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LIMIT THEOREMS WITH MALLIAVIN CALCULUS AND STEIN'S METHOD

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Organisation of the manuscript

This PhD dissertation is organized in the following way.

In Chapter 1, we give an overview of the main topics we will adress. Sections 1.1 and 1.2 consist in a detailed presentation of the fractional Brownian motion and Hermite processes. These processes are the main focus of the two articles [14] and [15].

In Section 1.3, we gather some relevant facts about Stein's method, Malliavin calculus, and the way to combine them to build the so-called Malliavin-Stein's approach. This recent and flourishing topic provides us a with a set of tools which is used in the three research articles that constitutes this thesis, by allowing us to prove normal and non-normal convergences in different settings. In Section 1.4 we give a summary of the three articles on which constitute this dissertation is based.

Finally, Chapters 2, 3 and 4 will consist in a copy of the three articles aformentioned.

Chapter 1

Introduction

1.1 Fractional Brownian motion

1.1.1 Historical background

In 1951, Hurst released his paper [24] describing the fluctuations of the levels of the Nile river. The observations did not appear to verify the independence assumption; instead, they turned out to be *positively* correlated, with the variance of their (renormalized) partial sum behaving like $n^{0.72}$ (with n the sample size). This phenomenon was surprising at the time, because scientists were more used to the "$\sqrt{n}$-type behaviour" observed for the variance of the sum in case of independent summands. More details on this can be found in [56] and [59]. A few years later, Mandelbrot had the idea to utilize a then relatively overlooked object introduced by Kolmogorov [27] to model such phenomenon. Together with Van Ness, they popularized the name "fractional Brownian motion" in a seminal paper [34].

Since its introduction in [34], the popularity and range of applications of the fractional Brownian motion has literaly exploded. What makes this mathematical object beautiful is that it can model numerous phenomena with very different behaviours, simply by fitting its self-similarity exponent H (called "Hurst exponent"). For instance, it can be used to describe highly irregular, negatively correlated observations, such as the time series of the log-volatility of some financial assets, see [16], as well as phenomena exhibiting long range dependence, such as the example of the Nile river's fluctuations described above. The fractional Brownian motion also plays a pivotal role in the *rough path theory* introduced by Lyons in [31], which studies differential equations perturbed by irregular noises.

In the present dissertation, we exclusively work with the case $H \geq \frac{1}{2}$ (which encompass both the "standard Brownian case" ($H = \frac{1}{2}$) and the "regular case" ($H > \frac{1}{2}$)), which is already a very rich object from which a whole range of remarkable mathematical phenomena can be derived.

1.1.2 Definition

We start with the following proposition.

Proposition 1.1.1. *Let $H \in (0, \infty)$. The function F defined on $\mathbb{R}_+ \times \mathbb{R}_+$ by: $\forall s, t \in \mathbb{R}_+$,*

$$F(s,t) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right) \tag{1.1}$$

is positive semidefinite if and only if $H \leq 1$.

For a proof of this fact, the reader is refered to e.g [36], Proposition 1.6. The fractional Brownian can be defined in the following way.

Definition 1.1.2. Let $H \in (0,1]$. We call fractional Brownian of index H any centered Gaussian process $B = (B_t)_{t \geq 0}$ such that

- $B_0 = 0$;
- B has almost surely continuous sample paths;
- $\forall t, s \in \mathbb{R}_+,\ \mathrm{Cov}(B_t, B_s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$

H is called the *Hurst index* of B.

Remark 1.1.3. When $H = 1$, we have $(B_t)_{t\geq 0} \overset{law}{=} (tG)_{t \geq 0}$ where $G \sim \mathcal{N}(0,1)$. Since this case is too simple to be interesting, we will exclude it from now on.

Remark 1.1.4. When $H = \frac{1}{2}$, we can notice that

$$\forall t, s > 0,\ \mathrm{Cov}(B_t, B_s) = \frac{1}{2}\left(t + s - |t-s|\right) = \min(t,s).$$

Then, it turns out that the standard Brownian motion is the fractional Brownian motion of index $\frac{1}{2}$.

1.1.3 Some basic properties

In the following statement, we gather some elementary properties of the fractional Brownian motion which are useful throughout this dissertation and may help the reader to get a better grasp of this object.

Proposition 1.1.5. *Let B be a fractional Brownian motion with Hurst index $H \in (0,1)$.*

(a) B is H-self similar, i.e. $\forall c > 0$, $(B_{ct})_{t\geq 0} \overset{law}{=} (c^H B_t)_{t\geq 0}$.

(b) B has stationary increments, i.e., $\forall h \geq 0$,

$$(B_{t+h} - B_h)_{h\geq 0} \overset{law}{=} (B_t)_{t\geq 0}.$$

(c) If $H > \frac{1}{2}$, disjoint increments are positively correlated, i.e.

$$\mathbb{E}[(B_{t_1} - B_{s_1})(B_{t_2} - B_{s_2})] \geq 0$$

for all $0 \leq s_1 \leq t_1 \leq s_2 \leq t_2$. If $H < \frac{1}{2}$, disjoint increments are negatively correlated, i.e.

$$\mathbb{E}[(B_{t_1} - B_{s_1})(B_{t_2} - B_{s_2}] \leq 0$$

for all $0 \leq s_1 \leq t_1 \leq s_2 \leq t_2$. If $H = \frac{1}{2}$, disjoint increments are independent.

(d) Let $(r(k))_{k\in\mathbb{N}}$ be the sequence defined as $r(k) = \mathbb{E}[(B_{k+1} - B_k)B_1]$.

- *(long memory) if $H > \frac{1}{2}$,*

$$\sum_{k\in\mathbb{N}} |r(k)| = \infty;$$

- *(short memory) if $H < \frac{1}{2}$,*

$$\sum_{k\in\mathbb{N}} |r(k)| < \infty.$$

(e) Increments of B have finite moments, which are controlled in the following way: $\forall s, t \geq 0$, $\forall p > 0$,

$$\mathbb{E}[|B_t - B_s|^p] \leq K|t-s|^{pH},$$

with $K = \mathbb{E}[|B_1|^p] = 2^p \frac{\Gamma(\frac{p+1}{2})}{\sqrt{\pi}}$.

(f) *For any $0 \leq \alpha < H$, B admits an α-Hölder continuous version on each compact interval $[0, T]$.*

Proof. Points (a)-(c) can be proved through easy computations, using the covariance function (1.1). The proof of (d) follows from the fact that $r(k) \sim_{k\to\infty} H(2H-1)k^{2H-2}$. Finally, $\forall s, t \geq 0$,

$$B_t - B_s \overset{law}{=} |t-s|^H G$$

with $G \sim \mathcal{N}(0,1)$. From this, we have that $\mathbb{E}[|B_t - B_s|^p] = \mathbb{E}[|G|^p]|t - s|^{pH}$ and $\mathbb{E}[|G|^p] = 2^p \frac{\Gamma(\frac{p+1}{2})}{\sqrt{\pi}}$, by [63]. This proves (e). Then, (f) follows immediately from the Kolmogorov-Censov criterion. □

Another important property of the fractional Brownian motion is that it can be represented as a *Volterra process*, i.e a sequence of Wiener integrals with respect to a standard Brownian motion.

Proposition 1.1.6. *For $H \in (0,1)$, we have that:*

$$(B_t)_{t\geq 0} \overset{law}{=} \left(\int_0^t K_H(t,s) dW_s \right)_{t \geq 0}, \tag{1.2}$$

where W is a standard Brownian motion and

$$\begin{aligned} K_H(t,s) &= \frac{c(H)}{\Gamma\left(H - \frac{1}{2}\right)} s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{1}{2}} (u-s)^{H-\frac{3}{2}} du \quad \left(\textit{if } H > \frac{1}{2} \right), \\ K_H(t,s) &= \frac{c(H)}{\Gamma\left(H - \frac{1}{2}\right)} \left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} \\ &\quad - \frac{c(H)}{\Gamma\left(H - \frac{1}{2}\right)} s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du \quad \left(\textit{if } H < \frac{1}{2} \right), \end{aligned}$$

with

$$c(H) = \left(\frac{2H\Gamma\left(H + \frac{1}{2}\right)\Gamma(\frac{3}{2} - H)}{\Gamma(2-2H)} \right)^{\frac{1}{2}}.$$

A proof of this fact can be found in either [11, Corollary 3.1] or [45, Proposition 5.1.3]. Another useful representation due to Mandelbrot and Van Ness involves a two-sided Brownian motion (see [34]); we will discuss it later in Section 1.2.

Except when $H = \frac{1}{2}$, B does not verify the properties required to be an integrand in an Itô-type integral, because it is not a semimartingale, as explained in e.g [50, Section 2]. To build an integral with respect to the fractional Brownian motion, in the present thesis we will use either the divergence operator of Malliavin calculus or the Young integral (when $H > \frac{1}{2}$).

1.1.4 Isonormal Gaussian process and Malliavin derivative

Malliavin calculus is a powerful set of tools first introduced by Paul Malliavin in [33] in order to give a probabilistic proof of Hormander's theorem for parabolic SDEs. It can be thought of as an infinite dimensional integro-differential calculus operating on Gaussian fields (although it has also been extensively studied in the setting of Poisson processes, see e.g [28]). In the three following sections, we give a brief introduction to this topic and some relevant applications in the fractional Brownian motion setting. For more details, the reader is refered to the seminal monography [45] or to [41], Chapter 1-2 for a more compact presentation.

The framework for (Gaussian) Malliavin calculus is a general object called *isonormal Gaussian field.* For simplicity, we restrict the presentation to the case of $\mathbb{R}$-valued fields (one can also consider $\mathbb{R}^d$-valued fields). Fix an Hilbert space $\mathcal{H}$ endowed with a scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 1.1.7. An isonormal Gaussian field over $\mathcal{H}$ is a centered Gaussian family $X = \{X(h),\ h \in \mathcal{H}\}$ defined on Ω such that

$$\forall f, g \in \mathcal{H},\ \mathrm{Cov}(X(f), X(g)) = \langle f, g \rangle_{\mathcal{H}}.$$

Let $T > 0$. If $\mathcal{H} = L^2([0,T])$ is endowed with its usual scalar product, and if X is the associated isonormal Gaussian process, then the (standard) Brownian motion B over $[0,T]$ can be embedded in X:

$$(B_t)_{t \in [0,T]} \overset{law}{=} \{X(\mathbb{I}_{[0,t]}), t \in [0,T]\}. \tag{1.3}$$

Moreover, (1.3) provides a convenient representation of the Wiener integral with respect to B.

If $f \in L^2([0,T])$, then the Wiener integral of f with respect to B has the same law as $X(f)$. Fortunately, such a representation also exists for the fractional Brownian motion of any index $H \in (0,1)$, but the procedure to obtain it is not trivial, see Section 1.1.6.

With these definitions in mind, we are now ready to define the main operators of Malliavin calculus.

Definition 1.1.8. Let X be an isonormal Gaussian process over $\mathcal{H}$. We define $\mathcal{S}$ as the class of functions $f \in \cup_{m\in\mathbb{N}}\mathcal{C}^\infty(\mathbb{R}^m, \mathbb{R})$ such that f and its derivatives have at most polynomial growth.

Definition 1.1.9 (Malliavin derivative)**.** Let X be an isonormal Gaussian process, $f \in \mathcal{S}$ and $F = f(X(h_1), \dots, X(h_m))$ with $h_1, \dots, h_m \in \mathcal{H}$. Let $p \in \mathbb{N}^*$. Then, the p-th Malliavin derivative of F with respect to X is the element of $L^2(\Omega, \mathcal{H}^{\odot p})$ defined by

$$D^pF = \sum_{i_1,\dots,i_p=1}^{m} \frac{\partial^p f}{\partial x_{i_1} \dots \partial x_{i_p}}(X(h_1), \dots, X(h_p))h_{i_1} \otimes \dots \otimes h_{i_p}.$$

(Here, $\mathcal{H}^{\odot p}$ denotes the subset of the tensor product space $\mathcal{H}^{\otimes p}$ formed of the elements which are symmetric, see Definition 1.1.14).

The Malliavin derivative can then be extended to the whole space $\mathcal{H}$ in the following way.

Proposition 1.1.10. *Let the notations of Definition 1.1.9 prevail. For any $p \in \mathbb{N}^*$, the operator D^p defined above is closable with respect to the norms*

$$\|F\|_{p,q} = \left(\mathbb{E}[|F|^q] + \sum_{k=1}^{p} \mathbb{E}[\|D^kF\|^q_{\mathcal{H}^{\otimes k}}]\right)^{\frac{1}{q}}$$

for any $q \geq 1$. The closure of $\mathcal{S}$ with respect to $\|\cdot\|_{p,q}$ is denoted $\mathbb{D}^{p,q}$.

Hence, the Malliavin derivative can be viewed as an infinite dimensional derivative operator.

1.1.5 The Skorokhod integral

Let us now explain how to build an integral with respect to an isonormal Gaussian process.

Let X be an isonormal Gaussian process, let $p \in \mathbb{N}^*$ and let $u \in L^2(\Omega, \mathcal{H}^{\otimes p})$. If there is a constant $K > 0$ such that $\mathbb{E}[\langle D^pF, u\rangle_{\mathcal{H}^{\otimes p}}] \leq K\sqrt{\mathbb{E}[F^2]}$ for all cylindrical functional $F \in \mathcal{S}$, it means that the linear operator $F \to \mathbb{E}[\langle D^pF, u\rangle_{\mathcal{H}^{\otimes p}}]$ is continuous, and then, by Riesz Theorem, there is a unique element $\delta^p(u) \in L^2(\Omega)$ such that $\mathbb{E}[\langle D^pF, u\rangle_{\mathcal{H}^{\otimes p}}] = \mathbb{E}[F\delta^p(u)]$. We then say that u belongs to the domain of δ^p.

Definition 1.1.11 (Skorokhod integral)**.** The operator δ^p is called the p-th multiple Skorokhod integral with respect to X. By construction, it is the adjoint of the p-th Malliavin derivative D^p. Its domain (consisting of the elements described above) is denoted by $Dom\delta^p$.

Let us gather some important properties of Skorokhod integrals.

Proposition 1.1.12. *(a) If $p < q$, then $Dom\delta^q \subset Dom\delta^p$.*

(b) If $u, v \in Dom\delta^1$, then

$$\mathbb{E}[\delta^1(u)\delta^1(v)] = \mathbb{E}[\langle u, v\rangle_{\mathcal{H}}] + \mathbb{E}[\langle D.u.., D..v.\rangle_{\mathcal{H}\otimes\mathcal{H}}]. \tag{1.4}$$

(c) If $u \in Dom\delta^1$, then

$$\mathbb{E}[(\delta^1(u))^2] \leq M\left(\mathbb{E}[\|u\|^2_{\mathcal{H}}] + \mathbb{E}[\|Du\|^2_{\mathcal{H}\otimes\mathcal{H}}]\right). \tag{1.5}$$

Point (c) is a particular case of the more general *Meyer's inequalities* and will be used in Chapter 2, as well as Point (b).

Notice that if B is a standard Brownian motion (that is, if $\mathcal{H} = L^2([0,T])$) and if u, v are two processes in $\mathbb{D}^{1,2}$ that are progressively measurable with respect to B, then $D_s u_t = 0$ if $s > t$. From (1.4), we deduce that

$$\mathbb{E}[\delta^1(u)\delta^1(v)] = \int_0^T \mathbb{E}[u_s v_s]ds,$$

and we recover the isometry property of the Itô integral. Actually, the Skorokhod integral coincides with the Itô integral when $H = \frac{1}{2}$ and the integrand is progressively measurable.

Proposition 1.1.13. *Let B be a standard Brownian motion and u be a square integrable, progressively measurable process over $[0,T]$. Then, $u \in Dom\delta^1$. Moreover, $\delta^1(u)$ coincides with the Itô integral of u with respect to B over $[0,T]$:*

$$\delta(u) = \int_0^T u_x dB_x, \tag{1.6}$$

and $\forall s, t \in [0,T]$,

$$\delta(u\mathbb{I}_{[s,t]}) = \int_s^t u_x dB_x. \tag{1.7}$$

We end this section with a discussion about multiple Wiener-Itô integrals.

Definition 1.1.14 (Symmetric elements)**.** Let $e = (e_i)_{i\in\mathbb{N}}$ be an orthonormal basis of the Hilbert space $\mathcal{H}$. Let $p \in \mathbb{N}^*$. Every element $h \in \mathcal{H}^{\otimes p}$ can be written as

$$h = \sum_{i_1,\ldots,i_p\in\mathbb{N}} a_{i_1,\ldots,i_p} e_1 \otimes \ldots \otimes e_p,$$

with $a_{i_1,\dots i_p} \in \mathbb{R}$.
The symmetrization of h is the element:

$$\tilde{h} := \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_q} \sum_{i_1,\dots,i_p \in \mathbb{N}} a_{i_1,\dots,i_p} e_{\sigma(1)} \otimes \dots \otimes e_{\sigma(p)}.$$

The space $\mathcal{H}^{\odot p}$ is the space of elements $h \in \mathcal{H}^{\otimes p}$ such that $h = \tilde{h}$.

Definition 1.1.15. Let $p \in \mathbb{N}^*$ and consider an element $f_p \in \mathcal{H}^{\odot p}$. We have that $f_p \in Dom\delta^p$. The multiple Wiener-Itô integral of f_p with respect to X is the Skorokhod interal $\delta^p(f_p)$. In this case, we will use the more classical notation $I_p(f_p) := \delta^p(f_p)$.

Remark 1.1.16. When $p = 1$, we have $I_1(f_1) = X(f_1)$, so the Wiener-Itô integrals of order 1 are Gaussian.

Remark 1.1.17. When $\mathcal{H} = L^2([0,T])$ (so that X generates a Brownian motion B), multiple Wiener-Itô integrals coincide with iterated Itô integrals. Indeed, if $f_p \in \mathcal{H}^{\odot p}$, we have:

$$I_p(f_p) = p! \int_0^T \int_0^{t_1} \dots \int_0^{t_n} f_p(t_1, \dots, t_n) dB_{t_n} \dots dB_{t_1}$$

(see [41], Exercice 2.7.6 for a proof of this fact).

Multiple Wiener integrals are fundamental because they form a "basis" of $L^2(\Omega, \mathfrak{F})$ (where $\mathfrak{F}$ is the σ-algebra generated by X) as shown by the following proposition.

Theorem 1.1.18 (Chaotic decomposition)**.** *Let $F \in L^2(\Omega, \mathfrak{F})$. Then, there is a unique sequence of elements $f_p \in \mathcal{H}^{\odot p}$ such that*

$$F = \mathbb{E}[F] + \sum_{p=1}^{\infty} I_p(f_p),$$

where the previous series converges in $L^2(\Omega)$.

Multiple Wiener-Itô integrals possess rich properties that we enumerate below.

Proposition 1.1.19. *(a) (Isometry) For all integers $k, l \geq 1$, all $f \in \mathcal{H}^{\odot k}$ and all $g \in \mathcal{H}^{\odot l}$,*

$$\mathbb{E}[\delta^k(f)\delta^l(g)] = k! \langle f, g \rangle_{\mathcal{H}^{\otimes k}} \mathbb{I}_{\{k=l\}}.$$

(b) *(Hypercontractivity) For all $r \geq 2$ and all integer $k \geq 1$, we have that for all $f \in \mathcal{H}^{\odot k}$,*

$$\mathbb{E}\left[|\delta^k(f)|^r\right] \leq (r-1)^{\frac{rk}{2}} \mathbb{E}[|\delta^k(f)|^2]^{\frac{r}{2}}. \tag{1.8}$$

(c) *(Malliavin derivative) If $u_s = \delta^k(f(.,s))$ with $f \in \mathcal{H}^{\otimes(k+1)}$ symmetric in the k first variables, then $u \in \mathbb{D}^{1,2}(\mathcal{H})$, with*

$$D_s u_t = k\delta^{k-1}(f(.,t,s)).$$

(d) *(Product formula) Fix $f \in \mathcal{H}^{\odot k}$ and $g \in \mathcal{H}^{\odot l}$ and, as usual, let $\otimes_r$ (resp. $\widetilde{\otimes}_r$) denote the contraction operator (resp. the symmetrization of the contraction operator) of order r, see [41, Appendix B] for a precise definition. Then,*

$$\delta^k(f)\delta^l(g) = \sum_{r=0}^{k\wedge l} r! \binom{k}{r}\binom{l}{r} I_{k+l-2r}(f\widetilde{\otimes}_r g).$$

We will provide further properties of multiple Wiener-Itô integrals in the forthcoming Section 1.2.2. Before, we end this section dedicated to the fractional Brownian motion with a description of the space $\mathcal{H}$ associated to it, followed by a short discussion on the Young integral.

1.1.6 The spaces $\mathcal{H}$ and $|\mathcal{H}|$ in the case of the fractional Brownian motion.

Let B be a fractional Brownian motion with index $H \in (0,1)\backslash\{\frac{1}{2}\}$. Similarly to Brownian motion, the fractional Brownian motion can be represented by means of an isonormal Gaussian field. The most comprehensive survey on this subject is the paper [48], from which we extract the following relevant facts.

• When $H < \frac{1}{2}$, let us define

$$\mathcal{H} := \left\{ f : \exists \phi_f \in L^2([0,T]), \forall u \in [0,T], f(u) = u^{\frac{1}{2}-H}\left(I_{T-}^{\frac{1}{2}-H} s^{H-\frac{1}{2}}\phi_f(s)\right)(u) \right\} \tag{1.9}$$

endowed with the scalar product:

$$\langle f, g\rangle_{\mathcal{H}} := \frac{2\pi H(H-\frac{1}{2})}{\Gamma(2-2H)\sin(\pi(H-\frac{1}{2}))}\int_0^T \phi_f(s)\phi_g(s)ds.$$

Here, I^α_{T-} stands for the fractional integration operator, i.e.

$$I^\alpha_{T-}f(s) = \frac{1}{\Gamma(\alpha)}\int_0^T f(u)(u-s)^{\alpha-1}du.$$

Then, $(B_t)_{t\in[0,T]} \stackrel{law}{=} \{X(\mathbb{I}_{[0,t]}), t \in [0,T]\}$, with X an isonormal Gaussian process with respect to $\mathcal{H}$.

- When $H > \frac{1}{2}$, let us define first the scalar product

$$\langle f, g\rangle_{\mathcal{H}} := c_H \int_0^T \int_0^T f(u)g(s)|u-s|^{2H-2}duds$$

with $c_H = H(2H-1)$. Let $\|.\|_{\mathcal{H}}$ be the norm induced by $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and let $\mathcal{H}_1$ be the space of measurable functions f over $[0,T]$ such that $\|f\|_{\mathcal{H}} < \infty$. Finally, let $\mathcal{H}$ be the completion of $\mathcal{H}_1$ with respect to $\langle\cdot,\cdot\rangle_{\mathcal{H}}$, that is,

$$\mathcal{H} = \overline{\mathcal{H}_1}^{\langle\cdot,\cdot\rangle_{\mathcal{H}}}. \tag{1.10}$$

Then, if X is an isonormal Gaussian process with respect to $\mathcal{H}$, we have that $(B_t)_{t\in[0,T]} \stackrel{law}{=} \{X(\mathbb{I}_{[0,t]}), t \in [0,T]\}$.

In the case $H > \frac{1}{2}$, complications arise from the fact that $\mathcal{H}$ contains distributions (i.e, $\mathcal{H}_1$ is not complete). This fact was proven in [48]. In order to only work with functions, one often introduces the following subspace $|\mathcal{H}|$ of $\mathcal{H}_1$:

$$|\mathcal{H}| = \left\{ f,\ \|f\|^2_{|\mathcal{H}|} := H(2H-1)\int_{[0,T]^2} |f(u)||f(v)||u-v|^{2H-2}dudv < \infty \right\}.$$

It turns out that $(|\mathcal{H}|, \|\cdot\|_{|\mathcal{H}|})$ is a Banach space.

For the fractional Brownian motion B, the Malliavin derivative and Skorokhod integrals defined above are obtained by taking $\mathcal{H}$ as in (1.9) (when $H < \frac{1}{2}$) or as in (1.10) (when $H > \frac{1}{2}$). In Chapter 2, we often restrict the domain of these operators in order to always work with elements in $|\mathcal{H}|$.

This discussion provides the first type of integrals with respect to the fractional Brownian motion B that we shall use in this dissertation. The case $H < \frac{1}{2}$ will not be used further and was only given for informative purpose. Finally, for convenience, notice that we will sometimes use the notation

$$\int f(x_1,\ldots,x_q)dB_{x_1}\ldots dB_{x_q} := I_q(f)$$

in the sequel (with f any element in $\mathcal{H}^{\odot q}$).

1.1.7 The Young integral

We now turn to the second type of integral (beside the Skorokhod integral) commonly used in the fractional Brownian motion setting, and also throughout this dissertation, the *Young integral*. It was introduced first in [64]. Contrary to the Skorokhod integral, it is obtained through a deterministic procedure, and can be seen as a generalization of the usual Riemann integral. Its existence is in general not guaranteed, especially when $H \leq \frac{1}{2}$ (see Remark 1.1.25). A nice overview of this topic (as well as an introduction to the more advanced theory of rough paths) can be found in [2].

Definition 1.1.20 (p-variation)**.** Let $p > 0$ and fix an horizon $T > 0$. A function $f : [0,T] \to \mathbb{R}$ is said to have finite p-variation if the quantity:

$$\|f\|_{p-var} := \left(\sup_{0=t_0 \leq \ldots \leq t_n = T} \sum_{k=1}^{n} |f(t_i) - f(t_{i-1})|^p \right)^{\frac{1}{p}}$$

is finite (where the supremum is taken over all subdivisions of the interval $[0,T]$). The space of functions with finite p-variations over $[0,T]$ is denoted by $\mathcal{C}^{p-var}([0,T])$. Endowed with the norm $f \mapsto \|f(0)\| + \|f\|_{p-var}$, $\mathcal{C}^{p-var}([0,T])$ is a Banach space.

A classical example of functions with finite p-variations is the class of Hölder continuous functions.

Proposition 1.1.21. *Let $f : [0,T] \to \mathbb{R}$ be an α-Hölder continuous function for some $\alpha \in (0,1]$. Then, f belongs to $\mathcal{C}^{\frac{1}{\alpha}-var}$.*

Proof. Let $\{0 = t_0 \leq \ldots \leq t_n = T\}$ be a subdivision of $[0,T]$. Then, $\forall i \in \{1, \ldots, n\}$, $|f(t_i) - f(t_{i-1})|^{\frac{1}{\alpha}} \leq K^{\frac{1}{\alpha}}(t_i - t_{i-1})$, where K is the Hölder modulus of f. Then,

$$\|f\|_{\frac{1}{\alpha}-var} \leq KT^{\alpha} < \infty.$$

□

Theorem 1.1.22. *Let $f \in \mathcal{C}^{p-var}([0,T])$ and $g \in \mathcal{C}^{q-var}([0,T])$ with*

$$\frac{1}{p} + \frac{1}{q} > 1.$$

Let $0 \leq a \leq b \leq T$. Then, the sequence $(S_n)_{n\in\mathbb{N}}$ of weighted Riemann sums

$$S_n := \sum_{k=0}^{n} f\left(a + (b-a)\frac{k}{n}\right)\left(g\left(a + (b-a)\frac{k+1}{n}\right) - g\left(a + (b-a)\frac{k}{n}\right)\right)$$

converges to a quantity that is denoted $\int_a^b f dg$ *and called the Young integral of* f *against* g*.*

The Young integral possesses the following properties.

Proposition 1.1.23. *Let* f, g *be as in Theorem 1.1.22.*

- *(Chasles relation): Let* $0 \leq a \leq b \leq c \leq T$*. We have*

$$\int_a^c f dg = \int_a^b f dg + \int_b^c f dg.$$

- *(change of variables): Let* $0 \leq t \leq T$*. Assume that* $f \in \mathcal{C}^2$*, and that* $q < 2$*. Then,* $f' \circ g \in \mathcal{C}^{q-var}([0,T])$ *and*

$$f(g(t)) = f(g(0)) + \int_0^t f' \circ g \; dg. \tag{1.11}$$

- *(differentiable case): If* g *is an absolutely continuous function, then*

$$\int f dg = \int f g' d\lambda,$$

with λ *the usual Lebesgue measure.*

Recall now that the fractional Brownian motion B of index $H \in (0,1)$ is $(H-\epsilon)$-Hölder continuous for every $\epsilon > 0$. Then, we can define the Young integral

$$\int u dB$$

as long as the process u has finite $p-variation$ with p such that $\frac{1}{p} > 1-H+\epsilon$ for some $\epsilon > 0$ (for example, when u is $(1-H+\epsilon)$-Hölder continuous). In particular, if $H > \frac{1}{2}$ and f is Lipschitz, we can define in this way the integral $\int f(B)dB$.

Remark 1.1.24. The Young integral behaves in a very different way compared to the Itô integral or to the Skorokhod integral defined above. Indeed, it is actually a pathwise integral, and verify as such a first order Taylor expansion (instead of the Itô formula, which also involves the second order derivative)

Remark 1.1.25. The Young integral is of little use when $H \leq \frac{1}{2}$. Indeed, it is impossible to define simple expressions such as $\int f(B)dB$. When $H \leq \frac{1}{2}$, one can instead relies on the rough path theory, which involves a more complicated approximation scheme. Although we will briefly make use of the notions of Lvy area and controlled path in Chapter 2, the rough path theory itself will not be used in this dissertation.

1.2 Hermite processes

1.2.1 Historical motivation

Hermite processes form a relatively recent addition to the field of stochastic analysis. This family of processes share a lot of similarities with the fractional Brownian motion (which is actually the simplest example of Hermite process) except a crucial one: they are in general not Gaussian. Historically, they have been discovered in [13] and [18] as the limiting process arising in a functional central limit theorem in the context of long-range dependence (although Rosenblatt was the first to observe such phenomenon in [52]). More precisely, let us consider $(X_n)_{n\in\mathbb{N}}$, a centered *Gaussian stationary* (i.e $\forall k \in \mathbb{N}$, $\forall p \in \mathbb{N}$, $\mathbb{E}[X_{p+k}X_k] = \mathbb{E}[X_pX_0]$) *sequence*, with $\mathbb{E}[X_0^2] = 1$ and such that X has long-range dependence, i.e. there is $\alpha \in (0,1)$ and a slowly varying function $L : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\forall n \in \mathbb{N},\ r(n) := \mathbb{E}[X_0X_n] = n^{-\alpha}L(n).$$

Let us now consider the sequence

$$Y_n := \frac{1}{A_n}\sum_{k=0}^{n} f(X_k), \tag{1.12}$$

where A_n is an appropriate renormalization constant (which we will make explicit later) and where $f \in L^2(\mathbb{R},\gamma)$ with γ the standard Gaussian measure, that is, $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} |f(x)|e^{-\frac{x^2}{2}}dx < \infty.$$

Then, depending on α and some properties of the function f, the limit may or may not be Gaussian, see Theorem 1.2.7 for a more precise statement. A functional version of this result also provides a counterpart to the Donsker theorem. In the non-Gaussian case, the limit has been called a *Hermite process.*

Hermite processes have been the object of a growing interest in the recent literature, as they provide an interesting non-Gaussian alternative to the fractional Brownian motion. Among the relevant works, one can mention the aforementioned pioneering papers by Dobrushin, Major and others ([13], [18], [52]) as well as various recent additions, see for instance [1], [8], [32], [44], [53] or [60].

We would also like to mention the recent PhD dissertation [61] which has been an important source of material for the present section.

In the forthcoming sections, we introduce more rigorously the Hermite processes and review their main properties, as well as some connections with the fractional Brownian motion. We also give a short introduction to the stochastic calculus with respect to the Hermite processes.

1.2.2 Hermite polynomials

Hermite polynomials form an infinite family of polynomials of increasing degrees. They can be defined as below (both definitions given here are equivalent).

Definition 1.2.1 (Hermite Polynomials)**.**

1. The Hermite polynomial of order k is given by the *Rodrigues formula*:

$$\forall x \in \mathbb{R},\ H_k(x) = (-1)^k e^{\frac{x^2}{2}} \frac{d^k}{dx^k} e^{-\frac{x^2}{2}}.$$

2. Alternatively, the Hermite polynomials are the only polynomials verifying the following recursive relationship:

$$\begin{cases} H_0(X) & = & 1 \\ H_1(X) & = & X \\ \forall k \geq 2,\ H_k(X) & = & XH_{k-1}(X) - (k-1)H_{k-2}(X). \end{cases}$$

Since the Hermite polynomials are graduated, they define a basis of $\mathbb{R}[X]$. More remarkably, if X is an isonormal Gaussian field with respect to a Hilbert space $\mathcal{H}$, and if $\mathfrak{F}$ denotes the σ-algebra generated by X, then it is possible to build an orthonormal basis of $L^2(\Omega, \mathfrak{F})$ with the help of the Hermite polynomials. As a result, we can reformulate Theorem 1.1.18 in the following way.

Theorem 1.2.2 (Chaotic decomposition revisited)**.** *Let $F \in L^2(\Omega, \mathfrak{F})$. Let L_k be the closed linear subspace of $L^2(\Omega, \mathfrak{F})$ generated by the family*

$$\{H_k(X(h)),\ h \in \mathcal{H},\ \|h\|_{\mathcal{H}} = 1\}$$

(with H_k the Hermite polynomial of order k). Then, there is a unique sequence of random variables $F_k \in L_k$ such that:

$$F = \mathbb{E}[F] + \sum_{k=1}^{n} F_k$$

with the above serie converging in $L^2(\Omega)$.

A proof of this theorem can be found in [41, Section 2.2]. Notice that an equivalent decomposition exists in the one dimensional case, see the following proposition.

Proposition 1.2.3. *Let γ be the standard Gaussian measure over $\mathbb{R}$, i.e $\int_{\mathbb{R}} h d\gamma = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} h(x) e^{-\frac{x^2}{2}} dx$ for all positive Borel function $h : \mathbb{R} \to \mathbb{R}_+$. Let $f \in L^2(\gamma)$. Then, there is a unique sequence $(a_k)_{k\in\mathbb{N}}$ such that:*

$$f = \sum_{k=0}^{\infty} a_k H_k,$$

where the above serie converges in $L^2(\gamma)$.

Finally, Hermite polynomials are directly connected to multiple Wiener integrals introduced above in Section 1.1.5.

Proposition 1.2.4. *Let $h \in \mathcal{H}$ and $p \in \mathbb{N}$. Then,*

$$I_p(h^{\otimes p}) = H_p(X(h)). \tag{1.13}$$

A direct consequence of the equation (1.13) is the following formula, which is immediate but will be used many times in Chapter 2.

Corollary 1.2.5. *Let B be a fractional Brownian motion of Hurst index $H \in (0,1)$, and let $s \leq t$. Then,*

$$I_2(\mathbb{I}_{[s,t]}^{\otimes 2}) = (B_t - B_s)^2 - 1. \tag{1.14}$$

1.2.3 Definition of Hermite processes

With the definition of Hermite polynomials at hands, we are now ready to introduce the Hermite processes, using the notion of *Hermite rank*. There are more general ways to introduce the Hermite processes (e.g, by using the sequence (1.12) and the notion of slowly varying functions), but the definition we give here is more in line with the general spirit of this dissertation.

Definition 1.2.6. Let $f \in L^2(\gamma)$ (where once again, γ is the standard Gaussian measure). Let $(a_k)_{k\in\mathbb{N}}$ be the sequence involved in the Hermite decomposition of f given in Proposition 1.2.3. The Hermite rank of f is the integer $k_0 = \inf\{k > 0, a_k \neq 0\}$.

Hermite processes are then obtained as follows.

Theorem 1.2.7. *[Dobrushin-Major, 1979] Let B be a fractional Brownian motion of index $H' \in (0,1)$ and let $f \in L^2(\gamma)$ be such that $\int_{\mathbb{R}} f(x)e^{-\frac{x^2}{2}}dx = 0$. For all $n \in \mathbb{N}^*$, let us define the process $(Y_t^n)_{t>0}$ as:*

$$Y_\cdot^n := n^{q(1-H')-1} \sum_{k=0}^{\lfloor n\cdot \rfloor} f(B_{k+1} - B_k). \tag{1.15}$$

Assume that the Hermite rank q of f verifies $H' > 1 - \frac{1}{2q}$. Then, the sequence Y^n converges in $\mathcal{D}_{\mathbb{R}}(\mathbb{R}_+)$ to a process with values in the q-th Wiener chaos.

Definition 1.2.8. The limiting process of Theorem 1.2.7 is called *Hermite process of order q and self-similarity parameter $H = 1 - q(1 - H')$*. It is denoted by $Z^{q,H}$.

Here $\mathcal{D}_A(B)$ is the Skorokhod space of càdlàg functions $f : B \to A$ (for more information, see the forthcoming Section 1.3.1).

We end this section with a couple of important remarks, as well as a further definition.

Remark 1.2.9. In the case where $q = 1$, it turns out that $H = H'$ and that the Hermite process $Z^{q,H}$ is actually a fractional Brownian motion of Hurst index H. This is the only case where $Z^{q,H}$ is Gaussian.

Remark 1.2.10. Due to the condition on H' imposed in Theorem 1.2.7, the Hurst parameter H belongs to $(\frac{1}{2}, 1)$. In particular, a fractional Brownian motion of Hurst index $H \leq \frac{1}{2}$ is not a Hermite process.

Definition 1.2.11 (Rosenblatt process)**.** Let $q = 2$ and let $H \in (\frac{1}{2}, 1)$. The Hermite process $Z^{2,H}$ is called the *Rosenblatt process*.

The Rosenblatt process was actually discovered before the seminal works by Dobrushin, Major and others. This name was used for the first time in [58] as a tribute to Murray Rosenblatt.

After the fractional Brownian motion, the Rosenblatt process is the second most well known among the Hermite processes. In particular, the fact that the Rosenblatt process belongs to the second Wiener chaos make it more practical to study than the higher order Hermite processes (second order iterated integrals possess useful properties which are lost at higher orders, se e.g [41], Section 2.7.4). For a thorough review of the properties of the Rosenblatt process, we refer the reader to the paper [62] and to the dedicated section in the dissertation [61].

1.2.4 Properties of Hermite processes

Hermite processes differ significantly from the fractional Brownian motion in one regard: as soon as $q \geq 2$, their marginals laws are no longer Gaussian. Aside from this important caveat, Hermite processes share most of their properties with the fractional Brownian motion. Here is a list of such properties, which is very similar to Proposition 1.1.5.

Proposition 1.2.12. *Let $(Z^{q,H})_{t\in[0,T]}$ be a Hermite process of order $q \in \mathbb{N}^*$ and self-similarity parameter $H \in (\frac{1}{2}, 1)$.*

(a) $Z^{q,H}$ is H-self similar.

(b) $\forall t, s \geq 0$, $\mathbb{E}[Z_s^{q,H}] = 0$ and $\mathbb{E}[Z_s^{q,H} Z_t^{q,H}] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$.

(c) $Z^{q,H}$ has stationary increments.

(d) (long memory) Let $(r(k))_{k\in\mathbb{N}}$ be the sequence defined as

$$r(k) = \mathbb{E}[(Z_{k+1}^{q,H} - Z_k^{q,H}) Z_1^{q,H}].$$

Then,

$$\sum_{k\in\mathbb{N}} |r(k)| = \infty.$$

(e) The increments of $Z^{q,H}$ have finite moments, which are controlled in the following way: $\forall s, t \geq 0$, $\forall p > 0$,

$$\mathbb{E}[|Z_t^{q,H} - Z_s^{q,H}|^p] \leq (p-1)^{\frac{pq}{2}} |t-s|^{pH}.$$

(f) For any $0 \leq \alpha < H$, $Z^{q,H}$ admits an α-Hölder continuous version on each compact interval $[0, T]$.

(g) Let $t > 0$. There is a constant $c_t > 0$ such that $\mathbb{E}[|f(Z_t^{q,H})|] < \infty$ for all measurable function $f : \mathbb{R} \to \mathbb{R}$ satisfying that there exists $K > 0$ such that $\forall x \in \mathbb{R}, |f(x)| \leq K e^{-c_t |x|^{\frac{2}{q}}}$.

Proof. The items (a)-(c) can be proved by means of the forthcoming integral representation given in Proposition 1.2.14, the details have been written down in [61], Proposition 1.1.2.

For the item (d), the proof is exactly the same as in the fractional Brownian motion case (because the sequence $(r(k))_{k\in\mathbb{N}}$ is identical).

For all $0 \leq s,t$, the difference $Z_t^{q,H} - Z_s^{q,H}$ is a multiple Wiener integral of order q (thanks to Proposition 1.2.14) so we can use the self-similarity property and hypercontractivity property to obtain that $Z_t^{q,H} - Z_s^{q,H} \overset{law}{=} |t-s|^H Z_1^{q,H}$, and that

$$\mathbb{E}[|Z_t^{q,H} - Z_s^{q,H}|^p] \leq \mathbb{E}[(Z_1^{q,H})^2]^{\frac{p}{2}}(p-1)^{\frac{pq}{2}}|t-s|^{pH}.$$

The proof of (e) then follows from the fact that $\mathbb{E}[(Z_1^{q,H})^2] = 1$, and the proof of (f) follows from the Kolmogorov-Censov criterion.

Finally, let us prove (g). A power series developement provides

$$\mathbb{E}[|f(Z_t^{q,H})|] \leq \sum_{k=0}^{\infty} \frac{1}{k!}\mathbb{E}[c_t^k |Z_t^{q,H}|^{\frac{2}{q}k}].$$

Since $Z_t^{q,H}$ is an element of the $q-th$ Wiener chaos, the hypercontractivity property yields: $\forall k \geq \frac{q}{2}$,

$$\mathbb{E}[|Z^{q,H}|^{\frac{2}{q}k}] \leq g(k) = (k-1)^k t^{H\frac{2}{q}k}.$$

The Stirling formula then provides:

$$\frac{(c_t)^k g(k)}{k!} \sim_{k\to\infty} (c_t)^k \frac{(k-1)^k}{k^k} \frac{t^{H\frac{2}{q}k} e^k}{\sqrt{2\pi k}}, \tag{1.16}$$

and the associated series converges if $c_t < e^{-H\frac{2}{q}\log(t)-1}$. □

Remark 1.2.13. The item (g) is linked to a very important property of the Hermite processes: the law $Z_1^{q,H}$ is uniquely characterised by its moments if and only if $q \leq 2$. In the case $q = 1$, (g) is also a direct consequence of the more general *Fernique's theorem* for Gaussian measures, see [57].

Similarly to the fractional Brownian motion case, it is possible to represent Hermite processes as stochastic integrals with respect to the Brownian motion. The most common (and useful) representation makes use of a *two-sided Brownian motion* $(W_t)_{t\in\mathbb{R}}$ (i.e, $(W_t^1)_{t>0} := (W_t)_{t\geq 0}$ and $(W_t^2)_{t\geq 0} := (W_{-t})_{t\geq 0}$ are two independant standard Brownian motions). In this case, the Skorokhod and multiple Wiener integrals with respect to W can be obtained by specializing to the case $\mathcal{H} = L^2(\mathbb{R})$.

Proposition 1.2.14. *Let $(W_t)_{t\in\mathbb{R}}$ be a two sided Brownian motion, and for all $p \in \mathbb{N}$ and $f_p \in L^2(\mathbb{R})^{\odot p}$, let $I_p(f_p)$ be the multiple Wiener integral of f_p with respect to W. Then, the Hermite process $(Z^{q,H})$ has the same law as the process*

$$(I_q(f_q(t)))_{t\geq 0} \tag{1.17}$$

with

$$f_q(t,\cdot) : (x_1,\dots,x_q) \mapsto c(H,q)\int_0^t (s-x_i)_+^{H_0-\frac{3}{2}} ds,$$

$$H_0 = 1 - \frac{1-H}{q},$$

$$c(H,q) = \sqrt{\frac{H(2H-1)}{q!\beta(H_0-\frac{1}{2}, 2-2H_0)^q}}.$$

When $q = 1$, Proposition 1.2.14 provides an alternative "spatial" representation for the fractional Brownian motion, thus completing Proposition 1.1.6.

Another integral representation exists, which only involves a one sided Brownian motion (see e.g. [49]) but it won't be utilized in this thesis.

1.2.5 Wiener integral with respect to Hermite processes

It is possible to build a Skorokhod-type integral with respect to Hermite processes. It was done in [62, Section 6] in the specific case of the Rosenblatt process. In this thesis, we will only need to work with Wiener integrals, that we introduce now very briefly.

Let us extend the class $|\mathcal{H}|$ already introduced in Section 1.1.6 for functions defined on $[0,T]$ to functions $f : \mathbb{R} \to \mathbb{R}$. We say that $f : \mathbb{R} \to \mathbb{R}$ belongs to $|\mathcal{H}|$ if $\|f\|^2_{|\mathcal{H}|} := H(2H-1)\int_{\mathbb{R}}\int_{\mathbb{R}} |f(u)||f(v)||u-v|^{2H-2}dudv < \infty$. As noted above, $(|\mathcal{H}|, \|\cdot\|_{|\mathcal{H}|})$ is a Banach space. In [32], the authors considered integrals of simple functions, i.e

$$\int_{\mathbb{R}} \left(f := \sum_{i=0}^{p-1} a_i \mathbb{I}_{[t_i,t_{i+1}]} \right)(s) dZ_s^{q,H} = \sum_{i=0}^{p-1} a_i (Z^{q,H}_{t_{i+1}} - Z^{q,H}_{t_i}).$$

The Hermite process $Z^{q,H}$ itself can be viewed as a multiple Wiener integral, see Proposition 1.2.14, so the previous expression can be rewritten as:

$$\int_{\mathbb{R}} f(s) dZ_s^{q,H} = I_q(F)$$

with I_q the q-th multiple Wiener integral with respect to the two sided Brownian motion W and

$$F : (x_1, \dots, x_q) \to c(H,q) \int_{\mathbb{R}} f(u) \prod_{i=1}^{q} (u - x_i)_+^{H_0 - \frac{3}{2}} du.$$

Finally, using a classic isometry extension procedure, it is shown in [32] that the above integral (defined for simple functions) can be extended to the whole space $|\mathcal{H}|$, as a $L^2(\Omega)$-limit of integrals of simple functions. Furthermore, it coincides with the following multiple integral with respect to the two-sided Brownian motion W: $\forall f \in |\mathcal{H}|$,

$$\int_{\mathbb{R}} f(s) dZ_s^{q,H} = c(H,q) \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}} f(u) \prod_{i=1}^{q} (u - \xi_i)_+^{H_0 - \frac{3}{2}} du \right) dW(\xi_1) \dots dW(\xi_q). \tag{1.18}$$

1.3 Stein's method, Malliavin-Stein approach and other convergence results

This dissertation is all about establishing probabilistic convergence results for sequences of functionals of Gaussian fields (or, in Chapter 4, for functionals of log-concave distributions). In the previous two sections, we gave some background about the fractional Brownian motion and the Hermite processes. In the present section, we will review some of the techniques that we will use in order to establish these convergence results. We will first start by providing some reminders about the functional convergence of stochastic processes. We will then review the basics of Stein's method, before introducing the more recent Malliavin-Stein approach.

1.3.1 Convergence of processes

Let $((X_t^n)_{t\in I})_{n\in\mathbb{N}}$ be a sequence of stochastic processes (where I is either $\mathbb{R}_+$ or an interval of the form $[0,T]$ for some $T > 0$) whose marginals are $\mathbb{R}^d$-valued for some $d \in \mathbb{N}^*$. Assume further that both X^n and X take values in a complete metric space $\mathcal{X}$. We say that the sequence X^n converges in law to X in the space $\mathcal{X}$ if

$$\mathbb{E}[\phi(X^n)] \underset{n\to\infty}{\longrightarrow} \mathbb{E}[\phi(X)] \tag{1.19}$$

for all functionals $\phi : \mathcal{X} \to \mathbb{R}$ which are bounded and continuous with respect to the topology induced by the distance on $\mathcal{X}$. When dealing with stochastic processes, the two most commonly used spaces are the *Wiener space* and the *Skorokhod space*. A useful reference about the Skorokhod space is the short note [26].

Definition 1.3.1 (Wiener space)**.** Let us denote by $\mathcal{C}_{\mathbb{R}^d}(I)$ the space of continuous functions $f : I \to \mathbb{R}^d$. Let $\|\cdot\|_\infty$ be the uniform norm over I, i.e

$$\forall f \in \mathcal{C}_{\mathbb{R}^d}(I),\ \|f\|_\infty = \sup_{x\in I} \|f(x)\|$$

(where $\|\cdot\|$ is the usual Euclidian norm on $\mathbb{R}^d$). Then, $(\mathcal{C}_{\mathbb{R}^d}(I), \|\|_\infty)$ is called the *Wiener space*. It is a vector space and a Polish space.

Definition 1.3.2 (Skorokhod space)**.** Let $\mathcal{D}_{\mathbb{R}^d}(I)$ be the space of cádlág functions $f : I \to \mathbb{R}^d$ (i.e, the functions which are right continuous and admit a left limit in every point $x \in I$). Let us define the following map on $(\mathcal{D}_{\mathbb{R}^d}(I))^2$:

$$\sigma : (f, g) \mapsto \inf_{\lambda\in\Lambda} \{\|\lambda - Id\|_\infty + \|f - g \circ \lambda\|_\infty\},$$

where Λ is the set of all continuous and increasing bijections $\lambda : I \to I$. Let J_1 be the topology generated by the metric σ. Then, the space $\mathcal{D}_{\mathbb{R}^d}(I)$ endowed with the J_1 topology is called the *Skorokhod space*. It is a vector space and a Polish space.

We endow the space $\mathcal{D}_{\mathbb{R}^d}(I)$ with the topology J_1 rather than the topology of uniform convergence, because otherwise it wouldn't be a Polish space. That said, we have the following result, which will be useful in Chapter 2.

Lemma 1.3.3. *The topology J_1 and the topology of uniform convergence coincide on $\mathcal{C}_{\mathbb{R}^d}(I)$.*

To establish the convergence (1.19) in the Wiener space, we will mainly rely on the following classic result, which is an easy consequence of the *Prokhorov's Theorem*.

Proposition 1.3.4. *Let $(x_n)_{n\in\mathbb{N}^*}$ be a sequence of processes in $\mathcal{X}$ (here $\mathcal{X}$ is the Wiener space $\mathcal{C}_{\mathbb{R}^d}(I)$). Then the sequence $(X_n)_{n\in\mathbb{N}^*}$ converges (for the topology of $\mathcal{X}$) to a process $X \in \mathcal{X}$ if and only if the two following conditions holds.*

(1) The finite dimensionals distributions of the processes X^n converge to the finite dimensional distributions of the process X.

(2) *The sequence* $(X^n)_{n\in\mathbb{N}^*}$ *is tight, that is, for all* $\epsilon > 0$, *there is compact* K_ϵ *of* $\mathcal{X}$ *such that* $\forall n \in \mathbb{N}^*$,

$$\mathbb{P}[X^n \in \mathcal{X} \setminus K_\epsilon] \leq \epsilon.$$

Proof. The implication "$\Leftarrow$" is immediate. Let us now assume that both conditions (1) and (2) hold true. By condition (2) and thanks to the fact that $\mathcal{X}$ is a Polish space, the Prokhorov theorem implies that there is a subsequence $(X_{\phi(n)})_{n\in\mathbb{N}^*}$ such that $(X_{\phi(n)})_{n\in\mathbb{N}^*}$ converges weakly to a process $X_\phi \in \mathcal{X}$. Thanks to (1) (and the fact that the finite dimensional projections are continuous in $\mathcal{X}$), we have that the finite-dimensional distributions of the process X_ϕ are the finite dimensional distributions of X. By the uniqueness part in the Kolmogorov extension theorem, $X_\phi \stackrel{law}{=} X$ (and the limit does not depends on the extraction function ϕ). This prove the implication "$\Rightarrow$". ☐

Remark 1.3.5. The case of the Skorokhod space is more delicate because the finite dimensional projections are not continuous. More information on this can be found in [3, Section 13]

It is in general very inconvenient to directly verify the condition (2). Fortunately, in the case where I is a compact interval ($I = [0, T]$ in our case), (2) follows from criterions which are much easier to check. Below are the two criterions that will be used in this thesis.

Lemma 1.3.6. *Let* $\mathcal{X}$ *be either* $\mathcal{D}_\mathbb{R}([0,T])$ *or* $\mathcal{C}_\mathbb{R}([0,T])$ *and let* $(X^n)_{n\in\mathbb{N}^*} \in \mathcal{X}^{\mathbb{N}^*}$. *Then, we have the two following sufficient conditions.*

(a) *If* $\mathcal{X} = \mathcal{C}_\mathbb{R}([0,T])$, *and if there is* $\alpha, \beta > 0$ *and a constant* K *(depending only on* α, β *and* T*) such that*

$$\forall n \in \mathbb{N}^*, \forall\ 0 \leq s, t \leq T,\ \mathbb{E}\left[|X_t^n - X_s^n|^\alpha\right] \leq K|t-s|^{1+\beta}, \tag{1.20}$$

then the sequence is tight in $\mathcal{C}_\mathbb{R}([0,T])$.

