A LOCAL-GLOBAL PRINCIPLE FOR POLYQUADRATIC TWISTS OF ABELIAN SURFACES

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ABSTRACT. We say that two abelian varieties A and A' defined over a field F are polyquadratic twists if they are isogenous over a Galois extension of F whose Galois group has exponent dividing 2. Let A and A' be abelian varieties defined over a number field K of dimension $g \geqslant 1$. In this article we prove that, if $g \leqslant 2$, then A and A' are polyquadratic twists if and only if for almost all primes $\mathfrak{p}$ of K their reductions modulo $\mathfrak{p}$ are polyquadratic twists. We exhibit a counterexample to this local-global principle for $g = 3$. This work builds on a geometric analogue by Khare and Larsen, and on a similar criterion for quadratic twists established by Fité, relying itself on the works by Rajan and Ramakrishnan.

1. INTRODUCTION

Let K denote a number field and let A and A' be abelian varieties defined over K of dimension $g \geqslant 1$. Call Σ_K the set of nonzero prime ideals of the ring of integers $\mathcal{O}_K$ of K. Only finitely many primes in Σ_K are of bad reduction for A and A'. In this article, by "almost all $\mathfrak{p} \in \Sigma_K$" we will mean all primes of Σ_K outside a zero density subset containing the finite set of primes of bad reduction.

Faltings isogeny theorem [Fal83] asserts that A and A' are isogenous if and only if their reductions $A_\mathfrak{p}$ and $A'_\mathfrak{p}$ modulo $\mathfrak{p}$ are isogenous for almost all $\mathfrak{p} \in \Sigma_K$. We say that two abelian varieties are *geometrically isogenous* if their base changes to an algebraic closure of their field of definition are isogenous. Building on Faltings isogeny theorem and a result by Pink [Pin98], Khare and Larsen [KL20, Thm. 1] have shown that A and A' are geometrically isogenous if and only if their reductions $A_\mathfrak{p}$ and $A'_\mathfrak{p}$ modulo $\mathfrak{p}$ are geometrically isogenous for almost all $\mathfrak{p} \in \Sigma_K$ (in fact, they show that it suffices to require this property for a set of primes of sufficiently large density).

In the context of the result of Khare and Larsen, Faltings isogeny theorem asserts that the property of A and A' admitting an isogeny defined over K can be read purely from local data. There is a variety of results in the literature studying whether certain global properties of isogenies can be inferred from local information. For example, when $g = 1$ and m is a positive integer, the articles [Sut12], [Ann14], [Vog20] study the problem of characterizing the existence of a degree m isogeny of A in terms of the existence of degree m isogenies for $A_\mathfrak{p}$ for almost all $\mathfrak{p} \in \Sigma_K$. The analogous problem when $g = 2$ has been recently explored by [Ban21] and [LV22]. Rather than on the degree of the isogeny, in the present article we focus on its field of definition.

We say that A and A' are *quadratic twists* if A' is isogenous to A_χ, where A_χ denotes the twist of A by a quadratic character χ of G_K, regarded as an element in $H^1(G_K, \{\pm 1\}) \subseteq H^1(G_K, \mathrm{Aut}(A_{\overline{\mathbb{Q}}}))$. Notice that the condition of being quadratic twists is finer than that of being isogenous over a quadratic extension. We say that A and A' are *locally quadratic twists* if $A_\mathfrak{p}$ and $A'_\mathfrak{p}$ are quadratic twists for almost all $\mathfrak{p} \in \Sigma_K$. In the spirit of the previously mentioned results, the main theorem of [Fit21] shows that, for $g \leqslant 3$, A and A' are quadratic twists if and only if they are locally quadratic twists. Moreover, a counterexample to this local-global principle in dimension $g = 4$ is presented.

In this article we are concerned with a notion that unwinds that of being quadratic twists. We say that a Galois field extension is polyquadratic if its Galois group has exponent dividing 2. Note that a polyquadratic extension of a finite field is either trivial or quadratic. We say that two abelian varieties are *polyquadratic twists* if their base changes to a polyquadratic extension are isogenous. We say that A and A' are *locally polyquadratic twists* if $A_\mathfrak{p}$ and $A'_\mathfrak{p}$ are polyquadratic twists for almost all $\mathfrak{p} \in \Sigma_K$.

It is easy to see (cf. Remark 16) that if A and A' are polyquadratic twists, then A and A' are locally polyquadratic twists. The goal of this article is to study the converse of this implication. For $g = 1$, this converse follows easily from results of Rajan [Raj98, Thm. B] and Ramakrishnan [Ram00, Thm. 2]. The main result of this article is the following.

Theorem 1. *Suppose that A and A' are abelian surfaces defined over a number field K. Then they are polyquadratic twists if and only if they are locally polyquadratic twists.*

To complement the above theorem, in §5 we exhibit two abelian threefolds that are locally polyquadratic twists but are not polyquadratic twists. In §2, we translate our problem into a problem about ℓ-adic Galois representations and reinterpret the notions defined in this introduction in group theoretic terms. We also recall the main results of [KL20] and [Fit21], and we prove some technical results that will be used in later sections. In §3 we carry out the proof of Theorem 1 in the most complicated case, that is, when A is either geometrically isogenous to the square of an elliptic curve or has geometric quaternionic multiplication. A crucial input in this case is the tensor decomposition of the Tate module $T_\ell(A)$ provided by [FG22]. This remarkably allows us to translate our problem concerning ℓ-adic representations of degree 4 into a problem concerning Artin representations of degree 2 (see Theorem 12 for a solution of the latter problem). The proof of the remaining cases of Theorem 1 takes place in §4, where we benefit from the classification of Sato-Tate groups of abelian surfaces of [FKRS12]. We remark however that our proof is independent of the Sato-Tate conjecture.

Notation and terminology. All algebraic extensions of K are contained in some fixed algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, and we denote by G_K the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$. For a field extension L/K, we write A_L to denote the base change of A from K to L. Given a representation ϱ, we call $\varrho^\vee$ its contragredient representation, $\mathrm{ad}(\varrho) \simeq \varrho \otimes \varrho^\vee$ its adjoint representation, and $\mathrm{ad}^0(\varrho)$ the subrepresentation of $\mathrm{ad}(\varrho)$ on the trace 0 subspace. If ϱ is a representation of G_K with coefficients in a field E and L/K an algebraic extension, we denote by $\varrho|_L$ the restriction of ϱ to G_L; we say that ϱ is absolutely irreducible if $\varrho \otimes \overline{E}$ is irreducible; and we say that ϱ is strongly absolutely irreducible if $\varrho|_L$ is absolutely irreducible for every finite extension L/K.

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2. Preliminaries

2.1. **Group theoretic descriptions.** Let E be a topological field, and fix an algebraic closure $\overline{E}$. Let G be a compact topological group, and let $\varrho, \varrho' : G \to \mathrm{GL}_r(E)$ be semisimple continuous group representations, where r is a positive integer.

Definition 2. We call ϱ and ϱ' *locally quadratic twists* if for every $s \in G$ there exists $\epsilon_s \in \{\pm 1\}$ such that

$$\det(1 - \varrho(s)T) = \det(1 - \epsilon_s \varrho'(s)T)\,.$$

We call them *locally polyquadratic twists* if for every $s \in G$ we have

$$\det(1 - \varrho(s^2)T) = \det(1 - \varrho'(s^2)T)\,.$$

Definition 3. We call ϱ and ϱ' *quadratic twists* if $\varrho' \simeq \chi \otimes \varrho$ holds for some quadratic character χ of G. We call them *polyquadratic twists* if

$$\varrho \simeq \bigoplus_{i=1}^{t} \varrho_i \qquad \text{and} \qquad \varrho' \simeq \bigoplus_{i=1}^{t} \varrho'_i \tag{1}$$

holds for representations $\varrho_i, \varrho'_i : G \to \mathrm{GL}_{n_i}(E)$ such that $\varrho'_i \simeq \chi_i \otimes \varrho_i$, where n_i is a positive integer and χ_i is a quadratic character of G.

