PhD-FSTM-2022-060
The Faculty of Science, Technology and Medicine

Faculté des sciences et technologie

DISSERTATION

Defense held on 20/05/2022 in Luxembourg
to obtain the degree of

DOCTEUR DE L'UNIVERSITÉ DU LUXEMBOURG EN MATHEMATIQUES
AND
DOCTEUR DE L'UNIVERSITÉ PARIS-EST CRÉTEIL EN MATHEMATIQUES

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FRACTAL DIMENSION AND POINT-WISE PROPERTIES OF TRAJECTORIES OF FRACTIONAL PROCESSES

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To the soul of my grandfather ..

Abstract

The topics of this thesis lie at the interference of probability theory with dimensional and harmonic analysis, accentuating the geometric properties of random paths of Gaussian and non-Gaussian stochastic processes. Such line of research has been rapidly growing in past years, paying off clear local and global properties for random paths associated to various stochastic processes such as Brownian and fractional Brownian motion. In this thesis, we start by studying the level sets associated to fractional Brownian motion using the macroscopic Hausdorff dimension. Then as a preliminary step, we establish some technical points regarding the distribution of the Rosenblatt process for the purpose of studying various geometric properties of its random paths. First, we obtain results concerning the Hausdorff (both classical and macroscopic), packing and intermediate dimensions, and the logarithmic and pixel densities of the image, level and sojourn time sets associated with sample paths of the Rosenblatt process. Second, we study the pointwise regularity of the generalized Rosenblatt and prove the existence of three kinds of local behavior: slow, ordinary and rapid points.

In the last chapter, we illustrate several methods to estimate the macroscopic Hausdorff dimension, which played a key role in our results. In particular, we build the potential theoretical methods. Then, relying on this, we show that the macroscopic Hausdorff dimension of the projection of a set $E \subset \mathbb{R}^2$ onto almost all straight lines passing through the origin in $\mathbb{R}^2$ depends only on E, that is, they are almost surely independent of the choice of straight line.

Keywords: Fractional Brownian motion, Rosenblatt process, Image set, Level set, Sojourn times, Wavelet series, Slow/Ordinary/Rapid points, Fractal dimensions, Macroscopic Hausdorff dimension, Potential theory for dimensions, Projection theorem.

Acknowledgements

Throughout the writing of this dissertation I have received a great deal of support and assistance.

First and foremost I would like to deeply thank my thesis advisors, Ivan Nourdin and Stéphane Seuret, whose sincerity and encouragement I will never forget. They have been my source of inspiration as I hurdled through the path of this PhD degree. This thesis would not have been possible without them. Their guidance from the initial step in research enabled me to develop an understanding of the subject. I am thankful for the extraordinary experiences they arranged for me and for providing opportunities for me to grow professionally. It is an honor to learn from Prof. Nourdin and Prof. Seuret.

I express my deep gratitude to all the jury members, Ivan Nourdin, Stéphane Seuret, Giovanni Peccati, Stéphane Jaffard, Céline Esser and Yimin Xiao. Thank you for agreeing to read the manuscript and to participate in the defense of this thesis. It is a great honor for me to present my work in front of you.

My appreciation also extends to my colleagues in Luxembourg: thanks to my office mates, Valentin and Leonardo: sharing the office with you guys made my days more enjoyable. I thank Guenda, James, George, Massimo, Luca, Christian, Alexandre, Nicola, Robert, Mikolaj, Krzysztof, Tara, Xiaochuan, Arturo: on both a professional and a personal level, I am grateful for the time we spent together inside and outside the department, our lunch breaks, discussions, and ping-pong matches (even though I am a bad player). I would like also to thank my colleagues in Paris for the enjoyable time we spent during my frequent visits: in particular Elie, Edouard, and Guillaume: thank you guys for always encouraging and pushing me to speak French.

I thank our secretaries, Marie and Katharina, together with Catherine from the doctoral office for maintaining the best possible work environment by taking care of the administration procedures.

I want to thank my friends, I could not have completed this dissertation without your support guys: Ghina, Sara, Rancy, Elie, Joseph, Nadine, Nada, Adnan, Bob, Rim and Vita who provided stimulating discussions as well as happy distractions to rest my mind outside of my research.

Last but not least, I am grateful for my parents, Raif and Sanaa, whose constant love and support keep me motivated and confident. My accomplishments and success are because they believed in me. Deepest thanks to my siblings, Taher and Sara, who keep me grounded, remind me of what is important in life, and are always supportive of my adventures. Finally, I owe my deepest gratitude to Ryan, my love. I am forever thankful for the unconditional love and support throughout the entire thesis process and every day.

I acknowledge support from the FNR grant APOGee (R-AGR-3585-10) at the University of Luxembourg, in the framework of which this thesis was elaborated.

Table of Contents

List of Publications

The following list of publications contains the work presented in this thesis:

1. L. Daw. **"A uniform result for the dimension of fractional Brownian motion level sets."** Statistics and Probability Letters 169 (2021): 108984.
2. L. Daw and G. Kerchev. **"Fractal dimensions of the Rosenblatt process."** Submitted.
3. L. Daw and L. Looseveldt. **"Wavelet methods to study the pointwise regularity of the generalized Rosenblatt process."** Submitted.
4. L. Daw and S. Seuret. **"Potential method and projection theorems for Macroscopic Hausdorff Dimension."** This work is at an advanced stage and is to be submitted in the near future.

Chapter I

Introduction

I.1 Preliminaries

In this section, we provide an overview of the main theoretical tools that will be used in this manuscript. Our exposition is divided into 3 main building blocks:

- In Section I.1.1, we introduce the class of self-similar stochastic processes with stationary increments (SSSI processes). Section I.1.1.1 presents the *fractional Brownian motion* and its local times, whereas Section I.1.1.2 is dedicated to define the *Generalized Rosenblatt process* which is investigated intensively in our work, and finally Section I.1.1.3 introduces the *Hermite processes*.
- In Section I.1.2, we define several fractal dimensions that are key tools in this manuscript. We mainly divide them onto two groups depending on the properties they reflect, local or global properties. Section I.1.2.2.2 is dedicated to the so-called *macroscopic Hausdorff dimension*, which plays a pivotal role in our work.
- In Section I.1.3, we expose basic wavelet tools and illustrate the so-called *wavelet leaders* methods.

The goal behind this expository part is to present the necessary material in a self-contained way, hopefully allowing the reader to follow it easily without further referencing.

I.1.1 Self-similar processes with stationary increments

Here and throughout the thesis, every random object is defined on an appropriate probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The symbols '$\mathbb{E}$', '$Var$' and '$Cov$' denote, respectively, the expectation, the variance and the covariance associated with $\mathbb{P}$.
A stochastic process $(X_t)_{t\geq 0}$ is a $\mathbb{R}$-valued random function on $\mathbb{R}$. Two stochastic processes X and Y have the same distribution (noted $X \overset{(d)}{=} Y$) if they have the same finite-dimensional distributions.

Self-similar processes are stochastic processes that are invariant in distribution under suitable scaling of time and space. Formally speaking, a stochastic process $(X_t)_{t\geq 0}$ is said to be self-similar with exponent $H > 0$ if

$$(X_{ct})_{t\geq 0} \overset{(d)}{=} (c^H X_t)_{t\geq 0} \text{ for all } c > 0,$$

and has stationary increments if

$$(X_{t+t_0} - X_{t_0})_{t\geq 0} \overset{(d)}{=} (X_t)_{t\geq 0} \text{ for all } t_0 \in \mathbb{R}.$$

Self-similar processes with stationary increments (SSSI processes) appear as limits in various normalization procedures [65, 102, 111]. In applications, they occur in various fields such as finance, hydrology, biomedicine and image processing. The simplest SSSI processes are the Brownian motion and, more generally, Lévy stable motions. A broad class of SSSI processes which belongs to the homogeneous Wiener chaos of an arbitrary order $N \geq 1$ are *Hermite processes* of rank N. They generalize the *fractional Brownian motion* and *the Rosenblatt process*. In the following proposition, we discuss some properties of SSSI processes.

Proposition I.1.1. *[112] Fix $H \in (0,1]$ and let $(X_t)_{t\geq 0}$ be an H-self-similar stochastic process with stationary increments. Then the following properties hold:*

1. *$X_0 = 0$ almost surely.*
2. *If $H \neq 1$, then $\mathbb{E}[X_t] = 0$, for all $t \in \mathbb{R}$.*
3. *If $H = 1$, then $X_t = tX_1$ almost surely for $t \in \mathbb{R}$.*
4. *If $\mathbb{E}[X_1^2] < \infty$, then the covariance function of the process X is given by*

$$\mathbb{E}[X_t X_s] = \frac{\mathbb{E}[X_1^2]}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

SSSI processes, in particular the Hermite processes, are defined with the aid of a multiple stochastic integral called Wiener-Itô integral. One mentions that two classical books on Wiener chaoses, multiple Wiener integrals and related topics are [53, 89]. First of all, let us define the multiple Wiener-Itô integrals.

Definition I.1.2. The multiple Wiener-Itô integral of order $k \geq 1$ is defined for any $f \in L^2(\mathbb{R}^k)$ as

$$I_k(f) = \int'_{\mathbb{R}^k} f(x_1, ..., x_k) dB(x_1)...dB(x_k),$$

where B is Brownian motion viewed as a random integrator, and $\int'_{\mathbb{R}^k}$ denotes integration over $\mathbb{R}^k$ excluding the diagonals.

Remark 1. The set of random variables $I_k(f)$ forms the k-th Wiener chaos when f varies in $L^2(\mathbb{R}^k)$. Moreover, $I_k(f)$ has the following properties:

1. $I_k(.)$ is a linear mapping from $L^2(\mathbb{R}^k)$ to $L^2(\Omega)$.
2. If $f_\sigma(x_1, ..., x_k) := f(x_{\sigma(1)}, ..., x_{\sigma(k)})$, where σ is a permutation, then $I_k(f_\sigma) = I_k(f)$. As a result if we denote by $\hat{f}$ the symmetrization of f, namely
$$\hat{f}(x_1, ..., x_k) := \frac{1}{k!} \sum_{\sigma \in \mathcal{S}_d} f_\sigma(x_1, ..., x_k),$$
then $I_k(f) = I_k(\hat{f})$ for all $f \in L^2(\mathbb{R}^k)$.
3. For $f \in L^2(\mathbb{R}^q)$ and $g \in L^2(\mathbb{R}^p)$, one has
$$\mathbb{E}\left[I_q(f) I_p(g)\right] = \begin{cases} \frac{1}{p!} \int_{\mathbb{R}^k} \hat{f}(x)\hat{g}(x)dx, & \text{if } q = p \\ 0 & \text{if } q \neq p \end{cases}.$$

In the next three sections, we define three SSSI process. We start with the fractional Brownian motion which is the simplest Gaussian Hermite process, then we move to the Rosenblatt proces which is the simplest non-Gaussian Hermite process, and finally we introduce the class of Hermite processes. Our aim is to provide definitions and properties of these processes allowing one to have all the essential tools for the coming chapters.

I.1.1.1 Fractional Brownian Motion

Brownian motion is a random phenomenon, of central theoretical importance. Nevertheless it often appears as too restrictive for applications. Brownian motion is the unique Gaussian process, which has stationary increments that are independent and of finite variance with mean 0. To obtain a less restrictive model, it is necessary to relax one or more of these conditions.

Fractional Brownian motion is a generalization of Brownian motion, which has stationary increments that are normally distributed but no longer independent. Fractional Brownian motion, which was first itroduced by Kolmogorov [62] and further developed by Mandelbrot and Van Ness [74], is defined as follows.

