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Some Prevalent Sets in Multifractal Analysis: How Smooth is Almost Every Function in $T_p^\alpha(x)$

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Abstract
We present prevalent results concerning generalized versions of the T_p^α spaces, initially introduced by Calderón and Zygmund. We notably show that the logarithmic correction appearing in the quasi-characterization of such spaces is mandatory for almost every function; it is in particular true for the Hölder spaces, for which the existence of the correction was showed necessary for a specific function. We also show that almost every function from $T_p^\alpha(x_0)$ has α as generalized Hölder exponent at x_0.

Keywords Wavelets · Multifractal analysis · Prevalence · Pointwise smoothness · Generalized smoothness

Mathematics Subject Classification 42C40 · 26A16 · 46E35

1 Introduction

The determination of the multifractal properties of a function is a problem with countless applications (see e.g., [3, 10, 11, 42] and references therein). Most often, it relies on the notion of Hölder exponent.

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Definition 1.1 Let f be a locally bounded function, $x_0 \in \mathbb{R}^d$ and $\alpha \geq 0$; f belongs to the Hölder space $\Lambda^\alpha(x_0)$ if there exists $R > 0$, $C > 0$ and a polynomial of degree less than α such that $|x - x_0| < R$ implies

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha. \tag{1}$$

The Hölder exponent of f at x_0 is

$$h_f^{(\infty)}(x_0) = \sup\{\alpha \geq 0 : f \in \Lambda^\alpha(x_0)\}.$$

The Hölder exponent of a function can present a berserk behavior (it can be everywhere discontinuous [19]). Therefore instead of trying to determine the function $h_f^{(\infty)}$ associated to f, one rather tries to perform a multifractal analysis. This consists in determining the spectrum of singularity d_f of the function.

Definition 1.2 Let f be a locally bounded function; its isoHölder sets are the sets $E_H = \{x_0 : h_f^{(\infty)}(x_0) = H\}$. The spectrum of singularity d_f of f is the function

$$d_f : [0, \infty] \to \mathbb{R}^+ \cup \{-\infty\} \quad H \mapsto \dim(E_H),$$

where dim denotes the Hausdorff dimension (with the standard convention $\dim(\emptyset) = -\infty$).

The spectrum of singularities of signals obtained through the registering of real-life data cannot be estimated in the case of multifractal signals: the determination of their Hölder exponent is numerically unstable because it usually jumps at every point. Moreover, the acquisition of the whole spectrum would require the determination of an infinite number of Hausdorff dimensions, the computation of such a quantity being a challenge in itself [31]. Therefore, in practice, one uses formulas referred to as “multifractal formalism” in order to derive the spectrum of singularities of a signal from numerically computable quantities. All of them are variants of seminal derivation which was proposed by Parisi and Frish [44]. Though they are based on the same thermodynamic argument, their numerical performance can vary a lot. A first step in the numerical improvement was performed by A. Arneodo and his collaborators, when they introduced formulas based on wavelet analysis [2, 41]. A second improvement was made by Jaffard et al.: they provided a mathematical framework for these wavelet-based methods [18, 22]. Among other things, this allowed to theoretically prove the efficiency of such methods [17, 18].

However, formalisms relying on Hölder spaces present several drawbacks. First, Definition 1.1 cannot supply a sensible notion of pointwise regularity if the natural setting for f is L^p_{loc} instead of L^∞_{loc} [20]. To tackle this problem, Jaffard and Melot proposed the T_p^α spaces to supplant the Hölder spaces Λ^α [21]. This basically consists in replacing the L^∞-norm in (1) by a L^p-norm.

Definition 1.3 Let f be a function from L^p_{loc}, $x_0 \in \mathbb{R}^d$ and $\alpha \geq -d/p$; f belongs to the space $T_p^{\alpha(x_0)}$) if there exists $R > 0$, $C > 0$ and a polynomial of degree less than α

such that, for all $0 < r < R$,

$$r^{-d/p} \| f - P(\cdot - x_0) \|_{L^p(B(x_0,r))} \leq C r^{\alpha}, \tag{2}$$

where $B(x_0, r)$ denotes the ball of radius r centered at x_0.

These spaces were originally introduced by Calderòn and Zygmund in [5] for studying PDE. Obviously, one can consider Λ^{α} as being T^{α}_{∞}. From this base, one defines the associated Hölder exponent

$$h_f^{(p)}(x_0) = \{\alpha \geq -d/p : f \in T^{\alpha}_p(x_0)\},$$

which gives rise to a multifractal formalism similar to the one associated to the Hölder spaces.

A second problem resides in the fact that such spaces are inadequate for capturing the precise behavior of the function f at a given point x_0: the T^{α}_p spaces are unable to grasp the presence of irregular comportments that are more subtle than a power function. For example, such spaces cannot detect the law of the iterated logarithm in a stochastic process [25, 27] (the classical case of a process displaying such a logarithmic correction is the celebrated Brownian motion [13, 39]). To take into account such a limitation, the authors proposed to consider the space of functions satisfying

$$r^{-d/p} \| f - P(\cdot - x_0) \|_{L^p(B(x_0,r))} \leq C \omega(r) \tag{3}$$

instead of (2), where ω is basically a germ satisfying some properties [29, 35] (see Sect. 3). It has been shown that these spaces naturally define a multifractal formalism which is a generalization of the methods relying on the usual spaces T^{α}_p [35]. Moreover, they also provide a natural framework for studying PDE [34].

In the present context of the pointwise regularity, the advantage of this extended approach is twofold. As these spaces are closely related to the wavelet representation of their elements, when looking at the regularity of a function, the study of the wavelet coefficients could help to discriminate between different behaviors occuring in the function. For example, it well known that decomposing the Brownian notion in a well-chosen basis lead to so-called fast and slow points, which roughly differ in the nature of the function ω appearing in (3) [24, 38]. It could also be the case with the Brjuno function, which is not locally bounded. In [20], the authors extends the definition of the T^{α}_p spaces in a way similar to (3) in practice and use wavelets to obtain the spectrum of singularity of the function. One could try to expand their work in order to determine the possible existence of fast and slow points, as in the Brownian motion. On a more applied domain, the notion p-leader induced by these generalized spaces [35] can be considered as more natural than the usual p-leaders [21]. For example, they naturally generalize the usual case $p = \infty$ (see [32, 35] for more details). It is therefore conceivable that the related multifractal formalism could help to address numerical issues encoutered in the applications (see for example [30]). That being said, it is important to bear in mind that considering such a generalization only brings minor technical modification to the proofs. The results provided here are valid in the usual setting. For example, the reader only interested in the case of the Hölder spaces can consider Theorem 4.1 instead of Theorem 4.4.

This work is based on the following remark. Exploring the regularity of a function through its wavelet transform offers several advantages, both practical and theoretical. However, there is no wavelet characterization of the associated regularity spaces (such as the pointwise Hölder spaces [18, 29], the T_p^α spaces [21] or their generalized version obtained using (3) [35]). In general, if a function belongs to such a space, its wavelet coefficients must satisfy some growth condition. On the other hand, if such a condition is met, the corresponding function does not necessarily belongs to the former space, but rather to a larger one involving some logarithmic correction [18, 21, 29, 35]. Typically, for the T_p^α spaces, condition (2) must be replaced by (3) with $\omega(r) = r^\alpha |\log r|$. One uses the term "quasi-characterization" when referring to such a result, as they still lead to a characterization of the corresponding Hölder exponent; such results are epitomized by Theorem 3.8.

In this paper, we consider the following question: what are the classical elements of the generalized spaces? This could be considered as an idiosyncratic problem at first glance. But as these spaces were recently introduced, we cannot speculate on the influence of this apparatus. Moreover, the obtained results can be applied to the classical spaces T_p^α (and so to Λ^α). First, we need to define what we mean by "classical element". To this end, we use the notion of prevalence [6, 15] in Sect. 2, which is a natural generalization of the locution "almost everywhere" in an infinite-dimensional setting. We then focus on the logarithmic correction appearing in the quasi-characterization of the regularity spaces. Let us recall that the necessity of such a correction in the case of the poinwise Hölder spaces has been obtained in [16], where a rather artificial counter-example is given. Roughly speaking, we show that almost every function satisfying the condition on the wavelet coefficients displays such a logarithmic correction, revealing that the counter-example is more a typical function in regard to such a wavelet condition, thus refining and generalizing the result in [16]. In particular, one cannot get rid of this correction in the general framework, so that Theorem 3.8 is optimal. Next, by shaping a function that belongs to a given generalized space, we show (from a prevalence point of view) that these are not a small extension of the usual T_p^α spaces. Finally, using such a construction, we also establish that under very general conditions, almost every function of this space has everywhere the same Hölder exponent, determined by ω. In particular, almost every function f from $T_p^\alpha(x_0)$ is such that $h_f^{(p)}(x_0) = \alpha$. This is a generalization of a result by Hunt concerning the Hölder spaces [14].

