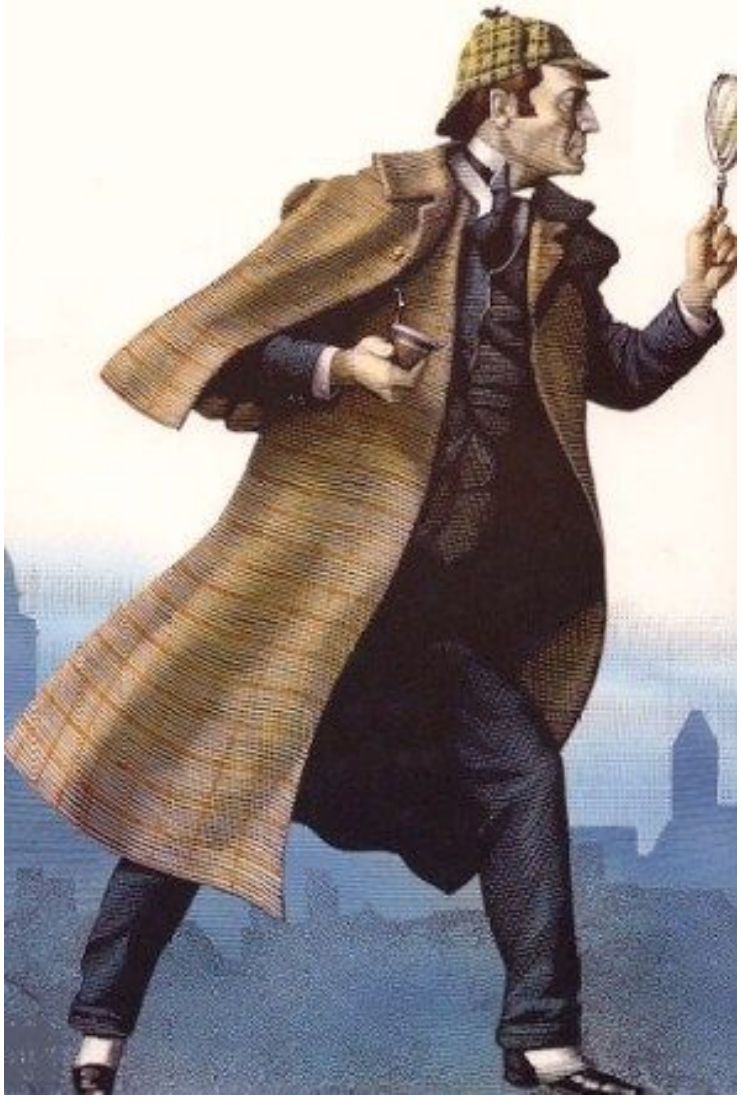
# Unlikely Revelations?



The Hidden Lattice Problem

Gabor Wiese

Cogent Seminar

24 January 2022

Joint work with

Luca Notarnicola

with some help from

Jean-Sébastien Coron,

based on an original idea by

Phong Nguyen and

Jacques Stern (1999).

Luxembourg National Research Fund:
PRIDE15/10621687/SPsquared.

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$$v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

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$$v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

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$$v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \in \mathbb{Z} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \mathbb{Z} \cdot \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

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$$v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 2 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \in \mathbb{Z} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \mathbb{Z} \cdot \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} =: \mathcal{L}_3$$

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Maybe bigger dimensions?

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Maybe larger numbers?

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$$v = \begin{pmatrix} -29281502594572513868 \\ 569947941397951101611 \\ 234712501947420023306 \\ -58498212031054212374 \\ -122970365805580085081 \\ -34790870569381838941 \\ -447984097826439363496 \\ -201452809418164273238 \\ -30963977588258181389 \\ 130919139518165120564 \end{pmatrix} \in \mathcal{L} \subset \mathbb{Z}^{10} \text{ with } \mathcal{L} \text{ of rank 5 and (relatively) small.}$$

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because $\mathcal{L}$ is small, and so is $\mathcal{L}^\perp$.

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$\Rightarrow \mathcal{L}^\perp \subset \langle v \rangle^\perp$ small lattice of rank $m - n$ in lattice of rank $n - 1$.

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$\Rightarrow \mathcal{L}^\perp \subset \langle v \rangle^\perp$ small lattice of rank $m - n$ in lattice of rank $n - 1$.

$\Rightarrow$ uniquely identify $\mathcal{L}^\perp$

$\Rightarrow \mathcal{L} \subseteq (\mathcal{L}^\perp)^\perp = \overline{\mathcal{L}} = (\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{L}) \cap \mathbb{Z}^m$ the saturation.

