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Beware Proportions

We collect here some reflections around proportions that stem from our discussions while developing a very user-friendly proportionality calculator. They might be included in a future publication on the subject.

Proportionality relations

Talented pupils or curious pupils may appreciate exploring and knowing all there is to know around proportions, while weaker students should be warned about selected facts. We thus recommend the following questions, recalling that two quantities can be either directly proportional, inversely proportional, or not proportional.

- *If some quantity A is directly proportional to some quantity B, and B is directly proportional to some quantity C, then A is directly proportional to C. True or false?*
- *If some quantity A is inversely proportional to some quantity B, and B is inversely proportional to some quantity C, then A is inversely proportional to C. True or false?*
- *If some quantity A is not proportional to some quantity B, and B is not proportional to some quantity C, then A is not proportional to C. True or false?*

These questions are about transitivity for the relations “directly proportional”, “inversely proportional” and “not proportional” and suppose that no quantity is identically zero. The first is an equivalence relation, the second and the third are not, as they are symmetric but neither reflexive nor transitive. In fact, if both pairs A,B and B,C are inversely proportional quantities, then A,C are directly proportional.

It is quite easy to handle these concepts if one sees a direct/inverse proportionality between quantities A,B as an equation of the form $A = kB$ or $AB=k$ respectively, where k is the proportionality constant (swapping A and B has the effect of inverting the proportionality constant in the former case).

Are “pizza” and “children” proportional?

All pupils could be challenged with the following dilemma:

Consider the scenario of children parties. Do you consider the amount of pizza and the number of children as quantities that are directly proportional, inversely proportional, or not proportional?

All possibilities occur, depending on the context. Assume that each child eats the same amount of pizza, and that all pizzas are the same.

- Suppose that you *order* pizzas depending on the number of children, fixing the amount of pizza that you want to give to each child and ordering what you strictly need. Will you have direct proportionality? Yes, if a child eats an integer amount of (small) pizzas. No in general, because you order full pizzas hence you may have a fraction of pizza as a left-over.
- Suppose that you *share* the pizza you have according to the number of children, being absolutely fair. If you are able to cut pizza with infinite precision (say, you are able to cut 1/23 of a pizza) then you have inverse proportionality. If you can do e.g. at most eights of pizza, then you won't have proportionality, because you may have many slices as a left-over (possibly more than a full pizza, but the amount of slices is strictly less than the number of children).
- Suppose that you *compare* the pizza you have and the number of children. The two quantities are in general not proportionally related because one party may clearly have more pizza per child than another party. But of course if the rules are e.g. that each child brings three (small) pizzas, then you have direct proportionality.

The above innocent-looking question uncovers a world of possibilities that can be explored in class with fantasy and a think-out-of-the-box attitude.

Now suppose that "amount of pizza" and "number of children" are proportional. Then it depends on the scenario which quantity determines the other. Indeed, if you share a given amount of pizza or you order pizza, then the number of children determines the amount of pizza. But it is the other way round if you have pizza at your disposal and you are inviting children accordingly, or if you distribute the same amount of pizza to queuing children until the pizza is over. However, mathematically speaking, given two proportional quantities (provided that the proportionality constant is known), then any of it determines the other. For example, if you ordered a certain amount of pizza (and you know how many pizza you wanted pro child), then you can deduce from it the number of children. Mathematically speaking, the "dependent quantity" must have come from a specific value of the "independent quantity" hence the former also determines the latter. This is because you cannot find two distinct pairs of values for two proportional quantities where one quantity remains the same (excluding zero values for the moment).

Zero does not count. Or maybe it does?