2 Reminders and Notations About Wavelets and Prevalence

Let us first briefly recall some definitions and notations about wavelets (for more precisions, see e.g., [9, 36, 37]). Under some general assumptions, there exist a function φ and $2^d - 1$ functions $(\psi^{(i)})_{1 \leq i < 2^d}$, called wavelets, such that

$$\{\varphi(x-k) : k \in \mathbb{Z}^d\} \cup \{\psi^{(i)}(2^j x - k) : 1 \leq i < 2^d, k \in \mathbb{Z}^d, j \in \mathbb{N}\}$$

form an orthogonal basis of L^2. Any function $f \in L^2$ can be decomposed as follows,

$$f(x) = \sum_{k\in\mathbb{Z}^d} C_k \varphi(x-k) + \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{Z}^d} \sum_{1\le i<2^d} c_{j,k}^{(i)} \psi^{(i)}(2^j x - k),$$

where

$$c_{j,k}^{(i)} = 2^{dj} \int_{\mathbb{R}^d} f(x)\psi^{(i)}(2^j x - k)\, dx$$

and

$$C_k = \int_{\mathbb{R}^d} f(x)\varphi(x-k)\, dx. \tag{4}$$

Let us remark that we do not choose the L^2 normalization for the wavelets, but rather an L^∞ normalization, which is better fitted to the study of the Hölderian regularity. For the sake of simplicity, we will only consider here compactly supported wavelet basis; such wavelets are constructed in [8].

As in [18, 22], let $\lambda_{j,k}^{(i)}$ denote the dyadic cube

$$\lambda_{j,k}^{(i)} := \frac{i}{2^{j+1}} + \frac{k}{2^j} + [0, \frac{1}{2^{j+1}})^d.$$

In the sequel, we will often omit any reference to the indices i, j and k for such cubes by writing $\lambda = \lambda_{j,k}^{(i)}$. We will also index the wavelet coefficients of a function f with the dyadic cubes λ so that c_λ will refer to the quantity $c_{j,k}^{(i)}$. The notation Λ_j will stand for the set of dyadic cubes λ of $\mathbb{R}^d$ with side length 2^{-j} and the unique dyadic cube from Λ_j containing the point $x_0 \in \mathbb{R}^d$ will be denoted $\lambda_j(x_0)$. The set of the dyadic cubes is $\Lambda := \cup_{j\in\mathbb{N}} \Lambda_j$. Two dyadic cubes λ and λ' are adjacent if there exists $j \in \mathbb{N}$ such that $\lambda, \lambda' \in \Lambda_j$ and $\operatorname{dist}(\lambda, \lambda') = 0$. The set of the 3^d dyadic cubes adjacent to λ will be denoted by 3λ.

Now, we very briefly introduce the notion of prevalence (see [6, 14, 15] for more details). Lebesgue genericity plays a crucial role in the Euclidean space, since generic sets are defined as the complement of a set of vanishing Lebesgue measure. However this approach fails for infinite dimensional Banach spaces, as there is no σ-finite measure that is translation-invariant in a infinite dimensional normed space. A solution to extend this notion of genericity to a more general setting consists in turning a characterization of the sets of vanishing Lebesgue measure into a definition. It is well known that one can associate a compactly-supported probability measure μ to a Borel set B such that $\mu(B + x)$ vanishes for every $x \in \mathbb{R}^d$ if and only if the Lebesgue measure $\mathcal{L}(B)$ of B also vanishes. We are thus led to the following definition.

Definition 2.1 Let E be a complete metric vector space; a Borel set B of E is Haar-null if there exists a compactly-supported probability measure μ such that $\mu(B + x) = 0$ for every $x \in E$. A subset of E is Haar-null if it is contained in a Haar-null Borel set; the complement of a Haar-null set is a prevalent set.

The features one can expect from a notion of genericity are preserved with this framework. For example, if E is finite-dimensional, B is Haar-null if and only if $\mathcal{L}(B) = 0$; if E is infinite-dimensional, the compact sets of E are Haar-null. Moreover, it can be shown that a translation of a Haar-null set is Haar-null and that a prevalent set is dense in E. Finally, the intersection of a countable collection of prevalent sets is still prevalent. As an application, it can be shown that given $\alpha \in (0, 1)$, from the prevalence point of view, almost every function f that belongs to $\Lambda^\alpha(\mathbb{R}^d)$ is such that $h_f^{(\infty)}(x) = \alpha$ for any x [14].

Let us make some remarks about how to show that a set is Haar-null. A common choice for the measure in Definition 2.1 is the Lebesgue measure on the unit ball of a finite-dimensional subset E' of E. For such a choice, one has to show that $\mathcal{L}(B \cap (E' + x))$ vanishes for every x. Such a subspace is called a probe. If E is a function space, one can choose a random process X whose sample paths almost surely belong to E. In this case, one can show that a property only holds on a Haar-null set by showing that the sample path X is such that, for any $f \in E$, $X_t + f$ almost surely does not satisfy the property.

If a property holds on a prevalent set, we will say that it holds almost everywhere from the prevalence point of view.

3 Generalized Spaces of Pointwise Smoothness

In [35], we introduced generalized spaces of pointwise smoothness using admissible sequences.

Definition 3.1 A sequence $\boldsymbol{\sigma} = (\sigma_j)_j$ of real positive numbers is called admissible if there exists a positive constant C such that

$$C^{-1}\sigma_j \leq \sigma_{j+1} \leq C\sigma_j, \tag{5}$$

for any $j \in \mathbb{N}$.

If $\boldsymbol{\sigma}$ is such a sequence, we set

$$\underline{\sigma}_j = \inf_{k\in\mathbb{N}} \frac{\sigma_{j+k}}{\sigma_k} \quad \text{and} \quad \overline{\sigma}_j = \sup_{k\in\mathbb{N}} \frac{\sigma_{j+k}}{\sigma_k}$$

and define the lower and upper Boyd indices as follows,

$$\underline{s}(\boldsymbol{\sigma}) = \lim_j \frac{\log_2 \underline{\sigma}_j}{j} \quad \text{and} \quad \overline{s}(\boldsymbol{\sigma}) = \lim_j \frac{\log_2 \overline{\sigma}_j}{j}.$$

Since $(\log \underline{\sigma}_j)_j$ is a subadditive sequence, such limits always exist. The following relations about such sequences are well known (see e.g., [28]). Let $\epsilon > 0$; if $\boldsymbol{\sigma}$ is an admissible sequence, there exists a positive constant C such that

$$C^{-1}2^{j(\underline{s}(\boldsymbol{\sigma})-\epsilon)} \leq \underline{\sigma}_j \leq \frac{\sigma_{j+k}}{\sigma_k} \leq \overline{\sigma}_j \leq C2^{j(\overline{s}(\boldsymbol{\sigma})+\epsilon)}, \tag{6}$$

for any $j, k \in \mathbb{N}$. In this paper, $\boldsymbol{\sigma}$ will always stand for an admissible sequence and, given $u > 0$, we set $\boldsymbol{u} = (2^{ju})_j$. Of course, we have $\underline{s}(\boldsymbol{u}) = \overline{s}(\boldsymbol{u}) = u$.

As Eq. (6) suggests, Boyd indices are good indicators to measure the growth of an admissible sequence. For instance, they give some conditions to bound sums in which dyadic, admissible and ℓ^q sequences appear. The following lemma is proved in [35].

Lemma 3.2 *Let $m \in \mathbb{N}$, $\boldsymbol{\sigma}$ be an admissible sequence such that $\underline{s}(\boldsymbol{\sigma}^{-1}) > m$ and $\boldsymbol{\epsilon} \in \ell^q$ with $q \in [1, \infty]$; there exists a sequence $\boldsymbol{\xi} \in \ell^q$ such that*

$$\sum_{j=J}^{\infty} \epsilon_j 2^{jm} \sigma_j \leq \xi_J 2^{Jm} \sigma_J,$$

for all $J \in \mathbb{N}$.

An easy way to build admissible sequences is to use slowly varying functions.

Definition 3.3 A strictly positive function ψ is a *slowly varying function* if

$$\lim_{t \to 0} \frac{\psi(rt)}{\psi(t)} = 1,$$

for any $r > 0$.

