BEWARE PROPORTIONS

In proportionality exercises pupils are usually presented with two proportional quantities in two given situations. In the former situation, pupils know both quantities and in the latter they know only one. A proportion then allows to compute the unknown value. Some variants are possible, for example pupils may know the proportionality constant and one of the two quantities.

To solve a proportionality exercise, the first step involves *extracting mathematical information* from a text (in a multilingual environment, the text should be simple or translated). One must first understand the problem and determine whether the given quantities are *directly proportional or inversely proportional*. Then one has to write the proportion.
The second step is merely *computational* (a computer can solve the proportion provided that the pupil is able to input the numbers correctly).
Pupils should also make a *plausibility test* because it is a common mistake to get the reciprocal of the correct value (for example, saying that two bags together have half the weight of one bag). Some common sense suffices to determine whether the unknown value is expected to be larger than or smaller than (or equal to) the corresponding known value.

Remark that proportionality exercises might give only part of the information for the sake of brevity. These "unspoken conventions" force pupils to make use of assumptions that are not stated. In fact, the simplified situations or the impossible scenarios may distract certain pupils, and care is required.

One exercise to rule them all. We can *classify the proportionality exercises up to equivalence*. For example it does not matter whether each child receives 5 candies or 3 apples, as the logic of sharing is the same.
One could list all quantities in physics that are known by the pupils and investigate which among them are (or can be under certain circumstances) directly proportional, e.g. length, area, volume, weight, temperature, pressure, space, time, speed. For example, volume and weight for some homogeneous material, or time and space when the speed is constant are directly proportional. For inverse proportional quantities, one can think about time and speed (indeed, twice the same constant speed means half of the time). Moreover, densities in physics can be seen as proportionality constants.
There can also be quantities from finance (price) and "number of objects", where the objects could be anything from ants to spaceships. Moreover, we have the quantity "productivity" (e.g. some output per day). And often the proportionality constant is an "amount pro capita". Remark that we could have various speeds (movement, filling, working, downloading,..): the constant "production" (with a quantity that decreases or increases) is directly proportional to the speed, while speed and time to ensure a fixed quantity change are inversely proportional.

There are quantities that remind us of space and speed, the former quantity being an integral of the latter. For example, if for an amusement park the children pay proportionally to the time of their stay, then to compute the price the number of children has to be combined with the duration of their stay in a weighted sum that reminds of an integral. One could then speak of the real-valued quantity "children presence", whose unit could be "1 child for 1 hour". In these examples, the "objects" are counted with a weight and one needs the "impact" of the objects.

It is often the case that the two quantities in a proportionality exercise are distinct, and with different unit measures (for example, "hours" and "kilometres"). However, this may not always be the case. For example, if you take red paint and yellow paint to make a certain shade of orange you have liters of paints in both cases. Also notice that two values for a given quantity could be expressed in different units of measurement, hence a conversion is required.

Some words about *transitivity*. If the quantities A and B are directly (respectively, inversely) proportional, and the quantities B and C are directly (respectively, inversely) proportional, then A and C are directly proportional. One can have many quantities that are pairwise directly proportional (for example, some weight expressed in grams, kilograms, pounds, ounce,..). Moreover, there are no three quantities such that any two of them are inversely proportional. However, one may produce many quantities (all directly proportional) that are inversely proportional to one same quantity. As a curiosity, it could happen that A is not proportional to B and B is not proportional to C, however A is proportional to C (for example, take any two quantities A,B that are not proportional and set A=C).

Pizza and Children. Consider the scenario of children parties. *Do you consider the amount of pizza and the number of children as quantities that are directly proportional, inversely proportional, or not proportional?* Assume that each child eats the same amount of pizza, and that all pizzas are the same.

• Suppose that you *order* pizzas depending on the number of children, fixing the amount of pizza that you want to give to each child and ordering what you strictly need. Will you have direct proportionality? Yes, if a child eats an integer amount of (small) pizzas. No in general, because you order full pizzas hence you may have some pizza as a left-over.
• Suppose that you *share* the pizza you have according to the number of children, being fair. If you are able to cut pizza with arbitrary precision, then you have inverse proportionality. If you can do e.g. at most eights of pizza, then you won't have proportionality, because you might have many slices as a left-over (possibly more than a full pizza, however the amount of slices is strictly less than the number of children).
• Suppose that you *compare* the pizza you have and the number of children. The two quantities are in general not proportionally related because one party may have more pizza per child than another party. But of course if the rules are e.g. that each child brings two (small) pizzas, then you have direct proportionality.