Remark 4. *Suppose that $r = 2$, and let ϱ and ϱ' be polyquadratic twists. Then either ϱ and ϱ' are quadratic twists or there exist characters χ_1, χ_2 of G and quadratic characters φ_1, φ_2 of G such that $\varrho \simeq \chi_1 \oplus \chi_2$ and $\varrho' \simeq \varphi_1\chi_1 \oplus \varphi_2\chi_2$.*

We have the following characterization of polyquadratic twists.

Proposition 5. *The representations ϱ, ϱ' are polyquadratic twists if and only if $\varrho|_H \simeq \varrho'|_H$ holds for some normal subgroup $H \subseteq G$ such that G/H is a finite abelian group of exponent dividing 2.*

Proof. Supposing that ϱ, ϱ' are polyquadratic twists, we may take for H the intersection of the kernels of $\chi_1, \dots, \chi_r$. For the converse implication, let θ be an irreducible constituent of ϱ. By the argument in the proof of [Fit12, Thm. 3.1], the representation θ is an irreducible constituent of $\mathrm{Hom}_H(\theta, \varrho') \otimes \varrho'$, where $\mathrm{Hom}_H(\theta, \varrho')$ denotes the space of H-equivariant homomorphisms from θ to ϱ'. Since $\mathrm{Hom}_H(\theta, \varrho')$, as a representation of G/H, is a sum of quadratic characters of G/H, there exists an irreducible constituent θ' of ϱ' and a quadratic character χ of G/H such that $\theta' \simeq \chi \otimes \theta$. We can apply the same argument to the complement of θ in ϱ and the complement of θ' in ϱ', and the proposition follows by iterating the above step. □

Remark 6. *By the above proposition, in Definition 3 we may replace E by $\overline{E}$.*

Remark 7. *If ϱ and ϱ' are quadratic (respectively, polyquadratic) twists, then clearly they are locally quadratic (respectively, polyquadratic) twists.*

Remark 8. *Let ϱ_i and ϱ_i', for $i \in \{1,2,3\}$, be semisimple representations of G satisfying*

$$\varrho_1 \simeq \varrho_2 \oplus \varrho_3\,, \qquad \varrho_1' \simeq \varrho_2' \oplus \varrho_3'\,.$$

Suppose that ϱ_i and ϱ_i' are polyquadratic twists (resp. locally polyquadratic twists) for all $i \in I$, where I is a subset of $\{1,2,3\}$ with $|I| = 2$. It then follows immediately from the definitions that ϱ_i and ϱ_i' are polyquadratic twists (resp. locally polyquadratic twists) for all i. The previous property is not true in general if we replace "polyquadratic" by "quadratic".

Theorem 9 (Ramakrishnan)**.** *Suppose that $r = 2$. If ϱ and ϱ' are locally quadratic twists, then they are quadratic twists.*

Proof. The hypothesis is equivalent to having $\wedge^2\varrho \simeq \wedge^2\varrho'$ and $\varrho \otimes \varrho \simeq \varrho' \otimes \varrho'$. Since $\varrho^\vee \otimes \wedge^2\varrho \simeq \varrho$ (and the same holds for ϱ') we obtain $\mathrm{ad}^0(\varrho) \simeq \mathrm{ad}^0(\varrho')$. By [Ram00, Thm. B] (the result is stated for ℓ-adic representations, but the proof is valid in our context) there exists a character χ of G such that $\varrho' \simeq \chi \otimes \varrho$. Morever, χ must be quadratic because $\wedge^2\varrho \simeq \wedge^2\varrho'$. $\square$

Remark 10. *The above result also holds for r odd by* [Fit21, Cor. 4.3].

Example 11 (Chidambaram)**.** For $r = 4$ and G with [GAP] identifier $\langle 12, 1\rangle$, there are locally quadratic twists that are not quadratic twists. Indeed, let θ denote the only rational 2-dimensional irreducible representation of G up to isomorphism, let ε be the rational nontrivial character of G, and let χ denote any character of G of order 4. It is a straighforward computation to verify that $\mathbf{1} \oplus \varepsilon \oplus \chi \otimes \theta$ and $\chi \oplus \varepsilon\chi \oplus \theta$ are locally quadratic twists, but are not quadratic twists.

Theorem 12. *Suppose that $r = 2$. If ϱ and ϱ' are locally polyquadratic twists, then they are polyquadratic twists.*

Proof. By Remark 6 we may assume that E is algebraically closed. For every $s \in G$, let $\alpha_s, \beta_s \in E$ be such that

$$\det(1 - \varrho(s)T) = (1 - \alpha_s T)(1 - \beta_s T)\,.$$

By assumption, there exist $\psi_s, \varphi_s \in \{\pm 1\}$ such that

$$\det(1 - \varrho'(s)T) = (1 - \psi_s\alpha_s T)(1 - \varphi_s\beta_s T)\,.$$

The map $\varepsilon : G \to \{\pm 1\}$ mapping $s \in G$ to

$$\varepsilon(s) := \frac{\psi_s}{\varphi_s} = \frac{\det(\varrho')}{\det(\varrho)}(s) \tag{2}$$

is a quadratic character.

Suppose that ε is trivial. Then $\mathrm{ad}^0(\varrho) \simeq \mathrm{ad}^0(\varrho')$, and by [Ram00, Thm. B] there exists a character χ of G such that $\varrho' \simeq \chi \otimes \varrho$. Since $\det(\varrho') = \chi^2 \cdot \det(\varrho)$, we find that χ is quadratic.

Now suppose that ε is nontrivial. We claim that both ϱ and ϱ' are reducible. Thus there exist characters $\chi_1, \chi_2, \chi'_1, \chi'_2 : G_K \to E^\times$ such that $\varrho \simeq \chi_1 \oplus \chi_2$ and $\varrho' \simeq \chi'_1 \oplus \chi'_2$. By comparing traces, one immediately verifies that

$$\frac{\chi_1}{\chi_2} \oplus \frac{\chi_2}{\chi_1} \simeq \frac{\varepsilon\chi'_1}{\chi'_2} \oplus \frac{\varepsilon\chi'_2}{\chi'_1} \,.$$

Up to renaming χ'_1, χ'_2, we may assume $\chi_1/\chi_2 = \varepsilon\chi'_1/\chi'_2$. Also, by (2) we have that $\chi_1\chi_2 = \varepsilon\chi'_1\chi'_2$. Multiplying the last two equations we get $(\chi_1)^2 = \varepsilon^2(\chi'_1)^2 = (\chi'_1)^2$, which implies that χ_1 and χ'_1 differ by a quadratic character. The same holds for χ_2 and χ'_2. Hence ϱ and ϱ' are polyquadratic twists.

We prove the claim for ϱ' (the proof for ϱ being the same) by showing that the multiplicity of the trivial representation $\mathbf{1}$ in $\mathrm{ad}^0(\varrho')$ is positive. Denoting by μ the (normalized) Haar measure of G, this multiplicity is

$$\begin{aligned} \langle \mathbf{1}, \mathrm{ad}^0(\varrho')\rangle &= \int_{s\in G}\left(1+\varepsilon(s)\frac{\alpha_s}{\beta_s}+\varepsilon(s)\frac{\beta_s}{\alpha_s}\right)\mu \\ &= 1+\int_{s\in G}\varepsilon(s)\left(1+\frac{\alpha_s}{\beta_s}+\frac{\beta_s}{\alpha_s}\right)\mu = 1+\langle \varepsilon, \mathrm{ad}^0(\varrho)\rangle \geqslant 1\,, \end{aligned}$$

where we have used that, by the orthogonality of characters, $\int_{s\in G}\varepsilon(s)\mu = 0$ if ε is nontrivial. □

Example 13. For $r = 3$ there are locally polyquadratic twists that are not polyquadratic twists. A straightforward computation shows that such an example is given by any pair of nonisomorphic faithful irreducible degree 3 representations ϱ and ϱ' of the group G with [GAP] identifier $\langle 48, 3\rangle$. This also yields counterexamples for any $r \geqslant 4$: if θ is any representation of degree $r-3$, consider $\varrho \oplus \theta$ and $\varrho' \oplus \theta$.