(b) *If* $\mathcal{X} = \mathcal{D}_\mathbb{R}([0,T])$, *and if there is* $\alpha, \beta > 0$ *and a constant* K *(depending only on* α, β *and* T*) such that*

$$\forall n \in \mathbb{N}^*, \forall\ 0 \leq s, t \leq T,\ \mathbb{E}\left[|X_t^n - X_s^n|^\alpha\right] \leq K|t_n - s_n|^{1+\beta} \tag{1.21}$$

(where $t_n = \frac{\lfloor nt \rfloor}{n}$, $s_n = \frac{\lfloor ns \rfloor}{n}$*), then the sequence is tight in* $\mathcal{D}_\mathbb{R}([0,T])$.

The first criterion (a) is the classic and well known result proved by Billingsley in the seminal book [3]. A proof of (b) can be found in [12]

Finally, we provide a way to check the tightness for a sequence $X^n \in \mathcal{C}_{\mathbb{R}}([0,T])$, which will prove particularly convenient in Chapter 2. It consists in decomposing the sequence X into two sequences of processes belonging to $\mathcal{D}_{\mathbb{R}}([0,T])$ and checking the tightness of each separate sequence.

Lemma 1.3.7. *Let $(X^n)_{n\in\mathbb{N}^*} \in \mathcal{C}_{\mathbb{R}}([0,T])^{\mathbb{N}^*}$. Assume that for all $n \in \mathbb{N}^*$, we can write $X^n = A^n + C^n$, with*

1. *$\forall n \in \mathbb{N}^*,\ A^n, C^n \in \mathcal{D}_{\mathbb{R}}([0,T])$*
2. *$(A^n)_{n\in\mathbb{N}^*}$ verifies the criterion (b) from Lemma 1.3.6 for some $\alpha, \beta > 0$.*
3. *$\lim_{n\to\infty} \|C^n\|_\infty = 0$.*

Then, the sequence $(X^n)_{n\in\mathbb{N}^*}$ is tight in $\mathcal{C}_{\mathbb{R}}([0,T])$

Proof. By Lemma 1.3.6, the sequence $(A^n)_n$ is tight in $\mathcal{D}_{\mathbb{R}}([0,T])$. By hypothesis, $(C^n)_n$ is also tight in $\mathcal{D}_{\mathbb{R}}([0,T])$ and converges in $\mathcal{D}_{\mathbb{R}}([0,T])$ to 0, which is a continuous process. By [25, Lemma 2.2], the sequence $(A^n, C^n)_n$ is tight in $\mathcal{D}_{\mathbb{R}^2}([0,T])$ and since the map $(x, y) \to x + y$ is continuous from $\mathcal{D}_{\mathbb{R}^2}([0,T])$ to $\mathcal{D}_{\mathbb{R}}([0,T])$, the sequence $(X^n)_n$ is then tight in $\mathcal{D}_{\mathbb{R}}([0,T])$. Finally, by Lemma 1.3.3, the sequence $(X^n)_n$ is tight in $\mathcal{C}_{\mathbb{R}}([0,T])$. □

1.3.2 Distances between probability laws

Let μ, ν be two probability measures on $\mathbb{R}^d$ for some $d \in \mathbb{N}^*$. In this section, we are interested in quantifying the discrepancy between the laws μ and ν. More precisely, we want to endow the space $\mathcal{M}$ of probability measures over $\mathbb{R}^d$ with a distance, in order to make it a complete metric space.

There are several options which are commonly used in the litterature. A review can be found in [41, Appendix C].

Definition 1.3.8 (Fortet-Mourier distance)**.** The application

$$(\mu, \nu) \mapsto \sup_f \left| \int f d\mu - \int f d\nu \right|,$$

where the supremum runs over all Lipschitz functions $f : \mathbb{R}^d \to \mathbb{R}$ such that $\|f\|_\infty + \|f\|_{Lip} \leq 1$, is a distance on $\mathcal{M}$, called the *Fortet-Mourier distance.*

Definition 1.3.9 (Kolmogorov distance)**.** The application

$$(\mu,\nu) \mapsto \sup_{x_1\ldots,x_d\in\mathbb{R}} \big|\mu((-\infty,x_1]\times\ldots\times(-\infty,x_d])-\nu((-\infty,x_1]\times\ldots\times(-\infty,x_d])\big|$$

is a distance on $\mathcal{M}$, called the *Kolmogorov distance.*

Definition 1.3.10 (Total variation distance)**.** The application

$$(\mu,\nu) \mapsto \sup_{A\in\mathcal{B}(\mathbb{R}^d)} |\mu(A)-\nu(A)|$$

is a distance on $\mathcal{M}$, called the *total variation distance.*

Proposition 1.3.11. *Let μ,ν be two probability measures on $\mathbb{R}^d$. The three distances defined above are related as follows:*

$$d_{FM}(\mu,\nu) \leq d_{Kol}(\mu,\nu) \leq d_{TV}(\mu,\nu). \tag{1.22}$$

Remark 1.3.12. An important feature of the Fortet-Mourier is that it metrizes the convergence in law. In other words, a sequence $(\mu_n)_{n\in\mathbb{N}}$ converges weakly to a measure μ if and only if $d_{FM}(\mu_n,\mu) \underset{n\to\infty}{\longrightarrow} 0$. A consequence of this fact and the inequalities (1.22) is that the convergence of a sequence of measures for the Kolmogorov and total variation distances implies the weak convergence of this sequence.

1.3.3 The basics of Stein's method

Introduced first in the seminal work [54], Stein's method is a set of tools aiming at quantifying the distance between probability measures. It has been the object of a wide number of subsequent investigations generalizing it in various settings. An extensive presentation with references can be found in the book [6] for the topic of Stein's method with Gaussian target law (which is the only setting we consider in the present dissertation). A more compact and simplified presentation can be found in [41, Chapters 3-4], which already encompass all the material we need for this dissertation. Although it is not limited to it, Stein's method is mostly known for its contributions to normal approximation. The root of Stein's method is the following simple lemma, also known as *Stein's lemma*, which establishes an integration by part formula with respect to the Gaussian measure.

Lemma 1.3.13. *Let N be a real valued random variable. Let again γ be the standard Gaussian measure. Then, N follows the standard Gaussian* *distribution γ if and only if, for all absolutely continuous functions $f \in L^1(\gamma)$ such that $f' \in L^1(\gamma)$,*

$$\mathbb{E}[|f'(N)|] + \mathbb{E}[|Nf(N)|] < \infty$$

and

$$\mathbb{E}[Nf(N)] = \mathbb{E}[f'(N)].$$

A multidimensional counterpart of Lemma 1.3.13 exists, this time involving second order differential operators.

Lemma 1.3.14. *Let C be a $d \times d$ non-negative definite matrix, and let N be a random vector with values in $\mathbb{R}^d$. Then, N has the $\mathcal{N}(0, C)$ distribution if and only if*

$$\mathbb{E}[|\langle N, \nabla f(N)\rangle_{\mathbb{R}^d}|] + \mathbb{E}[|\langle C, Hessf(N)\rangle_{HS}|] < \infty$$

and

$$\mathbb{E}[\langle N, \nabla f(N)\rangle_{\mathbb{R}^d}] = \mathbb{E}[\langle C, Hessf(N)\rangle_{HS}],$$

for every function $f \in \mathcal{C}_b^2(\mathbb{R}^d, \mathbb{R})$, where $Hess\ f$ is the Hessian matrix of f and $\langle \cdot, \cdot \rangle_{HS}$ is the Hilbert-Schmidt scalar product on the $d \times d$ square matrix space, i.e

$$\forall A, B \in \mathcal{M}(\mathbb{R}^d),\ \langle A, B\rangle_{HS} = Tr(AB^T).$$

Proof of these results can be found in [41], p. 60 and pp. 80–81. It should be noted that similar results can be obtained in non-Gaussian settings as long as the target law possess a suitable form. In Chapter 4, we will make use of the following variation in the case of *regular Gibbs measure.*

Lemma 1.3.15. *Let μ be a probability law whose density is given by*

$$g_\mu := \frac{1}{K} e^{-\Phi}, \tag{1.23}$$

with $\Phi \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R})$ such that

$$K = \int_{\mathbb{R}^d} e^{-\Phi(x)} dx < \infty.$$

Then, for all absolutely continuous function $f : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\mathbb{E}_\mu[|f|] + \mathbb{E}_\mu[|\langle f, \nabla\Phi\rangle_{\mathbb{R}^d}|] + \mathbb{E}_\mu[|Tr(\nabla f)|] < \infty, \tag{1.24}$$

we have

$$\mathbb{E}_\mu[\langle f, \nabla\Phi\rangle_{\mathbb{R}^d}] = \mathbb{E}_\mu[Tr(\nabla f)]. \tag{1.25}$$

Proof. We start the proof with the case where f has compact support. We have, thanks to Fubini's theorem and the fact that $|f_i(x)|e^{-\Phi(x)} \underset{\|x\|\to\infty}{\longrightarrow} 0$ for all $i \in \{1, \ldots, n\}$,

$$\begin{aligned}
\mathbb{E}_\mu[\langle f, \nabla\Phi\rangle_{\mathbb{R}^d}] &= \int_{\mathbb{R}^d} \left(\sum_{i=1}^{d} f_i(x) \frac{\partial \Phi}{\partial x_i}(x) \right) e^{-\Phi(x)} dx \\
&= \sum_{i=1}^{n} \int_{\mathbb{R}^{d-1}} \left(\int_{\mathbb{R}} f_i(x) \frac{\partial \Phi}{\partial x_i}(x) e^{-\Phi(x)} dx_i \right) \prod_{j\neq i} dx_j \\
&= \sum_{i=1}^{n} \int_{\mathbb{R}^{d-1}} \left(\int_{\mathbb{R}} \frac{\partial f_i}{\partial x_i}(x) e^{-\Phi(x)} dx_i \right) \prod_{j\neq i} dx_j \\
&= \mathbb{E}_\mu[Tr(\nabla f)].
\end{aligned}$$

Let now f be a function satisfying the hypotheses of Lemma 1.3.15, and let $(p_n)_{n\in\mathbb{N}^*}$ be a sequence of smooth functions from $\mathbb{R}^d$ to $\mathbb{R}^d$ such that $p_n(x) = 1$ if $\|x\| \leq n$, $p_n(x) = 0$ if $\|x\| \geq n+1$ and $\sup_n \|\nabla p_n\|_\infty < \infty$. Then, the sequence $u_n = (fp_n)_{n\in\mathbb{N}^*}$ verifies

$$\mathbb{E}_\mu[\langle u_n, \nabla\Phi\rangle_{\mathbb{R}^n}] = \mathbb{E}_\mu[Tr(\nabla u_n)].$$

Thanks to (1.24), we can apply the dominated convergence theorem from which the desired conclusion follows. $\square$

The other main ingredient of Stein's method is the utilization of the following diffential equation, known as *Stein's equation of unknown* f (here stated in the one dimensional setting)

$$f'(x) - xf(x) = h(x) - \mathbb{E}[h(N)], \tag{1.26}$$

where N is a standard Gaussian random variable and $h : \mathbb{R} \to \mathbb{R}$ is a given Borel function in $L^1(\gamma)$.
It is not difficult to show that (1.26) admits a unique solution $f = f_h \in L^1(\gamma)$ verifying the decay condition $f_h(x)e^{-\frac{x^2}{2}} \underset{\|x\|\to\infty}{\longrightarrow} 0$. By Stein's lemma 1.3.13, we also know that $\mathbb{E}[f_h'(N) - Nf_h(N)] = 0$. Putting these two facts together leads to the identity

$$\mathbb{E}[h(F) - h(N)] = \mathbb{E}[f_h'(F) - Ff_h(F)], \tag{1.27}$$

for every real valued random variable F such that $\mathbb{E}[|h(F)|] < \infty$.

Crucially, the identity (1.27) means that the distance between any random variable F and the standard Gaussian distribution can be bounded by a quantity which does not depend on the Gaussian distribution γ. More formally, let us observe that the total variation distance between F and N given in Definition 1.3.10 can be expressed as

$$d_{TV}(F, N) = \sup_h |\mathbb{E}[h(N) - h(F)]|,$$

where the supremum is taken over the set $\mathcal{G} = \{\mathbb{I}_A,\ A \in \mathcal{B}(\mathbb{R})\}$. We deduce from (1.27) that

$$d_{TV}(F, N) = \sup_{h \in \mathcal{G}} \left|\mathbb{E}[f_h'(F) - F f_h(F)]\right|,$$

where the supremum is taken on the same set $\mathcal{G}$. To bound the total variation distance, it is then enough to find the "image" of the set $\mathcal{G}$ in the space of the solutions to the Stein's equation.

Proposition 1.3.16. *Let F be a real valued random variable. The total variation distance between F and N can be bounded as follows:*

$$d_{TV}(F, N) \leq \sup_f |\mathbb{E}[f'(F) - F f(F)]|, \tag{1.28}$$

where the supremum runs over the set of absolutely continuous functions f such that $\|f\|_\infty \leq \sqrt{\frac{\pi}{2}},\ \|f'\|_\infty \leq 2$.

Similar estimates also exist for the Fortet-Mourier and Kolmogorov distances, but they will not be used in this thesis.

In order to establish a bound on $d_{TV}(F, N)$ or to prove convergence in total variation, it remains to exhibit a bound on the quantity

$$S_{TV}(F) := \sup_f |\mathbb{E}[f'(F) - F f(F)]|,$$

known as the *Stein's discrepancy.* There is no general way to establish such a bound, so the method has to be taylored on the type of random variables studied. A well known tool is the *method of exchangeable pairs* used first in [10]. Another method, mostly used in the multidimensional setting, is to use Stein's kernel as a way to minimize Stein's discrepancy, see [55, Lecture 6] for an introduction. When the random variable F can be represented as a functional of a Gaussian field, a powerful tool is the Malliavin-Stein approach, which we will introduce in the forthcoming section.

1.3.4 The Malliavin-Stein approach

The Malliavin-Stein approach is a recent and active topic of research. Its foundations rests on two important articles: the paper [46] establishing the *fourth moment theorem* and the paper [37] linking for the first time Stein's method and Malliavin calculus for functionals of a Gaussian field. The main reference on this topic is the dedicated book [41] and particularly the chapters 5-6. The basic idea is that when the random variable G belongs to a space $\mathbb{D}^{p,q}$, integration by parts formulas involving the operators of Malliavin calculus can be used to bound the Stein's discrepancy for the total variation distance. This approach has been succesfully used, among other examples, to study universality phenomenons (see [38]), to extend the central limit theorem in total variation (see [42]), to obtain a central limit theorem for the solution of the stochastic heat equation (see [23]) and have been extended in the Poisson setting in e.g. [29]
We need first to introduce two additional operators.

Definition 1.3.17. Let X be an isonormal Gaussian process on a Hilbert space $\mathcal{H}$. Let $F \in L^2(\Omega, \mathfrak{F})$ with chaotic expansion $F = \mathbb{E}[F]+\sum_{k=1}^{\infty} I_p(f_p(F))$ for a unique sequence of elements $f_p(F) \in \mathcal{H}^{\odot p}$. Then, the Ornstein-Uhlenbeck semigroup $(P_t)_{t\geq 0}$ is defined as

$$\forall t \geq 0,\ P_t F = \sum_{p=0}^{\infty} e^{-pt} I_p(f_{p+1}(F)).$$

Its infinitesimal generator L is defined by

$$LF = -\sum_{p=1}^{\infty} p I_p(f_p(F))$$

for all variables $F \in Dom(L)$, with

$$Dom(L) := \left\{ F,\ \sum_{p=1}^{\infty} p^2 \mathbb{E}[I_p(f_p(F))^2] < \infty \right\}.$$

Lemma 1.3.18. *Let $F \in \mathbb{D}^{1,2}$ with chaotic expansion as above and $\mathbb{E}[F] = 0$. Let us define $L^{-1}F$ as*

$$L^{-1}F := \sum_{p=1}^{\infty} -\frac{1}{p} I_p(f_p(F)).$$

Then, $L^{-1}F \in \mathbb{D}^{1,2}$ *and*

$$-DL^{-1}F = \int_0^\infty e^{-t} P_t DF dt. \tag{1.29}$$

We are now ready to present the main result of this section, which was obtained first by Nourdin and Peccati in [37].

Theorem 1.3.19. *Let* $F \in \mathbb{D}^{1,2}$ *with* $\mathbb{E}[F] = 0$ *and* $Var(F) = \sigma^2 > 0$. *Then, with* $N \sim \mathcal{N}(0, \sigma^2)$

$$d_{TV}(F, N) \le S_{TV}(F) \le \frac{2}{\sigma^2} \mathbb{E}[|\sigma^2 - \langle DF, -DL^{-1}F\rangle_{\mathcal{H}}|]. \tag{1.30}$$

To get a better understanding of this formula in a less abstract setting, we can look at the particular (and easier) case when F is a functional of a Gaussian vector. This case has been investigated in [19, Theorem A.1], and will be extended in Chapter 4 to a non Gaussian setting. The proof is similar in spirit (but simpler) to that of Theorem 1.3.19, so we will reproduce it almost to the identical to illustrate the type of techniques utilized.

Proposition 1.3.20. *Let* g *be a standard normal vector in* $\mathbb{R}^d$ *and let* $H : \mathbb{R}^d \to \mathbb{R}$ *be square integrable with respect to* γ *the* d*-dimensional Gaussian measure. Assume further that* $H \in \mathbb{D}^{1,2}$. *Set* $m = \mathbb{E}[H(g)]$, $\sigma^2 = Var(H(g))$ *and*

$$F = \frac{H(g) - m}{\sigma}.$$

Moreover, for $t \ge 0$, *set* $\hat{g}_t = e^{-t} g + \sqrt{1 - e^{-2t}}\hat{g}$ *with* $\hat{g}$ *an independent copy of* g. *Let* $\hat{\mathbb{E}}$ *be the expectation with respect to* $\hat{g}$ *and* $\mathbf{E} = \mathbb{E} \otimes \hat{\mathbb{E}}$. *Then,*

$$d_{TV}(F, N) \le \frac{2}{\sigma^2} \mathbb{E}\left[\left|\sigma^2 - \int_0^\infty e^{-t} \langle DH(g), \hat{\mathbb{E}}[DH(\hat{g}_t)]\rangle_{\mathbb{R}^d} dt\right|\right]. \tag{1.31}$$

Remark 1.3.21. The *Mehler formula* gives an equivalent characterisation of the Ornstein-Uhlenbeck semigroup in the d-dimensional setting as follows: Let $F = H(g)$, then $\forall t \ge 0, \forall x \in \mathbb{R}^d$,

$$(P_t F)(x) = \int_{\mathbb{R}^d} H(e^{-t} x + \sqrt{1 - e^{-2t}} y) d\gamma(y).$$

Combining this representation with the equation (1.29) ensures that the formula (1.31) indeed coincide with (1.30).

Proof of Proposition 1.3.20: Without loss of generality, we may assume that $m = 0$ and $\sigma^2 = 1$. The random vector $g_t = \sqrt{1-e^{-2t}}g - e^{-t}\hat{g}$ is an independant copy of $\hat{g}_t$, and $g = e^{-t}\hat{g}_t + \sqrt{1-e^{-2t}}g_t$.
By a standard approximation argument, it is sufficient to show the result for $H \in \mathcal{C}^1$ with H and its derivatives having subexponential growth at infinity. Let $\mathbf{E} = \mathbb{E}\otimes\hat{\mathbb{E}}$. If $\phi : \mathbb{R} \to \mathbb{R}$ is $\mathcal{C}^1$, then using the growth conditions imposed on H to carry out the interchange of expectation and integration-by-parts, one has

$$\begin{aligned}
& \mathbb{E}[F\phi(F] \\
= & \mathbb{E}[(H(g) - H(\hat{g}))\phi(H(g))] = -\int_0^\infty \frac{d}{dt}\mathbf{E}[H(\hat{g}_t)\phi(H(g))]dt \\
= & \int_0^\infty \left(e^{-t}\mathbf{E}\langle \nabla H(\hat{g}_t), g\rangle \phi(H(g)) - \frac{e^{-2t}}{\sqrt{1-e^{-2t}}}\mathbf{E}\langle \nabla H(\hat{g}_t), \hat{g}\rangle \phi(H(g)) \right) dt \\
= & \int_0^\infty \frac{e^{-t}}{\sqrt{1-e^{-2t}}}\mathbf{E}\langle \nabla H(\hat{g}_t), g_t\rangle \phi(H(e^{-t}\hat{g}_t + \sqrt{1-e^{-2t}}g_t))dt \\
= & \int_0^\infty e^{-t}\mathbf{E}\langle \nabla H(\hat{g}_t), \nabla H(e^{-t}\hat{g}_t + \sqrt{1-e^{-2t}}g_t)\rangle \\
& \times\phi'(H(e^{-t}\hat{g}_t + \sqrt{1-e^{-2t}}g_t))dt \\
= & \int_0^\infty e^{-t}\langle \nabla H(g), \hat{\mathbb{E}}(\nabla H(\hat{g}_t))\rangle \phi'(H(g))dt. \qquad (1.32)
\end{aligned}$$

Applying the identity (1.32) to (1.28) yields the stated result. □

1.3.5 Fourth moment and Breuer-Major theorems

Another milestone of the Malliavin-Stein approach is the *fourth moment theorem* initially proved in [46] in a one dimensional setting and then extended to multivariate setting in [47].
A usual method to prove the convergence in distribution of a sequence of random variables to a Gaussian distribution is to use the *method of moments* (i.e, checking the convergence of every moment of the elements of the sequence to those of a Gaussian random variable). This method might be overly tedious in some instances. If the elements of the sequence belongs to a fixed Wiener chaos, the fourth moment theorem states that one only has to check the convergence of the second and fourth moments of each element of the sequence to the second and the fourth moments of a Gaussian random variable. We state the multi-dimensional version below:

Theorem 1.3.22 (Peccati-Tudor, [47])**.** *Let X be an isonormal Gaussian field (with respect to $\mathcal{H}$), fix $d \geq 2$ and let $q_1, \ldots, q_d$ be a family of fixed integers. Consider the random vector*

$$F^n = (I_{q_1}(f_1^n), \ldots, I_{q_d}(f_d^n)),$$

with $f_{q_i}^n \in \mathcal{H}^{\odot q_i}$ for all $i \in \{1, \ldots, d\}$ and all $n \in \mathbb{N}^*$. Assume that $\forall 1 \leq i, j \leq d$,

$$Cov(F_i^n, F_j^n) \underset{n\to\infty}{\longrightarrow} C(i,j),$$

where C is a covariance matrix. Then, the two following statements are equivalent:

(1) For all $i \in \{1, \ldots, d\}$, $\mathbb{E}[(F_i^n)^4] \underset{n\to\infty}{\longrightarrow} 3C(i,i)^2$.

(2) $(F^n)_{n\in\mathbb{N}^*}$ converges in law to a Gaussian vector $F \sim \mathcal{N}(0, C)$.

The most celebrated application of this result is a modern proof of the *Breuer-Major* theorem. Initially proved in [5] using the method of moments, this result establishes a Gaussian counterpart to the non-central limit theorem mentioned in Section 1.2.1 in the case where the stationary sequence $(X_n)_{n\in\mathbb{N}^*}$ possesses short-range memory. The fourth moment theorem drastically simplify the proof, and allowed to improve the result in several directions.

Another application is the following result which is of foremost importance in Chapter 2, where we investigate the asymptotic behaviour of the quadratic variation of the multidimensional fractional Brownian motion. The proof in the case $H = \frac{1}{2}$ is done in Chapter 2, the case $H > \frac{1}{2}$ was already studied in [22, Section 5] and is much more intricate. Reading the already very computational proof given in [22] gives a sense of how incredibly hard it would be to try to establish the result with the method of moments!

Definition 1.3.23 (matrix-valued Brownian motion)**.** Assume that $H \in \left[\frac{1}{2}, \frac{3}{4}\right]$. For $H \notin \{\frac{1}{2}, \frac{3}{4}\}$, define

$$\begin{aligned} q_H &= \sum_{p\in\mathbb{Z}} \int_0^1 \int_0^t \int_p^{p+1} \int_p^v |s-u|^{2H-2} |v-t|^{2H-2} ds dv du dt, \\ r_H &= \sum_{p\in\mathbb{Z}} \int_0^1 \int_t^1 int_p^{p+1} \int_p^v |s-u|^{2H-2} |v-t|^{2H-2} ds dv du dt, \end{aligned}$$

and let $q_{\frac{1}{2}} = \frac{1}{2}$, $r_{\frac{1}{2}} = 0$ and $q_{\frac{3}{4}} = r_{\frac{3}{4}} = \frac{1}{2}$. We have $q_H \geq r_H$ by [22, Lemma 2.1]. Let $\{W^{0,i,j}\}_{1\leq i\leq j\leq d}$ and $\{W^{1,i,j}\}_{1\leq i,j\leq d}$ be two independent families of independent standard Brownian motions, both independent of our underlying process B. We set $W^{0,i,j} = W^{0,j,i}$ for $j < i$. The *matrix-valued Brownian motion* $(W^{i,j})_{1\leq i,j\leq d}$ is then defined as follows:

$$W^{i,j} = \begin{cases} c_H\sqrt{q_H + r_H}\, W^{1,i,j} & \text{if } i = j, \\ c_H\sqrt{q_H - r_H} W^{1,i,j} + c_H\sqrt{r_H} W^{0,i,j} & \text{if } i \neq j, \end{cases} \tag{1.33}$$

with the convention that $c_{\frac{1}{2}} = 1$.

Proposition 1.3.24. *Let B be a d-dimensional fractional Brownian motion of Hurst index $H \in [\frac{1}{2}, \frac{3}{4}]$. Let $\nu_H(n) = \sqrt{n}$ if $H < \frac{3}{4}$ and $\nu_{\frac{3}{4}}(n) = \sqrt{\frac{n}{\ln(n)}}$ for all $n \geq 2$. For all $n \geq 2$, $1 \leq i, j \leq d$, let us consider the process*

$$\Theta^{i,j}_{\cdot,n} := \sum_{k=0}^{\lfloor n\cdot\rfloor -1} \int_{\frac{k}{n}}^{\frac{k+1}{n}\wedge\cdot} (B^i_s - B^i_{\frac{k}{n}})\delta B^j_s$$

(here, the integral is understood as a Skorokhod integral with respect to the field generated by the d-dimensional Brownian motion B). Then, for any B-measurable random variable F, the following convergence holds in the Wiener space $\mathcal{C}_{\mathbb{R}^{d\times d}}([0,T])$:

$$(F, (\Theta_n)_{n\geq 2}) \underset{n\to\infty}{\longrightarrow} (F, W) \tag{1.34}$$

where W is the independent matrix-valued Brownian motion defined above in Definition 1.3.23.

1.4 Overview of Chapters 2 to 4, which constitutes the new results obtained in this thesis

1.4.1 Chapter 2

Work based on the paper [14], currently in revision for the Electronic Journal of Probability, entitled "Asymptotic error distribution for the Riemann approximation of integrals driven by fractional Brownian motions" and written in collaboration with Ivan Nourdin and Pierre Vallois.

Let $B = (B_1, \ldots, B_d)$ be a d-dimensional fractional Brownian motion with Hurst index $H \in [\frac{1}{2}, 1)$ over $[0,T]$ and let $u = (u_1, \ldots, u_m)$ be a B-measurable stochastic process. We assume that the integral $\int_0^\cdot u dB$ is well defined (either as an Itô integral if $H = \frac{1}{2}$ or as a Young integral if $H > \frac{1}{2}$). In [14], we investigate the following problem: under which conditions over the process u does the approximation of $\int udB$ by its Riemann sum verifies a limit theorem? More precisely, is there a sequence $(a_n)_{n\in\mathbb{N}}$, a process X and a mode of convergence $\longrightarrow$ such that:

$$a_n\left(\int_0^{\cdot} u_s^i dB_s^j - \sum_{k=0}^{\lfloor n\cdot\rfloor} u_{\frac{k}{n}}^i \left(B_{\frac{k+1}{n}\wedge\cdot}^j - B_{\frac{k}{n}}^j\right)\right)_{i\leq m, j\leq d} \underset{n\to\infty}{\longrightarrow} (X_{\cdot}^{i,j})_{i\leq m,j\leq d}? \tag{1.35}$$

Since computing stochastic integrals requires this kind of approximation schemes, the problem (1.35) is natural from a practical point of view. The first paper to tackle this problem was [51] in the (standard, one dimensional) Brownian setting for processes u of the form $u_s = f(B_s)$, with f a regular enough function. It established the following convergence result.

Theorem 1.4.1.

$$\sqrt{n}\left(\int_0^{\cdot} f(B_s)dB_s - \sum_{k=0}^{\lfloor n\cdot\rfloor} f(B_{\frac{k}{n}})\left(B_{\frac{k+1}{n}\wedge\cdot} - B_{\frac{k}{n}}\right)\right) \overset{\mathcal{C}([0,T])}{\underset{n\to\infty}{\longrightarrow}} \frac{1}{\sqrt{2}}\int_0^{\cdot} f'(B_s)dW_s, \tag{1.36}$$

with W a Brownian motion independent of B.

Surprisingly, the problem (1.35) has received relatively little attention after [51] (especially in the fractional Brownian motion case), even though a substantial amount of litterature has been dedicated to the related question of error quantification for approximation schemes of SDEs, both in the Brownian motion setting (e.g [25]) and, more recently, in the fractional Brownian motion setting (see e.g [22] and [35]).

In Chapter 2, we establish a general framework to tackle the problem (1.35), which we summarize below (without entering too much here into the technical details).
Inspired by the notion of *controlled paths*, we define a notion of pseudo derivative P for the process u, whose role is to mimick the process $f'(B_s)$ which appear in the specific case described by Theorem 1.4.1. This process P naturally appears in the limit, provided we can bound the quantity $u_t - u_s - P_s(B_t - B_s)$ for all $s, t \in [0, T]$.

Definition 1.4.2 (See Definition 2.1.1)**.** Fix $a \in \{1, 2\}$. We say that the pair (u, P) belongs to $\mathbb{C}_a$ if

- $P = (P_t^{i,j})_{t\in[0,T],1\leq i\leq m,1\leq j\leq d}$ is a $\sigma\{B\}$-measurable $m \times d$-dimensional process ;

- $\int_s^t u_r^i dB_r^j$ is well-defined for any $1 \le i \le m$ and $1 \le j \le d$;
- $$\mathbb{E}\left[L_{s,t}^{i,j} L_{x,y}^{i,j}\right] = o(f_a(s,t,x,y)) \tag{1.37}$$
  for all $1 \le i \le m$ and $1 \le j \le d$, uniformly on $(s,t,x,y) \in [0,T]^4$ such that $s \le t$ and $x \le y$ as $|t-s| + |x-y| \to 0$,

where
$$L_{s,t}^{i,j} = \int_s^t \left\{ u_r^i - u_s^i - \sum_{k=1}^d P_s^{i,k}(B_r^k - B_s^k) \right\} dB_r^j. \tag{1.38}$$

Here, f_1, f_2 are two functions from $[0,T]^4$ to $\mathbb{R}_+$ whose precise definition will be given in the introduction of Chapter 2.

We also rely on (and extend when necessary) various results established regarding the weighted power variations of the fractional Brownian in order to establish the limit X. These results describe the asymptotic behaviour of the quantity
$$a_n \sum_{k=0}^{\lfloor n\cdot \rfloor} x_{\frac{k}{n}} f\left(B_{\frac{k+1}{n}\wedge \cdot} - B_{\frac{k}{n}}\right),$$
where x is a stochastic process, $(a_n)_{n\in\mathbb{N}^*}$ a normalization sequence and f a suitable function (most of the time a polynomial).

For $n \in \mathbb{N}^*$, let $(M_t^{n,i,j})_{t\in[0,T], i\le m, j\le d}$ be the matrix-valued process whose entries are
$$M_t^{n,i,j} := n^{2H-1}\left(\int_0^t u_s^i dB_s^j - \sum_{k=0}^{\lfloor nt \rfloor} u_{\frac{k}{n}}^i \left(B_{\frac{k+1}{n}\wedge t}^j - B_{\frac{k}{n}}^j\right)\right).$$

In [14], we establish the following two results:

Theorem 1.4.3 (See Theorem 2.1.2)**.** *Fix $H \in (\frac{1}{2}, 1)$ and let $(u, P) \in \mathbb{C}_1$ be such that u and P are $\sigma\{B\}$-measurable, where P is a.s. continuous and satisfies $\mathbb{E}\left[\|P\|_\infty^{2+\gamma}\right] < +\infty$ for some $\gamma > 0$. Then, uniformly on $[0,T]$ in probability,*
$$\left\{M_\cdot^{n,i,j}\right\}_{1\le i\le m, 1\le j\le d} \underset{n\to\infty}{\longrightarrow} \left\{\frac{1}{2}\int_0^\cdot P_s^{i,j} ds\right\}_{1\le i\le m, 1\le j\le d}. \tag{1.39}$$
Moreover, this convergence also holds in $L^2(\Omega)$ for any fixed $t \in [0,T]$.

Let us define the function ν_H by:

$$\nu_H(n) := \begin{cases} \sqrt{n} & \text{if } H \in [\frac{1}{2}, \frac{3}{4}) \\ \sqrt{n/\ln n} & \text{if } H = \frac{3}{4} \\ n^{2-2H} & \text{if } H \in (\frac{3}{4}, 1) \end{cases}, \quad n \geq 1.$$

Theorem 1.4.4 (See Theorem 2.1.3)**.** *Fix $H \in [\frac{1}{2}, 1)$, and let $Z = (Z^{k,j})_{1\leq k,j\leq d}$ (resp. $W = (W^{k,j})_{1\leq k,j\leq d}$) denote the matrix-valued Rosenblatt process measurable with respect to B (resp. the matrix-valued Brownian motion independent from B) constructed in the Section 2.2.5 in Chapter 2 (see also the Section 1.3.5 above).*

(A) [non-Brownian case $H > \frac{1}{2}$] *Assume $(u, P) \in \mathbb{C}_2$ is such that u is α-Hölder continuous for some $\alpha > 1-H$ and P is β-Hölder continuous over $[0, T]$ for some $\beta > \frac{1}{2}$.*

- *If $\frac{1}{2} < H \leq \frac{3}{4}$ then, stably in $\mathcal{C}_{\mathbb{R}^{m\times d}}([0, T])$,*

$$\left\{\nu_H(n)\left(M_\cdot^{n,i,j} - \frac{1}{2}\int_0^\cdot P_s^{i,j} ds\right)\right\}_{1\leq i\leq m, 1\leq j\leq d} \underset{n\to\infty}{\longrightarrow} \left\{\sum_{k=1}^{d}\int_0^\cdot P_s^{i,k} dW_s^{k,j}\right\}_{1\leq i\leq m, 1\leq j\leq d},$$

  where the integrals in the right-hand side are understood as a Wiener integrals.

- *If $\frac{3}{4} < H < 1$, assume in addition that $\sum_{j=1}^{d}\sum_{i=1}^{m}\mathbb{E}\|P^{i,j}\|_\beta^{2+\gamma} < \infty$ for some $\gamma > 0$. Then, uniformly on $[0, T]$ in probability,*

$$\left\{\nu_H(n)\left(M_\cdot^{n,i,j} - \frac{1}{2}\int_0^\cdot P_s^{i,j} ds\right)\right\}_{1\leq i\leq m, 1\leq j\leq d} \underset{n\to\infty}{\longrightarrow} \left\{\sum_{k=1}^{d}\int_0^\cdot P_s^{i,k} dZ_s^{k,j}\right\}_{1\leq i\leq m, 1\leq j\leq d},$$

  where the integrals in the right-hand side are understood as Young integrals. Moreover, this convergence also holds in $L^2(\Omega)$ for any fixed $t \in [0, T]$.

(B) [Brownian case $H = \frac{1}{2}$] *Assume $(u, P) \in \mathbb{C}_2$ is such that u and P are progressively measurable, and P is a.s. piecewise continuous with*

$\mathbb{E}\left[\|P\|_\infty^{2+\gamma}\right] < +\infty$ *for some* $\gamma > 0$*. Then, stably in* $\mathcal{C}_{\mathbb{R}^{m\times d}}([0,T])$*,*

$$\left\{\nu_H(n)M_\cdot^{n,i,j}\right\}_{1\leq i\leq m, 1\leq j\leq d} \underset{n\to\infty}{\longrightarrow} \left\{\sum_{k=1}^d \int_0^\cdot P_s^{i,k} dW_s^{k,j}\right\}_{1\leq i\leq m, 1\leq j\leq d},$$

where the integrals in the right-hand side are understood as Wiener integrals.

From a heuristic point of view, the threshold $H = \frac{3}{4}$ appearing above was somehow expected, as it already appears when dealing with the quadratic variation of the fractional Brownian motion, see the paper [39] for a synthesis. The main innovation of our work is the introduction of the more general framework defined above, with the spaces $\mathbb{C}_1$ and $\mathbb{C}_2$.

We then explore various illustative examples.

- We use an estimate taken from [20] to prove that processes of the form

  $$u_\cdot^i = \sum_{i=} \int_0^\cdot a_s^{i,j} dB_s^j + \int_0^\cdot b_s^j ds, \ i \in \{1, \dots, m\}$$

  verifies the hypothesis of Theorem 1.4.4 with $P^{i,j} = a^{i,j}$ (provided that a is regular enough). Depending on the value of H, an additional term involving b^j might also appear in the limit. As a particular case, we recover the case $u = F(B)$ where F is a regular function $\mathbb{R}^d \to \mathbb{R}^m$.

- We use Malliavin calculus to tackle the case where the marginals of u can be expressed as multiple Wiener integrals.

- We establish a sufficient criterion for Theorem 1.4.4 to be verified in the specific case where B is a standard Brownian motion.

- Finally, we study the "limit" case $u_s = F(B_s)$ where F is a convex function (so not necessarily $\mathcal{C}^1$). Although we can not directly use Theorems 1.4.3 and 1.4.4 to study this case, we were able to prove the following result.

  Proposition 1.4.5 (See Proposition 2.3.9)**.** *Fix* $H \in (\frac{1}{2}, \frac{2}{3})$*. Let* $u_s = F(B_s)$*,* $s \in [0,T]$*, with* F *a real convex function such that, for some* $K > 0$ *and* $\gamma \in (0,2)$*,*

  $$|F(x)| + |F'(x)| + \int_{-|x|}^{|x|} (|a|+1) dF''(a) \leq Ke^{|x|^\gamma}, \quad x \in \mathbb{R},$$

where F' is the right derivative of F and F'' denotes its second derivative in the distributional sense (a simple 'non-smooth' example is given by $x \to |x|$). Then, for all $t \in [0,T]$,

$$\begin{aligned} M_t^n &:= n^{2H-1}\left(\int_0^t F(B_s)dB_s - \sum_{k=0}^{\lfloor nt \rfloor} F(B_{\frac{k}{n}})(B_{\frac{k+1}{n}\wedge t} - B_{\frac{k}{n}})\right) \\ &\xrightarrow[n\to\infty]{L^2(\Omega)} \frac{1}{2}\int_0^t F'(B_s)ds. \end{aligned}$$

1.4.2 Chapter 3

Work based on the paper [15], entitled "Limit theorem of integral functionals of Hermite-driven process" and written in collaboration with Ivan Nourdin, David Nualart and Majid Salamat. Bernoulli **27** *(2021), no. 3, pp. 1764–1788.*

We consider the following *moving average* process

$$X(t) := \int_{-\infty}^t x(t-u)dZ_u, \ t \geq 0, \tag{1.40}$$

where x is a sufficiently integrable function and Z is a Hermite process of parameter $q \geq 1$ and Hurst index $H \in (\frac{1}{2}, 1)$. Here, the integral should be understood in the sense defined in Section 1.2.5. We are interested in studying the fluctuations as $T \to \infty$ of the sequence

$$t \to \int_0^{tT} P(X(s))ds, \ t \in [0,1] \tag{1.41}$$

when P is a polynomial. The study of these fluctuations are useful, from a statistical point of view, in order to derive parameter estimation methods. For example, the paper [44] establishes the consistency of estimators of the parameters of a Hermite-driven generalization of the Vasicek model. Our work [15] is a continuation of the particular case of (1.41) studied in the paper [60] with $P(x) = x^2$. When general polynomials are considered, and depending on the parity of the coefficients of P, a rather surprising behavior arises, which contrasts with the "Breuer-Major" type behavior observed in the Gaussian case. We also establish convergence in $\mathcal{C}([0,1])$ whereas [60] only looked at finite dimensional distributions.

Let d be the centered Hermite rank of P (see Section 1.2.3) and

$$H_0 = 1 - \frac{1-H}{q}.$$

The following three theorems are the main results established in this part of the thesis.

Theorem 1.4.6 (See Chapter 3, Proposition 11)**.** *Let Z be a fractional Brownian motion of Hurst index $H \in (\frac{1}{2}, 1)$ and let $x \in L^1(\mathbb{R}_+) \cap L^{\frac{1}{H}}(\mathbb{R}_+)$. Consider the moving average process X defined by* (1.40) *and assume without loss of generality that $Var(X(0)) = 1$ (if not, it suffices to multiply x by a constant). Assume that $q \geq 2$ and the following condition holds: $\forall r \in \{1, \ldots, q-1\}$, $\lim_{s\to\infty} \|f_0 \otimes_r f_s\|_{L^2(\mathbb{R}^{2q-2r})} = 0$, with*

$$f_s(y_1, \ldots, y_q) := \mathbb{I}_{(-\infty,s]^q(y_1,\ldots,y_q)} \int_{y_1\vee\ldots\vee y_q}^{s} x(s-u) \prod_{i=1}^{q} (u - x_i)^{H_0-\frac{3}{2}} du.$$

Then, for every measurable function f such that $\mathbb{E}[|f(X_0)|] < \infty$,

$$\frac{1}{T} \int_0^{Tt} f(X(s))ds \underset{T\to\infty}{\longrightarrow} t\mathbb{E}[f(X_0)] \ a.s.$$

Theorem 1.4.7 (See Chapter 3, Theorem 1)**.** *Let Z be a fractional Brownian motion of Hurst index $H \in (\frac{1}{2}, 1)$ and let $x \in L^1(\mathbb{R}_+) \cap L^{\frac{1}{H}}(\mathbb{R}_+)$. Consider the moving average process X defined by* (1.40) *and assume without loss of generality that $Var(X(0)) = 1$ (if not, it suffices to multiply x by a constant).*

(1) If $d \geq 2$ and $H \in (\frac{1}{2}, 1 - \frac{1}{2d})$ then

$$T^{-\frac{1}{2}} \left\{ \int_0^{Tt} (P(X(s)) - \mathbb{E}[P(X(s))])ds \right\}_{t\in[0,1]}$$

converges in distribution in $\mathcal{C}([0,1])$ to a standard Brownian motion W, up to some multiplicative constant C_1 which is explicit and depends only on x, P and H.

(2) If $H \in (1 - \frac{1}{2d}, 1)$ then

$$T^{d(1-H)-1} \left\{ \int_0^{Tt} (P(X(s)) - \mathbb{E}[P(X(s))])ds \right\}_{t\in[0,1]}$$

converges in distribution in $\mathcal{C}([0,1])$ to a Hermite process of index d and Hurst parameter $1 - d(1-H)$, up to some multiplicative constant C_2 which is explicit and depends only on x, P and H.

Theorem 1.4.8 (See Chapter 3, Theorem 2)**.** *Let Z be a Hermite process of order $q \geq 2$ and Hurst parameter $H \in (\frac{1}{2}, 1)$. Let $x \in \mathcal{S}_L$ for some $L > 1$ (where S_L is the set of bounded functions l such that $y^L l(y) \underset{|y|\to\infty}{\longrightarrow} 0$). Consider the moving average process X defined by (1.40). Finally, let $P(x) = \sum_{n=0}^{N} a_n x^n$ be a real valued polynomial function. Then, one and only one of the following two situations takes place as $T \to \infty$:*

(1) If q is odd and if $a_n \neq 0$ for at least one odd $n \in \{1, \dots, N\}$, then

$$T^{-H_0} \left\{ \int_0^{Tt} (P(X(s)) - \mathbb{E}[P(X(s))]) ds \right\}_{t\in[0,1]}$$

converges in distribution in $\mathcal{C}([0,1])$ to a fractional Brownian motion of parameter $H_1 := H_0$, up to some multiplicative constant K_1 which is explicit and depends only on x, P, q and H (the constant possesses an intricate expression and is explicitely computed in the paper).

(2) If q is even, or if q is odd and $a_n = 0$ for all odd $n \in \{1, \dots, N\}$, then

$$T^{1-2H_0} \left\{ \int_0^{Tt} (P(X(s)) - \mathbb{E}[P(X(s))]) ds \right\}_{t\in[0,1]}$$

converges in distribution in $\mathcal{C}([0,1])$ to a Rosenblatt process of Hurst parameter $H_2 := 2H_0 - 1$, up to some multiplicative constant K_2 which is explicit and depends on x, P, q and H (similar remark as above).

The proof technique relies on a fine study of the chaotic decomposition of the process X viewed as a multiple Wiener integral. As a by-product of this analysis, we introduce the novel notation $\otimes_\alpha(h_1, \dots, h_n)$ for multi index contractions of symetric elements $(h_i)_{i\in\mathbb{N}^*}$ of $L^2(\mathbb{R}^q)$, and establishes the following multi-index version of the product formula.

Lemma 1.4.9 (See Chapter 3, Lemma 5)**.** *Let $n, q \geq 2$ be some integrers and let $h_i \in L^2_s(\mathbb{R}^q)$ for $i = 1, \dots, n$. We have*

$$\prod_{k=1}^{n} I_q(h_K) = \sum_{\alpha\in A_{n,q}} C_\alpha I_{nq-2|\alpha|}(\otimes_\alpha(h_1, \dots, h_n)),$$

where I_q is the q-th multiple Wiener integral with respect to the standard Brownian motion, C_α is a constant which depends on $\alpha \in A_{n,q}$ and $A_{n,q}$ is a suitable set of multi-index $\alpha = (\alpha_{i,j},\ 1 \leq i < j \leq n)$ which is made explicit in Chapter 3.

Finally, we apply our result to a generalized version of the stationary Ornstein-Uhlenbeck process, i.e. the case where the driving function x can be written as $x : s \to e^{-\alpha s}\mathbb{I}_{\mathbb{R}_+}(s)$ for some $\alpha > 0$. This process, relatively well studied in the case where the driving process is a fractional Brownian motion (see [7] or the seminal paper [9]) had received a lot of attention attention very recently, with papers initiating a study about parameter estimation (see [1], [44] or [60]). We also combine the well-known Birkhoff ergodic theorem with a criterion established in [43] to prove a *first order ergodic theorem* for the Hermite-Ornstein-Uhlenbeck process.

Remark 1.4.10. Shortly after the release of our paper, the paper [17] was also released, independantly proving similar results in a more general (though also more abstract) setting. This paper also contain a nice application to homogeneization of differential equations, thus illustrating the potential scope of this result.

1.4.3 Chapter 4

Work in progress currently titled "fluctuation of the Hadwiger-Wills information content", written in collaboration with Ivan Nourdin.

In this study, as a departure from the aforementioned works, we no longer investigate the asymptotic behaviour of a functional of a *Gaussian* field. Rather, we focus on a log-concave functional which was introduced by Hadwiger in [21] in a geometric context.

Definition 1.4.11 (Distance law)**.** Let K be a convex body in $\mathbb{R}^d$ (i.e a convex compact set) for some $d \in \mathbb{N}^*$. The distance law with respect to K is then the probability measure μ_K on $\mathbb{R}^d$ with density given by

$$f_{\mu_K} : x \to \frac{1}{W(K)} e^{-\pi d^2(x,K)},$$

where

$$W(K) = \int_{\mathbb{R}^d} e^{-\pi d^2(x,K)} dx,$$

and $d(\cdot, K)$ is the Euclidian distance with respect to K.

Definition 1.4.12 (Information content)**.** If X is a $\mathbb{R}^d$ valued random variable with density f_{μ_K}, its information content is the random variable

$$H_{\mu_K} = -\log(f_{\mu_K}(X)).$$

The notion of information content is of particular interest in information theory. In this paper, we will study the information content of the distance law with respect to a convex body K, that is, the random variable $H_{\mu_K} := \pi d^2(X, K) + \ln(W(K))$ with $X \sim \mu_K$. The functional $H_K := H_{\mu_K}$ is of particular interest for its connections with geometric invariants of the body K. Indeed, there is a correspondance between H_{μ_K} and the distribution of the *intrinsic volumes* of the body K through the *Steiner formula.* This fact is in particular used in the paper [30] to establish concentration bounds for the variance of the intrinsic volumes distribution of the convex body K.

In our work, we establish a central limit theorem for the information content of the distance law H_{μ_K} (as the dimension d of the space goes to infinity). The main result is a quantitative bound on the total variation distance obtained with the use of Stein's method. Although in some particular case (for example, when K is a cube), the central limit theorem can be obtained through elementary computations, the general case requires a more sophisticated methodology.

Theorem 1.4.13 (See Theorem 4.1.1)**.** *Consider a sequence $(K_n)_{n\geq 1}$ of non-empty convex bodies and suppose, for each n, that*

- *$K_n \subset \mathbb{R}^{d_n}$ with $d_n \to \infty$;*
- *the boundary ∂K_n of K_n is $\mathcal{C}^2$;*
- *K_n is symmetric in the sense that there exists $y \in K_n$ such that $x \in K_n \Rightarrow 2y - x \in K_n$;*
- *the quantity $\lambda_1^n := \min_{x\in\partial K_n} \lambda_1^{K_n}(x)$, where $\lambda_1^{K_n}(x)$ denotes the minimal principal curvature of ∂K_n at x (see Section 4.3.5), satisfies $0 < \lambda_1^n \leq 1$ (in particular, K_n is strictly convex) and $\frac{1}{\lambda_1^n} = O(d_n^\gamma)$ as $n \to \infty$, for some $\frac{1}{4} > \gamma > 0$ independent of n.*

Then, there exists $\alpha, \beta > 0$ independent of n such that

$$d_{TV}\left(\frac{H_{K_n} - \mathbb{E}H_{K_n}}{\sqrt{\text{Var}(H_{K_n})}}, N(0,1)\right) = O_{n\to\infty}\left(d_n^{2\gamma - d_n}\right) \tag{1.42}$$

as $n \to \infty$. In particular, H_{K_n} satisfies a central limit theorem:

$$\frac{H_{K_n} - \mathbb{E}H_{K_n}}{\sqrt{\text{Var}(H_{K_n})}} \to N(0,1) \quad \text{as } n \to \infty.$$

The proof of the above theorem is inspired by the paper [19], which established, among other results, a central limit theorem for the projection of a Gaussian vector over a convex *cone* using Stein's method. Though, the non-Gaussian character of our setting involves different and more complicated techniques.

The proof of Theorem 1.4.13 relies on the following ingredients.

(1) We establish a generalization of the Malliavin-Stein formula (1.31) to the case where the target random variable F can be expressed as a functional of a *continuous Gibbs measure*. This formula applies in particular to the functional H_K.

Proposition 1.4.14 (See Proposition 4.4.1 and Remark 4.4.2)**.** *Let $d \in \mathbb{N}^*$ and let X be a random variable with values in $\mathbb{R}^d$. Assume further that X has a density satisfying*

$$f_X(x) = \frac{1}{K} e^{-\Phi(x)},$$

with Φ a twice differentiable, absolutely continuous function such that $e^{-\Phi}$ is integrable.

Let $F = \frac{f(X)-\mu}{\sigma}$ with $\sigma^2 = Var(f(X)) > 0, \mu = \mathbb{E}[f(X)]$ and $f \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R})$ a function such that ∇f is absolutely continuous and

$$\mathbb{E}[f(X)^4 + \|\nabla f(X)\|^4] \leq \infty.$$

Let $\hat{X}$ be an independent copy of X, $\hat{\mathbb{E}}$ the expectation with respect to $\hat{X}$ and $\mathbf{E} = \mathbb{E} \otimes \hat{\mathbb{E}}$. Finally, for all $t \in \mathbb{R}_+$, define

$$X_t = e^{-t} X + \sqrt{1 - e^{-2t}} \hat{X}.$$

Then, we have, for all $\gamma > 0$:

$$\begin{aligned} & d_{TV}(F, N) \\ \leq \; & \frac{2}{\gamma\sigma^2} \sqrt{Var\left(\int_0^\infty e^{-t} \langle \nabla f(X), \hat{\mathbb{E}}[\nabla f(X_t)] \rangle dt\right)} \\ & + \frac{3}{\sigma} \sup_{g \in \mathcal{G}} \left| \int_0^\infty \mathbf{E}\left[g\left(\frac{f(X)}{\sigma}\right) \left\langle \nabla f(X_t), e^{-t}(X - \frac{1}{\gamma}\nabla\phi(X)) \right.\right.\right. \\ & \left.\left.\left. - \frac{e^{-2t}}{\sqrt{1-e^{-2t}}} (\hat{X} - \frac{1}{\gamma}\nabla\phi(\hat{X})) \right\rangle \right] dt \right|, \end{aligned} \tag{1.43}$$

where $N \sim \mathcal{N}(0,1)$, d_{TV} *is the total variation distance and*

$$\mathcal{G} := \left\{ g \in \mathcal{C}^1(\mathbb{R}) | \forall x \in \mathbb{R}, |g(x)| \leq |x| + \sqrt{\frac{\pi}{2}}, |g'(x)| \leq 2 \right\}.$$

(2) We then use the Brascamp-Lieb inequality from [4] to bound the variance in the expression (1.43).