Zero denominators constitute a mathematical danger hence it is not totally safe to write a proportionality relation as an equality of fractions. Moreover, swapping the quantities or the situations, the fractions may get inverted hence one is only truly safe if both numerators and denominators are non-zero. Writing proportionality relations as an equality of integers (representing an equality of fractions) might give a false sense of security. Indeed, although there are no denominators in sight, the zero issue is still there. Namely, for non-zero values, any three real values in a proportionality relation mathematically determine the fourth one (you cannot have minus 3.5 children, however this is not a mathematical issue). But if you have zero values around, then these may force further values to be zero. And some triples of values may be impossible, for example the proportionality equation 0 x … = 1 x 2 cannot be solved.

Notice that having an inverse proportionality relation automatically excludes zero values (for some non-zero proportionality constant) or forces at least one quantity to be zero. Indeed, the subset of the Euclidean plane consisting of the pairs of values for two inversely proportional quantities is: for a given non-zero proportionality constant, an hyperbola having as asymptotes the coordinate axes; for the zero proportionality constant (namely, the equation AB=0), the union of the two coordinate axes.

The issue with zero values for direct proportionality is also geometrically clear, as the values for pairs of two real quantities that are directly proportional is a line through the origin. For a line which is neither vertical or horizontal (namely, if the proportionality constant is non-zero), one quantity is zero if and only if the other one is zero. A vertical or horizontal line means that one quantity is always zero.
Notice that here we are taking all mathematically possible real values, else we should restrict the above geometrical sets and get for example only one hyperbola branch, or a line segment, or an isolated set of points (for example if both quantities can only take integer values).

It is convenient to exclude quantities that are the constant zero from the theory of proportionality because any quantity is directly and inversely proportional to that zero quantity (with proportionality constant zero) and for example we would lose transitivity for direct proportionality.

How many quantities can we deal with?

It is easy to produce many quantities that are directly proportional with all distinct proportionality constants. Imagine for example that each dwarf has one head, two hands, three hats (in his closet), five apples (in his lunch bag),… Any two of these quantities are directly proportional to the number of dwarves. It is much harder to come up with many inverse proportionality quantities, also because no three of them can be pairwise inversely proportional.

One could have two sets of directly proportional quantities such that each quantity in the first set is inversely proportional to any quantity in the second set. A not very brilliant example would be considering two inverse proportional quantities, like volume and density (for homogeneous material of a fixed weight) and then considering the same quantities expressed with different measure units.

Proportionality exercises usually involve two quantities, although more proportional quantities could still make sense. Also notice that in many (if not, most) of proportionality exercises, the fact that the involved quantities are proportional is only a simplification w.r.t. reality. For example that all children eat the same amount of pizza.

Representing real values for several quantities requires a multidimensional space. However, one only compares two quantities at a time and hence works on planes (where, if there is proportionality, one needs values on a line through the origin or on a rectangular hyperbola whose asymptotes are the coordinate axes). However, several quantities can be direct proportional, which means that we have values on a line through the origin in a higher dimensional space.

Great expectations

In the classical proportionality exercises you are given two proportional quantities in two situations. In one situation you know both quantities and in the second situation you know only one. The proportionality relation allows you to compute the missing value. Alternatively, instead of knowing both quantities in one situation, you may know the proportionality constant. From a mathematical point of view, all these exercises are basically the same, and they are all totally straight-forward. Then why all the mistakes, which can be dangerous e.g. for people working in a lab (proportions are used e.g. when diluting liquids)?

For example if you swap a proportionality constant with its inverse, or multiply instead of dividing, then you find a quantity multiplied by 3/2 rather than 2/3. Such a mistake would be like saying e.g. that if you dilute wine with water, then the alcoholic percentage of the wine increases, which does not make any sense.

The fact that implausible results are so common suggests that the outcome is, sadly, not reflected. But this can be avoided, in part, with a dedicated training aimed to "think before you compute". Prior to solving a proportionality exercise one should write expectations as part of the solution, for example:

- We expect a larger (respectively smaller value) than …
- We expect a positive value.
- The requested result should be an integer value. We need a round-up (respectively, round down).