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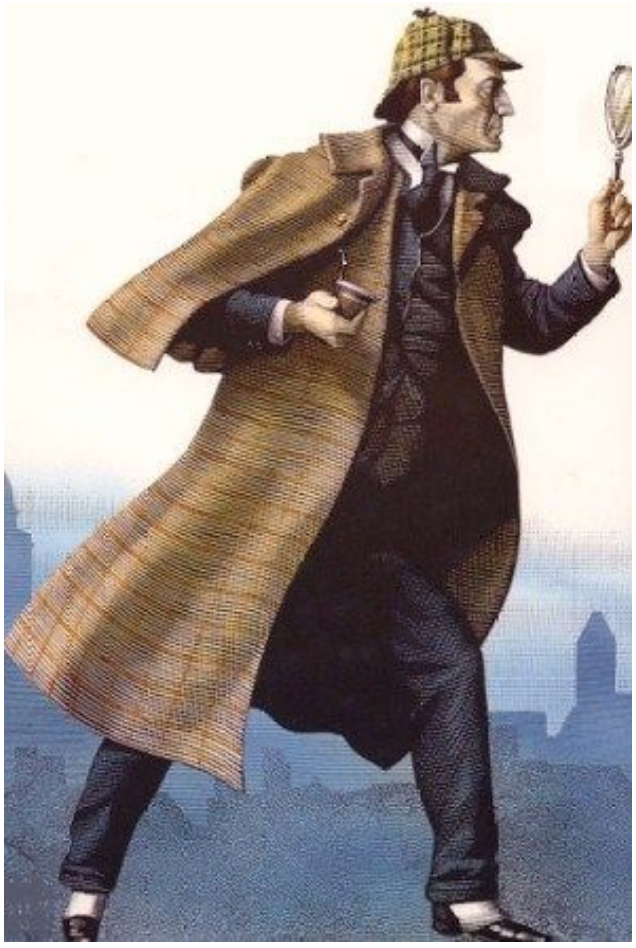
Given a large vector $v \in \mathbb{Z}^m$.
Find the unique saturated small sublattice $\mathcal{L} \subset \mathbb{Z}^m$ of rank $n < m$ containing $v$.



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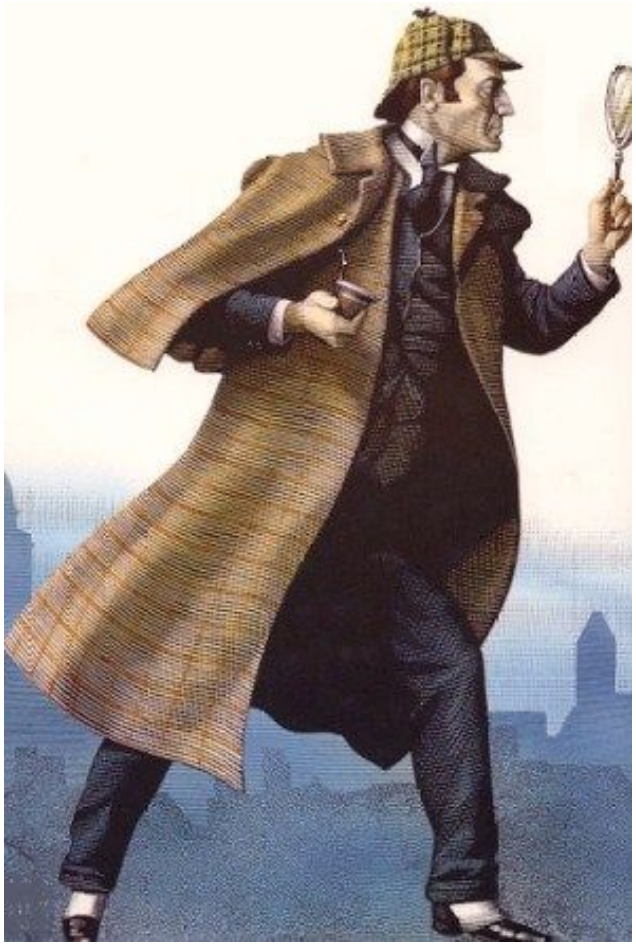