Example 3.4 If ψ is a slowly varying function and $u \in \mathbb{R}$, the sequence $\boldsymbol{\sigma} = (2^{ju}\psi(2^j))_j$ is admissible with $\underline{s}(\boldsymbol{\sigma}) = \overline{s}(\boldsymbol{\sigma}) = u$.

We will heavily use the finite differences in the sequel (see e.g., [4, 23]). Given a function f defined on $\mathbb{R}^d$ and $x, h \in \mathbb{R}^d$, the finite difference $\Delta_h^n f$ of f is defined as follows,

$$\Delta_h^1 f(x) = f(x+h) - f(x) \qquad \text{and} \qquad \Delta_h^{n+1} f(x) = \Delta_h^1 \Delta_h^n f(x),$$

for any $n \in \mathbb{N}$. It is easy to check that the following formula holds:

$$\Delta_h^n f(x) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} f(x + (n-j)h). \tag{7}$$

If $n \leq 0$, we agree that $\Delta_h^n f = f$.

The generalized spaces of pointwise smoothness are then defined in the following way.

Definition 3.5 Let $p \in [1, \infty]$, $\boldsymbol{\sigma} = (\sigma_j)_j$ be an admissible sequence such that $\underline{s}(\boldsymbol{\sigma}) > -d/p$, $f \in L^p_{\mathrm{loc}}$ and $x_0 \in \mathbb{R}^d$; f belongs to $T_p^{\boldsymbol{\sigma}}(x_0)$ whenever

$$(\sigma_j 2^{jd/p} \sup_{|h| \leq 2^{-j}} \|\Delta_h^{\lfloor \overline{s}(\sigma) \rfloor + 1} f\|_{L^p(B_h(x_0, 2^{-j}))})_j \in \ell^\infty,$$

where, given $r > 0$,

$$B_h(x_0, r) = \{x : [x, x + (\lfloor \overline{s}(\boldsymbol{\sigma}) \rfloor + 1)h] \subset B(x_0, r)\}$$

if $\overline{s}(\boldsymbol{\sigma}) > 0$ and $B_h(x_0, r) = B(x_0, r)$ otherwise.

It is easy to check that $T^{\boldsymbol{\sigma}}_\infty(x_0)$ is the generalized Hölder space $\Lambda^{\boldsymbol{\sigma}}(x_0)$ introduced in [29]. These spaces can also be seen as a generalization of the spaces $T^u_p(x_0)$ introduced by Calderón and Zygmund in [5]: when considering the regularity, one can suppose having $T^u_p(x_0) = T^{\boldsymbol{u}}_p(x_0)$. This aspect is studied in details in [34], where the generalized regularity of the solutions of elliptic partial differential equations is explored. Let us also mention that we can equip $T^{\boldsymbol{\sigma}}_p(x_0)$ with the natural norm

$$\|f\|_{T^{\sigma}_p(x_0)} = \|f\|_{L^p(B(x_0,1))} + \|(\sigma_j 2^{jd/p} \sup_{|h|\leq 2^{-j}} \|\Delta_h^{\lfloor \overline{s}(\sigma)\rfloor+1} f\|_{L^p(B_h(x_0,2^{-j}))})_j\|_{\ell^\infty}$$

and, from the completeness of the L^p spaces, $(T^{\boldsymbol{\sigma}}_p(x_0), \|\cdot\|_{T^{\sigma}_p(x_0)})$ is a Banach space.

In fact, these spaces can be seen as a pointwise version of some Besov spaces of generalized smoothness $B^{\boldsymbol{\sigma}}_{p,q}$ considered in [33]: if $p, q \in [1,\infty]$ and $0 < \underline{s}(\boldsymbol{\sigma}) < n$,

$$B^{\boldsymbol{\sigma}}_{p,q} = \{f \in L^p : (\sigma_j \sup_{|h|\leq 2^{-j}} \|\Delta^n_h f\|_{L^p})_j \in \ell^q\}. \tag{8}$$

This definition is due to Moura [40]. Here, we will mainly use the characterization of Besov spaces of generalized smoothness in terms of discrete wavelet transform (see [1]): if $\boldsymbol{\sigma}$ is an admissible sequence and $p, q \in [1,\infty]$, a tempered distribution f belongs to $B^{\boldsymbol{\sigma}}_{p,q}$ if and only if the sequence $(C_k)_k$ defined by (4) belongs to ℓ^q and if

$$(\sigma_j 2^{-jd/p}\|(c_\lambda)_{\lambda\in\Lambda_j}\|_{\ell^p})_j \in \ell^q. \tag{9}$$

We can also characterize the $T^{\boldsymbol{\sigma}}_p(x_0)$-regularity by the mean of polynomial approximation: a function f belongs to $T^{\boldsymbol{\sigma}}_p(x_0)$ if and only if there exists a sequence of polynomials $(P_{j,x_0})_j$ of degree less or equal to $\lfloor \overline{s}(\boldsymbol{\sigma})\rfloor$ such that

$$(\sigma_j 2^{jd/p}\|f - P_{j,x_0}\|_{L^p(B(x_0,2^{-j}))})_j \in \ell^\infty. \tag{10}$$

Moreover, if $\boldsymbol{\sigma}$ is an admissible sequence such that $0 \leq n := \lfloor \overline{s}(\boldsymbol{\sigma})\rfloor < \underline{s}(\boldsymbol{\sigma})$, one can find a unique polynomial P_{x_0} of degree less or equal to n such that

$$(\sigma_j 2^{jd/p}\|f - P_{x_0}\|_{L^p(B(x_0,2^{-j}))})_j \in \ell^\infty. \tag{11}$$

In order to obtain a nearly-characterization of the generalized pointwise regularity in terms of wavelet transform, we introduce the following coefficients.

Definition 3.6 Given a dyadic cube $\lambda \in \Lambda_j$ at scale j, the p-wavelet leader of λ ($p \in [1, \infty]$) is defined by

$$d_\lambda^p = \sup_{j' \geq j} \Big(\sum_{\lambda' \in \Lambda_{j'}, \lambda' \subset \lambda} (2^{(j-j')d/p} |c_{\lambda'}|)^p \Big)^{1/p}.$$

Given $x_0 \in \mathbb{R}^d$, we set

$$d_j^p(x_0) = \sup_{\lambda \in 3\lambda_j(x_0)} d_\lambda^p.$$

To state our result, we need to define the "logarithmic corrected" space.

Definition 3.7 Let $p \in [1, \infty]$, $x_0 \in \mathbb{R}^d$ and f be a function from L^p_{loc}; if $\boldsymbol{\sigma}$ is an admissible sequence such that $2^{-jd/p}\sigma_j^{-1}$ tends to 0 as j tends to infinity, we says that f belongs to $T^{\boldsymbol{\sigma}}_{p,\log}(x_0)$ if there exists $J \in \mathbb{N}$ for which

$$\Big(\frac{2^{jd/p}\sigma_j}{|\log_2(2^{-jd/p}\sigma_j^{-1})|} \sup_{|h| \leq 2^{-j}} \|\Delta_h^{\lfloor \bar{s}(\sigma^{-1}) \rfloor + 1} f\|_{L^p(B_h(x_0, 2^{-j}))}\Big)_{j \geq J} \in \ell^\infty.$$

In [35], we establish the following nearly-characterization of the generalized pointwise regularity, given by the p-wavelet leaders.

Theorem 3.8 *Let $\boldsymbol{\sigma}$ be an admissible sequence, $p \in [1, \infty]$ and $x_0 \in \mathbb{R}^d$. If f belongs to the space $T^{\boldsymbol{\sigma}}_p(x_0)$, then*

$$(\sigma_j d_j^p(x_0))_j \in \ell^\infty. \tag{12}$$

Conversely, if $\boldsymbol{\sigma}$ is such that $2^{-jd/p}\sigma_j^{-1}$ tends to 0 as j tends to infinity with $\underline{\sigma}_1 > 2^{-d/p}$ and if f belongs to $B^{\boldsymbol{\eta}}_{p,\infty}(\mathbb{R}^d)$ for some $\eta > 0$, then the belonging (12) implies $f \in T^{\boldsymbol{\sigma}}_{p,\log}(x_0)$.

4 Prevalence of the Logarithmic Correction

In this section, we aim at showing that, from the prevalence point of view, for almost every function with some minimum regularity (we ask slightly more than the continuity), the logarithmic correction occuring in Theorem 3.8 is necessary.

