tilde{Y}(a), c_{m-k}\right)$ when $a \notin K_{\tilde{Y}}$.

It is easily checked that $\tilde{Y}^* \succcurlyeq \tilde{Y}$ and that $Q_e(\tilde{Y}^*) = c_{m+k}$.

But we obtain, furthermore, that if $\tilde{Y}$ is a solution of Equation System 21 then $\tilde{Y}^*$ is also a solution of System 21.

This, however, contradicts that $\tilde{Y}$ is a maximal sharp solution of Equation System 21, i.e. $\tilde{Y} \notin \mathcal{F}^{\text{dom}}(G^L)$.

Whence it follows necessarily that $Q_e(\tilde{Y}) = Q^{\text{good}}(K_{\tilde{Y}})$ for all $\tilde{Y} \in \mathcal{F}^{\text{dom}}(G^L)$. □

5. Conclusion

In this paper we have, first, introduced clearly good, bad and ambiguous choices and shown their equivalence with corresponding dominant and/or absorbent kernels (see Proposition 1).

In a second part, we have extended these notions to an ordinal valued setting, with the result that mainly the initially crisp ambiguous choices are here split into three classes: more good than bad, clearly ambiguous and more bad than good choices. The median-level cut operation installs a formal link, shown in Proposition 2, between crisp and valued choices which allows us to compute rather efficiently the set of all possible good and bad choices in an ordinal valued outranking graph.

Finally, we have extended the dominant and absorbent kernel defining equation systems to the ordinal valued case. Main results, Theorems 1 and 2, establish the formal equivalence between valued good and bad choices and the corresponding valued dominant and absorbent kernels.

Despite this equivalence and contrary to the crisp case, valued kernels, through their element-wise qualification, deliver a complement information to the subset-wise qualification of valued choices. An issue which we are still investigating and that we like to reserve for a further publication.

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