(3) Since the Brascamp-Lieb inequality only applies to *strongly* log-concave random variables (which is not the case of a random vector X following the distance law as soon as $K \neq \{0\}$), we need to introduce a modified version $\bar{X}$ of X which is strongly log-concave, and such that

$$d_{TV}(d^2(X, K_n), d^2(\bar{X}, K_n)) \underset{d_n \to \infty}{\longrightarrow} 0.$$

We then prove that the log-concavity index of the random vector $\bar{X}$ is tied to the curvature properties of the boundary ∂K of K, hence the condition in Theorem 1.4.13.

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Chapter 2

Asymptotic error distribution for the Riemann approximation of integrals driven by fractional Brownian motion

Reproduction of the paper [14], currently in revision for the Electronic Journal of Probability, entitled "Asymptotic error distribution for the Riemann approximation of integrals driven by fractional Brownian motions" and written in collaboration with Ivan Nourdin and Pierre Vallois

2.1 Introduction

Fractional Brownian motion was introduced by Kolmogorov [16] in the 40's. Mandelbrot and Van Ness [20] popularized it and gave some quantitative properties. Since then, its range of applications has been steadily growing: for example, nowadays it can serve to recreate certain natural landscapes (such as submarine floors, see [29]) or to model rainfalls (see [35]). It also often serves as a model in hydrology (e.g. [22]), telecommunications (e.g. [17,21]), finance (e.g. [4]) or physics (e.g. [36]), to name but a few. Since the explicit calculation of stochastic integrals driven by fractional Brownian motion is impossible except in very particular cases, it is natural to try to approximate these integrals by Riemann sums and to study their convergence.

In [32], Rootzén considered the Itô integral $\int_0^t u_s dB_s$ of an adapted integrand u with respect to a standard Brownian motion B, and investigated the asymptotic behavior of the approximation error $\int_0^t u_s dB_s - \int_0^t u_s^n dB$ when u^n are approximating integrands (for instance, we can choose u^n so that $\int_0^t u_s^n dB_s$ corresponds to the Riemann sum associated with $\int_0^t u_s dB_s$). Using Itô stochastic calculus, Rootzén [32] exhibits after proper normalisation a *stable* limit of the form $\int_0^t a_s dW_s$, with W a Brownian motion *independent* of B. As an illustration, he applied his abstract result to prove a functional central limit-type theorem in the space $\mathcal{D}_{\mathbb{R}}([0,T])$ of càdlàg functions equipped with the Skorohod topology, and with $u_s = f(B_s)$ (provided f is smooth and bounded enough):

$$\sqrt{n}\left(\int_0^t f(B_s)dB_s - \sum_{k=0}^{\lfloor nt\rfloor -1} f(B_{\frac{k}{n}})(B_{\frac{k+1}{n}} - B_{\frac{k}{n}})\right)_{t\in[0,T]} \xrightarrow[n\to\infty]{\text{stably}} \left(\sqrt{\frac{1}{2}}\int_0^t f'(B_s)dW_s\right)_{t\in[0,T]}. \tag{2.1}$$

Rootzén's work [32] paved the way for a new area of research on the subject and related topics. For example, we can mention multidimensional extensions (see [18]), generalizations to the case of random discretisation times (see [9]), applications in finance (see [11]) and approximation schemes of stochastic differential equations (SDEs) driven by semimartingales (see [14]). The recent paper [1] provides an asymptotic expansion for the weak discretization error of Itô's integrals.

Approximation schemes for SDEs driven by a fractional Brownian motion has been addressed in [13, 23]. But Riemann sums approximations of stochastic integrals with respect to fractional Browian motion, as done by Rootzén [32] in the case of the standard Brownian motion, had not yet been studied; the aim of this article is to fill this gap.

In the present paper, we deal with a fractional Brownian motion B of Hurst index $H \in \left[\frac{1}{2}, 1\right)$. All the processes considered in this paper will always be implicitly assumed to be measurable with respect to B. Also, note that the range of H includes $\frac{1}{2}$ (corresponding to Brownian motion), which will allow us to compare our results with those of [32]. Our goal is to analyze the fluctuations around the approximation by Riemann sums of stochastic integrals with respect to a fractional Brownian motion. We will set up an approach based on two main steps.

- *Step 1: weighted limit theorem.* Let (u^n) be a sequence of processes of the form $u^n = \sum_{k=1}^{\lfloor n\cdot \rfloor} X_k^n$ for which a functional convergence $u^n \to w$ holds. We extend this convergence to

$$\sum_{k=1}^{\lfloor n\cdot \rfloor} h_{\frac{k}{n}} X_k^n \longrightarrow \int_0^{\cdot} h_s dw_s$$

for a given class of appropriate random processes h, and where the nature of the integral with respect to w (Itô, Young, etc.) is chosen according to the features of w. When the sequence (X_k^n) is built from the increments of a fractional Brownian motion, this type of questions has received some important contributions in recent years, see e.g. [19] and the references therein. We also mention [13], which was actually our main inspiration for this step.

- *Step 2: Taylor expansion.* To perform Step 1, we assume that our integrand u is 'controlled' by the increments of the integrator B, in the sense that there is a process h and a remainder r such that $u_t = u_s + h_s(B_t - B_s) + r_{s,t}$ for any $t \geq s$. These types of Taylor-like expansions are strongly related with the notion of controlled paths studied in the rough path theory, see [12]. We will characterize precisely the set of such processes below.

The statement of the two main Theorems 2.1.2 and 2.1.3 require the introduction of notations:

(i) a d-dimensional fractional Brownian motion $B = (B^1, \ldots, B^d)$ of Hurst index $H \in \left[\frac{1}{2}, 1\right)$ (as already mentioned, all the processes considered in this paper are implicitly assumed to be measurable with respect to B);

(ii) an m-dimensional process u, with the property that the stochastic integrals $\int_0^t u_s^i dB_s^j$, $1 \leq i \leq m$, $1 \leq j \leq d$, are well-defined. At this stage, we note that the integrals $\int u^i dB^j$ must be understood in the Young sense when $H > \frac{1}{2}$ and in the Itô sense when $H = \frac{1}{2}$. Precise statements will be given later on.

(iii) our quantity of interest: for $t \in [0, T]$, $1 \leq i \leq m$, $1 \leq j \leq d$,

$$\begin{aligned} M_t^{n,i,j} &= n^{2H-1} \left(\int_0^t u_s^i dB_s^j - \sum_{k=0}^{nt_n} u_{\frac{k}{n}}^i \left(B^j_{\frac{k+1}{n} \wedge t} - B^j_{\frac{k}{n}} \right) \right) \\ &= n^{2H-1} \int_0^t (u_s^i - u_{s_n}^i) dB_s^j. \end{aligned} \tag{2.2}$$

In (2.2) and in all what follows, we write $t_n = \frac{\lfloor nt \rfloor}{n}$ when $t \in \mathbb{R}_+$ and $n \in \mathbb{N} \setminus \{0\}$.

(iv) the correlation function: for all $t \geq s$ and all $y \geq x$,

$$\begin{aligned} r_H(s,t,x,y) &= \mathbb{E}\left[(B_t^1 - B_s^1)(B_y^1 - B_x^1)\right] \\ &= \frac{1}{2}\left(|t-x|^{2H} + |s-y|^{2H} - |s-x|^{2H} - |t-y|^{2H}\right); \end{aligned}$$

(v) the rate function at zero

$$\kappa_H(v) := \begin{cases} \sqrt{v} & \text{if } H \in [\frac{1}{2}, \frac{3}{4}) \\ \sqrt{v \ln \frac{1}{v}} & \text{if } H = \frac{3}{4} \\ v^{2-2H} & \text{if } H \in (\frac{3}{4}, 1) \end{cases}, \quad v \in (0,1];$$

(vi) the rate function at infinity

$$\nu_H(n) := \begin{cases} \sqrt{n} & \text{if } H \in [\frac{1}{2}, \frac{3}{4}) \\ \sqrt{n/\ln n} & \text{if } H = \frac{3}{4} \\ n^{2-2H} & \text{if } H \in (\frac{3}{4}, 1) \end{cases}, \quad n \geq 1.$$

In addition, we assume that the process u considered in point **(ii)** satisfies a *structural condition*, that we describe now. Set

$$\begin{aligned} f_1(s,t,x,y) &= |t-s|^{2H-1}|x-y|^{2H-1} r_H(s,t,x,y); \\ f_2(s,t,x,y) &= f_1(s,t,x,y)\kappa_H(|t-s|)\kappa_H(|x-y|). \end{aligned}$$

We introduce the two following spaces $\mathbb{C}_1$ and $\mathbb{C}_2$ of *pseudo-controlled paths.*

Definition 2.1.1. *Fix $a \in \{1,2\}$. We say that the pair (u,P) belongs to $\mathbb{C}_a$ if:*

- $P = (P_t^{i,j})_{t \in [0,T], 1 \leq i \leq m, 1 \leq j \leq d}$ *is an $(m \times d)$-dimensional process ;*
- $\int_s^t u_r^i dB_r^j$ *is well-defined for any $1 \leq i \leq m$ and $1 \leq j \leq d$;*
- $$\mathbb{E}\left[L_{s,t}^{i,j} L_{x,y}^{i,j}\right] = o(f_a(s,t,x,y)) \quad (2.3)$$

  for all $1 \leq i \leq m$ and $1 \leq j \leq d$, uniformly on $(s,t,x,y) \in [0,T]^4$ such that $s \leq t$ and $x \leq y$ as $|t-s| + |x-y| \to 0$, where

  $$L_{s,t}^{i,j} = \int_s^t \left\{ u_r^i - u_s^i - \sum_{k=1}^d P_s^{i,k}(B_r^k - B_s^k) \right\} dB_r^j. \quad (2.4)$$

We note the obvious inclusion $\mathbb{C}_2 \subset \mathbb{C}_1$. We give two examples to understand Definition 2.1.1. For the first one, we consider the case where each component u^i of u is a "fractional semimartingale", namely

$$u_t^i = u_0^i + \sum_{j=1}^{d} \int_0^t a_s^{i,j} dB_s^j + \int_0^t b_s^i ds, \quad t \in [0,T].$$

Then, under certain assumptions on a and b (see Section 2.3.1 for precise statements), the pair (u,a) belongs to $\mathbb{C}_2$ with $a = P$.

For the second one, we assume that $m = d = 1$ (for simplicity) and that u has the form of a multiple Wiener-Itô integral of order $q \geq 1$; then, with $P_s = D_s u_s$ (where D indicates the Malliavin derivative) and under some conditions, the pair (u,P) belongs to $\mathbb{C}_2$, see Section 2.3.2 for precise statements.

We can now state our two main results. The framework of Theorem 2.1.2 is general (assuming that the pair (u,P) belongs to $\mathbb{C}_1$ and satisfies other technical conditions) and concerns the convergence of $M^{n,i,j}$ as $n \to \infty$ in *probability*, towards an identified limit. The situation where $H > \frac{1}{2}$ differs significantly from $H = \frac{1}{2}$, because in this latter case $M^{n,i,j}$ converges *in law* (but not in probability, because of the creation of an independent alea, see e.g. (2.1)).

Theorem 2.1.2. *(First order convergence) Fix $H \in (\frac{1}{2},1)$ and let $(u,P) \in \mathbb{C}_1$ be such that P is a.s. continuous and satisfies $\mathbb{E}\left[\|P\|_\infty^{2+\gamma}\right] < +\infty$ for some $\gamma > 0$. (Here and throughout the paper, we write $\|\cdot\|_\infty$ to indicate the uniform norm over $[0,T]$.) Then, uniformly on $[0,T]$ in probability,*

$$\left\{M_\cdot^{n,i,j}\right\}_{1\leq i\leq m, 1\leq j\leq d} \underset{n\to\infty}{\longrightarrow} \left\{\frac{1}{2}\int_0^\cdot P_s^{i,j} ds\right\}_{1\leq i\leq m, 1\leq j\leq d}. \tag{2.5}$$

Moreover, this convergence also holds in $L^2(\Omega)$ for any fixed $t \in [0,T]$.

Theorem 2.1.2 give sufficient conditions for (2.5) to take place. These conditions are however not necessary: we develop in Section 2.3.4 an example where the assumptions of Theorem 2.1.2 are not satisfied whereas the convergence (2.5) holds.

Let us now study the fluctuations of $M_\cdot^{n,i,j}$ around its limit.

Theorem 2.1.3 (Second order convergence)**.** *Fix $H \in [\frac{1}{2},1)$, and let $Z = (Z^{k,j})_{1\leq k,j\leq d}$ (resp. $W = (W^{k,j})_{1\leq k,j\leq d}$) denote the matrix-valued Rosenblatt process measurable with respect to B (resp. the matrix-valued Brownian motion independent from B) constructed in Section 2.2.5.*

(A) [non-Brownian case $H > \frac{1}{2}$] *Assume* $(u, P) \in \mathbb{C}_2$*,* u *is* α*-Hölder continuous for some* $\alpha > 1 - H$ *and* P *is* β*-Hölder continuous over* $[0, T]$ *for some* $\beta > \frac{1}{2}$*.*

- *If* $\frac{1}{2} < H \leq \frac{3}{4}$ *then, stably in* $\mathcal{C}_{\mathbb{R}^{m\times d}}([0, T])$*,*

$$\left\{\nu_H(n)\left(M_\cdot^{n,i,j} - \frac{1}{2}\int_0^\cdot P_s^{i,j} ds\right)\right\}_{1\leq i\leq m, 1\leq j\leq d} \underset{n\to\infty}{\longrightarrow} \left\{\sum_{k=1}^d \int_0^\cdot P_s^{i,k} dW_s^{k,j}\right\}_{1\leq i\leq m, 1\leq j\leq d},$$

  where the integrals in the right-hand side are understood as Wiener integrals.

- *If* $\frac{3}{4} < H < 1$*, assume in addition that* $\sum_{j=1}^d \sum_{i=1}^m \mathbb{E}\|P^{i,j}\|_\beta^{2+\gamma} < \infty$ *for some* $\gamma > 0$ *where, here and throughout the paper,* $\|\cdot\|_\beta$ *indicates the usual* β*-Hölder seminorm (see also (2.6)). Then, uniformly on* $[0, T]$ *in probability,*

$$\left\{\nu_H(n)\left(M_\cdot^{n,i,j} - \frac{1}{2}\int_0^\cdot P_s^{i,j} ds\right)\right\}_{1\leq i\leq m, 1\leq j\leq d} \underset{n\to\infty}{\longrightarrow} \left\{\sum_{k=1}^d \int_0^\cdot P_s^{i,k} dZ_s^{k,j}\right\}_{1\leq i\leq m, 1\leq j\leq d},$$

  where the integrals in the right-hand side are understood as Young integrals. Moreover, this convergence also holds in $L^2(\Omega)$ *for any fixed* $t \in [0, T]$*.*

(B) [Brownian case $H = \frac{1}{2}$] *Assume that* $(u, P) \in \mathbb{C}_2$*, that* u *and* P *are progressively measurable, and that* P *is a.s. piecewise continuous with* $\mathbb{E}\left[\|P\|_\infty^{2+\gamma}\right] < +\infty$ *for some* $\gamma > 0$*. Then, stably in* $\mathcal{C}_{\mathbb{R}^{m\times d}}([0, T])$*,*

$$\left\{\nu_H(n) M_\cdot^{n,i,j}\right\}_{1\leq i\leq m, 1\leq j\leq d} \underset{n\to\infty}{\longrightarrow} \left\{\sum_{k=1}^d \int_0^\cdot P_s^{i,k} dW_s^{k,j}\right\}_{1\leq i\leq m, 1\leq j\leq d},$$

where the integrals in the right-hand side are understood as Wiener integrals.

In Theorem 2.1.3, we could have considered non-uniform or even random subdivisions (like done in [9] in the semimartingale context) but this would have led to significant technical complications due to the non-stationarity of the resulting sequence of increments. Similarly, we could also have replace the fractional Brownian motion by a general Gaussian processes with a covariance function assumed to behave locally as that of the fractional Brownian motion.

The rest of the paper is organized as follows. Section 2 contains some reminders and useful results about Malliavin calculus and fractional integration. In Section 3, we discuss in details some examples. Finally, the proofs of the main results are given in Section 4.

2.2 Preliminaries

2.2.1 Notation

In the sequel, $\mathbb{N}$ (resp $\mathbb{N}^*$) will denote the space of nonnegative (resp strictly positive) integers, $\mathcal{C}^k([0,T])$ (resp $\mathcal{C}^k_b([0,T])$) the space of k-times continuously differentiable functions (resp k-times continuously differentiable with bounded derivatives) over $[0,T]$, and $\mathcal{C}^\theta([0,T])$ the space of θ-Hölder continuous functions (with $\theta \in (0,1)$) endowed with the θ-Hölder seminorm, i.e

$$\|f\|_\theta = \sup_{0\leq s<t\leq T} \frac{|f(t)-f(s)|}{|t-s|^\theta}. \tag{2.6}$$

We also consider the space $\mathcal{C}_{\mathbb{R}^p}([0,T])$ of functions $[0,T] \to \mathbb{R}^p$ endowed with the norm $\|\cdot\|_\infty$ of uniform convergence over $[0,T]$, the space $\mathcal{D}_{\mathbb{R}^p}([0,T])$ of càdlàg functions endowed with the Skorokod topology J_1 and, for $p>0$, the space $L^p(\Omega)$ of random variables endowed with the $L^p(\Omega)$-norm $\|\cdot\|_p$.

2.2.2 Reminders of Malliavin calculus

This section is a condensed summary of some notions presented in [26,27,30]. It is the occasion to fix the notation used in the paper. For more details or missing proofs, we refer the reader to the aforementioned references.

Starting from now, we fix once for all an horizon time $T>0$ and a complete filtered probability space $\left(\Omega, (\mathfrak{F}_t)_{t\in[0,T]}, \mathfrak{F}=\mathfrak{F}_T, \mathbb{P}\right)$. We consider a d-dimensional fractional Brownian motion $(B_t)_{t\in[0,T]} = (B^1_t,\dots,B^d_t)_{t\in[0,T]}$ defined on Ω. We assume that the filtration $(\mathfrak{F}_t)_{t\in[0,T]}$ is generated by B.

Let $\mathcal{B}$ be the Gaussian space spanned by the (one-dimensional) fractional Brownian motion B^1. Let $\mathcal{E}$ be the linear space of step functions over $[0,T]$

and let $\mathcal{H}$ be the Hilbert space obtained as the completion of $\mathcal{E}$ with respect to the inner product induced from B^1:

$$\langle \mathbb{I}_{[0,t]}, \mathbb{I}_{[0,s]} \rangle_{\mathcal{H}} = \mathbb{E}[B_t^1 B_s^1], \quad 0 \leq s, t \leq T.$$

The linear map defined on $\mathcal{E}$ by $\Phi : \mathbb{I}_{[0,t]} \to B_t^1$ is an isometry from $(\mathcal{E}, \langle ., . \rangle_{\mathcal{H}})$ to $(\mathcal{B}, \mathbb{E}[., .])$, and can thus be extended to an isometry from the whole space $\mathcal{H}$.

For $H = \frac{1}{2}$, we have $\mathcal{H} = L^2([0,T])$. When $H > \frac{1}{2}$, it is well known that $\mathcal{H}$ contains distributions, and therefore is not a subspace of some convenient functional space, see [30]. This is why we introduce the subspace $|\mathcal{H}|$ of $\mathcal{H}$, which is defined as the set of measurable functions $f : [0,T] \to \mathbb{R}$ such that

$$\int_{[0,T]^2} |f(x)||f(y)| \mu_H(dxdy) < +\infty,$$

with

$$\mu_H(dxdy) = H(2H-1)|x-y|^{2H-2} dxdy.$$

From [30], we have that $(|\mathcal{H}|, \| \cdot \|_{|\mathcal{H}|})$ is a Banach space with respect to the norm $\| \cdot \|_{|\mathcal{H}|}$, defined as

$$\|f\|_{|\mathcal{H}|}^2 = \int_{[0,T]^2} |f(x)||f(y)| \mu_H(dxdy).$$

We observe that $\|f\|_{|\mathcal{H}|} \leq \|f\|_{\mathcal{H}}$ for all $f \in |\mathcal{H}|$.

Still for $H > \frac{1}{2}$, we define $|\mathcal{H}|^{\otimes p}$, $p \in \mathbb{N}^*$, to be the Banach space of measurable functions $f : [0,T]^p \to \mathbb{R}$ such that

$$\int_{[0,T]^{2p}} |f(x_1, \ldots, x_p)||f(y_1, \ldots, y_p)| \prod_{i=1}^{p} \mu_H(dx_i dy_i) < +\infty,$$

and we observe that $|\mathcal{H}|^{\otimes p} \subset \mathcal{H}^{\otimes p}$.

Let $n \in \mathbb{N}^*$ and let $\mathcal{S}_n$ be the space of infinitely differentiable functions $f : \mathbb{R}^{nd} \to \mathbb{R}$ such that f and all its derivatives have at most polynomial growth. We consider the *Schwartz space* $\mathcal{C}$ composed of all cylindrical random variables, that is, of all random variables F of the form $F = f(B_{t_1}, \ldots, B_{t_n})$, with $n \in \mathbb{N}^*$, $f \in \mathcal{S}_n$, and $t_1, \ldots, t_n \in [0,T]$.

The pth-order Malliavin derivative of $F \in \mathcal{C}$ is the element

$$D^p F = \{ D^{p, j_1, \ldots, j_p}_{l_1, \ldots, l_p} F : l_1, \ldots, l_p \in [0,T] \}_{1 \leq j_1, \ldots, j_p \leq d}$$

belonging to $\cap_{r\geq 1} L^r(\Omega, (\mathcal{H}^{\otimes p})^{\otimes dp})$ defined as

$$D^{p,j_1,\dots,j_p}_{l_1,\dots,l_p} F = \sum_{k_1,\dots,k_p=1}^{n} \frac{\partial^p f}{\partial x^{k_1,j_1} \dots \partial x^{k_p,j_p}}(B_{t_1},\dots,B_{t_n}) \prod_{i=1}^{p} \mathbb{I}_{[0,t_{k_i}]}(l_i).$$

Since these operators are closable in $L^r(\Omega, (\mathcal{H}^{\otimes p})^{\otimes dp})$ for all $r \geq 1$, we can consider the *Sobolev space* $\mathbb{D}^{p,r}$ as the closure of $\mathcal{C}$ with respect to the norm

$$\|F\|^r_{\mathbb{D}^{p,r}} = \mathbb{E}[|F|^r] + \sum_{m=1}^{p} \sum_{j_1,\dots,j_m=1}^{d} \mathbb{E}\left[\|D^{m,j_1,\dots,j_m} F\|^r_{\mathcal{H}^{\otimes m}}\right].$$

In the same way, it is possible to define the Malliavin derivative for step processes u of the form $u = \sum_{i=0}^{n-1} F_i \mathbb{I}_{[t_i,t_{i+1}]}$ (where $n \in \mathbb{N}^*$, $t_0 = 0, t_1, \dots, t_n \in [0,T]$ and $F_1, \dots, F_n \in \mathcal{C}$), and to consider the associated spaces $\mathbb{D}^{p,r}(\mathcal{H})$. In order to only deal with functions (and not distributions), we consider the subspace $\mathbb{D}^{p,r}(|\mathcal{H}|)$ of $\mathbb{D}^{p,r}(\mathcal{H})$, which is by definition the set of $u \in \mathbb{D}^{p,r}(\mathcal{H})$ that are such that $u \in |\mathcal{H}|$ a.s., $D^1 u \in (|\mathcal{H}|^{\otimes 2})^{\otimes d}$ a.s., …, $D^p u \in (|\mathcal{H}|^{\otimes p+1})^{\otimes dp}$ a.s.. This subspace is endowed with the norm

$$\|u\|^r_{\mathbb{D}^{p,r}(|\mathcal{H}|)} = \mathbb{E}\|u\|^r_{|\mathcal{H}|} + \sum_{m=1}^{p} \sum_{j_1,\dots,j_m=1}^{d} \mathbb{E}\left[\|D^{m,j_1,\dots,j_m} u\|^r_{|\mathcal{H}|^{\otimes m+1}}\right].$$

Let $u \in L^2(\Omega, \mathcal{H}^{\otimes d})$ be such that $|\mathbb{E}[\langle D^1 F, u\rangle_{\mathcal{H}^{\otimes d}}]| \leq K_u \sqrt{\mathbb{E}[F^2]}$ for all $F \in \mathcal{C}$, for some constant K_u depending only on u. We then say that u belongs to the domain $\mathrm{Dom}(\delta^1)$, and we define the Skorohod integral δ^1 as the adjoint of D^1, that is, $\delta^1(u)$ is the uniquely determined random variable in $L^2(\Omega)$ verifying the duality relationship:

$$\mathbb{E}[\langle D^1 F, u\rangle_{\mathcal{H}^{\otimes d}}] = \mathbb{E}[F\delta^1(u)] \text{ for all } F \in \mathbb{D}^{1,2}. \tag{2.7}$$

In the same way, if u is an element of $L^2(\Omega, (\mathcal{H}^{\otimes p})^{\otimes dp})$ ($p \geq 2$) we define $\delta^p = (\delta^{p,j_1,\dots,j_p})_{1\leq j_1,\dots,j_p\leq d}$ as the adjoint of $D^p = (D^{p,j_1,\dots,j_p})_{1\leq j_1,\dots,j_p\leq d}$ through the identity:

$$\mathbb{E}[\langle D^p F, u\rangle_{(\mathcal{H}^{\otimes p})^{\otimes dp}}] = \mathbb{E}[F\delta^p(u)] \text{ for all } F \in \mathbb{D}^{p,2}.$$

We can show that $\mathbb{D}^{p,2}(\mathcal{H}) \subset \mathrm{Dom}(\delta^p)$.

The following two results will be also useful. The first one is a straightforward consequence of the Hardy-Littlewood-Sobolev inequality (see [3, Theorem 6]), whereas the second one corresponds to [27, Proposition 1.3.1].

Proposition 2.2.1. *1. Fix an integer $k \geq 1$. There exists $M > 0$ such that, for all $u \in L^2(\Omega, L^2([0,T]^k))$,*

$$\mathbb{E}\left[\|u\|^2_{|\mathcal{H}|^{\otimes k}}\right] \leq M\mathbb{E}\left[\|u\|^2_{L^2([0,T]^k)}\right]. \tag{2.8}$$

2. For all $u, v \in \mathbb{D}^{1,2}(\mathcal{H})$ and $j \in \{1, \dots, d\}$, we have

$$\mathbb{E}[\delta^{1,j}(u)\delta^{1,j}(v)] = \mathbb{E}\left[\langle u, v\rangle_{\mathcal{H}}\right] + \mathbb{E}\left[\langle D^{1,j}_{\cdot} u_*, D^{1,j}_* v_{\cdot}\rangle_{\mathcal{H}\otimes\mathcal{H}}\right]. \tag{2.9}$$

2.2.3 Multiple Wiener-Itô integrals

Throughout all this section, we assume for simplicity that the underlying fractional Brownian motion is one-dimensional, i.e. that $d = 1$. We write D^k (resp. δ^k) instead of $D^{k,1,\dots,1}$ (resp. $\delta^{k,1,\dots,1}$).

When the process u is *deterministic* in $\mathcal{H}^{\otimes k}$, its Skorohod integral $\delta^k(u)$ is called the *kth-order Wiener-Itô integral* of u. If $\widetilde{u}$ denotes the symmetrization of u (see the footnote [1]), we have $\delta^k(u) = \delta^k(\widetilde{u})$; we can therefore assume without loss of generality that u is symmetric. In what follows, we denote by $\mathcal{H}^{\odot k}$ the set of symmetric elements in $\mathcal{H}^{\otimes k}$.

The following statement summarizes what is needed about multiple Wiener-Itô integrals in this paper. We refer e.g. to [26] for the proofs.

Proposition 2.2.2. *1. (Isometry) For all integers $k, l \geq 1$, all $f \in \mathcal{H}^{\odot k}$ and all $g \in \mathcal{H}^{\odot l}$,*

$$\mathbb{E}[\delta^k(f)\delta^l(g)] = k!\langle f, g\rangle_{\mathcal{H}^{\otimes k}}\mathbb{I}_{\{k=l\}}.$$

2. (Hypercontractivity) For all $r \geq 2$ and all integer $k \geq 1$, there exists $C_{k,r} > 0$ such that, for all $f \in \mathcal{H}^{\odot k}$,

$$\mathbb{E}\left[|\delta^k(f)|^r\right] \leq C_{k,r}\mathbb{E}[|\delta^k(f)|^2]^{\frac{r}{2}}.$$

3. (Malliavin derivative) If $u_s = \delta^k(f(.,s))$ with $f \in \mathcal{H}^{\otimes(k+1)}$ symmetric in the k first variables, then $u \in \mathbb{D}^{1,2}(\mathcal{H})$, with

$$D_s u_t = k\delta^{k-1}(f(.,t,s)).$$

[1] If $\{e_j\}_{j\geq 1}$ denotes an orthonormal basis of $\mathcal{H}$ and if u is given by $u = \sum_{j_1,\dots,j_k\geq 1} a_{j_1,\dots,j_k} e_{j_1}\otimes\dots\otimes e_{j_k}$, then $\widetilde{u} = \frac{1}{k!}\sum_\sigma \sum_{j_1,\dots,j_k\geq 1} a_{j_1,\dots,j_k} e_{j_{\sigma(1)}}\otimes\dots\otimes e_{j_{\sigma(k)}}$, where the first sum runs over all permutation σ of $\{1,\dots,k\}$.

4. *(Product formula) Fix $f \in \mathcal{H}^{\odot k}$ and $g \in \mathcal{H}^{\odot l}$ and, as usual, let $\otimes_r$ (resp. $\widetilde{\otimes}_r$) denote the contraction operator (resp. the symmetrization of the contraction operator) of order r, see [26, Appendix B] for a precise definition. Then,*

$$\delta^k(f)\delta^l(g) = \sum_{r=0}^{k\wedge l} r!\binom{k}{r}\binom{l}{r} f\delta^{k+l-2r}(f\widetilde{\otimes}_r g).$$

2.2.4 Fractional Integration

This section gives a brief summary of the useful properties related to the Young integral when the Hurst index H is strictly bigger than $\frac{1}{2}$, see [37,39] for more details.

The following result extends the Riemann integral to a larger class of integrands and integrators. For $p > 0$, we use the classical notations $\mathcal{C}^{p-var}([0,T])$ to denote the space of functions $f : [0,T] \to \mathbb{R}$ with finite p-variations. It is well known that θ-Hölder continuous functions have $\frac{1}{\theta}$-finite variations.

Proposition 2.2.3. *Suppose $p, q > 0$ are such that $\frac{1}{p} + \frac{1}{q} > 1$. If $f \in \mathcal{C}^{p-var}([0,T])$ and $g \in \mathcal{C}^{q-var}([0,T])$ (with g continuous), then the limit of Riemann sums*

$$\sum_{k=0}^{n-1} f\left(\frac{kT}{n}\right)\left(g\left(\left(\frac{(k+1)T}{n} \vee a\right) \wedge b\right) - g\left(\left(\frac{kT}{n} \vee a\right) \wedge b\right)\right)$$

exists for all $0 \le a < b \le T$, and is called the Young integral $\int_a^b f dg$ of f against g. It is compatible in the sense that, if $0 \le a < c < d < b \le T$, then $\int_c^d f dg = \int_a^b f \mathbb{I}_{[c,d]} dg$. Moreover, it satisfies the chain rule and the change of variable formula.
Moreover, if f (resp g) are $\frac{1}{p}$-Hölder continuous (resp $\frac{1}{q}$-Hölder continuous), we have the Young-Loeve estimates:

$$\left|\int_a^b f dg - f(a)(g(b) - g(a))\right| \le c_{\mu,\beta}\|f\|_{\frac{1}{p}}\|g\|_{\frac{1}{q}}|b-a|^{\frac{1}{p}+\frac{1}{q}},$$

$$\left|\int_a^b f dg\right| \le c_{\mu,\beta}\left(\|f\|_\infty\|g\|_{\frac{1}{q}}|b-a|^{\frac{1}{q}} + \|f\|_{\frac{1}{p}}\|g\|_{\frac{1}{q}}|b-a|^{\frac{1}{p}+\frac{1}{q}}\right),$$

where $c_{\mu,\beta}$ is a constant depending only on p and q.

When $f : [0,T]^2 \to \mathbb{R}$ is such that $f(t,t) = 0$, we write $f \in \mathcal{C}^\kappa([0,T]^2)$ if

$$\|f\|_\kappa := \sup_{0\le s\ne t\le T} \frac{|f(s,t)|}{|t-s|^\kappa} < \infty. \tag{2.10}$$

Recall that, for each i, the fractional Brownian motion B^i has a.s. κ-Hölder continuous paths for every $\kappa < H$. Therefore, if the process u has a.s. finite p-variations for some $\frac{1}{p} > 1-H$, it is an immediate consequence of Proposition 2.2.3 that the Young integral $\int_0^\cdot u dB^i$ is well-defined pathwise on $[0,T]$; this makes the Young integral a suitable integral when $H > \frac{1}{2}$. In contrast, it is not a suitable integral when $H = \frac{1}{2}$ because, for instance, we cannot deal with integrals as simple as $\int B^j dB^i$.

Another way to define the Young integral is to make use of the forward integration *à la Russo-Vallois* [33]. Their *forward integral* is defined, for fixed j, as

$$\int_0^\cdot u_s dB_s^j = \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_0^\cdot u_s \left(B^j_{s+\epsilon\wedge\cdot} - B^j_s \right) ds, \tag{2.11}$$

provided the limit exists uniformly in probability over the interval $[0,T]$. When $H > \frac{1}{2}$ and $u \in C^\theta([0,T])$ with $\theta > 1-H$, then the limit (2.11) exists and coincides with the Young integral. When $H = \frac{1}{2}$ and u is progressively measurable, then the limit (2.11) exists and coincides with the Itô integral.

In [27], the following relationship between the forward and Skorohod integrals is shown.

Proposition 2.2.4. *Assume that $H > \frac{1}{2}$, and let $u \in \mathbb{D}^{1,2}(|\mathcal{H}|)$ be a scalar process. In addition, suppose that u verifies the following condition:*

$$\forall j \in \{1,\dots,d\}, \int_0^T \int_0^T |D_s^{1,j} u_r| \mu_H(dsdr) < \infty \ \text{a.s.}. \tag{2.12}$$

Then, the limit (2.11) *exists and verifies the relation*

$$\int_0^T u_s dB_s^j = \delta^{1,j}(u) + \int_0^T \int_0^T D_s^{1,j} u_r \mu_H(dsdr), \tag{2.13}$$

where the integral in the left-hand side is in the Russo-Vallois sense.

2.2.5 Matrix-valued Brownian motion and matrix-valued Rosenblatt process

We introduce some probabilistic objects, taken from [13, Sections 2.4 and 2.5] when $H > \frac{1}{2}$, which we complete when $H = \frac{1}{2}$. For more information about the Rosenblatt process, one can e.g. refer to [34].

(*a*) Assume first that $H \in \left[\frac{1}{2}, \frac{3}{4}\right]$. For $H \notin \{\frac{1}{2}, \frac{3}{4}\}$, define

$$\begin{aligned} q_H &= \sum_{p\in\mathbb{Z}} T^{4H} \int_0^1 \int_p^{p+1} \int_0^t \int_p^s \mu_H(dvdu)\mu_H(dsdt) \\ r_H &= \sum_{p\in\mathbb{Z}} T^{4H} \int_0^1 \int_p^{p+1} \int_t^1 \int_p^s \mu_H(dvdu)\mu_H(dsdt), \end{aligned}$$

and let $q_{\frac{1}{2}} = \frac{1}{2}$, $r_{\frac{1}{2}} = 0$ and $q_{\frac{3}{4}} = r_{\frac{3}{4}} = \frac{1}{2}$. We have $q_H \geq r_H$ by [13, Lemma 2.1]. Let $\{W^{0,i,j}\}_{1\leq i\leq j\leq d}$ and $\{W^{1,i,j}\}_{1\leq i,j\leq d}$ be two independent families of independent standard Brownian motions, both independent of our underlying process B. We set $W^{0,i,j} = W^{0,j,i}$ for $j < i$. The *matrix-valued Brownian motion* $(W^{i,j})_{1\leq i,j\leq d}$ is then defined as follows:

$$W^{i,j} = \begin{cases} \sqrt{q_H + r_H}\, W^{1,i,j} & \text{if } i = j \\ \sqrt{q_H - r_H} W^{1,i,j} + c_H \sqrt{r_H} W^{0,i,j} & \text{if } i \neq j \end{cases}, \tag{2.14}$$

with the convention that $c_{\frac{1}{2}} = 1$.

(*b*) Assume now that $H \in (\frac{3}{4}, 1)$. For any fixed $t \in [0, T]$, the sequence of $(d \times d)$-matrix-valued processes

$$\left(n \sum_{k=0}^{\lfloor nt \rfloor -1} \delta^{1,i} \left(\left(B^j_{\cdot} - B^j_{\frac{k}{n}} \right) \mathbb{I}_{\left[\frac{k}{n}, \frac{k+1}{n}\right]}(.) \right) \right)_{1\leq i,j\leq d}$$

converges for all fixed $t \in [0, T]$ to some Z_t. The continuous version of the process $(Z_t)_{t\in[0,T]}$ is called the *matrix-valued Rosenblatt proces of order H*. Each component of this matrix-valued process is α-Hölder continuous for every $\alpha < 2H - 1$. Moreover, the diagonals elements are independent Rosenblatt processes with selfsimilarity index $2H - 1$.

2.3 Examples

We start by defining the notion of controlled process. This notion plays a key role because such a process verifies the conditions of Definition 2.1.1. We then give two classes of examples: fractional semimartingales (i.e. processes with decomposition (2.18)) and multiple Wiener-Itô integrals.

2.3.1 Controlled process

Throughout all this section, we assume $H > \frac{1}{2}$.
Recall that $\mathcal{C}^\kappa([0,T])$ denotes the set of κ-Hölder continuous functions $f : [0,T] \to \mathbb{R}$, whereas $\mathcal{C}^\kappa([0,T]^2)$ denotes the set of κ-Hölder continuous functions $f : [0,T]^2 \to \mathbb{R}$ such that $f(t,t) = 0$ for all t, see (2.10). The class of controlled path, introduced first by Gubinelli in [12], is then defined as follows.

Definition 2.3.1 (Controlled process)**.** *Consider $\kappa \in (\frac{1}{2}, 1)$. The set $\mathcal{D}^{2\kappa}([0,T])$ is defined as the set of pairs (u,P) with u (resp. P) an m-dimensional process (resp. $(m \times d)$-dimensional process) belonging a.s. to $\mathcal{C}^\kappa([0,T])$ and such that the m-dimensional remainder process R defined by*

$$R^i_{s,t} = u^i_t - u^i_s - \sum_{j=1}^{d} P^{i,j}_s (B^j_t - B^j_s), \quad 0 \le s \le t \le T, \tag{2.15}$$

belongs a.s. to $\mathcal{C}^{2\kappa}([0,T]^2)$.

For all $(u,P) \in \mathcal{D}^{2\kappa}([0,T])$, for all $s,t \in [0,T]$ and all $j \in \{1,\ldots,d\}$, Theorem 4.10 in [8] implies:

$$|L^{i,j}_{s,t}| \le C\left(\|B\|_\kappa \|R\|_{2\kappa} + \|P\|_\kappa \|\mathbb{B}\|_{2\kappa}\right)|t-s|^{3\kappa} \tag{2.16}$$

where $\mathbb{B}$ is defined as $\mathbb{B}^{k,j}_{s,t} = \int_s^t (B^k_l - B^k_s) dB^j_l$, L is given by $L^{i,j}_{s,t} = \int_s^t R^i_{s,r} dB^j_r$ (or equivalently by (2.4)), and C is a constant depending only on κ and T. The following proposition gives an explicit link between the notion of controlled path *à la Gubinelli* [12] (Definition 2.3.1) and our notion of pseudo-controlled path (Definition 2.1.1).

Proposition 2.3.2. *Assume that $\kappa > \frac{2(H \wedge \frac{3}{4})}{3} + \frac{1}{6}$, $(u,P) \in \mathcal{D}^{2\kappa}([0,T])$ and, for some $\theta > 0$ and all $j \in \{1,\cdots,d\}$,*

$$\sum_{i=1}^{m} \mathbb{E}\left[\|R^i\|^{2+\theta}_{2\kappa} + \sum_{j=1}^{d} \|P^{i,j}\|^{2+\theta}_\kappa\right] < \infty, \tag{2.17}$$

with R defined by (2.15). Then $(u,P) \in \mathbb{C}_2$.

Proof. The proof is a straightforward combination of the identity (2.16), the Hölder inequality and the forthcoming Lemmas 2.4.1 and 2.4.2. □

As a consequence of Theorem 2.1.3 and Proposition 2.3.2, we deduce the following statement.

Proposition 2.3.3. *Fix $H > \frac{1}{2}$, and let*

$$u_t^i = u_0^i + \sum_{j=1}^d \int_0^t a_s^{i,j} dB_s^j + \int_0^t b_s^i ds, \quad t \in [0,T],\, i \in \{1,\dots,m\}, \tag{2.18}$$

where the $a^{i,j}$ are a.s. κ-Hölder continuous for some $\kappa > \frac{2(H\wedge\frac{3}{4})}{3} + \frac{1}{6}$ and the b^j are β-Hölder continuous for some $\beta > H - \frac{1}{2}$. Assume moreover that there exists $\theta > 0$ such that

$$\sum_{j=1}^d \mathbb{E}\left[|b_0^j|^{2+\theta} + \|b^j\|_\beta^{2+\theta} + \sum_{i=1}^m \|a^{i,j}\|_\kappa^{2+\theta} \right] < \infty.$$

Then, with $M^{n,i,j}$ defined by (2.2) and W and Z the matrix-valued processes of Section 2.2.5,

- *if $H \leq \frac{3}{4}$, then, stably in $\mathcal{C}_{\mathbb{R}^{m\times d}}([0,T])$,*

$$\left\{ \nu_H(n) \left(M_\cdot^{n,i,j} - \frac{1}{2} \int_0^\cdot a_s^{i,j} ds \right) \right\}_{i,j} \underset{n\to\infty}{\longrightarrow} \left\{ \sum_{k=1}^d \int_0^\cdot a_s^{i,k} dW_s^{k,j} \right\}_{i,j}.$$

- *if $H > \frac{3}{4}$, then, uniformly on $[0,T]$ in probability,*

$$\left\{ \nu_H(n) \left(M_\cdot^{n,i,j} - \frac{1}{2} \int_0^\cdot a_s^{i,j} ds \right) \right\}_{i,j} \underset{n\to\infty}{\longrightarrow} \left\{ \sum_{k=1}^d \int_0^\cdot a_s^{i,k} dZ_s^{k,j} + \frac{1}{2} \int_0^\cdot b_s^i dB_s^j \right\}_{i,j}.$$

Proof. Set $v_t^i = u_t^i - \int_0^t b_s^i ds = u_0^i + \sum_{j=1}^d \int_0^t a_s^{i,j} dB_s^j$. For any i,j, we have

$$\nu_H(n) \left(M_\cdot^{n,i,j} - \frac{1}{2} \int_0^\cdot a_s^{i,j} ds \right) = A_\cdot^{n,i,j} + C_\cdot^{n,i,j}$$

with, for $t \in [0,T]$,

$$\begin{aligned} A_t^{n,i,j} &= \nu_H(n) \Bigg\{ n^{2H-1} \left(\int_0^t v_s^i dB_s^j - \sum_{k=0}^{nt_n} v_{\frac{k}{n}}^i \left(B_{\frac{k+1}{n}\wedge t}^j - B_{\frac{k}{n}}^j \right) \right) \\ &\qquad - \frac{1}{2} \int_0^t a_s^{i,j} ds \Bigg\}, \\ C_t^{n,i,j} &= \nu_H(n) n^{2H-1} \left(\int_0^t \int_0^s b_r^i dr dB_s^j - \sum_{k=0}^{nt_n} \int_0^{\frac{k}{n}} b_r^i dr \left(B_{\frac{k+1}{n}\wedge t}^j - B_{\frac{k}{n}}^j \right) \right). \end{aligned}$$

We will show that $(v,a) \in \mathbb{C}_2$ and we will deduce from Theorem 2.1.3 the convergence of $(A^{n,i,j})_{i,j}$. Then we will prove that $(C^{n,i,j})_{i,j}$ converges either to 0 in $\mathcal{C}^{m\times d}([0,T])$ (when $H \le \frac{3}{4}$) or uniformly in probability to $\frac{1}{2}\int_0^\cdot b_s dB_s$ (when $H > \frac{3}{4}$). The continuous mapping theorem will then allow to conclude.

We start by showing that $(v,a) \in \mathbb{C}_2$. For $0 \le s \le t \le T$, set

$$R^i_{s,t} = v^i_t - v^i_s - \sum_{j=1}^d a^{i,j}_s (B^j_t - B^j_s) = \sum_{j=1}^d \int_s^t (a^{i,j}_r - a^{i,j}_s) dB^j_r.$$

Using the Young-Loeve inequality (Proposition 2.2.3), we have

$$\begin{aligned} |R^i_{s,t}| &\le |t-s|^{2\kappa} \times c_\kappa \sum_{j=1}^d \|a^{i,j}\|_\kappa \|B^j\|_\kappa, \\ \|v^i\|_\kappa &\le \sum_{j=1}^d \left(\|a^{i,j}\|_\infty \|B^j\|_\kappa + c_{\kappa,\kappa}\|a^{i,j}\|_\kappa \|B^j\|_\kappa T^\kappa\right) \\ &\le \sum_{j=1}^d \left((1+c_{\kappa,\kappa})T^\kappa \|a^{i,j}\|_\kappa + |a^{i,j}_0|\right) \|B^j\|_\kappa, \end{aligned}$$

where the last inequality comes from the fact that $\|a^{i,j}\|_\infty \le |a^{i,j}_0| + T^\kappa \|a^{i,j}\|_\kappa$. Thus, v verifies the condition of Proposition 2.3.2, with $P^{i,j} = a^{i,j}$. We deduce that $(v,a) \in \mathbb{C}_2$, and we can apply Theorem 2.1.3 to (v,a), after observing that v is α-Hölder continuous for all $\alpha = \kappa > \frac{1}{2} > 1-H$. This shows the convergence of $(A^{n,i,j})_{i,j}$.

We now study the convergence of $C^{n,i,j}$. Set $s_n = \lfloor ns \rfloor / n$. We have

$$\begin{aligned} C^{n,i,j}_t &= \nu_H(n) n^{2H-1} \left(\int_0^t \int_0^s b^i_r dr dB^j_s - \int_0^t \int_0^{s_n} b^i_r dr dB^j_s \right) \\ &= \nu_H(n) n^{2H-1} \Bigg(\int_0^t \int_{s_n}^s \left(b^i_r - b^i_{s_n}\right) dr dB^j_s \\ &\qquad\qquad + \sum_{k=0}^{nt_n} b^i_{\frac{k}{n}} \int_{\frac{k}{n}}^{\frac{k+1}{n}\wedge t} (s - s_n) dB^j_s \Bigg) \\ &=: R^{n,i,j}_t + D^{n,i,j}_t. \end{aligned}$$

Lemma 2.4.11 provides the desired convergence for $D^{n,i,j}$. It remains to show that $R^{n,i,j}$ is negligible. We have

$$R^{n,i,j}_t = \nu_H(n) n^{2H-1} \sum_{k=0}^{nt_n} \int_{\frac{k}{n}}^{\frac{k+1}{n}\wedge t} \left(\int_{\frac{k}{n}}^s (b^i_r - b^i_{\frac{k}{n}}) dr \right) dB^j_s.$$

Fix $\varepsilon > 0$ small enough. We can write, using the Young-Loeve inequalities (Proposition 2.2.3) and denoting by c a constant independent of n (whose value can change from line to another)

$$\begin{aligned}
&\left| \int_{\frac{k}{n}}^{\frac{k+1}{n}\wedge t} \left(\int_{\frac{k}{n}}^{s} (b_r^i - b_{\frac{k}{n}}^i) dr \right) dB_s^j \right| \\
\leq\ & c\, n^{-H+\varepsilon} \left\| \int_{\frac{k}{n}}^{\cdot} (b_r^i - b_{\frac{k}{n}}^i) dr \right\|_{\infty, \left[\frac{k}{n}, \frac{k+1}{n}\wedge t\right]} \|B^j\|_{H-\varepsilon} \\
+\ & c\, n^{-1-H+\varepsilon} \left\| \int_{\frac{k}{n}}^{\cdot} (b_r^i - b_{\frac{k}{n}}^i) dr \right\|_{1, \left[\frac{k}{n}, \frac{k+1}{n}\wedge t\right]} \|B^j\|_{H-\varepsilon} \\
\leq\ & c\, n^{-1-H-\beta+\varepsilon} \, \|b^i\|_\beta \, \|B^j\|_{H-\varepsilon}.
\end{aligned}$$

We deduce that

$$\left| R_t^{n,i,j} \right| \leq c\, \nu_H(n) n^{H-1-\beta+\varepsilon} \, \|b^i\|_\beta \, \|B^j\|_{H-\varepsilon},$$

and then $\mathbb{E}\left[\sup_{t\in[0,T]} (R_t^{n,i,j})^2\right] \to 0$ (chosing ε small enough), proving the convergence of this remainder to zero uniformly in probability. This concludes the proof of Proposition 2.3.3. □

We now state a corollary of Proposition 2.3.3, which extends to the case $H > \frac{1}{2}$ a similar statement proved in [18] when $H = \frac{1}{2}$.

Corollary 2.3.4. *Fix $H > \frac{1}{2}$, and let $F : \mathbb{R}^d \to \mathbb{R}^m$ be a $\mathcal{C}^2$-function satisfying the following growth condition: for some $K_1, K_2 > 0$ and some $0 < \gamma < 2$, one has, for all $x \in \mathbb{R}^d$,*

$$\max_{i\in\{1,\ldots,m\}} \max_{j,k\in\{1,\ldots,d\}} \max\left\{ |F^i(x)|, \left| \frac{\partial F^i}{\partial x_j}(x) \right|, \left| \frac{\partial^2 F^i}{\partial x_k \partial x_j} \right| \right\} \leq K_1 e^{K_2 \|x\|_{\mathbb{R}^d}^\gamma}. \tag{2.19}$$

Let $u_t = F(B_t)$. We have, with W and Z the matrix-valued processes of Section 2.2.5:

- *if $H \leq \frac{3}{4}$, then, stably in $\mathcal{C}_{\mathbb{R}^{m\times d}}([0,T])$,*

$$\left\{ \nu_H(n) \left(M_\cdot^{n,i,j} - \frac{1}{2} \int_0^\cdot \frac{\partial F^i}{\partial x_j}(B_s) ds \right) \right\}_{i,j} \underset{n\to\infty}{\longrightarrow} \left\{ \sum_{k=1}^d \int_0^\cdot \frac{\partial F^i}{\partial x_k}(B_s) dW_s^{k,j} \right\}_{i,j} ;$$

- *if $H > \frac{3}{4}$, then, uniformly on $[0,T]$ in probability,*

$$\left\{\nu_H(n)\left(M^{n,i,j}_\cdot - \frac{1}{2}\int_0^\cdot \frac{\partial F^i}{\partial x_j}(B_s)ds\right)\right\}_{i,j} \underset{n\to\infty}{\longrightarrow} \left\{\sum_{k=1}^d \int_0^\cdot \frac{\partial F^i}{\partial x_k}(B_s)dZ^{k,j}_s\right\}_{i,j}.$$

Proof. The change of variable formula for the Young integral leads to

$$u^i_t = F^i(0) + \sum_{j=1}^d \int_0^t \frac{\partial F^i}{\partial x_j}(B_s)dB^j_s, \quad 1 \le i \le m.$$

Then, u is of the type (2.18), with $a^{i,j}_\cdot = \frac{\partial F^i}{\partial x_j}(B_\cdot)$ and $b^i \equiv 0$. The regularity condition (2.19) implies that $a^{i,j}$ is α-Hölder continous for every $\alpha < H$ and that

$$\|a^{i,j}\|_\alpha \le K_1 \prod_{j=1}^d e^{K_2 T^\gamma (\|B^j\|_\alpha)^\gamma} \sum_{k=1}^d \|B^k\|_\alpha.$$

Lemma 2.4.1 then guarantees the existence of moments of any order for this random variable, so that the desired conclusion follows from Proposition 2.3.3. □

2.3.2 Multiple Wiener-Itô integrals

Assume $H > \frac{1}{2}$ and, for simplicity, $d = m = 1$. Let $k \ge 1$ be an integer and let $f_k : [0,T]^{k+1} \to \mathbb{R}$ be measurable and symmetric in the first k variables (this latter condition is of course immaterial when $k = 1$). Assume finally that $f_k(x_1, \ldots, x_k, s) = 0$ if $x_l > s$ for at least one l. In that setting, Theorems 2.1.2 and 2.1.3 apply.