Definition I.1.3. Let $H \in (0, 1]$. A fractional Brownian motion (FBM) of Hurst index H is a centered continuous Gaussian process $B^H = \left(B_t^H\right)_{t>0}$ with covariance function
$$R_H(t, s) := \mathbb{E}[B_t^H B_s^H] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t - s|^{2H}\right). \tag{I.1.1}$$

Fractional Brownian motion exists for all $H \in (0, 1]$. Moreover, it admits a version with continuous paths, and for every $t \geq 0$ and $s > 0$ the increment $B_{t+s}^H - B_t^H$ has normal distribution with mean zero and variance s^{2H}, so that
$$\mathbb{P}(B_{t+s}^H - B_t^H \leq x) = \frac{1}{s^H \sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(\frac{-u^2}{2s^{2H}}\right) du,\ x \in \mathbb{R}.$$
Figure I.1 shows sample paths of fractional Brownian motion for various H. The $\frac{1}{2}$-indexed fractional Brownian motion is simply the standard Brownian motion. As we can see in the figure, the smoothness of the path increases with H.

Proposition I.1.4. *[86] Let B^H be a fractional Brownian motion of Hurst parameter $H \in (0,1]$. Then B^H has the following properties:*

(1) Self-similarity: *The processes $\left\{B^H_{ct},\, t \geq 0\right\}$ and $\left\{c^H B^H_t,\, t \geq 0\right\}$ have the same distribution.*

(2) Stationary increments: *The distribution of the process $\left\{B^H_{t+s} - B^H_s,\, t \geq 0\right\}$ does not depend on $s \geq 0$.*

(3) Time inversion: *The processes $\left\{B^H_t,\, t \geq 0\right\}$ and $\left\{t^{2H} B^H_{1/t},\, t \geq 0\right\}$ have the same distribution.*

(4) Brownian filtration: *The natural filtration associated to a fractional Brownian motion is Brownian, i.e., there is a Brownian motion $(B_t)_{t\geq 0}$ defined on the same probability space than B^H such that its filtration satisfies*

$$\sigma\left\{B^H_s \,:\, s \leq t\right\} \subset \sigma\left\{B_s \,:\, s \leq t\right\}, \tag{I.1.2}$$

for all $t > 0$.
Conversely, any continuous Gaussian process $B^H = (B^H_t)_{t\geq 0}$ with $B^H_0 = 0$, and $Var(B^H_1) = 1$, and such that (1) and (2) hold, is a fractional Brownian motion of index H.

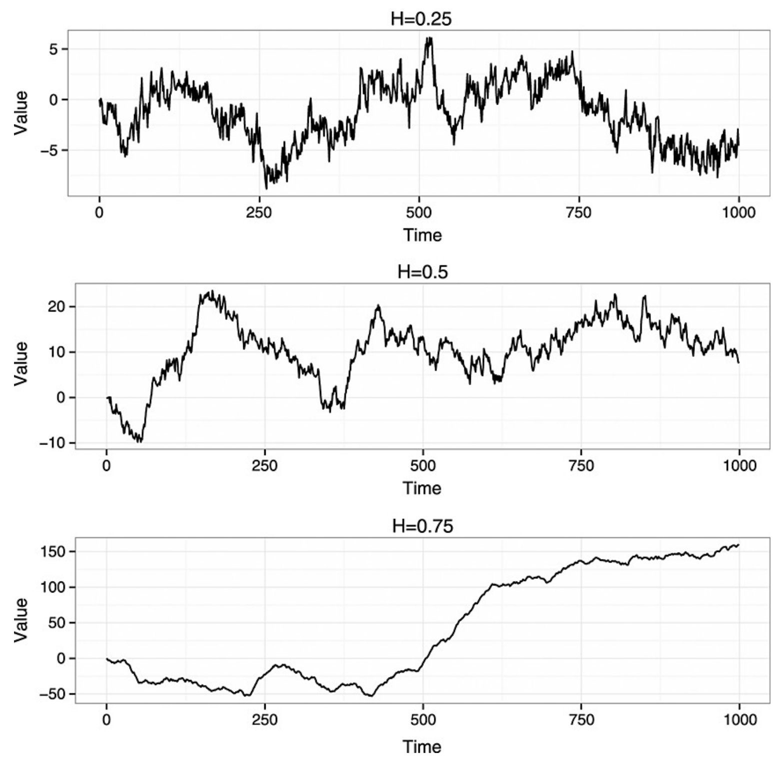

Figure I.1: Simulation of a Fractional Brownian motion of Hurst index $H = 0.25, 0.5, 0.75$. [108]

A fractional Brownian motion with Hurst index H admits Hölder continuous paths for all exponents less than H (see e.g. [86]).

Proposition I.1.5. *Let B^H be a fractional Brownian motion of Hurst parameter $H \in (0,1]$. If $0 < \delta < H$ and $T > 0$ then, with probability 1, there exists a random constant C_T such that*

$$|B^H_{t+s} - B^H_t| \leq C_T |s|^\delta$$

for all $t, s \in [0,T]$.

As we will see, the use of the local time will play a key role throughout Chapter II. Provided it exists, the local time $x \mapsto L(x,t)$ of a process $(X_t)_{t\geq 0}$ is, for each t, the density of the occupation measure $\mu_t(A) = \lambda(\{s \in [0,t] : X_s \in A\})$ associated with X, where λ stands for the Lebesgue measure; otherwise stated, one has $L(.,t) = \frac{d\mu_t}{d\lambda}$.
The case of Gaussian (and centered, say) processes has been widely studied in the literature. For instance, one of the main striking results in the Gaussian framework (see e.g. Dozzi [34]), in particular for fractional Brownian motion, is the following condition ensuring the existence of $(L^x_t)_{t\in[0,T],x\in\mathbb{R}}$ in $L^2(\Omega)$.

$$I := \int\int_{[0,T]^2} \frac{ds\,dt}{\sqrt{R_H(s,s)R_H(t,t) - R_H(s,t)^2}} < +\infty, \tag{I.1.3}$$

where $R_H(s,t) = \mathbb{E}\left(B^H_s B^H_t\right)$; moreover, in this case we have the Fourier type representation:

$$L(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} dy \int_0^t du\, e^{iy(B^H_u - x)}. \tag{I.1.4}$$

As B^H is selfsimilar and satisfies (I.1.3), then it is immediate from (I.1.4) that its local time at level x also has some selfsimilarity properties in time with index $1-H$ but with a different level, as stated below.

Proposition I.1.6. *Let $c > 0$. Assume B^H is a fractional Brownian motion of Hurst index $H \in (0,1)$ and consider its local time $(L(x,t))_{t\geq 0,x\in\mathbb{R}}$. One has*

$$(L(x,ct))_{t\geq 0,x\in\mathbb{R}} \overset{(d)}{=} c^{1-H}(L(c^{-H}x,t))_{t\geq 0,x\in\mathbb{R}}. \tag{I.1.5}$$

Finally, the local time is Hölder continuous in both time and space. In particular

Proposition I.1.7. *[17] For every $x \in \mathbb{R}$, almost surely, the local time $L(x,t)$ is β-Hölder continuous in t for every $\beta \in [0, 1-H]$.*

Proposition I.1.8. *[44, Theorem 26.1] Assume B^H is a fractional Brownian motion of Hurst index $H \in (0,1)$ and consider its local time $(L(x,t))_{x\in K}$, where K is a given compact interval in $\mathbb{R}$. Then, for all $\beta \in \left(0, \frac{1}{2}\left(\frac{1}{H} - 1\right)\right)$ and for all $t \geq 0$,*

$$\mathbb{P}\left(\sup_{x,y\in K} \frac{|L(x,t) - L(y,t)|}{|x-y|^\beta} < \infty\right) = 1. \tag{I.1.6}$$

I.1.1.2 Generalized Rosenblatt Process

Like the fractional Brownian motion, the Rosenblatt process is a selfsimilar stochastic process with stationary increments. Both processes belong to the class of Hermite processes, fractional Brownian motion being of order 1 while Rosenblatt process is of order 2. However, unlike the fractional Brownian motion, the Rosenblatt process is not Gaussian.

Before giving a formal definition of the Rosenblatt process, let us recall some important notions related to the Hermite polynomials which are essential for our coming definitions. For $m \geq 0$, the Hermite polynomial of degree m is given by

$$H_m(x) := (-1)^m e^{\frac{x^2}{2}} \frac{d^m}{dx^m} e^{-\frac{x^2}{2}}.$$

Definition I.1.9. Given a function $f \in L^2(\mathbb{R})$, we say that f has Hermite rank equal to k if $\mathbb{E}[f(\xi)H_m(\xi)] = 0$ for $m \leq k-1$ and $\mathbb{E}[f(\xi)H_k(\xi)] \neq 0$, where $\xi \sim N(0,1)$.

The Rosenblatt process appears in the limit of Non-Central Limit Theorem of [19]. Formally speaking, consider a stationary Gaussian sequence $(\xi_n)_{n\geq 0}$ with mean zero and variance 1 such that, for all $n \geq 0$, one has

$$\mathbb{E}(\xi_0 \xi_n) = n^{H-1} L(n),$$

where $H \in (\frac{1}{2}, 1)$ and L is a slowly varying [1] function at infinity. Let f be a function such that $\mathbb{E}(f(\xi_0)) = 0$ and $\mathbb{E}(f(\xi_0)^2) < \infty$. Suppose that f has Hermite rank equal to 2. Then the Non-Central Limit Theorem of [19] asserts that

$$\frac{1}{n^H} \sum_{j=1}^{\lfloor nt \rfloor} f(\xi_j)$$

converges as $n \to \infty$ in the sense of finite dimensional distributions to the process

$$R_t^H = c_H \int_{\mathbb{R}^2}' \int_0^t (s-x_1)_+^{H_0 - \frac{3}{2}} (s-x_2)_+^{H_0 - \frac{3}{2}} \, ds \, dB(x_1) dB(x_2), \tag{I.1.7}$$

where

$$x_+ = \max(x, 0) \text{ and } H_0 = \frac{H+1}{2}.$$

The above integral is Wiener-Itô stochastic integrals with respect to Brownian motion $(B(t))_{t\in\mathbb{R}}$ (see Definition I.1.2), and the constant c_H is positive and satisfies $\mathbb{E}((R_1^H)^2) = 1$. The process $(R_t^H)_{t\geq 0}$ is called *the Rosenblatt process* (it was actually been named in this way by Taqqu in [113]) and it is non-Gaussian, H-selfsimilar, with stationary increments. In addition it has the same second order properties as fractional Brownian motion, namely,

$$\mathbb{E}[(R_t^H)^2] = t^{2H}, \quad \mathbb{E}[R_t^H R_s^H] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}). \tag{I.1.8}$$

[1] A positive measurable function L is said to be slowly varying if $\lim_{t\to+\infty} \frac{L(xt)}{L(t)} = 1$ for all $x > 0$.

We call (I.1.7) the *time-domain representation*. It is known that Rosenblatt process admits other representations in terms of Wiener-Itô integrals, among which we note the spectral-domain representation (see [113] and [33]):

$$R^H_{\cdot} \overset{(d)}{=} C(H) \int_{\mathbb{R}^2} \frac{e^{i(x+y)\cdot} - 1}{i(x+y)} Z_G(dx) Z_G(dy), \tag{I.1.9}$$

where the double Wiener-Ito integral is taken over $x \neq \pm y$ and $Z_G(dx)$ is a complex-valued random white noise with control measure G satisfying $G(tA) = t^{1-H} G(A)$ for all $t \in \mathbb{R}$ and $G(dx) = |x|^{-H} dx$. The constant $C(H)$ in (I.1.9) is such that $\mathbb{E}((R^H_1)^2) = 1$.

Remark 2. Note that in the notation of [113], $Z_G(dx) = |x|^{-H/2} d\hat{B}(x)$, with $(B(t))_{t\in\mathbb{R}}$ the Brownian motion and $d\hat{B}(x)$ is viewed as the complex-valued Fourier transform of $dB(x)$. For more details, see [111].