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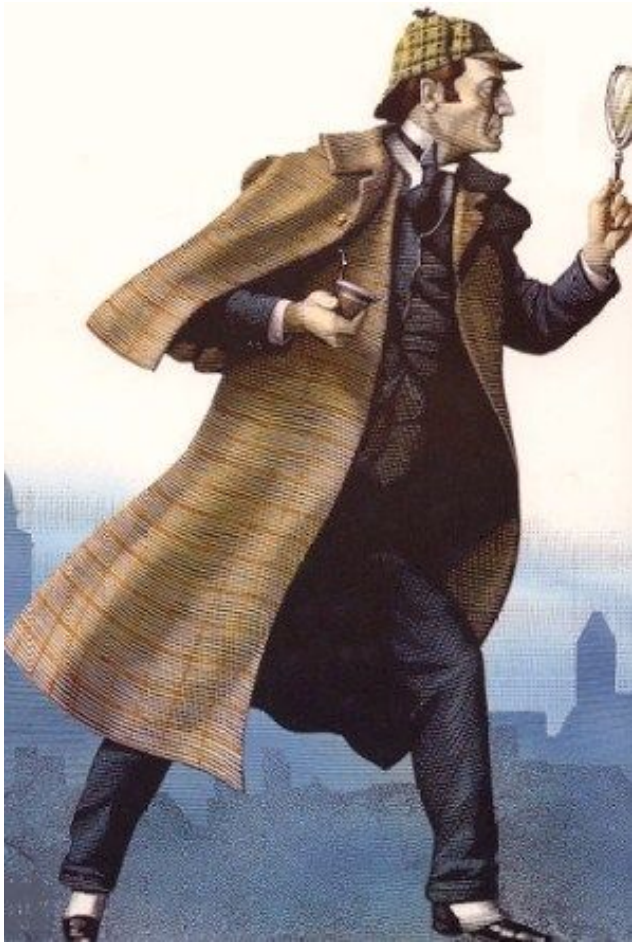

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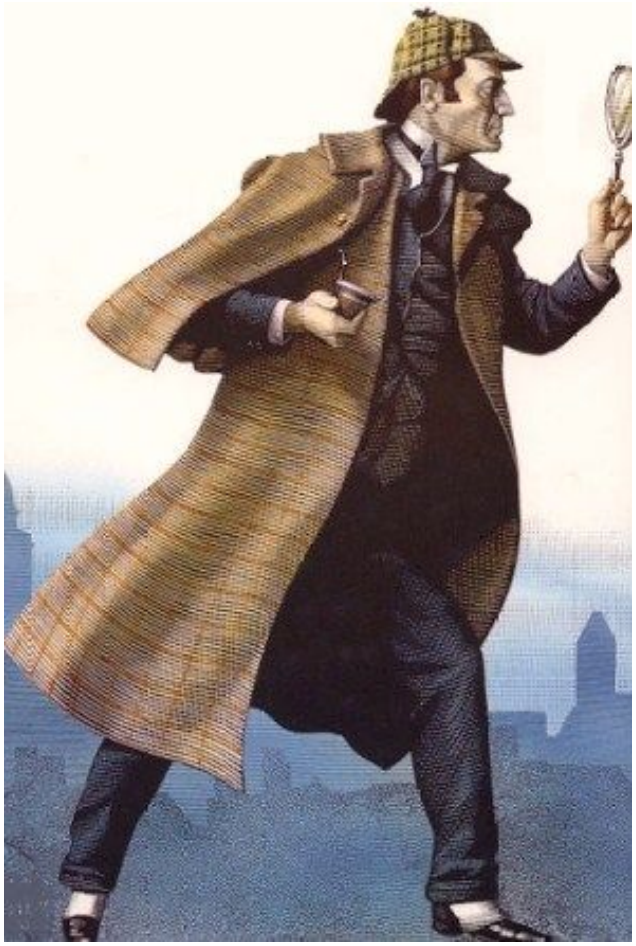

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Now:
General set-up and a new algorithm.

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Continue by hand...

# Theorems

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Let $\mathcal{H}(\mathfrak{B}) = \{\sum_{i=1}^n a_i v_i \mid a \in (\mathbb{Z}/N\mathbb{Z})^n\}$.
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**Theorem.** *Let $\varepsilon \in (0, 1)$ such that*

$$\begin{aligned} \log(N\varepsilon) \quad > \quad & \frac{mn}{2}\log(c) + n(n+1)\log(\mu) \qquad (1) \\ & + \frac{n(m-n)}{2}\log((2/3)(m-n)) + n\log(3\sqrt{n}) + 1 \end{aligned}$$

*At least $(1-\varepsilon)\#\mathcal{H}(\mathfrak{B})$ of the hidden lattice problems from $\mathcal{H}(\mathfrak{B})$ are solvable by the Orthogonal Lattice Attack using $\delta$-LLL.*