Proposition 2.3.5. *Let the previous notation prevail, as well as the notation from Section 2.2.*

1. *Assume that f_k is α-Hölder continuous on*

$$\mathfrak{D} = \{(x_1, \ldots, x_k, s) \in [0,T]^{k+1}, s \ge \max(x_1, \ldots, x_k)\},$$

for some $\alpha > H$. Set $u_s = \delta^k(f_k(\cdot, s))$. Then, uniformly on $[0,T]$ in probability,

$$M^n_\cdot \underset{n\to\infty}{\longrightarrow} \frac{k}{2}\int_0^\cdot \delta^{k-1}(f_k(., s, s))ds.$$

2. *Assume $\frac{1}{2} < H \leq \frac{3}{4}$. Assume moreover that the hypothesis of the previous point holds, and that in addition*

$$f_k(x_1, \ldots, x_k, s) = g_k(x_1, \ldots, x_k)\mathbb{I}_{[0,s]^k}(x_1, \ldots, x_k)$$

with g_k symmetric and β-Hölder continuous for some $\beta > \frac{1}{2}$. Then, stably in $\mathcal{C}_{\mathbb{R}}([0,T])$ and with W an independent standard Brownian motion,

$$\nu_H(n)\left(M^n_{\cdot} - \frac{k}{2}\int_0^{\cdot} \delta^{k-1}(f_k(\cdot, s, s))ds\right)$$
$$\underset{n\to\infty}{\longrightarrow} \quad H(2H-1)\sqrt{q_H + r_H}\int_0^{\cdot} \delta^{k-1}(f_k(\cdot, s, s))dW_s,$$

where q_H and r_H as defined in Section 2.2.5.

Remark 2.3.6. Before making the proof of Proposition 2.3.5, let us stress that (u, P) with $u_s = \delta^k(f_k(\cdot, s))$ and $P_s = D_s u_s$ does not *a priori* belong to $\mathcal{D}^{2\kappa}$ for some $\kappa > \frac{1}{2}$, and therefore we cannot directly apply the results of Section 2.3.1. Indeed, assuming $g_k = 1$, i.e $f_k(x_1, \ldots, x_{k+1}) = \mathbb{I}_{[0,x_{k+1}]^k}(x_1, \ldots, x_k)$ we can write

$$C_{s,t} = \delta^{k-2}\left(\mathbb{I}_{[0,s]^{k-2}}(\cdot)\int_0^s\int_s^t |l-r|^{2H-2}dldr\right) = r^H(0,s,s,t)\delta^{k-2}(\mathbb{I}_{[0,s]^{k-2}}).$$

Since $r^H(0,s,s,t) > s|t-s|$ thanks to Lemma 2.4.2, we have $\left|\frac{C_{s,t}}{|t-s|^{2\kappa}}\right| \geq \frac{s|\delta^{k-2}(\mathbb{I}_{[0,s]^{k-2}})|}{|t-s|^{2\kappa-1}}$ for any $\kappa > \frac{1}{2}$. We have $\delta^{k-2}(\mathbb{I}_{[0,s]^{k-2}}) = H_{k-2}(B_s)$ (with H_k the k-th Hermite polynomial). Since B as a Gaussian law, there is a real number $l > 0$ and a set $\Omega_0 \subset \Omega$ such that $\mathbb{P}(\Omega_0) > 0$ and $\forall \omega \in \Omega_0, |\delta^{k-2}(\mathbb{I}_{[0,s]^{k-2}})(\omega)| > l$. Then $\left|\frac{C_{s,t}(\omega)}{|t-s|^{2\kappa}}\right| \underset{s\to t}{\longrightarrow} +\infty$ for all fixed $s > 0$ and $\omega \in \Omega_0$.

Proof of Proposition 2.3.5. We only do the proof of point (2), since the proof of point (1) (which requires to show that (u, P) with $P_s := D_s u_s$ verifies the assumptions of Theorem 2.1.2) is very similar and easier. Before going into the details, let us explain the main steps we are going to follow:

- in the first step, we show that u and P are β'-Hölder continuous for some $\beta' > \frac{1}{2} > 1 - H$;

- in the second step, we provide a suitable decomposition of $L_{s,t}L_{x,y}$. We recall that $L_{s,t}$ is defined as

$$L_{s,t} = \int_s^t (u_l - u_s - D_s u_s (B_l - B_s))\, dB_l; \qquad (2.20)$$

- finally, in the remaining steps, we analyze each term of the previous decomposition and show that the stuctural condition (2.3) is verified, i.e, for all $0 \le s \le t \le T$ and all $0 \le x \le y \le T$,

$$\mathbb{E}\left[L_{s,t}L_{x,y}\right] = o_{|s-t|+|x-y|\to 0}(f_2(s,t,x,y)) \quad \text{uniformly in } s,t \in [0,T]. \qquad (2.21)$$

<u>*Step 1: Hölder continuity*</u>. The process u is adapted with respect to B and belongs to $\mathbb{D}^{1,2}(|\mathcal{H}|)$ with $D_s u_t = k\delta^{k-1}\left(f_k(.,s,t)\right)\mathbb{I}_{s\le t}$ by Propostion 2.2.2. Using the hypercontractivity and isometry properties (again Proposition 2.2.2), we obtain, for $a > 1$ and $s \le t$,

$$\begin{aligned}
\mathbb{E}[|u_s - u_t|^a] &\le C_{k,a}\mathbb{E}[(u_s - u_t)^2]^{\frac{a}{2}} \\
&= C_{k,a}\|f_k(\cdot,s) - f_k(\cdot,t)\|^a_{\mathcal{H}^{\otimes k}} \\
&\le C_{k,a}\|f_k(\cdot,s) - f_k(\cdot,t)\|^a_{|\mathcal{H}|^{\otimes k}},
\end{aligned}$$

thanks to the continuous embedding $|\mathcal{H}|^{\otimes k} \subset \mathcal{H}^{\otimes k}$ in the last line.

Let $\Delta_{s,t}f_k(\cdot) = f_k(\cdot,t) - f_k(\cdot,s)$. We have

$$\begin{aligned}
&\|f_k(\cdot,s) - f_k(\cdot,t)\|^2_{|\mathcal{H}|^{\otimes k}} \\
\le\ & \int_{[0,t]^{2k}} |\Delta_{s,t}f_k(x)||\Delta_{s,t}f_k(y)| \prod_{m=1}^k \mu_H(dx_m dy_m) \\
\le\ & \sum_{i,j=1}^k \int_{[0,t]^{2k}} \mathbb{I}_{[s,t]}(x_i)\mathbb{I}_{[s,t]}(y_j)|g_k(x)||g_k(y)| \prod_{m=1}^k \mu_H(dx_m dy_m) \\
=\ & \sum_{\substack{i,j=1\\ i\ne j}}^k \int_{[0,t]^{2k}} \mathbb{I}_{[s,t]}(x_i)\mathbb{I}_{[s,t]}(y_j)|g_k(x)||g_k(y)| \\
& \times \left(\prod_{\substack{m=1\\ m\ne i,j}}^k \mu_H(dx_m dy_m)\right)\mu_H(dx_i dy_i)\mu_H(dx_j dy_j) \\
& + \sum_{i=1}^k \int_{[0,t]^{2k}} \mathbb{I}_{[s,t]}(x_i)\mathbb{I}_{[s,t]}(y_i)|g_k(x)||g_k(y)| \left(\prod_{\substack{m=1\\ m\ne i}}^k \mu_H(dx_m dy_m)\right)\mu_H(dx_i dy_i).
\end{aligned}$$

From Lemma 2.4.2, we have

$$\int_{[0,t]\times[s,t]} \mu_H(dxdy) \leq K|t-s|$$

for some constant K. Note that we take the liberty to change the value of K from line to line in the rest of the proof. We deduce, for $i \neq j$, that

$$\begin{aligned} & \int_{[0,t]^{2k}} \mathbb{I}_{[s,t]}(x_i)\mathbb{I}_{[s,t]}(y_j)|g_k(x)||g_k(y)| \prod_{\substack{m=1 \\ m\neq i,j}}^{k} \mu_H(dx_m dy_m) \\ \leq \quad & \|g_k\|_\infty^2 |t-s|^2 \int_{[0,t]^{2k-2}} \prod_{\substack{m=1 \\ m\neq i,j}}^{k} \mu_H(dx_m dy_m). \end{aligned}$$

As a result,

$$\|f_k(\cdot,s)-f_k(\cdot,t)\|^2_{|\mathcal{H}|^{\otimes k}} \leq K\|g_k\|^2_\infty \big((k-1)^2|t-s|^2 t^{2H(k-2)} + k|t-s|^{2H} t^{2H(k-1)}\big).$$

Since $|t-s| \leq K|t-s|^H$ on $[0,T]^2$, this leads to

$$\mathbb{E}[|u_s - u_t|^a] \leq K\|g_k\|^a_\infty |t-s|^{Ha}.$$

We can show a similar bound for the derivative Du:

$$\begin{aligned} & \mathbb{E}[|D_s u_s - D_t u_t|^a] \leq \mathbb{E}[|D_s u_s - D_s u_t|^a] + \mathbb{E}[|D_s u_t - D_t u_t|^a] \\ \leq \quad & C_{k-1,a}\left(\|f_k(\cdot,s,s) - f_k(\cdot,s,t)\|^a_{|\mathcal{H}|^{\otimes k-1}} + \|f_k(\cdot,s,t) - f_k(\cdot,t,t)\|^a_{|\mathcal{H}|^{\otimes k-1}}\right) \end{aligned}$$

and

$$\begin{aligned} & \|f_k(\cdot,s,s) - f_k(\cdot,s,t)\|^2_{|\mathcal{H}|^{\otimes k-1}} + \|f_k(\cdot,s,t) - f_k(\cdot,t,t)\|^2_{|\mathcal{H}|^{\otimes k-1}} \\ \leq \quad & K\|g_k\|^2_\infty (k|t-s|^{2H} t^{2H(k-1)} + (k-1)^2 t^{2H(k-2)}|t-s|^2) \\ & + t^{2Hk}\|g_k\|^2_\beta |t-s|^{2\beta}, \end{aligned}$$

where $\|g_k\|_\beta$ is the Hölder seminorm of g_k over $[0,T]^k$. Then,

$$\mathbb{E}[|D_s u_s - D_t u_t|^a] \leq K(\|g_k\|^a_\infty |t-s|^{Ha} + \|g_k\|^a_\beta |t-s|^{\beta a}).$$

Finally, for all $a > 1$ we have

$$\begin{aligned} \mathbb{E}[|u_s - u_t|^a + |D_s u_s - D_t u_t|^a] \quad & \leq \quad C\left(|t-s|^{aH} + |t-s|^{a\beta}\right) \\ & = \quad C\left(|t-s|^{a'+1} + |t-s|^{a''+1}\right), \end{aligned}$$

with $a' = aH - 1, a'' = a\beta - 1$ and the constant C depending on $k, a, \|g_k\|_\infty$ and T.
Observe that $\frac{a'}{a} \to H$ and $\frac{a''}{a} \to \beta$ when $a \to \infty$. The Kolmogorov-Censov criterion applies and yields that u and $s \to D_s u_s$ verifies the Hölder semi-norm condition in Theorem 2.1.3, namely: u and P are β'-Hölder continuous for all β' such that $\beta \wedge H > \beta' > \frac{1}{2} > 1 - H$.

Step 2: Decomposition of $L_{s,t} L_{x,y}$ (recall the definition of L from (2.20)). The product formula (3.14) yields, for $s \leq t$,

$$\begin{aligned} D_s u_s (B_t - B_s) &= k\delta^{k-1}\left(f_k(\cdot, s, s)\right)\delta(\mathbb{I}_{[s,t]}) \\ &= k\delta^k\left(f_k(\cdot, s, \widetilde{s) \otimes \mathbb{I}_{[s,t]}}(\cdot\cdot)\right) \\ &+ k(k-1)\delta^{k-2}\left(f_k(\cdot, s, s) \otimes_1 \mathbb{I}_{[s,t]}\right). \end{aligned}$$

Then, $u_t - u_s - D_s u_s(B_t - B_s) = A_{s,t} - C_{s,t}$, with

$$\begin{cases} A_{s,t} = \delta^k\left(f_k(\cdot, t) - f_k(\cdot, s) - k f_k(\cdot, s, \widetilde{s) \otimes \mathbb{I}_{[s,t]}}(\cdot\cdot)\right) \\ C_{s,t} = k(k-1)\delta^{k-2}\left(f_k(\cdot, s, s) \otimes_1 \mathbb{I}_{[s,t]}\right). \end{cases} \tag{2.22}$$

Notice that $C_{s,t} = 0$ when $k = 1$. We also use the convention that $A_{s,t} = C_{s,t} = 0$ if $s > t$.

To prove that $(u, P) \in \mathbb{C}_2$ we will proceed as follows.

The hypothesis of Proposition 2.2.4 are verified by A and C. Indeed, $A_{s,\cdot}, C_{s,\cdot} \in \mathbb{D}^{1,2}(|\mathcal{H}|)$ for all $s \in [0, T]$. Moreover, using the same arguments as in Step 1, one can show that $DA_{s,\cdot}$ and $DC_{s,\cdot}$ have almost continuous paths in $[0, T]^2$, implying in turn that

$$\int_0^T \int_0^T (|D_w A_{s,l}| + |D_w C_{s,l}|)\mu_H(dldw) < \infty \quad \text{a.s.}$$

for all $s \in [0, T]$.

Formula (2.13) allows to write

$$\int_s^t C_{s,l} dB_l = \delta\left(C_{s,\cdot} \times \mathbb{I}_{[s,t]}(\cdot)\right) + \int_s^t \int_0^t D_w C_{s,l}\, \mu_H(dwdl) \tag{2.23}$$

as well as

$$\int_s^t A_{s,l} dB_l = \delta\left(A_{s,\cdot} \times \mathbb{I}_{[s,t]}(\cdot)\right) + \int_s^t \int_0^t D_w A_{s,l}\, \mu_H(dwdl). \tag{2.24}$$

For any $0 \le s \le t \le T$ and $0 \le x \le y \le T$, we can then write $L_{s,t}L_{x,y} = \sum_{i,j=1}^{4} R^{i,j}(s,t,x,y)$, with

$$
\begin{aligned}
R^{1,1}(s,t,x,y) &= \delta\left(C_{s,\cdot}\mathbb{I}_{[s,t]}(\cdot)\right)\delta\left(C_{x,\cdot}\mathbb{I}_{[x,y]}(\cdot)\right) \\
R^{1,2}(s,t,x,y) &= \int_s^t\int_0^t \mu_H(dldw)\int_x^y\int_0^y \mu_H(drdz)D_wC_{s,l}D_zC_{x,r} \\
R^{1,3}(s,t,x,y) &= R^{1,4}(x,y,s,t) \\
&= \delta\left(C_{s,\cdot}\mathbb{I}_{[s,t]}(\cdot)\right)\int_x^y\int_0^y D_wC_{x,l}\mu_H(dwdl) \\
R^{2,1}(s,t,x,y) &= \delta\left(A_{s,\cdot}\mathbb{I}_{[s,t]}(\cdot)\right)\delta\left(A_{x,\cdot}\mathbb{I}_{[x,y]}(\cdot)\right) \\
R^{2,2}(s,t,x,y) &= \int_s^t\int_0^t \mu_H(dldw)\int_x^y\int_0^y \mu_H(drdz)D_wA_{s,l}D_zA_{x,r} \\
R^{2,3}(s,t,x,y) &= R^{2,4}(x,y,s,t) \\
&= \delta\left(A_{s,\cdot}\mathbb{I}_{[s,t]}(\cdot)\right)\int_x^y\int_0^y D_wA_{x,l}\mu_H(dwdl) \\
R^{3,1}(s,t,x,y) &= R^{4,1}(x,y,s,t) \\
&= \delta\left(A_{s,\cdot}\mathbb{I}_{[s,t]}(\cdot)\right)\delta\left(C_{x,\cdot}\mathbb{I}_{[x,y]}(\cdot)\right) \\
R^{3,2}(s,t,x,y) &= R^{4,2}(x,y,s,t) \\
&= \delta\left(A_{s,\cdot}\mathbb{I}_{[s,t]}(\cdot)\right)\int_x^y\int_0^y D_wC_{x,l}\mu_H(dwdl) \\
R^{3,3}(s,t,x,y) &= R^{4,3}(x,y,s,t) \\
&= \delta\left(C_{s,\cdot}\mathbb{I}_{[s,t]}(\cdot)\right)\int_x^y\int_0^y D_wA_{x,l}\mu_H(dwdl) \\
R^{3,4}(s,t,x,y) &= R^{4,4}(x,y,s,t) \\
&= \int_s^t\int_0^t \mu_H(dldw)\int_x^y\int_0^y \mu_H(drdz)D_wC_{s,l}D_zA_{x,r}.
\end{aligned}
$$

We can easily check that

$$
\mathbb{E}\left[R^{1,3}\right] = \mathbb{E}\left[R^{2,3}\right] = \mathbb{E}\left[R^{3,1}\right] = \mathbb{E}\left[R^{3,2}\right] = \mathbb{E}\left[R^{3,4}\right] = 0.
$$

Indeed, these expectations reduce to a sum of expectations of products of two multiple Wiener integrals of different orders, which are orthogonal in $L^2(\Omega)$ by Proposition 2.2.2. More precisely, Lemma 2.4.3 allows to show that all the expectations in play vanish. For example,

$$
\mathbb{E}[R^{1,3}] = \int_x^y\int_0^y \mathbb{E}[\delta(C_{s,\cdot}\mathbb{I}_{[s,t]}(\cdot))D_wC_{x,l}]\mu_H(dwdl),
$$

which corresponds exactly to a term of the form (2.32).

We will now apply Proposition 2.2.4, together with several inequalities, to show that all the remaining terms satisfy the condition (2.3), namely

$$\mathbb{E}[R^{i,j}(s,t,x,y)] = o_{|t-s|+|x-y|\to 0}(f_2(s,t,x,y))$$

for all $(i,j) \in \{(1,1),(1,2),(2,1),(2,2),(3,3),(4,3)\}$ and uniformly in $[0,T]^2$. (Starting from now, note that every time we write $o_{|t-s|+|x-y|\to 0}(f_2(s,t,x,y))$, it is implicitely assumed that it takes place uniformly in $s,t \in [0,T]$.)

Whatever the value of $(i,j) \in \{(1,1),(1,2),(2,1),(2,2),(3,3),(4,3)\}$, deriving a bound for $\mathbb{E}[R^{i,j}]$ requires similar arguments. For this reason, in what follows we will fully develop the cases $(i,j)=(1,1)$, $(i,j)=(1,2)$ and $(i,j)=(2,1)$, then we will only explain the differences for the remaining cases.

For notational simplicity, we will also write $R^{i,j}$ instead of $R^{i,j}(s,t,x,y)$.

Step 3: Bound on $\mathbb{E}[R^{1,2}]$. First, we give an upper bound for $\mathbb{E}[(D_wC_{s,l})^2]$: for all $w \in [0,t]$ and $l \in [s,t]$,

$$\begin{aligned}
\mathbb{E}\left[(D_wC_{s,l})^2\right] &= k^2(k-1)^2(k-2)^2\,\mathbb{E}\left[\left(\delta^{k-3}\left(f_k(\cdot,w,s,s)\otimes_1 \mathbb{I}_{[s,l]}\right)\right)^2\right] \\
&= k!k(k-1)(k-2)\left\|f_k(\cdot,w,s,s)\otimes_1 \mathbb{I}_{[s,l]}\right\|^2_{\mathcal{H}^{\otimes k-3}} \\
&\leq k!k(k-1)(k-2)\|g_k\|^2_\infty \left\|\mathbb{I}_{[0,T]^{k-2}}\otimes_1 \mathbb{I}_{[s,l]}\right\|^2_{|\mathcal{H}|^{\otimes k-3}},
\end{aligned}$$

where, in the last inequality, we have used that $|h_1\otimes_1 h_2| \leq |h_1|\otimes_1|h_2|$ for all $h_1,h_2 \in \mathbb{D}^{1,2}(|\mathcal{H}|)$. Moreover, according to Lemma 2.4.2,

$$\left|\mathbb{I}_{[0,T]^{k-2}}\otimes_1 \mathbb{I}_{[s,l]}\right| = \left|\mathbb{E}[B_T(B_s-B_l)]\mathbb{I}_{[0,T]^{k-3}}\right| \leq K|s-l|\mathbb{I}_{[0,T]^{k-3}}.$$

Plugging this identity into (2.25) leads to

$$\mathbb{E}\left[(D_wC_{s,l})^2\right] \leq K|s-l|^2.$$

As a result, and using the Hölder inequality, we have, for all $s\leq t$ and $x \leq y$,

$$\begin{aligned}
&\left|\mathbb{E}\left[\int_s^t\int_0^t D_wC_{s,l}\mu_H(dwdl)\int_x^y\int_0^y D_wC_{x,l}\mu_H(dwdl)\right]\right| \\
\leq\ & \int_s^t\int_0^t\int_x^y\int_0^y \mathbb{E}\left[(D_wC_{s,l})^2\right]^{\frac{1}{2}}\mathbb{E}\left[(D_zC_{x,r})^2\right]^{\frac{1}{2}}\mu_H(dzdr)\mu_H(dwdl) \\
\leq\ & K|t-s|^2|x-y|^2 = o_{|t-s|+|x-y|\to 0}(f_2(s,t,x,y))
\end{aligned}$$

where, in the last identity, we made use of the following two facts: on one hand $|t-s||x-y| \leq r_H(s,t,x,y)$ according to Lemma 2.4.2; on the other hand, and since $H \leq \frac{3}{4}$,

$$|t-s||x-y| = o_{|t-s|+|x-y|\to 0}\big(|t-s|^{2H-1}|x-y|^{2H-1}\kappa_H(|x-y|)\kappa_H(|t-s|)\big).$$

Step 4: Bound on $\mathbb{E}[R^{1,1}]$. This term can be handled similarly, with the help of Proposition 2.2.1:

$$\begin{aligned}
&\left|\mathbb{E}\left[\delta(C_{s,\cdot}\mathbb{I}_{[s,t]}(\cdot))\delta(C_{x,\cdot}\mathbb{I}_{[x,y]}(\cdot))\right]\right| \\
\leq\ & \int_{[s,t]\times[x,y]} \mathbb{I}_{[s,t]}(l)\mathbb{I}_{[x,y]}(r)|\mathbb{E}[C_{s,l}C_{x,r}]|\mu_H(drdl) \\
& + \int_{[0,t]\times[x,y]}\int_{[s,t]\times[0,y]} |\mathbb{E}[D_w C_{s,l} D_z C_{x,r}]|\mu_H(dzdl)\mu_H(drdw) \\
\leq\ & K\|g_k\|_\infty^2 |t-s||x-y|(|t-s||x-y| + r_H(s,t,x,y)),
\end{aligned}$$

where $\mathbb{E}[C_{s,l}C_{x,r}]$ and $\mathbb{E}[D_u C_{s,l} D_v C_{x,r}]$ are computed by means of Proposition 2.2.2. Again,

$$|t-s||x-y|(|t-s||x-y| + r_H(s,t,x,y)) = o_{|t-s|+|x-y|\to 0}(f_2(s,t,x,y)).$$

Step 5: Bound on $\mathbb{E}[R^{2,1}]$. Using Proposition 2.2.1, we can write

$$\begin{aligned}
&|\mathbb{E}[R^{2,1}]| = \left|\mathbb{E}\left[\delta(A_{s,\cdot}\mathbb{I}_{[s,t]}(\cdot))\delta(A_{x,\cdot}\mathbb{I}_{[x,y]}(\cdot))\right]\right| \\
\leq\ & \int_{[s,t]\times[x,y]} \mu_H(dldr)\mathbb{I}_{[s,t]}(l)\mathbb{I}_{[x,y]}(r)\left|\mathbb{E}[A_{s,l}A_{x,r}]\right| \\
+\ & \int_{[0,t]\times[x,y]} \mu_H(dwdr)\int_{[s,t]\times[0,y]} \mu_H(dldz)\left|\mathbb{E}[D_w A_{s,l} D_z A_{x,r}]\right|.
\end{aligned}$$

Let us define the following function:

$$\begin{aligned}
& h_k^s(x_1,\dots,x_k,l) \\
:=\ & \sum_{i=1}^k \mathbb{I}_{[0,s]^{k-1}}(x_1,\dots,x_{i-1},x_{i+1},\dots,x_k)\mathbb{I}_{[s,l]}(x_i) \\
& \times (g_k(x_1,\dots,x_k) - g_k(x_1,\dots,x_{i-1},x_{i+1},\dots,x_k,s)) \\
& + g_k(x_1,\dots,x_k)\sum_{i=1}^k \mathbb{I}_{[s,l]}(x_i)\mathbb{I}_{[0,l]^{k-1}\setminus[0,s]^{k-1}}(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n).
\end{aligned}$$

Since $s \leq l$, we have:

$$\begin{aligned} & f_k(x_1, \dots, x_k, l) - f_k(x_1, \dots, x_k, s) \\ = \; & g_k(x_1, \dots, x_k) \sum_{i=1}^{k} \mathbb{I}_{[s,l]}(x_i) \mathbb{I}_{[0,s]^{k-1}}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k) \\ & + g_k(x_1, \dots, x_k) \sum_{i=1}^{k} \mathbb{I}_{[s,l]}(x_i) \mathbb{I}_{[0,l]^{k-1} \setminus [0,s]^{k-1}}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k) \end{aligned}$$

and

$$\begin{aligned} & k f_k(\cdot, \widetilde{s, s) \otimes} \mathbb{I}_{[s,l]}(x_1, \dots, x_k) \\ = \; & \sum_{i=1}^{k} \mathbb{I}_{[s,l]}(x_i) \mathbb{I}_{[0,s]^{k-1}} g_k(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k, s). \end{aligned}$$

We obtain, for all $x_1, \dots, x_k \in [0, T]$, that

$$\begin{aligned} & f_k(x_1, \dots, x_k, l) - f_k(x_1, \dots, x_k, s) - k f_k(\cdot, \widetilde{s, s) \otimes} \mathbb{I}_{[s,l]}(x_1, \dots, x_k) \\ = \; & h_k^s(x_1, \dots, x_k, l). \end{aligned}$$

Then, $A_{s,l} = \delta^k(h_k^s(\cdot, l))$, $A_{x,r} = \delta^k(h_k^x(\cdot, r))$ and, by Proposition 2.2.2 (isometry),

$$\begin{aligned} & \int_{[s,t] \times [x,y]} \mathbb{I}_{[s,t]}(l) \mathbb{I}_{[x,y]}(r) \left| \mathbb{E}[A_{s,l} A_{x,r}] \right| |l - r|^{2H-2} dr dl \\ \leq \; & k! \int_s^t \int_x^y \mu_H(dl dr) \int_{[0,t]^k \times [0,y]^k} \prod_{i=1}^{k} \mu_H(dx_i dy_i) \\ & \times |h_k^s(x_1, \dots, x_k, l)| \, |h_k^x(y_1, \dots, y_k, r)| \, . \end{aligned}$$

On the other hand, observe the following facts:

- $h_k^s(x_1, \dots, x_k) = 0$ if $(x_1, \dots, x_k) \in [0, s]^k$;
- if there is a unique index i such that $x_i \in [s, l]$, then

$$\begin{aligned} & |h_k^s(x_1, \dots, x_k)| = |g_k(x_1, \dots, x_k) - g_k(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k, s)| \\ \leq \; & \|g_k\|_\beta |s - l|^\beta; \end{aligned}$$

- if there is more than one index i such that $x_i \in [s,l]$, then

$$|h_k^s(x_1,\dots,x_k,l)| \leq \|g_k\|_\infty \mathbb{I}_{[0,l]^k}(x_1,\dots,x_k) \sum_{i\neq j=1}^{k} \mathbb{I}_{[s,l]^2}(x_i,x_j).$$

As a result,

$$\begin{aligned}
& |h_k^s(x_1,\dots,x_k,l)| \\
\leq\ & \sum_{i=1}^{k-1} \mathbb{I}_{[0,s]^k}(x_1,\dots,x_{i-1},x_{i+1},\dots,x_k)\mathbb{I}_{[s,l]}(x_i)|x_i-s|^\beta \|g_k\|_\beta \\
& +\|g_k\|_\infty \sum_{i\neq j=1}^{k} \left|g_k(x_1\dots,x_k)\right|\mathbb{I}_{[s,l]^2}(x_i,x_j)\mathbb{I}_{[0,l]^k}(x_1\dots,x_k).
\end{aligned}$$

We then have

$$\begin{aligned}
& \int_s^t\int_x^y \mu_H(drdl)\int_{[0,l]^k\times[0,v]^k}\prod_{i=1}^k \mu_H(dx_idy_i)|h_k^s(x_1,\dots,x_k,l) \\
& \times|\,|h_k^x(y_1,\dots,y_k,r)| \\
\leq\ & (A+B+C+D)r_H(s,t,x,y),
\end{aligned}$$

with

$$\begin{aligned}
A =\ & \|g_k\|_\beta^2|t-s|^\beta|x-y|^\beta \sum_{i,j=1}^k \int_{[0,t]^k\times[0,y]^k}\prod_{m=1}^k \mu_H(dx_mdy_m) \\
& \times\mathbb{I}_{[s,t]}(x_i)\mathbb{I}_{[x,y]}(y_j) \\
B =\ & \|g_k\|_\infty^2 \sum_{i_i\neq i_2,j_1\neq j_2=1}^k \int_{[0,t]^k\times[0,y]^k}\prod_{m=1}^k \mu_H(dx_mdy_m) \\
& \times\mathbb{I}_{[s,t]^2}(x_{i_1},x_{i_2})\mathbb{I}_{[x,y]^2}(y_{j_1},y_{j_2}) \\
C =\ & |x-y|^\beta\|g_k\|_\beta\|g_k\|_\infty \sum_{i_i\neq i_2,j=1}^k \int_{[0,t]^k\times[0,y]^k}\prod_{m=1}^k \mu_H(dx_mdy_m) \\
& \times\mathbb{I}_{[s,t]^2}(x_{i_1},x_{i_2})\mathbb{I}_{[x,y]}(y_j) \\
D =\ & |t-s|^\beta\|g_k\|_\beta\|g_k\|_\infty \sum_{i,j_1\neq j_2=1}^k \int_{[0,t]^k\times[0,y]^k}\prod_{m=1}^k \mu_H(dx_mdy_m) \\
& \times\mathbb{I}_{[s,t]}(x_i)\mathbb{I}_{[x,y]^2}(y_{j_1},y_{j_2}).
\end{aligned}$$

We only write down the details for the upper bound of A, since the technique is similar for the three other terms.

Two cases should then be analyzed to handle the integral A:

- $i \neq j$:

$$\begin{aligned}&\int_{[0,t]^k\times[0,y]^k} \mathbb{I}_{[s,t]}(x_i)\mathbb{I}_{[x,y]}(y_j)\prod_{m=1}^{k}\mu_H(dx_m dy_m)\\ =\ & \mathbb{E}[B_tB_y]^{k-2}\mathbb{E}[B_y(B_t-B_s)]\mathbb{E}[B_t(B_y-B_x)] \leq K_T^2T^{2H(k-2)}|t-s||x-y|,\end{aligned}$$

where the last inequality follows from Lemma 2.4.2.

- $i = j$:

$$\begin{aligned}&\int_{[0,t]^k\times[0,y]^k} \mathbb{I}_{[s,t]}(x_i)\mathbb{I}_{[x,y]}(y_j)\prod_{m=1}^{k}\mu_H(dx_m dy_m)\\ =\ & \mathbb{E}[B_tB_y]^{k-1}\mathbb{E}[(B_x-B_y)(B_t-B_s)]\\ \leq\ & T^{2H(k-1)}r_H(s,t,x,y) \leq T^{2H(k-1)}|t-s|^H|x-y|^H,\end{aligned}$$

where the last inequality comes from Lemma 2.4.2. We then have

$$A \leq K(|t-s||x-y| + |t-s|^H|x-y|^H)|t-s|^\beta|x-y|^\beta.$$

Similar arguments for handling the integrals B, C, D lead to

$$\begin{aligned}B &\leq K(|t-s|^{2H}|x-y|^{2H} + |t-s|^H|x-y|^H|t-s||x-y| + |t-s|^2|x-y|^2)\\ C &\leq K|x-y|^\beta(|x-y|^H|t-s|^{1+H} + |x-y||t-s|^2)\\ D &\leq K|t-s|^\beta(|t-s|^H|x-y|^{1+H} + |t-s||x-y|^2).\end{aligned}$$

Since $\beta, H > \frac{1}{2}$, we have

$$\begin{aligned}&\left|\int_{[s,t]\times[x,y]} \mathbb{I}_{[s,t]}(l)\mathbb{I}_{[x,y]}(r)|\mathbb{E}[A_{s,l}A_{x,r}]|\mu_H(drdl)\right|\\ \leq\ & r_H(s,t,x,y)(A+B+C+D) = o_{|t-s|+|x-y|\to 0}(f_2(s,t,x,y)).\end{aligned}$$

We have $D_wA_{s,l} = \delta^{k-1}(h_k^s(x_1,\ldots,x_{k-1},u,l))$. Similar computations allow to treat the trace term:

$$\begin{aligned}&\int_{[0,t]\times[x,y]}\mu_H(dwdr)\int_{[s,t]\times[0,y]}\mu_H(dldz)\left|\mathbb{E}[D_wA_{s,l}D_zA_{x,r}]\right|\\ =\ & o_{|t-s|+|x-y|\to 0}(f_2(s,t,x,y)).\end{aligned}$$

Putting all these facts together, we obtain

$$\mathbb{E}[R^{2,1}] = o_{|t-s|+|x-y|\to 0}(f_2(s,t,x,y)).$$

<u>*Step 6: Bound on* $\mathbb{E}[R^{2,2}+R^{3,3}+R^{4,3}]$</u>. We use similar arguments here as in Step 5: we can obtain trough easy but tedious computations, and distinguishing again several cases,

$$\mathbb{E}\left[R^{2,2}+R^{3,3}+R^{4,3}\right] = o_{|t-s|,|x-y|\to 0}(f_2(s,t,x,y)).$$

<u>*Step 7: Conclusion*</u>. We have shown that

$$\mathbb{E}\left[L_{s,t}L_{x,y}\right] = o_{|t-s|,|x-y|\to 0}(f_2(s,t,x,y))$$

implying that $(u,P)\in\mathbb{C}_2$. □

2.3.3 Examples in the Brownian motion case

Since this section only concerns the standard Brownian motion case, in the following $H=\frac{1}{2}$. We give below a criterion which generalizes the examples developped in [32].

In Proposition 2.3.3, we considered fractional semimartingales of the form (2.18). Here, we take advantage of the standard Brownian framework, to consider processes of the form (2.25). Note that the integrand $V^{i,j}_{s,t}$ is allowed to depend on t in (2.25), making useless to consider a drift term as in (2.18).

Let $\big((u^i_t)_{t\in[0,T]}\big)_{1\le i\le m}$ be a collection of square integrable and progressively measurable processes, i.e. $\mathbb{E}\big[(u^i_t)^2\big]<\infty$ for all i and t. According to the representation theorem for square integrable random variables, for all i and t there exists progressively measurable processes $\big((V^{i,j}_{s,t})_{0\le s\le t}\big)_{1\le j\le d}$ such that, for all i and t:

$$u^i_t = \mathbb{E}[u^i_t] + \sum_{j=1}^{d}\int_0^t V^{i,j}_{s,t}\,dB^j_s \quad \text{a.s.}, \tag{2.25}$$

and $\mathbb{E}\big[\int_0^t (V^{i,j}_{s,t})^2 ds\big]<\infty$. We assume moreover:

($\mathfrak{H}_1$) $(V^{i,j}_{s,t})_{0\le s\le t\le T}$ is measurable for all i and j, and (i) $(s,t)\mapsto V_{s,t}$ has a progressively measurable version, (ii) $\mathbb{E}[|V^{i,j}_{s,s}-V^{i,j}_{s,t}|^2]+\mathbb{E}[|V^{i,j}_{s,s}-V^{i,j}_{t,t}|^2]\underset{s\to t-}{\longrightarrow} 0$ for all i,j uniformly in $s\le t\in[0,T]$ and (iii) $(V^{i,j}_{s,s})_{s\in[0,T]}$ is piecewise continuous.

($\mathfrak{H}_2$) For all i, j, the family $\left(|V_{s,t}^{i,j}|\right)_{s,t\in[0,T]}$ is bounded by a square integrable random variable S such that $\mathbb{E}[S^{2+\gamma}] < \infty$ for some $\gamma > 0$.

($\mathfrak{H}_3$) One has, for all $0 \le s \le t \le T$ and all $i \le m$ and $j \le d$

$$\mathbb{E}\left[\int_0^s (V_{l,s}^{i,j} - V_{l,t}^{i,j})^2 dl\right] + \left(\mathbb{E}[u_s^i - u_t^i]\right)^2 \le |s-t|\mu(s,t),$$

where μ is a bounded function which is continuous on $[0,T]^2$ and such that $\mu(s,s) = 0$ for all $s \in [0,T]$.

As an application of Theorem 2.1.3 (with $P_s = V_{s,s}$), we can state the following proposition.

Proposition 2.3.7. *Assume* ($\mathfrak{H}_1$) $-$ ($\mathfrak{H}_3$) *and recall that* $H = \frac{1}{2}$*. Then, stably in* $\mathcal{C}_{\mathbb{R}^{m\times d}}([0,T])$*,*

$$\{\sqrt{n}M_\cdot^{n,(i,j)}\}_{1\le i\le m, 1\le j\le d} \underset{n\to\infty}{\longrightarrow} \left\{\sum_{k=1}^d \int_0^\cdot V_{s,s}^{i,k} dW^{k,j}(s)\right\}_{1\le i\le m, 1\le j\le d},$$

where W *is the independent matrix-valued Brownian motion of Section 2.2.5.*

Proof. To simplify, without loss of generality we assume that $m = 1$. We then write $P^j = P^{1,j}$, $V^j = V^{1,j}$ and $L^j = L^{1,j}$ for all $1 \le j \le d$.

Given ($\mathfrak{H}_1$), (iii) and ($\mathfrak{H}_2$), we have that $s \mapsto P_s$ is piecewise continuous over $[0,T]$, with $\mathbb{E}\left[\|P_\cdot\|_\infty^{2+\gamma}\right] < +\infty$. Thus, it remains to check that $(u,P) \in \mathbb{C}_2$. Since we are dealing with the standard Brownian case and since $s \le t$ and $x \le y$, we note that $r_H(s,t,x,y) = ((t\wedge y) - (s\vee x))_+$. Thanks to the independence of increments, we are then left to check that $\forall j \in \{1\dots,d\}$,

$$\mathbb{E}[L_{s,t}^j L_{x,y}^j] = \sqrt{|t-s||x-y|} \times o_{|t-s|+|x-y|\to 0}\big(((t\wedge y) - (s\vee x))_+\big).$$

We have, for all $1 \le j \le d$ and with $\mathbb{B}_{s,t}^{i,j} = \int_s^t (B_l^i - B_s^i) dB_l^j$,

$$\begin{aligned}
L_{s,t}^j &= \int_s^t u_l dB_l^j - u_s(B_t^j - B_s^j) - \sum_{i=1}^d P_s^i \mathbb{B}_{s,t}^{i,j} \\
&= \int_s^t \left(\mathbb{E}\left[u_l\right] - \mathbb{E}\left[u_s\right]\right) dB_l^j \\
&+ \int_s^t \left(\sum_{i=1}^d \int_0^l \left(\left(V_{r,l}^i - V_{r,s}^i\right)\mathbb{I}_{[0,s]}(r) + \left(V_{r,l}^i - V_{s,s}^i\right)\mathbb{I}_{[s,l]}(r)\right) dB_r^i\right) dB_l^j \\
&=: L_{s,t}^{1,j} + L_{s,t}^{2,j}.
\end{aligned}$$

Let $s \le t$ and $x \le y$ be such that $s \vee x \le t \wedge y$. The hypothesis $(\mathfrak{H}_3)$ allows us to write

$$\begin{aligned}
\mathbb{E}[L^{1,j}_{s,t} L^{1,j}_{x,y}] &= \int_{s\vee x}^{t\wedge y} \mathbb{E}[u_l - u_s]\mathbb{E}[u_l - u_x] dl \\
&\le \sqrt{\int_{s\vee x}^{t\wedge y} (\mathbb{E}[u_l - u_s])^2 \, dl \int_{s\vee x}^{t\wedge y} (\mathbb{E}[u_l - u_x])^2 \, dl} \\
&\le ((t\wedge y) - (s \vee x))_+ \sqrt{|t-s||x-y|} \sqrt{\sup_{l\in[s,t]} \mu(s,l) \sup_{l\in[x,y]} \mu(x,l)}.
\end{aligned} \tag{2.26}$$

We also have:

$$\begin{aligned}
\mathbb{E}[L^{2,j}_{s,t} L^{2,j}_{x,y}] &= \int_{s\vee x}^{t\wedge y} dl \sum_{i=1}^{d} \mathbb{E}\left[\int_0^l ((V^i_{r,l} - V^i_{r,s})\mathbb{I}_{[0,s]}(r) + (V^i_{r,l} - V^i_{s,s})\mathbb{I}_{[s,l]}(r)) dB^i_r \right. \\
&\quad \left. \times \int_0^y ((V^i_{r,l} - V^i_{r,x})\mathbb{I}_{[0,x]}(r) + (V^i_{r,l} - V^i_{x,x})\mathbb{I}_{[s,l]}(r)) dB^i_r \right].
\end{aligned}$$

Moreover, thanks to the isometry property, the Cauchy-Schwarz inequality and the assumption $(\mathfrak{H}_3)$, we can write, for all $i \le d$,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t ((V^i_{r,l} - V^i_{r,s})\mathbb{I}_{[0,s]}(r) + (V^i_{r,l} - V^i_{s,s})\mathbb{I}_{[s,l]}(r)) dB^i_r \right. \\
&\quad \left. \times \int_0^y ((V^i_{r,l} - V^i_{r,x})\mathbb{I}_{[0,s]}(r) + (V^i_{r,l} - V^i_{x,x})\mathbb{I}_{[s,l]}(r)) dB^i_r \right] \\
\le\ & \sqrt{|t-s|\mu(s,t) + \sup_{r\in[s,l]} \mathbb{E}[V^i_{r,l} - V^i_{s,s}]^2} \sqrt{|x-y|\mu(x,y) + \sup_{r\in[x,l]} \mathbb{E}[V^i_{r,l} - V^i_{x,x}]^2}.
\end{aligned} \tag{2.27}$$

Using the Cauchy-Schwarz inequality and then (2.26) and (2.27), we finally obtain

$$\begin{aligned}
&\mathbb{E}[L^{1,j}_{s,t} L^{2,j}_{x,y} + L^{2,j}_{s,t} L^{1,j}_{x,y}] \\
\le\ & ((t\wedge y) - (s\vee x))_+ \sqrt{|t-s|(\mu(s,t) + \sup_{l\in[s,t], r\in[s,l]} \mathbb{E}[V^i_{r,l} - V^i_{s,s}]^2)} \\
&\qquad \times \sqrt{|x-y| \sup_{l\in[x,y]} \mu(x,l)} \\
&+ ((t\wedge y) - (s\vee x))_+ \sqrt{|t-s|(\mu(s,t) + \sup_{l\in[s,t], r\in[s,l]} \mathbb{E}[V^i_{r,l} - V^i_{s,s}]^2)} \\
&\qquad \times \sqrt{|x-y| \sup_{l\in[x,y]} \mu(x,l)}.
\end{aligned}$$

Thanks to $(\mathfrak{H}_3)$ we have that the function $(s,t) \to \sup_{x\in[s,t]} \mu(s,t)$ is uniformly continuous on $[0,T]^2$ and since $\mu(t,t)=0$ for all t,

$$\sup_{s,t\in[0,T],|s-t|\leq\delta} \sup_{x\in[s,t]} \mu(s,t) \underset{\delta\to 0}{\longrightarrow} 0.$$

On the other hand, we have thanks to $(\mathfrak{H}_1)$,

$$\sup_{s,t\in[0,T],s\leq t,|s-t|\leq\delta} \sup_{x\in[s,t]} \mathbb{E}[(V^i_{x,l} - V^i_{s,s})^2] \underset{\delta\to 0}{\longrightarrow} 0.$$

Finally, $(u,P)\in\mathbb{C}_2$. □

We obtain a result analogous to Proposition 2.3.3 for semimartingale processes but with weaker hypotheses on the volatility a and the drift b.

Corollary 2.3.8. *Assume $m=1$, and consider*

$$u_t = u_0 + \sum_{j=1}^d \int_0^t a^j_s dB^j_s + \int_0^t b_s ds.$$

Assume that a^j is progressively measurable and piecewise continuous for any j, that b is progressively measurable, that $g(s,t) = \sum_{k=1}^d \mathbb{E}\left[(a^k_s - a^k_t)^2\right]$ is continuous as a function of two variables, that u_0 is independent of B and that for some $\gamma>0$,

$$\mathbb{E}\left[\max_{1\leq j\leq d} \|a^j\|_\infty^{2+\gamma}\right] + \mathbb{E}\left[\|b\|_\infty^{2+\gamma}\right] < +\infty.$$

Then, with $M^{n,1,j}$ defined by (2.2), we have, stably in $\mathcal{C}_{\mathbb{R}^d}([0,T])$

$$\left\{\sqrt{n} M^{n,1,j}_{\cdot}\right\}_{1\leq j\leq d} \underset{n\to\infty}{\longrightarrow} \left\{\sum_{i=0}^d \int_0^{\cdot} a^i_s dW^{i,j}_s\right\}_{1\leq j\leq d}$$

where W is the independent matrix-valued Brownian motion of Section 2.2.5, see (2.14).

Proof. We have that the function $f : t \to \int_0^t b_s ds$ is a.s. continuous and satisfies $\mathbb{E}[\|f\|_\infty^{2+\gamma}] < \infty$. Using Jensen inequality and the isometry property, we easily see that

$$\left|\mathbb{E}\left[\int_s^t \left(\int_s^l b_u du\right) dB^j_l \int_x^y \left(\int_x^l b_u du\right) dB^j_l\right]\right|$$
$$\leq |x-y||t-s|(t\wedge y - s\vee x)_+ \sup_{l\in[0,T]} \mathbb{E}[b_l^2],$$

that is, $(\int_0^{\cdot} b_s ds, 0) \in \mathbb{C}_2$. Then, Theorem 2.1.3 applies, and

$$\forall j \in \{1 \ldots, d\}, \int_0^t dl \int_0^l b_s \, dB_s^i - \sum_{k=1}^{nt_n} \int_0^{\frac{k}{n}} b_l dl (B^j_{\frac{k+1}{n} \wedge t} - B^j_{\frac{k}{n}}) \overset{\mathcal{C}([0,T])}{\underset{n\to\infty}{\longrightarrow}} 0.$$

Moreover, we can apply Proposition 2.3.7 to $\int_0^{\cdot} a_s dB_s$ with $V_{s,t}^{1,j} = a_s^j \mathbb{I}_{[0,t]}(s)$ (all its assumptions are satisfied). Slutsky's lemma allows finally to conclude. □

Unlike the case $H > \frac{1}{2}$, here we can allow the volatility process a to be discontinuous. An illustration of this fact is given by choosing $d = 1$, $(T_i)_{i\geq 1}$ a sequence of increasing stopping times such that $T_i \underset{i\to\infty}{\longrightarrow} \infty$ a.s, a sequence $(x^i)_{i\geq 1} \in \mathbb{R}^{N^*}$ of progressively measurable processes on $[0,T]$ such that $\sum_i \|x^i\|_\infty^2 < \infty$, and

$$u_t = \sum_{i\geq 1} \int_0^{t\wedge T_i} x_s^i dB_s.$$

We then have, stably in $\mathcal{C}_{\mathbb{R}}([0,T])$,

$$\sqrt{n} M_{\cdot}^n \longrightarrow \frac{1}{\sqrt{2}} \int_0^{\cdot} \sum_i x_s^i \mathbb{I}_{[0,T_i]}(s) dW_s.$$

2.3.4 Irregular processes

In this section, $H \in (\frac{1}{2}, \frac{2}{3})$. We state a first order convergence for a general class of processes possessing mild regularity properties. Notice that related problems have been studied in the papers [2] and [6] (the latter establishing existence of Local time and Tanaka's formula for fractional Brownian motion).

Although the process u considered in Proposition 2.3.9 is of the form $u_s = F(B_s)$, the fact that F is supposed to be convex allows potential discontinuities for F', and it becomes hopeless to expect a second order result as obtained in Corollary 2.3.4 in a seemingly similar framework.

Proposition 2.3.9. *Let $u_s = F(B_s)$, $s \in [0,T]$, with F a real convex function such that, for some $K > 0$ and $\gamma \in (0,2)$,*

$$|F(x)| + |F'(x)| + \int_{-|x|}^{|x|} (|a|+1) dF''(a) \leq K e^{|x|^\gamma}, \quad x \in \mathbb{R},$$

where F' is the right derivative of F and F'' denotes its second derivative in the distributional sense (a simple 'non-smooth' example is given by $x \to |x|$). Then, for all $t \in [0,T]$,

$$\begin{aligned} M_t^n &:= n^{2H-1}\left(\int_0^t F(B_s)dB_s - \sum_{k=0}^{\lfloor nt \rfloor} F(B_{\frac{k}{n}})(B_{\frac{k+1}{n}\wedge t} - B_{\frac{k}{n}})\right) \\ &\xrightarrow[n\to\infty]{L^2(\Omega)} \frac{1}{2}\int_0^t F'(B_s)ds. \end{aligned}$$

Proof. The proof is divided into two steps: in the first one, we will first show that $u_s = F(B_s)$ belongs to $\mathbb{D}^{1,2}(|\mathcal{H}|)$ and give a suitable expression for its Malliavin derivative. This is then in Step 2 that we will show the $L^2(\Omega)$-convergence of M^n, with the help of Proposition 2.2.4 and of Lemma 2.4.6.

Step 1: u belongs to $\mathbb{D}^{1,2}(|\mathcal{H}|)$. Consider the truncated function

$$F^n : x \to F(x)\mathbb{I}_{|x|\leq n} + F(n)\mathbb{I}_{x>n} + F(-n)\mathbb{I}_{x<-n}.$$

Every convex function is locally Lipschitz continuous so the previous sequence is Lipschitz continuous. Then, by a slight extention of [26, Proposition 2.3.8], we know that the process $u_s^n = F^n(B_s)$ belongs to $\mathbb{D}^{1,2}(|\mathfrak{H}|)$, and $D_s u_t^n = (F^n)'(B_t)\mathbb{I}_{s\leq t}$. Moreover, $F^n \to F$ and $(F^n)' \to F'$ pointwise as $n \to \infty$, and the growth condition on F and F' ensures that, for all $p > 2$, the sequences $F^n(B_s)$ and $(F^n)'(B_s)$ are bounded in $L^p(\Omega, |\mathcal{H}|)$ and $L^p(\Omega, |\mathcal{H}| \times |\mathcal{H}|)$ respectively. Then, these sequences are both uniformly integrable in L^2, and the bounded convergence theorem ensures that, as $n \to \infty$,

$$\begin{aligned} &u^n \to u \quad \text{in } L^p(\Omega, |\mathcal{H}|) \\ &\mathbb{I}_{\{\cdot\leq *\}}\, D_\cdot u_*^n \to \mathbb{I}_{\{\cdot\leq *\}}\, F'(B_*) \quad \text{in } L^p(\Omega, |\mathcal{H}| \times |\mathcal{H}|). \end{aligned}$$

Then, $u \in \mathbb{D}^{1,2}(|\mathcal{H}|)$, with $D_s u_t = \mathbb{I}_{s\leq t}F'(B_t)$. Since F' is locally bounded, the process u verifies the assumptions of Proposition 2.2.4.

Step 2: L^2 convergence. By e.g. [31, page 224], we know that, for all $k \in \mathbb{N}^*$, there exist $\alpha_k, \beta_k \in \mathbb{R}$ such that

$$F(x) = \alpha_k + \beta_k x + \frac{1}{2}\int_{-k}^{k} |x-a| dF''(a), \quad x \in [-k,k].$$

Then, for all $x \in \mathbb{R}$,

$$\begin{aligned} F(x) &= F(0)\mathbb{I}_{\{0\}}(x) \\ &+ \sum_{k=0}^{+\infty}\left(\alpha_{k+1}+\beta_{k+1}x+\frac{1}{2}\int_{-k-1}^{k+1}|x-a|dF''(a)\right)\mathbb{I}_{[-k-1,-k)\cup(k,k+1]}(x). \end{aligned}$$

Since F is convex, dF'' can be identified with a Radon measure, which is σ-finite. This allows us to interchange the integrals and derivatives. Since $D_{\cdot}u_* = F'(B_*)\mathbb{I}_{\cdot\leq *}$ we can then rewrite Du as:

$$D_s u_t = \mathbb{I}_{t\geq s}\sum_{k=0}^{+\infty}\left(\beta_{k+1}+\frac{1}{2}\int_{-k-1}^{k+1}\operatorname{sign}(B_t-a)dF''(a)\right)\mathbb{I}_{[-k-1,-k)\cup(k,k+1]}(B_t), \tag{2.28}$$

where sign is the left derivative of $x \to |x|$.