In Chapter III, our main interest consists in studying the geometric properties of the random paths of the Rosenblatt process. In this analysis, the local time of the Rosenblatt process plays a pivotal role. Its existence was shown in [106] together with this L^2 representation:

$$L(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_0^t e^{i\xi(x - R^H_s)} ds d\xi. \tag{I.1.10}$$

As mentioned before, since R^H is selfsimilar of index H, its local time at level x also has some self-similarity properties in time with index $1-H$, but with a different level. More precisely, one has, for every $c > 0$:

$$(L(x,ct))_{t\geq 0, x\in\mathbb{R}} \overset{(d)}{=} c^{1-H} (L(c^{-H}x, t))_{t\geq 0, x\in\mathbb{R}}. \tag{I.1.11}$$

In Chapter IV we study the generalized Rosenblatt process which is a generalization of the Rosenblatt process introduced in [71], and is defined as follows:

Definition I.1.10. Given two parameters $H_1, H_2 \in (\frac{1}{2}, 1)$ such that $H_1 + H_2 > \frac{3}{2}$, the generalized Rosenblatt process $\{R_{H_1,H_2}(t,\cdot)\}_{t\in\mathbb{R}_+}$ is defined as a double Wiener-Itô integral of a kernel function K_{H_1,H_2} with respect to a given Brownian motion. More precisely, consider a standard two-sided Brownian motion B, and set

$$R_{H_1,H_2}(t,\cdot) = \int'_{\mathbb{R}^2} K_{H_1,H_2}(t, x_1, x_2)\, dB(x_1) dB(x_2), \tag{I.1.12}$$

where $\int'_{\mathbb{R}^2}$ denotes integration over $\mathbb{R}^2$ excluding the diagonal. The kernel function in (I.1.12) is expressed, for all (t, x_1, x_2) on $\mathbb{R}_+ \times \mathbb{R}^2$, by

$$K_{H_1,H_2}(t,x_1,x_2) = \frac{1}{\Gamma\left(H_1 - \frac{1}{2}\right)\Gamma\left(H_2 - \frac{1}{2}\right)} \int_0^t (s - x_1)_+^{H_1 - \frac{3}{2}} (s - x_2)_+^{H_2 - \frac{3}{2}}\, ds,$$

where Γ stands for the usual Gamma Euler function, and where for $(x, \alpha) \in \mathbb{R}^2$

$$x_+^\alpha = \begin{cases} x^\alpha & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Note that the (standard) Rosenblatt process defined in (I.1.7) is the process $\{R_{H,H}(t,\cdot)\}_{t\in\mathbb{R}_+}$ for $H \in (3/4, 1)$. The generalized Rosenblatt process $\{R_{H_1,H_2}(t,\cdot)\}_{t\in\mathbb{R}_+}$ is non-Gaussian, belongs to the second Wiener chaos, has stationary increments, and is $(H_1 + H_2 - 1)$-self-similar.

I.1.1.3 Hermite processes

Hermite processes are SSSI processes that naturally arise as limits of normalized sums of long-range dependent random variables [33]. Since the seminal works of Taqqu [110, 111], the class of Hermite processes has attracted considerable interest in probability and statistics. A Hermite process can be defined by any of its equivalent representations. Here by equivalent representations, we mean that the represented processes share the same finite-dimensional distributions. The most well known representation is the *time domain* representation in terms of multiple Wiener-Itô integrals.

Definition I.1.11. Fix an integer $N \geq 1$ and a real number $H \in (1 - 1/(2N), 1)$. The Hermite process of rank N and parameter H is defined through the multiple Wiener integral with respect to Brownian motion $(B(t))_{t\in\mathbb{R}}$:

$$Z_H^N(t) = c_{N,H} \int'_{\mathbb{R}^N} \left(\int_0^t \prod_{p=1}^N (s - x_p)_+^{H-3/2} ds \right) dB(x_1), ..., dB(x_N), \tag{I.1.13}$$

where $x_+ = \max(x, 0)$, and $c_{N,H}$ is some positive constant that makes $\mathrm{Var}(Z_H^N(1)) = 1$.

Remark 3. The Hermite process Z_H^N has stationary increments and is self-similar with Hurst index H. When the rank $N = 1$, we recover the classical (Gaussian) fractional Brownian motion. When $N \geq 2$, the law of Z_H^N is non-Gaussian, and if $N = 2$, the process is also known as the Rosenblatt process.

Another important representation is the *spectral domain* representation which is given by

$$Z_H^N(t) = C_{N,H} \int''_{\mathbb{R}^N} \frac{e^{it(x_1+...+x_N)} - 1}{i(x_1 + ... + x_N)} \prod_{p=1}^N |x_p|^{1/2-H}\, d\hat{B}(x_1), ..., d\hat{B}(x_N), \tag{I.1.14}$$

where $\hat{B}$ is a complex-valued Hermitian Gaussian random measure (see [96, Definition B.1.3]) with Lebesgue control measure, the double prime " at the top of the integral sign indicates the exclusion of the hyperdiagonals $x_i = \pm x_j$, $i \neq j$, in the N-tuple stochastic integral, and $C_{N,H}$ is a constant such that $Var(Z_H^N(1)) = 1$.

Remark 4. All Hermite processes with Hurst index H, regardless of the order, share the same covariance structure as a standard fractional Brownian motion, that is,

$$\mathbb{E}[Z_H^N(t) Z_H^N(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).$$

I.1.2 Fractal Dimensions

From an early age, we learned that straight lines and curves have dimension 1, planes and surfaces have dimension 2, solids such as a ball have dimension 3, and so on. More properly, we say that a set is n-dimensional if we need n independent variables to describe a neighborhood of any point. However, one can map a real line into a plane bijectively and continuously. In other words, a one-dimensional curve can cover a two-dimensional plane completely, which is known by space-filling curve. Fractal geometry generalizes this notion of dimension to a wider class of sets by defining non-integer dimensions. Roughly speaking, these fractal dimensions measures how much space is occupied near each point of a set. In this section we give an overall summary of all fractal dimensions used in this manuscript. We split this section into three subsections:

1. Dimensions reflecting microscopic properties: We introduce Hausdorff, box, packing, and intermediate dimensions, following [38, 39, 22, 40, 48].

2. Dimensions reflecting macroscopic properties: We introduce another group of dimensions which are logarithmic density, pixel density, and macroscopic Hausdorff dimension, following [59, 61, 12, 11].

3. Overview of the types of fractal dimensions: We compare all dimensions mentioned above and give a few relations between them to give the reader an intuition.

Our main bibliographic sources serving as guiding inspiration for this section is the book by Falconer [38]. In this section, we let $(\mathbb{R}^d, \|\|_2)$ be the d-dimension Euclidean space equipped with its usual L^2-norm.

I.1.2.1 Dimensions reflecting microscopic properties

I.1.2.1.1 Box dimensions

What is the relation between an object length (area or volume) and its diameter? This question leads us to think about dimensions from different perspectives. To clarify our idea, let us consider some examples. When we want to cover a unit square with little squares of side δ, we obviously need $1/\delta^2$ little squares. Whereas, if we want to cover a unit cube, we will need exactly $1/\delta^3$ of little cubes of diameter δ. We note that the exponents we got here are the dimensions of the objects that we are covering, which is not a coincidence. This was the main idea behind the box dimension, also known as Minkowski–Bouligand dimension, that determines the fractal dimension of a set $F \subset \mathbb{R}^d$ using box-counting analysis. Formally speaking, given a non-empty set $F \subset \mathbb{R}^d$, we call $\{U_i\}$ an exact δ-cover of F if $\{U_i\}$ is a countable or finite collection of sets with diameter equal to δ covering F, i.e. $F \subset \cup_i U_i$. Recall that for a set $U \subset \mathbb{R}^d$, the *diameter* of U is defined as $|U| = \sup\{\|x-y\|_2 : x, y \in U\}$. Moreover, $N_\delta(F)$ denotes the smallest number of exact δ-covers of F. The lower and upper box dimension of F are defined respectively by:

$$\underline{\dim}_B(F) = \liminf_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta} \tag{I.1.15}$$

$$\overline{\dim}_B(F) = \limsup_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta}. \tag{I.1.16}$$

Obviously, $\underline{\dim}_B(F) \le \overline{\dim}_B(F)$, and if these are equal, their common value refers to the box dimension and is denoted by

$$\dim_B(F) = \lim_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta}. \tag{I.1.17}$$

If $s = \dim_B(F)$, then (I.1.17) roughly states that $N_\delta(F) \sim \delta^{-s}$ for δ small enough, or more precisely one has

$$\lim_{\delta\to 0} N_\delta(F)\delta^s = \begin{cases} \infty & \text{if } s < \dim_B(F) \\ 0 & \text{if } s > \dim_B(F) \end{cases}$$

Motivated by the above limit, we want to mention another equivalent definition for the lower and upper box dimensions, which is more convenient to use.

Definition I.1.12. For a given set $F \subset \mathbb{R}^d$, the *lower box dimension* is given by:

$$\underline{\dim}_B(F) := \inf\left\{ s \ge 0 : \begin{array}{c} \forall\varepsilon > 0, \exists \text{ cover } \{U_i\}_{i=1}^\infty \text{ of } F, \text{ s.t.} \\ |U_i| = |U_j|\ \forall i,j \text{ and } \sum_{i=1}^\infty |U_i|^s \le \varepsilon \end{array} \right\}. \tag{I.1.18}$$

Similarly, we define the *upper box dimension*:

$$\overline{\dim}_B(F) := \inf\left\{ s \ge 0 : \begin{array}{c} \forall\varepsilon > 0, \exists\delta > 0, \forall \text{ cover } \{U_i\}_{i=1}^\infty \text{ of } F, \text{ s t.} \\ |U_i| \le \delta,\ |U_i| = |U_j|\ \forall i,j \text{ and } \sum_{i=1}^\infty |U_i|^s \le \varepsilon \end{array} \right\}. \tag{I.1.19}$$

Releasing the constraints on the covers of F, we get another fractal dimension, called Hausdorff dimension, that we introduce in the next section.

I.1.2.1.2 Hausdorff dimensions

The Hausdorff dimension is one of the oldest fractal dimensions. It can be defined for any set, and its definition is based on a measure which gives it some advantages on other dimensions. For a given non-empty set $F \subset \mathbb{R}^d$, we call $\{U_i\}$ an δ-cover of F if $\{U_i\}$ is a countable or finite collection of sets with diameter at most δ covering F, i.e. $F \subset \cup_i U_i$. To define the Hausdorff dimension we start by defining the Hausdorff measure. For $F \subset \mathbb{R}^d$, $s \ge 0$, and $\delta > 0$, define

$$\mathcal{H}^s_\delta(F) = \inf\left\{ \sum_{i=1}^\infty |U_i|^s \,:\, \{U_i\}_{i=1}^\infty \text{ is a } \delta\text{-cover of } F \right\}, \tag{I.1.20}$$

where the infimum is taken over all possible covers with diameter at most δ. Moreover, as δ decreases, the set of possible covers decreases too, and so $\delta \mapsto \mathcal{H}^s_\delta(F)$ is an non-decreasing function. On the other hand, for a given $\delta < 1$, $s \mapsto \mathcal{H}^s_\delta(F)$ is non-increasing, and for $s < t$, one has

$$\mathcal{H}^t_\delta(F) \leq \delta^{t-s}\mathcal{H}^s_\delta(F). \tag{I.1.21}$$

We define the s-dimensional Hausdorff measure by

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F), \tag{I.1.22}$$

where the value of the limit belongs to $[0, \infty]$. Moreover, by (I.1.21) if $\mathcal{H}^s(F)$ is finite, then letting $\delta \to 0$, one can observe that $\mathcal{H}^t(F) = 0$. As a result, for a given set $F \subset \mathbb{R}^d$, there exists a critical value of s at which $\mathcal{H}^s(F)$ jumps from ∞ to 0. This critical value is called the Hausdorff dimension of F. Rigorously speaking

$$\dim_H(F) = \inf\{s \geq 0 \,:\, \mathcal{H}^s(F) = 0\} = \sup\{s \geq 0 \,:\, \mathcal{H}^s(F) = +\infty\} \tag{I.1.23}$$

with $\sup \emptyset = 0$ by convention. Then, one has

$$\mathcal{H}^s(F) = \begin{cases} \infty & \text{if } 0 \leq s < \dim_H(F) \\ 0 & \text{if } s > \dim_H(F) \end{cases}.$$

As illustrated in Figure I.2, the Hausdorff measure jumps from ∞ to zero. If $s = \dim_H(F)$, $\mathcal{H}^s(F) \in [0, +\infty]$. Moreover, if $0 < \mathcal{H}^s(F) < +\infty$, F is said to be an s-set.