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*At least $(1 - \varepsilon)\#\mathcal{H}(\mathfrak{B})$ of the hidden lattice problems from $\mathcal{H}(\mathfrak{B})$ are solvable by the new algorithm using $\delta$-LLL.*

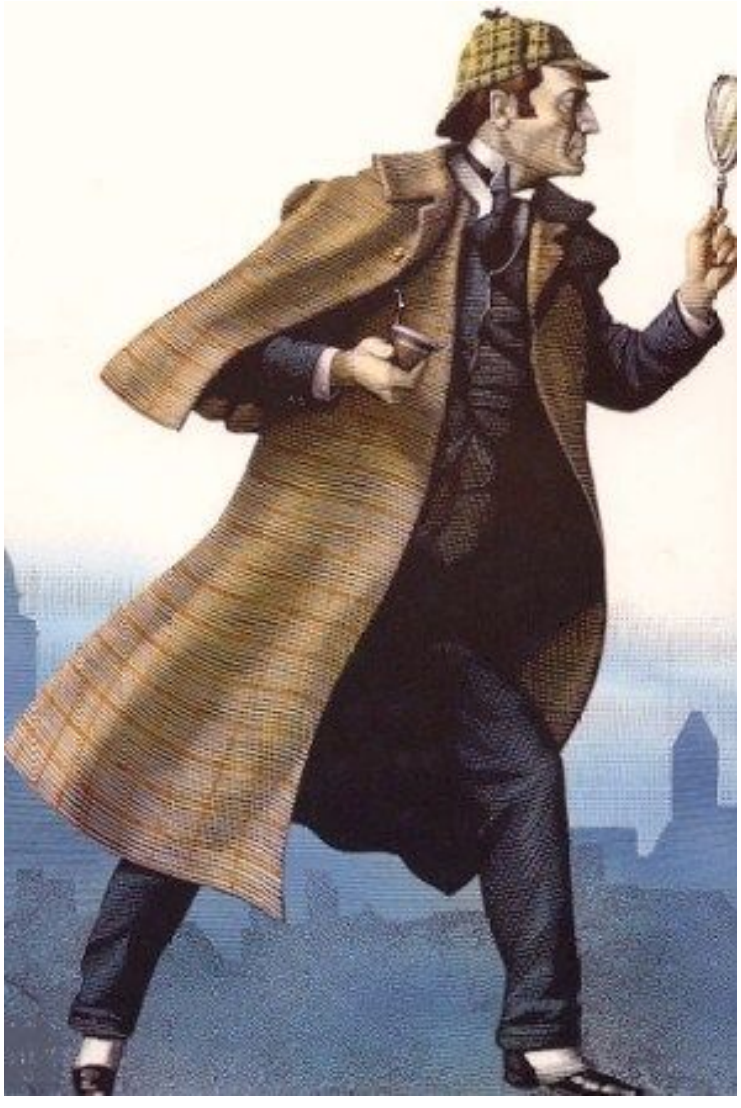
# Running times

| | | | | Algorithm I | | Algorithm II | | |
|---|---|---|---|---|---|---|---|---|
| $r$ | $n$ | $m$ | $\log(N)$ | Step 1 | Step 2 | Step 1 | Step 2 (S) | Step 2 (M) |
| 60 | 150 | 200 | 140 | 7 min 13 s | 1 min 20 s | 10 min 2 s | 3 min 4 s | 0.37 s |
| 110 | 150 | 200 | 90 | 6 min 20 s | 1 min 29 s | 4 min | 1 min 33 s | 0.24 s |
| 175 | 180 | 200 | 140 | 6 min 56 s | 1 min 24 s | 1 min 39 s | 20 s | 0.19 s |
| 80 | 100 | 300 | 75 | 3 min 51 s | 30 min 17 s | 22 min 51 s | 30 s | 0.12 s |
| 150 | 200 | 300 | 75 | 145 min 29 s | 22 min 23 s | 116 min 14 s | 6 min 19 s | 0.56 s |
| 75 | 150 | 400 | 80 | 75 min 16 s | 326 min 44 s | 414 min 51 s | 5 min 13 s | 0.61 s |
| 235 | 275 | 400 | 80 | 527 min 43 s | 117 min 2 s | 304 min 10 s | 15 min 39 s | 0.95 s |

Entries of $\mathcal{L}$ in $[-2^{10}, 2^{10}] \cap \mathbb{Z}$.

S=Sage, M=Magma.

# Unlikely Revelations?



Thank you for your attention!