Let $0 \leq t \leq T$. We have, thanks to Proposition 2.2.4 and recalling that $s_n = \frac{1}{n}\lfloor ns\rfloor$,

$$\begin{aligned} & M_t^n - \frac{1}{2}\int_0^t F'(B_s)ds \\ = \; & n^{2H-1}\Big[\int_0^t (F(B_s)-F(B_{s_n}))\delta B_s \\ & + \int_0^t\int_0^t \left(F'(B_s)\mathbb{I}_{[0,s]}(l) - F'(B_{s_n})\mathbb{I}_{[0,s_n]}(l)\right)\mu_H(dlds)\Big] \\ & -\frac{1}{2}\int_0^t F'(B_s)ds \\ = \; & \frac{1}{2}\int_0^{t_n}\left(F'(B_{s_n})-F'(B_s)\right)ds \\ & -\frac{1}{2}\int_{t_n}^t F'(B_s)ds + \frac{(nt-\lfloor nt\rfloor)^{2H-1}}{2}\int_{t_n}^t F'(B_{s_n})ds \\ & + n^{2H-1}\int_0^t\int_0^{s_n}\left(F'(B_s)-F'(B_{s_n})\right)\mu_H(dlds) \\ & + n^{2H-1}\int_0^t\int_{s_n}^{s}\left(F'(B_s)-F'(B_{s_n})\right)\mu_H(dlds) \\ & + n^{2H-1}\int_0^t (F(B_s)-F(B_{s_n}))\delta B_s, \end{aligned}$$

where we used the fact that $\int_{k/n}^{(k+1)/n}\int_{k/n}^{s}\mu_H(dlds) = \frac{1}{2}n^{-2H}$.

We can see easily that for all $0 \leq t \leq T$,

$$\mathbb{E}\left[\left(\frac{1}{2}\int_{t_n}^{t} F'(B_s)ds + \frac{(nt-\lfloor nt\rfloor)^{2H-1}}{2}\int_{t_n}^{t} F'(B_{s_n})ds\right)^2\right] = O_{n\to\infty}\left(\frac{1}{n^2}\right).$$

We also have

$$\begin{aligned}
&\mathbb{E}\left[\left(n^{2H-1}\int_0^t\int_0^{s_n}\left(F'(B_s)-F'(B_{s_n})\right)\mu_H(dlds)\right)^2\right]\\
\leq\ & Kn^{4H-2}\mathbb{E}\left[\left(\int_0^t (F'(B_s)-F'(B_{s_n}))ds\right)^2\right],
\end{aligned}$$

where we used the fact that

$$\int_0^{s_n} |l-s|^{2H-2}dl \leq (2-2H)s^{2H-1} \leq (2-2H)T^{2H-1},$$

and

$$\begin{aligned}
&\mathbb{E}\left[\left(n^{2H-1}\int_0^t\int_{s_n}^{s}\left(F'(B_s)-F'(B_{s_n})\right)\mu_H(dlds)\right)^2\right]\\
=\ & n^{4H-2}\int_0^t\int_{s_n}^{s}\mu_H(d\theta ds)\int_0^t\int_{x_n}^{x}\mu_H(d\mu dx)\\
&\times\mathbb{E}\left[(F'(B_s)-F'(B_{s_n})(F'(B_x)-F'(B_{x_n})\right]\\
\leq\ & n^{2H-2}\mathbb{E}\left[\left(\int_0^t (F'(B_s)-F'(B_{s_n}))ds\right)^2\right],
\end{aligned}$$

(where we used that $\int_{s_n}^{s} |l-s|^{2H-2}dl \leq |s-s_n|$, thanks to Lemma 2.4.2), and finally

$$\begin{aligned}
& \mathbb{E}\left[\left(n^{2H-1}\int_0^t (F(B_s)-F(B_{s_n}))\delta B_s\right)^2\right] \\
\leq\ & Kn^{4H-2}\mathbb{E}\left[\int_0^t (F(B_s)-F(B_{s_n}))^2 ds\right] \\
& +Kn^{4H-2}\mathbb{E}\left[\int_0^t\int_0^t (D_l u_s - D_l u_{s_n})^2 dlds\right] \\
\leq\ & Kn^{4H-2}\mathbb{E}\left[\int_0^t (F(B_s)-F(B_{s_n}))^2 ds\right] \\
& +Kn^{4H-2}\left(\mathbb{E}\left[\int_0^t\int_0^{s_n} (F'(B_s)-F'(B_{s_n}))^2 dlds\right]\right. \\
& \left. +\mathbb{E}\left[\int_0^t\int_{s_n}^s F'(B_s)^2 dlds\right]\right)
\end{aligned}$$

with K depending only on T. We used (2.8) then (2.9) in Proposition 2.2.1, and the fact that $D_l u_s - D_l u_{s_n} = F'(B_s)\mathbb{I}_{l\leq s} - F'(B_{s_n})\mathbb{I}_{l\leq s_n}$ to obtain the last inequality.

We have

$$n^{4H-2}\mathbb{E}\left[\int_0^t\int_{s_n}^s F'(B_s)^2 dlds\right] \leq \frac{n^{4H-3}}{2}\mathbb{E}\left[\int_0^t F'(B_s)^2 ds\right] \underset{n\to\infty}{\longrightarrow} 0$$

(because $H < \frac{2}{3}$). We have

$$\begin{aligned}
& \mathbb{E}\left[\int_0^t\int_0^{s_n} (F'(B_s)-F'(B_{s_n}))^2 dlds\right] \\
=\ & \mathbb{E}\left[\int_0^t\int_0^{s_n} (F'(B_s)-F'(B_{s_n}))^2 \mathbb{I}_{\lfloor B_s\rfloor=\lfloor B_{s_n}\rfloor} dlds\right] \\
& +\mathbb{E}\left[\int_0^t\int_0^{s_n} (F'(B_s)-F'(B_{s_n}))^2 \mathbb{I}_{\lfloor B_s\rfloor\neq\lfloor B_{s_n}\rfloor} dlds\right].
\end{aligned}$$

Using (2.28), we have:

$$\begin{aligned}
& \mathbb{E}\left[\int_0^t\int_0^{s_n} (F'(B_s)-F'(B_{s_n}))^2 \mathbb{I}_{\lfloor B_s\rfloor=\lfloor B_{s_n}\rfloor} dlds\right] \\
=\ & \frac{1}{4}\sum_{k=0}^{\infty}\mathbb{E}\left[\int_0^t\int_0^{s_n} dlds\,\mathbb{I}_{[-k-1,-k)\cup(k,k+1]}(B_s)\right. \\
& \left.\left(\int_{-k-1}^{k+1}(\mathrm{sign}(B_s-a)-\mathrm{sign}(B_{s_n}-a))dF''(a)\right)^2 \mathbb{I}_{\lfloor B_s\rfloor=\lfloor B_{s_n}\rfloor}\right].
\end{aligned}$$

Moreover, since dF'' is σ-finite, we can use Fubini's theorem and Jensen's inequality to get that

$$\mathbb{E}\left[\int_0^t\int_0^{s_n}(F'(B_s)-F'(B_{s_n}))^2dlds\right]\leq t(C_t^n+D_t^n),$$

with

$$\begin{aligned}C_t^n &= \sum_{k=0}^{+\infty}F''([-k-1,k+1))\\ &\times\int_{-k-1}^{k+1}\int_0^t\mathbb{E}\left[(\text{sign}(B_s-a)-\text{sign}(B_{s_n}-a))^2\right.\\ &\times\left.\mathbb{I}_{[-k-1,-k)\cup(k,k+1]}(B_s)\mathbb{I}_{\lfloor B_s\rfloor=\lfloor B_{s_n}\rfloor}\right]dsdF''(a);\\ D_t^n &= \mathbb{E}\left[\int_0^t(F'(B_s)-F'(B_{s_n}))^2\mathbb{I}_{\lfloor B_s\rfloor\neq\lfloor B_{s_n}\rfloor}ds\right].\end{aligned}$$

Let $\gamma>0$ and let $p,q>0$ be two conjugate exponents such that $\frac{H-\gamma}{p}>4H-2$. (Notice that γ,p,q exist if and only if $H<\frac{2}{3}$.) We apply Hölder's inequality to obtain:

$$\begin{aligned}&\mathbb{E}\left[(\text{sign}(B_s-a)-\text{sign}(B_{s_n}-a))^2\,\mathbb{I}_{[-k-1,-k)\cup(k,k+1]}(B_s)\mathbb{I}_{\lfloor B_s\rfloor=\lfloor B_{s_n}\rfloor}\right]\\ \leq\;&\mathbb{E}\left[|\text{sign}(B_s-a)-\text{sign}(B_{s_n}-a)|^{2p}\right]^{\frac{1}{p}}\mathbb{E}\left[\mathbb{I}_{[-k-1,-k)\cup(k,k+1]}(B_s)\right]^{\frac{1}{q}}.\end{aligned}$$

We know that $\mathbb{E}\left[\mathbb{I}_{[-k-1,-k)\cup(k,k+1]}(B_s)\right]^{\frac{1}{q}}=O_{k\to\infty}(e^{-\frac{k^2}{2qT^{2H}}})$ for all $s\in[0,T]$. We also have that (by hypothesis on F''),

$$\sum_{k=0}^{\infty}F''([-k-1,k+1])e^{-\frac{k^2}{2qT^{2H}}}<\infty.$$

By Lemma 2.4.6, for all $a\in\mathbb{R}$,

$$\int_0^t\mathbb{E}\left[(\text{sign}(B_s-a)-\text{sign}(B_{s_n}-a))^{2p}\right]^{\frac{1}{p}}ds=o_{n\to\infty}(n^{2-4H}),$$

where the o does not depend on t.

We then obtain:

$$n^{4H-2}C_t^n=o_{n\to\infty}(1). \tag{2.29}$$

A similar use of Lemma 2.4.6 shows that, for all t,

$$n^{4H-2}D_t^n + n^{4H-2}\mathbb{E}\left[\left(\int_0^t (F(B_s) - F(B_{s_n}))ds\right)^2\right] = o_{n\to\infty}(1).$$

Putting these facts together leads to:

$$M_t^n \underset{n\to\infty}{\overset{L^2(\Omega)}{\longrightarrow}} \frac{1}{2}\int_0^t F'(B_s)ds.$$

□

2.4 Proofs of the main Theorems and other results

Throughout all this section, we denote by B a fractional Brownian motion of Hurst index H.

2.4.1 Miscellaneous

We start by giving a collection of technical results that are used throughout the paper.

The following lemma is an easy consequence of Fernique's theorem (see e.g. [38] and the references therein), and represents a very useful tool for proving the existence of moments for Hölder modulus of Gaussian functionals.

Lemma 2.4.1. ***(Fernique)*** *Assume that $H > \frac{1}{2}$ and let $\mathbb{B}$ be the associated Lévy area of B, defined as $\mathbb{B}_{s,t}^{k,j} = \int_s^t (B_l^k - B_s^k)dB_l^j$. For all $\gamma \in (0,2)$ and all $\kappa \in (0,H)$, and for all function f satisfying the growth condition $|f(x)| \le \exp|x|^\gamma$, we have*

$$\mathbb{E}[f(\|B\|_\kappa + \sqrt{\|\mathbb{B}\|_{2\kappa}})] < \infty,$$

where $\|\cdot\|_\theta$ is the Hölder seminorm, see (2.6) and (2.10).

We also have the following elementary lemma.

Lemma 2.4.2. *Assume $H > \frac{1}{2}$. There exists a constant $k_T > 0$ such that, for all $x, y, s, t \in [0,T]^4$ such that $t \ge s$ and $y \ge x$,*

$$k_T|t-s||y-x| \le r_H(s,t,x,y) \le |t-s|^H|x-y|^H, \qquad (2.30)$$

$$|\mathbb{E}[B_t(B_x - B_y)]| \le |x-y|. \qquad (2.31)$$

Proof. For the sake of simplicity, we will consider $T = 1$ (which only modifies the constants). In the expression (2.30), the right inequality is a simple consequence of the Cauchy-Schwarz inequality. For the proof of the left inequality, six cases must be analyzed carefully.

(i) case where $t \geq s \geq y \geq x$. For fixed s, y, x, let

$$f(t) = (2H-1)(t-s)(y-x) \text{ and } g(t) = r_H(s,t,x,y).$$

We have $f(s) = g(s) = 0$ and

$$g'(t) - f'(t) = 2H\left(-(t-y)^{2H-1} + (t-x)^{2H-1}\right) - (2H-1)(y-x).$$

We see that $-(t-y)^{2H-1} + (t-x)^{2H-1} \geq (2H-1)(y-x)$ thanks to an elementary function study, so

$$g'(t) - f'(t) \geq (2H-1)^2(y-x) \geq 0,$$

so $g(t) \geq f(t)$ and then $(2H-1)|t-s||y-x| \leq r_H(s,t,x,y)$.

(ii) case where $t \geq y > s \geq x$. For fixed t, y, x, we see (thanks to an elementary function study) that the quantity $r_H(s,t,x,y) - (t-s)(y-x)$ decreases with s and then reaches its minimum for $s = y$. Assume then $s = y$ and let $\delta = t - x$ and $a = y - x$. Then

$$\begin{aligned} &r_H(s,t,x,y) - (t-s)(y-x) \\ &\geq h(a) = \delta^{2H} - (a^{2H} + (\delta-a)^{2H} + a(\delta-a)). \end{aligned}$$

We have $h(\delta) = h(0) = 0$, and the function h is increasing over $(0, \frac{\delta}{2})$ then decreasing, so is always positive.

(iii) case $y > t \geq x > s$. This is similar to (ii).

(iv) case $y \geq x > t \geq s$. This is similar to (i).

(v) case $t \geq y \geq x > s$. Write $B_t - B_s = (B_t - B_y) + (B_y - B_x) + (B_x - B_s)$ and then combine the inequalities from (i) and (iv).

(vi) case $y > t \geq s \geq x$. This is similar to (v).

Finally, the proof of inequality (2.31) can be found in [24, Lemma 6]. □

The following lemma is used in Step 2 of the proof of Proposition 2.3.5.

Lemma 2.4.3. *Let $m, n \in \mathbb{N}$ with $m > n$, $f \in \mathcal{H}^{\odot m} \otimes \mathcal{H}$, $g \in \mathcal{H}^{\odot n}$, $h \in \mathcal{H}^{\odot n} \otimes \mathcal{H}$. Let $x \in [0,T]$, $F_x = \delta^m(f(\cdot, x)), G = \delta^n(g), H_x = \delta^n(h(\cdot, x))$. Then, for all $s \leq t$ and $u \leq v$,*

$$\mathbb{E}[\delta(F.\mathbb{I}_{[s,t]}(\cdot))G] = 0 \tag{2.32}$$

$$\mathbb{E}[\delta(F.\mathbb{I}_{[s,t]}(\cdot))\delta(H.\mathbb{I}_{[u,v]}(\cdot))] = 0. \tag{2.33}$$

Proof. If $n = 0$, the result is immediate. Otherwise, thanks to Proposition 2.2.1, we can write:

$$\begin{aligned} &\mathbb{E}[\delta(F.\mathbb{I}_{[s,t]}(\cdot))G] \\ = \; &\int_{[0,T]^2} \mathbb{E}[F_x \mathbb{I}_{[s,t]}(x)\delta^{n-1}(g(\cdot, y))]\mu_H(dxdy) \\ &+ \int_{[0,T]^4} \mathbb{E}[D_w(F_x \mathbb{I}_{[s,t]}(x))D_z(\delta^{n-1}(g(\cdot, y)))]\mu_H(dxdy)\mu_H(dwdz). \end{aligned}$$

Thanks to Proposition 2.2.2, we have, for all $x, y \in [0,T]$,

$$\mathbb{E}[F_x \delta^{n-1}(g(\cdot, y))] = 0.$$

Moreover, $D_w(F_x \mathbb{I}_{[s,t]}(x)) = m\delta^{m-1}(f(\cdot, w, x))\mathbb{I}_{[s,t]}(x)$, and $D_z(\delta^{n-1})(g(\cdot, y)) = (n-1)\delta^{n-2}(g(\cdot, z, y))$ if $n \geq 2$ and $D_z(\delta^{n-1}(g(\cdot, y))) = 0$ otherwise. In any case, we have thanks to Proposition 2.2.2,

$$\mathbb{E}[D_w(F_x \mathbb{I}_{[s,t]}(x))D_z(\delta^{n-1}(g(\cdot, y)))] = 0.$$

Equality (2.33) can be obtained by the same way. □

The following lemma provides a tightness criterion for two sequences of processes in $\mathcal{D}([0,T])$ whose sum belongs to $\mathcal{C}([0,T])$. Recall the notation $s_n = \frac{1}{n}\lfloor ns \rfloor$ and $t_n = \frac{1}{n}\lfloor nt \rfloor$.

Lemma 2.4.4. *Let $(X^n) \subset \mathcal{C}_{\mathbb{R}}([0,T])$ be a sequence of continous stochastic processes such that $X^n_t = A^n_t + C^n_t$ for all $t \in [0,T]$, where $A^n, C^n \in \mathcal{D}_{\mathbb{R}}([0,T])$. Assume also the existence of $\alpha_0, \beta_0 > 0$ such that*

$$\mathbb{E}\left[|A^n_t - A^n_s|^{\beta_0}\right] \leq K|t_n - s_n|^{1+\alpha_0}, \quad 0 \leq s, t \leq T, \tag{2.34}$$

and

$$\sup_{t\in[0,T]} |C_t^n| \xrightarrow[n\to\infty]{\mathbb{P}} 0. \tag{2.35}$$

Then the sequence X^n is tight in $\mathcal{C}_{\mathbb{R}}([0,T])$.

Proof. In [7], it is proved that the sequence (A^n) is tight in $\mathcal{D}_{\mathbb{R}}([0,T])$. Moreover, the sequence (C^n) is also tight in $\mathcal{D}_{\mathbb{R}}([0,T])$. by [25, Lemma 2.2], the sequence (A^n, C^n) is tight in $\mathcal{D}_{\mathbb{R}^2}([0,T])$ and since the map $(x,y)\to x+y$ is continuous from $\mathcal{D}_{\mathbb{R}^2}([0,T])$ to $\mathcal{D}_{\mathbb{R}}([0,T])$, the sequence $(X^n)_n$ is then tight in $\mathcal{D}_{\mathbb{R}}([0,T])$. Since the uniform and the Skorohod topologies coincide on $\mathcal{C}_{\mathbb{R}}([0,T])$, we deduce that (X^n) is tight in $\mathcal{C}_{\mathbb{R}}([0,T])$. □

The following lemma is used in the proof of the forthcoming (2.41).

Lemma 2.4.5. *Let $(X^n)\subset\mathcal{C}_{\mathbb{R}}([0,T])$ be a tight sequence of continous stochastic processes such that $\forall t\in[0,T]$, $X_t^n \xrightarrow{\mathbb{P}} 0$ as $n\to\infty$. Then, as $n\to\infty$,*

$$\sup_{t\in[0,T]} |X_t^n| \xrightarrow{\mathbb{P}} 0. \tag{2.36}$$

Proof. Let $1>\epsilon>0$. Since the function defined by $x\to 1\wedge\sup_{t\in[0,T]}|x_t|$ on $\mathcal{C}_{\mathbb{R}}([0,T])$ is continuous and bounded, we deduce that $\mathbb{E}\left[1\wedge\sup_{t\in[0,T]}|X_t^n|\right]\to 0$ as $n\to\infty$. Then, by Markov's inequality and as $n\to\infty$,

$$\mathbb{P}\left[\sup_{t\in[0,T]}|X_t^n|>\epsilon\right]=\mathbb{P}\left[\sup_{t\in[0,T]}|X_t^n|\wedge 1>\epsilon\right]\le\frac{1}{\epsilon}\mathbb{E}\left[\sup_{t\in[0,T]}|X_t^n|\wedge 1\right]\longrightarrow 0.$$

□

The following lemma gives technical estimates used in the proof of Propositions 2.3.7 and 2.3.9.

Lemma 2.4.6. *Assume $H>\frac{1}{2}$. Then, for all $0\le t\le T$, $a\in\mathbb{R}$, $\gamma>0$, $p>0$ and $\theta\ge 1$, we have*

$$\int_0^t \mathbb{E}\left[|\mathrm{sign}(B_s-a)-\mathrm{sign}(B_{s_n}-a)|^p\right]^{\frac{1}{\theta}}\,ds\le Kn^{\frac{-H+\gamma}{\theta}}, \tag{2.37}$$

and

$$\int_0^t \mathbb{P}\left(\lfloor B_s\rfloor\ne\lfloor B_{s_n}\rfloor\right)^{\frac{1}{\theta}}\,ds\le Kn^{\frac{-H+\gamma}{\theta}} \quad \textit{(where } s_n=\tfrac{1}{n}\lfloor ns\rfloor\textit{)}, \tag{2.38}$$

with K depending only on T,p,θ and where sign *is the left derivative of the function $x\to|x|$.*

Proof. We only do the proof of the first inequality, the proof of the second one being similar. Moreover, for simplicity we reduce to $a = 0$ and $\theta = 1$.

We have

$$\mathrm{sign}(B_s) - \mathrm{sign}(B_{s_n}) = 2\mathbb{I}_{\{B_s>0, B_{s_n}\leq 0\}} - 2\mathbb{I}_{\{B_s\leq 0, B_{s_n}>0\}}.$$

Plugging this identity into the integral yields

$$\int_0^t \mathbb{E}\left[|\mathrm{sign}(B_s) - \mathrm{sign}(B_{s_n})|^p\right] ds \leq 2^{p+1} \int_0^t \mathbb{P}\left(B_s > 0, B_{s_n} \leq 0\right) ds.$$

On the other hand, for all $k \in \{2, \dots, nT_n\}$ and $s \in [\frac{k}{n}, \frac{k+1}{n} \wedge T)$,

- if $s = \frac{k}{n}$, then $\mathbb{P}\left[B_s > 0, B_{s_n} \leq 0\right] = 0$
- else,

$$\begin{aligned}
&\mathbb{P}\left[B_s > 0, B_{s_n} \leq 0\right] = \mathbb{P}\left[B_{\frac{k}{n}} \leq 0,\ B_s - B_{\frac{k}{n}} > -B_{\frac{k}{n}}\right] \\
\leq\ & \sum_{i=0}^{+\infty} \mathbb{E}\left[\mathbb{I}_{\left\{B_{\frac{k}{n}} \in [-\frac{i+1}{n}, -\frac{i}{n}]\right\}} \mathbb{I}_{\left\{B_s - B_{\frac{k}{n}} > \frac{i}{n}\right\}}\right] \\
=\ & \sum_{i=0}^{+\infty} \mathbb{E}\left[\mathbb{I}_{\left\{B_1 \in \left[-\frac{i+1}{n^{1-H}\ k^H}, -\frac{i}{n^{1-H}\ k^H}\right)\right\}} \mathbb{I}_{\left\{B_{\frac{ns}{k}} - B_1 > \frac{i}{n^{1-H}\ k^H}\right\}}\right] \\
&\text{(using the self-similarity of } B) \\
=\ & \frac{1}{2\pi\sqrt{\det(\Sigma)}} \sum_{i=0}^{+\infty} \int_{\frac{i}{k^H\ n^{1-H}}}^{\frac{i+1}{k^H\ n^{1-H}}} \int_{-\infty}^{\frac{-i}{k^H n^{1-H}}} e^{-\frac{(\frac{ns}{k}-1)^{2H}}{2\det(\Sigma)}x^2 - \frac{1}{2\det(\Sigma)}y^2 + \frac{c_{s,k}}{\det(\Sigma)}xy} dydx \\
&\text{with } c_{s,k} = \mathbb{E}\left[B_1\left(B_{\frac{ns}{k}} - B_1\right)\right] > 0 \text{ and } \Sigma = \begin{pmatrix} 1 & c_{s,k} \\ c_{s,k} & \left(\frac{ns}{k} - 1\right)^{2H} \end{pmatrix}.
\end{aligned}$$

Since $\frac{c_{s,k}}{\det(\Sigma)}xy \leq 0$ for all $(x, y) \in \left[\frac{i}{k^H\ n^{1-H}}, \frac{i+1}{k^H\ n^{1-H}}\right] \times \left(-\infty, \frac{-i}{k^H n^{1-H}}\right]$, we deduce

$$\begin{aligned}
&\mathbb{P}\left[B_s > 0, B_{s_n} \leq 0\right] \qquad (2.39) \\
\leq\ & \frac{1}{2\pi\sqrt{\det(\Sigma)}} \sum_{i=0}^{+\infty} \int_{\frac{i}{k^H\ n^{1-H}}}^{\frac{i+1}{k^H\ n^{1-H}}} \int_{-\infty}^{\frac{-i}{k^H n^{1-H}}} e^{-\frac{(\frac{ns}{k}-1)^{2H}}{2\det(\Sigma)}x^2 - \frac{1}{2\det(\Sigma)}y^2} dydx.
\end{aligned}$$

Let us now estimate the three terms appearing in the right-hand side of the previous inequality.

1st term: We have $\det(\Sigma) = (\frac{ns}{k} - 1)^{2H} - c_{s,k}^2$. According to Lemma 2.4.2, $c_{s,k} \le \frac{ns}{k} - 1$. If $k > 2$, we have $\frac{ns}{k} - 1 \le \frac{1}{2}$ and then

$$\begin{aligned} \left(\frac{ns}{k} - 1\right)^{2H} &\ge \det(\Sigma) \ge \left(\frac{ns}{k} - 1\right)^{2H} \left(1 - \left(\frac{ns}{k} - 1\right)^{2-2H}\right) \\ &\ge \left(\frac{ns}{k} - 1\right)^{2H} \left(1 - \frac{1}{2^{2-2H}}\right). \end{aligned}$$

That is, $\frac{1}{2\pi\sqrt{\det(\Sigma)}} \le \frac{1}{2\pi\sqrt{(1-2^{2H-2})\left(\frac{ns}{k}-1\right)^{2H}}}$.

2nd term: We have

$$\int_{\frac{i}{k^H n^{1-H}}}^{\frac{i+1}{k^H n^{1-H}}} e^{-\frac{(\frac{ns}{k}-1)^{2H}}{2\det(\Sigma)}x^2} dx \le \frac{1}{n^{1-H}k^H} e^{-\frac{1}{2}(\frac{i}{k^H n^{1-H}})^2}.$$

3rd term: We have, using that $s \in [\frac{k}{n}, \frac{k+1}{n} \wedge T)$,

$$\begin{aligned} \int_{-\infty}^{\frac{-i}{k^H n^{1-H}}} e^{-\frac{y^2}{2\det(\Sigma)}} dy \le \int_{-\infty}^{\frac{-i}{k^H n^{1-H}}} e^{-\frac{y^2}{2\left(\frac{ns}{k}-1\right)^{2H}}} dy &\le \int_{-\infty}^{\frac{-i}{k^H n^{1-H}}} e^{-k^{2H}y^2} dy \\ &= \frac{1}{k^H} \int_{-\infty}^{\frac{-i}{n^{1-H}}} e^{-y^2} dy. \end{aligned}$$

By plugging these three estimates into (2.39) and by using the fact that

$$e^{-\frac{1}{2}(\frac{i}{k^H n^{1-H}})^2} \le \frac{(n^{1-H}k^H)^\alpha}{i^\alpha} \quad \text{and} \quad \int_{-\infty}^{\frac{-i}{n^{1-H}}} e^{-y^2} dy \le \frac{n^{1-H}}{i}$$

for all $i \in \mathbb{N}^*$ and all $\alpha \in (0,1)$, we get, by choosing α so that $\alpha(2-H) < \gamma$,

$$\mathbb{P}\left[B_s > 0, B_{s_n} \le 0\right] \le \frac{n^{\alpha(1-H)}}{2\pi k^{2H-\alpha}\sqrt{(1-2^{2H-2})}(\frac{ns}{k}-1)^H} \sum_{i=1}^{\infty} \frac{1}{i^{1+\alpha}}.$$

Finally,

$$\begin{aligned} &\int_0^t \mathbb{E}\left[\left|\operatorname{sign}(B_s) - \operatorname{sign}(B_{\frac{\lfloor ns \rfloor}{n}})\right|^p\right] ds \\ \le\ & 2^{p+1}\left(\frac{2}{n} + \int_{\frac{2}{n}}^t \mathbb{P}[B_s \ge 0, B_{s_n} < 0] ds\right) \end{aligned}$$

and

$$
\begin{aligned}
&\int_{\frac{2}{n}}^{t} \mathbb{P}[B_s > 0, B_{s_n} \le 0] ds \\
\le\ & \frac{1}{2\pi} \sum_{k=1}^{nt_n} \frac{1}{k^{2H-\alpha}} \int_{\frac{k}{n}}^{\frac{k+1}{n}\wedge t} \frac{1}{\sqrt{(1-2^{2H-2})(\frac{ns}{k}-1)^H}} ds \sum_{i=1}^{\infty} \frac{1}{i^{1+\alpha}} n^{\alpha(1-H)} \\
\le\ & \frac{1}{2\pi\sqrt{(1-2^{2H-2})}} \left(\frac{(t-t_n)^H}{(nt_n)^{2H-\alpha}} + \frac{1}{1-H} \sum_{k=2}^{nt_n} \frac{1}{nk^{H-\alpha}} \right) \sum_{i=1}^{\infty} \frac{1}{i^{1+\alpha}} n^{\alpha(1-H)} \\
\le\ & K n^{-H+\alpha(2-H)}
\end{aligned}
$$

This provides the desired estimate. □

Remark 2.4.7. In [2], the author obtained for all $s \le t \in [0,T]$ and for all $a \in \mathbb{R}$ the following bound:

$$\mathbb{P}[B_s > a, B_t \le a] \le C(a)(t-s)^H s^{-2H},$$

for some constant $C(a) > 0$. On the other hand, the computations in the proof of Lemma 2.4.6 give the estimate

$$\mathbb{P}[B_s > a, B_t \le a] \le C(a,\gamma)(t-s)^{H+\gamma} s^{-H},$$

which is weaker when $s \ge 1$ but better when $s << 1$ (this improvement is necessary for the inequalities 2.37 and 2.38 to hold).

2.4.2 Weighted quadratic variations of the fractional Brownian motion

In the proofs of Theorems 2.1.2 and 2.1.3, we will see that the announced convergences are determined by the asymptotic behaviour of the weighted quadratic variations of the fractional Brownian motion. These variations have already been extensively studied, for example in [13, 24] and especially in [5]. In the next three lemmas, we gather the results that are relevant to us, and we extend them when necessary.

Lemma 2.4.8. *Let x be a scalar process over $[0,T]$, and assume it is a.s. continuous and satisfies $\mathbb{E}\left[\|x\|_\infty^{2+\gamma}\right] < +\infty$ for some $\gamma > 0$. Let $H > \frac{1}{2}$. Then,* *1. For all $j \le d$, for all $t \in [0,T]$*

$$S^{n,j}_{t,x} = n^{2H-1}\sum_{k=0}^{\lfloor nt \rfloor} x_{\frac{k}{n}}\left(B^j_{\frac{k+1}{n}\wedge t} - B^j_{\frac{k}{n}}\right)^2 \underset{n\to\infty}{\overset{L^2(\Omega)}{\longrightarrow}} \int_0^t x_s ds. \tag{2.40}$$

2. For all $i \neq j$,

$$n^{2H-1}\sum_{k=0}^{\lfloor nt \rfloor} x_{\frac{k}{n}}\delta^{1,i}\left(\left(B^j_{\cdot} - B^j_{\frac{k}{n}}\right)\mathbb{I}_{[\frac{k}{n},\frac{k+1}{n}\wedge t]}(\cdot)\right) \underset{n\to\infty}{\overset{L^2(\Omega)}{\longrightarrow}} 0. \tag{2.41}$$

These two convergences also holds UCP as a process over $[0,T]$.

Proof. Step 1: Proof of (2.40). It is well known that (2.40) is true in the a.s. sense if $x = \mathbb{I}_{[0,t]}$ (see e.g [15]) and then (by substraction) for every process of the type $x = \mathbb{I}_{[s,t]}$ for $s \le t$. Now, consider $0 = a_0 \le \ldots \le a_p \le T$ and let $(\alpha_0, \ldots, \alpha_{p-1})$ be a collection of $\mathfrak{F}$-measurable random variables. For all $1 \le i \le p$, let Ω_i be the subset of Ω on which (2.40) holds true for the process $\alpha_i\mathbb{I}_{[a_i,a_{i+1}]}$. Then $\mathbb{P}(\cap_{i=1}^p \Omega_i) = 1$, and (2.40) holds (pointwise) for the step process $x = \sum_{i=0}^{p-1}\alpha_i\mathbb{I}_{[a_i,a_{i+1}]}$ on $\cap_{i=1}^p\Omega_i$.

Moreover, if a process f is bounded for $\|\cdot\|_\infty$ in $L^{2+\gamma}$ then the sequence $\{(S^{n,j}_{t,f})^2\}_{n=1}^\infty$ is uniformly integrable. Indeed, let $0 \le \mu \le \gamma$. We have, thanks to the Minkowski inequality,

$$\|S^{n,j}_{t,f}\|_{L^{2+\mu}} \le n^{2H-1}\sum_{k=0}^{\lfloor nt \rfloor}\left\| f_{\frac{k}{n}}\left(B^j_{\frac{k+1}{n}\wedge t} - B^j_{\frac{k}{n}}\right)^2\right\|_{L^{2+\mu}}.$$

Then, using the Hölder inequality, we have

$$\begin{aligned}
&\left\| f_{\frac{k}{n}}\left(B^j_{\frac{k+1}{n}\wedge t} - B^j_{\frac{k}{n}}\right)^2\right\|^{2+\mu}_{L^{2+\mu}} \\
\le\ & \mathbb{E}[|f_{\frac{k}{n}}|^{2+\gamma}]^{\frac{2+\mu}{2+\gamma}}\mathbb{E}\left[\left(B^j_{\frac{k+1}{n}\wedge t} - B^j_{\frac{k}{n}}\right)^{2q(2+\mu)}\right]^{\frac{1}{q}} \\
\le\ & \mathbb{E}[\|f\|_\infty^{2+\gamma}]^{\frac{2+\mu}{2+\gamma}} n^{2H(2+\mu)},
\end{aligned}$$

with q the conjugate of $\frac{2+\gamma}{2+\mu}$. This implies that $\sup_n \mathbb{E}\left[|S^{n,j}_{t,f}|^{2+\mu}\right] < \infty$, and then the sequence $\{(S^{n,j}_{t,f})^2\}_{n=1}^\infty$ is uniformly integrable.

Back to the initial process x, we know, by uniform continuity of x on $[0,T]$, that $\|x-x^m\|_\infty \underset{m\to\infty}{\longrightarrow} 0$ a.s. (where x^m is the sampled process $x_{\frac{\lfloor m\cdot\rfloor}{m}}$). As a result,

$$
\begin{aligned}
&\mathbb{E}\left[\left(S^{n,j}_{t,x}-\int_0^t x_s ds\right)^2\right]\\
=\ &\mathbb{E}\left[\left((S^{n,j}_{t,x}-S^{n,j}_{t,x^m})+\left(S^{n,j}_{t,x^m}-\int_0^t x^m_s ds\right)+\left(\int_0^t x^m_s ds-\int_0^t x_s ds\right)\right)^2\right]\\
\leq\ &C\left\{n^{4H-2}\mathbb{E}\left[\left(\sum_{k=0}^{nt_n}\|x-x^m\|_\infty\left(B_{\frac{k+1}{n}\wedge t}-B_{\frac{k}{n}}\right)^2\right)^2\right]\right.\\
&\left.+\mathbb{E}\left[\left(S^{n,j}_{t,x^m}-\int_0^t x^m_s ds\right)^2\right]+T^2\mathbb{E}\left[\|x-x^m\|^2_\infty\right]\right\},
\end{aligned}
$$

where C is a positive constant. The previous arguments, an appropriate choice of $n,m\in\mathbb{N}^*$ and the fact that $\|x-x^m\|_\infty \overset{L^{2+\gamma}(\Omega)}{\underset{n\to\infty}{\longrightarrow}} 0$ allow to conclude.

Step 2: UCP convergence of $S^{n,j}_{\cdot,x}$. According to Lemma 2.4.5, the UCP convergence of $S^{n,j}_{\cdot,x}$ to $\int_0^\cdot x_s ds$ follows from the convergence in probability of $S^{n,j}_{t,x}$ for fixed t and the tightness of the sequence $(S^{n,j}_{\cdot,x})_n$ in $\mathcal{C}([0,T])$. The convergence in probability for fixed t is shown in Step 1. For the tightness, this can be checked with Lemma 2.4.4 applied to $S^{n,j}_{t,x}=A^n_t+C^n_t$, with

$$
\begin{aligned}
A^n_t &= n^{2H-1}\sum_{k=0}^{nt_n-1} x_{\frac{k}{n}}\left(B^j_{(\frac{k+1}{n})\wedge t}-B^j_{\frac{k}{n}}\right)^2\\
C^n_t &= n^{2H-1}x_{t_n}\left(B^j_t-B^j_{t_n}\right)^2\\
\alpha_0 &= 1, \beta_0=2.
\end{aligned}
$$

Indeed, using the Hölder inequality, we have for $s\leq t$:

$$
\begin{aligned}
&\mathbb{E}[|A^n_t-A^n_s|^2]\\
\leq\ &n^{4H-2}\sum_{k,l=ns_n}^{nt_n-1}\left(\mathbb{E}\left[\|x\|^{2+\gamma}_\infty\right]\right)^{\frac{1}{1+\frac{\gamma}{2}}}\left(\mathbb{E}\left[\left|B^j_{\frac{k+1}{n}}-B^j_{\frac{k}{n}}\right|^{2p'}\left|B^j_{\frac{l+1}{n}}-B^j_{\frac{l}{n}}\right|^{2p'}\right]\right)^{\frac{1}{p'}}\\
\leq\ &n^{4H}|t_n-s_n|^2\left(\mathbb{E}\left[\|x\|^{2+\gamma}_\infty\right]\right)^{\frac{1}{1+\frac{\gamma}{2}}}\left(\mathbb{E}\left[|B_{\frac{1}{n}}|^{4p'}\right]\right)^{\frac{1}{p'}}\leq K|t_n-s_n|^2
\end{aligned}
$$

with p' the conjugate of $1+\frac{\gamma}{2}$ and K some constant depending only on γ and x.

On the other hand, B has $(H-\epsilon)$-Hölder continuous paths for every $\epsilon>0$, so that, for all $t\in[0,T]$, $|C_t^n|\leq K_\epsilon n^{2\epsilon-1}\|x\|_\infty$ a.s. for some random variable $K_\epsilon>0$. Taking ϵ small enough, we have $\sup_{t\in[0,T]}|C_t^n| \xrightarrow[n\to\infty]{\mathbb{P}} 0$.

Step 3: Proof of (2.41). We now turn to the case $i\neq j$. Similarly to the proof of (2.40) (Step 1), we first show (2.41) for x the function identically one, in other words:

$$S_{t,1}^{n,i,j}=n^{2H-1}\sum_{k=0}^{nt_n}\delta^{1,i}\left(\left(B_\cdot^j-B_{\frac{k}{n}}^j\right)\mathbb{I}_{[\frac{k}{n},\frac{k+1}{n}\wedge t]}(\cdot)\right)\xrightarrow{L^2(\Omega)}0. \tag{2.42}$$

Using Proposition 2.2.1 and taking into account that $D^{1,i}B^j=0$ if $i\neq j$, we have:

$$\begin{aligned}
&\mathbb{E}\left[(S_{t,1}^{n,i,j})^2\right]\\
=\;& n^{4H-2}\sum_{k,l=0}^{nt_n}\mathbb{E}\left[\left\langle\left(B_\cdot^j-B_{\frac{k}{n}}^j\right)\mathbb{I}_{[\frac{k}{n},\frac{k+1}{n}\wedge t]}(.),\left(B_\cdot^j-B_{\frac{l}{n}}^j\right)\mathbb{I}_{[\frac{l}{n},\frac{l+1}{n}\wedge t]}(.)\right\rangle_{\mathcal{H}}\right]\\
=\;& n^{4H-2}\sum_{k,l=0}^{nt_n}\int_{\frac{k}{n}}^{\frac{k+1}{n}\wedge t}\int_{\frac{l}{n}}^{\frac{l+1}{n}\wedge t}\mathbb{E}[(B_x-B_{\frac{k}{n}})(B_y-B_{\frac{l}{n}})]\mu_H(dydx)\\
\leq\;& n^{2H-2}\sum_{k,l=0}^{nt_n}\langle\mathbb{I}_{[\frac{k}{n},\frac{k+1}{n}\wedge t]}(.),\mathbb{I}_{[\frac{l}{n},\frac{l+1}{n}\wedge t]}(\cdot)\rangle_{\mathcal{H}},
\end{aligned}$$

where the last inequality follows from the fact that: for all $x\in\left[\frac{k}{n},\frac{k+1}{n}\right]$ and all $y\in\left[\frac{l}{n},\frac{l+1}{n}\right]$,
$|\mathbb{E}[(B_x-B_{\frac{k}{n}})(B_y-B_{\frac{l}{n}})]|\leq\left(\mathbb{E}[(B_x-B_{\frac{k}{n}})^2]\right)^{\frac12}\left(\mathbb{E}[(B_y-B_{\frac{l}{n}})^2]\right)^{\frac12}\leq\frac{1}{n^{2H}}$.
We also have

$$\sum_{k,l=0}^{nt_n}\langle\mathbb{I}_{[\frac{k}{n},\frac{k+1}{n}\wedge t]}(\cdot),\mathbb{I}_{[\frac{l}{n},\frac{l+1}{n}\wedge t]}(.)\rangle_{\mathcal{H}}=t^{2H},$$

and then $\mathbb{E}\left[(S_{t,1}^{n,i,j})^2\right]=O_{n\to\infty}(n^{2H-2})$, implying (2.42). To prove (2.41) in the general case for x, we can then proceed exactly as in the proof of (2.40), that is, we show first that (2.41) holds true for step processes and then, by an approximation argument, to x. Tightness in $\mathcal{C}([0,T])$ can also be obtained as for (2.40). By Lemma 2.4.5, this proves the UCP convergence to 0 of each $S_{\cdot,x}^{n,i,j}$ with $i\neq j$. $\square$

The study of the fluctuations (which are required for the proof of Theorem 2.1.3) being more delicate, more stringent assumptions on the process x are required (except when $H = \frac{1}{2}$, see the first point in the proposition below).

Lemma 2.4.9. *Let $x = (x^{i,e})_{1 \le i \le m, 1 \le e \le d}$ be an $(m \times d)$-dimensional process, and recall the matrix-valued processes W and Z from Section 2.2.5. For any $1 \le i \le m$ and $1 \le e, j \le d$, set*

$$S^{n,i,j,e}_{t,x} = \sum_{k=0}^{nt_n} x^{i,e}_{\frac{k}{n}} \int_{\frac{k}{n}}^{\frac{k+1}{n} \wedge t} \left(B^e_s - B^e_{\frac{k}{n}} \right) \delta B^j_s.$$

We have

1. *If $H = \frac{1}{2}$ and if, for all (i,e), $x^{i,e}$ is adapted to B, piecewise continuous and satisfies $\mathbb{E}[\sup_{i,e} \|x^{i,e}\|_\infty^{2+\gamma}] < \infty$ for some $\gamma > 0$, then, stably in $\mathcal{C}_{\mathbb{R}^{d^2 \times m}}([0,T])$,*

$$\left(\sqrt{n} S^{n,i,j,e}_{\cdot,x} \right)_{i,j,e} \underset{n \to \infty}{\longrightarrow} \left(\int_0^\cdot x^{i,e}_s dW^{e,j}_s \right)_{i,j,e}. \tag{2.43}$$

2. *If $\frac{1}{2} < H \le \frac{3}{4}$ and if x is β-Hölder continuous for some $\beta > \frac{1}{2}$, then, stably in $\mathcal{C}_{\mathbb{R}^{d^2 \times m}}([0,T])$,*

$$\left(n^{2H-1} \nu_H(n) S^{n,i,j,e}_{\cdot,x} \right)_{i,j,e} \underset{n \to \infty}{\longrightarrow} \left(\int_0^\cdot x^{i,e}_s dW^{e,j}_s \right)_{i,j,e}. \tag{2.44}$$

3. *If $H > \frac{3}{4}$ and if x verifies that $\mathbb{E}\left[\|x\|_\beta^{2+\gamma} \right] < +\infty$ for some $\beta > \frac{1}{2}$ and $\gamma > 0$ then, in probability uniformly on $[0,T]$ (and also in $L^2(\Omega)$ for fixed t),*

$$\left(n^{2H-1} \nu_H(n) S^{n,i,j,e}_{\cdot,x} \right)_{i,j,e} \underset{n \to \infty}{\longrightarrow} \left(\int_0^\cdot x^{i,e}_s dZ^{e,j}_s \right)_{i,j,e}. \tag{2.45}$$

Proof. Even if they are not stated in exactly the same way, the limits (2.44) and (2.45) follow from [5, 13] (see especially [13, Sections 4,5,7]) by means of fractional integration techniques. This is why we only concentrate on the case $H = \frac{1}{2}$ and the proof of (2.43), not covered by [5, 13].

<u>*Proof of (2.43).*</u> We divide the proof into three steps. In the sequel, 'f.d.d.' is shorthand for finite dimensional distributions.

Step 1: Convergence of the f.d.d. when x is a step process: Let us first sketch the proof in the case where x is constant over an interval, without going too much into the details, since the approach is very similar to that in [13, Section 5]. Let $0 \leq s \leq t \leq T$, let $q \in \mathbb{N}^*$, let $0 = a_0 \leq a_1 \leq \ldots \leq a_q \leq t$ and let x be the matrix function whose entries are all equal to $\mathbb{I}_{[s,t]}$. It is immediate that the $\mathbb{R}^{d^2 m}$-valued random vector

$$X_x^n = ((\sqrt{n} S_{a_{l+1},x}^{n,i,j,e} - \sqrt{n} S_{a_l,x}^{n,i,j,e})_{i,j,e})_{l \in \{0,\ldots,q-1\}}$$

has independent entries. We can also check that

$$\mathbb{E}[(X_x^n)_l^{i_1,j_1,e_1} (X_x^n)_l^{i_2,j_2,e_2}] = 0$$

for all i_1, i_2, all $(j_1, e_1) \neq (j_2, e_2)$ and all $l \in \{0, \ldots, q-1\}$. Finally, we can easily show that

$$\mathbb{E}[\Big((X_x^n)_l^{i_1,j_1,e_1}\Big)^4] \underset{n\to\infty}{\longrightarrow} \frac{3}{4}(a_{l+1} \vee s - a_l \vee s)^2.$$

Peccati and Tudor's *fourth moment theorem* [28] applies, and shows the stable convergence

$$X_x^n \xrightarrow[n\to\infty]{\mathcal{L}} \left(\left(W_{a_{l+1}\vee s}^{e,j} - W_{a_l \vee s}^{e,j}\right)\right)_{i,j,e}\Bigg)_l .$$

Since the increments are independent, this gives the convergence of the finite dimensional distributions in (2.43) when $x = \mathbb{I}_{[s,t]}$.

Now, let $[a_1, b_1], \ldots, [a_q, b_q]$ be q mutually disjoint intervals. Due to the independence of Brownian increments, the process

$$\left(\sqrt{n} S_{\cdot,\mathbb{I}_{[a_1,b_1]}}^{n,i,j,e}, \ldots, \sqrt{n} S_{\cdot,\mathbb{I}_{[a_q,b_q]}}^{n,i,j,e}\right)_{i,j,e}$$

has independent entries, so we have the stable convergence of its f.d.d. to the f.d.d. of the process $\left(\int_0^T \mathbb{I}_{[a_1,b_1]} dW_s^{e,j}, \ldots, \int_0^T \mathbb{I}_{[a_q,b_q]} dW_s^{e,j}\right)_{i,j,e}$. This implies in turn the convergence of the f.d.d. of $\sqrt{n} S_{\cdot,x}^n$ for processes x of the form:

$$x = \sum_{l=0}^{q-1} F_l \mathbb{I}_{[a_l, a_{l+1}]},$$

where $q \in \mathbb{N}^*$ and F_l is a $\mathbb{R}^{m\times d}$ valued and $\mathfrak{F}_{a_l}$-measurable random variable.

Step 2: Convergence of the f.d.d in the general case: We now turn to the general case. Let x be an adapted, almost surely piecewise continuous process such that $\mathbb{E}[\sup_{i,e} \|x^{i,e}\|_\infty^2] < \infty$, and set

$$\Delta_{e,j,k,n}(t) = \int_{\frac{k}{n}}^{\frac{k+1}{n}\wedge t} \left(B_s^e - B_{\frac{k}{n}}^e\right) \delta B_s^j.$$

As is the proof of Lemma 2.4.8, we can rely on the *small blocks / big blocks* technique by considering the approximation

$$\sqrt{n} S_{t,x}^{n,i,j,e} = \sqrt{n} S_{t,x^m}^{n,i,j,e} + \sqrt{n}\left(S_{t,x}^{n,i,j,e} - S_{t,x^m}^{n,i,j,e}\right) = \sqrt{n} S_{t,x^m}^{n,i,j,e} + R_{t,m,n,x}^{i,j,e}, \tag{2.46}$$

with $m \leq n$ and x^m the sampled process $x_{\frac{\lfloor m \cdot \rfloor}{m}}$.
Fix $m \in \mathbb{N}^*$. Since x^m is a step process, we have by Step 1 that

$$\text{f.d.d.} - \lim_{n\to\infty} \left(\sqrt{n} S_{\cdot,x^m}^{n,i,j,e}\right)_{i,j,e} = \left(\int_0^{\cdot} (x^m)_s^{i,e} dW_s^{e,j}\right)_{i,j,e}.$$

Morever, for all $t \in [0,T]$,

$$L^2(\Omega) - \lim_{m\to\infty} \left(\int_0^t (x^m)_s^{i,e} dW_s^{e,j}\right)_{i,j,e} = \left(\int_0^t x_s^{i,e} dW_s^{e,j}\right)_{i,j,e},$$

thanks to the isometry property of Itô integral. Putting these two facts together , we deduce that

$$\text{f.d.d.} - \lim_{m\to\infty} \lim_{n\to\infty} \left(\sqrt{n} S_{\cdot,x^m}^{n,i,j,e}\right)_{i,j,e} = \left(\int_0^{\cdot} x_s^{i,e} dW_s^{e,j}\right)_{i,j,e}.$$

To conclude that $\text{f.d.d.} - \lim_{n\to\infty} \left(\sqrt{n} S_{\cdot,x}^{n,i,j,e}\right)_{i,j,e} = \left(\int_0^{\cdot} x_s^{i,e} dW_s^{e,j}\right)_{i,j,e}$, and given the decomposition (2.46), it remains to show that

$$\lim_{m\to\infty} \sup_{n\geq m} \sup_{t\in[0,T]} \mathbb{E}[(R_{t,m,n,x}^{i,j,e})^2] = 0, \tag{2.47}$$

which we do now.

We have, for all t,

$$
\begin{aligned}
& \mathbb{E}[(R^{i,j,e}_{t,m,n,x})^2] \\
= \ & n \sum_{l,k=1}^{nt_n} \mathbb{E}\left[\left(x^{i,e}_{\frac{k}{n}} - (x^m)^{i,e}_{\frac{k}{n}}\right)\left(x^{i,e}_{\frac{l}{n}} - (x^m)^{i,e}_{\frac{l}{n}}\right)\Delta_{e,j,k,n}(t)\Delta_{e,j,l,n}(t)\right] \\
= \ & n \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}\left[\left(x^{i,e}_{\frac{k}{n}} - (x^m)^{i,e}_{\frac{k}{n}}\right)^2 \Delta_{e,j,k,n}(t)^2\right] \\
& \text{(since } x \text{ is adapted and the increments of the} \\
& \text{Brownian motion are independent)} \\
\leq \ & n \sum_{k=1}^{\lfloor nT \rfloor} \mathbb{E}\left[\left(x^{i,e}_{\frac{k}{n}} - (x^m)^{i,e}_{\frac{k}{n}}\right)^2\right] \mathbb{E}\left[\Delta_{e,j,k,n}(t)^2\right] \\
\leq \ & \frac{T}{2} \max_{k \in \{1,\dots,\lfloor nT \rfloor\}} \mathbb{E}\left[\left(x^{i,e}_{\frac{k}{n}} - (x^m)^{i,e}_{\frac{k}{n}}\right)^2\right].
\end{aligned}
$$

Let

$$
N^{i,e} = \mathrm{Card}\left\{t \in [0,T], |x^{i,e}_t - x^{i,e}_{t-}| + |x^{i,e}_t - x^{i,e}_{t+}| > 0\right\},
$$

which is almost surely finite because x is piecewise continuous. Let $T^{i,e}_l$ be the l-th (random) discontinuity of $x^{i,e}$ ($T^{i,e}_l(\omega) = +\infty$ if $x^{i,e}(\omega)$ has less than l discontinuities over $[0,T]$). It is clear that $T^{i,e}_l$ is measurable as a stopping time. Let $E^m_{i,e} = \cup_{l \in \mathbb{N}^*}(T^{i,e}_l - \frac{1}{m}, T^{i,e}_l + \frac{1}{m}) \cap [0,T]$. Then,

$$
\lim_{m\to\infty} \sup_{n>m} \frac{\mathrm{Card}\{k, \frac{k}{n} \in E^m_{i,e}\}}{n} = \lim_{m\to\infty} (\frac{2N^{i,e}}{m} \wedge 1) = 0 \quad \text{a.s..}
$$

Observe that $x^{i,e}$ is a.s. uniformly continuous on $[0,T] \setminus \cup_l\{T^{i,e}_l\}$. Moreover, if $s \in (E^m_{i,e})^c$ for some m, then there is no discontinuities between $s_m = \frac{\lfloor ms \rfloor}{m}$ and s. Then,

$$
|x^{i,e}_s - (x^m_s)^{i,e}| \leq X^{i,e}_m \mathbb{I}_{(E^m_{i,e})^c}(s) + 2\|x^{i,e}\|_\infty \mathbb{I}_{E^m_{i,e}}(s),
$$

with

$$
X^{i,e}_m = \sup_{s \in (E^m_{i,e})^c} |x^{i,e}_s - x^{i,e}_{s_m}|.
$$

Note that $X^{i,e}_m$ is a sequence of square integrables random variables, which converges a.s. to 0 as $m \to \infty$ and is bounded by the square integrable random variable $2\|x^{i,e}\|_\infty$. Finally, we can write

$$\mathbb{E}[(R^{i,j,e}_{t,m,n,x})^2] \leq \frac{T}{2}\mathbb{E}\left[(X^{i,e}_m)^2 + (\frac{4N^{i,e}}{m} \wedge 1)\|x^{i,e}\|^2_\infty\right].$$

The sequence $\left((X^{i,e}_m)^2 + (\frac{2N^{i,e}}{m} \wedge 1)\|x^{i,e}\|^2_\infty\right)_m$ converges to 0 as $m \to \infty$, and is bounded by a square integrable random variable. The conclusion (2.47) then follows by dominated convergence.