We conclude this section by giving purely group theoretic characterizations of locally polyquadratic twists and locally quadratic twists when $r = 4$.

Lemma 14. *The representations ϱ, ϱ' are locally polyquadratic twists if and only if*

$$\mathrm{Sym}^2\,\varrho - \wedge^2\varrho \simeq \mathrm{Sym}^2\,\varrho' - \wedge^2\varrho' \tag{3}$$

as virtual representations.

If $r = 4$, they are locally quadratic twists if and only if

$$\mathrm{Sym}^2\,\varrho \simeq \mathrm{Sym}^2\,\varrho' \qquad \textit{and} \qquad \wedge^2\,\varrho \simeq \wedge^2\varrho'\,.$$

Proof. For the first assertion, it suffices to verify that both sides of (3) have the same virtual character. Indeed, if for $s \in G$ we denote by α_i the eigenvalues of $\varrho(s)$, then we have

$$\mathrm{Tr}\,\mathrm{Sym}^2\,\varrho(s) - \mathrm{Tr}\wedge^2\varrho(s) = \sum_i \alpha_i^2 = \mathrm{Tr}\,\mathrm{Sym}^2\,\varrho'(s) - \mathrm{Tr}\wedge^2\varrho'(s)\,.$$

Now consider the second assertion. As $r = 4$, we know that ϱ and ϱ' are locally quadratic twists if and only if we have

$$\varrho\otimes\varrho \simeq \varrho'\otimes\varrho', \quad \wedge^2\varrho \simeq \wedge^2\varrho', \quad \varrho\otimes\wedge^3\varrho \simeq \varrho'\otimes\wedge^3\varrho', \quad \det(\varrho)\simeq\det(\varrho')\,. \tag{4}$$

Then it suffices to show that the third and fourth of the above isomorphisms follow from the first two. Taking determinants, the first (respectively, second) isomorphism implies $\det(\varrho)^8 \simeq \det(\varrho')^8$ (respectively, $\det(\varrho)^3 \simeq \det(\varrho')^3$) and we deduce $\det(\varrho) \simeq \det(\varrho')$. As ϱ and ϱ' are locally quadratic twists, we know $\varrho \otimes \varrho^\vee \simeq \varrho' \otimes \varrho'^\vee$. Considering that $\wedge^3 \varrho \simeq \det(\varrho) \otimes \varrho^\vee$ (and similarly for ϱ'), we obtain $\varrho \otimes \wedge^3 \varrho \simeq \varrho' \otimes \wedge^3 \varrho'$. $\square$

2.2. **Polyquadratic twists of abelian varieties.** In this section, we consider ℓ-adic representations of abelian varieties. For some prime ℓ, let

$$\varrho_{A,\ell} : G_K \to \mathrm{Aut}(V_\ell(A))$$

denote the ℓ-adic representation attached to A, where $V_\ell(A) = T_\ell(A) \otimes \mathbb{Q}$ and $T_\ell(A)$ is the ℓ-adic Tate module of A (use the analogous notation for A'). For $\mathfrak{p} \in \Sigma_K$, let $\mathrm{Fr}_\mathfrak{p}$ denote an arithmetic Frobenius at $\mathfrak{p}$. From Weil, we know that, for every $\mathfrak{p} \nmid \ell$ of good reduction for A, the polynomial $\det(1 - \varrho_{A,\ell}(\mathrm{Fr}_\mathfrak{p})T)$ is well defined and does not depend on the choice of ℓ.

Proposition 15. *The abelian varieties A and A' are polyquadratic twists if and only if $\varrho_{A,\ell}$ and $\varrho_{A',\ell}$ are polyquadratic twists for some ℓ (equivalently, for all ℓ). Moreover, they are locally polyquadratic twists if and only if $\varrho_{A,\ell}$ and $\varrho_{A',\ell}$ are locally polyquadratic twists for some ℓ (equivalently, for all ℓ).*

Proof. The first assertion is an immediate consequence of Faltings isogeny theorem [Fal83] and Proposition 5, so consider the second assertion. Since A and A' being locally polyquadratic twists does not depend on ℓ, it suffices to prove the statement for some fixed prime ℓ. By the Chebotarev density theorem it suffices to show the following: if $\mathfrak{p} \in \Sigma_K$ is a prime of good reduction for A and A' with $\mathfrak{p} \nmid \ell$, then $A_\mathfrak{p}$ and $A'_\mathfrak{p}$ are polyquadratic twists if and only

$$\det(1 - \varrho_{A,\ell}(\mathrm{Fr}_\mathfrak{p}^2)T) = \det(1 - \varrho_{A',\ell}(\mathrm{Fr}_\mathfrak{p}^2)T) \tag{5}$$

holds. And this is clear because, by [Tat66, Thm. 1], (5) amounts to saying that the base changes of $A_\mathfrak{p}$ and $A'_\mathfrak{p}$ to the quadratic extension of the residue field $K(\mathfrak{p})$ of K at $\mathfrak{p}$ are isogenous. $\square$

By the above proposition and Remark 7, we see that if A and A' are polyquadratic twists, then they are locally polyquadratic twists. This may be seen more directly by means of the following remark.

Remark 16. *Suppose that A and A' are polyquadratic twists. Then A and A' are locally polyquadratic twists at almost all primes of Σ_K. Indeed, choose a polyquadratic extension L/K and an isogeny $f : A'_L \to A_L$. There exists a finite set $S \subseteq \Sigma_K$ such that for every $\mathfrak{p} \in \Sigma_K \setminus S$ the abelian varieties A and A' have good reduction at $\mathfrak{p}$ and, if $\mathfrak{P} \in \Sigma_L$ lies over $\mathfrak{p}$, then there is an isogeny $f_\mathfrak{P} : A'_{L,\mathfrak{P}} \to A_{L,\mathfrak{P}}$ making the diagram*

$$\begin{array}{ccc} A_L & \xrightarrow{\;f\;} & A'_L \\ {\scriptstyle \mathrm{red}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{red}} \\ A_{L,\mathfrak{P}} & \xrightarrow[f_\mathfrak{P}]{} & A'_{L,\mathfrak{P}} \end{array}$$

commutative.

Remark 17. *Let ϱ and ϱ' be as in §2. The argument of the proof of Proposition 5 shows that there exists a normal subgroup $H \subseteq G$ of index 2 such that $\varrho|_H \simeq \varrho'|_H$ if and only if ϱ and ϱ' admit decompositions as in* (1) *such that each of the characters χ_i is either the trivial or the nontrivial character of G/H. Hence, if A and A' are abelian varieties that become isogenous over a quadratic extension L/K, then each of the characters χ_i relating their ℓ-adic representations is either the trivial or the nontrivial character of* $\mathrm{Gal}(L/K)$. *According to our definition, A and A' are called quadratic twists precisely when all of the χ_i can be taken to be equal.*

In §4, we will make use of the following lemma to prove Theorem 1.