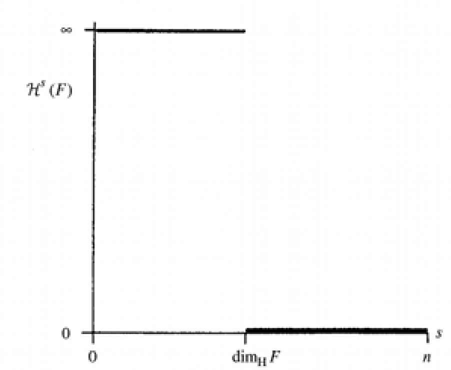

Figure I.2: The Hausdorff measure $\mathcal{H}^s(F)$ of a set F as a function of s [38].

Using the definition of limit together with (I.1.22, I.1.23), one can also define the Hausdorff dimension of F as follows

Definition I.1.13. For $F \subset \mathbb{R}^d$, the Hausdorff dimension of F is given by

$$\dim_H(F) := \inf\left\{ s \geq 0 \,:\, \forall \varepsilon > 0, \exists \text{ cover } \{U_i\}_{i=1}^\infty \text{ of } F, \text{ s.t. } \sum_{i=1}^\infty |U_i|^s \leq \varepsilon \right\}. \tag{I.1.24}$$

If we compare the Hausdorff dimension with the box dimension (see Definition I.1.12), we see that the box dimension is more restrictive with the covering of the set, i.e. all covers should have same diameter.

At the end of this section, we introduce two well-known techniques for calculating the Hausdorff dimension. Firstly we start by the potential theoretical methods which are mainly based on integral analysis. To this end we define, for $s \geq 0$, the s-potential at a point $x \in \mathbb{R}^d$ resulting from a mass distribution [2] μ on $\mathbb{R}^d$:

$$\phi_s(x) = \int \frac{d\mu(y)}{\|x-y\|^s},$$

and the s-energy of μ:

$$I_s(\mu) = \int \phi_s(x) d\mu(x) = \int \frac{d\mu(x) d\mu(y)}{\|x-y\|^s}.$$

The following theorem states the potential theoretical methods which are usually used to bound the Hausdorff dimension from below.

Theorem I.1.14. *[40, Theorem 4.13] Let F be a subset of $\mathbb{R}^d$.*

1. *If there exists a mass distribution μ on F such that $I_s(\mu) < \infty$, then $H^s(F) = \infty$ and $dim_H(F) \geq s$.*
2. *If F is a Borel set with $0 < H^s(F) \leq \infty$ then, for all $0 < t < s$, there exists a mass distribution μ of F with $I_t(\mu) < \infty$.*

The second technique, called the mass distribution principle, is based on a given measure μ, and estimating the μ-mass of small sets in order to bound the Hausdorff dimension from below too.

Theorem I.1.15. *[40, Section 4] Let μ be a mass distribution on F and suppose that for some $s > 0$, there is a number $c > 0$ and $\varepsilon > 0$ such that*

$$\mu(U) \leq c|U|^s$$

for all sets $|U| \leq \varepsilon$. Then $H^s(E) \geq \mu(F)/c$ and $dim_H(F) \geq s$.

I.1.2.1.3 Packing dimensions

With the Hausdorff dimension we are able to outpace most dimensions based on the fact that it is defined in terms of measures. In fact this is not the case for the box dimension (see definition I.1.12), though one can construct a measure based dimension, the packing dimension, which is in some sense "dual" to the Hausdorff dimension. To this end, for $F \subset \mathbb{R}^d$ and $s > 0$, let us recall the definition of the $s-$dimensional packing measure of F

[2] A mass distribution μ on $\mathbb{R}^d$ is a measure such that $0 < \mu(\mathbb{R}^d) < \infty$.

$$\mathcal{P}^s(F) := \inf\left\{\sum_n \mathcal{P}_0^s(F_n) : F \subseteq \bigcup_n F_n\right\},$$

where for $F \subset \mathbb{R}^d$,

$$\mathcal{P}_0^s(F) := \lim_{\varepsilon\to 0}\sup\left\{\sum_i (2r_i)^s : \overline{B}(x_i, r_i) \text{ are disjoint }, x_i \in F, r_i < \varepsilon\right\}.$$

Definition I.1.16. Given $F \subset \mathbb{R}^d$ and a Borel measure μ on $\mathbb{R}^d$, the packing dimension of F is

$$\dim_P(F) := \inf\{s > 0 : \mathcal{P}^s(F) = 0\}, \tag{I.1.25}$$

and the packing dimension of μ is defined by

$$\dim_P(\mu) := \inf\{\dim_P(F) \ : \ \mu(F) > 0 \text{ and } F \subset \mathbb{R}^d \text{ is a Borel set}\}.$$

Next, we recall the concept of packing dimension profiles first conceived by Falconer and Howroyd in [40] and [48]. For finite Borel measures μ on $\mathbb{R}^d$ and for any $s > 0$, let

$$F_s^\mu(x, r) = \int_{\mathbb{R}} \psi_s\left(\frac{x-y}{r}\right) d\mu(y)$$

be the potential with respect to the kernel $\psi_s(x) = \min\{1, \|x\|^{-s}\}$, $\forall x \in \mathbb{R}^d$.

Definition I.1.17. Let μ be a Borel measure on $\mathbb{R}^d$. The packing dimension profile of μ is defined as

$$\dim_{P,s}(\mu) = \sup\left\{\beta \geq 0 : \ \liminf_{r\to 0} \frac{F_s^\mu(x,r)}{r^\beta} = 0 \text{ for } \mu - \text{a.e.}\, x \in \mathbb{R}^d\right\}.$$

Now for any Borel set $F \subset \mathbb{R}^d$, we define $\mathcal{M}_c^+(F)$ to be the family of finite Borel measures with compact support in F. Then an equivalent definition of the packing dimension can be established.

Proposition I.1.18. *Given $F \subset \mathbb{R}^d$, the packing dimension of F is equal to*

$$dim_P(F) = \sup\left\{dim_P(\mu) \ : \ \mu \in \mathcal{M}_c^+(F)\right\}.$$

Motivated by this, Falconer and Howroyd [40] define s-dimensional packing dimension profile of $F \subset \mathbb{R}^d$ by

$$\dim_{P,s}(F) = \sup\left\{\dim_{P,s}(\mu) \ : \ \mu \in \mathcal{M}_c^+(F)\right\}.$$

It is easy to see that $0 \leq \dim_{P,s}(F) \leq s$, and $\dim_{P,s}(F) = \dim_P(F)$ for any $s \geq d$.

I.1.2.1.4 Intermediate dimensions

By comparing the covering restrictions between the box and the Hausdorff dimensions, Falconer et al [39] introduced a new continuum of dimensions intermediate between the box and the Hausdorff dimension, named intermediate dimensions. As we will see later in details, for a bounded and non-empty set $F \subset \mathbb{R}^d$, $\theta \in (0,1]$ and $s \in [0,d]$, they defined

$$H^s_{r,\theta}(F) = \inf\left\{\sum_i |U_i|^s \,:\, \{U_i\}_i \text{ is a cover of } F \text{ such that } r \leq |U_i| \leq r^\theta \text{ for all } i\right\}. \tag{I.1.26}$$

In particular, for $\theta = 0$, $H^s_{r,0}(F)$ is the s-dimensional Hausdorff measure of f. The following lemma enables us to define the intermediate dimensions.

Lemma I.1.19. *[22, Lemma 2.1] Let $\theta \in (0,1)$ and $F \subset \mathbb{R}^d$. For each $0 < r < 1$ and all $0 \leq t \leq s \leq d$,*

$$-(s-t) \leq \frac{\log(H^s_{r,\theta}(F))}{-\log r} - \frac{\log(H^t_{r,\theta}(F))}{-\log r} \leq -\theta(s-t)$$

In particular, there is a unique $s \in [0,d]$ such that $\liminf_{r\to 0} \dfrac{\log(H^s_{r,\theta}(F))}{-\log r} = 0$ and a unique $s \in [0,d]$ such that $\limsup_{r\to 0} \dfrac{\log(H^s_{r,\theta}(F))}{-\log r} = 0$. As a consequence, the intermediate dimensions are defined as in [22].

Definition I.1.20. Let $F \subset \mathbb{R}^d$ be bounded. For $0 \leq \theta \leq 1$, the lower θ-intermediate dimension is

$$\underline{\dim}_\theta(F) = \left(\text{the unique } s \in [0,d] \text{ such that } \liminf_{r\to 0} \frac{\log H^s_{r,\theta}(F)}{-\log r} = 0\right). \tag{I.1.27}$$

Similarly, the upper θ-intermediate dimension of E is defined by

$$\overline{\dim}_\theta(F) = \left(\text{the unique } s \in [0,d] \text{ such that } \limsup_{r\to 0} \frac{\log H^s_{r,\theta}(F)}{-\log r} = 0\right). \tag{I.1.28}$$

When $\underline{\dim}_\theta(F) = \overline{\dim}_\theta(F)$, we refer to the θ-intermediate dimension $\dim_\theta(F) = \underline{\dim}_\theta(F) = \overline{\dim}_\theta(F)$.

Thus, the classical Hausdorff (I.1.24) and box dimensions (I.1.18), (I.1.19) can be viewed as the extremes of a continuum of dimensions with increasing restrictions on the relative sizes of covering sets. Indeed, for every bounded $F \subset \mathbb{R}$,

$$\underline{\dim}_0 F = \overline{\dim}_0 F = \dim_H(F), \quad \underline{\dim}_1 F = \underline{\dim}_B(F) \quad \text{and} \quad \overline{\dim}_1 F = \overline{\dim}_B(F).$$

Moreover, the intermediate dimensions can be defined in terms of capacities with respect to an appropriate kernel denoted by $\phi_{r,\theta}^{s,m}$ (see [22]). For each collection of parameters $\theta \in (0,1]$, $m \in \{1,...,d\}$, $0 \le s \le m$ and $0 < r < 1$, let $\phi_{r,\theta}^{s,m} : \mathbb{R}^d \to \mathbb{R}$ be the function

$$\phi_{r,\theta}^{s,m}(x) := \begin{cases} 1 & 0 \le |x| < r, \\ \left(\frac{r}{|x|}\right)^s & r \le |x| < r^\theta, \\ \frac{r^{\theta(m-s)+s}}{|x|^m} & r^\theta \le |x|. \end{cases} \tag{I.1.29}$$

Using this kernel we define the *capacity* of a compact set $F \subset \mathbb{R}^d$ as

$$C_{r,\theta}^{s,m}(F) := \left(\inf_{\mu \in \mathcal{M}(F)} \int\int \phi_{r,\theta}^{s,m}(x-y)d\mu(x)d\mu(y) \right)^{-1}, \tag{I.1.30}$$

where $\mathcal{M}(F)$ is the set of probability measures supported in F. The following lemma, which is similar to Lemma I.1.19, allows us to define the intermediate dimension profiles.