Let us show how such results can be transposed in the classical cases. For the Hölder spaces, Theorem 3.8 leads to the following classical result. If f belongs to $\Lambda^\alpha(x_0)$ for some $x_0 \in \mathbb{R}^d$ and $\alpha > 0$, then there exists a constant $C > 0$ such that

$$d_j^\infty(x_0) \leq C 2^{-j\alpha} \quad \forall j. \tag{13}$$

Conversely, if inequality (13) is satisfied for a function f that belongs to $\Lambda^\eta(\mathbb{R}^d)$ for some $\eta > 0$, then f also belongs to $\Lambda^\alpha_{\log}(x_0)$, where $\Lambda^\alpha_{\log}(x_0) = T^{\boldsymbol{\alpha}}_{\infty,\log}(x_0)$ is the set of functions f such that there exists $C > 0$ and a polynomial P of degree at most $\lfloor \alpha \rfloor$ such that

$$|f(x) - P(x)| \le C|x - x_0|^\alpha \log(\frac{1}{|x - x_0|}), \tag{14}$$

for x in neighborhood of x_0. In [16], it is shown that one cannot hope for an exact characterization of $\Lambda^\alpha(x_0)$, since an example of function $f \in \Lambda^\eta(\mathbb{R}^d)$ that satisfies (13) but does not belong to $\Lambda^\alpha(x_0)$ is proposed. In this section, we show that, from the prevalence point of view, it is the case for most function f. More precisely, we have the following result:

Theorem 4.1 *If $x_0 \in \mathbb{R}^d$, almost every function of $\Lambda^\eta(\mathbb{R}^d)$ satisfying (13) belongs to $\Lambda^\alpha_{\log}(x_0) \setminus \Lambda^\alpha(x_0)$.*

Indeed, we obtain a more precise result, since we show that the logarithmic correction is optimum: almost every function satisfying the hypothesis of the previous theorem belongs to $\Lambda^\alpha_{\log}(x_0) \setminus \Lambda^\alpha_{/_s\log}(x_0)$, where $\Lambda^\alpha_{/_s\log}(x_0)$ is a class of spaces between $\Lambda^\alpha_{\log}(x_0)$ and $\Lambda^\alpha(x_0)$ (see Definition 4.3 for the details). We will establish this result in a more general setting.

Let us first consider the following lemma which gives a way to define the probe we will use afterwards.

Lemma 4.2 *Let $\boldsymbol{\sigma} = (\sigma_j)_j$ be an admissible sequence, $x_0 \in \mathbb{R}^d$ and E be a complete metrisable space of functions defined on $\mathbb{R}^d$ such that*

$$S = E \cap T_p^{\boldsymbol{\sigma}}(x_0)$$

is a Borel set of E. If there exists $f \in E$ such that for all $M \in \mathbb{N}$ one can find $j \in \mathbb{N}$ for which

$$\sigma_j 2^{jd/p} \sup_{|h| \le 2^{-j}} \|\Delta_h^{\lfloor \overline{s}(\boldsymbol{\sigma}) \rfloor + 1} f\|_{L^p(B_h(x_0, 2^{-j}))} \ge M,$$

then S is Haar-null in E.

Proof Let us fix $\widetilde{f} \in E$, $N \in \mathbb{N}$ and consider the set

$$S_N = \{g \in E \ : \ \sigma_j 2^{jd/p} \sup_{|h| \le 2^{-j}} \|\Delta_h^{\lfloor \overline{s}(\boldsymbol{\sigma}) \rfloor + 1} g\|_{L^p(B_h(x_0, 2^{-j}))} \le N \ \forall \, j \in \mathbb{N}\}.$$

Assume that there exist $a, b \in \mathbb{R}$ such that both $\widetilde{f} + af$ and $\widetilde{f} + bf$ belong to S_N. Given $M \in \mathbb{N}$, there exists $j \in \mathbb{N}$ for which

$$\sigma_j 2^{jd/p} \sup_{|h| \le 2^{-j}} \|\Delta_h^{\lfloor \overline{s}(\boldsymbol{\sigma}) \rfloor + 1} f\|_{L^p(B_h(x_0, 2^{-j}))} \ge M.$$

It follows that

$$|a - b| = \frac{|a - b| 2^{jd/p} \sup_{|h| \le 2^{-j}} \|\Delta^{\lfloor \overline{s}(\boldsymbol{\sigma}) \rfloor + 1} f\|_{L^p(B_h(x_0, 2^{-j}))}}{2^{jd/p} \sup_{|h| \le 2^{-j}} \|\Delta^{\lfloor \overline{s}(\boldsymbol{\sigma}) \rfloor + 1} f\|_{L^p(B_h(x_0, 2^{-j}))}}$$

$$\leq \frac{2^{jd/p}}{M\sigma_j^{-1}} \big(\sup_{|h|\leq 2^{-j}} \|\Delta_h^{\lfloor \overline{s}(\boldsymbol{\sigma})\rfloor+1} \widetilde{f} + af\|_{L^p(B_h(x_0,2^{-j}))}$$
$$+ \sup_{|h|\leq 2^{-j}} \|\Delta_h^{\lfloor \overline{s}(\boldsymbol{\sigma})\rfloor+1} \widetilde{f} + bf\|_{L^p(B_h(x_0,2^{-j}))}\big)$$
$$\leq \frac{2N}{M}$$

and so, as M is arbitrary, $a = b$. As a consequence, the set

$$\{a \in \mathbb{R} \,:\, \widetilde{f} + af \in S_N\}$$

contains at most one point. Therefore, the set

$$\{a \in \mathbb{R} \,:\, \widetilde{f} + af \in S\} = \bigcup_{N\in\mathbb{N}} \{a \in \mathbb{R} \,:\, \widetilde{f} + af \in S_N\}$$

is countable and thus of Lebesgue-measure zero. The conclusion follows.

Now, let us fix an admissible sequence $\boldsymbol{\sigma} = (\sigma_j)_j$ such that $2^{-jd/p}\sigma_j^{-1}$ tends to 0 as j tends to infinity and $\underline{\sigma}_1 > 2^{-d/p}$. We define, for all $k \in \mathbb{N}$, the admissible sequence $\boldsymbol{\sigma}^{(k)} = (j^{1-1/k}\sigma_j)_j$. As inequalities (6) ensure that the sequence $(|\log_2(2^{-jd/p}\sigma_j)|/j)_j$ is bounded, we define the spaces of “under-log” corrected functions in the following way.

Definition 4.3 If $x_0 \in \mathbb{R}^d$, a *function* $f \in L^\infty_{\mathrm{loc}}$ belongs to $T^{\boldsymbol{\sigma},p}_{/_s\log}(x_0)$ if there exists $k \in \mathbb{N}$ such that $f \in T_p^{\boldsymbol{\sigma}^{(k)}}(x_0)$.

This space can be interpreted as follows: a function belongs to $T^{\boldsymbol{\sigma},p}_{/_s\log}(x_0)$ if its pointwise behavior at x_0 is obtained from $\boldsymbol{\sigma}$ with a correction that is asymptotically weaker than the absolute value of the logarithm of $\mathbf{2^{-\cdot d/p}}\boldsymbol{\sigma}$.

Let us first consider the case where $p = \infty$ and exhibit a function which satisfies the condition of Theorem 3.8 (with $p = q = \infty$) but which does not belong to $T^{\boldsymbol{\sigma},\infty}_{/_s\log}(0)$. This construction is based on an example from [16], but some substantive modifications have to be made in order to correct some mistakes and adapt it to our context. For the sake of clarity, we take $d = 1$ (in the general setting, one can replace the intervals by balls with collinear dyadics, chosen so that there is no overlapping).