Lemma 18. *Let A and A' be abelian surfaces and let $\mathfrak{p}$ be a prime of good reduction for A and A'. If $A_\mathfrak{p}$ and $A'_\mathfrak{p}$ are polyquadratic twists and for some prime ℓ the traces of $\varrho_{A,\ell}(\mathrm{Fr}_\mathfrak{p})$ and $\varrho_{A',\ell}(\mathrm{Fr}_\mathfrak{p})$ are both zero, then $A_\mathfrak{p}$ and $A'_\mathfrak{p}$ are quadratic twists.*

Proof. Let $\alpha, \overline{\alpha}, \beta, \overline{\beta}$ be the eigenvalues of $\varrho_{A,\ell}(\mathrm{Fr}_\mathfrak{p})$. Then there exist $\epsilon, \gamma \in \{\pm 1\}$ such that $\epsilon\alpha, \epsilon\overline{\alpha}, \gamma\beta, \gamma\overline{\beta}$ are the eigenvalues of $\varrho_{A',\ell}(\mathrm{Fr}_\mathfrak{p})$. The vanishing of the traces yields the equation

$$\begin{pmatrix} 1 & 1 \\ \epsilon & \gamma \end{pmatrix} \begin{pmatrix} \alpha + \overline{\alpha} \\ \beta + \overline{\beta} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

This implies that $\epsilon = \gamma$ or $\alpha + \overline{\alpha} = \beta + \overline{\beta} = 0$. Either of the possibilities shows that $A_\mathfrak{p}$ and $A'_\mathfrak{p}$ are quadratic twists. ☐

We conclude this section by recalling a result that will be used multiple times in the proof of Theorem 1. In §2.1 we observed that Theorem 9 does not remain true in general for representations of degree $r \geqslant 4$. However, the following result asserts that Theorem 9 remains true in degrees 4 and 6 if one restricts to ℓ-adic representations attached to abelian surfaces or threefolds.

Theorem 19 ([Fit21])**.** *Suppose that A and A' have dimension at most* 3. *If $\varrho_{A,\ell}$ and $\varrho_{A',\ell}$ are locally quadratic twists, then they are quadratic twists.*

2.3. **A theorem by Khare and Larsen.**

We denote by $\overline{A}_\mathfrak{p}$ the base change of $A_\mathfrak{p}$ to the algebraic closure of $K(\mathfrak{p})$, and similarly define $\overline{A}'_\mathfrak{p}$. The following result of Khare and Larsen [KL20] will play an important role in our investigation.

Theorem 20 (Khare–Larsen)**.** *The abelian varieties $A_{\overline{\mathbb{Q}}}$ and $A'_{\overline{\mathbb{Q}}}$ are isogenous if and only if for almost all primes $\mathfrak{p} \in \Sigma_K$ the abelian varieties $\overline{A}_\mathfrak{p}$ and $\overline{A}'_\mathfrak{p}$ are isogenous.*

Proof. For the "only if" implication we may reason as in Remark 16. Now suppose that for almost all $\mathfrak{p}$ the abelian varieties $\overline{A}_\mathfrak{p}$ and $\overline{A}'_\mathfrak{p}$ are isogenous. We invoke [KL20, Thm. 1], which asserts:

$$\text{If } \mathrm{Hom}(\overline{A}_\mathfrak{p}, \overline{A}'_\mathfrak{p}) \neq 0 \text{ for almost all } \mathfrak{p} \in \Sigma_K, \text{ then } \mathrm{Hom}(A_{\overline{\mathbb{Q}}}, A'_{\overline{\mathbb{Q}}}) \neq 0 \,. \tag{6}$$

The theorem follows from (6) by induction on the number n of simple isogeny factors of $A_{\overline{\mathbb{Q}}}$. Note that the assumption implies, in particular, that A and A' have the same dimension. Hence if $n=1$, the existence of a nontrivial homomorphism from $A_{\overline{\mathbb{Q}}}$ to $A'_{\overline{\mathbb{Q}}}$ is equivalent to $A_{\overline{\mathbb{Q}}}$ and $A'_{\overline{\mathbb{Q}}}$ being isogenous. For general n, (6) implies that $A_{\overline{\mathbb{Q}}}$ and $A'_{\overline{\mathbb{Q}}}$ have a common simple isogeny factor. Let B and B' be the complements of this simple isogeny factor in $A_{\overline{\mathbb{Q}}}$ and $A'_{\overline{\mathbb{Q}}}$, respectively. Then, for almost all $\mathfrak{p}$ in Σ_K, the reductions of B and B' modulo $\mathfrak{p}$ are isogenous. Since the number of simple isogeny factors in B is $n-1$, by induction we find that B and B' are isogenous. Hence so are $A_{\overline{\mathbb{Q}}}$ and $A'_{\overline{\mathbb{Q}}}$. □

3. Squares of elliptic curves and quaternionic multiplication

This section is devoted to proving Theorem 1 in the following particular case:

> Suppose that the abelian surfaces A and A' are locally polyquadratic twists, and that $A_{\overline{\mathbb{Q}}}$ is either isogenous to the square of an elliptic curve or has quaternionic multiplication. Then A and A' are polyquadratic twists.

By Theorem 20, the assumptions imply that there exists a finite Galois extension L/K such that either:

- There is an elliptic curve E defined over L such that $A_L \sim E^2$ and $A'_L \sim E^2$; or
- $\mathrm{End}(A_L)\otimes\mathbb{Q}\simeq \mathrm{End}(A'_L)\otimes\mathbb{Q}$ is a quaternion algebra.

We divide the first case into three subcases, according to whether E has CM or not, and in the former case according to whether the imaginary quadratic field M by which E has CM is contained in K or not. Notice that by Proposition 15 it suffices to show that for some ℓ the representations $\varrho_{A,\ell}$ and $\varrho_{A',\ell}$ are polyquadratic twists. Alternatively, by Theorem 19, it suffices to show that $A_{\mathfrak{p}}$ and $A'_{\mathfrak{p}}$ are quadratic twists for almost all primes $\mathfrak{p}$.

3.1. **Non CM or quaternionic multiplication.** By [FG22, Thm. 1.1] (see also [Fit21, Thm. 4.5 (i)] for a restatement of this result in our situation) and after enlarging L/K if necessary, there exist a number field F, Artin representations $\theta,\theta' : \mathrm{Gal}(L/K)\to \mathrm{GL}_2(F)$, a prime ℓ totally split in F, and strongly absolutely irreducible F-rational ℓ-adic representations ϱ,ϱ' of G_K of degree 2 such that

$$\varrho_{A,\ell}\simeq \theta\otimes\varrho\,,\qquad \varrho_{A',\ell}\simeq\theta'\otimes\varrho'\,,\qquad \varrho|_L\simeq\varrho'|_L\,.$$

In particular, there exists a character χ of $\mathrm{Gal}(L/K)$ such that $\varrho'\simeq\chi\otimes\varrho$. Let $\alpha_{1,\mathfrak{p}},\alpha_{2,\mathfrak{p}}$ be the eigenvalues of $\varrho(\mathrm{Fr}_{\mathfrak{p}})$; $\beta_{1,\mathfrak{p}},\beta_{2,\mathfrak{p}}$ the eigenvalues of $\theta(\mathrm{Fr}_{\mathfrak{p}})$; and $\gamma_{1,\mathfrak{p}},\gamma_{2,\mathfrak{p}}$ the eigenvalues of $\chi\otimes\theta'(\mathrm{Fr}_{\mathfrak{p}})$. By [Fit21, Lem. 4.7], there exists a density 1 subset Σ of Σ_K of primes of good reduction for A and A' such that for every $\mathfrak{p}\in\Sigma$ the quotient $\alpha_{1,\mathfrak{p}}/\alpha_{2,\mathfrak{p}}$ is not a root of unity, and there exist $\epsilon_{ij,\mathfrak{p}}\in\{\pm1\}$ such that

$$\prod_{i,j=1}^{2}(1-\alpha_{i,\mathfrak{p}}\beta_{j,\mathfrak{p}}T)=\prod_{i,j=1}^{2}(1-\epsilon_{ij,\mathfrak{p}}\alpha_{i,\mathfrak{p}}\gamma_{j,\mathfrak{p}}T)\,.$$

Hence, for every $\mathfrak{p} \in \Sigma$ and for $i \in \{1,2\}$, we have

$$\prod_{j=1}^{2}(1-\beta_{j,\mathfrak{p}}T)=\prod_{j=1}^{2}(1-\epsilon_{ij,\mathfrak{p}}\gamma_{j,\mathfrak{p}}T)\,.$$

By the Chebotarev density theorem, this implies that θ and $\chi \otimes \theta'$ are locally polyquadratic twists and hence polyquadratic twists by Theorem 12. By Remark 4 and the fact that $\varrho_{A',\ell} \simeq \theta' \otimes \chi \otimes \varrho$ and $\varrho_{A,\ell} \simeq \theta \otimes \varrho$, we conclude that $\varrho_{A,\ell}$ and $\varrho_{A',\ell}$ are locally quadratic twists.