Lemma I.1.21. *[22, Lemma 3.2] Let $\theta \in (0,1)$ and $F \subset \mathbb{R}^d$. For each $0 < r < 1$ and all $0 \le t \le s \le d$,*

$$-(s-t) \le \left(\frac{\log(C_{r,\theta}^{s,m}(F))}{-\log r} - s \right) - \left(\frac{\log(C_{r,\theta}^{t,m}(F))}{-\log r} - t \right) \le -\theta(s-t)$$

In particular, there is a unique $\underline{s} \in [0,d]$ such that $\liminf_{r\to 0} \frac{\log(C_{r,\theta}^{\underline{s},m}(F))}{-\log r} = \underline{s}$ and a unique $\bar{s} \in [0,d]$ such that $\limsup_{r\to 0} \frac{\log(C_{r,\theta}^{\bar{s},m}(F))}{-\log r} = \bar{s}$. Now for $m \in \{1,...,d\}$, the *lower intermediate dimension profiles* of $F \subset \mathbb{R}^d$ are

$$\underline{\dim}_{\theta,m}(F) = \left(\text{the unique } s \in [0,m] \text{ such that } \liminf_{r\to 0} \frac{\log C_{r,\theta}^{s,m}(F)}{-\log r} = s \right), \tag{I.1.31}$$

and the *upper intermediate dimension profiles* are

$$\overline{\dim}_{\theta,m}(F) = \left(\text{the unique } s \in [0,m] \text{ such that } \limsup_{r\to 0} \frac{\log C_{r,\theta}^{s,m}(F)}{-\log r} = s \right). \tag{I.1.32}$$

The intermediate dimension profiles are increasing in m and for $F \subset \mathbb{R}^d$,

$$\underline{\dim}_{\theta,d}(F) = \underline{\dim}_\theta(F) \quad \text{and} \quad \overline{\dim}_{\theta,d}(F) = \overline{\dim}_\theta(F).$$

I.1.2.2 Dimensions reflecting macroscopic properties

In this section, we would like to note that all our definitions are independent of the choice of the norm on $\mathbb{R}^d$. But for the coherence of the definitions, we continue to work with the d-dimension Euclidean space $(\mathbb{R}^d, \|\|_2)$. For $x \in \mathbb{R}^d$ and $r > 0$, $B(x,r)$ denotes the Euclidean ball with center x and radius r. For $F \subset \mathbb{R}^d$, the diameter of a set F is denoted by $|E|$.

I.1.2.2.1 Logarithmic and pixel densities

In the scope of measuring macroscopic properties of a given set F, we recall the definitions of logarithmic and pixel densities. But, first, we have to introduce some notions.
For all $x = (x_1, ..., x_d) \in \mathbb{R}^d$, define

$$Q(x) = [x_1, x_1 + 1) \times ... \times [x_d, x_d + 1).$$

One defines the *pixelization* of a set $F \subset \mathbb{R}^d$ as

$$\text{pix}(F) = \{x \in \mathbb{Z}^d \,:\, F \cap Q(x) \neq \emptyset\}. \tag{I.1.33}$$

It is clear that $\text{pix}(F) = F$ if $F \subset \mathbb{Z}^d$, and $\text{pix}(\mathbb{R}^d) = \mathbb{Z}^d$.

Definition I.1.22. (see [59, 61]) For $F \subset \mathbb{R}^d$, the pixel density of F is

$$\text{Den}_{pix}(F) \coloneqq \limsup_{n\to\infty} \frac{\log_2 \#\text{pix}(F \cap B(0, 2^n))}{n},$$

where # denotes cardinality. The logarithmic density of F is given by

$$\text{Den}_{\log}(F) \coloneqq \limsup_{n\to\infty} \frac{\log_2 \text{Leb}(F \cap B(0, 2^n))}{n},$$

where 'Leb' is the d-dimensional Lebesgue measure.

Note that for any $F \subset \mathbb{R}^d$, both $\text{Den}_{pix}(F)$ and $\text{Den}_{\log}(F)$ range between 0 and d.

I.1.2.2.2 Macroscopic Hausdorff dimensions

The macroscopic Hausdorff dimension $\text{Dim}_H(F)$ of a set $F \subset \mathbb{R}^d$ was introduced by Barlow and Taylor [12, 11] to define the notion of fractals in a discrete setup. It is a discrete analog of Hausdorff dimension, and the word macroscopic comes from the fact that this dimension ignores the local structure of the sets. In this section we aim to define this macroscopic Hausdorff dimension. To this end, define for all integer $n \in \mathbb{N}$, the n-th shell of $\mathbb{R}^d$ by

$$S_0 = B(0, 1) \quad \text{and} \quad S_n := B(0, 2^n) \setminus B(0, 2^{n-1}) \text{ for all } n \geq 1. \tag{I.1.34}$$

Both standard Hausdorff dimension and macroscopic Hausdorff dimension describe how a set F can be efficiently covered by balls. Nevertheless, the macroscopic Hausdorff dimension is concerned only with large scale behaviors, and so Barlow and Taylor proposed to cover the intersections $F \cap S_n$ by balls with diameters at least 1. In this capacity, let us introduce, for $F \subseteq \mathbb{R}^d$, the set of *covers* of F restricted to S_n defined by

$$\mathcal{I}_n(F) = \left\{ \{B(x_i, r_i)\}_{i=1}^m \,:\; m \in \mathbb{N},\, x_i \in S_n,\, r_i \geq 1,\, F \cap S_n \subset \bigcup_{i=1}^m B(x_i, r_i) \right\}.$$

Finally, for $s \geq 0$ and $n \in \mathbb{N}$, set

$$\nu_n^s(F) = \inf\left\{\sum_{i=1}^m \left(\frac{r_i}{2^n}\right)^s : \{B_i = B(x_i, r_i)\}_{i=1}^m \in \mathcal{I}_n(F)\right\}. \tag{I.1.35}$$

Observe that ν_n^s is sub-additive, i.e. $\nu_n^s(A \cup B) \leq \nu_n^s(A) + \nu_n^s(B)$ for every sets A and B, but is not a measure (because of the constraints on r_i).

Now we define the Barlow-Taylor macroscopic Hausdorff dimension.

Definition I.1.23. For every $s \geq 0$ and $F \subset \mathbb{R}^d$, define

$$\nu^s(F) = \sum_{n \geq 1} \nu_n^s(F).$$

The macroscopic Hausdorff dimension of $F \subset \mathbb{R}^d$ is defined by

$$\mathrm{Dim}_H(F) = \inf\{s \geq 0 : \nu^s(F) < +\infty\}. \tag{I.1.36}$$

One easily checks that $\mathrm{Dim}_H(F) \in [0, d]$ for all $F \subset \mathbb{R}^d$, $\mathrm{Dim}_H(F) = 0$ when F is bounded, and an alternative definition for $\mathrm{Dim}_H(F)$ is

$$\mathrm{Dim}_H(F) = \sup\{s \geq 0 : \nu^s(F) = +\infty\},$$

where $\sup \emptyset = 0$ by convention.

Remark 5. Both $\mathrm{Dim}_H(F)$ and $\mathrm{Den}_{pix}(F)$ (resp. $\mathrm{Den}_{\log}(E)$) give an intuition about the macroscopic geometry of F. The main difference is that $\mathrm{Dim}_H(F)$ not only counts the number of points of $F \cap S_n$ as $\mathrm{Den}_{pix}(F)$ (resp. measures $F \cap S_n$ as $\mathrm{Den}_{\log}(F)$) but also takes into account the geometry of the set F, in particular by considering the most efficient covering of $F \cap S_n$. Thus, as an intuition, the value of $\nu_n^s(F)$ is larger when the points $F \cap S_n$ are scattered all over S_n, while it is smaller when these points are all located in the same region. For instance, for $0 < \alpha < 1$, define the two sets A_α and B_α by for all $n \geq 1$,

$$A_\alpha \cap S_n = \left\{2^{n-1} + k\frac{2^{n-1}}{2^{n\alpha}} : k \in \{0, ..., 2^{n\alpha} - 1\}\right\};$$
$$B_\alpha \cap S_n = \left\{2^{n-1} + \frac{k}{2^{n\alpha}} : k \in \{0, ..., 2^{n\alpha} - 1\}\right\}.$$

Even though both sets have same cardinality, we have $\mathrm{Dim}_H A_\alpha = \alpha$ whereas $\mathrm{Dim}_H B_\alpha = 0$.

I.1.2.3 Overview of the types of fractal dimensions

Throughout this thesis, we use various fractal dimensions. In this section we aim to compare between the dimensions mentioned so far to give the reader some intuition. Table I.1 compares the covering procedure between the mentioned fractal dimensions, while Table I.2 compares how these dimensions measures different types of sets.

Dimension	Name	Cover	Size of covers	Values	Limit
$\dim_H(\cdot)$	Hausdorff	Covering	$(0, \delta]$	$[0, d]$	$\delta \to 0$
$\dim_B(\cdot)$	Box	Upper - Covering Lower - Packing	δ	$[0, d]$	$\delta \to 0$
$\dim_P(\cdot)$	Packing	Packing	$(0, \delta]$	$[0, d]$	$\delta \to 0$
$\dim_\theta(\cdot)$	Intermediate		$\in (\delta^{1/\theta}, \delta)$	$[0, d]$	$\delta \to 0$
$\mathrm{Den}_{\log}(\cdot)$	Logarithmic density	Balls	$[1, 2^n]$	$[0, d]$	$n \to \infty$
$\mathrm{Den}_{pix}(\cdot)$	Pixel density	Balls	$[1, 2^n]$	$[0, d]$	$n \to \infty$
$\mathrm{Dim}_H(\cdot)$	Macroscopic Hausdorff	Collections of balls in $B(0, 2^n)/B(0, 2^{n-1})$	$[1, 2^n]$	$[0, d]$	$n \to \infty$

Table I.1: Overview of the types of fractal dimensions. For the pixel density the cover consists of the integer points in the ball at distance less than 1 from F.

Dimension	Name	Discrete sets	Bounded sets	$\mathbb{R}^d$	$\mathbb{Z}^d$
$\dim_H(\cdot)$	Classical Hausdorff	0	$[0, d]$	d	0
$\dim_B(\cdot)$	Box	0	$[0, d]$	d	0
$\dim_P(\cdot)$	Packing	0	$[0, d]$	d	0
$\dim_\theta(\cdot)$	Intermediate	0	$[0, d]$	d	0
$\mathrm{Den}_{\log}(\cdot)$	Logarithmic density	$[0, d]$	0	d	d
$\mathrm{Den}_{pix}(\cdot)$	Pixel density	$[0, d]$	0	d	d
$\mathrm{Dim}_H(\cdot)$	Macroscopic Hausdorff	$[0, d]$	0	d	d

Table I.2: A summary of the fractal dimensions of discrete and bounded sets.

Remark 6. We also mention a few relations between the dimensions mentioned so far to give the reader some intuition:

$$\dim_H(F) \leq \underline{\dim}_B(F) \leq \overline{\dim}_B(F); \quad \dim_H(F) \leq \underline{\dim}_\theta(F) \leq \overline{\dim}_\theta(F) \leq \overline{\dim}_B(F);$$
$$\dim_P(F) \leq \overline{\dim}_B(F); \quad \mathrm{Den}_{\log}(F) \leq \mathrm{Den}_{pix}(F); \quad \mathrm{Dim}_H(F) \leq \mathrm{Den}_{pix}(F)$$

I.1.3 Wavelets tools and wavelet leaders method

Another field of study we were interested in is studying precisely the path behavior, and in particular regularity, of stochastic processes. To this aim, wavelet analysis allowed to obtain series expansions for many stochastic processes which made it a key tool in studying point-wise properties.

There seems to be no agreement in the literature on one unique definition of a wavelet. Nevertheless, the following conditions are commonly used.

Definition I.1.24. We say $\psi : \mathbb{R} \to \mathbb{R}$ is a *wavelet* if $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ satisfying the so-called admissibility condition

$$\int_{\mathbb{R}} \frac{|\widehat{\psi}(\xi)|}{|\xi|}\, d\xi < \infty, \tag{I.1.37}$$

where $\widehat{\psi}$ is the Fourier transform of ψ.

Remark 7. An immediate but important consequence of the admissibility condition is that ψ has (at least) 1 vanishing moment, i.e. $\int_{\mathbb{R}} \psi(x)dx = 0$, which makes ψ orthogonal to polynomials of degree 0. In many situations, it is preferable to use wavelets that are orthogonal to all low-order polynomials. Therefore, it is generally required that ψ has $M\,(M \in \mathbb{N})$ vanishing moments, i.e. for each $m \in \mathbb{N}$ such that $m < M$, the function $x \to x^m\psi(x)$ belongs to $L^1(\mathbb{R})$ and

$$\int_{\mathbb{R}} x^m \psi(x)dx = 0.$$

The regularity properties of a function can be studied by decomposing it in an orthonormal wavelet basis of the space $L^2(\mathbb{R})$.