Consider a wavelet ψ of regularity $r > \lfloor \overline{s}(\boldsymbol{\sigma})\rfloor + 1$ such that $\operatorname{supp}(\psi) \subseteq [-N, N]$ for some $N > 0$ and $\psi(0) = C$ with $C \neq 0$. Define the sequence $(\varepsilon_m)_{m\in\mathbb{N}}$ by $\varepsilon_m = 2^{-2^{m-1}}$ for all $m \geq 1$. For such a m, define the function

$$f_m = \sigma_{2^{m-1}}^{-1} \sum_{2^m \leq j < 2^{m+1}} \psi(2^j(\cdot - \varepsilon_m)).$$

As $\operatorname{supp}(\psi(2^j(\cdot - \varepsilon_m))) \subseteq [\frac{-N}{2^j} + \varepsilon_m, \varepsilon_m + \frac{N}{2^j}]$, we have

$$\operatorname{supp}(f_m) \subseteq [-N\varepsilon_{m+1} + \varepsilon_m, \varepsilon_m + N\varepsilon_{m+1}],$$

so $f_m(0) = 0$ as soon as $m \geq \log_2(|\log_2(N)|) + 1$. Moreover, let us remark that there exists $M(N)$ such that, for all $m \geq M(N)$

$$2^{32^{m-1}} - (N+1)2^{2^m} \geq N$$

which implies that the supports of the functions in the family $(f_m)_{m \geq M(N)}$ are disjoint. It follows that $f_m(\varepsilon_k) = \delta_{m,k} C 2^m \sigma_{2^{m-1}}^{-1}$ for all $m, k \geq M(N)$. That being said, one can also find $M(N, \boldsymbol{\sigma}) \in \mathbb{N}$ so that

$$\varepsilon_m + N\varepsilon_{m+1} < l\,\varepsilon_m < -N\varepsilon_m + \varepsilon_{m-1} \quad \forall m \geq M(N, \boldsymbol{\sigma}),\ l \in \{2, ..., \lfloor \overline{s}(\boldsymbol{\sigma}) \rfloor + 1\}.$$

As a consequence, if $m \geq M(N, \boldsymbol{\sigma})$, then $\Delta_{\varepsilon_k}^{\lfloor \overline{s}(\boldsymbol{\sigma}) \rfloor + 1} f_m(0) = \delta_{m,k} C 2^m \sigma_{2^{m-1}}^{-1}$. Let us finally consider the function f defined by

$$f = \sum_{m \geq M(N, \boldsymbol{\sigma})} f_m,$$

with convergence in L^∞. Its wavelet coefficients are given by

$$c_{j,k} := \begin{cases} \sigma_{2^{m-1}}^{-1} & \text{if } j \geq M(N, \boldsymbol{\sigma}) \text{ and } k = \varepsilon_m 2^j \\ 0 & \text{otherwise.} \end{cases}$$

For any scale $j \in [2^m, 2^{m+1})$ (with $m \geq M(N, \boldsymbol{\sigma})$), there is only one non-vanishing wavelet coefficient whose value is $\sigma_{2^{m-1}}^{-1}$. Using (6) with $\varepsilon > 0$ small enough so that $\underline{s}(\boldsymbol{\sigma}) - \varepsilon > 0$, we find

$$|c_{j,k}| \leq 2^{-2^{m-1}(\underline{s}(\boldsymbol{\sigma}) - \varepsilon)} \leq 2^{-\frac{(\underline{s}(\boldsymbol{\sigma}) - \varepsilon)}{4} j}.$$

From (9), this guarantees that f belongs to $B_{\infty,\infty}^{\frac{(\underline{s}(\boldsymbol{\sigma}) - \varepsilon)}{4}}$, so that the minimal regularity assumption of Theorem 3.8 is ensured.

For all $j \in \mathbb{N}$, a dyadic cube $[\varepsilon_m, \varepsilon_m + 2^{-j'})$ at scale $j' \in [2^m, 2^{m+1})$ is involved in the computation of $d_j^\infty(0)$ if the distance between ε_m and the origin is less than $2^{-(j-1)}$, which is equivalent to $j \leq 2^{m-1} + 1$. As $\underline{\sigma}_1 > 1$, the sequence $\boldsymbol{\sigma}$ is increasing and, using Eq. (5), we can conclude that

$$(\sigma_j d_j^\infty(0))_j \in \ell^\infty.$$

Now, from what precedes, we also have, for all $m \geq M(N, \boldsymbol{\sigma})$,

$$|\Delta_{\varepsilon_m}^{\lfloor \overline{s}(\boldsymbol{\sigma}) \rfloor + 1} f(0)| = C 2^m \sigma_{2^{m-1}}^{-1} \geq C' |\log(\sigma_{2^{m-1}})| \sigma_{2^{m-1}}^{-1},$$

which shows that f cannot belong to $T_{/s\log}^{\boldsymbol{\sigma},\infty}(0)$. Of course, using a translation, this construction can be generalized to any $x_0 \in \mathbb{R}^d$.

Using this last function and Lemma 4.2, one can establish a first prevalence result concerning the logarithmic correction. Given $0 < \varepsilon < \underline{s}(\boldsymbol{\sigma})/4$ and $x_0 \in \mathbb{R}^d$, let us set

$$E^{\varepsilon}_{\infty}(x_0) = \{f \in B^{\varepsilon}_{\infty,\infty}(\mathbb{R}^d) \ : \ (\sigma_j d^{\infty}_j(x_0))_j \in \ell^{\infty}\}.$$

We can equip this space with the norm

$$\| \cdot \|_{E^{\varepsilon}_{\infty}(x_0)} \ : \ E^{\varepsilon}_{\infty}(x_0) \to [0, +\infty) \ : \ f \mapsto \|f\|_{B^{\varepsilon}_{\infty,\infty}} + \|(\sigma_j d^{\infty}_j(x_0))_j\|_{\ell^{\infty}},$$

so that $E^{\varepsilon}_{\infty}(x_0)$ is a complete normed space.

Theorem 4.4 *If $x_0 \in \mathbb{R}^d$, for all $0 < \varepsilon < \underline{s}(\boldsymbol{\sigma})/4$, almost every function of $E^{\varepsilon}_{\infty}(x_0)$ belongs to $T^{\boldsymbol{\sigma}}_{\infty,\log}(x_0) \setminus T^{\boldsymbol{\sigma},\infty}_{/_s\log}(x_0)$.*

Proof We already know that every function of $E^{\varepsilon}_{\infty}(x_0)$ belongs to $T^{\boldsymbol{\sigma}}_{\infty,\log}(x_0)$. For all $k \in \mathbb{N}$, let us check that the set

$$B_k = \{g \in E^{\varepsilon}_{\infty}(x_0) \ : \ f \in T^{\boldsymbol{\sigma}^{(k)}}_{\infty}\}$$

is Borel. For all $L \in \mathbb{N}$, we define

$$B_{L,k} = \{g \in E^{\varepsilon}_{\infty}(x_0) \ : \ \sigma^{(k)}_j \sup_{|h| \leq 2^{-j}} \|\Delta^{\lfloor \overline{s}(\boldsymbol{\sigma}) \rfloor + 1}_h g\|_{L^{\infty}(B_h(x_0, 2^{-j}))} \leq L \ \forall \, j \in \mathbb{N}\}.$$

The set $B_{N,k}$ is closed as if $(g_m)_{m \in \mathbb{N}}$ is a sequence of functions of $B_{L,k}$ that converges to g in $E^{\varepsilon}_{\infty}(x_0)$, then $\|g - g_m\|_{B^{\varepsilon}_{\infty,\infty}}$ tends to 0 and for all $m, j \in \mathbb{N}$, we have, from (8),

$$\begin{aligned}
&\sup_{|h| \leq 2^{-j}} \|\Delta^{\lfloor \overline{s}(\boldsymbol{\sigma}) \rfloor + 1}_h g\|_{L^{\infty}(B_h(x_0, 2^{-j}))} \\
&\quad \leq \sup_{|h| \leq 2^{-j}} \|\Delta^{\lfloor \overline{s}(\boldsymbol{\sigma}) \rfloor + 1}_h g_m\|_{L^{\infty}(B_h(x_0, 2^{-j}))} + C\|g - g_m\|_{B^{\varepsilon}_{\infty,\infty}} \\
&\quad \leq L(\sigma^{(k)}_j)^{-1} + C\|g - g_m\|_{B^{\varepsilon}_{\infty,\infty}}.
\end{aligned}$$

Taking the limit for $m \to \infty$, we conclude that g belongs to $B_{L,k}$. It follows that

$$B_k = \bigcup_{L \in \mathbb{N}} B_{L,k}$$

is a Borel set. The function f built above belongs to $E^{\varepsilon}_{\infty}(x_0)$ but, for all $M \in \mathbb{N}$, there exists $j \in \mathbb{N}$ for which

$$\sigma^{(k)}_j \sup_{|h| \leq 2^{-j}} \|\Delta^{\lfloor \overline{s}(\boldsymbol{\sigma}) \rfloor + 1} f\|_{L^{\infty}(B_h(x_0, 2^{-j}))} \geq M$$

and we conclude from Proposition 4.2 that B_k is Haar-null. As we have

$$E^{\varepsilon}_{\infty}(x_0) \cap T^{\boldsymbol{\sigma},\infty}_{/_s\log}(x_0) = \bigcup_{k\in\mathbb{N}} B_k,$$

we conclude that almost every function of $E^{\varepsilon}_{\infty}(x_0)$ belongs to $T^{\boldsymbol{\sigma}}_{\infty,\log}(x_0)\backslash T^{\boldsymbol{\sigma},\infty}_{/_s\log}(x_0)$. □