3.2. **CM over** K**.** By [FG22, Thm. 1.1] (see also [Fit21, Thm. 4.5 (ii)]), after enlarging L/K if necessary, there exist a Galois number field F containing M, Artin representations $\theta, \theta' : \mathrm{Gal}(L/K) \to \mathrm{GL}_2(F)$, a prime ℓ totally split in F, and continuous F-rational ℓ-adic characters ψ, ψ' of G_K such that

$$\varrho_{A,\ell} \simeq (\theta\otimes\psi)\oplus(\overline{\theta}\otimes\overline{\psi})\,, \quad \text{and} \quad \varrho_{A',\ell}\simeq(\theta'\otimes\psi')\oplus(\overline{\theta}'\otimes\overline{\psi}')\,. \tag{7}$$

Here $\overline{\theta}$ (resp. $\overline{\theta}'$, $\overline{\psi}$, $\overline{\psi}'$) stands for the "complex conjugate" of θ (resp. θ', ψ, ψ'). By this, more precisely we mean the following. Fix a complex comjugation $\bar{\cdot}$ in $\mathrm{Gal}(F/\mathbb{Q})$, that is, a lift of the nontrivial element of $\mathrm{Gal}(M/\mathbb{Q})$. That ψ is an F-rational ℓ-adic character implies that $\psi(\mathrm{Fr}_\mathfrak{p}) \in F$ for almost all $\mathfrak{p} \in \Sigma_K$. For any such $\mathfrak{p}$, the ℓ-adic character $\overline{\psi}$ satisfies $\overline{\psi}(\mathrm{Fr}_\mathfrak{p}) = \overline{\psi(\mathrm{Fr}_\mathfrak{p})}$. Similar descriptions hold for $\overline{\theta}'$, $\overline{\psi}$, and $\overline{\psi}'$.

After reordering ψ and $\overline{\psi}$ if necessary, we may assume that $\psi|_L \simeq \psi'|_L$. Hence there exists a character φ of $\mathrm{Gal}(L/K)$ such that $\psi' \simeq \varphi\psi$. Let $\beta_{1,\mathfrak{p}}, \beta_{2,\mathfrak{p}}$ be the eigenvalues of $\theta(\mathrm{Fr}_\mathfrak{p})$ and $\gamma_{1,\mathfrak{p}}, \gamma_{2,\mathfrak{p}}$ the eigenvalues of $\varphi \otimes \theta'(\mathrm{Fr}_\mathfrak{p})$. By [Fit21, Lem. 4.7], there exists a density 1 subset Σ of Σ_K of primes of good reduction for A and A' such that for every $\mathfrak{p} \in \Sigma$ the quotient $\psi(\mathrm{Fr}_\mathfrak{p})/\overline{\psi}(\mathrm{Fr}_\mathfrak{p})$ is not a root of unity, and there exist $\epsilon_{j,\mathfrak{p}}, \delta_{j,\mathfrak{p}} \in \{\pm 1\}$ such that

$$\prod_{j=1}^{2}(1-\psi(\mathrm{Fr}_\mathfrak{p})\beta_{j,\mathfrak{p}}T)(1-\overline{\psi}(\mathrm{Fr}_\mathfrak{p})\overline{\beta}_{j,\mathfrak{p}}T)=\prod_{j=1}^{2}(1-\epsilon_{j,\mathfrak{p}}\psi(\mathrm{Fr}_\mathfrak{p})\gamma_{j,\mathfrak{p}}T)(1-\delta_{j,\mathfrak{p}}\overline{\psi}(\mathrm{Fr}_\mathfrak{p})\overline{\gamma}_{j,\mathfrak{p}}T)\,.$$

Hence for every $\mathfrak{p} \in \Sigma$ we have

$$\prod_{j=1}^{2}(1-\beta_{j,\mathfrak{p}}T)=\prod_{j=1}^{2}(1-\epsilon_{j,\mathfrak{p}}\gamma_{j,\mathfrak{p}}T) \quad \text{and} \quad \prod_{j=1}^{2}(1-\overline{\beta}_{j,\mathfrak{p}}T)=\prod_{j=1}^{2}(1-\delta_{j,\mathfrak{p}}\overline{\gamma}_{j,\mathfrak{p}}T)\,.$$

By the Chebotarev density theorem, each of the above equations implies that θ and $\varphi \otimes \theta'$ are locally polyquadratic twists and we conclude as in the previous case.

3.3. **CM not over** K**.** Given a representation ϱ of G_L for a finite extension L/K, we write $\mathrm{Ind}_L^K(\varrho)$ to denote the induction of ϱ from G_L to G_K. In this section, to shorten the notation, we omit the subindex and superindex fields in $\mathrm{Ind}_{KM}^K(\varrho)$. The last statement of [Fit21, Thm. 4.5 (ii)] implies that there exist Artin representations θ, θ' of G_{KM}, an ℓ-adic character ψ of G_{KM}, and a character φ of $\mathrm{Gal}(L/KM)$ satisfying

$$\varrho_{A,\ell} \simeq \mathrm{Ind}(\theta\otimes\psi)\,, \qquad \varrho_{A',\ell}\simeq \mathrm{Ind}(\varphi\otimes\theta'\otimes\psi)\,.$$

Moreover, by the construction of θ and ψ (see [FG22, §2] and especifically [FG22, Lem. 2.10]), one has that if ξ is a constituent of θ, then

$$\mathrm{Ind}(\xi\otimes\psi)|_L \simeq \xi\otimes\psi\oplus\overline{\xi}\otimes\overline{\psi}\,. \tag{8}$$

By the previous case, one of the following holds:

i) There exists a quadratic character χ of $\mathrm{Gal}(L/KM)$ such that $\theta'\otimes\varphi\simeq\chi\otimes\theta$.
ii) There exist characters ξ_1,ξ_2 and quadratic characters χ_1,χ_2 of $\mathrm{Gal}(L/KM)$ such that

$$\theta\simeq\xi_1\oplus\xi_2\,,\qquad \theta'\otimes\varphi\simeq\chi_1\xi_1\oplus\chi_2\xi_2\,.$$

We first consider case ii). Then, we have

$$\varrho_{A,\ell}\simeq\mathrm{Ind}(\xi_1\psi)\oplus\mathrm{Ind}(\xi_2\psi)\,,\qquad \varrho_{A',\ell}\simeq\mathrm{Ind}(\chi_1\xi_1\psi)\oplus\mathrm{Ind}(\chi_2\xi_2\psi)\,.$$

Observe that $\mathrm{Ind}(\xi_i\psi)$ and $\mathrm{Ind}(\chi_i\xi_i\psi)$ are locally quadratic twists for $i=1,2$ (this can be seen by comparing traces on an element $s\in G_K$: if $s\in G_L$, then use (8), and if $s\notin G_L$, then notice that both traces are 0). By Theorem 9, they are quadratic twists, which finishes the proof in this case.