Proposition I.1.25. *Under some general assumptions ([27, 79, 73]), it is possible to build a wavelet ψ such that*

$$\{\psi(2^j \cdot -k) : j \in \mathbb{Z}, k \in \mathbb{Z}\}$$

forms an orthogonal basis of $L^2(\mathbb{R})$. Therefore, any function $f \in L^2(\mathbb{R})$ can be decomposed as

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} c_{j,k}\psi(2^j \cdot -k),$$

where

$$c_{j,k} = 2^j \int_{\mathbb{R}} f(x)\psi(2^j x - k)\, dx.$$

Remark 8. The coefficients $c_{j,k}$ are called the wavelet coefficients of f. For a given scale j ($j \in \mathbb{N}$) and position k ($k \in \mathbb{Z}$) the wavelet coefficients $c_{j,k}$ are usually associated with the dyadic cube $\lambda_{j,k}$ of $\mathbb{R}$ defined as

$$\lambda_{j,k} := \left[\frac{k}{2^j}, \frac{k+1}{2^j}\right).$$

The notation Λ_j will stand for the set of dyadic intervals λ of $\mathbb{R}$ with side length 2^{-j}. The unique dyadic interval from Λ_j containing the point $t \in \mathbb{R}$ will be denoted $\lambda_j(t)$. The set of dyadic intervals is $\Lambda := \cup_{j\in\mathbb{N}}\Lambda_j$. Two dyadic intervals λ and λ' are adjacent if there exist $j \in \mathbb{N}$ such that $\lambda, \lambda' \in \Lambda_j$ and $\text{dist}(\lambda, \lambda') = 0$.

These dyadic cubes allow an easy geometric visualization of the concepts of wavelet leaders related to a point and to a cube defined in the following.

Definition I.1.26. The wavelet leader of $t_0 \in \mathbb{R}$ at the scale j is defined as

$$d_j(t_0) = \max_{\lambda \in 3\lambda_j(t_0)} \sup_{\lambda' \subseteq \lambda} |c'_\lambda|, \tag{I.1.38}$$

where $\lambda_j(t_0)$ is the unique dyadic cube at the scale j containing t_0, and $3\lambda_j(t_0)$ is the set of dyadic intervals adjacent to $\lambda_j(t_0)$.

The wavelet leaders $(d_j(t_0))_{j\in\mathbb{N}}$ of t_0 are key quantities to study the pointwise regularity of t_0, as we will see in Chapter III when studying the pointwise regularity of the generalized Rosenblatt process R_{H_1,H_2}.

I.2 Main contributions and structure of the thesis

In this section, we take a closer look at the contributions of this thesis. We give some of our results in a simplified and informal way summing-up all our main findings. Nevertheless, complete formulations and technical details can be find in referred respective chapters. Chapters II-IV deal with describing the geometric properties of sample paths, which played a significant role in modern stochastic analysis, and have been investigated using various methods. Two of the most relevant tools utilized are multifractal analysis and harmonic analysis. These are the approaches that we are going to develop in the majority of this thesis. In Chapter V we investigate the macroscopic Hausdorff dimension which played a pivotal role in our research. In particular, we develop a potential theoretical method to estimate this dimension. Then, we apply this method to obtain projection theorems that link between the macroscopic Hausdorff dimension of a set in $\mathbb{R}^2$, and its projections on almost every straight line passing through the origin.

This line of research started by studying the regularity and irregularity of the real valued Brownian motion. Paley, Wiener and Zygmund [93] have shown that its local Hölder regularity cannot be larger than 1/2. Followed by investigating the behavior of a Brownian motion on a given point, Khinchin [58] introduced a new notion of *ordinary points* by proving that the law of iterated logarithm holds almost surely. Later, Oray and Taylor [92] proved that there exist exceptional points, called *rapid points*, where the law of the iterated logarithm fails. Furthermore, Kahane [54] obtained the existence of a third category of points, presenting a slower oscillation. These points are called *slow points*.

Another natural question is studying the graph of a Brownian motion using various fractal dimensions, such as box, packing and Hausdorff dimension, which lead to some global and local geometric properties (see [84, 114, 115]). A further approach for understanding the features of a random path was through assessing the proportion of time spent by a Brownian motion in a given region, which is known by *sojourn times*. Many authors studied sojourn times associated to Brownian motion (see [24, 97, 122, 105]). As a consequence of these remarkable efforts, the Brownian case is well understood, and many authors have tried to extend these results to more general stochastic processes such as fractional Brownian motion and Rosenblatt process.

Fractional Brownian motion is a generalization of the Brownian motion. Mathematicians were motivated to extend the Brownian motion properties mentioned above, and to understand how much those findings rely on the specific features of Brownian motion, such as the (strong and weak) Markov properties. In this capacity, the law of iterated logarithm and the study of the set of fast points has naturally been studied and extended for more general classes of Gaussian processes, such as the fractional Brownian motion, see e.g. [75, 91, 18, 76, 83, 60]. In 1999, Meyer, Sellan and Taqqu introduced their famous decomposition of the fractional Brownian motion using the Lemarié-Meyer wavelet [80], which can be used to generalize the notion of *ordinary, rapid, and slow points* for Gaussian wavelets series [36], in particular for fractional Brownian motion.

One the other hand, fractal analysis played a major role in studying the random path of the fractional Brownian motion. Various random sets such as graph, level sets and sojourn times associated to fractional Brownian motion were assessed using packing dimension [129], Hausdorff dimension [54, 38], and macroscopic Hausdorff dimension [87]. In Chapter II, we assess the level sets associated to the fractional Brownian motion using the macroscopic Hausdorff dimension. Our results recover Seuret-Yang's results [105] for Brownian motion and can be considered as an addendum to Nourdin-Peccati-Seuret's work [87].

In this thesis, we mainly focus on generalizing all the results mentioned above to the Rosenblatt process case. Our investigations are essentially motivated by the fact that, unlike the fractional Brownian motion, the Rosenblatt process is not Gaussian. A natural question is how much this property impact random paths? In Chapter III, we study the fractal properties of the random sets and measures determined by the sample paths of a Rosenblatt process, and in Chapter IV we assess its pointwise regularity where the existence of three types of points is proved: *slow, ordinary and rapid.*

The last chapter of this thesis develops various methods for estimating the macroscopic Hausdorff dimension. Recalling the fact that the macroscopic Hausdorff dimension is a discrete analog of the Hausdorff dimension, we developed similar estimating methods used for the Hausdorff dimension. The two usual methods are the mass distribution principle and the potential theoretic method. The potential theoretic method is based on an integral analysis, and it is a practical tool with various applications. As an application of the new potential theoretic method, we obtain a Marstrand-like projection theorem, describing the dimension of almost all projections on lines of sets in $\mathbb{R}^d$.

Here below we give a global outline of each of the chapters.

Chapter II: A uniform result for the dimension of fractional Brownian motion level sets

This chapter is concerned with estimating the size of level sets of the fractional Brownian motion $(B_t^H)_{t\geq 0}$, which are defined for any $x \in \mathbb{R}$ as

$$\mathcal{L}_B(x) = \{t \geq 0 \,:\, B_t^H = x\}.$$

Due to self-similarity property of B^H, $\mathcal{L}_B(x)$ may look like a fractal, so in order to describe it quantitatively one can use a type of fractal dimension. In this aim, the macroscopic Hausdorff dimension Dim_H has proven to be relevant in the present chapter because it gives an intuition about the geometry of the set, precisely whether it is scattered or not. Our work add supplementary results to [87]. In particular, in [87] they proved

$$\forall x \in \mathbb{R},\, \mathbb{P}(\mathrm{Dim}_H \mathcal{L}_B(x) = 1 - H) = 1.$$

Our aim is to extend this result *uniformly* for all x which is a non-trivial mission. Formally, we proved:

Theorem I.2.1 (L. Daw (2021))**.**

$$\mathbb{P}(\forall x \in \mathbb{R} :\, \mathrm{Dim}_H \mathcal{L}_B(x) = 1 - H) = 1. \tag{I.2.1}$$

We note that Theorem I.2.1 also recovers Seuret-Yang's result [105, Theorem 2] (Brownian motion), using a more natural approach in our opinion, where the local time of the fractional Brownian motion plays a crucial role.

Chapter III: Fractal dimensions of the Rosenblatt process

In this chapter we focus on the fractal properties of the random sets and measures determined by the sample paths of the Rosenblatt process Z, i.e., we study the function $Z_t = Z_t(\omega)$, for a fixed $\omega \in \Omega$. Some (random) sets of interest are then:

$$\text{Image set: } Z(E) := \{Z(t) : t \in E\}\,; \tag{I.2.2}$$

$$\text{Graph set: } Gr_Z(E) := \{(t, Z(t)) \in E \times \mathbb{R} : t \in E\}\,; \tag{I.2.3}$$

$$\text{Level set: } \mathcal{L}_Z(x) := \{t \in \mathbb{R}_+ : Z(t) = x\}\,, x \in \mathbb{R}; \tag{I.2.4}$$

$$\text{Sojourn set: } E_Z(\gamma) := \{t \in \mathbb{R}_+ : |Z(t)| \leq t^\gamma\}\,, \gamma > 0; \tag{I.2.5}$$

$$\text{Inverse image: } Z^{-1}(E') := \{t \in \mathbb{R}_+ : Z(t) \in E'\}\,, \tag{I.2.6}$$

where $E \subset \mathbb{R}_+$ and $E' \subset \mathbb{R}$ are Borel sets. For instance, by self-similarity of Z, these sets may look like a fractal. As a result in this chapter, we measure the above sets using the Hausdorff (both classical and macroscopic), packing and intermediate dimensions, and the logarithmic and pixel densities (see Section I.1.2 for exact definitions). Our results can be collected in three theorems. First, we assess the image sets $Z(E)$, for all $E \subset \mathbb{R}^+$, using intermediate dimension.

Theorem I.2.2 (L. Daw, G. Kerchev (2021))**.** *Let $\theta \in (0, 1]$ and $E \subset \mathbb{R}^+$ be compact. Then almost surely:*

$$\underline{dim}_\theta\,(Z(E)) = \frac{1}{H}\underline{dim}_{\theta,H}\,(E)\,, \tag{I.2.7}$$

and

$$\overline{dim}_\theta\left(Z(E)\right) = \frac{1}{H}\overline{dim}_{\theta,H}\left(E\right), \tag{I.2.8}$$

where $\underline{dim}_{\theta,H}(\cdot)$ *and* $\underline{dim}_{\theta,H}(\cdot)$ *are the lower and upper* θ*-intermediate dimension profiles respectively. For the precise technical definitions of these two objects see* (I.1.31) *and* (I.1.32) *in Section I.1.2.1.4.*

Then, we describe the size of the level sets $\mathcal{L}_Z(x)$ in terms of intermediate dimensions and macroscopic Hausdorff dimension. The following holds:

Theorem I.2.3 (L. Daw, G. Kerchev (2021))**.** *For* $E \subset \mathbb{R}$ *and* $\theta \in [0,1]$*, let* $dim_\theta(E)$ *and* $Dim_H(E)$ *denote the* θ*-intermediate and macroscopic Hausdorff dimensions of* E*. Then, for any* $x \in \mathbb{R}$ *and* $0 < \varepsilon < 1$*,*

$$\forall x \in \mathbb{R},\ \mathbb{P}\left(dim_\theta\left(\mathcal{L}_Z(x) \cap [\varepsilon, 1]\right) = 1 - H\right) = 1 \tag{I.2.9}$$

$$\forall x \in \mathbb{R},\ \mathbb{P}(dim_P\left(\mathcal{L}_Z(x) \cap [\varepsilon, 1]\right) = 1 - H) = 1 \tag{I.2.10}$$

$$\mathbb{P}(\forall x \in \mathbb{R} : Dim_H\left(\mathcal{L}_Z(x)\right) = 1 - H) = 1. \tag{I.2.11}$$

Finally, we establish a result for the sojourn times $E_Z(\gamma)$.