Step 3: Tightness. Let $0 < \mu < \gamma$. We have, for all i, j, e, all $s \leq t$ and all $n \in \mathbb{N}^*$,

$$\begin{aligned}\mathbb{E}\left[\left|S^{n,i,j,e}_{t,x} - S^{n,i,j,e}_{s,x}\right|^{2+\mu}\right] &\leq |t-s|^{\frac{\mu}{2}} \int_s^t \mathbb{E}\left[\left|x^{n,i,j}_{s_n}(B^{j,e}_s - B^{j,e}_{s_n})\right|^{2+\mu}\right] ds \\ &\leq K|t-s|^{1+\frac{\mu}{2}}\mathbb{E}[\|x^{i,j}\|^{2+\gamma}]^{\frac{2+\gamma}{2+\mu}}\end{aligned}$$

where the first inequality is obtained by applying the Burkholder and Jensen inequalities, and the second inequality is obtained by applying the Hölder inequality. This prove the tightness in $\mathcal{C}_\mathbb{R}([0,T])$ of each component of S^n_x, and conclude the proof of (2.43).

□

Remark 2.4.10. In the case $H = \frac{1}{2}$, notice that the hypothesis $\mathbb{E}[\|x\|^{2+\gamma}_\infty] < \infty$ is only needed to obtain the tightness of the process. For the convergence of the f.d.d., the hypothesis $\mathbb{E}[\|x\|^2_\infty] < \infty$ is sufficient.

Finally, the following lemma is used in the proof of Proposition 2.3.3.

Lemma 2.4.11. *Let b be a piecewise continuous process such that $\mathbb{E}[\|b\|^{2+\gamma}_\infty] < \infty$ for some $\gamma > 0$. Then:*

- *For $H > \frac{3}{4}$, in probability uniformly on $[0,T]$,*

$$\nu_H(n)n^{2H-1}\sum_{k=0}^{\lfloor n\cdot\rfloor} b_{\frac{k}{n}} \int_{\frac{k}{n}}^{\frac{k+1}{n}\wedge\cdot} (s - s_n)dB^i_s \longrightarrow \frac{1}{2}\int_0^\cdot b_s dB^i_s.$$

- *For $\frac{1}{2} \leq H \leq \frac{3}{4}$, in probability uniformly on $[0,T]$,*

$$\nu_H(n)n^{2H-1}\sum_{k=0}^{\lfloor n\cdot\rfloor} b_{\frac{k}{n}} \int_{\frac{k}{n}}^{\frac{k+1}{n}\wedge\cdot} (s - s_n)dB^i_s \longrightarrow 0.$$

Proof. The proof in the case $b = \mathbb{I}_{[0,t]}$ is done in [13]. Similar arguments as in Lemma 2.4.8 allow to conclude. □

2.4.3 Proof of Theorem 2.1.2 and 2.1.3

Proof of Theorem 2.1.2: For $s \in [0, T]$, recall that $s_n = \frac{\lfloor ns \rfloor}{n}$ and

$$M_t^{n,i,j} = n^{2H-1} \left(\int_0^t u_s^i dB_s^j - \sum_{k=0}^{\lfloor nt \rfloor} u_{\frac{k}{n}}^i \left(B_{\frac{k+1}{n} \wedge t}^j - B_{\frac{k}{n}}^j \right) \right).$$

We have

$$\begin{aligned} M_t^{n,i,j} &= n^{2H-1} \int_0^t (u_s^i - u_{s_n}^i) dB_s^j \\ &= n^{2H-1} \int_0^t \sum_{e=1}^d P_{s_n}^{i,e} (B_s^e - B_{s_n}^e) dB_s^j \\ &\quad + n^{2H-1} \int_0^t \left(u_s^i - u_{s_n}^i - \sum_{e=1}^d P_{s_n}^{i,e} (B_s^e - B_{s_n}^e) \right) dB_s^j \\ &= A_t^{n,i,j} + \sum_{e \neq j} R_t^{n,e} + R_t^{n,i,j}, \end{aligned}$$

with

$$\begin{aligned} A_t^{n,i,j} &= \frac{1}{2} n^{2H-1} \sum_{k=0}^{nt_n} P_{\frac{k}{n}}^{i,j} \left(B_{\frac{k+1}{n} \wedge t}^j - B_{\frac{k}{n}}^j \right)^2 \\ R_t^{n,e} &= n^{2H-1} \sum_{k=0}^{nt_n} P_{\frac{k}{n}}^{i,e} \int_{\frac{k}{n}}^{\frac{k+1}{n} \wedge t} \left(B_s^e - B_{\frac{k}{n}}^e \right) dB_s^j, \quad e \neq j \\ R_t^{n,i,j} &= n^{2H-1} \int_0^t \left(u_s^i - u_{s_n}^i - \sum_{e=1}^d P_{s_n}^{i,e} (B_s^e - B_{s_n}^e) \right) dB_s^j. \end{aligned}$$

Lemma 2.4.8 implies the $L^2(\Omega)$-convergence of $A_t^{n,i,j}$ to $\frac{1}{2} \int_0^t P_s^{i,j} ds$. We show that all the additional terms converge to 0 in $L^2(\Omega)$-norm as $n \to \infty$. If $e \neq j$, $D^{1,j} B^e = 0$, so according to Proposition 2.2.4

$$\int_{\frac{k}{n}}^{\frac{k+1}{n} \wedge t} \left(B_s^e - B_{\frac{k+1}{n}}^e \right) dB_s^j = \int_{\frac{k}{n}}^{\frac{k+1}{n} \wedge t} \left(B_s^e - B_{\frac{k+1}{n}}^e \right) \delta B_s^j.$$

Lemma 2.4.8 then implies the $L^2(\Omega)$ and UCP convergence of every $R_t^{n,e}$ to 0 for all $e \neq j$ and $t \in [0, T]$. Moreover, $(u, P) \in \mathbb{C}_1$, so the equation (2.3) implies that

$$\begin{aligned}\mathbb{E}\left[\left(R_t^{n,i,j}\right)^2\right] &= n^{4H-2}\mathbb{E}\left[\left(\sum_{k=0}^{nt_n} L^{i,j}_{\frac{k}{n},\frac{k+1}{n}\wedge t}\right)^2\right] \\ &= n^{4H-2}\sum_{l=0}^{nt_n}\sum_{k=0}^{nt_n}\mathbb{E}\left[L^{i,j}_{\frac{k}{n},\frac{k+1}{n}\wedge t}L^{i,j}_{\frac{l}{n},\frac{l+1}{n}\wedge t}\right] \\ &= \epsilon(n)\sum_{l=0}^{nt_n}\sum_{k=0}^{nt_n} r_H\left(\frac{k}{n},\frac{k+1}{n}\wedge t,\frac{l}{n},\frac{l+1}{n}\wedge t\right) \leq T^{2H}\epsilon(n),\end{aligned}$$

with $\epsilon(n) \underset{n\to\infty}{\longrightarrow} 0$.

Thanks to Lemma 2.4.5, we can now show that, for all $i,j \in \{1,\dots,d\}$, the sequences $(R_t^{n,i,j})_n$ converges UCP to 0 as $n \to \infty$, by checking their tightness in $\mathcal{C}_{\mathbb{R}}([0,T])$. We have

$$R_t^{n,i,j} = n^{2H-1}\sum_{k=0}^{nt_n-1} L^{i,j}_{\frac{k}{n},\frac{k+1}{n}} + n^{2H-1}L^{i,j}_{t_n,t} \tag{2.48}$$

Thanks to (2.3), we have

$$\begin{aligned}\mathbb{E}\left[\left(n^{2H-1}\sum_{k=ns_n}^{nt_n-1} L^{i,j}_{\frac{k}{n},\frac{k+1}{n}}\right)^2\right] &\leq K\sum_{l,k=ns_n}^{nt_n-1} r_H\left(\frac{k}{n},\frac{k+1}{n},\frac{l}{n}\frac{l+1}{n}\right) \\ &= K(t_n-s_n)^{2H},\end{aligned}$$

for some $K>0$. Moreover, let $\epsilon \in (0,\alpha-(1-H))$ be small enough (let us recall that α (resp β) is the Hölder exponent of u (resp P)). The second term in the right-hand side of (2.48) verifies (due to the regularity and integrability assumptions on u and P, as well as the Young-Loeve inequality):

$$\begin{aligned}&\sup_{t\in[0,T]}\left(n^{2H-1}|L^{i,j}_{t_n,t}|\right) \\ \leq\ & c_{\alpha-\frac{\epsilon}{2},H-\frac{\epsilon}{2}}n^{2H-1}n^{-(H+\alpha-\epsilon)}\|B\|_{H-\frac{\epsilon}{2}}\|u^i\|_{\alpha-\frac{\epsilon}{2}} \\ &+c_{H-\frac{\epsilon}{2},H-\frac{\epsilon}{2}}n^{-1+\epsilon}\|P^{i,j}\|_\infty\|B\|^2_{H-\frac{\epsilon}{2}} \underset{n\to\infty}{\longrightarrow} 0 \quad \text{a.s..}\end{aligned}$$

Then, the sequence $R_t^{n,i,j}$ verifies the assumptions of Lemma 2.4.4, with $A_t^{n,i,j} = n^{2H-1}\sum_{k=0}^{nt_n-1} L^{i,j}_{\frac{k}{n},\frac{k+1}{n}}$, $C_t^{n,i,j} = n^{2H-1}L^{i,j}_{t_n,t}$, $\alpha_0 = 2H-1, \beta_0 = 2$, which proves the tightness. □

Proof of Theorem 2.1.3: 1. Let $H > \frac{1}{2}$. Again, we can write

$$M_t^{n,i,j} - \frac{1}{2}\int_0^t P_s^{i,j} ds = M_{j,t}^{n,i,j} + \sum_{e\neq j} M_{e,t}^{n,i,j} + R_{1,t}^{n,i,j} + R_{2,t}^{n,i,j},$$

where, for $1 \le i \le m$ and $1 \le j \neq e \le d$,

$$\begin{aligned}
M_{j,t}^{n,i,j} &= n^{2H-1}\sum_{k=0}^{nt_n} P_{\frac{k}{n}}^{i,j}\left(\int_{\frac{k}{n}}^{\frac{k+1}{n}\wedge t}\left(B_s^j - B_{\frac{k}{n}}^j\right) dB_s^j - \frac{1}{2}\left(\frac{k+1}{n}\wedge t - \frac{k}{n}\right)^{2H}\right)\\
M_{e,t}^{n,i,j} &= n^{2H-1}\sum_{k=0}^{nt_n} P_{\frac{k}{n}}^{i,e}\int_{\frac{k}{n}}^{\frac{k+1}{n}\wedge t}\left(B_s^e - B_{\frac{k}{n}}^e\right) dB_s^j\\
R_{1,t}^{n,i,j} &= \frac{1}{2}\left(\frac{1}{n}\sum_{k=0}^{nt_n-1} P_{\frac{k}{n}}^{i,j} + \frac{1}{n}P_{t_n}^{i,j}(nt - nt_n)^{2H} - \int_0^t P_s^{i,j} ds\right)\\
R_{2,t}^{n,i,j} &= n^{2H-1}\int_0^t\left(u_s^i - u_{s_n}^i - \sum_{e=1}^d P_{s_n}^{i,e}(B_s^e - B_{s_n}^e)\right) dB_s^j.
\end{aligned}$$

Since $(u, P) \in \mathbb{C}_2$, we have that $\nu_H(n)R_{2,t}^{n,i,j} \overset{L^2(\Omega)}{\underset{n\to\infty}{\longrightarrow}} 0$ by using again the formula (2.3). The tightness of the sequence $(\nu_H(n)R_{2,t}^{n,i,j})_n$ can be proved by using the same argument as in the previous proof.

On the other hand, since P is β-Hölder continuous for some $\beta > \frac{1}{2}$ we have that $\sup_{t\in[0,T]}\left|\nu_H(n)R_{1,t}^{n,i,j}\right| \to 0$ a.s., which guarantees the convergence of $\nu_H(n)R_{1,\cdot}^{n,i,j}$ to 0 in $\mathcal{C}_{\mathbb{R}}([0,T])$. When $H > \frac{3}{4}$, since we have the additional hypothesis that $\sum_{i,j}\mathbb{E}[\|P^{i,j}\|_\beta^{2+\gamma}] < \infty$ for some $\gamma > 0$, we can further prove the $L^2(\Omega)$ convergence: for each $t \le T$, $\nu_H(n)R_{1,t}^{n,i,j} \overset{L^2(\Omega)}{\underset{n\to\infty}{\longrightarrow}} 0$.

Finally, using Proposition 2.2.4 we observe that

$$\int_{\frac{k}{n}}^{\frac{k+1}{n}\wedge t}\left(B_s^j - B_{\frac{k}{n}}^j\right) dB_s^j - \frac{(nt - nt_n)}{2n^{2H}} = \int_{\frac{k}{n}}^{\frac{k+1}{n}\wedge t}\left(B_s^j - B_{\frac{k}{n}}^j\right)\delta B_s^j$$

and, if $e \neq j$,

$$\int_{\frac{k}{n}}^{\frac{k+1}{n}\wedge t}\left(B_s^e - B_{\frac{k}{n}}^e\right) dB_s^j = \int_{\frac{k}{n}}^{\frac{k+1}{n}\wedge t}\left(B_s^e - B_{\frac{k}{n}}^e\right)\delta B_s^j.$$

Since P verifies the regularity assumptions of Lemma 2.4.9, we get the stated convergence for all values of $H > \frac{1}{2}$:

- If $\frac{1}{2} < H \leq \frac{3}{4}$,

$$\left\{\nu_H(n)\left(M_{\cdot}^{n,i,j} - \frac{1}{2}\int_0^{\cdot} P_s^{i,j} ds\right)\right\}_{i,j} \underset{n\to\infty}{\longrightarrow} \left\{\int_0^{\cdot} P_s^{i,e} dW_s^{e,j}\right\}_{e,i,j}$$

  where the convergence holds in $\mathcal{C}_{\mathbb{R}^{d\times m}}([0,t])$.

- If $H > \frac{3}{4}$,

$$\left\{\nu_H(n)\left(M_{\cdot}^{n,i,j} - \frac{1}{2}\int_0^{\cdot} P_s^{i,j} ds\right)\right\}_{i,j} \underset{n\to\infty}{\longrightarrow} \left\{\int_0^{\cdot} P_s^{i,e} dZ_s^{e,j}\right\}_{e,i,j}$$

  where the convergence holds UCP (and in $L^2(\Omega)$ for fixed t).

2. Once the necessary modifications are made, the proof is the same for Brownian motion. □

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Chapter 3

Limit theorem for integral functionals of Hermite driven processes

Reproduction of the paper [15], entitled "Limit theorem of integral functionals of Hermite-driven process" and written in collaboration with Ivan Nourdin, David Nualart and Majid Salamat. Bernoulli **27** *(2021), no. 3, pp. 1764–1788.*

3.1 Introduction

Hermite processes occur naturally when we consider limits of partial sums associated with long-range dependent stationary series. They have become increasingly popular in the recent literature, see for example the book [16] by Pipiras and Taqqu, in particular section 4.11, which contains bibliographical notes on their history and recent developments. They form a family of stochastic processes, indexed by an integer $q \geq 1$ and a self-similarity index $H \in (\frac{1}{2}, 1)$, called the Hurst parameter, that contains the fractional Brownian motion ($q = 1$) and the Rosenblatt process ($q = 2$) as particular cases. We refer the reader to Section 3.2.2 and the references therein for a precise definition of the Hermite processes. Of primary importance in the sequel is the parameter H_0, given in terms of H and q by

$$H_0 = 1 - \frac{1-H}{q} \in (1 - \frac{1}{2q}, 1). \tag{3.1}$$

The goal of the present paper is to investigate the fluctuations, as $T \to$

∞, of the family of stochastic processes

$$t \mapsto \int_0^{Tt} P(X(s))ds, \quad t \in [0,1] \text{ (say)}, \tag{3.2}$$

in the case where $P(x)$ is a polynomial function and X is a moving average process of the form

$$X(t) = \int_{-\infty}^{t} \varphi(t-u)dZ_u, \quad t \geq 0, \tag{3.3}$$

with Z a Hermite process and $\varphi : \mathbb{R}_+ \to \mathbb{R}$ a sufficiently integrable function. We note that integral functionals such as (3.2) are often encountered in the context of statistical estimation, see e.g. [21] for a concrete example.

Let us first consider the case where $q = 1$, that is to say the case where Z is the fractional Brownian motion. Note that this is the only case where Z is Gaussian, making the study *a priori* much simpler and more affordable. By linearity and passage to the limit, the process X is also Gaussian. Moreover, it is stationary, since the quantity $\mathbb{E}[X(t)X(s)] =: \rho(t-s)$ only depends on $t-s$. For simplicity and without loss of generality, assume that $\rho(0) = 1$, that is, $X(t)$ has variance 1 for any t. As is well-known since the eighties (see [5, 8, 19]), the fluctuations of (3.2) heavily depends on the centered Hermite rank of P, defined as the integer $d \geq 1$ such that P decomposes in the form

$$P = \mathbb{E}[P(X(0))] + \sum_{k=d}^{\infty} a_k H_k, \tag{3.4}$$

with H_k the kth Hermite polynomials and $a_d \neq 0$. (Note that the sum (3.4) is actually finite, since P is a polynomial, so that $\#\{k : a_k \neq 0\} < \infty$.)

The first result of this paper concerns the fractional Brownian motion. Even if it does not follow directly from the well-known results of Breuer-Major [5], Dobrushin-Major [8] and Taqqu [19], the limits obtained are somehow expected. In particular, the threshold $H = 1 - \frac{1}{2d}$ is well known to specialists. However, the proof of this result is not straightforward, and requires several estimations which are interesting in themselves.

Theorem 1. *Let Z be a fractional Brownian motion of Hurst index $H \in (\frac{1}{2}, 1)$, and let $\varphi \in L^1(\mathbb{R}_+) \cap L^{\frac{1}{H}}(\mathbb{R}_+)$. Consider the moving average process X defined by (3.3) and assume without loss of generality that* $\mathrm{Var}(X(0)) = 1$ *(if not, it suffices to multiply φ by a constant). Finally, let $P(x) = \sum_{n=0}^{N} a_n x^n$ be a real-valued polynomial function, and let $d \geq 1$ denotes its centered Hermite rank.*

1. *If $d \geq 2$ and $H \in (\frac{1}{2}, 1 - \frac{1}{2d})$ then*

$$T^{-\frac{1}{2}} \left\{ \int_0^{Tt} \big(P(X(s)) - \mathbb{E}[P(X(s))] \big) ds \right\}_{t \in [0,1]} \tag{3.5}$$

converges in distribution in $C([0,1])$ to a standard Brownian motion W, up to some multiplicative constant C_1 which is explicit and depends only on φ, P and H.

2. *If $H \in (1 - \frac{1}{2d}, 1)$ then*

$$T^{d(1-H)-1} \left\{ \int_0^{Tt} \big(P(X(s)) - \mathbb{E}[P(X(s))] \big) ds \right\}_{t \in [0,1]} \tag{3.6}$$

converges in distribution in $C([0,1])$ to a Hermite process of index d and Hurst parameter $1 - d(1-H)$, up to some multiplicative constant C_2 which is explicit and depends only on φ, P and H.

Now, let us consider the non-Gaussian case, that is, the case where $q \geq 2$. As we will see, the situation is completely different, both in the results obtained (rather unexpected) and in the methods used (very different from the Gaussian case). Let $L > 0$. We define $\mathcal{S}_L$ to be the set of bounded functions $l : \mathbb{R}_+ \to \mathbb{R}$ such that $y^L l(y) \to 0$ as $y \to \infty$. We observe that $\mathcal{S}_L \subset L^1(\mathbb{R}_+) \cap L^{\frac{1}{H}}(\mathbb{R}_+)$ for any $L > 1$. We can now state the following result.

Theorem 2. *Let Z be a Hermite process of order $q \geq 2$ and Hurst parameter $H \in (\frac{1}{2}, 1)$, and let $\varphi \in \mathcal{S}_L$ for some $L > 1$. Recall H_0 from (3.1) and consider the moving average process X defined by (3.3). Finally, let $P(x) = \sum_{n=0}^{N} a_n x^n$ be a real-valued polynomial function. Then, one and only one of the following two situations takes place at $T \to \infty$:*

(i) If q is odd and if $a_n \neq 0$ for at least one odd $n \in \{1, \dots, N\}$, then

$$T^{-H_0} \left\{ \int_0^{Tt} \big(P(X(s)) - \mathbb{E}[P(X(s))] \big) ds \right\}_{t \in [0,1]}$$

converges in distribution in $C([0,1])$ to a fractional Brownian motion of parameter $H_1 := H_0$, up to some multiplicative constant K_1 which is explicit and depends only on φ, P, q and H, see Remark 4.

(ii) If q is even, or if q is odd and $a_n = 0$ for all odd $n \in \{1, \dots, N\}$, then

$$T^{1-2H_0} \left\{ \int_0^{Tt} \big(P(X(s)) - \mathbb{E}[P(X(s))]\big) ds \right\}_{t \in [0,1]}$$

converges in distribution in $C([0,1])$ to a Rosenblatt process of Hurst parameter $H_2 := 2H_0 - 1$, up to some multiplicative constant K_2 which is explicit and depends only on φ, P, q and H, see Remark 4.

Remark 3. Whether in Theorem 1 or Theorem 2, the multiplicative constants appearing in the limit can be all given explicitly by following the respective proofs. For example, the constant K_1 and K_2 of Theorem 2 are given by the following intricate expressions:

$$\begin{aligned} K_1 &= \sum_{n=3, n \,\text{odd}}^{N} a_n c_{H,q}^n K_{\varphi,n,1} \\ K_2 &= \sum_{n=2}^{N} a_n c_{H,q}^n K_{\varphi,n,2} + a_1 \mathbb{I}_{\{q=2\}} \int_{\mathbb{R}_+} \varphi(\nu) d\nu \end{aligned}$$

with

$$K_{\varphi,n,i} = \sum_{\alpha \in A_{n,q}, nq - 2|\alpha| = i} \frac{C_\alpha K_{\varphi,\alpha,H_0}}{c_{H_i,i}}, \quad i = 1, 2,$$

where the sets and constants in the previous formula are defined in Sections 3.2, 3.3 and 3.4.

Remark 4. Note that, unlike the case of a fractional Brownian motion X, where the limit depends on the Hermite rank of the polynomial P, here the Hermite rank of P plays no role and the limit depends on the parity of the nonvanishing coefficients of P. This is not really surprising in our *non-Gaussian* context, since the Hermite rank of P is defined by means of its decomposition into Hermite polynomials, and these latter polynomials only have good probabilistic properties when evaluated in *Gaussian* random variables.

We note that our Theorem 2 contains as a very particular case the main result of [21], which corresponds to the choice $P(x) = x^2$ and thus situation (ii). Moreover, let us emphasize that our Theorem 2 not only studies the convergence of finite-dimensional distributions as in [21], but also provides a *functional* result.

Because the employed method is new, let us sketch the main steps of the proof of Theorem 2, by using the classical notation of the Malliavin calculus (see Section 2 for any unexplained definition or result); in particular we write $I_p^B(h)$ to indicate the pth multiple Wiener-Itô integral of kernel h with respect to the standard (two-sided) Brownian motion B.

(Step 1) In Section 3.3, we represent the moving average process X as a qth multiple Wiener-Itô integral with respect to B:

$$X(t) = c_{H,q}\, I_q^B(g(t,\cdot)),$$

where $c_{H,q}$ is an explicit constant and the kernel $g(t,\cdot)$ is given by

$$g(t,\xi_1,\ldots,\xi_q) = \int_{-\infty}^{t} \varphi(t-v) \prod_{j=1}^{q} (v-\xi_j)_+^{H_0-\frac{3}{2}} dv, \tag{3.7}$$

for $\xi_1,\ldots,\xi_q \in \mathbb{R}$, $t \geq 0$. Thanks to this representation, we compute in Lemma 5 the chaotic expansion of the nth power of $X(t)$ for any $n \geq 2$ and $t > 0$, and obtain an expression of the form

$$X^n(t) = c_{H,q}^n \sum_{\alpha \in A_{n,q}} C_\alpha I^B_{nq-2|\alpha|}(\otimes_\alpha(g(t,\cdot),\ldots,g(t,\cdot))),$$

where we have used the novel notation $\otimes_\alpha(g(t,\cdot),\ldots,g(t,\cdot))$ to indicate iterated contractions whose precise definition is given in Section 3.2.1, and where C_α are combinatorial constants and the sum runs over a family $A_{n,q}$ of suitable multi-indices $\alpha = (\alpha_{ij}, 1 \leq i < j \leq n)$. As an immediate consequence, we deduce that our quantity of interest can be decomposed as follows:

$$\int_0^{Tt} \big(P(X(s)) - \mathbb{E}[P(X(s))]\big) ds = a_0 \int_0^{Tt} X(s) ds \tag{3.8}$$
$$+ \sum_{n=2}^{N} a_n\, c_{H,q}^n \sum_{\alpha \in A_{n,q}, nq-2|\alpha| \geq 1} C_\alpha \int_0^{Tt} I^B_{nq-2|\alpha|}(\otimes_\alpha(g(s,\cdot),\ldots,g(s,\cdot)))ds.$$

(Step 2) In Proposition 6, we compute an explicit expression for the iterated contractions $\otimes_\alpha(g(t,\cdot),\ldots,g(t,\cdot))$ appearing in the right-hand side of (3.8), by using that g is given by (3.7).

To ease the description of the remaining steps, let us now set

$$F_{n,q,\alpha,T}(t) = \int_0^{Tt} I^B_{nq-2|\alpha|}(\otimes_\alpha(g(s,\cdot),\ldots,g(s,\cdot)))ds. \tag{3.9}$$

(Step 3) As $T \to \infty$, we show in Proposition 7 that, if $nq - 2|\alpha| < \frac{q}{1-H}$, then $T^{-1+(1-H_0)(nq-2|\alpha|)} F_{n,q,\alpha,T}(t)$ converges in distribution to a Hermite process (whose order and Hurst index are specified) up to some multiplicative constant. Similarly, we prove in Proposition 9 that, if $nq - 2|\alpha| \geq 3$, then $T^{\alpha_0} F_{n,q,\alpha,T}(t)$ is tight and converges in $L^2(\Omega)$ to zero, where α_0 is given in (3.28).

(Step 4) By putting together the results obtained in the previous steps, the two convergences stated in Theorem 2 follow immediately.

To illustrate a possible use of our results, we study in Section 3.6 an extension of the classical fractional Ornstein-Uhlenbeck process (see, e.g., Cheridito *et al* [7]) to the case where the driving process is more generally a Hermite process. To the best of our knowledge, there is very little literature devoted to this mathematical object, only [11, 17].

The rest of the paper is organized as follows. Section 3.2 presents some basic results about multiple Wiener-Itô integrals and Hermite processes, as well as some other facts that are used throughout the paper. Section 3.3 contains preliminary results. The proof of Theorem 1 (resp. Theorem 2) is given in Section 3.5 (resp. Section 3.4). In Section 3.6, we provide a complete asymptotic study of the Hermite-Ornstein-Uhlenbeck process, by means of Theorems 1 and 2 and of an extension of Birkhoff's ergodic Theorem. Finally, Section 3.7 contains two technical results: a power counting theorem and a version of the Hardy-Littelwood inequality, which both play an important role in the proof of our main theorems.

3.2 Preliminaries on multiple Wiener-Itô integrals and Hermite processes

3.2.1 Multiple Wiener-Itô integrals and a product formula

A function $f : \mathbb{R}^p \to \mathbb{R}$ is said to be *symmetric* if the following relation holds for all permutation $\sigma \in \mathfrak{S}(p)$:

$$f(t_1, \ldots, t_p) = f(t_{\sigma(1)}, \ldots, t_{\sigma(p)}), \quad t_1, \ldots, t_p \in \mathbb{R}.$$

The subset of $L^2(\mathbb{R}^p)$ composed of symmetric functions is denoted by $L^2_s(\mathbb{R}^p)$.

Let $B = \{B(t)\}_{t\in\mathbb{R}}$ be a two-sided Brownian motion. For any given $f \in L^2_s(\mathbb{R}^p)$ we consider the *multiple Wiener-Itô integral* of f with respect to B, denoted by

$$I_p^B(f) = \int_{\mathbb{R}^p} f(t_1, \dots, t_p) dB(t_1) \cdots dB(t_p).$$

This stochastic integral satisfies $\mathbb{E}[I_p^B(f)] = 0$ and

$$\mathbb{E}[I_p^B(f) I_q^B(g)] = \mathbf{1}_{\{p=q\}} p! \langle f, g \rangle_{L^2(\mathbb{R}^p)}$$

for $f \in L_s^2(\mathbb{R}^p)$ and $g \in L_s^2(\mathbb{R}^q)$, see [10] and [13] for precise definitions and further details.

It will be convenient in this paper to deal with multiple Wiener-Itô integrals of possibly nonsymmetric functions. If $f \in L^2(\mathbb{R}^p)$, we put $I_p^B(f) = I_p^B(\tilde{f})$, where $\tilde{f}$ denotes the symmetrization of f, that is,

$$\tilde{f}(x_1, \dots, x_p) = \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}(p)} f(x_{\sigma(1)}, \dots, x_{\sigma(p)}).$$

We will need the expansion as a sum of multiple Wiener-Itô integrals for a product of the form

$$\prod_{k=1}^n I_q^B(h_k),$$

where $q \geq 2$ is fixed and the functions h_k belong to $L_s^2(\mathbb{R}^q)$ for $k = 1, \dots, n$. In order to present this extension of the product formula and to define the relevant contractions between the functions h_i and h_j that will naturally appear, we introduce some further notation. Let $A_{n,q}$ be the set of multi-indices $\alpha = (\alpha_{ij}, 1 \leq i < j \leq n)$ such that, for each $k = 1, \dots, n$,

$$\sum_{1 \leq i < j \leq n} \alpha_{ij} \mathbf{1}_{k \in \{i,j\}} \leq q.$$

Set $|\alpha| = \sum_{1 \leq i < j \leq n} \alpha_{ij}$,

$$\beta_k^0 = q - \sum_{1 \leq i < j \leq n} \alpha_{ij} \mathbf{1}_{k \in \{i,j\}}, \qquad 1 \leq k \leq n$$

and

$$m := m(\alpha) = \sum_{k=1}^n \beta_k^0 = nq - 2|\alpha|. \tag{3.10}$$

For each $1 \leq i < j \leq n$, the integer α_{ij} will represent the number of variables in h_i which are contracted with h_j whereas, for each $k = 1, \dots, n$, the integer

β_k^0 is the number of variables in h_k which are not contracted. We will also write $\beta_k = \sum_{j=1}^k \beta_j^0$ for $k = 1, \dots, n$ and $\beta_0 = 0$. Finally, we set

$$C_\alpha = \frac{q!^n}{\prod_{k=1}^n \beta_k^0! \prod_{1\le i<j\le n} \alpha_{ij}!}. \tag{3.11}$$

With these preliminaries, for any element $\alpha \in A_{n,q}$ we can define the contraction $\otimes_\alpha(h_1, \dots, h_n)$ as the function of $nq - 2|\alpha|$ variables obtained by contracting α_{ij} variables between h_i and h_j for each couple of indices $1 \le i < j \le n$. Define the collection $(u^{i,j})_{1\le i,j\le n, i\neq j}$ in the following way:

$$u^{i,j} = \alpha_{\min(i,j),\max(i,j)}.$$

We then have

$$\begin{aligned} &\otimes_\alpha (h_1, \dots, h_n)(\xi_1, \dots, \xi_{nq-2|\alpha|}) \\ &= \int_{\mathbb{R}^{|\alpha|}} \prod_{k=1}^n h_k(s_1^{k,1}, \dots, s_{u^{k,1}}^{k,1}, \dots, s_1^{k,n}, \dots, s_{u^{k,n}}^{k,n}, \xi_{1+\beta_{k-1}}, \dots, \xi_{\beta_k}) \\ &\quad \times \prod_{1\le i<j\le n} ds_1^{i,j} \dots ds_{u^{i,j}}^{i,j} \end{aligned} \tag{3.12}$$

When $n = 2$, α has only one component $\alpha_{1,2}$ and $\otimes_\alpha(h_1, h_2) = h_1 \otimes_{\alpha_{1,2}} h_2$ is the usual contraction of $\alpha_{1,2}$ indices between h_1 and h_2. Notice that the function $\otimes_\alpha(h_1, \dots, h_n)$ is not necessarily symmetric.

Then, we have the following result.

Lemma 5. *Let $n, q \ge 2$ be some integers and let $h_i \in L_s^2(\mathbb{R}^q)$ for $i = 1, \dots, n$. We have*

$$\prod_{k=1}^n I_q^B(h_k) = \sum_{\alpha\in A_{n,q}} C_\alpha I_{nq-2|\alpha|}^B(\otimes_\alpha(h_1, \dots, h_n)). \tag{3.13}$$

Proof. The product formula for multiple stochastic integrals (see, for instance, [14, Theorem 6.1.1], or formula (2.1) in [3] for $n = 2$) says that

$$\prod_{k=1}^n I_q^B(h_k) = \sum_{\mathcal{P},\psi} I_{\beta_1^0+\dots+\beta_n^0}^B\left((\otimes_{k=1}^n h_k)_{\mathcal{P},\psi}\right), \tag{3.14}$$

where $\mathcal{P}$ denotes the set of all partitions $\{1, \dots, q\} = J_i \cup (\cup_{k=1,\dots,n,k\neq i} I_{ik})$, where for any $i, j = 1, \dots, n$, I_{ij} and I_{ji} have the same cardinality α_{ij}, ψ_{ij}

is a bijection between I_{ij} and I_{ji} and $\beta_k^0 = |J_k|$. Moreover, $(\otimes_{k=1}^n h_k)_{\mathcal{P},\psi}$ denotes the contraction of the indexes ℓ and $\psi_{ij}(\ell)$ for any $\ell \in I_{ij}$ and any $i, j = 1 \ldots, n$. Then, formula (3.13) follows form (3.14), by just counting the number of partitions, which is

$$\prod_{k=1}^{n} \frac{q!}{\prod_{i \text{ or } j \neq k} \alpha_{ij}! \beta_k^0!}$$

and multiplying by the number of bijections, which is $\prod_{1 \le i < j \le n} \alpha_{ij}!$. $\square$

Notice that when $n = 2$, formula (3.13) reduces to the well-known formula for the product of two multiple integrals. That is, for any two symmetric functions $f \in L_s^2(\mathbb{R}^p)$ and $g \in L_s^2(\mathbb{R}^q)$ we have

$$I_p^B(f) I_q^B(g) = \sum_{r=0}^{\min(p,q)} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}^B(f \otimes_r g).$$

where, for $0 \le r \le \min(p, q)$, $f \otimes_r g \in L^2(\mathbb{R}^{p+q-2r})$ denotes the contraction of r coordinates between f and g.

3.2.2 Hermite processes

Fix $q \ge 1$ and $H \in (\frac{1}{2}, 1)$. The Hermite process of index q and Hurst parameter H can be represented by means of a multiple Wiener-Itô integral with respect to B as follows, see e.g. [9]:

$$Z^{H,q}(t) = c_{H,q} \int_{\mathbb{R}^q} \int_{[0,t]} \prod_{j=1}^{q} (s - x_j)_+^{H_0 - \frac{3}{2}} ds dB(x_1) \cdots dB(x_q), \quad t \in \mathbb{R}. \tag{3.15}$$

Here, $x_+ = \max\{x, 0\}$, the constant $c_{H,q}$ is chosen to ensure that $\mathrm{Var}(Z^{H,q}(1)) = 1$, and

$$H_0 = 1 - \frac{1 - H}{q} \in \big(1 - \frac{1}{2q}, 1\big).$$

Note that $Z^{H,q}$ is self-similar of index H. When $q = 1$, the process $Z^{H,1}$ is Gaussian and is nothing but the fractional Brownian motion with Hurst parameter H. For $q \ge 2$, the processes $Z^{H,q}$ are no longer Gaussian: they belong to the qth Wiener chaos. The process $Z^{H,2}$ is known as the Rosenblatt process.

Let $|\mathcal{H}|$ be the following class of functions:

$$|\mathcal{H}| = \Big\{ f : \mathbb{R} \to \mathbb{R} \Big| \int_{\mathbb{R}} \int_{\mathbb{R}} |f(u)||f(v)||u-v|^{2H-2} du dv < \infty \Big\}.$$

Maejima and Tudor [9] proved that the stochastic integral $\int_{\mathbb{R}} f(u) dZ^{H,q}(u)$ with respect to the Hermite process $Z^{H,q}$ is well defined when f belongs to $|\mathcal{H}|$. Moreover, for any order $q \geq 1$, index $H \in (\frac{1}{2}, 1)$ and function $f \in |\mathcal{H}|$,

$$\begin{aligned} & \int_{\mathbb{R}} f(u) dZ^{H,q}(u) \qquad (3.16) \\ = \quad & c_{H,q} \int_{\mathbb{R}^q} \left(\int_{\mathbb{R}} f(u) \prod_{j=1}^{q} (u - \xi_j)_+^{H_0 - \frac{3}{2}} du \right) dB(\xi_1) \cdots dB(\xi_q). \end{aligned}$$

As a consequence of the Hardy-Littlewood-Sobolev inequality featured in [1], we observe that $L^1(\mathbb{R}) \cap L^{\frac{1}{H}}(\mathbb{R}) \subset |\mathcal{H}|$.

3.3 Chaotic decomposition of $\int_0^{Tt} P(X(s)) ds$

Assume $\varphi \in |\mathcal{H}|$ and $q \geq 1$. Using (3.16) and bearing in mind the notation and results from Section 3.2, it is immediate that X can be written as

$$X(t) = c_{H,q} I_q^B(g(t,.)), \qquad (3.17)$$

where $g(t,.)$ is given by

$$g(t, \xi_1, \ldots, \xi_q) = \int_{-\infty}^{t} \varphi(t-v) \prod_{j=1}^{q} (v - \xi_j)_+^{H_0 - 3/2} dv, \qquad (3.18)$$

and $c_{H,q}$ is defined as in (3.15).

3.3.1 Computing the chaotic expansion of $X(t)^n$ when $n \geq 2$

Let us denote by $A_{n,q}^0$ the set of elements $\alpha \in A_{n,q}$ such that $nq - 2|\alpha| = 0$ and $A_{n,q}^1$ will be the set of elements $\alpha \in A_{n,q}$ such that $nq - 2|\alpha| \geq 1$. Notice that when nq is odd, $A_{n,q}^0$ is empty. Using (3.13), we obtain the following formula for the expectation of the nth power ($n \geq 2$) of X given by (3.3):

$$\mathbb{E}[X(t)^n] = (c_{H,q})^n \sum_{\alpha \in A_{n,q}^0} C_\alpha I_{nq-2|\alpha|}^B (\otimes_\alpha (g(t,\cdot), \ldots, g(t,\cdot))). \qquad (3.19)$$

We observe in particular that $\mathbb{E}[X(t)^n] = 0$ whenever nq is odd. From (3.13) and (3.19), we deduce for $n \geq 2$ that

$$X(t)^n - \mathbb{E}[X(t)^n] = (c_{H,q})^n \sum_{\alpha \in A^1_{n,q}} C_\alpha I^B_{nq-2|\alpha|}(\otimes_\alpha(g(t,\cdot),\ldots,g(t,\cdot))). \quad (3.20)$$

To clarify this formula, let us write down detailed a expression in the cases $n = 2$ and $n = 3$. When $n = 2$, the right-hand side of (3.20) is

$$(c_{H,q})^2 \sum_{r=0}^{q-1} r! \binom{q}{r}^2 I^B_{2q-2r}(g(t,\cdot) \otimes_r g(t,\cdot)),$$

because α has just one component $\alpha_{1,2} =: r$ and condition $\alpha \in A^1_{n,q}$ means $0 \leq r \leq q-1$. For $n = 3$, we have

$$A_{3,q} = \{(\alpha_{1,2}, \alpha_{1,3}, \alpha_{2,3}) : \alpha_{1,2} + \alpha_{1,3} \leq q, \alpha_{1,2} + \alpha_{2,3} \leq q, \alpha_{1,3} + \alpha_{2,3} \leq q\}$$

and the right-hand side of (3.20) is

$$(c_{H,q})^3 \sum_{\alpha \in A_{3,q}: 3q-2|\alpha| \geq 1} C_\alpha I_{3q-2|\alpha|}(\otimes_\alpha(g(t,\cdot), g(t,\cdot), g(t,\cdot)),$$

where

$$C_\alpha = \frac{(q!)^3}{\alpha_{1,2}!\alpha_{1,3}!\alpha_{2,2}!(q-\alpha_{1,2}-\alpha_{1,3})!(q-\alpha_{1,2}-\alpha_{2,3})!(q-\alpha_{1,3}-\alpha_{1,3})!}.$$

In this case, the contraction $\otimes_\alpha(g(t,\cdot), g(t,\cdot), g(t,\cdot))$ is the function of $3q - 2|\alpha|$ variables defined by

$$\int_{\mathbb{R}^{|\alpha|}} g(\bullet, s, u) g(\star, s, v)) g(\circ, u, v) ds du dv,$$

with $s = (s_1, \ldots, s_{\alpha_{1,2}})$, $u = (u_1, \ldots, u_{\alpha_{1,3}})$ and $v = (v_1, \ldots, v_{\alpha_{2,3}})$.

From (3.20) we obtain

$$\int_0^{Tt} \big(P(X(s)) - \mathbb{E}[P(X(s))]\big) ds = a_1 \int_0^{Tt} X(s) ds \quad (3.21)$$

$$+ \sum_{n=2}^{N} a_n (c_{H,q})^n \sum_{\alpha \in A^1_{n,q}} C_\alpha \int_0^{Tt} I^B_{nq-2|\alpha|}(\otimes_\alpha(g(s,\cdot),\ldots,g(s,\cdot))) ds.$$

3.3.2 Expressing the iterated contractions of g

We now compute an explicit expression for the iterated contractions appearing in (3.21).

Proposition 6. *Fix $n \geq 2$, $q \geq 1$ and $\alpha \in A_{n,q}$. We have*

$$
\begin{aligned}
\otimes_\alpha (g(t,.),\ldots,g(t,.))(\xi) &= \beta(H_0 - \frac{1}{2}, 2-2H_0)^{|\alpha|} \\
&\times \int_{(-\infty,t]^n} dv_1 \ldots dv_n \prod_{k=1}^n \varphi(t-v_k) \prod_{1\leq i<j\leq n} |v_i - v_j|^{(2H_0-2)\alpha_{ij}} \\
&\times \prod_{k=1}^n \prod_{\ell=1+\beta_{k-1}}^{\beta_k} (v_k - \xi_\ell)_+^{H_0-\frac{3}{2}},
\end{aligned}
$$

with the convention $\beta_0 = 0$.

Proof. The proof is a straightforward consequence of the following identity

$$\int_{\mathbb{R}} (v-\xi)_+^{H_0-3/2}(w-\xi)_+^{H_0-3/2} d\xi = \beta(H_0 - \frac{1}{2}, 2-2H_0)|v-w|^{(2H_0-2)}, \quad (3.22)$$

whose proof is elementary, see e.g. [4]. □

3.4 Proof of Theorem 2

We are now ready to prove Theorem 2. To do so, we will mostly rely on the forthcoming Proposition 7, which might be a result of independent interest by itself, and which studies the asymptotic behavior of $F^B_{n,q,\alpha,T}$ given by (3.9). We will denote by f.d.d. the convergence in law of the finite-dimensional distributions of a given process. Notice that the hypothesis on φ is a bit weaker than the one in the main theorem, the fact that $\varphi \in \mathcal{S}_L$ being required in the forthcoming Proposition 9.

Proposition 7. *Fix $n \geq 2$, $q \geq 1$ and $\alpha \in A_{n,q}$. Assume the function φ belongs to $L^1(\mathbb{R}_+) \cap L^{\frac{1}{H}}(\mathbb{R}_+)$, recall H_0 from (3.1) and let m be defined as in (3.10). Finally, assume that $2m < \frac{q}{1-H}$ (which is automatically satisfied when $m = 1$ or $m = 2$). Then, as $T \to \infty$,*

$$(T^{-1+(1-H_0)m} F_{n,q,\alpha,T}(t))_{t\in[0,1]} \xrightarrow{f.d.d.} \left(\frac{C_\alpha K_{\varphi,\alpha,H_0}}{c_{H(m),m}} Z^{H(m),m}(t) \right)_{t\in[0,1]}, \quad (3.23)$$

where $Z^{H(m),m}$ denotes the mth Hermite process of Hurst index $H(m) = 1 - \frac{m}{q}(1-H)$ and the constants C_α and K_{φ,α,H_0} are defined in (3.11) and (3.24), respectively. Furthermore, $\{T^{-1+(1-H_0)m}(F_{n,q,\alpha,T}(t))_{t\in[0,1]}, T>0\}$ is tight in $C([0,1])$.

Remark 8. Note that for $m_1 < m_2$ the chaos of order m_1 dominates the chaos of order m_2.

Proof of Proposition 7. Let $n \geq 2$, $q \geq 1$ and $\alpha \in A_{n,q}$.

Step 1: We will first show the convergence (3.23). We will make several change of variables in order to transform the expression of $F_{n,q,\alpha,T}(t)$. By means of an application of stochastic's Fubini's theorem, we can write

$$F_{n,q,\alpha,T}(t) = C_\alpha \int_{\mathbb{R}^m} \Psi_T(\xi_1,\dots,\xi_m) dB(\xi_1)\cdots dB(\xi_m),$$

where

$$\Psi_T(\xi_1,\dots,\xi_m) := T^{-1+m(1-H_0)} \int_0^{Tt} ds \int_{(-\infty,s]^n} dv_1\cdots dv_n \prod_{k=1}^n \varphi(s-v_k) \times \prod_{1\leq i<j\leq n} |v_i - v_j|^{(2H_0-2)\alpha_{ij}} \prod_{k=1}^n \prod_{\ell=1+\beta_{k-1}}^{\beta_k} (v_k - \xi_\ell)_+^{H_0-\frac{3}{2}}.$$

Using the change of variables $s \to Ts$ and $v_k \to Ts - v_k$, $1 \leq k \leq n$, we obtain

$$\begin{aligned}\Psi_T(\xi_1,\dots,\xi_m) &:= T^{m(1-H_0)} \int_0^t ds \int_{(-\infty,Ts]^n} dv_1\cdots dv_n \prod_{k=1}^n \varphi(Ts-v_k) \\ &\quad \times \prod_{1\leq i<j\leq n} |v_i - v_j|^{(2H_0-2)\alpha_{ij}} \prod_{k=1}^n \prod_{\ell=1+\beta_{k-1}}^{\beta_k} (v_k - \xi_\ell)_+^{H_0-\frac{3}{2}} \\ &= T^{-\frac{m}{2}} \int_0^t ds \int_{[0,\infty)^n} dv_1\cdots dv_n \prod_{k=1}^n \varphi(v_k) \\ &\quad \times \prod_{1\leq i<j\leq n} |v_i - v_j|^{(2H_0-2)\alpha_{ij}} \prod_{k=1}^n \prod_{\ell=1+\beta_{k-1}}^{\beta_k} (s - \frac{v_k}{T} - \frac{\xi_\ell}{T})_+^{H_0-\frac{3}{2}}.\end{aligned}$$

By the scaling property of the Brownian motion, the processes

$$(F_{n,q,\alpha,T}(t))_{\alpha\in A_{n,q}, 2\leq n\leq N, t\in[0,1]}$$

and

$$(\widehat{F}_{n,q,\alpha,T}(t))_{\alpha\in A_{n,q},2\le n\le N,t\in[0,1]}$$

have the same probability distribution, where

$$\widehat{F}_{n,q,\alpha,T}(t) = C_\alpha \int_{\mathbb{R}^m} \widehat{\Psi}_T(\xi_1,\ldots,\xi_m) dB(\xi_1)\cdots dB(\xi_m)$$

$$\widehat{\Psi}_T(\xi_1,\ldots,\xi_m) := \int_0^t ds \int_{(-\infty,0]^n} dv_1\cdots dv_n \prod_{k=1}^n \varphi(v_k)$$
$$\times \prod_{1\le i<j\le n} |v_i - v_j|^{(2H_0-2)\alpha_{ij}} \prod_{k=1}^n \prod_{\ell=1+\beta_{k-1}}^{\beta_k} (s - \frac{v_k}{T} - \xi_\ell)_+^{H_0-\frac{3}{2}}.$$

Set

$$\widehat{\Psi}(\xi_1,\ldots,\xi_m) := K_{\varphi,\alpha,H_0} \int_0^t ds \prod_{\ell=1}^m (s-\xi_\ell)_+^{H_0-\frac{3}{2}},$$

where

$$K_{\varphi,\alpha,H_0} = \int_{\mathbb{R}_+^n} dv_1\cdots dv_n \prod_{k=1}^n \varphi(v_k) \prod_{1\le i<j\le n} |v_i-v_j|^{(2H_0-2)\alpha_{ij}}. \tag{3.24}$$

Notice that, by Lemma 15, K_{φ,α,H_0} is well defined. We claim that

$$\lim_{T\to\infty} \widehat{\Psi}_T = \widehat{\Psi}, \tag{3.25}$$

where the convergence holds in $L^2(\mathbb{R}^m)$. This will imply the convergence in $L^2(\Omega)$ of $\widehat{F}_{n,q,\alpha,T}(t)$, as $T\to\infty$ to a Hermite process of order m, multiplied by the constant $C_\alpha K_{\varphi,\alpha,H_0}$.

Proof of (3.25): It suffices to show that the inner products $\langle \widehat{\Psi}_T, \widehat{\Psi}_T\rangle_{L^2(\mathbb{R}^m)}$ and $\langle \widehat{\Psi}_T, \widehat{\Psi}\rangle_{L^2(\mathbb{R}^m)}$ converge, as $T\to\infty$, to

$$\|\widehat{\Psi}\|^2_{L^2(\mathbb{R}^m)} = K^2_{\varphi,\alpha,H_0} \beta(H_0-\frac{1}{2}, 2-2H_0)^m \int_{[0,t]^2} ds ds' |s-s'|^{(2H_0-2)m},$$

which is finite because $m < \frac{1}{2(1-H_0)} = \frac{q}{2(1-H)}$. We will show the convergence of $\langle \widehat{\Psi}_T, \widehat{\Psi}_T\rangle_{L^2(\mathbb{R}^m)}$ and the second term can be handled by the same arguments. We have

$$\|\widehat{\Psi}_T\|^2_{L^2(\mathbb{R}^m)} = \int_{[0,t]^2} ds ds' \int_{\mathbb{R}_+^{2n}} dv_1\cdots dv_n dv_1'\cdots dv_n'$$

$$\times \prod_{k=1}^{n} \varphi(v_k)\varphi(v'_k) \prod_{1\le i<j\le n} |v_i - v_j|^{(2H_0-2)\alpha_{ij}} |v'_i - v'_j|^{(2H_0-2)\alpha_{ij}}$$
$$\times \prod_{k=1}^{n} \beta(H_0 - \frac{1}{2}, 2 - 2H_0)^{\beta_k} |s - s' - \frac{v_k - v'_k}{T}|^{(2H_0-2)\beta_k}.$$

Let us first show that given $w_k \in \mathbb{R}$, $1 \le k \le n$,

$$\lim_{T\to\infty} \int_{[0,t]^2} dsds' \prod_{k=1}^{n} |s - s' - \frac{w_k}{T}|^{(2H_0-2)\beta_k} = \int_{[0,t]^2} dsds' |s - s'|^{(2H_0-2)m} \tag{3.26}$$

and, moreover,

$$\sup_{w_k\in\mathbb{R}, 1\le k\le n} \int_{[0,t]^2} dsds' \prod_{k=1}^{n} |s - s' - w_k|^{(2H_0-2)\beta_k} < \infty. \tag{3.27}$$

By the dominated convergence theorem and using Lemma 15, (3.26) and (3.27) imply (3.25).