In case i), we claim that A and A' are locally quadratic twists, which is enough for our purposes in virtue of Theorem 19. Let $\mathfrak{p}\in\Sigma_K$ be a prime of good reduction for A and A' of absolute residue degree 1. If $\mathfrak{p}$ is split in KM, it follows from (7) that the reductions $A_\mathfrak{p}$ and $A'_\mathfrak{p}$ are quadratic twists. If $\mathfrak{p}$ is inert in KM, then the same conclusion is attained by using the lemma below.

Lemma 21. *Let A and A' be abelian surfaces $\overline{\mathbb{Q}}$-isogenous to the square of an elliptic curve with CM, say by an imaginary quadratic field M. Then, for every $\mathfrak{p}\in\Sigma_K$ of good reduction for A and A', inert in KM, and of absolute residue degree 1, the reductions $A_\mathfrak{p}$ and $A'_\mathfrak{p}$ are isogenous if and only if they are polyquadratic twists.*

Proof. Let $\mathfrak{p}\in\Sigma_K$ be as in the statement, and let p denote its absolute norm. Define the set of polynomials

$$S:=\{(1-T^2)^2\,,1-T^2+T^4\,,1+T^4\,,1+T^2+T^4\,,(1+T^2)^2\}\,.$$

The fact that such a $\mathfrak{p}$ is of supersingular reduction for A, together with the Weil bounds, implies that $L_\mathfrak{p}(A,p^{-1/2}T)\in S$. We similarly have $L_\mathfrak{p}(A',p^{-1/2}T)\in S$. Consider the set

$$R:=\{(1-T)^4\,,(1-T+T^2)^2\,,(1-T^2)^2\,,(1+T+T^2)^2\,,(1+T)^4\}\,,$$

and observe that the map $\Phi:S\to R$ defined by $P(T)\mapsto\mathrm{Res}_Z(P(Z),Z^2-T)$ is a bijection. Notice that $A_\mathfrak{p}$ and $A'_\mathfrak{p}$ are polyquadratic twists if and only if the polynomials $L_\mathfrak{p}(A,p^{-1/2}T)$ and $L_\mathfrak{p}(A',p^{-1/2}T)$ have the same image under Φ. Since Φ is a bijection, this is equivalent to their equality, and we conclude by [Tat66, Thm. 1]. $\square$

4. Proof of Theorem 1

In this section we complete the proof of Theorem 1.

Notation. For an abelian surface A defined over K, denote by $\mathrm{End}^0(A)$ the endomorphism algebra $\mathrm{End}(A)\otimes\mathbb{Q}$, and by $\mathcal{X}(A)$ the *absolute type* of A, that is, the $\mathbb{R}$-algebra $\mathrm{End}(A_{\overline{\mathbb{Q}}})\otimes\mathbb{R}$.

We borrow from [FKRS12, §4.1] the labels $\mathbf{A}, \dots, \mathbf{F}$ for the different possibilities for $\mathcal{X}(A)$. These are in bijection with the possibilities for the connected component of the identity of the *Sato–Tate group* of A, denoted $\mathrm{ST}(A)$. This is a closed real Lie subgroup of $\mathrm{USp}(4)$, only defined up to conjugacy. It captures important arithmetic information of A and it is conjectured to predict the limiting distribution of the Frobenius elements attached to A. See [BK15, §2] or [FKRS12, §2] for its definition in our context; see [Ser12, Chap. 8] for a conditional definition in a more general context. We will use the notation settled in [FKRS12, §3] for Sato–Tate groups of abelian surfaces.

It follows from Theorem 19 (as explained in [Fit21, Rem. 2.10]) that two abelian surfaces which are locally quadratic twists share the same Sato–Tate group. There are however examples of abelian surfaces that are locally polyquadratic twists with distinct Sato–Tate groups. Indeed, let E be an elliptic curve defined over K without CM and χ a nontrivial quadratic character of G_K. Let E_χ denote the twist of E by χ. Then $E \times E$ has Sato–Tate group E_1, while $E \times E_\chi$ has Sato–Tate group $J(E_1)$. Despite the existence of these examples, the Sato-Tate group is preserved under locally polyquadratic twist in most of the cases. This is essentially the first step in the proof of Theorem 1 in the cases considered in this section.

From now on, let A and A' be abelian surfaces that are locally quadratic twists. By Theorem 20, A and A' are geometrically isogenous, and hence they have the same absolute type.

We will repeatedly use the following well known lemma, whose proof may be found split between the proofs of [Fit21, Cor. 2.5] and [Fit21, Cor. 2.7].

Lemma 22. *Let ϱ and ϱ' be strongly absolutely irreducible representations of G_K. If there exists a finite extension L/K such that $\varrho|_L \simeq \varrho'|_L$, then there exists a finite order character χ of G_K such that $\varrho' \simeq \chi \otimes \varrho$. In particular, if B and B' are abelian varieties defined over K such that $B_{\overline{\mathbb{Q}}}$ and $B'_{\overline{\mathbb{Q}}}$ are isogenous and such that $\mathcal{X}(B) \simeq \mathcal{X}(B') \simeq \mathbb{R}$, then B and B' are quadratic twists.*

4.1. **Absolute types E, F, A, C.** If A has absolute type **E** or **F**, namely, $\mathcal{X}(A) \simeq \mathrm{M}_2(\mathbb{R})$ or $\mathrm{M}_2(\mathbb{C})$, then Theorem 1 was proven in §3. If A has absolute type **A**, equivalently, $\mathcal{X}(A) \simeq \mathbb{R}$, then A and A' are quadratic twists by Lemma 22. Now suppose that A has absolute type **C**. By Lemma [Fit21, Lem. 4.13], A is isogenous to the product of an elliptic curve E_1 without CM and an elliptic curve E_2 with CM. We similarly define E_1' and E_2' for A'. By Theorem 20, E_1 and E_1' (respectively, E_2 and E_2') are geometrically isogenous, and hence, by Lemma 22, E_1 and E_1' are quadratic twists. Then Remark 8 implies that E_2 and E_2' are locally quadratic twists, and hence quadratic twists by Theorem 9. Thus A and A' are polyquadratic twists.

4.2. **Absolute type D.** Suppose that A has absolute type **D**, namely, $\mathcal{X}(A) \simeq \mathbb{C} \times \mathbb{C}$. Then precisely one of the following two conditions hold:

D.1) A is isogenous to the product of two nonisogenous elliptic curves E_1 and E_2 with CM;
D.2) $A_{\overline{\mathbb{Q}}}$ has CM by a quartic CM field.

Suppose that A falls in case D.1. By [Fit21, Lem. 4.13] and Theorem 6, there exist elliptic curves E_1' and E_2' such that A' is isogenous to $E_1' \times E_2'$. After possibly reordering E_1 and E_2, and using [Fit21, Lem. 4.14], we may assume that E_1 and E_1' are locally quadratic twists.

Hence they are quadratic twists by Theorem 9. As the same holds for E_2 and E_2', we conclude that A and A' are polyquadratic twists.

Now suppose that A falls in case D.2. Denote the CM field by M and its reflex field by M^*. As explained in [Fit21, §5.1.1] (see the references therein), precisely one of the following three cases occurs:

i) $M^* \subseteq K$ and $\mathrm{End}^0(A) \simeq M$;
ii) KM^*/K is quadratic and $\mathrm{End}^0(A)$ is a real quadratic field;
iii) KM^*/K is cyclic of degree 4 and $\mathrm{End}^0(A) \simeq \mathbb{Q}$.

Moreover, which case occurs depends on whether A has Sato–Tate group F, F_{ab} or F_{ac}, respectively. Notice that, as it is clear from the case distinction, A and A' fall in the same case.