Theorem I.2.4 (L. Daw, G. Kerchev (2021))**.** *For* $E \subset \mathbb{R}$*, let* $Den_{pix}(E)$ *and* $Den_{\log}(E)$ *denote the pixel and logarithmic densities of* E*. Then, for all* $\gamma \in [0, H]$*,*

$$Den_{pix}\left(E_Z(\gamma)\right) = Den_{\log}\left(E_Z(\gamma)\right) = \gamma + 1 - H, \quad a.s. \tag{I.2.12}$$

$$Dim_H\left(E_Z(\gamma)\right) = 1 - H \quad a.s. \tag{I.2.13}$$

Many of the results listed above rely on Hölder regularity conditions for the sample paths, and more precisely, for the local time of the process. The existence of local time of Z was first established in [106]. Hölder regularity was then recovered in the recent paper [57]. For our analysis, as a preliminary step we also establish the time inversion property of the Rosenblatt process:

Proposition I.2.5 (L. Daw, G. Kerchev (2021))**.** *The inverse time process*

$$t \mapsto \tilde{Z}_t := t^{2H} Z_{1/t}, \tag{I.2.14}$$

is also a Rosenblatt process.

In addition to that, a few properties of the density for the joint process (Z_{t_1}, Z_{t_2}) are needed. Using techniques from [57] we were able to prove the following:

Proposition I.2.6 (L. Daw, G. Kerchev (2021))**.** *(i) The probability density function* $f : \mathbb{R} \to \mathbb{R}_+$ *of* Z_1 *is continuous and* $f(x) > 0$ *for* $x \geq 0$*.*

(ii) For every $t_1, \ldots, t_n \geq 0$*, the vector* $(Z_{t_1}, \ldots, Z_{t_n})$ *has a continuous density.*

Chapter IV: Wavelet methods to study the pointwise regularity of the generalized Rosenblatt process

In this chapter we prove that the generalized Rosenblatt process $(R_{H_1,H_2}(t))_{t\geq 0}$ presents three kinds of local behaviors: slow, ordinary and rapid points. On this purpose, fine bounds on the increments of this process are needed, both from above and below. For the upper bounds, we take advantage of the wavelet-type representation of the generalized Rosenblatt process established in [7] by the means of the Meyer's wavelet. For the lower bounds we use the compactly supported Daubechies wavelets basis, and our main tools are the wavelet leaders. These representations are our key tools to prove the following Theorem I.2.7 which is the main result of this chapter.

Theorem I.2.7 (L. Daw, L. Loosveldt (2022)). *For all $H_1, H_2 \in (\frac{1}{2}, 1)$ such that $H_1+H_2 > \frac{3}{2}$, there exists an event Ω_{H_1,H_2} of probability 1 satisfying the following assertions for all $\omega \in \Omega_{H_1,H_2}$ and every non-empty interval I of $\mathbb{R}$.*

- *For almost every $t \in I$,*

$$0 < \limsup_{s\to t} \frac{|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)|}{|t-s|^{H_1+H_2-1}\log\log|t-s|^{-1}} < +\infty. \tag{I.2.15}$$

  Such points are called ordinary points*.*

- *There exists a dense set of points $t \in I$ such that*

$$0 < \limsup_{s\to t} \frac{|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)|}{|t-s|^{H_1+H_2-1}\log|t-s|^{-1}} < +\infty. \tag{I.2.16}$$

  Such points are called rapid points*.*

- *There exists a dense set of points $t \in I$ such that*

$$\limsup_{s\to t} \frac{|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)|}{|t-s|^{H_1+H_2-1}} < +\infty. \tag{I.2.17}$$

  Such points are called slow points*.*

In [36], Esser and Loosveldt proved the existence of slow, ordinary and rapid points for Gaussian wavelet series, in particular for the fractional Brownian motion. Theorem I.2.7 shows in particular that slow, ordinary and rapid points are not specific to Gaussian processes.

Chapter V: Potential methods and projection theorems for macroscopic Hausdorff dimension

In this chapter we build various methods for estimating the macroscopic Hausdorff dimension (see Section I.1.2.2.2 for formal definitions), which is a discrete analog of the standard Hausdorff dimension. A natural approach is to extend the estimating methods used for the standard Hausdorff dimension. In general, when assessing the standard Hausdorff dimension of a given set $E \subset \mathbb{R}^d$ the challenging part is finding a lower bound. To this end, a famous approach is the potential theoretical method, which is based on integral analysis. Our aim in this chapter is to establish the potential theoretical method for the macroscopic Hausdorff dimension which requires careful analysis. Let us first introduce the macroscopic s-energy of a measure.

Definition I.2.8. Let $s \geq 0$, and let μ be a finite mass distribution on $\mathbb{R}^d$. The macroscopic (μ, s)-potential at a point x is defined as

$$\phi_\mu^s(x) := \int_{\mathbb{R}^d} \frac{d\mu(y)}{\|x-y\|_2^s \vee 1}. \tag{I.2.18}$$

The macroscopic s-energy of μ is

$$I_s(\mu) := \int_{\mathbb{R}^d} \phi_\mu^s(x) d\mu(x) = \iint_{(\mathbb{R}^d)^2} \frac{d\mu(x)d\mu(y)}{\|x-y\|_2^s \vee 1}. \tag{I.2.19}$$

This result is quite comparable to the standard Hausdorff dimension (see Section I.1.2.1.2), except that in the integrals (I.2.18) and (I.2.19), the quantity $\|x-y\|_2^s \vee 1$ is simply $\|x-y\|_2^s$. This modification is justified by the fact that Dim_H is not concerned with local behavior, so we are not interested in small interactions $\|x-y\|_2 < 1$. The following theorem illustrates the potential theoretical methods for the macroscopic Hasudorff dimension, and is one of the main results in this chapter.

Theorem I.2.9 (L. Daw, S. Seuret (2022))**.** *Let E be a subset of $\mathbb{R}^d$.*

1. *If there exists a Radon measure μ on $\mathbb{R}^d$ such that $\mu(E) = +\infty$ and if*

$$\sum_{n \geq 0} 2^{ns} I_s(\mu_{|S_n}) < +\infty,$$

   then $\nu^s(E) = +\infty$ and $Dim_H(E) \geq s$.

2. *If $\nu^s(E) = +\infty$, then for all $0 < \varepsilon < s$ there exists a Radon measure μ^ε on $\mathbb{R}^d$ such that $\mu^\varepsilon(E) = +\infty$ and $\sum_{n \geq 0} 2^{n(s-\varepsilon)} I_{s-\varepsilon}(\mu^\varepsilon_{|S_n}) < +\infty$.*

Although the potential theoretic methods are very comparable to the ones established for the standard Hausdorff dimension [38, Theorem 4.13], for the macroscopic Hausdorff dimension we consider the measure μ which is defined on $\mathbb{R}^d$, and we focus on the restriction of μ on every annulus S_n. For this reason, we deal with sums over n.

A key ingredient in proving Theorem I.2.9 is the existence of macroscopic s-sets which can be defined as follows.

Definition I.2.10. Let $s \geq 0$. A set $E \subset \mathbb{R}^d$ is called a macroscopic s-set when $\mathrm{Dim}_H(E) = s$ and $\nu^s(E) < +\infty$.

We prove the existence of macroscopic s-sets.

Theorem I.2.11 (L. Daw, S. Seuret (2022))**.** *Let $E \subset \mathbb{R}^d$ be such that $\nu^s(E) = +\infty$. Then there exists a macroscopic s-set $\widetilde{E}$ such that $\widetilde{E} \subset E$.*

This extraction theorem is a key ingredient at various places in our proofs, in particular for the projection theorems which are an application of the potential theoretic methods we demonstrate in Theorem I.2.9.

Theorem I.2.12 (L. Daw, S. Seuret (2022))**.** *Let $E \subset \mathbb{R}^2$ be a Borel set. Define L_θ as the straight line passing through 0 with angle θ, and $proj_\theta E$ as the orthogonal projection of E onto L_θ.*

(a) If $Dim_H(E) < 1$, then $Dim_H(proj_\theta E) = Dim_H(E)$ for Lebesgue almost every $\theta \in [0, \pi]$.

(b) If $Dim_H(E) \geq 1$, then $Dim_H(proj_\theta E) = 1$ for Lebesgue almost every $\theta \in [0, \pi]$.

It is natural to seek projection results for fractal dimensions, hence it is quite satisfactory to obtain the projection theorems for the macroscopic Hausdorff dimension. We expect that Theorem I.2.12 can be extended in higher dimensional spaces, and Theorem I.2.9 is useful in this situation.

Chapter II

A Uniform Result for the Dimension of Fractional Brownian Motion Level Sets

The content of this chapter is a copy of the paper entitled "A Uniform Result for the Dimension of Fractional Brownian Motion Level Sets", and published in "Statistics and Probability Letters".

II.1 Introduction

Let $B = \{B_t : t \geq 0\}$ be a fractional Brownian motion of index $H \in (0,1)$, that is, a centered, real-valued Gaussian process with covariance function

$$R(s,t) = \mathbb{E}\left(B_s B_t\right) = \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |s-t|^{2H}\right), \quad s,t \geq 0. \tag{II.1.1}$$

Since $\mathbb{E}\big[\,(B_s - B_t)^2\,\big] = |s-t|^{2H}$, it is an immediate consequence of the Kolmogorov–Centsov continuity theorem that B admits a continuous modification. Throughout this note, we will always assume that B is continuous. It is also immediate (see, e.g., [86]) that B is a self-similar process of exponent H, that is, for any $a > 0$,

$$\{B_{at} : t \geq 0\} \stackrel{d}{=} \left\{a^H B_t : t \geq 0\right\},$$

where $X \stackrel{d}{=} Y$ means that two processes X and Y have the same distribution. Moreover, B has stationary increments, that is, for every $s \geq 0$,

$$\{B_{t+s} - B_s : t \geq 0\} \stackrel{d}{=} \{B_t : t \geq 0\}.$$

This article is concerned with estimating the size of the level sets of B, which are defined for any $x \in \mathbb{R}$ as

$$\mathcal{L}_x = \{t \geq 0 : B_t = x\}. \tag{II.1.2}$$

This line of research started with the seminal work of Taylor [116], who was the first to study the Hausdorff dimensions of the level sets in the case of a standard Brownian motion. His results were extended later on by Perkins [94] who showed that, with probability one, the level sets $\mathcal{L}_x$ have a Hausdorff dimension $\frac{1}{2}$ for all $x \in \mathbb{R}$. Hence, the local structure of the level sets in the Brownian case is well understood.

Another method to describe the geometric properties of the sample paths of a given process is in terms of its sojourn times. Here, the goal is to study the dimension of the amount of time spent by the stochastic process inside a moving boundary, that is, of the form

$$E(\phi) := \{t \geq 0 \,:\, |B_t| \leq \phi(t)\}\,,$$

where $\phi : \mathbb{R}_+ \to \mathbb{R}$ is an appropriate function.

Strongly related to our note, we mention the recent work of Nourdin, Peccati and Seuret [87], in which a specific large scale dimension is computed for the sojourn times

$$E_\gamma := \{t \geq 0 \,:\, |B_t| \leq t^\gamma\}\,, \quad 0 < \gamma < H, \tag{II.1.3}$$

of the fractional Brownian motion B. Note that this choice for ϕ is completely natural here because, on the one hand, the fractional Brownian motion is selfsimilar (hence the choice of a power function for ϕ) and, on the other hand, it satisfies a law of iterated logarithm as $t \to \infty$ (hence the range $(0, H)$ for γ). Actually, [87] extended to the fractional Brownian motion the results given by Seuret and Yang [105] in the framework of the standard Brownian case.

In general, defining a notion of fractal dimension for a subset of $\mathbb{R}^d$ involves taking into consideration the microscopic (i.e. local) properties of this set. However, many models in statistical physics are based on the Euclidean lattice $\mathbb{Z}^d$; in this case, it may look more natural to rely on the macroscopic (i.e. global) properties of the set to define a notion of dimension. This is what Barlow and Taylor proposed in [11, 12]. Their dimension, called *macroscopic Hausdorff dimension*, has proven to be relevant in many contexts. This is the one that was used in [87, 105], and also the one we will use in the present note, because it can give a good intuition about the geometry of the set into consideration, precisely whether it is scattered or not. Precise definitions will be given in Section II.2.1. At this stage, we only mention that we denote this macroscopic Hausdorff dimension by Dim_H.