Also notice that proportionality exercises only give part of the requested information for the sake of brevity. This "unspoken convention" forces pupils to assume assumptions that are not stated, like "of course the text tacitly supposes this and that". This can be practical for training proportions but it could be a killer of scientific reasoning and thinking out of the box. To counteract this potential danger, the teacher must at random times give pupils the task of finding all "unspoken assumptions" of a proportionality exercise. And the talented pupil should always state in the solution something like e.g. "Following the usual conventions, we suppose that all cows give the same amount of milk, and that this does not depend on the day."

An unusual formulation

We found online (https://tasks.illustrativemathematics.org/content-standards/tasks/1527) the following formulation to describe directly proportional quantities, which we find a bit misleading: "if one quantity goes up by a certain percentage, the other quantity goes up by the same percentage as well".

Mathematically speaking, a percentage is a number, so this seems like saying that if A changes to A+0,2 then B changes to B+0,2, which would mean that A and B differ by a constant. Of course the quoted text meant that if A becomes A + 20%A, then B becomes B+20%B, and with this interpretation it is correct. However, formulating a proportion with an addition is quite misleading.

Proportions regard multiplication and have nothing to do with addition, so why not saying instead "if one quantity gets rescaled by a factor, then the the other quantity also gets rescaled by the same factor", where "factor" underlines a multiplicative scenario.

One exercise to rule them all

From a mathematical point of view, there are only finitely many (and very few) proportionality exercises. But there are thousands of them everywhere in the world because the scenarios can be the most unusual.

Let us consider an equivalence relation on these exercises, namely identifying those that deal with the same quantities. For example if you deal with "the price of apples at the train station" or "the price of pears at the town market", then they are about "price of objects". How many different exercises do we have then?

First of all, there are the finitely many classical physical quantities (e.g. length, area, volume, weight, temperature, pressure, space, time, speed, various densities,…) and chemical quantities (e.g. concentration of a substance). Then we have some quantities from finance (e.g. price). Then we have the quantity "amount of some objects", where "objects" depends on the scenario, and that could be anything from ants to spaceships. Moreover, we have the quantity "productivity" (e.g. some output per day).
And often the proportionality constant is an "amount pro capita", if one quantity represents a positive integer amount of "objects". Notice that we could have various constant "speeds" (movement, filling, working, downloading,..), basically for any quantity that is linear in time: the constant "production" (with a quantity that decreases or increases) is directly proportional to the speed, while speed and time to ensure a fixed change are inverse proportional.

As an aside remark, there are several quantities that remind of the physical quantities "space and speed" or "work and power", the former quantity being an integral of the latter. For example if a child pays his visit to some amusement park proportionally to the time of their stay, then the number of children has to be combined with the duration of their stay in a weighted sum that reminds of an integral. One could then speak of the real-valued quantity "children presence", whose unit could be "1 child for 1 hour". Similarly, one could have workers and the time they work (possibly expressed with the percentage w.r.t. the full-time job). But one could also have for example several cars going at different speeds and one would compute the total amount of driven kilometres in a given amount of time. In these examples, the "objects" are counted with a weight and one needs the "presence" or "impact" of the objects. Attaching a weight to each object is in fact comparable to working with a multiset where the frequencies can be real numbers. To take care of the fact that the "shareholders" in the given situation are not all alike.

To summarize, suitably grouping proportionality exercises into categories will not give a very long list after all.

Notice that it is fortunately often the case that the two quantities in a proportionality exercise are distinct, with unit measures for example "hours" and "kilometres". However, this may not be always the case, for example if you mix "liters" of red paint and "liters" of yellow paint to make a certain shade of orange paint.

Proportional prior to rounding

If you prepare children presents with 3 candies, the number of candy bags you prepare with a given amount of candies is roughly proportional to it, namely it is the floor function of a direct proportional quantity. Mathematically speaking, we should use the terminology "proportional prior to rounding". What one would usually do is allowing a non-integer amount of bags, to be corrected with a final rounding.