Now suppose that "amount of pizza" and "number of children" are proportional. Then it depends on the scenario which quantity determines the other. Indeed, if you order pizza, then the number of children determines the amount of pizza. But it is the other way round if you have pizza at your disposal and you are inviting children accordingly, or if you distribute the same amount of pizza to queuing children until the pizza is over (and you count the children that got pizza). Mathematically speaking, given two proportional quantities (provided that the proportionality constant is non-zero), then any of them determines the other.

Proportional prior to rounding. The answer to a proportionality exercise could be the result of the proportion after a suitable rounding, for example when the result needs to be an integer.

- Suppose that you buy items of the same type without discount (the total price is proportional to the number of items). Then the number of items that you can buy is only roughly proportional to the money in your wallet. Indeed, you must round with the floor function, as you cannot buy an item if you have less money than its price.
- Suppose that you want to give 3 pencils to each child and the pencils are sold in boxes of 10. In this case you must round the value of the proportion with the ceiling function to determine the number of boxes that you need to buy.

Considering the rounding, the line representing the proportionality becomes a staircase function, while for inverse proportionality one gets a staircase hyperbola.

Two exercises dealing with proportions and rounding are the following:

Your class is preparing for an excursion and your teachers are filling lunch bags with sandwiches, one lunch bag for each pupil. The lunch bags and the sandwiches are all alike. Each sandwich can be cut into 2 or 3 equal parts. You know that with 14 sandwiches, one can fill 9 lunch bags but not 10 lunch bags. How many sandwiches were used to prepare lunch bags for the 24 pupils? (Solution: The amount of sandwich in each bag is in particular a rational number of the form X/6. From the information about 14 sandwiches, we know that $14/10 \leq X/6 \leq 14/9$. We deduce that $8 < X < 10$, hence X=9. So each bag contains 9/6=3/2 sandwiches hence for 24 bags one needs $24 \cdot 3/2 = 36$ sandwiches.)

Your grandparents' cake recipe for 10 people uses 6 eggs and 600 grams of flour. To be faithful to the recipe, it is important to preserve the ratio between eggs and flour. You need to use an integer amount of eggs. How many grams of flour do you use for making a cake that suffices for 8 people? (Solution: For 8 people you should use $6 \times \frac{8}{10} = 4{,}8$ eggs, so you take 5 eggs. As you are using $\frac{5}{6}$ of the eggs, then you need to use $\frac{5}{6}$ of the flour, namely 500 grams.)

Rounding the proportion result is closely related to solving *proportion inequalities*. Namely, suppose that two quantities are proportional. Then which values of the first quantity ensures that the second quantity is larger (respectively, smaller) than a given value? Exercises with a proportional inequality could ask, for example: *Will the basin be full after 5 hours? How many workers suffice to complete a task in at most 3 days?*

Per per. Usually, there are two proportional quantities, and it is common to have the proportionality constant e.g. “price per liter”. In proportionality exercises involving three quantities, one can have more proportionality constants because one compares the quantities pairwise. However, one may also have the *per per*, for example the “litres of milk per cow per day”. In such an exercise, the amount of milk is directly proportional to the number of cows (if we fix the days) and to the number of days (if we fix the cows). Moreover, “cows” and “days” are inversely proportional (if we fix the milk). Mathematically speaking, the function M (milk) is given by $M=kCD$, namely it is the product of C (cows) and D (days) times a constant k that is the “milk per cow per day” and which is simply the value for M when $C=1$ and $D=1$. Remark that the “per per” only makes sense if the corresponding quantities can be simultaneously set to 1.

Conclusion. While preparing or choosing proportionality exercises, educators have to pay attention that: occasionally the unspoken conventions are challenged; the different types of proportionality exercises are considered; pupils familiarise themselves with “proportionality up to rounding” and with proportion inequalities; in a multilingual environment the exercise text is simplified or translated; the (idealised) scenario does not distract sensible pupils. Finally, educators should encourage the plausibility check of the result after solving any proportion.