Suppose that A and A' fall in case i). Let ℓ be a prime inert in M, let λ be the prime of M lying over ℓ, and let M_λ denote the completion of M at λ. Then $V_\ell(A)$ is an M_λ-module of dimension 1, and hence strongly absolutely irreducible. Therefore there exists a character φ of G_K such that $\varrho_{A',\ell} \simeq \varphi \otimes \varrho_{A,\ell}$. Since in this case the Sato–Tate group is F, we can apply [Fit21, Prop. 2.11], which implies that A and A' are quadratic twists.

Now suppose that A and A' fall in case ii) or iii). We choose a prime ℓ totally split in M, so that the four distinct embeddings $\lambda_i\colon M \hookrightarrow \overline{\mathbb{Q}}_\ell$, for $i = 1, \dots, 4$, take values in $\mathbb{Q}_\ell$. Define

$$V_{\lambda_i}(A) := V_\ell(A) \otimes_{M\otimes\mathbb{Q}_\ell,\lambda_i} \mathbb{Q}_\ell\,,$$

where $\mathbb{Q}_\ell$ is being regarded as an $M \otimes \mathbb{Q}_\ell$-module via λ_i. It is a 1-dimensional vector space over $\mathbb{Q}_\ell$. We equip it with an action of G_K by letting this group act naturally on $V_\ell(A)$ and trivially on $\mathbb{Q}_\ell$.

Let $n = [\mathrm{End}^0(A) : \mathbb{Q}]$. By [Fit21, (5.2)], we have an isomorphism

$$V_\ell(A) \simeq \bigoplus_{i=1}^{n} \mathrm{Ind}_{KM^*}^{K}(V_{\lambda_i})\,, \tag{9}$$

where we assume that $\lambda_1, \dots, \lambda_4$ have been ordered so that the restrictions of $\lambda_1, \dots, \lambda_n$ to $\mathrm{End}^0(A) \subseteq M$ are all distinct.

Let $\mathfrak{p} \in \Sigma_K$ be a prime of good reduction for A and A'. If $\mathfrak{p}$ is totally split in KM^*, then $A_\mathfrak{p}$ and $A'_\mathfrak{p}$ are quadratic twists by case i). If $\mathfrak{p}$ is not totally split in KM^*, then (9) implies that $\mathrm{Tr}(\varrho_{A,\ell}(\mathrm{Fr}_\mathfrak{p})) = \mathrm{Tr}(\varrho_{A',\ell}(\mathrm{Fr}_\mathfrak{p})) = 0$. Hence we can apply Lemma 18 which implies that $A_\mathfrak{p}$ and $A'_\mathfrak{p}$ are quadratic twists. It follows that $\varrho_{A,\ell}$ and $\varrho_{A',\ell}$ are locally quadratic twists, and hence they are quadratic twists by Theorem 19.

4.3. **Absolute type B.** Suppose that A has absolute type **B**, which means that $\mathcal{X}(A) \simeq \mathbb{R} \times \mathbb{R}$. Then, as explained in [Fit21, §5.1.2], there exists a prime ℓ so that precisely one of the following two conditions holds:

B.1) $\varrho_{A,\ell} \simeq \varrho_1 \oplus \varrho_2$, where $\varrho_1, \varrho_2 : G_K \to \mathrm{GL}_2(\mathbb{Q}_\ell)$ are strongly absolutely irreducible ℓ-adic representations that do not become isomorphic after restriction to G_L for any finite extension L/K;

B.2) $\varrho_{A,\ell} \simeq \mathrm{Ind}_K^L(\varrho)$, where L/K is quadratic and $\varrho : G_L \to \mathrm{GL}_2(\mathbb{Q}_\ell)$ is a strongly absolutely irreducible ℓ-adic representation.

Whether B.1 or B.2 holds depends on whether $\mathrm{ST}(A)$ is isomorphic to $\mathrm{SU}(2) \times \mathrm{SU}(2)$ or $N(\mathrm{SU}(2) \times \mathrm{SU}(2))$.

Lemma 23. *If A falls in case B.1 (respectively, B.2), then so does A'.*

Proof. Let ϱ be the tautological representation of $\mathrm{ST}(A)$, and similarly define ϱ' for $\mathrm{ST}(A')$. Let θ denote $\mathrm{Sym}^2 \varrho - \wedge^2 \varrho$ and θ_ℓ denote $\mathrm{Sym}^2 \varrho_{A,\ell} - \wedge^2 \varrho_{A,\ell}$. Define similarly θ' and θ'_ℓ. The same argument used in the proof of [Saw16, Thm. 1] shows that the virtual multiplicity of the trivial representation in θ coincides with the virtual multiplicity of the ℓ-adic cyclotomic character χ_ℓ in θ_ℓ. By Lemma 14, we have

$$\langle \mathbf{1}, \theta \rangle = \langle \chi_\ell, \theta_\ell \rangle = \langle \chi_\ell, \theta'_\ell \rangle = \langle \mathbf{1}, \theta' \rangle \,.$$

Suppose that the statement of the lemma were false. Without loss of generality we may assume that A falls in case B.1 and A' in case B.2. However, [FKRS12, Table 8] yields

$$\langle \mathbf{1}, \theta \rangle = \langle \mathbf{1}, \varrho \otimes \varrho \rangle - 2\langle \mathbf{1}, \wedge^2 \varrho \rangle = -2 \,, \quad \langle \mathbf{1}, \theta' \rangle = \langle \mathbf{1}, \varrho' \otimes \varrho' \rangle - 2\langle \mathbf{1}, \wedge^2 \varrho' \rangle = -1 \,,$$

which gives a contradiction. □

Lemma 24. *If A falls in case B.1, then $\wedge^2 \varrho_{A,\ell}$ has no 1-dimensional constituent of the form $\psi\chi_\ell$, where ψ is a nontrivial finite order character and χ_ℓ is the ℓ-adic cyclotomic character.*

Proof. Write $\varrho_{A,\ell}$ as $\varrho_1 \oplus \varrho_2$, where ϱ_1, ϱ_2 are as in B.1. By [Rib76, Thm. 4.3.1], we have $\det(\varrho_i) = \chi_\ell$ and hence

$$\wedge^2 \varrho_{A,\ell} \simeq \chi_\ell \oplus (\varrho_1 \otimes \varrho_2) \oplus \chi_\ell \,.$$

Suppose that $\varrho_1 \otimes \varrho_2$ has a 1-dimensional constituent of the form $\psi\chi_\ell$ and let L/K be the field extension cut out by ψ. Then the subspace affording ψ is contained in

$$(\varrho_1 \otimes \varrho_2 \otimes \chi_\ell^{-1})^{G_L} \simeq (\varrho_1^\vee \otimes \varrho_2)^{G_L} \simeq \mathrm{Hom}_{G_L}(\varrho_1, \varrho_2) \,.$$

This contradicts that ϱ_1 and ϱ_2 are strongly absolutely irreducible and $\varrho_1|_L \not\simeq \varrho_2|_L$. □

Suppose that A and A' fall in case B.1. Write accordingly $\varrho_{A,\ell}$ as $\varrho_1 \oplus \varrho_2$ and $\varrho_{A',\ell}$ as $\varrho'_1 \oplus \varrho'_2$. After possibly reordering ϱ_1 and ϱ_2, Lemma 22 shows that there exist finite order characters χ_1 and χ_2 of G_K such that $\varrho'_1 \simeq \chi_1 \otimes \varrho_1$ and $\varrho'_2 \simeq \chi_2 \otimes \varrho_2$. Since

$$\wedge^2 \varrho_{A',\ell} \simeq \chi_1^2\chi_\ell \oplus \chi_1\chi_2(\varrho_1 \otimes \varrho_2) \oplus \chi_2^2\chi_\ell \,,$$

Lemma 24 implies that the characters χ_1 and χ_2 must be quadratic.