To show (3.26), choose ϵ such that $|w_k|/T < \epsilon$, $1 \le k \le n$, for T large enough (depending on the fixed w_k's). Then, we can write

$$\begin{aligned}
&\int_{[0,t]^2} dsds' \left| \prod_{k=1}^{n} |s - s' - \frac{w_k}{T}|^{(2H_0-2)\beta_k} - |s - s'|^{(2H_0-2)m} \right| \\
&\le t \int_{|\xi|>2\epsilon} d\xi \left| \prod_{k=1}^{n} |\xi - \frac{w_k}{T}|^{(2H_0-2)\beta_k} - |\xi|^{(2H_0-2)m} \right| \\
&\quad + 2t \sup_{|w_k|<\epsilon} \int_{|\xi|\le 2\epsilon} d\xi \prod_{k=1}^{n} |\xi - w_k|^{(2H_0-2)\beta_k} \\
&:= B_1 + B_2.
\end{aligned}$$

The term B_1 tends to zero a $T \to \infty$, for each $\epsilon > 0$. On the other hand, the term B_2 tends to zero as $\epsilon \to 0$. Indeed,

$$B_2 = 2t\epsilon^{(2H_0-2)m+1} \sup_{|w_k|<1} \int_{|\xi|\le 2} d\xi \prod_{k=1}^{n} |\xi - w_k|^{(2H_0-2)\beta_k}.$$

Note that the above supremum is finite because the function $(w_1, \dots, w_k) \to \int_{|\xi|\le 2} d\xi \prod_{k=1}^{n} |\xi - w_k|^{(2H_0-2)\beta_k}$ is continuous.

Property (3.27) follows immediately from the fact that the function

$$(w_1, \dots, w_k) \to \int_{[0,t]^2} dsds' \prod_{k=1}^{n} |s - s' - w_k|^{(2H_0-2)\beta_k}$$

is continuous and vanishes as $|(w_1,\dots,w_k)|$ tends to infinity.

We have $H_0 = 1 - \frac{1-H}{q} = 1 - \frac{1-H(m)}{m}$ with $H(m)$ as above. As a result, we obtain the convergence of the finite-dimensional distributions of $T^{-1+(1-H_0)m}F_{n,q,\alpha,T}(t)$ to those the mth Hermite process $Z^{H(m),m}$ multiplied by the constant $\frac{C_\alpha K_{\varphi,\alpha,H_0}}{c_{H(m),m}}$.

Step 2: *Tightness*. Fix $0 \le s < t \le 1$. To check that tightness holds in $C([0,1])$, let us compute the squared $L^2(\Omega)$-norm

$$\Phi_T := T^{-1+(1-H_0)m}\mathbb{E}(|F_{n,q,\alpha,T}(t) - F_{n,q,\alpha,T}(s)|^2).$$

Proceeding as in the first step of the proof, we obtain

$$\begin{aligned}
\Psi_T &= \mathbb{E}\Bigg(\Bigg|\int_{\mathbb{R}^m} dB(\xi_1)\cdots dB(\xi_m)\int_s^t du\int_{\mathbb{R}_+^n} dv_1\cdots dv_n\prod_{k=1}^n\varphi(v_k)\\
&\quad\times\prod_{1\le i<j\le n}|v_i-v_j|^{(2H_0-2)\alpha_{ij}}\prod_{k=1}^n\prod_{\ell=1+\beta_{k-1}}^{\beta_k}(u-\frac{v_k}{T}-\xi_\ell)_+^{H_0-\frac{3}{2}}\Bigg|^2\Bigg)\\
&\le m!\int_{\mathbb{R}^m} d\xi_1\cdots d\xi_m\Bigg|\int_s^t du\int_{\mathbb{R}_+^n} dv_1\cdots dv_n\prod_{k=1}^n\varphi(v_k)\\
&\quad\times\prod_{1\le i<j\le n}|v_i-v_j|^{(2H_0-2)\alpha_{ij}}\prod_{k=1}^n\prod_{\ell=1+\beta_{k-1}}^{\beta_k}(u-\frac{v_k}{T}-\xi_\ell)_+^{H_0-\frac{3}{2}}\Bigg|^2.
\end{aligned}$$

Using (3.22) yields

$$\begin{aligned}
\Psi_T &\le m!\int_{[s,t]^2} du du'\int_{\mathbb{R}_+^{2n}} dv_1\cdots dv_n dv_1'\cdots dv_n'\\
&\quad\times\prod_{k=1}^n\varphi(v_k)\varphi(v_k')\prod_{1\le i<j\le n}|v_i-v_j|^{(2H_0-2)\alpha_{ij}}|v_i'-v_j'|^{(2H_0-2)\alpha_{ij}}\\
&\quad\times\prod_{k=1}^n\beta(H_0-\frac{1}{2},2-2H_0)^{\beta_k}|u-u'-\frac{v_k-v_k'}{T}|^{(2H_0-2)\beta_k}\\
&\le m!(t-s)\int_{-1}^1 d\xi\int_{\mathbb{R}_+^{2n}} dv_1\cdots dv_n dv_1'\cdots dv_n'\\
&\quad\times\prod_{k=1}^n\varphi(v_k)\varphi(v_k')\prod_{1\le i<j\le n}|v_i-v_j|^{(2H_0-2)\alpha_{ij}}|v_i'-v_j'|^{(2H_0-2)\alpha_{ij}}
\end{aligned}$$

$$\times \prod_{k=1}^{n} \beta(H_0 - \frac{1}{2}, 2 - 2H_0)^{\beta_k} |\xi - \frac{v_k - v_k'}{T}|^{(2H_0-2)\beta_k}$$
$$\leq C(t-s).$$

Then the equivalence of all $L^p(\Omega)$-norms, $p \geq 2$, on a fixed Wiener chaos, also known as the hypercontractivity property, allows us to conclude the proof of the tightness. □

We will make use of the notation

$$\alpha_0 = (1 - 2H_0)\mathbf{1}_{\{nq \text{ is even}\}} - H_0 \mathbf{1}_{\{nq \text{ is odd}\}}. \tag{3.28}$$

Proposition 9. *Fix $n, q \geq 2$ and $\alpha \in A_{n,q}$, assume that the function φ belongs to $\mathcal{S}_L$ for some $L > 1$ and that $m \geq 3$. Then for any $t \in [0, 1]$, $T^{\alpha_0} F_{n,q,\alpha,T}(t)$ converge in $L^2(\Omega)$ to zero as $T \to \infty$; furthermore, the family $\{(F_{n,q,\alpha,T}(t))_{t\in[0,1]}, T > 0\}$ is tight in $C([0, 1])$.*

Proof. If $(2H_0 - 2)m > -1$, we know that $T^{-1+m(1-H_0)} F_{n,q,\alpha,T}(t)$ converges to zero in $L^2(\Omega)$ as $T \to \infty$ (by Proposition 7). This implies the convergence to zero in $L^2(\Omega)$ as $T \to \infty$ of $T^{-\alpha_0} F_{n,q,\alpha,T}(t)$ because $-1+m(1-H_0) > \alpha_0$. We should then concentrate on the case $(2H_0 - 2)m \leq -1$. Once again, we shall divide the proof in two steps:

Step 1: Let us first prove the convergence in $L^2(\Omega)$. Fix $\alpha \in A_{n,q}$. We are going to show that

$$\lim_{T\to\infty} T^{2\alpha_0} \mathbb{E}\left(|F_{n,q,\alpha,T}(t)|^2\right) = 0.$$

We know that

$$T^{2\alpha_0} \mathbb{E}\left(|F_{n,q,\alpha,T}(t)|^2\right) = T^{2\alpha_0} m! \times \left\| \int_{[0,Tt]} ds \otimes_\alpha (g(s,\cdot), \dots, g(s,\cdot)) \right\|^2_{L^2(\mathbb{R}^m)}.$$

In view of the expression for the contractions obtained in Proposition 7, it suffices to show that

$$\lim_{T\to\infty} T^{2\alpha_0} \int_{[0,Tt]^2} ds ds' \int_{\mathbb{R}^m} \int_{(-\infty,s]^n} \int_{(-\infty,s']^n} dv_1 \cdots dv_n dv_1' \cdots dv_n' d\xi_1 \cdots d\xi_m$$
$$\times \prod_{k=1}^{n} \varphi(s - v_k)\varphi(s' - v_k') \prod_{1\leq i<j\leq n} |v_i - v_j|^{(2H_0-2)\alpha_{ij}} |v_i' - v_j'|^{(2H_0-2)\alpha_{ij}}$$

$$\times \prod_{k=1}^{n} \prod_{\ell=1+\beta_{j-1}}^{\beta_j} (v_k - \xi_\ell)_+^{H_0-\frac{3}{2}} \prod_{\ell=1+\beta_{j-1}}^{\beta_j} (v'_k - \xi_\ell)_+^{H_0-\frac{3}{2}} = 0.$$

Integrating in the variables ξ's and using (3.22), it remains to show that

$$\lim_{T\to\infty} T^{2\alpha_0} \int_{[0,Tt]^2} ds ds' \int_{(-\infty,s]^n} \int_{(-\infty,s']^n} dv_1 \cdots dv_n dv'_1 \cdots dv'_n$$
$$\times \prod_{k=1}^{n} \varphi(s-v_k)\varphi(s'-v'_k) \prod_{1\le i<j\le n} |v_i - v_j|^{(2H_0-2)\alpha_{ij}} |v'_i - v'_j|^{(2H_0-2)\alpha_{ij}}$$
$$\times \prod_{k=1}^{n} |v_k - v'_k|^{(2H_0-2)\beta_k} = 0.$$

Set

$$\Phi_T := T^{2\alpha_0} \int_{[0,Tt]^2} ds ds' \int_{(-\infty,s]^n} \int_{(-\infty,s']^n} dv_1 \cdots dv_n dv'_1 \cdots dv'_n$$
$$\times \prod_{k=1}^{n} \varphi(s-v_k)\varphi(s'-v'_k) \prod_{1\le i<j\le n} |v_i - v_j|^{(2H_0-2)\alpha_{ij}} |v'_i - v'_j|^{(2H_0-2)\alpha_{ij}}$$
$$\times \prod_{k=1}^{n} |v_k - v'_k|^{(2H_0-2)\beta_k}.$$

Making the change of variables $w_k = s - v_k$, $w'_k = s' - v'_k$ for $k = 1, \ldots, n$, yields

$$\Phi_T = T^{2\alpha_0} \int_{[0,Tt]^2} ds ds' \int_{\mathbb{R}_+^n} \int_{[0,\infty)^n} dw_1 \cdots dw_n dw'_1 \cdots dw'_n$$
$$\times \prod_{k=1}^{n} \varphi(w_k)\varphi(w'_k) \prod_{1\le i<j\le n} |w_i - w_j|^{(2H_0-2)\alpha_{ij}} |w'_i - w'_j|^{(2H_0-2)\alpha_{ij}}$$
$$\times \prod_{k=1}^{n} |s - s' - w_k + w'_k|^{(2H_0-2)\beta_k}.$$

Now we use Fubini's theorem and make the change of variables $s - s' = \xi$ to obtain

$$\Phi_T = tT^{2\alpha_0+1} \int_{\mathbb{R}_+^{2n}} dw_1 \cdots dw_n dw'_1 \cdots dw'_n$$
$$\times \prod_{k=1}^{n} |\varphi(w_k)\varphi(w'_k)| \prod_{1\le i<j\le n} |w_i - w_j|^{(2H_0-2)\alpha_{ij}} |w'_i - w'_j|^{(2H_0-2)\alpha_{ij}}$$

$$\times \int_{-tT}^{tT} d\xi \prod_{k=1}^{n} |\xi - w_k + w_k'|^{(2H_0-2)\beta_k}.$$

We shall distinguish again two subcases:

Case $(2H_0 - 2)m < -1$: Notice that the exponent $2\alpha_0 + 1$ is negative:

(i) If nq is even, then $\alpha_0 = 1 - 2H_0$ and

$$2\alpha_0 + 1 = 3 - 4H_0 < 0$$

because $H_0 > \frac{3}{4}$.

(ii) If nq is odd, then $\alpha_0 = -H_0$ and

$$2\alpha_0 + 1 = 1 - 2H_0 < 0.$$

Therefore, in order to show that $\lim_{T\to\infty} \Phi_T = 0$, it suffices to check that

$$\begin{aligned} J := \int_{\mathbb{R}^{2n}} & dw_1 \cdots dw_n dw_1' \cdots dw_n' \prod_{k=1}^{n} |\varphi(w_k)\varphi(w_k')| \\ & \times \prod_{1\le i<j\le n} |w_i - w_j|^{(2H_0-2)\alpha_{ij}} |w_i' - w_j'|^{(2H_0-2)\alpha_{ij}} \\ & \times \int_{\mathbb{R}} d\xi \prod_{k=1}^{n} |\xi - w_k + w_k'|^{(2H_0-2)\beta_k} < \infty, \end{aligned} \tag{3.29}$$

where, by convention $\varphi(w) = 0$ if $w < 0$. We will apply the Power Counting Theorem 14 to prove that this integral is finite. We consider functions on $\mathbb{R}^{2n+1}$ with variables $\{(w_k)_{k\le n}, (w_k')_{k\le n}, \xi\}$. The set of linear functions is

$$\begin{aligned} T = & \{\omega_k, \omega_k', 1 \le k \le n\} \cup \{w_i - w_j, w_i' - w_j', 1 \le i < j \le n\} \\ & \cup \{\xi - w_k + w_k', 1 \le k \le n\}. \end{aligned}$$

The corresponding exponents (μ_M, ν_M) for each $M \in T$ are $(0, -L)$ for the linear functions w_k and w_k' (taking into account that $\varphi \in \mathcal{S}_L$), $(2H_0 - 2)\alpha_{ij}$ for each function of the form $w_i - w_j$ or $w_i' - w_j'$ and $(2H_0 - 2)\beta_k$ for each function of the form $\xi - w_k + w_k'$.

Then $J < \infty$, provided conditions (a) and (b) are satisfied.

• *Verification of (b)*: Let $W \subset T$ be a linearly independent proper subset of T, and

$$d_\infty = 2n + 1 - \dim(\text{Span}(W)) + \sum_{M\in T\setminus(\text{Span}(W)\cap T)} \nu_M.$$

Let S be the following subset of T: $S = \{w_k, w'_k, 1 \le k \le n\}$. Let $e = \mathrm{Card}(S \cap \mathrm{Span}(W))$. Consider the following two cases:

(i) There exists $k \le n$ such that $\xi - w_k + w'_k \in \mathrm{Span}(W) \cap T$. Then $\dim(\mathrm{Span}(W)) \ge e + 1$. As a consequence,

$$d_\infty \le 2n + 1 - (e+1) - (2n - e)L < 0,$$

because $L > 1$ and in this case, we should have $e < 2n$ because W is a proper subset of T.

(ii) Otherwise,

$$d_\infty \le 2n + 1 - e - (2n - e)L + (2H_0 - 2)m < 0,$$

because $L > 1$ and $(2H_0 - 2)m < -1$.

• *Verification of (a)*: A direct verification would require to solve a seemingly difficult combinatorial problem. We can simply remark that

$$\begin{aligned}
&\int_{[-1,1]^{2n}} dw_1 \cdots dw_n dw'_1 \cdots dw'_n \\
&\quad \times \prod_{1 \le i < j \le n} |w_i - w_j|^{(2H_0-2)\alpha_{ij}} |w'_i - w'_j|^{(2H_0-2)\alpha_{ij}} \\
&\quad \times \int_{-1}^{1} d\xi \prod_{k=1}^{n} |\xi - w_k + w'_k|^{(2H_0-2)\beta_k} \\
=& m! \frac{1}{\beta(H_0 - \frac{1}{2}, 2 - 2H_0)^{|\alpha|}} \mathbb{E}\left[\left(\int_0^1 I^B_{nq-2|\alpha|}(f(s,\cdot), \ldots, f(s,\cdot)) \right)^2 \right] < \infty
\end{aligned}$$

where $f(s, \xi_1, \ldots \xi_q) = \int_{-\infty}^{+\infty} \mathbb{I}_{[-1,1]}(s - v) \prod_{j=1}^{q} (v - \xi_j)_+^{H_0 - 3/2} dv$. Since $\varphi \in \mathcal{S}_L$, φ is bounded on $[-1, 1]$. This implies that (a) is verified by the converse side of the Power Counting Theorem.

Case $(2H_0 - 2)m = -1$: In this case, we can apply Hölder and Jensen inequalities to Φ_T in order to get

$$\Phi_T \le T^{2\alpha_0 + 1} A^{\frac{\epsilon}{1+\epsilon}} B^{\frac{1}{1+\epsilon}},$$

with $2\alpha_0 + 1 < 0$, $A = \left(\int_{\mathbb{R}} |\varphi(w)| dw \right)^{2n}$ and

$$B = \int_{\mathbb{R}^{2n}} dw_1 \cdots dw_n dw'_1 \cdots dw'_n \prod_{k=1}^{n} |\varphi(w_k)\varphi(w'_k)|$$

$$\times \prod_{1\le i<j\le n} |w_i - w_j|^{(2H_0'-2)\alpha_{ij}} |w_i' - w_j'|^{(2H_0'-2)\alpha_{ij}}$$
$$\times \int_{\mathbb{R}} d\xi \prod_{k=1}^{n} |\xi - w_k + w_k'|^{(2H_0'-2)\beta_k},$$

where $H_0' = H_0(1+\epsilon) - \epsilon$. If ϵ is small enough, H_0' can still be expressed as $1 - \frac{1-H'}{q}$ for some $\frac{1}{2} < H' < H$. Moreover, in this case $(2H_0' - 2)m < -1$ so we are exactly in the situation of the previous case, and the integral B is finite.

<u>*Step 2*</u>: Using the same arguments as previously and the hypercontractivity property, we deduce that there exists a constant $K > 0$ such that for all $0 \le s < t \le 1$,

$$\mathbb{E}\left(|F_{n,q,\alpha,T}(t) - F_{n,q,\alpha,T}(s)|^4\right) \le K|t-s|^2,$$

which proves the tightness in $C([0,1])$.

□

It remains to study what happens when $n = 1$. The proof of Proposition 10 is very similar to that of Proposition 7 (although much simpler) and details are left to the reader.

Proposition 10. *Fix $q \ge 1$ and assume the function φ belongs to $L^1(\mathbb{R}_+) \cap L^{\frac{1}{H}}(\mathbb{R}_+)$. Then the finite-dimensional distributions of the process*

$$G_T(t) := T^{q(1-H_0)-1} \int_0^{Tt} ds I_q^B(g(s,\cdot)), \quad t \in [0,1], \tag{3.30}$$

where $g(s,\cdot)$ is defined in (3.18), converge in law to those of a qth Hermite process of Hurst parameter $1 - q(1-H_0)$ multiplied by the constant $c_{H_0,q}^{-1} \int_0^\infty \varphi(w)dw$, and the family $\{(G_T(t))_{t\in[0,1]}, T > 0\}$ is tight in $C([0,1])$.

We are now ready to make the proof of Theorem 2.

Proof of Theorem 2. It suffices to consider the decomposition (3.21) and to apply the results shown in Propositions 7 and 9. □

3.5 Proof of Theorem 1

Let Z be a fractional Brownian motion of Hurst index $H \in (\frac{1}{2}, 1)$, and let $\varphi \in L^1(\mathbb{R}_+) \cap L^{\frac{1}{H}}(\mathbb{R}_+)$. Consider the moving average process X defined by

$$X(t) = \int_{-\infty}^{t} \varphi(t-u) dZ_u, \quad t \geq 0,$$

which is easily checked to be a stationary centered Gaussian process. Denote by $\rho : \mathbb{R} \to \mathbb{R}$ the correlation function of X, that is, $\rho(t-s) = \mathbb{E}[X(t)X(s)]$, $s, t \geq 0$. By multiplying the function φ by a constant if necessary, we can assume without loss of generality that $\rho(0) = 1 (= \mathrm{Var}(X(t))$ for all $t)$. Let $P(x) = \sum_{n=0}^{N} a_n x^n$ be a real-valued polynomial function, and let d denotes its centered Hermite rank.

3.5.1 Proof of (3.6)

In this section, we assume that $d \geq 1$ and that $H \in (1 - \frac{1}{2d}, 1)$, and our goal is to show that

$$T^{d(1-H)-1} \left\{ \int_0^{Tt} \big(P(X(s)) - \mathbb{E}[P(X(s))]\big) ds \right\}_{t \in [0,1]}$$

converges in distribution in $C([0,1])$ to a Hermite process of index d and Hurst parameter $1 - d(1-H)$, up to some multiplicative constant C_2. Since P has centered Hermite rank d, it can be rewritten as

$$P(x) = \mathbb{E}[P(X(s)] + \sum_{l=d}^{N} b_l H_l(x),$$

for some $b_d, \dots, b_N \in \mathbb{R}$, with $b_d \neq 0$ and H_l the lth Hermite polynomial. As a result, we have

$$\int_0^{Tt} \big(P(X(s)) - \mathbb{E}[P(X(s))]\big) ds \quad = \quad \sum_{l=d}^{N} b_l (c_{H,1})^l \int_0^{Tt} I_l^B(g(s,\cdot)^{\otimes l}) ds,$$

and the desired conclusion follows thanks to Propositions 7 and 10.

3.5.2 Proof of (3.5)

In this section, we assume that $d \geq 2$ and that $H \in (\frac{1}{2}, 1 - \frac{1}{2d})$, and our goal is to show that

$$T^{-\frac{1}{2}} \left\{ \int_0^{Tt} \big(P(X(s)) - \mathbb{E}[P(X(s))]\big) ds \right\}_{t \in [0,1]}$$

converges in distribution in $C([0,1])$ to a standard Brownian motion W, up to some multiplicative constant C_1. To do so, we will rely on the Breuer-Major theorem, which asserts that the desired conclusion holds as soon as

$$\int_{\mathbb{R}} |\rho(s)|^d ds < \infty, \tag{3.31}$$

where $\rho(s) = \mathbb{E}[X(s)X(0)]$ (see, e.g., [6] for a continuous version of the Breuer-Major theorem).

The rest of this section is devoted to checking that (3.31) holds true. Let us first compute ρ:

$$\begin{aligned} & \rho(t-s) = \mathbb{E}[X(t)X(s)] \\ = \quad & H(2H-1) \iint_{\mathbb{R}^2} \varphi(t-v)\mathbf{1}_{(-\infty,t]}(v)\varphi(s-u)\mathbf{1}_{(-\infty,s]}(u)|v-u|^{2H-2}dudv \\ = \quad & H(2H-1) \iint_{\mathbb{R}^2} \varphi(u)\varphi(v)|t-s-v+u|^{2H-2}dudv, \end{aligned}$$

with the convention that $\varphi(u) = 0$ if $u < 0$. This allows us to write

$$\rho(s) = c_H[\tilde{\varphi} * (I^{2H-1}\varphi)](s),$$

where $\tilde{\varphi}(u) = \varphi(-u)$, I^{2H-1} is the fractional integral operator of order $2H-1$ and c_H is a constant depending on H. As a consequence, applying Young's inequality and Hardy-Littlewood's inequality (see [18, Theorem 1]) yields

$$\|\rho\|_{L^d(\mathbb{R})} \leq c_H \|\varphi\|_{L^p(\mathbb{R})} \|I^{2H-1}\varphi\|_{L^q(\mathbb{R})} \leq c_{H,p} \|\varphi\|^2_{L^p(\mathbb{R})},$$

where $\frac{1}{d} = \frac{1}{p} + \frac{1}{q} - 1$ and $\frac{1}{q} = \frac{1}{p} - (2H-1)$. This implies $p = (H + \frac{1}{2d})^{-1}$ and we have $\|\varphi\|_{L^p(\mathbb{R})} < \infty$, because $p \in (1, \frac{1}{H})$ and $\varphi \in L^1(\mathbb{R}) \cap L^{\frac{1}{H}}(\mathbb{R})$. The proof of (3.5) is complete. □

3.6 The Stationary Hermite-Ornstein-Uhlenbeck process

We dedicate this section to the study of the extension of the Ornstein Uhlenbeck process to the case where the driving process is a Hermite process. To our knowledge, there is not much literature about this object. Among the few existing references, we mention [17] and [11]. The special case in which the driving process is a fractional Brownian motion has been, in contrast, well studied, see for instance [7]. In what follows, we will prove a first-order ergodic theorem for the stationary Hermite-Ornstein-Uhlenbeck process. Then, we will use Theorem 2 to study its second order fluctuations.

Let $\alpha > 0$. Consider the function $\varphi(s) = e^{-\alpha s}\mathbb{I}_{s>0}$ and let $Z^{H,q}$ be a Hermite process of order $q \geq 1$ and Hurst index $H > \frac{1}{2}$. Then $\varphi \in S_L$ for all $L > 0$, and we can define the stationary Hermite-Ornstein-Uhlenbeck process as:

$$(U_t)_{t\geq 0} = \int_{-\infty}^{t} \varphi(t-s)dZ_s^{H,q}. \tag{3.32}$$

As its name suggests, this process is strongly stationary, that is, for any $h > 0$ the processes $(U_t)_{t\geq 0}$ and $(U_{t+h})_{t\geq 0}$ have the same finite-dimensional distributions. We then state the following general ergodic type result.

Proposition 11. *Let $(u_t)_{t\geq 0}$ be a real valued process of the form $u_t = I_q^B(f_t)$, where $f_t \in L_s^2(\mathbb{R}^q)$ for each $t \geq 0$. Assume that u is strongly stationary, has integrable sample paths and satisfies, for each $1 \leq r \leq q$,*

$$\|f_0 \otimes_r f_s\|_{L^2(\mathbb{R}^{2q-2r})} \underset{s\to\infty}{\longrightarrow} 0.$$

Then, for all measurable function such that $\mathbb{E}[|f(u_0)|] < +\infty$,

$$\frac{1}{T}\int_0^T f(u_s)ds \xrightarrow[T\to\infty]{a.s} \mathbb{E}\left[f(u_0)\right].$$

Proof. According to Theorem 1.3 in [12], the process u is strongly mixing if for all $t > 0$ and $1 \leq r \leq q$, the following convergence holds

$$\|f_t \otimes_r f_{t+s}\|_{L^2(\mathbb{R}^{2q-2r})} \underset{s\to\infty}{\longrightarrow} 0.$$

Taking into account that u is strongly stationary, we can write

$$\|f_t \otimes_r f_{t+s}\|_{L^2(\mathbb{R}^{2q-2r})} = \|f_0 \otimes_r f_s\|_{L^2(\mathbb{R}^{2q-2r})},$$

and the conclusion follows immediately from Birkhoff's continuous ergodic theorem. □

We can now particularize to the Hermite-Ornstein-Uhlenbeck process.

Theorem 12. *Let U be the Hermite-Ornstein-Uhlenbeck process defined by (3.32). Let f be a measurable function such that $|f(x)| \leq \exp(|x|^\gamma)$ for some $\gamma < \frac{2}{q}$. Then,*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(U_s)ds = \mathbb{E}\left[f(U_0)\right] \ \ a.s.$$

Proof. We shall prove that the process U verifies the conditions of Proposition 11. We have $U_t = I_q^B(f_t)$ with

$$f_t(x_1,\dots,x_q) = c_{H,q}\mathbb{I}_{[-\infty,t]^q}(x_1,\dots,x_q)\int_{x_1\vee\cdots\vee x_q}^t e^{-\alpha(t-u)}\prod_{i=1}^q (u-x_i)^{H_0-\frac{3}{2}}du.$$

Step 1. Let us first show the mixing condition, that is

$$\lim_{s\to\infty}\|f_0\otimes_r f_s\|_{L^2(\mathbb{R}^{2q-2r})} = 0$$

for all $r\in\{1,\dots,q\}$. We can write

$$\begin{aligned}
&f_0\otimes_r f_s(y_1,\dots,y_{2q-2r})\\
&= c_{H,q}^2\int_{(-\infty,0]^r}\left(\int_{x_1\vee\cdots\vee x_r\vee y_1\cdots\vee y_{q-r}}^0 e^{\alpha u}\prod_{i=1}^r\prod_{j=1}^{q-r}(u-x_i)^{H_0-\frac{3}{2}}(u-y_j)^{H_0-\frac{3}{2}}du\right)\\
&\quad\times\left(\int_{x_1\vee\cdots\vee x_r\vee y_{q-r+1}\cdots\vee y_{2q-2r}}^s e^{-\alpha(s-u)}\right.\\
&\quad\left.\times\prod_{i=1}^r\prod_{j=q-r+1}^{2q-2r}(u-x_i)^{H_0-\frac{3}{2}}(u-y_j)^{H_0-\frac{3}{2}}du\right)dx_1\cdots dx_r\\
&= c_{H,q}^2\int_{y_1\vee\cdots\vee y_{q-r}}^0 e^{\alpha u}\int_{y_{q-r+1}\vee\cdots\vee y_{2q-2r}}^s e^{-\alpha(s-v)}\\
&\quad\times\left(\int_{(-\infty,u\wedge v]}(u-x)^{H_0-\frac{3}{2}}(v-x)^{H_0-\frac{3}{2}}dx\right)^r\\
&\quad\times\prod_{j=1}^{q-r}\prod_{l=q-r+1}^{2q-2r}(u-y_j)^{H_0-\frac{3}{2}}(v-y_l)^{H_0-\frac{3}{2}}dvdu\\
&= c_{H,q}^2\beta(H_0-\frac{1}{2},2-2H_0)^r\int_{y_1\vee\cdots\vee y_{q-r}}^0 e^{\alpha u}\int_{y_{q-r+1}\vee\cdots\vee y_{2q-2r}}^s e^{-\alpha(s-v)}\\
&\quad\times|u-v|^{r(2H_0-2)}\prod_{j=1}^{q-r}\prod_{l=q-r+1}^{2q-2r}(u-y_j)^{H_0-\frac{3}{2}}(v-y_l)^{H_0-\frac{3}{2}}dvdu,
\end{aligned}$$

where we used again the identity (3.22). We then have

$$\|f_0\otimes_r f_s\|_{L^2(\mathbb{R}^{2q-2r})}^2 = c_{H,q}^4\beta(H_0-\frac{1}{2},2-2H_0)^{2q}$$

$$\times \int_{(-\infty,0]^2} \int_{(-\infty,s]^2} e^{\alpha(u+u_1)} e^{-\alpha(2s-(v+v_1))} |u-u_1|^{(q-r)(2H_0-2)}$$
$$\times |v-v_1|^{(q-r)(2H_0-2)} |u-v|^{r(2H_0-2)} |u_1-v_1|^{r(2H_0-2)} dv_1 dv du_1 du$$
$$\leq c_{H,q}^4 \beta(H_0 - \frac{1}{2}, 2-2H_0)^{2q} A_0 A_s R_s^2,$$

with

$$A_x = \left(\int_{(-\infty,x]^2} e^{-q\alpha(2x-(u+u_1))} |u-u_1|^{q(2H_0-2)} du du_1 \right)^{\frac{1}{a}}$$

and

$$R_s = \left(\int_{-\infty}^0 \int_{-\infty}^s e^{-q\alpha(s-(u+v))} |u-v|^{q(2H_0-2)} dv du \right)^{\frac{1}{b}},$$

where we used the Hölder inequality with $a = \frac{q}{q-r}, b = \frac{q}{r}$. Making the change of variable $x-u=v$, $x-u_1=v_1$, we obtain

$$\int_{(-\infty,x]^2} e^{-q\alpha(2x-(u+u_1))} |u-u_1|^{q(2H_0-2)} du du_1$$
$$= \int_{[0,\infty)^2} e^{-q\alpha(v+v_1)} |v-v_1|^{q(2H_0-2)} du du_1 < \infty.$$

On the other hand, we have $q(2H_0-2) = 2H-2$, so

$$R_s^b = \mathrm{Cov}(U_0^H, U_s^H)$$

where U^H is a stationary Ornstein Uhlenbeck process driven by a fractional Brownian motion of index H (and with $\alpha^H = q\alpha$). According to [7, Lemma 2.2], one has $R_s^b = O_{s\to\infty}(s^{-H})$, implying in turn that $\lim_{s\to\infty} R_s = 0$ and concluding the proof of the mixing condition.

Step 2. We now show the integrability condition $\mathbb{E}[|f(U_0)|] < \infty$. From the results of [7], we have $\mathbb{E}[U_0^2] = \frac{1}{\alpha^{2H}}\Gamma(2H)$. A power series development yields

$$\mathbb{E}[|f(U_0)|] \leq \sum_{k=0}^{\infty} \frac{1}{k!} \mathbb{E}[|U_0|^{\gamma k}],$$

where U_0 is an element of the qth Wiener chaos. By the hypercontractivity property, for all $k \geq \frac{2}{\gamma}$,

$$\mathbb{E}[|U_0|^{\gamma k}] \leq g(k) := (k-1)^{\frac{\gamma q k}{2}} \left(\frac{1}{\alpha^{2H}} \Gamma(2H) \right)^{\frac{\gamma k}{2}}.$$

Stirling formula allows us to write

$$\frac{g(k)}{k!} \sim_{k\to\infty} \frac{(k-1)^{\frac{\gamma q k}{2}}}{k^k} \frac{\left(\frac{1}{\alpha^{2H}}\Gamma(2H)\right)^{\frac{\gamma k}{2}} e^k}{\sqrt{2\pi k}}, \tag{3.33}$$

and the associated series converges if $\gamma q < 2$. □

The next result analyzes the fluctuations in the ergodic theorem proved in Theorem 12.

Theorem 13. *(A)* [Case $q = 1$] *Let f be in $L^2(\mathbb{R}, \gamma)$ for $\gamma = \mathcal{N}(0, \frac{\Gamma(2H)}{\alpha^{2H}})$. We denote by $(a_i)_{i\geq 0}$ the coefficients of f in its Hermite expansion, and we let d be the centered Hermite rank of f. Then,*

- *if $\frac{1}{2} < H < 1 - \frac{1}{2d}$,*

$$\frac{1}{\sqrt{T}} \int_0^{Tt} (f(U_s) - \mathbb{E}[f(U_0)])\, ds \xrightarrow[T\to\infty]{f.d.d} c_{f,H} W_t,$$

- *if $H = 1 - \frac{1}{2d}$,*

$$\frac{1}{\sqrt{T\log T}} \int_0^{Tt} (f(U_s) - \mathbb{E}[f(U_0)])\, ds \xrightarrow[T\to\infty]{f.d.d} c_{f,H} W_t,$$

- *if $H > 1 - \frac{1}{2d}$,*

$$T^{q(1-H)-1} \int_0^{Tt} (f(U_s) - \mathbb{E}[f(U_0)])\, ds \xrightarrow[T\to\infty]{f.d.d} c_{f,H} Z_t^{d,H},$$

where $Z^{d,H}$ is a Hermite process of order d and index $d(H-1)+1$, W is a Brownian motion and

$$c_{f,H} = \begin{cases} \sqrt{\sum_{k\geq d} k! a_k^2 \int_{\mathbb{R}_+} |\rho(s)|^k} & \text{if } H < 1 - \frac{1}{2d} \\ a_d\sqrt{d! \frac{3}{16\alpha^2}} & \text{if } H = 1 - \frac{1}{2d} \\ a_d\sqrt{d! \frac{H^d\Gamma(2H)^d}{\alpha^{2Hd}}} & \text{if } H > 1 - \frac{1}{2d} \end{cases} \tag{3.34}$$

with

$$\rho(s) = \mathbb{E}[U_s U_0] = \int_{-\infty}^0 \int_{-\infty}^s e^{-\alpha(s-(u+v))} |u-v|^{2H-2} du dv.$$

Moreover, if $f \in L^p(\mathbb{R}, \gamma)$ for some $p > 2$, the previous convergences holds true in the Banach space $C([0,1])$.
(B) [Case $q > 1$] *Let P be a real valued polynomial. Then, the conclusions of Theorem 2 apply to U.*

Proof. Except for $H = 1 - \frac{1}{2d}$ in Part A, this is a direct consequence of Theorems 1 and 2. The convergence in the critical case can be checked through easy but tedious computations, by reducing to the case where f is the dth Hermite polynomial. Details are left to the reader. $\square$

3.7 Appendix

In this section we present two technical lemmas that play an important role along the paper. First, we shall reproduce a very useful result from [20]:

Theorem 14 (Power Counting Theorem)**.** *Let $T = \{M_1, \ldots, M_K\}$ a set of linear functionals on $\mathbb{R}^n$, $\{f_1, \ldots, f_K\}$ a set of real measurable functions on $\mathbb{R}^n$ such that there exist real numbers $(a_i, b_i, \mu_i, \nu_i)_{1 \le i \le K}$, satisfying for each $i = 1, \ldots, K$,*

$$\begin{aligned}
&0 < a_i \le b_i,\\
&|f_i(x)| \le |x|^{\mu_i} \text{ if } |x| \le a_i,\\
&|f_i(x)| \le |x|^{\nu_i} \text{ if } |x| \ge b_i,\\
&f_i \text{ is bounded over } [a_i, b_i].
\end{aligned}$$

For a linearly independent subset of W of T, we write $S_T(W) = \mathrm{Span}(M) \cap T$. We also define

$$d_0(W) = \dim(\mathrm{Span}(W)) + \sum_{i : M_i \in S_T(W)} \mu_i,$$

$$d_\infty(W) = n - \dim(\mathrm{Span}(W)) + \sum_{i : M_i \in T \setminus S_T(W)} \nu_i.$$

Assume $\dim(\mathrm{Span}(T)) = n$. *Then, the two conditions $(a) : d_0(W) > 0$ for all linearly independent subsets $W \subset T$, $(b) : d_\infty(W') < 0$ for all linearly independent proper subsets $W' \subset T$, imply*

$$\int_{\mathbb{R}^n} \prod_{i=1}^{K} |f_i(M_i(x))| dx < \infty \tag{3.35}$$

Moreover, assume that $|f_i(x)| = |x|^{\mu_i}$ if $|x| \le a_i$, Then $\int_{[-1,1]^n} \prod_{i=1}^{K} |f_i(M_i(x))| dx < \infty$, if an only if for any linearly independent subset $W \subset T$ condition (a) holds.

The next lemma is an application of the Hardy-Littlewood-Sobolev inequality,

Lemma 15. *Fix $n, q \geq 2$ and $\alpha \in A_{n,q}$. Recall H and H_0 from (3.1). Assume $\varphi \in L^1(\mathbb{R}) \cap L^{\frac{1}{H}}(\mathbb{R})$ Then*

$$\int_{\mathbb{R}^n} \prod_{k=1}^{n} |\varphi(\eta_k)| \prod_{1 \leq i < j \leq n} |\eta_i - \eta_j|^{(2H_0 - 2)\alpha_{ij}} d\eta_1 \dots d\eta_n < \infty.$$

Proof. We are going to use the multilinear Hardy-Littlewood-Sobolev inequality, that we recall here for the convenience of the reader (see [1, Theorem 6]): if $f : \mathbb{R} \to \mathbb{R}$ is a measurable function, if $p \in (1, n)$ and if the $\gamma_{ij} \in (0, 1)$ are such that $\sum_{1 \leq i < j \leq n} \gamma_{ij} = 1 - \frac{1}{p}$, then there exists $c_{p,\gamma} > 0$ such that

$$\int_{\mathbb{R}^n} \prod_{k=1}^{n} |f(u_k)| \prod_{1 \leq i < j \leq n} |u_i - u_j|^{-\gamma_{ij}} du_1 \dots du_n \leq c_{p,\gamma} \left(\int_{\mathbb{R}} |f(u)|^p du \right)^{\frac{n}{p}}. \tag{3.36}$$

Set $p = 1/(1 - (1 - H)\frac{2|\alpha|}{nq})$. Since $2|\alpha| \leq nq$, we have that $p > 1$. On the other hand, since $H > \frac{1}{2}$, one has $nH > \frac{n}{2} \geq 1$; this implies that $(1 - H)\frac{2|\alpha|}{q} < (1 - H)n < n - 1$, that is, $p < n$. Moreover, set $\gamma_{ij} = (2 - 2H_0)\alpha_{ij} = (1 - H)\frac{2\alpha_{ij}}{q} \in (0, 1)$; we have $\sum_{1 \leq i < j \leq n} \gamma_{ij} = 2(1 - H)\frac{|\alpha|}{q} \leq (1 - H)n < n - 1$. We deduce from (3.36) that

$$\int_{\mathbb{R}^n} \prod_{k=1}^{n} |\varphi(\eta_k)| \prod_{1 \leq i < j \leq n} |\eta_i - \eta_j|^{(2H_0 - 2)\alpha_{ij}} d\eta_1 \dots d\eta_n \leq c_{p,\gamma} \left(\int_{-\infty}^{\infty} |x(u)|^p du \right)^{\frac{n}{p}}.$$

But $p \in (1, \frac{1}{H})$ and $x \in L^1(\mathbb{R}) \cap L^{\frac{1}{H}}(\mathbb{R})$, so the claim follows. ☐

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Chapter 4

Fluctuation of the Hadwiger-Wills information content

This chapter is based on a work in progress in collaboration with Ivan Nourdin.

4.1 Introduction

4.1.1 Convex body and intrinsic volumes

Throughout this paper, K denotes a non-empty convex body in $\mathbb{R}^d$. Its dimension, noted $\dim K$ and taking values in $\{0, 1, 2, \ldots, d\}$, is the dimension of the affine hull of K. When K has dimension j, we define the j-dimensional volume $\mathrm{Vol}_j(K)$ to be the Lebesgue measure of K, computed relative to its affine hull. We also write $\mathbb{B}^j$ for the Euclidean unit ball of $\mathbb{R}^j$. The *Steiner formula* (e.g. [10, Section 1]) asserts that $\mathrm{Vol}_d(K + r\mathbb{B}^d)$ is a polynomial in $r > 0$ given by

$$\mathrm{Vol}_d(K + r\mathbb{B}^d) = \sum_{k=0}^{d} \kappa_{d-k} r^{d-k} V_k(K), \tag{4.1}$$

where the multiplicative constant $\kappa_j = \mathrm{Vol}_j(\mathbb{B}^j) = \frac{\pi^{\frac{j}{2}}}{\Gamma(1+\frac{j}{2})}$ are here to guarantee that the kth intrinsic volume $V_k(K)$ is really intrinsic to K, in the sense that it does not depend on the dimension of the underlying space.

4.1.2 Hadwiger-Wills information content

In [8, Corollary 2.5] the following link between the intrinsic volumes of K and a distance integral was established as a consequence of the Steiner formula (4.1): for any absolutely integrable function $g : \mathbb{R}_+ \to \mathbb{R}$, we have

$$\begin{aligned} &\int_{\mathbb{R}^d} g(\pi\,\mathrm{dist}^2(x,K)) e^{-\pi\,\mathrm{dist}^2(x,K)} dx \qquad (4.2) \\ = \;& g(0)V_d(K) + \sum_{j=0}^{d-1} \left(\frac{1}{\Gamma((d-j)/2)} \int_0^\infty g(r) r^{-1+(d-j)/2} e^{-r} dr \right) V_j(K). \end{aligned}$$

In particular, taking $g \equiv 1$ in (4.2) yields that the Wills functional [12] defined as

$$W(K) := \int_{\mathbb{R}^d} e^{-\pi\,\mathrm{dist}^2(x,K)} dx,$$

is equal to the *total* intrinsic volume, that is, $W(K) = \sum_{j=0}^{d} V_j(K)$.

Consider now the log-concave density

$$\mu_K(x) := \frac{1}{W(K)} e^{-\pi\,\mathrm{dist}^2(x,K)}, \quad x \in \mathbb{R}^d, \qquad (4.3)$$

that we name *Hadwiger-Wills density associated to* K, in honor of the influential papers [7] and [12]. Let $X_K : \Omega \to \mathbb{R}^d$ be a random vector distributed according to μ_K. It follows from (4.2) with $g(r) = e^{(1-\lambda^2)r}$ that the real-valued random variable

$$H_K := \pi\,\mathrm{dist}^2(X_K, K)$$

satisfies

$$\mathbb{E}[e^{(1-\lambda^2)H_K}] = \frac{1}{W(K)} \sum_{j=0}^{d} \lambda^{j-d} V_j(K)$$

for all $\lambda > 0$, and thus its distribution is intimately related and fully characterized by the intrinsic volumes of K.

To explain the title of this section and of our paper, and although we will not follow an information-theoretic approach in this paper, we call H_K the *Hadwiger-Wills information content* to highlight that H_K represents the *information content* (also called *Shannon entropy*) of X_K, a property that was crucially used by Lotz *et al* [8] to prove that H_K displays a form of concentration.

4.1.3 Our main result

In [8], concentration properties for H_K have been investigated. In the present paper, we study the fluctuations of H_K around its mean. Let F and G denote two rel valued random variables. The total variation distance between F and G is defined as

$$d_{TV}(F,G) = \sup_{A\in\mathcal{B}(\mathbb{R})} |\mathbb{P}(G) - \mathbb{P}(F)|.$$

Our main result is the following statement.

Theorem 4.1.1. *Consider a sequence $(K_n)_{n\geq 1}$ of non-empty convex bodies and suppose, for each n, that*

- $K_n \subset \mathbb{R}^{d_n}$ *with* $d_n \to \infty$*;*
- *the boundary* ∂K_n *of* K_n *is* $\mathcal{C}^2$*;*
- K_n *is symmetric in the sense that there exists* $y \in K_n$ *such that* $x \in K_n \Rightarrow 2y - x \in K_n$*;*
- *the quantity* $\lambda_1^n := \min_{x\in\partial K_n} \lambda_1^{K_n}(x)$*, where* $\lambda_1^{K_n}(x)$ *denotes the minimal principal curvature of* ∂K_n *at* x *(see Section 4.3.5), satisfies* $0 < \lambda_1^n \leq 1$ *(in particular,* K_n *is strictly convex) and* $\frac{1}{\lambda_1^n} = O(d_n^\gamma)$ *as* $n \to \infty$*, for some* $\frac{1}{4} > \gamma > 0$ *independent of* n*.*

Then, there exists $\alpha, \beta > 0$ *independent of* n *such that*

$$d_{TV}\left(\frac{H_{K_n} - \mathbb{E}H_{K_n}}{\sqrt{\mathrm{Var}(H_{K_n})}}, N(0,1)\right) = O_{n\to\infty}\left(d_n^{2\gamma - d_n}\right) \tag{4.4}$$

as $n \to \infty$*. In particular,* H_{K_n} *satisfies a central limit theorem:*

$$\frac{H_{K_n} - \mathbb{E}H_{K_n}}{\sqrt{\mathrm{Var}(H_{K_n})}} \to N(0,1) \quad \textit{as } n \to \infty.$$

We note that Theorem 4.1.1 parallels recent fluctuation results proved in the context of *conic* intrinsic volumes, see [5] and more precisely Theorems 1.1, 2.1 and 3.1 therein.

In section 4.2, we will show that a weaker version of Theorem 4.1.1 can be obtained relatively easily by exploiting the formula (4.2). The main improvement of Theorem 4.1.1 is to demonstrate the convergence in the total variation metric, which is notoriously difficult to obtain. Theorem 4.1.1 provide an example of central limit theorem in total variation for a dependant, non stationary sequence whose representation in the Wiener chaos is not known, thus falling outside the scope of the existing references such as [1] or [9]. Theorem 4.1.1 is also an illustration of the flexibility of Stein's method, a powerful tool in quantitative Gaussian approximation which is actually the main ingredient of the proof.

Our method does not allow to prove a central limit theorem for the intrinsic volumes of the sequence K_n themselves, in variance with the setting of conic intrinsic volumes studied in [5]. However, it is likely that this central limit theorem holds at least in some circumstances. This could be a further interesting question to study.

4.1.4 Case where K_n is an hypercube

To better understand the scope of Theorem 4.1.1, let us analyze for the sake of comparison the case where K_n is a hypercube. Even if it does not verify the hypotheses of Theorem 4.1.1 (as it is not regular and therefore we cannot speak of its principal curvatures), it seems to be the only[1] case where the fluctuations of H_{K_n} can be analyzed by hand (thanks to the induced independence), helping us to better understanding the structure of this random variable in general.

More specifically and for simplicity, assume that K_n is a hypercube of the form $[-T_n, T_n]^{d_n}$ with $T_n > 0$. It is immediate to check that

$$\operatorname{dist}^2(x, K_n) = \sum_{k=1}^{d_n} (|x_k| - T_n)_+^2 \tag{4.5}$$

for all $x = (x_1, \ldots, x_{d_n}) \in \mathbb{R}^{d_n}$, where $(\ldots)_+^2$ is shorthand for $[(\ldots)_+]^2$. By plugging (4.5) into (4.3), we deduce that the marginals of X_{K_n} are independent, with a common density given by $u \mapsto \frac{e^{-\pi\,(|u|-T_n)_+^2}}{1+2T_n}$.

The simplest case is when we choose $T_n = 1$, that is, $K_n = [-1, 1]^{d_n}$. The usual CLT then applies and yields that $\frac{H_{K_n} - \mathbb{E}H_{K_n}}{\sqrt{\operatorname{Var}(H_{K_n})}} \to N(0, 1)$, in agreement with the conclusion of Theorem 4.1.1.

At the opposite, let us now choose $d_n = T_n = n$, that is, $K_n = [-n, n]^n$. Straightforward calculations show that $\mathbb{E}H_{K_n} = \frac{2\pi n}{1+2n} \to \pi$ and $\operatorname{Var}(H_{K_n}) = \frac{8\pi^2 n(1+3n)}{(1+2n)^2} \to 6\pi^2$ as $n \to \infty$. If we had $\frac{H_{K_n} - \mathbb{E}H_{K_n}}{\sqrt{\operatorname{Var}(H_{K_n})}} \to N(0, 1)$, we would

[1]More precisely, we could have considered hyperrectangles as well.

deduce that $H_{K_n} \to N(\pi, 6\pi^2)$ and in particular $\mathbb{P}(H_{K_n} = 0) \to 0$. But $\mathbb{P}(H_{K_n} = 0) = \left(\frac{1}{1+\frac{1}{2n}}\right)^n \to \frac{1}{\sqrt{e}}$, meaning that $\frac{H_{K_n} - \mathbb{E}H_{K_n}}{\sqrt{\mathrm{Var}(H_{K_n})}} \not\to N(0,1)$.

We learn from this analysis in the easy case where K_n is a hypercube that $\frac{H_{K_n} - \mathbb{E}H_{K_n}}{\sqrt{\mathrm{Var}(H_{K_n})}}$ may or may not satisfy a CLT, and that is seems to depend on the asymptotic size of K_n. It therefore does not appear unreasonable to impose a condition on the minimal principal curvature for the conclusion of Theorem 4.1.1 to be valid.

4.1.5 Sketch of the proof

To prove Theorem 4.1.1, we rely on several steps.

The first step is to show (see Proposition 4.4.1) by means of Stein's method the following bound on the total variation distance between H_K properly normalized and the standard Gaussian distribution:

$$d_{TV}\left(\frac{H_K - \mathbb{E}H_K}{\sqrt{\mathrm{Var}(H_K)}}, N(0,1)\right) \leq \frac{\sqrt{\mathrm{Var}(U_K(X_K))}}{\pi \mathrm{Var}(H_K)} + \text{remainder}. \tag{4.6}$$

In (4.6), $U_K : \mathbb{R}^d \to \mathbb{R}$ is the smooth function given by (4.17).

A second step is then to bound $\mathrm{Var}(U_K(X_K))$ in (4.6). For this, we rely on the classical *Brascamp-Lieb inequality*, according to which

$$\mathrm{Var}(V(Y)) \leq \frac{1}{k}\mathbb{E}\big[\|\nabla V(Y)\|^2\big] \tag{4.7}$$

when $V : \mathbb{R}^d \to \mathbb{R}$ is smooth enough and $Y : \Omega \to \mathbb{R}^d$ admits a density of the form $e^{-\theta}$ where θ is k-strongly convex (that is, satisfies $\langle(\mathrm{Hess}\,\theta)u, u\rangle \geq k\|u\|^2$ for all $u \in \mathbb{R}^d$). But $\pi\,\mathrm{dist}^2(\cdot, K)$ being convex but not *strongly* convex, we cannot apply directly (4.7) to $V = U_K$ and $Y = X_K$. This is why we first approximate X_K by a strongly convex random variable Y_K (with density given in Definition 4.4.3) and we then estimate the difference between $\mathrm{Var}(U_K(X_K))$ and $\mathrm{Var}(U_K(Y_K))$ (see Proposition 4.4.5).

Finally, in a third step we control the remainder term of (4.6), before concluding that (4.4) takes place.

4.1.6 Organisation of the paper

The rest of the paper is organised as follows. In Section 4.2, we prove a weaker version of Theorem 4.1.1. Section 4.3 contains a few preliminaries to prepare the proof of Theorem 4.1.1, which is finally done in Section 4.4.