Let us now focus on the case $p = 1$. In this setting, the required condition in Theorem 3.8 for the admissible sequence $\boldsymbol{\sigma}$ is that $2^{-jd}\sigma_j^{-1}$ tends to 0 as j tends to infinity with $\underline{\sigma}_1 > 2^{-d}$. Once again, we will work with $d = 1$. Let us take the same compactly supported wavelet with the additional assumption that[1] $\int_0^N \psi(x)\,dx \neq 0$ and define the sequence $(f_m)_m$ by

$$f_m = \sigma^{-1}_{2^{m-1}}\varepsilon_m \sum_{2^m\leq j<2^{m+1}} 2^j\psi(2^j(\cdot-\varepsilon_m)).$$

The probe function f can be defined as previously but with convergence in L^1 this time. In this case, for $j \in [2^m, 2^{m+1})$, with $m \geq M(N,\boldsymbol{\sigma})$, the only non-vanishing coefficient at scale j is now $\sigma^{-1}_{2^{m-1}}\varepsilon_m 2^j$. First of all, if $\varepsilon > 0$ is now chosen such that $\underline{s}(\boldsymbol{\sigma}) - \varepsilon > -1$, we have

$$2^{-j}\sigma^{-1}_{2^{m-1}}\varepsilon_m 2^j \leq 2^{-2^{m-1}(\underline{s}(\boldsymbol{\sigma})+1-\varepsilon)} \leq 2^{-j\frac{\underline{s}(\boldsymbol{\sigma})+1-\varepsilon}{4}}$$

and $f \in B_{1,\infty}^{\frac{\underline{s}(\boldsymbol{\sigma})+1-\varepsilon}{4}}$. Next, if $[\varepsilon_m, \varepsilon_m + 2^{-j'})$ is a dyadic cube at scale $j' \in [2^m, 2^{m+1})$ for which $\varepsilon_m \leq 2^{-(j-1)}$, we have

$$2^{-(j'-j)}\sigma^{-1}_{2^{m-1}}\varepsilon_m 2^{j'} \leq C'\sigma_j^{-1},$$

which ensures that $(\sigma_j d^1_j(0))_j$ belongs to ℓ^∞. Finally, let us remark that, by increasing $M(N,\boldsymbol{\sigma})$ if necessary, one can suppose that $2\varepsilon_m \geq N\varepsilon_{m+1}$ and, for all $l \in \{1, ..., \lfloor\overline{s}(\boldsymbol{\sigma})\rfloor + 1\}$ and $x \in [\varepsilon_m - N\varepsilon_{m+1}, \varepsilon_m + N\varepsilon_{m+1}]$, we have

$$\varepsilon_m + N\varepsilon_{m+1} < x + l\varepsilon_m < \varepsilon_{m-1} - N\varepsilon_m,$$

so that $f(x + l\varepsilon_m) = \delta_{l,0} f_m(x)$. It follows that for all such large enough m,

$$\begin{aligned}
&(3\varepsilon_m)^{-1}\|\Delta^{\lfloor\overline{s}(\sigma)\rfloor+1}_{\varepsilon_m} f\|_{L^1(B_{\varepsilon_m}(0,3\varepsilon_m))}\\
&\quad\geq C_1\sigma^{-1}_{2^{m-1}}\int_{\varepsilon_m}^{\varepsilon_m+N\varepsilon_{m+1}} |\sum_{2^m\leq j<2^{m+1}} 2^j\psi(2^j(x-\varepsilon_m))|\,dx\\
&\quad\geq C_1\sigma^{-1}_{2^{m-1}}\left|\sum_{2^m\leq j<2^{m+1}} 2^j\int_0^{N\varepsilon_{m+1}}\psi(2^jx)\,dx\right|
\end{aligned}$$

[1] This assumption is satisfied for Daubechies wavelets for instance.

$$= C_1 \sigma_{2^{m-1}}^{-1} \left| \sum_{2^m \le j < 2^{m+1}} \int_0^{N 2^j \varepsilon_{m+1}} \psi(x)\, dx \right|$$
$$= C_2 \sigma_{2^{m-1}}^{-1} 2^m.$$

and, as for all $2^m \le j < 2^{m+1}$, $2^j \varepsilon_{m+1} \ge 1$, and because ψ vanishes on $]N, +\infty[$, one has

$$\int_0^{N 2^j \varepsilon_{m+1}} \psi(x)\, dx = \int_0^N \psi(x)\, dx$$

which is different from 0, by assumption on the wavelet, and, finally

$$(3\varepsilon_m)^{-1} \| \Delta_{\varepsilon_m}^{\lfloor \bar{s}(\sigma) \rfloor + 1} f \|_{L^1(B_{\varepsilon_m}(0, 3\varepsilon_m))} \ge C_2 \sigma_{2^{m-1}}^{-1} 2^m.$$

For $0 < \varepsilon < \frac{\underline{s}(\boldsymbol{\sigma}) + d}{4}$ and $x_0 \in \mathbb{R}^d$, define the space

$$E_1^\varepsilon(x_0) = \{ f \in B_{1,\infty}^\varepsilon(\mathbb{R}^d) \ : \ (\sigma_j d_j^1(x_0))_j \in \ell^\infty \}.$$

The norm defined for $E_\infty^\varepsilon(x_0)$ can be trivially adjusted so that E_1^ε is a complete normed space and Theorem 4.4 can be easily adapted in this setting (for $p = 1$).

Theorem 4.5 *If $x_0 \in \mathbb{R}^d$, for all $0 < \varepsilon < \frac{\underline{s}(\boldsymbol{\sigma}) + d}{4}$, almost every function of $E_1^\varepsilon(x_0)$ belongs to $T_{1,\log}^{\boldsymbol{\sigma}}(x_0) \setminus T_{/_s \log}^{\boldsymbol{\sigma},1}(x_0)$.*

Now, for $1 < p < \infty$, a judicious choice to obtain the desired probe seems to be

$$f_m = \sigma_{2^{m-1}}^{-1} \varepsilon_m^{\frac{1}{p}} \sum_{2^m \le j < 2^{m+1}} 2^{\frac{j}{p}} \psi(2^j(\cdot - \varepsilon_m)).$$

Once again, it is easy to check that the obtained function f satisfies the two first desired properties

$$(\sigma_j d_j^p(0))_j \in \ell^\infty \quad \text{and} \quad f \in B_{p,\infty}^{\frac{\underline{s}(\boldsymbol{\sigma}) + 1/p}{4}}.$$

But, unfortunately, the L^p norm of f_m (see [37]) is proportional to

$$\left(\int_{\mathbb{R}} \left(\sum_{2^m \le j < 2^{m+1}} (\sigma_{2^{m-1}}^{-1} \varepsilon_m^{1/p} 2^{j/p})^2 \chi_{[\varepsilon_m, \varepsilon_m + 2^{-j})} \right)^{p/2} dx \right)^{1/p}$$
$$= \sigma_{2^{m-1}}^{-1} \varepsilon_m^{1/p} \left(\sum_{2^m \le j < 2^{m+1}} 2^{-j} \left(\sum_{k=2^m}^{j} 2^{2k/p} \right)^{p/2} \right)^{1/p}$$

and this last term is itself proportional to $\sigma_{2m-1}^{-1}\varepsilon_m^{1/p}2^{m/p}$, which is not sufficient to establish a theorem comparable to Theorems 4.4 and 4.5 in this case. As the condition $(\sigma_j d_j^p(0))_j \in \ell^\infty$ is optimal, one cannot add a multiplicative term of order $2^{m/q}$ without altering it. Also, by increasing the number of terms in the sum defining f_m up to 2^{mp}, one loses the belonging to a uniform Besov space.

The functions f exhibited here guarantee the necessity of a correction of order $(|\log_2(2^{-jd/p}\sigma_j)|)^{1/p}$ for almost every function in E_p^ε (defined in an obvious way) with $0 < \varepsilon < \frac{\underline{s}(\sigma)+d/p}{4}$, but cannot be used to prove any further result.

5 Functions Providing a Given Generalized Pointwise Regularity

In this section, we start by giving, for any admissible sequence $\boldsymbol{\sigma}$ and $p \in [1,\infty]$, an example of function that belongs to $T_p^{\boldsymbol{\sigma}}(x_0)$. This example leads to discussions concerning the contribution of pointwise spaces of generalized smoothness. This leads to results concerning the Hölder regularity of almost every function belonging to such spaces.