Now suppose that A and A' fall in case B.2. Then we write $\varrho_{A,\ell} \simeq \mathrm{Ind}_K^L(\varrho)$ and $\varrho_{A',\ell} \simeq \mathrm{Ind}_K^{L'}(\varrho')$, where L/K and L'/K are quadratic extensions, and ϱ and ϱ' are strongly absolutely irreducible representations. Observe that A_L falls in case B.1. Then Lemma 23 implies that A'_L also falls in case B.1. This can only occur if $L = L'$.

Let τ denote an element of G_K projecting onto the nontrivial element of $\mathrm{Gal}(L/K)$. Let ϱ^τ denote the representation of G_L defined by $\varrho^\tau(s) = \varrho(\tau s \tau)$ for every $s \in G_L$. Then we have

$$\varrho_{A,\ell}|_L \simeq \varrho \oplus \varrho^\tau \,, \qquad \varrho_{A',\ell}|_L \simeq \varrho' \oplus \varrho'^\tau \,. \tag{10}$$

After possibly reordering ϱ and ϱ^τ, by the B.1. case of the proof, there exist quadratic characters χ and ψ of G_L such that

$$\varrho' \simeq \chi \otimes \varrho\,, \qquad \varrho'^\tau \simeq \psi \otimes \varrho^\tau\,. \tag{11}$$

We claim that the characters χ and ψ coincide. Assuming the claim, the proof of Theorem 1 takes the following steps. Let $\mathfrak{p} \in \Sigma_K$ be a prime of good reduction for A and A'. If $\mathfrak{p}$ is split in L/K, the isomorphisms (10) and (11) show that $A_\mathfrak{p}$ and $A'_\mathfrak{p}$ are quadratic twists. The same holds, by Lemma 18, if $\mathfrak{p}$ is inert in L/K. Then by Theorem 19, A and A' are quadratic twists.

We now turn to prove the claim. There are two cases to distinguish: either A is an abelian surface such that $\mathrm{End}^0(A_L)$ is isomorphic to a real quadratic field F or A_L is isogenous to the product $E \times E^\tau$, where E is an elliptic curve defined over L without complex multiplication, and E^τ is the Galois conjugate of E by τ.

In the first case, ℓ is split in F. Let $\bar{\cdot}$ denote the nontrivial element of $\mathrm{Gal}(F/\mathbb{Q})$. After possibly renaming the primes λ and $\overline{\lambda}$ of F lying over ℓ, we have that ϱ (resp. ϱ^τ) is the F-rational representation afforded by

$$V_\lambda(A) := V_\ell(A) \otimes_{F\otimes\mathbb{Q}_\ell,\lambda} \mathbb{Q}_\ell \qquad \Big(\text{resp. } V_{\overline{\lambda}}(A) := V_\ell(A) \otimes_{F\otimes\mathbb{Q}_\ell,\overline{\lambda}} \mathbb{Q}_\ell\Big)\,.$$

From [FG22, Lem. 2.10], we deduce that $\varrho^\tau \simeq \overline{\varrho}$, where $\overline{\varrho}$ is the F-rational ℓ-adic representation characterized by the property that, for almost all $\mathfrak{p} \in \Sigma_L$, the traces of $\overline{\varrho}(\mathrm{Fr}_\mathfrak{p})$ and $\varrho(\mathrm{Fr}_\mathfrak{p})$ are Galois conjugates in F. Since χ is quadratic and takes rational values, using the first isomorphism of (11), we find

$$\varrho'^\tau \simeq \overline{\varrho}' \simeq \chi \otimes \overline{\varrho} \simeq \chi \otimes \varrho^\tau\,,$$

which together with the second isomorphism of (11) concludes the proof of the claim in the first case.

In the second case, by Theorem 20, there is an elliptic curve E' defined over L such that A'_L is isogenous to $E' \times E'^\tau$, and we may take

$$\varrho = \varrho_{E,\ell}\,, \quad \varrho^\tau = \varrho_{E^\tau,\ell}\,, \quad \varrho' = \varrho_{E',\ell}\,, \quad \varrho'^\tau = \varrho_{E'^\tau,\ell}\,.$$

Then, the first isomorphism of (11) expresses the fact that E' is the quadratic twist of E by χ. Choosing Weierstrass equations $y^2 = x^3 + ax + b$ and $dy^2 = x^3 + ax + b$, with $a, b, d \in \mathcal{O}_L$, for E and E' respectively, we see that for almost all $\mathfrak{p} \in \Sigma_L$ whether $\chi(\mathrm{Fr}_\mathfrak{p})$ is 1 or -1 depends on whether d is a square or not in $L(\mathfrak{p})^\times$. The same reasonning using E^τ and E'^τ instead, shows that for almost all $\mathfrak{p} \in \Sigma_L$ whether $\psi(\mathrm{Fr}_\mathfrak{p})$ is 1 or -1 depends on whether $\tau(d)$ is a square or not in $L(\mathfrak{p})^\times$. Since d is a square in $L(\mathfrak{p})^\times$ if and only if so is $\tau(d)$ in $L(\tau(\mathfrak{p}))^\times \simeq L(\mathfrak{p})^\times$, we deduce that χ and ψ coincide, as desired.

5. A COUNTEREXAMPLE IN DIMENSION 3

Let ϱ and ϱ' be as in Example 13, namely, a pair of nonisomorphic faithful irreducible degree 3 representations of the group G with [GAP] identifier $\langle 48, 3\rangle$. They are realizable over the quadratic ring $\mathbb{Z}[i]$. Let F be the number field $\mathbb{Q}[T]/(f)$, where

$$f(T) = T^{12} + 2T^{10} - 82T^8 + 50T^6 + 595T^4 + 500T^2 + 25\,.$$

Let $\tilde F$ be the Galois closure of F. According to [LMFDB], the [GAP] identifier of $\mathrm{Gal}(\tilde F/\mathbb{Q})$ is $\langle 48, 3\rangle$. A straightforward computation shows that $\tilde F$ does not contain the quadratic field $K = \mathbb{Q}(i)$. Let L denote the compositum of $\tilde F$ and K, and choose an isomorphism between G and $\mathrm{Gal}(L/K)$. Use this isomorphism to regard ϱ and ϱ' as Artin representations of the group $\mathrm{Gal}(L/K)$.

Let E be an elliptic curve with CM by $\mathbb{Z}[i]$ defined over K, and let A and A' be the abelian varieties $\varrho \otimes_{\mathbb{Z}[i]} E$ and $\varrho' \otimes_{\mathbb{Z}[i]} E$ defined over K as described in [Mil72, §2]. By [MRS07, Thm. 2.2 (iii)], we have that there exists a K-rational ℓ-adic character ψ of G_K such that

$$\varrho_{A,\ell} \simeq (\varrho \otimes \psi) \oplus (\overline{\varrho} \otimes \overline{\psi})\,, \quad \text{and} \quad \varrho_{A',\ell} \simeq (\varrho' \otimes \psi) \oplus (\overline{\varrho}' \otimes \overline{\psi})\,. \tag{12}$$

Observe that this is compatible with (7): by our choices of A and A', the Artin representations θ and θ' in that formula can be taken as ϱ and ϱ'. From (12), we see that the fact that ϱ and ϱ' are locally polyquadratic twists implies that so are A and A'.

We claim however that the threefolds A and A' are not polyquadratic twists. If the contrary were true, then as in §3.2 the existence of a density 1 subset Σ of Σ_K of primes of good reduction for A and A' such that for every prime $\mathfrak{p} \in \Sigma$ the quotient $\psi(\mathrm{Fr}_\mathfrak{p})/\overline{\psi}(\mathrm{Fr}_\mathfrak{p})$ is not a root of unity, would imply, via the Chebotarev density theorem, that $\varrho \otimes \psi$ and $\varrho' \otimes \psi$ are polyquadratic twists. This contradicts the fact that ϱ and ϱ' are not polyquadratic twists.

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