The geometrical representation for values of a pair of quantities like "candies, bags" is a stairway function close to a line through the origin, which consists of horizontal segments (all congruent and regularly placed). This still looks quite familiar, however for inverse proportionality quantities one would have instead a stairway hyperbola. Do ever pupils see this? Maybe they could, at least the talented ones.

The usual proportionality exercises do not reveal the issue of quantities that are proportional prior to rounding, namely the fact that e.g. the number of bags only gives us with some approximation the number of candies we have (as there could be up to 2 candies left-over). The number of bags is the quotient of the number of candies after division by 3, while the number of left-over candies is the remainder.

A non-standard exercise would be determining the proportionality constant for integer quantities that are proportional prior to rounding. This means that some pairs "dividend, quotient" are given and one should determine the divisor. This question becomes tricky if we allow the dividend to be a fraction, e.g. if each bag contains 2 candies and a half. In this case, one should get a bound for the denominator of the proportionality constant and sufficiently many rounded values (or few values, but carefully chosen to get conclusive bounds for the proportionality constant).

Proportional inequalities

Suppose that two quantities are proportional. Then which values of the first quantity ensure that the corresponding value of the second quantity is larger (respectively, smaller) than a given value? For directly proportional quantities, this gives a linear inequality which is easy to solve. And for indirectly proportional quantities this gives a "hyperbolic inequality": this may look more difficult at first sight, but as the corresponding function (whose graph is an hyperbola) is again strictly monotone (at least on each hyperbola branch), then one can proceed in the same way. Indeed, one first solves the corresponding equality and then one only has to choose the correct "direction".

There are modifications to the above question that include a final rounding if e.g. the first quantity only assumes integer values. However, this is only a minor modification, as this does not affect the initial part of the solution.

Examples of exercises with a proportional inequality would be: How many candies do we need to make sure that we can make children presents if we expect up to 20 children? Will the basin be full after 5 hours? Is it possible that 5 workers complete the task in 3 days?

Proportional powers

There are several quantities that have related scaling. For example, if we double the length (in every direction), then we have four times the surface and eight times the volume. The quantities length, area, and volume are not proportional but some of their powers are directly proportional. For a homogenous material one could also relate length and weight in this way.

One could then work with the usual proportions for suitable powers of the quantities and then deduce results for the quantities of interest.

Per per

In usual proportionality exercises there are two proportional quantities, and it is very common to have the proportionality constant described e.g. as “price per liter”. In proportionality exercises involving three quantities, you can have more proportionality constants because you compare the quantities pairwise. But you may also have something more, namely the “per per”, for example the “litres of milk per cow per day”. In such an exercise (with a simplified reality), the amount of milk is directly proportional to the number of cows (if we fix the days) and to the number of days (if we fix the cows). Moreover, “cows” and “days” are inversely proportional (if we fix the milk). Thus we can establish the proportionality relationships by fixing one quantity and looking at the others.

Mathematically speaking, the function M (milk) is given by the product of C (cows) and D (days) times a constant k that is the “milk per cow per day” and which is simply the value for M when C=1 and D=1. And then the proportionality happens between M and C (by fixing D) and between M and D (by fixing C). One could fix, mathematically speaking, any pair of values for C and D. The function relating C,D,M is a graph (in the three dimensional space) over the real CD-plane with values at the M-axis.

Strictly speaking, the above equality M=kCD is a “multivariable proportion”. Notice that the “per per per…” would only make sense if the corresponding quantities can all be simultaneously set to 1. It would be different having three direct proportional quantities like in the example “each dwarf has two hats and three apples”, with three proportionality constants. In this case, it would not make sense to consider the amount of apples per dwarf per hat (because we will never have one dwarf and one hat). Notice that the “per per” is only for mathematical use, so it could refer to values that in reality do not make sense (e.g. the price per quarter-candy would make sense mathematically although nobody has quarters of candies).