4.2 Theorem without bounds

Theorem 4.2.1. *Consider a sequence* $(K_n)_{n\geq 1}$ *of non-empty convex bodies such that* $K_n \subset \mathbb{R}^{d_n}$ *for all* n*, with* $d_n \to \infty$*. To each* K_n *let us associate the discrete random variable* I_{K_n} *defined, for any* $j \in \{0, \dots, d_n\}$*, as*

$$\mathbb{P}(I_{K_n} = j) = \frac{V_{d_n - j}(K_n)}{W(K_n)}.$$

Assume that $\mathbb{E}(I_{K_n}) \to \infty$ *and* $\mathrm{Var}(I_{K_n}) = o(\mathbb{E}(I_{K_n}))$ *as* $n \to \infty$*. Then* $\mathrm{Var}(H_{K_n}) \to \infty$ *and* H_{K_n} *satisfies a central limit theorem:*

$$\frac{H_{K_n} - \mathbb{E}H_{K_n}}{\sqrt{\mathrm{Var}(H_{K_n})}} \to N(0,1) \quad \textit{as } n \to \infty.$$

Proof. We deduce from (4.2) that $H_{K_n} \overset{\text{law}}{=} \sum_{j=1}^{I_{K_n}} \gamma_j$, where the $\gamma_j \sim \Gamma(\frac{1}{2}, 1)$ are independent copies, also independent from I_{K_n}. In particular,

$$\mathbb{E}H_{K_n} = \frac{1}{2}\mathbb{E}I_{K_n} \tag{4.8}$$

$$\mathrm{Var}(H_{K_n}) = \frac{1}{2}\mathbb{E}I_{K_n} + \frac{1}{4}\mathrm{Var}(I_{K_n}) \to \infty. \tag{4.9}$$

Writing $\sigma_n^2 = \mathrm{Var}(H_{K_n})$ for simplicity, we deduce that

$$\frac{1}{\sigma_n}\big(H_{K_n} - \mathbb{E}H_{K_n}\big) = \sqrt{\frac{\lfloor \mathbb{E}I_{K_n} \rfloor}{\sigma_n^2}}\, S_n + \frac{R_n}{\sigma_n}, \tag{4.10}$$

where

$$S_n = \frac{1}{\sqrt{\lfloor \mathbb{E}I_{K_n} \rfloor}} \sum_{j=1}^{\lfloor \mathbb{E}I_{K_n} \rfloor} \big(\gamma_j - \frac{1}{2}\big)$$

$$R_n = \frac{1}{2}\big(\lfloor \mathbb{E}I_{K_n} \rfloor - \mathbb{E}I_{K_n}\big) + \begin{cases} \sum_{j=\lfloor \mathbb{E}I_{K_n} \rfloor + 1}^{I_{K_n}} \gamma_j & \text{if } \lfloor \mathbb{E}I_{K_n} \rfloor < I_{K_n} \\ 0 & \text{if } \lfloor \mathbb{E}I_{K_n} \rfloor = I_{K_n} \\ -\sum_{j=I_{K_n}+1}^{\lfloor \mathbb{E}I_{K_n} \rfloor} \gamma_j & \text{if } \lfloor \mathbb{E}I_{K_n} \rfloor > I_{K_n} \end{cases}.$$

Since $\mathbb{E}(I_{K_n}) \to \infty$ and $\mathrm{Var}(I_{K_n}) = o(\mathbb{E}(I_{K_n}))$, we have that $\frac{\lfloor \mathbb{E}I_{K_n} \rfloor}{\sigma_n^2} \to 2$ and $S_n \to N(0, \frac{1}{2})$ (by the usual CLT). Moreover,

$$\mathbb{E}|R_n| \leq \frac{1}{2} + \sum_{l=\lfloor \mathbb{E}I_{K_n} \rfloor + 1}^{d} \mathbb{P}(I_{K_n} = l) \sum_{j=\lfloor \mathbb{E}I_{K_n} \rfloor + 1}^{l} \mathbb{E}\gamma_j$$

$$+\sum_{l=0}^{\lfloor \mathbb{E}I_{K_n}\rfloor -1}\mathbb{P}(I_{K_n}=l)\sum_{j=l+1}^{\lfloor \mathbb{E}I_{K_n}\rfloor}\mathbb{E}\gamma_j.$$

so that, using that $\mathbb{E}\gamma_j \le 1$,

$$\begin{aligned}\mathbb{E}|R_n| &\le \frac{1}{2}+\mathbb{E}\big|I_{K_n}-\lfloor \mathbb{E}I_{K_n}\rfloor\big| \le \frac{3}{2}+\mathbb{E}\big|I_{K_n}-\mathbb{E}I_{K_n}\big| \\ &\le \frac{3}{2}+\sqrt{\mathrm{Var}(I_{K_n})}.\end{aligned}$$

As a consequence,

$$\frac{\mathbb{E}|R_n|}{\sigma_n}\le \frac{3}{2\sigma_n}+\sqrt{\frac{\mathrm{Var}(I_{K_n})}{\frac{1}{2}\mathbb{E}I_{K_n}}}\to 0.$$

The desired conclusion follows by plugging $\frac{\lfloor \mathbb{E}I_{K_n}\rfloor}{\sigma_n^2}\to 2$, $S_n\to N(0,\frac{1}{2})$ and $\frac{\mathbb{E}|R_n|}{\sigma_n}\to 0$ in (4.10). □

We now give a sufficient condition implying both that $\mathbb{E}(I_{K_n})\to\infty$ and $\mathrm{Var}(I_{K_n})=o(\mathbb{E}(I_{K_n}))$.

Proposition 4.2.2. *Let $\gamma<\frac{1}{2}$ and let us assume that for all $n\in\mathbb{N}^*$, K_n belongs to the scaled ball $\mathbb{B}^d(0,d_n^\gamma)$. Then,*

1. $\liminf_n \frac{\sigma^2}{d_n}>0$
2. $\mathbb{E}(I_{K_n})\to\infty$ *and* $\mathrm{Var}(I_{K_n})=o(\mathbb{E}(I_{K_n}))$ *as* $n\to\infty$.

Proof. It is well known that intrinsic volumes form an ultra log-concave distribution, see e.g. [3] for a proof of this fact. In [6], it was shown in Theorem 1.5 and Lemma 5.3 that ultra-log concave random variables X valued in $\mathbb{N}$ verifies $\mathrm{Var}(X)\le\frac{1}{c}$ with

$$\frac{\mathbb{P}[\{X=1\}]}{\mathbb{P}[\{X=0\}]}=\frac{1}{c}\ge\mathbb{E}[X].$$

Moreover we have, thanks to (4.9), that

$$\sigma^2\ge\frac{1}{2}\mathbb{E}[I_{K_n}]\ge\frac{d-\frac{V_1(K)}{V_0(K)}}{2}\ge\frac{d-d^\gamma V_1(\mathbb{B}^d)}{2},$$

with $V_1(\mathbb{B})$ is the first intrinsic volume of the unit ball in $\mathbb{R}^d$ (given by $V_1(\mathbb{B}^d)=d\frac{Vol_d(\mathbb{B}^{d-1})}{Vol_d(\mathbb{B}^d)}$, see [8]), and where the last inequality follows from the fact that for all i, $V_i(K) \leq V_i(C)$ for any convex bodies such that $K \subset C$, and $V_0(K) = 1$.

We have $V_1(\mathbb{B}) = \binom{d}{1}\frac{Vol(\mathbb{B}^d)}{Vol(\mathbb{B}^{d-1})}$ with $Vol(\mathbb{B}^d)$ given in Section 4.3.1. We have $V_1(\mathbb{B}^d) = O_{d\to\infty} d^{\frac{1}{2}}$ (see (4.11)) and then $\frac{\sigma^2}{d_n} \geq \frac{1}{2d_n}\mathbb{E}[I_{K_n}] \underset{n\to\infty}{\longrightarrow} \frac{1}{2}$, which proves item (1). Moreover, we have

$$\mathrm{Var}(I_{K_n}) = O_{n\to\infty} d_n^{\frac{1}{2}+\gamma} = o_{n\to\infty} d_n = o_n \mathbb{E}[I_{K_n}],$$

which proves item (2). □

4.3 A few preliminaries

This section gathers a few preliminaries, to prepare the proof of Theorem 4.1.1. In what follows, we note $\|\cdot\|$ (resp. $\langle\cdot,\cdot\rangle$) the Euclidean norm (resp. scalar product) in $\mathbb{R}^d$.

4.3.1 Volume of the unit ball and of the unit sphere

Let us recall the classical expressions for the volumes of the unit ball $\mathbb{B}^d$ and of the unit sphere $\mathbb{S}^{d-1}$:

$$\mathrm{Vol}_d(\mathbb{B}^d) = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)} \quad \text{and} \quad \mathrm{Vol}_{d-1}(\mathbb{S}^{d-1}) = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}.$$

Since $\Gamma(m) \sim \sqrt{\frac{2\pi}{m}}\left(\frac{m}{e}\right)^m$ as $m \to \infty$, we deduce that

$$\mathrm{Vol}_d(\mathbb{B}^d) + \mathrm{Vol}_{d-1}(\mathbb{S}^{d-1}) = O(d^\alpha \beta^d d^{-\frac{d}{2}}) \quad \text{as } d \to \infty,$$

for some $\alpha, \beta > 0$, whose value is not important and can change from one line to another in what follows. We also have that

$$\frac{\mathrm{Vol}_{d-1}(\mathbb{B}^{d-1})}{\mathrm{Vol}_d(\mathbb{B}^d)} = O_{d\to\infty}\sqrt{d} \tag{4.11}$$

4.3.2 A useful lemma

The following easy lemma will be used in the proof of the forthcoming Lemma 4.4.4. We prove it for completeness.

Lemma 4.3.1. *Let $c \in [0,1]$ and let A be a $d \times d$ real symmetric matrix satisfying $\|Ax\| \le c\|x\|$ for all $x \in \mathbb{R}^d$. Then*

$$\langle (I_d - A)u, u\rangle \ge (1-c)\|u\|^2$$

for all $u \in \mathbb{R}^d$, with I_d the $d \times d$-identity matrix.

Proof. Since A is real symmetric, there is an orthonormal basis $e_1, \dots, e_d$ of $\mathbb{R}^d$ consisting of eigenvectors of A. Let $\mu_1, \dots, \mu_d$ be the corresponding eigenvalues. Fix $u \in \mathbb{R}^d$. We can write $u = u_1 e_1 + \dots + u_d e_d$, and thus

$$\langle (I_d - A)u, u\rangle = \sum_{i=1}^{d} (1-\mu_i) u_i^2.$$

But $|\mu_i| \le c$ for all i given the assumption on A, so the desired conclusion follows.
□

Starting from now, we let the notation of Sections 4.1.1 and 4.1.2 prevail.

4.3.3 In Theorem 4.1.1, the symmetry center of K_n can be assumed to be zero

The following lemma justifies why, without loss of generality, we may and will assume that the symmetry center of K_n is $0 \in \mathbb{R}^{d_n}$ in Theorem 4.1.1.

Lemma 4.3.2. *The law of H_K is invariant by translation. In other words, we have $H_{K+c} \overset{\text{law}}{=} H_K$ for any $c \in \mathbb{R}^d$.*

Proof. For any bounded Borel function $h : \mathbb{R}^d \to \mathbb{R}$, we can write

$$\begin{aligned}
\mathbb{E}h(H_{K+c}) &= \mathbb{E}h(\pi \operatorname{dist}^2(X_{K+c}, K+c)) \\
&= \int_{\mathbb{R}^d} h(\pi \operatorname{dist}^2(x, K+c)) \frac{e^{-\pi \operatorname{dist}^2(x,K+c)}}{W(K+c)} dx \\
&= \int_{\mathbb{R}^d} h(\pi \operatorname{dist}^2(x+c, K+c)) \frac{e^{-\pi \operatorname{dist}^2(x+c,K+c)}}{W(K+c)} dx \\
&= \int_{\mathbb{R}^d} h(\pi \operatorname{dist}^2(x, K)) \frac{e^{-\pi \operatorname{dist}^2(x,K)}}{W(K)} dx = \mathbb{E}h(H_K),
\end{aligned}$$

where in the last line we have used that $\operatorname{dist}^2(x+c, K+c) = \operatorname{dist}^2(x, K)$. The desired conclusion follows. □

4.3.4 Nearest point projection

To each $x \in \mathbb{R}^d$ one can associate a unique point $\Pi_K(x)$ of K such that $\|x - \Pi_K(x)\| = \operatorname{dist}(x, K)$. The map $\Pi_K : \mathbb{R}^d \to K$ is called the *nearest point projection*. If $x \in K$ then $\Pi_K(x) = x$. The map Π_K is 1-Lipschitz. We also have

$$\nabla \operatorname{dist}^2(x, K) = 2(x - \Pi_K(x)), \quad x \in \mathbb{R}^d, \tag{4.12}$$

see e.g. [5, Lemma 2.2].

When $x \in \partial K$, we denote by $n(x)$ the outward pointing unit normal to ∂K at x. We have, for all $x \in \mathbb{R}^d$,

$$x = \Pi_K(x) + \operatorname{dist}(x, K)\, n(\Pi_K(x)).$$

We deduce that $\Phi : \partial K \times (0, \infty) \to \mathbb{R}^d \setminus K$ defined as

$$\Phi(x, r) = x + r\, n(x)$$

is a homeomorphism, whose inverse is given by

$$\Phi^{-1}(y) = (\Pi_K(y), \operatorname{dist}(y, K)).$$

4.3.5 Principal curvatures

The *Gauss map* of ∂K is the map $G : \partial K \to \mathbb{S}^{d-1}$ defined by the inward unit normal, that is, $G(x) = -n(x)$. The shape operator of ∂K at x is $S_x = -DG_x$, where $DG_x : T_x\partial K \to T_{G(x)}\mathbb{S}^{d-1}$ is the differential of the Gauss map at x. The eigenvalues of S_x, denoted $\lambda_1^K(x), \ldots, \lambda_{d-1}^K(x)$ are the *principal curvatures* of ∂K at x. It is the usual convention to order them so that $0 \leq \lambda_1^K(x) \leq \ldots \leq \lambda_{d-1}^K(x)$, and to say that $\lambda_1(x)$ (resp. $\lambda_{d-1}(x)$) is the *minimal* (resp. *maximal*) principal curvature of ∂K at x.

Lemma 4.3.3. *For all $y \in \mathbb{R}^d$, we have*

$$\|\nabla \Pi_K(y)\| \leq \frac{1}{\operatorname{dist}(y, K)\, \lambda_1^K(\Pi_K(y)) + 1}. \tag{4.13}$$

Proof. Let $x \in \partial K$. The shape operator S_x being selfadjoint, there is an orthonormal basis $e = (e_1, \ldots, e_{d-1})$ of $T_x\partial K$ in which the matrix representing S_x is diagonal with entries $\lambda_1^K(x), \ldots, \lambda_{d-1}^K(x)$. Then, in the orthonormal basis $(e_1, \ldots, e_{d-1}, n(x))$ the matrix of the differential of Φ at (x,r) is given by

$$\nabla\Phi(x,r) = \begin{pmatrix} 1 + r\lambda_1^K(x) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 + r\lambda_{d-1}^K(x) & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

If $y \in K$, (4.13) is obviously satisfied because Π_K is 1-Lipschitz. Consider now $y \in \mathbb{R}^d \setminus K$. We can write

$$\Pi_K(y) = \gamma \circ \Phi^{-1}(y),$$

with γ the projection

$$\gamma = \left\{ \begin{array}{ccc} \partial K \times \mathbb{R} & \to & \partial K \\ (x,r) & \mapsto & x \end{array} \right. .$$

We deduce that the matrix of the gradient of Π_K at y in the basis $(e_1, \ldots, e_{d-1}, n(x))$ is given by

$$\nabla\Pi_K(y) = \nabla(\theta \circ \gamma \circ \Phi^{-1})(y) = \begin{pmatrix} \frac{1}{1+r\lambda_1^K(\Pi_K(y))} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \frac{1}{1+r\lambda_{d-1}^K(\Pi_K(y))} & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}.$$

The conclusion (4.13) follows from the fact that $r = \mathrm{dist}(y,K)$ and that $\lambda_1^K(\Pi_K(y)) \le \ldots \le \lambda_{d-1}^K(\Pi_K(y))$.

□

4.3.6 Blaschke's Rolling Theorem

Set

$$\lambda_1^K := \min_{x \in \partial K} \lambda_1^K(x) > 0. \tag{4.14}$$

From Blaschke's Rolling Theorem [4], we have that K is entirely contained in a ball (not necessarily centered at 0) of radius $\frac{1}{\lambda_1^K}$. By Lemma 4.3.2, we can assume without loss of generality that $0 \in K$; if so, we get that

$$K \subset \mathbb{B}^d(0, \frac{2}{\lambda_1^K}), \tag{4.15}$$

a fact that we label as it will be used many times in the sequel. We also deduce from (4.15) that

$$\|x\| \le \frac{2}{\lambda_1^K} + \mathrm{dist}(x,K), \quad x \in \mathbb{R}^d. \tag{4.16}$$

4.4 Proof of Theorem 4.1.1

We are now ready to proceed with the proof of Theorem 4.1.1. It is decomposed into several steps. By Lemma 4.3.2, we assume that $0 \in K_n$ for all n.

4.4.1 Step 1: Stein's method

As stated in the introduction, the central ingredient of the proof is Stein's method. Introduced first in [11], this method relies on astute integration by parts formulas to bound the total variation distance between any given random variables and the standard normal law (although other target variables and other distances can also be considered).

We write K for K_n in this step to simplify the notation. We also let the notation introduced in Sections 4.1.1 and 4.1.2 prevail, in particular the definition of X_K and H_K. We start by applying Stein's method to prove the following estimate for the distance in total variation between $\frac{H_K - \mathbb{E}H_K}{\sqrt{\mathrm{Var}(H_K)}}$ and the standard Gaussian distribution.

Proposition 4.4.1. *Write $\phi(x) = \pi\,\mathrm{dist}^2(x,K)$, $x \in \mathbb{R}^d$, set*

$$U_K(x) = \left\langle \int_0^\infty e^{-2t}\mathbb{E}\big[\nabla\phi(e^{-t}x + \sqrt{1-e^{-2t}}X_K)\big]dt, \nabla\phi(x) \right\rangle, \tag{4.17}$$

set $F_K = \frac{H_K - \mathbb{E}H_K}{\sqrt{\mathrm{Var}(H_K)}}$, and let $N \sim N(0,1)$. We have

$$d_{TV}(F_K, N) \le \frac{\sqrt{\mathrm{Var}(U_K(X_K))}}{\pi \mathrm{Var}(H_K)} + \frac{3}{\sqrt{\mathrm{Var}(H_K)}} \sup \big|B_1(h) - B_2(h)\big|, \tag{4.18}$$

where

$$B_1(h) = \int_0^\infty e^{-t}\mathbb{E}\left[\langle \nabla\phi(X_{K,t}), \Pi_K(X_K)\rangle h(F_K)\right] dt \tag{4.19}$$

$$B_2(h) = \int_0^\infty \frac{e^{-2t}}{\sqrt{1-e^{-2t}}}\mathbb{E}\left[\langle \nabla\phi(X_{K,t}), \Pi_K(\widehat{X}_K)\rangle h(F_K)\right] dt, \tag{4.20}$$

with $X_{K,t} = e^{-t}X_K + \sqrt{1-e^{-2t}}\widehat{X}_K$ *and* $\widehat{X}_K$ *an independent copy of* X_K*, and where the supremum in the right-hand side of (4.18) runs over the functions* $h : \mathbb{R} \to \mathbb{R}$ *that are* $\mathcal{C}^1$*, 2-Lipschitz and such that* $|h(x)| \le \sqrt{\frac{\pi}{2}} + |x|$*.*

Proof. To simplify the presentation of the proof, we remove the subscript K from the quantities considered, that is, we write F for F_K, H for H_K, X for X_K, X_t for $X_{K,t}$, U for U_K, etc. Let $g : \mathbb{R} \to \mathbb{R}$ be $\mathcal{C}^1$ and Lipschitz. We can write, using $\nabla\phi(x) = 2\pi(x - \Pi_K(x))$ (see (4.12)) in the last equality,

$$\begin{aligned}
\mathbb{E}[Fg(F)] &= \frac{1}{\sqrt{\operatorname{Var}(H)}}\mathbb{E}\left[(\phi(X) - \phi(\widehat{X}))g(F)\right] \\
&= -\frac{1}{\sqrt{\operatorname{Var}(H)}}\int_0^\infty \mathbb{E}\left[\frac{d}{dt}\big(\phi(X_t)\big)g(F)\right]dt \\
&= \frac{1}{\sqrt{\operatorname{Var}(H)}}\int_0^\infty e^{-t}\mathbb{E}\left[\langle\nabla\phi(X_t), X\rangle\, g(F)\right]dt \\
&\quad -\frac{1}{\sqrt{\operatorname{Var}(H)}}\int_0^\infty \frac{e^{-2t}}{\sqrt{1-e^{-2t}}}\mathbb{E}\left[\langle\nabla\phi(X_t), \widehat{X}\rangle g(F)\right]dt \\
&= \frac{1}{\sqrt{\operatorname{Var}(H)}}(A_1(g) - A_2(g) + B_1(g) - B_2(g)),
\end{aligned}$$

where

$$\begin{aligned}
A_1(g) &= \frac{1}{2\pi}\int_0^\infty e^{-t}\mathbb{E}\left[\langle\nabla\phi(X_t), \nabla\phi(X)\rangle g(F)\right]dt \\
A_2(g) &= \frac{1}{2\pi}\int_0^\infty \frac{e^{-2t}}{\sqrt{1-e^{-2t}}}\mathbb{E}\left[\langle\nabla\phi(X_t), \nabla\phi(\widehat{X})\rangle g(F)\right]dt
\end{aligned}$$

and $B_1(g)$ and $B_2(g)$ are given by (4.19) and (4.20) respectively. We have

$$\begin{aligned}
&\mathbb{E}\left[\langle\nabla\phi(X_t), \nabla\phi(X)\rangle g(F)\right] \\
&= \sum_{i=1}^d \mathbb{E}\Bigg[\int_{\mathbb{R}^{d-1}}\prod_{j\neq i}dx_j\int_{\mathbb{R}}dx_i\frac{\partial\phi}{\partial x_i}(e^{-t}x + \sqrt{1-e^{-2t}}\widehat{X}) \\
&\qquad \times\frac{\partial\phi}{\partial x_i}(x)\, g\left(\frac{\phi(x) - \mathbb{E}H}{\sqrt{\operatorname{Var}(H)}}\right)e^{-\phi(x)}\Bigg] \\
&= -\sum_{i=1}^d \mathbb{E}\Bigg[\int_{\mathbb{R}^{d-1}}\prod_{j\neq i}dx_j\int_{\mathbb{R}}dx_i\frac{\partial\phi}{\partial x_i}(e^{-t}x + \sqrt{1-e^{-2t}}\widehat{X}) \\
&\qquad \times\frac{\partial e^{-\phi}}{\partial x_i}(x)\, g\left(\frac{\phi(x) - \mathbb{E}H}{\sqrt{\operatorname{Var}(H)}}\right)\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&= e^{-t}\sum_{i=1}^{d}\mathbb{E}\left[\int_{\mathbb{R}^d}\frac{\partial^2\phi}{\partial x_i^2}(e^{-t}x+\sqrt{1-e^{-2t}}\widehat{X})g\left(\frac{\phi(x)-\mathbb{E}H}{\sqrt{\mathrm{Var}(H)}}\right)e^{-\phi(x)}dx\right] \\
&\quad +\frac{1}{\sqrt{\mathrm{Var}(H)}}\sum_{i=1}^{d}\mathbb{E}\left[\int_{\mathbb{R}^d}\frac{\partial\phi}{\partial x_i}(e^{-t}x+\sqrt{1-e^{-2t}}\widehat{X})\frac{\partial\phi}{\partial x_i}(x)\right. \\
&\quad \left.\times g'\left(\frac{\phi(x)-\mathbb{E}H}{\sqrt{\mathrm{Var}(H)}}\right)e^{-\phi(x)}dx\right] \\
&= e^{-t}\,\mathbb{E}\left[\Delta\phi(X_t)\,g(F)\right]+\frac{1}{\sqrt{\mathrm{Var}(H)}}\mathbb{E}\left[\langle\nabla\phi(X_t),\nabla\phi(X)\rangle g'(F)\right],
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}\left[\langle\nabla\phi(X_t),\nabla\phi(\widehat{X})\rangle g(F)\right] \\
&= \sum_{i=1}^{d}\mathbb{E}\left[\int_{\mathbb{R}^{d-1}}\prod_{j\neq i}dx_j\int_{\mathbb{R}}dx_i\frac{\partial\phi}{\partial x_i}(e^{-t}X+\sqrt{1-e^{-2t}}x)\right. \\
&\quad \left.\times\frac{\partial\phi}{\partial x_i}(x)\,g\left(\frac{\phi(X)-\mathbb{E}H}{\sqrt{\mathrm{Var}(H)}}\right)e^{-\phi(x)}\right] \\
&= -\sum_{i=1}^{d}\mathbb{E}\left[\int_{\mathbb{R}^{d-1}}\prod_{j\neq i}dx_j\int_{\mathbb{R}}dx_i\frac{\partial\phi}{\partial x_i}(e^{-t}X+\sqrt{1-e^{-2t}}x)\right. \\
&\quad \left.\times\frac{\partial e^{-\phi}}{\partial x_i}(x)\,g\left(\frac{\phi(X)-\mathbb{E}H}{\sqrt{\mathrm{Var}(H)}}\right)\right] \\
&= \sqrt{1-e^{-2t}}\sum_{i=1}^{d}\mathbb{E}\left[\int_{\mathbb{R}^d}\frac{\partial^2\phi}{\partial x_i^2}(e^{-t}X+\sqrt{1-e^{-2t}}x)\right. \\
&\quad \left.\times g\left(\frac{\phi(X)-\mathbb{E}H}{\sqrt{\mathrm{Var}(H)}}\right)e^{-\phi(x)}dx\right] \\
&= \sqrt{1-e^{-2t}}\,\mathbb{E}\left[\Delta\phi(X_t)\,g(F)\right].
\end{aligned}
$$

We deduce that

$$
\begin{aligned}
\mathbb{E}[Fg(F)-g'(F)] &= \frac{1}{2\pi\mathrm{Var}(H)}\mathbb{E}\left[(U(X)-2\pi\mathrm{Var}(H))\,g'(F)\right] \\
&\quad +\frac{1}{\sqrt{\mathrm{Var}(H)}}\left(B_1(g)-B_2(g)\right)
\end{aligned}
$$

and (with $g(x) = x$ the identity function)

$$2\pi \mathrm{Var}(H) = \mathbb{E}\left[U(X)\right] + 2\pi\sqrt{\mathrm{Var}(H)}(B_1(\mathrm{id}) - B_2(\mathrm{id})).$$

By combining the two previous identities, we get

$$\begin{aligned}\mathbb{E}[Fg(F) - g'(F)] &= \frac{1}{2\pi \mathrm{Var}(H)}\mathbb{E}\left[(U(X) - \mathbb{E}[U(X)])\, g'(F)\right] \\ &\quad - \frac{1}{\sqrt{\mathrm{Var}(H)}}\mathbb{E}[(B_1(\mathrm{id}) - B_2(\mathrm{id}))g'(F)] \\ &\quad + \frac{1}{\sqrt{\mathrm{Var}(H)}}(B_1(g) - B_2(g)).\end{aligned}$$

The desired conclusion then follows from Stein's lemma, according to which

$$d_{TV}(F, N) \leq \sup \left|\mathbb{E}[Fg(F) - g'(F)]\right|,$$

where the supremum runs over the functions $g : \mathbb{R} \to \mathbb{R}$ that are $\mathcal{C}^1$ and such that $\|g'\|_\infty \leq 2$ and $\|g\|_\infty \leq \sqrt{\frac{\pi}{2}}$. □

Remark 4.4.2. To keep things simple, the previous result was stated in the particular case of the information content of the distance law H_K. However, using exactly the same proof technique, it is actually possible to generalize this result to the case where the random variable F can be expressed as

$$F = \frac{f(X) - \mathbb{E}[f(X)]}{\sigma},$$

with $f : \mathbb{R}^d \to \mathbb{R}$ an absolutely continuous and sufficiently integrable function, X is a $\mathbb{R}^d$ valued random vector with density proportional to $e^{-\phi}$ with ϕ an absolutely continuous, sufficiently integrable function (continuous Gibbs measure) and $\sigma^2 = \mathrm{var}(f(X))$. In this case, with the same notations as in the statement of Proposition 4.4.1, we have for all $\gamma > 0$:

$$\begin{aligned}&d_{TV}(F, N) \\ \leq\ & \frac{2}{\gamma\sigma^2}\sqrt{\mathrm{var}\left(\int_0^\infty e^{-t}\langle \nabla f(X), \mathbb{E}_\infty[\nabla f(X_t)]\rangle dt\right)} \\ & + \frac{3}{\sigma}\sup_g \left| \int_0^\infty \mathbf{E}\left[g\left(\frac{f(Y)}{\sigma}\right) \left\langle \nabla f(X_t), e^{-t}(X - \frac{1}{\gamma}\nabla\phi(X)) \right.\right.\right. \\ & \left.\left.\left. - \frac{e^{-2t}}{\sqrt{1 - e^{-2t}}}(\hat{X} - \frac{1}{\gamma}\nabla\phi(\hat{X})) \right\rangle \right] dt \right|.\end{aligned}$$

where $\hat{\mathbb{E}}$ is the expectation with respect to $\hat{X}$ and $\mathbf{E} := \mathbb{E} \otimes \hat{\mathbb{E}}$.

Taking $\phi(x) = \frac{\|x\|^2}{2}$ and with an easy approximation argument, we actually recover the result of Theorem A.1 in [5].

4.4.2 Step 2: Preparation to the use of the Brascamp-Lieb inequality

In this step again, we let the notation introduced in Sections 4.1.1 and 4.1.2 prevail.

First, we note that

$$W(K) \geq \int_{\mathbb{R}^d} e^{-\pi\|x\|^2} dx = \left(\int_{\mathbb{R}} e^{-\pi u^2} du \right)^d = 1.$$

The polynomial $P(x) = (1-10x^3+15x^4-6x^5)^2$ satisfies $P \geq 0$, $P(0) = 1$, $P(1) = 0$ and $P'(0) = P''(0) = P'(1) = P''(1) = 0$. Let $\kappa \in (0,1)$ be such that $\inf_{y\in[0,\kappa]} P(y) \geq \frac{1}{2}$ and $\sup_{y\in[0,\kappa]} |P'(y)| + |P''(y)| \leq \frac{1}{24}$. Existence of κ is ensured by a continuity argument and its exact value is not important for the sequel. Recall the definition of λ_1^K from (4.14), define $\xi : \mathbb{R}^d \to [0, \infty)$ as

$$\xi(x) = \mathbf{1}_{[0,1]}(\frac{1}{2}\lambda_1^K\|x\|) + P(\frac{1}{2}\lambda_1^K\|x\| - 1)\mathbf{1}_{[1,2]}(\frac{1}{2}\lambda_1^K\|x\|)$$

and let $C > 0$ be given by

$$C = \frac{12}{\kappa} \left(\max_{y\in[0,1]} |P'(y)| + \max_{y\in[0,1]} |P''(y)| \right) + \frac{3}{2\kappa}.$$

Definition 4.4.3. The *modified distance law with respect to* K is the density

$$x \mapsto \frac{e^{-\left(\pi \operatorname{dist}^2(x,K) + \frac{1}{C}\xi(x)\|x\|^2\right)}}{\overline{W}(K)}, \quad x \in \mathbb{R}^d,$$

with $\overline{W}(K) = \int_{\mathbb{R}^d} e^{-\left(\pi \operatorname{dist}^2(x,K) + \frac{1}{C}\xi(x)\|x\|^2\right)} dx$.

Note that

$$\overline{W}(K) \geq W(K) \geq 1.$$

We have the following lemma, justifying in part why we introduced this modified distance law.

Lemma 4.4.4. *The modified distance law is strongly log-concave. More precisely, with* $k = \frac{(4\pi-1-2\kappa)\kappa}{1+2\kappa} \wedge \frac{1}{2C}$ *and* $\psi(x) = \pi \operatorname{dist}^2(x,K) + \frac{1}{C}\xi(x)\|x\|^2$, *we have, for all* $u, x \in \mathbb{R}^d$,

$$\langle \operatorname{Hess}(\psi(x))u, u \rangle \geq k\|u\|^2.$$

Proof. Using (4.12) we can write

$$\mathrm{Hess}(\psi(x)) = A(x) + B(x),$$

where

$$\begin{aligned} A(x) &= 2\pi(I_d - \nabla\Pi_K(x)) + \frac{2}{C}\xi(x)I_d \\ B(x) &= \mathbf{1}_{[1,2]}(\frac{1}{2}\lambda_1^K\|x\|)\left[\frac{3\lambda_1^K P'(\frac{1}{2}\lambda_1^K\|x\|-1)}{C\|x\|}xx^T\right. \\ &\quad \left. + \frac{\lambda_1^K\|x\|P'(\frac{1}{2}\lambda_1^K\|x\|-1)}{2C}I_d + \frac{(\lambda_1^K)^2P''(\frac{1}{2}\lambda_1^K\|x\|-1)}{4C}xx^T\right], \end{aligned}$$

with I_d the $d \times d$ identity matrix.

Fact 1. We claim that $\langle A(x)u, u\rangle \geq \frac{2}{C}\xi(x)\|u\|^2$ for all $u, x \in \mathbb{R}^d$. Indeed, from $\nabla\Pi_K(x)y = \lim_{t\to 0}\frac{1}{t}\big(\Pi_K(x+ty) - \Pi_K(x)\big)$ and $\|\Pi_K(x) - \Pi_K(y)\| \leq \|x-y\|$, we deduce that $\|\nabla\Pi_K(x)y\| \leq \|y\|$ for all $y \in \mathbb{R}^d$. Using Lemma 4.3.1 with $c = 1$, we get that $\langle (I_d - \nabla\Pi_K(x))u, u\rangle \geq 0$ for all $u, x \in \mathbb{R}^d$ and the claim follows from the definition of $A(x)$.

Fact 2. We claim that $|\langle B(x)u, u\rangle| \leq \kappa\|u\|^2$ for all $u, x \in \mathbb{R}^d$. Indeed,

$$\begin{aligned} & |\langle B(x)u,u\rangle| \\ = \;& \mathbf{1}_{[1,2]}(\frac{1}{2}\lambda_1^K\|x\|)\left|\frac{3\lambda_1^K P'(\frac{1}{2}\lambda_1^K\|x\|-1)}{C\|x\|}\langle u,x\rangle^2\right. \\ & \left. + \frac{\lambda_1^K\|x\|P'(\frac{1}{2}\lambda_1^K\|x\|-1)}{2C}\|u\|^2 + \frac{(\lambda_1^K)^2P''(\frac{1}{2}\lambda_1^K\|x\|-1)}{4C}\langle u,x\rangle^2\right| \\ \leq \;& \mathbf{1}_{[1,2]}(\frac{1}{2}\lambda_1^K\|x\|)(\frac{2\lambda_1^K\|x\|}{C}|P'(\frac{1}{2}\lambda_1^K\|x\|-1)| \qquad (4.21) \\ & + \frac{(\lambda_1^K)^2\|x\|^2}{4C}|P''(\frac{1}{2}\lambda_1^K\|x\|-1)|)\|u\|^2 \\ & \qquad (4.22) \\ \leq \;& \kappa\|u\|^2, \end{aligned}$$

where, in the last line, we used that $\lambda_1^K\|x\| \leq 4$ and that $|P'(y)|, |P''(y)| \leq \frac{C\kappa}{12}$ for $y \in [0,1]$ by definition of C.

Fact 3. We claim that $|\langle B(x)u, u\rangle| \leq \frac{1}{2C}\|u\|^2$ for all $u \in \mathbb{R}^d$ and all $x \in \mathbb{R}^d$ such that $\frac{1}{2}\lambda_1^K\|x\| \in [1, 1+\kappa]$. It is indeed an immediate consequence of (4.22) and the fact that $|P'(y)|, |P''(y)| \leq \frac{1}{24}$ for all $y \in [0, \kappa]$.

Fact 4. We claim that $\langle (I_d - \nabla\Pi_K(x))u, u\rangle \geq \frac{2\kappa}{1+2\kappa}\|u\|^2$ for all $u \in \mathbb{R}^d$ and all $x \in \mathbb{R}^d$ such that $\frac{1}{2}\lambda_1^K\|x\| \in [1+\kappa, \infty)$. Indeed, we deduce from (4.16) that $d(x, K) \geq \frac{2\kappa}{\lambda_1^K}$, implying in turn from Lemma 4.3.3 that $\|\nabla\Pi_K(x)\| \leq \frac{1}{1+2\kappa}$. Finally, the claim follows from Lemma 4.3.1.

We are now ready to prove Lemma 4.4.4. We have, for all $u, x \in \mathbb{R}^d$,

$$\langle \mathrm{Hess}(\psi(x))u, u\rangle = \langle A(x)u, u\rangle + \langle B(x)u, u\rangle.$$

First case: $\frac{1}{2}\lambda_1^K\|x\| \leq 1$. We have $\xi(x) = 1$ and we deduce from Facts 1 and 2 that

$$\langle \mathrm{Hess}(\psi(x))u, u\rangle \geq \left(\frac{2}{C} - \kappa\right)\|u\|^2 \geq \frac{1}{2C}\|u\|^2 \geq k\|u\|^2.$$

Second case: $\frac{1}{2}\lambda_1^K\|x\| \in [1, 1+\kappa]$. In this case, $\xi(x) \geq \frac{1}{2}$ and we get, from Facts 1 and 3,

$$\langle \mathrm{Hess}(\psi(x))u, u\rangle \geq \left(\frac{1}{C} - \frac{1}{2C}\right)\|u\|^2 = \frac{1}{2C}\|u\|^2 \geq k\|u\|^2.$$

Third case: $\frac{1}{2}\lambda_1^K\|x\| \geq 1+\kappa$. We can write, using Facts 2 and 4,

$$\begin{aligned}\langle \mathrm{Hess}(\psi(x))u, u\rangle &\geq 2\pi\langle (I_d - \nabla\Pi_K(x))u, u\rangle - \kappa\|u\|^2 \\ &\geq (2\pi\frac{2\kappa}{1+2\kappa} - \kappa)\|u\|^2 \geq k\|u\|^2.\end{aligned}$$

The proof is complete.

□

Proposition 4.4.5. *Fix $\gamma \in (0, \frac{1}{4})$, $c, q, M > 0$ and an integer $p \geq 1$, all independent of d. Assume that $1 \leq \frac{1}{\lambda_1} \leq cd^\gamma$. Let $Y_K : \Omega \to \mathbb{R}^d$ be distributed according to the modified distance law with respect to K, see Definition 4.4.3. Let $F : \mathbb{R}^d \to \mathbb{R}_+$ be a positive map such that $|F(x)| \leq M(\|x\|^q + 1)$ for all $x \in \mathbb{R}^d$. We then have the existence of $\alpha, \beta > 0$ (independent of d) such that*

$$\left|(\mathbb{E}[F(X_K)])^p - (\mathbb{E}[F(Y_K)])^p\right| = O(d^\alpha \beta^d d^{-d(\frac{1}{2}-2\gamma)}) \to 0 \quad \textit{as } d \to \infty.$$

Proof.
First case: $p = 1$. We can write

$$\Big|\mathbb{E}[F(X)] - \mathbb{E}[F(Y)]\Big| \leq A + B,$$

with

$$\begin{aligned} A &= \left|\mathbb{E}\left[F(X)\mathbf{1}_{\|X\| \leq \frac{2}{\lambda_1}}\right] - \mathbb{E}\left[F(Y)\mathbf{1}_{\|Y\| \leq \frac{2}{\lambda_1}}\right]\right| \\ B &= \left|\mathbb{E}\left[F(X)\mathbf{1}_{\|X\| > \frac{2}{\lambda_1}}\right] - \mathbb{E}\left[F(X)\mathbf{1}_{\|Y\| > \frac{2}{\lambda_1}}\right]\right|. \end{aligned}$$

Now, the proof is divided into three steps.

Step 1: Upper bound for A. We have $c_\mu \geq 1$ and $c_\nu \geq (1 + \frac{\|P\|_\infty}{\pi C})^{-\frac{d}{2}}$. Thanks to a polar change of coordinates, we can then write, for some α and β whose value can change from one line to another,

$$\begin{aligned} A &= \left|\int_{\|x\| \leq \frac{2}{\lambda_1}} F(x) e^{-\pi d^2(x,K)} \left(\frac{1}{c_\mu} - \frac{1}{c_\nu} e^{-\frac{1}{C}\xi(x)\|x\|^2}\right) dx\right| \\ &\leq \left\{1 + \left(1 + \frac{\|P\|_\infty}{\pi C}\right)^{\frac{d}{2}}\right\} M \left(\left(\frac{2}{\lambda_1}\right)^q + 1\right) \mathrm{Vol}(\mathbb{B}(0, \frac{2}{\lambda_1})) \\ &= O(d^\alpha \beta^d d^{-d(\frac{1}{2}-\gamma)})). \end{aligned}$$

Step 2: Upper bound for B. Since $c_\mu \geq 1$ and $c_\nu \geq (1 + \frac{\|P\|_\infty}{\pi C})^{-\frac{d}{2}}$, we have

$$\begin{aligned} \left|\frac{1}{c_\mu} - \frac{1}{c_\nu}\right| &\leq \frac{c_\mu - c_\nu}{c_\nu} \\ &= \frac{\int_{\mathbb{R}^d} e^{-\pi d^2(x,K)}(1 - e^{-\frac{1}{C}\xi(x)\|x\|^2})dx}{c_\nu} \\ &\leq \mathrm{Vol}(\mathbb{B}^d(0, \frac{2}{\lambda_1}))(1 + \frac{\|P\|_\infty}{\pi C})^{-\frac{d}{2}} \\ &= O(d^\alpha \beta^d d^{-d(\frac{1}{2}-\gamma)}). \end{aligned}$$

Since $K \subset \mathbb{B}(0, \frac{2}{\lambda_1})$ and $\xi(x) = 0$ for $\|x\| > \frac{2}{\lambda_1}$, we deduce

$$B = \left|\left(\frac{1}{c_\mu} - \frac{1}{c_\nu}\right) \int_{\|x\| > \frac{2}{\lambda_1}} F(x) e^{-\pi d^2(x,K)} dx\right|$$

$$\leq M\left|\frac{1}{c_\mu}-\frac{1}{c_\nu}\right| \mathrm{Vol}(\mathbb{S}^{d-1}(0,1)) \int_{\frac{2}{\lambda_1}}^{d} e^{-\pi(r-\frac{2}{\lambda_1})^2}(1+r^q)r^{d-1}dr.$$

Now, we observe that

$$\begin{aligned}
&\int_{\frac{2}{\lambda_1}}^{d} e^{-\pi(r-\frac{2}{\lambda_1})^2}(1+r^q)r^{d-1}dr \\
\leq\ & \int_0^\infty e^{-\pi r^2}(1+(r+\frac{2}{\lambda_1})^q)r^{d-1}dr \\
\leq\ & 2\int_0^\infty e^{-\pi r^2}(r+2cd^\gamma)^{q+d-1}dr \\
\leq\ & 2(4cd^\gamma)^{q+d-1}\int_0^{2cd^\gamma} e^{-\pi r^2}dr + 2^{q+d}\int_{2cd^\gamma}^\infty e^{-\pi r^2}r^{q+d-1}dr \\
=\ & O(d^\alpha\beta^d d^{\gamma d}),
\end{aligned}$$

The proof of the proposition when $p=1$ is complete, by putting together the estimates for A and B.

Second case: $p \geq 1$. In the general case for p, it suffices to use the inequality

$$\begin{aligned}
&\Big|(\mathbb{E}[F(X)])^p - (\mathbb{E}[F(Y)])^p\Big| \\
\leq\ & p\Big|\mathbb{E}[F(X)]-\mathbb{E}[F(Y)]\Big|\Big(\Big|(\mathbb{E}[F(X)])^{p-1}\Big| + \Big|(\mathbb{E}[F(Y)])^{p-1}\Big|\Big).
\end{aligned}$$

The result obtained in the case $p=1$ plus similar computations allow then to conclude. □

4.4.3 Step 3: Upper bound for $\mathrm{Var}(U_K(X_K))$

.

We can write, thanks to Proposition 4.4.5 and with $Y \sim \nu(dx) = e^{-\psi(x)}dx$, with ν and ψ as in Definition 4.4.3,

$$\begin{aligned}
\mathrm{Var}(U_\phi(X)) &= \mathrm{Var}(U_\phi(Y)) + O(d^\alpha\beta^d d^{-d(\frac{1}{2}-2\gamma)}) \\
&\leq \frac{1}{k}\mathbb{E}\big[\|\nabla U_\phi(Y)\|^2\big] + O(d^\alpha\beta^d d^{-d(\frac{1}{2}-2\gamma)}) \\
&\leq \frac{1}{k}\mathbb{E}\big[\|\nabla U_\phi(X)\|^2\big] + O(d^\alpha\beta^d d^{-d(\frac{1}{2}-2\gamma)}).
\end{aligned}$$

Now, let us estimate $\mathbb{E}\big[\|\nabla U_\phi(X)\|^2\big]$. We have

$$\begin{aligned}
& \nabla U_\phi(x) \\
= \; & \mathbb{E}\int_0^\infty e^{-2t}(\mathrm{Hess}\phi)(e^{-t}x+\sqrt{1-e^{-2t}X})\nabla\phi(x)dt \\
& +\mathbb{E}\int_0^\infty e^{-t}(\mathrm{Hess}\phi)(x)\nabla\phi(e^{-t}x+\sqrt{1-e^{-2t}X})dt \\
= \; & 2\pi^2\mathbb{E}\int_0^\infty 2e^{-2t}\big(I-\nabla\Pi_K(e^{-t}x+\sqrt{1-e^{-2t}X})\big)(x-\Pi_K(x))dt \\
& +4\pi^2\mathbb{E}\int_0^\infty e^{-t}\big(I-\nabla\Pi_K(x)\big)\big((I-\Pi_K)(e^{-t}x+\sqrt{1-e^{-2t}X})\big)dt.
\end{aligned}$$

We deduce from Jensen and the feact that $\nabla\Pi_K(\cdot)$ are contracting operators that

$$\begin{aligned}
& \|\nabla U_\phi(x)\|^2 \\
\leq \; & 2(2\pi^2)^2\int_0^\infty 2e^{-2t}dt\ \|x-\Pi_K(x)\|^2 \\
& +2(4\pi^2)^2\int_0^\infty e^{-t}\mathbb{E}\|e^{-t}x+\sqrt{1-e^{-2t}X}\|^2dt \\
\leq \; & 2(2\pi^2)^2\|x-\Pi_K(x)\|^2+4(4\pi^2)^2\big(\|x\|^2+\mathbb{E}\|X\|^2\big) \\
\leq \; & 72\pi^4\|x\|^2+64\pi^4\mathbb{E}\|X\|^2.
\end{aligned}$$

As a result,

$$\mathrm{Var}(U_\phi(X)) \leq \frac{136\pi^4}{k}\,\mathbb{E}\|X\|^2+O(d^\alpha\beta^d d^{-d(\frac{1}{2}-2\gamma)}).$$

4.4.4 Step 4: Upper bound for $\sup\big|B_1(h)-B_2(h)\big|$

Using that $\nabla\phi(x)=2\pi(x-\Pi_K(x))$, we can write

$$\begin{aligned}
B_1(h)-B_2(h) \;=\; & 2\pi\int_0^\infty e^{-t}\mathbb{E}\langle X_t,\Pi_K(X)\rangle h(F)dt \\
& -2\pi\int_0^\infty \frac{e^{-2t}}{\sqrt{1-e^{-2t}}}\mathbb{E}\langle X_t,\Pi_K(\widehat{X})\rangle h(F)dt \\
& -2\pi\int_0^\infty e^{-t}\mathbb{E}\langle \Pi_K(X_t),\Pi_K(X)\rangle h(F)dt \\
& +2\pi\int_0^\infty \frac{e^{-2t}}{\sqrt{1-e^{-2t}}}\mathbb{E}\langle \Pi_K(X_t),\Pi_K(\widehat{X})\rangle h(F)dt.
\end{aligned}$$

Since $K \subset \mathbb{B}^d(0, \frac{2}{\lambda_1})$, we have $\|\Pi_K(x)\| \leq \frac{2}{\lambda_1}$ for all $x \in \mathbb{R}^d$. In particular, using also that $\mathbb{E}\big[h(F)^2\big] \leq \pi + 1$,

$$\begin{aligned} 2\pi\left|\int_0^\infty e^{-t}\mathbb{E}\langle \Pi_K(X_t), \Pi_K(X)\rangle h(F)dt\right| &\leq \frac{8\pi\sqrt{\pi+1}}{\lambda_1^2} \leq \frac{52}{\lambda_1^2} \\ 2\pi\left|\int_0^\infty \frac{e^{-2t}}{\sqrt{1-e^{-2t}}}\mathbb{E}\langle \Pi_K(X_t), \Pi_K(\widehat{X})\rangle h(F)dt\right| &\leq \frac{8\pi\sqrt{\pi+1}}{\lambda_1^2} \leq \frac{52}{\lambda_1^2} \end{aligned}$$

On the other hand, since K is symmetric with respect to 0, the function $x \mapsto \Pi_K(x)$ is antisymmetric and $X \overset{\text{law}}{=} -X$ and $\widehat{X} \overset{\text{law}}{=} -\widehat{X}$. We deduce that

$$\begin{aligned} \mathbb{E}\langle X_t, \Pi_K(X)\rangle h(F) &= e^{-t}\mathbb{E}\langle X, \Pi_K(X)\rangle h(F) \\ \mathbb{E}\langle X_t, \Pi_K(\widehat{X})\rangle h(F) &= \sqrt{1-e^{-2t}}\mathbb{E}\langle \widehat{X}, \Pi_K(\widehat{X})\rangle \mathbb{E}h(F). \end{aligned}$$

We deduce that

$$\begin{aligned} &\big|B_1(h) - B_2(h)\big| \\ \leq\;& \pi\left|\mathbb{E}\left[\left(\langle X, \Pi_K(X)\rangle - \mathbb{E}\langle X, \Pi_K(X)\rangle\right)h(F)\right]\right| + \frac{104}{\lambda_1^2} \\ \leq\;& 7\sqrt{\operatorname{Var}(\langle X, \Pi_K(X)\rangle)} + \frac{104}{\lambda_1^2}. \end{aligned}$$

To bound $\operatorname{Var}(\langle X, \Pi_K(X)\rangle)$, as in Step 1 we will rely on the Brascamp-Lieb inequality. Set $H(x) = \langle x, \Pi_K(x)\rangle$. We have, thanks to Proposition 4.4.5 and with $Y \sim \nu(dx) = e^{-\psi(x)}dx$, with ν and ψ as in Definition 4.4.3,

$$\begin{aligned} \operatorname{Var}(H(X)) &= \operatorname{Var}(H(Y)) + O(d^\alpha\beta^d d^{-d(\frac{1}{2}-2\gamma)}) \\ &\leq \frac{1}{k}\mathbb{E}\big[\|\nabla H(Y)\|^2\big] + O(d^\alpha\beta^d d^{-d(\frac{1}{2}-2\gamma)}) \\ &\leq \frac{1}{k}\mathbb{E}\big[\|\nabla H(X)\|^2\big] + O(d^\alpha\beta^d d^{-d(\frac{1}{2}-2\gamma)}). \end{aligned}$$

Since $\nabla H(x) = \langle x, \nabla\Pi_K(x)\rangle + \Pi_K(x)$, we deduce

$$\begin{aligned} &\operatorname{Var}(H(X)) \\ \leq\;& \frac{2}{k}\mathbb{E}\big[\|\langle X, \nabla\Pi_K(X)\rangle\|^2\big] + \frac{2}{k}\mathbb{E}\big[\|\Pi_K(X)\|^2\big] + O(d^\alpha\beta^d d^{-d(\frac{1}{2}-2\gamma)}). \end{aligned}$$

Again, $K \subset \mathbb{B}^d(0, \frac{2}{\lambda_1})$, implying $\|\Pi_K(x)\| \leq \frac{2}{\lambda_1}$ for all $x \in \mathbb{R}^d$, so that $\mathbb{E}\big[\|\Pi_K(X)\|^2\big] \leq \frac{4}{\lambda_1^2}$. On the other hand, by Lemma 4.3.3 we have $\|\nabla\Pi_K(x)\| \leq \frac{1}{1+\lambda_1\, d(x,K)}$, leading to

$$\mathbb{E}\big[\|\langle X, \nabla\Pi_K(X)\rangle\|^2\big] \leq \mathbb{E}\left[\frac{\|X\|^2}{(1+\lambda_1\, d(x,K))^2}\right]$$

But $\|x\| \leq \frac{2}{\lambda_1} + d(x,K)$ for all $x \in \mathbb{R}^d$ (by inclusion of K in $\mathbb{B}^d(0, \frac{2}{\lambda_1})$), so $\mathbb{E}\big[\|\langle X, \nabla\Pi_K(X)\rangle\|^2\big] \leq \frac{4}{\lambda_1^2}$. Finally, we get that

$$\big|B_1(h) - B_2(h)\big| \leq 7\sqrt{\frac{16}{k\lambda_1^2} + O(d^\alpha \beta^d d^{-d(\frac{1}{2}-2\gamma)})} + \frac{104}{\lambda_1^2}.$$

4.4.5 Step 5: Conclusion

Putting the results of Steps 3 and 4 together, we deduce that

$$d_{TV}(F,N) \quad = \quad O_{n\to\infty}\left(\frac{1}{\sigma(\lambda_1 + \lambda_1^2)}\right)$$

Thanks to the facts that $\frac{1}{\lambda_1} \leq d^\gamma$, $K \subset \mathbb{B}(0, 2d^\gamma)$ and thanks to item 1 in Proposition 4.2.2, we have that $\frac{1}{\sigma} = O_{n\to\infty}\left(\frac{1}{\sqrt{d_n}}\right)$. This concludes the proof.

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