Example 5.1 Given $p \in [1,\infty]$, let $\boldsymbol{\sigma}$ be an admissible sequence such that $\underline{s}(\boldsymbol{\sigma}) > -1/p$. If ψ is again a wavelet with compact support included in $[-N, N]$, we take $K(N) \in \mathbb{N}$ such that, for all $k \geq K(N)$,

$$2^{-k}N \leq \frac{2}{3}$$

and define the function $f_{\boldsymbol{\sigma}}$ by

$$f_{\boldsymbol{\sigma}} = \sum_{k \geq K(N)} \sigma_k^{-1} 2^{k/p} \psi(2^{2k}(\cdot - 2^{-k})), \tag{15}$$

with convergence in L^p. For all $k \geq 2$, $\psi(2^{2k}(\cdot - 2^{-k}))$ is supported in

$$[2^{-k}(1 - N2^{-k}), 2^{-k}(1 + N2^{-k})]$$

and, in particular, the condition on $K(N)$ implies that for all $k, k' \geq K(N)$ with $k \neq k'$,

$$\operatorname{supp}(\psi(2^{2k}(\cdot - 2^{-k}))) \cap \operatorname{supp}(\psi(2^{2k'}(\cdot - 2^{-k'}))) = \emptyset.$$

Therefore, for all $j \geq K(N)$, we have, with the usual modifications if $p = \infty$,

$$
\begin{aligned}
&2^{j/p}\|f_{\boldsymbol{\sigma}}\|_{L^p(B(0,2^{-j}))}\\
&= 2^{j/p}\left(\int_{2^{-j}(1-N2^{-j})}^{2^{-j}}|f_{\boldsymbol{\sigma}}(x)|^p\,dx+\sum_{k>j}\int_{2^{-k}(1-N2^{-k})}^{2^{-k}(1+N2^{-k})}|f_{\boldsymbol{\sigma}}(x)|^p\,dx\right)^{1/p}\\
&= 2^{j/p}\left(\sigma_j^{-p}2^{-j}\int_{-N}^{0}|\psi(x)|^p\,dx+\sum_{k>j}\sigma_k^{-p}2^{-k}\int_{-N}^{N}|\psi(x)|^p\,dx\right)^{1/p}.
\end{aligned}
$$

Finally, from Lemma 3.2, it is clear that we can find constants $C_1, C_2 > 0$ such that, for all $j \geq 2$,

$$
C_1 \leq 2^{j/p}\sigma_j\|f_{\boldsymbol{\sigma}}\|_{L^p(B(0,2^{-j}))} \leq C_2. \tag{16}
$$

Condition (10) (with $P_{j,0} = 0$, for all j) and (16) ensure that f_σ belongs to $T_p^{\boldsymbol{\sigma}}(0)$. Figure 1 gives a representation of $f_{\boldsymbol{\sigma}}$ for various values of p and admissible sequences $\boldsymbol{\sigma}$. Of course, up to a translation, all these affirmations still hold for an arbitrary point x_0 instead of 0. Again, for the sake of clarity, this example is made for $d = 1$ but can be adapted for general d.

Remark 5.2 The spaces $T_p^{\boldsymbol{\sigma}}(x_0)$ can be naturally generalized as follows (see [34, 35]): f belongs to $T_{p,q}^{\boldsymbol{\sigma}}(x_0)$ whenever

$$
(\sigma_j 2^{jd/p}\sup_{|h|\leq 2^{-j}}\|\Delta_h^{\lfloor\bar{s}(\sigma)\rfloor+1}f\|_{L^p(B_h(x_0,2^{-j}))})_j \in \ell^q.
$$

In order to obtain a function in $T_{p,q}^{\boldsymbol{\sigma}}(0)$ with $q \neq \infty$, it suffices to consider a sequence $(\varepsilon_k)_k \in \ell^q$ and, for all $k \geq K(N)$, to multiply the k^{th} term in the sum (15) by ε_k (the conclusion follows again by Lemma 3.2).

From this peculiar example, we can discuss the general interest of the spaces introduced by Definition 3.5 from the regularity point of view. First, given an admissible sequence $\boldsymbol{\sigma}$, there exists a function f_σ which belongs to $T_p^{\boldsymbol{\sigma}}(x_0)$ and this belonging is optimal since inequalities (16) are satisfied (see [7]). That being said, if Ψ and Φ are two distinct slowly varying functions which tends to 0 at infinity, given a number $u > -d/p$, let us consider the associated admissible sequences

$$
\boldsymbol{\sigma_{u,\Psi}} = (2^{ju}\Psi(2^j))_j \quad\text{and}\quad \boldsymbol{\sigma_{u,\Phi}} = (2^{ju}\Phi(2^j))_j.
$$

The functions $f_{\boldsymbol{\sigma_{u,\Psi}}}$ and $f_{\boldsymbol{\sigma_{u,\Phi}}}$ (defined as in Example 5.1) both belong to $\cap_{\varepsilon>0}T_p^{\boldsymbol{u-\varepsilon}}(x_0)$ but not to $T_p^{\boldsymbol{u}}(x_0)$. From this point of view, the usual spaces of Calderón and Zygmund fail to precisely characterize the regularity of $f_{\boldsymbol{\sigma_{u,\Psi}}}$ and $f_{\boldsymbol{\sigma_{u,\Phi}}}$ at x_0. On the other hand, we have $f_{\boldsymbol{\sigma_{u,\Psi}}} \in T_p^{\boldsymbol{\sigma_{u,\Psi}}}(x_0)$ and $f_{\boldsymbol{\sigma_{u,\Phi}}} \in T_p^{\boldsymbol{\sigma_{u,\Phi}}}(x_0)$. As soon as $\Psi(x) \in o(\Phi(x))$ as $x \to \infty$, $f_{\boldsymbol{\sigma_{u,\Psi}}} \in T_p^{\boldsymbol{\sigma_{u,\Psi}}}(x_0)\backslash T_p^{\boldsymbol{\sigma_{u,\Phi}}}(x_0)$. More generally, if $\boldsymbol{\sigma}$ and $\boldsymbol{\gamma}$ are two admissible sequences such that $\sigma_j \in o(\gamma_j)$ as $j \to \infty$, $f_{\boldsymbol{\sigma}} \in T_p^{\boldsymbol{\sigma}}(x_0)\setminus T_p^{\boldsymbol{\gamma}}(x_0)$.

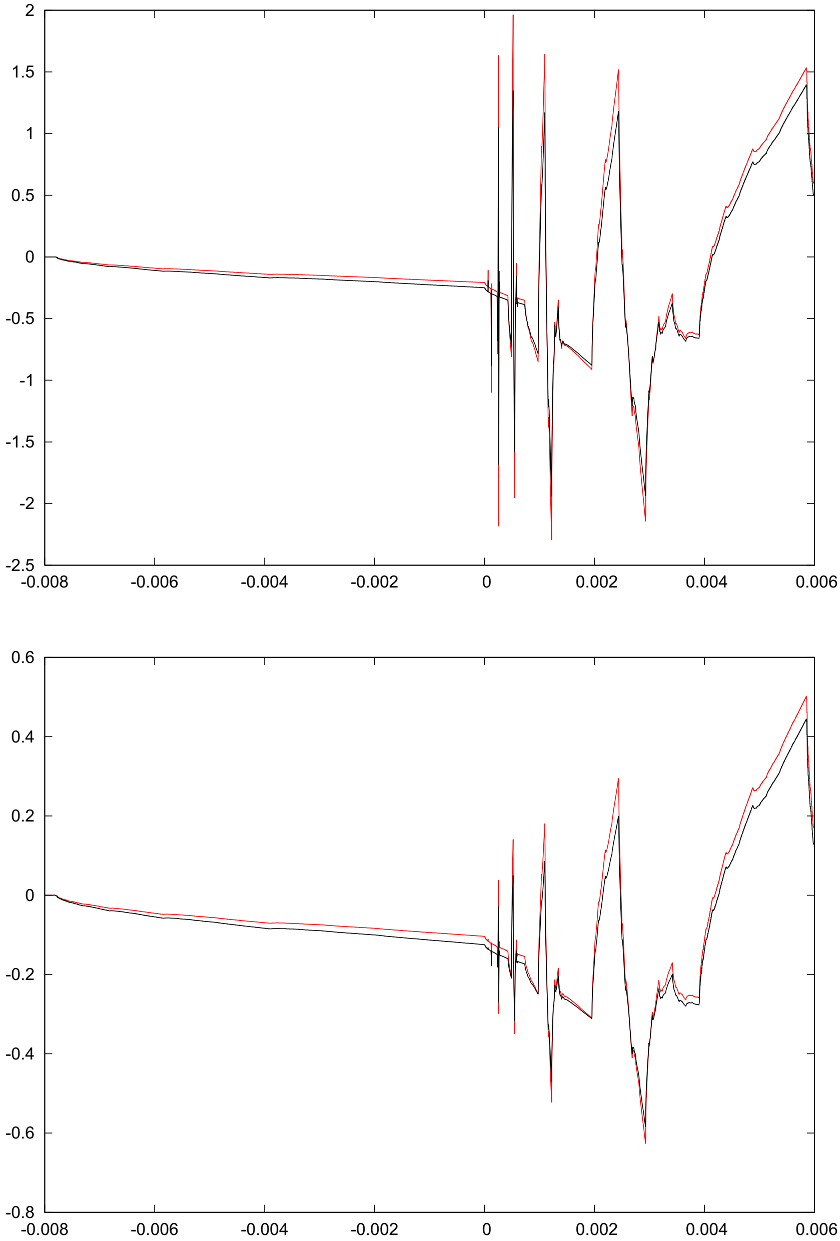

Fig. 1 Representation of some functions defined in Example 5.1 with $p = 2$ (upper panel) and $p = \infty$ (lower panel), using a dyadic sequence (black) and a dyadic sequence with a logarithmic correction (red). The wavelet considered is the Daubechies wavelet of order 2 (Color figure online)

For example, the usual spaces do not allow to capture the logarithmic correction in the regularity of the Brownian motion B [25–27]. More precisely, let $\boldsymbol{\sigma}$ be the admissible sequence $\boldsymbol{\sigma} = (2^{j/2}|\log j|^{-1/2})_j$; from the Khintchine Law of the iterated logarithm [13, 25, 27, 39], we know that, almost surely, for all ω and for almost every $x_0 \in \mathbb{R}$, $x \mapsto B_x(\omega)$ belongs to $T_\infty^{\boldsymbol{\sigma}}(x_0)$, while $B_\cdot(\omega) \notin T_\infty^{\mathbf{1/2}}(x_0)$. Being able to make a distinction between a Brownian motion and another process not displaying such logarithmic corrections is an important issue in practice (see [12, 26] for instance) and could be achieved through the use of admissible sequences [26, 29]. One could argue that in this case, uniform spaces would suffice. However, not every point of a Brownian motion displays such a behavior; a point for which the law of the iterated logarithm fails is qualified as fast [43] (there is also a third category of points, called slow [24]). Such a phenomenon can be expected in other processes or functions. Looking to which $T_p^{\boldsymbol{\sigma}}$ spaces they belong to could help investigating such comportments (determining the size of the set of the fast points is a natural example).

All these remarks lead to results of prevalence, using again Lemma 4.2.

Theorem 5.3 *Given $p \in [1, \infty]$, if $\boldsymbol{\sigma}$ and $\boldsymbol{\gamma}$ are two admissible sequences satisfying $\sigma_j \in o(\gamma_j)$ as $j \to \infty$ and $\overline{s}(\boldsymbol{\sigma}) \leq \overline{s}(\boldsymbol{\gamma})$ then, from the prevalence point of view, almost every function in $T_p^{\boldsymbol{\sigma}}(x_0)$ does not belong to $T_p^{\boldsymbol{\gamma}}(x_0)$.*

Proof The assumptions made on the admissible sequences ensure the inclusion of $T_p^{\boldsymbol{\gamma}}(x_0)$ in $T_p^{\boldsymbol{\sigma}}(x_0)$, see [29]. From the previous remarks made on $f_{\boldsymbol{\sigma}}$, we know that for all $M \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that

$$\gamma_j 2^{jd/p} \sup_{|h| \leq 2^{-j}} \|\Delta_h^{\lfloor \overline{s}(\boldsymbol{\gamma}) \rfloor + 1} f_{\boldsymbol{\sigma}}\|_{L^p(B_h(x_0, 2^{-j}))} \geq M.$$

Therefore, from Lemma 4.2, it suffices to show that $T_p^{\boldsymbol{\gamma}}(x_0)$ is a Borel set in $(T_p^{\boldsymbol{\sigma}}(x_0), \|\cdot\|_{T_p^{\boldsymbol{\sigma}}(x_0)})$. We proceed in the same way as for the proof of Theorem 4.4. For all $N \in \mathbb{N}$, define

$$B_N = \{f \in T_p^{\boldsymbol{\gamma}}(x_0) \, : \, \|f\|_{T_p^{\boldsymbol{\gamma}}(x_0)} \leq N\};$$

this set is closed in $(T_p^{\boldsymbol{\sigma}}(x_0), \|\cdot\|_{T_p^{\boldsymbol{\sigma}}(x_0)})$: if $(f_k)_k$ is a sequence of functions of B_N that converges to f then, for all $j, k \in \mathbb{N}$,

$$\begin{aligned} &\|f\|_{L^p(B(0,1))} + \gamma_j 2^{jd/p} \sup_{|h| \leq 2^{-j}} \|\Delta_h^{\lfloor \overline{s}(\boldsymbol{\gamma}) \rfloor + 1} f\|_{L^p(B_h(x_0, 2^{-j}))} \\ &\quad \leq \|f - f_k\|_{L^p(B(0,1))} + \gamma_j 2^{jd/p} \sup_{|h| \leq 2^{-j}} \|\Delta_h^{\lfloor \overline{s}(\boldsymbol{\gamma}) \rfloor + 1} (f - f_k)\|_{L^p(B_h(x_0, 2^{-j}))} \\ &\qquad + \|f_k\|_{L^p(B(0,1))} + \gamma_j 2^{jd/p} \sup_{|h| \leq 2^{-j}} \|\Delta_h^{\lfloor \overline{s}(\boldsymbol{\gamma}) \rfloor + 1} f_k\|_{L^p(B_h(x_0, 2^{-j}))}. \end{aligned}$$

Of course,

$$\|f_k\|_{L^p(B(0,1))} + \gamma_j 2^{jd/p} \sup_{|h| \leq 2^{-j}} \|\Delta_h^{\lfloor \overline{s}(\boldsymbol{\gamma}) \rfloor + 1} f_k\|_{L^p(B_h(x_0, 2^{-j}))} \leq N$$

and, using fundamental properties of finite differences,

$$\gamma_j 2^{jd/p} \sup_{|h|\le 2^{-j}} \|\Delta_h^{\lfloor \overline{s}(\boldsymbol{\gamma})\rfloor+1}(f-f_k)\|_{L^p(B_h(x_0,2^{-j}))}$$
$$\le C\frac{\gamma_j}{\sigma_j}\sigma_j 2^{jd/p} \sup_{|h|\le 2^{-j}} \|\Delta_h^{\lfloor \overline{s}(\boldsymbol{\sigma})\rfloor+1}(f-f_k)\|_{L^p(B_h(x_0,2^{-j}))}.$$

Taking the limit for $k\to\infty$, we have $\|f\|_{T_p^{\boldsymbol{\gamma}}(x_0)} \le N$, so that B_N is closed. As,

$$T_p^{\boldsymbol{\gamma}}(x_0) = \bigcup_{N\in\mathbb{N}} B_N,$$

the conclusion follows.

In [35], we use the generalized spaces of pointwise smoothness to define a generalized Hölder exponent in the following way. We consider a family $(\boldsymbol{\sigma}^{(\boldsymbol{h})})_{h>-d/p}$ which is decreasing in the following sense: for any $h>-d/p$,

- $\underline{s}(\boldsymbol{\sigma}^{(h)}) > -d/p$,
- $\underline{\sigma}_1^{(h)} > 2^{-d/p}$,
- $h<h'$ implies $T_p^{\boldsymbol{\sigma}^{(h')}}(x_0) \subset T_p^{\boldsymbol{\sigma}^{(h)}}(x_0)$.

When working with decreasing families of admissible sequences, the assumptions of Theorem 5.3 are often met (see [29]) and we can state the following corollary without too much restriction.

Corollary 5.4 *Given $p\in[1,\infty]$, if $(\boldsymbol{\sigma}^{(\boldsymbol{h})})_{h>-d/p}$ is a decreasing family of admissible sequences such that $h<h'$ implies $\sigma_j^{(h)}\in o(\sigma_j^{(h')})$ for $j\to\infty$ and $\overline{s}(\boldsymbol{\sigma}^{(\boldsymbol{h})}) \le \overline{s}(\boldsymbol{\sigma}^{(\boldsymbol{h'})})$ then, from the prevalence point of view, almost every function in $T_p^{\boldsymbol{\sigma}^{(\boldsymbol{h})}}(x_0)$ is of exponent h.*

Of course, this corollary can be applied to the classical spaces T_p^α to state the following generalization of a result of Hunt concerning the Hölder spaces [14].

Corollary 5.5 *Given $p\in[1,\infty]$ and $\alpha>-d/p$, from the prevalence point of view, almost every function f in $T_p^\alpha(x_0)$ satisfies $h_f^{(p)}(x_0)=\alpha$.*

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