Our note can be considered as an addendum to [87]. Let $\mathcal{L}_x$ be the level sets associated with a fractional Brownian motion. In [87], the following is shown.

Theorem II.1.1. Fix $x \in \mathbb{R}$. Then

$$\mathbb{P}(\mathrm{Dim}_H \mathcal{L}_x = 1 - H) = 1.$$

Our aim is to extend Theorem II.1.1 from "$\forall x$, $\mathbb{P}(\ldots) = 1$" to "$\mathbb{P}(\forall x : \ldots) = 1$". To this end, new and non-trivial arguments are required. We will prove the following.

Theorem II.1.2.

$$\mathbb{P}(\forall x \in \mathbb{R} : \operatorname{Dim}_H \mathcal{L}_x = 1 - H) = 1. \tag{II.1.4}$$

We note that our Theorem II.1.2 also recovers Seuret-Yang's result [105, Theorem 2] (Brownian motion), and provides a proof that we find more natural.

Throughout the note, every random object is defined on a common probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and $\mathbb{E}$ denotes the expectation with respect to $\mathbb{P}$.

II.2 Preliminaries

This section gathers the different tools that will be needed in order to prove Theorem II.1.2.

II.2.1 Macroscopic Hausdorff Dimension

Following the notations of [59, 61], we consider the intervals $S_{-1} = [0, 1/2)$ and $S_n = [2^{n-1}, 2^n)$ for $n \in \mathbb{N}$. For $E \subset \mathbb{R}^+$, we define the set of *proper covers* of E restricted to S_n by

$$\mathcal{I}_n(E) = \left\{ \begin{matrix} \{I_i\}_{i=1}^m \ : & I_i = [x_i, y_i] \text{ with } x_i, y_i \in \mathbb{N},\ y_i > x_i, \\ & I_i \subset S_n \ \text{ and } \ E \cap S_n \subset \bigcup_{i=1}^m I_i. \end{matrix} \right\}$$

For any set $E \subset \mathbb{R}^+$, $\rho \geq 0$ and $n \geq -1$, we define

$$\nu_\rho^n(E) = \inf \left\{ \sum_{i=1}^m \left(\frac{\operatorname{diam}(I_i)}{2^n} \right)^\rho \ : \ \{I_i\}_{i=1}^m \in \mathcal{I}_n(E) \right\}, \tag{II.2.1}$$

where $\operatorname{diam}([a, b]) = b - a$.

The key point in the definition of $\nu_\rho^n(E)$ is that the sets I_i are non-trivial intervals with *integer* boundaries; in particular, the infimum is reached.

Definition II.2.1. Let $E \subset \mathbb{R}^+$. The macroscopic Hausdorff dimension of E is defined by

$$\operatorname{Dim}_H E = \inf \left\{ \rho > 0 : \sum_{n \geq -1} \nu_\rho^n(E) < +\infty \right\}. \tag{II.2.2}$$

We observe that $\operatorname{Dim}_H E$ always belongs to $[0, 1]$, whatever $E \subset \mathbb{R}^+$. Indeed, consider the family $I_i = [2^{n-1} + i - 1, 2^{n-1} + i]$, $1 \leq i \leq 2^{n-1}$, which belongs to $\mathcal{I}_n(E)$ and satisfies $\sum_{i=1}^m \left(\frac{\operatorname{diam}(I_i)}{2^n} \right)^\rho \leq \frac{1}{2} 2^{n(1-\rho)}$. Thus, $\nu_{1+\varepsilon}^n(E) \leq 2^{-n\varepsilon}$ for all $\varepsilon > 0$, implying in turn that $\operatorname{Dim}_H E \leq 1 + \varepsilon$ for all $\varepsilon > 0$. As a result, we have that $\operatorname{Dim}_H E \in [0, 1]$.

In (II.2.1), the covers are chosen to have length larger than 1. This shows that the macroscopic Hausdorff dimension does not rely on the local structure of the underlying set.

The dimension of a set is unchanged when one removes any bounded subset, since the series in (II.2.2) converges if and only if its tail series converges. Consequently, the dimension of any bounded set E is zero. But the converse is not true, for example $\mathrm{Dim}_H(\{2^n,\, n \geq 1\}) = 0$.

The macroscopic Hausdorff dimension not only counts the number of covers of a set but also it gives an intuition about the geometry of the set. Precisely, the more the points of the set are spread-out, the larger its dimension. For instance for $0 < \alpha < 1$, define the two sets A_α and B_α by for all $n \geq 1$,

$$A_\alpha \cap S_n = \left\{ 2^{n-1} + k\frac{2^{n-1}}{2^{n\alpha}} : k \in \{0, ..., 2^{n\alpha} - 1\} \right\};$$
$$B_\alpha \cap S_n = \left\{ 2^{n-1} + \frac{k}{2^{n\alpha}} : k \in \{0, ..., 2^{n\alpha} - 1\} \right\}.$$

Even though both sets have same cardinality but $\mathrm{Dim}_H A_\alpha = \alpha$ whereas $\mathrm{Dim}_H B_\alpha = 0$.

These features make the macroscopic Hausdorff dimension an interesting quantity describing the large scale geometry of a set; in particular, it appears to be well suited for the study of the level sets $\mathcal{L}_x$.

As we will see in our upcoming analysis, it might be sometimes wise to slightly modify the way $\mathrm{Dim}_H E$ is defined, to get a definition that is more amenable to analysis. For this reason, let us introduce, for any $E \subset \mathbb{R}^+$, $\rho > 0$, $\xi \geq 0$, and $n \geq -1$, the quantity

$$\widetilde{\nu}^n_{\rho,\xi}(E) = \inf \left\{ \sum_{i=1}^{m} \left(\frac{\mathrm{diam}(I_i)}{2^n} \right)^\rho \left| \log_2 \frac{\mathrm{diam}(I_i)}{2^n} \right|^\xi : \{I_i\}_{i=1}^m \in \mathcal{I}_n(E) \right\}. \tag{II.2.3}$$

The difference between $\nu^n_\rho(E)$ and $\widetilde{\nu}^n_{\rho,\xi}(E)$ is that we introduce a logarithmic factor in the latter. This modification has actually no impact on the definition of $\mathrm{Dim}_H E$, as stated by the following lemma.

Lemma II.2.2. *Let $\xi \geq 0$. For every set $E \subset \mathbb{R}^+$,*

$$Dim_H E = \inf \left\{ \rho > 0 : \sum_{n \geq -1} \widetilde{\nu}^n_{\rho,\xi}(E) < +\infty \right\}. \tag{II.2.4}$$

Proof. Define $\tilde{d}_\xi = \inf \left\{ \rho > 0 : \sum_{n\geq -1} \widetilde{\nu}^{n,\xi}_\rho(E) < +\infty \right\}$. For $n \geq -1$, consider $\{I_i\}_{i=1}^m \in \mathcal{I}_n(E)$. As $I_i \subset S_n$, one has $\mathrm{diam}(I_i) \leq 2^{n-1}$, implying in turn that $\left| \log_2 \frac{\mathrm{diam}(I_i)}{2^n} \right|^\xi \geq 1$. Thus, $\widetilde{\nu}^n_{\rho,\xi}(E) \geq \nu^n_\rho(E)$ and then $\mathrm{Dim}_H E \leq \tilde{d}_\xi$.

If $\mathrm{Dim}_H E = 1$, the conclusion is straightforward. So, let us assume that $\mathrm{Dim}_H E < 1$ and let us fix $\varepsilon > 0$ small enough and $\rho < 1$ such that $\rho > \mathrm{Dim}_H E + \varepsilon$. Since the function

$x \mapsto x^\varepsilon |\log_2 x|^\xi$ is continuous on $(0,1]$ and tends to zero as x tends to zero, it follows that there exists $c > 0$ such that

$$|\log_2 x|^\xi \leq c x^{-\varepsilon}, \ \forall x \in (0,1]$$

We deduce that, for all $\{I_i\}_{i=1}^m \in \mathcal{I}_n(E)$,

$$\sum_{i=1}^m \left(\frac{\text{diam}(I_i)}{2^n}\right)^\rho \left|\log_2 \frac{\text{diam}(I_i)}{2^n}\right|^\xi \leq c \sum_{i=1}^m \left(\frac{\text{diam}(I_i)}{2^n}\right)^{\rho-\varepsilon}$$

By taking the infimum over all $\{I_i\}_{i=1}^m \in \mathcal{I}_n(E)$ and recalling the definitions (II.2.1) and (II.2.3), one deduces that $\widetilde{\nu}^n_{\rho,\xi}(E) \leq c\nu^n_{\rho-\varepsilon}(E)$, implying in turn $\widetilde{d_\xi} \leq \rho - \varepsilon$. Letting ρ tend to $\text{Dim}_H E + \varepsilon$ yields the result.

□

II.2.2 Local Time of Fractional Brownian Motion

As we will see, the use of the local time will play a key role throughout the proof of Theorem II.1.2.

Provided it exists, the local time $x \mapsto L^x(t)$ of a given process $(X_t)_{t\geq 0}$ is, for each t, the density of the occupation measure $\mu_t(A) = \lambda(\{s \in [0,t] \,:\, X_s \in A\})$ associated with X, where λ stands for the Lebesgue measure; otherwise stated, one has $L(t) = \frac{d\mu_t}{d\lambda}$. In what follows, we shall also freely use the notation $L^x([a,b])$ to indicate the quantity $L^x(b) - L^x(a)$.

The case where X is Gaussian (and centered, say) has been widely studied in the literature. For instance, we can refer to the survey by Dozzi [34]. One of the main striking results in the Gaussian framework is the following easy-to-check condition that ensures that $(L^x(t))_{t\in[0,T],x\in\mathbb{R}}$ exists in $L^2(\Omega)$:

$$I := \int\int_{[0,T]^2} \frac{ds\, dt}{\sqrt{R(s,s)R(t,t) - R(s,t)^2}} < +\infty, \tag{II.2.5}$$

where $R(s,t) = \mathbb{E}\left(X_s X_t\right)$; morever, in this case we have the Fourier type representation:

$$L^x(t) = \frac{1}{2\pi}\int_{\mathbb{R}} dy \int_0^t du\, e^{iy(B_u - x)}. \tag{II.2.6}$$

If X is Gaussian, selfsimilar of index H and satisfies (II.2.5), then it is immediate from (II.2.6) that its local time at level x also have some selfsimilarity properties in time with index $1-H$, but with a different level as stated below. More precisely, one has, for every $c > 0$:

$$(L^x(ct))_{t\geq 0, x\in\mathbb{R}} \overset{d}{=} c^{1-H}(L^{c^{-H}x}(t))_{t\geq 0,x\in\mathbb{R}}. \tag{II.2.7}$$

When X stands for the fractional Brownian motion B of Hurst index $H \in (0,1)$, it is immediate that (II.2.5) and (II.2.7) are satisfied. But we can go further. A consequence of Berman's work [17] is that the local time associated to B is $\beta-$Hölder continuous in t for every $\beta \leq 1-H$ and uniformly in x. On their side, German and Horowitz (see [44, Theorem 26.1]) proved that, for all fixed t, the local time $(L^x(t))_{x\in\mathbb{R}}$ admits the Hölder regularity in space stated in the following lemma.

Lemma II.2.3 (Spatial Hölder continuity of local time). *Assume X is a fractional Brownian motion of Hurst index $H \in (0,1)$ and consider its local time $(L^x(t))_{x\in K}$, where K is a given compact interval in $\mathbb{R}$. Then, for all $\beta \in \left(0, \frac{1}{2}\left(\frac{1}{H}-1\right)\right)$ and for all $t \geq 0$,*

$$\mathbb{P}\left(\sup_{x,y\in K} \frac{|L^x(t)-L^y(t)|}{|x-y|^\beta} < \infty\right) = 1. \tag{II.2.8}$$

As we will see, Lemma II.2.3 will be one of our main key tools in order to prove Lemma II.3.3 (which is one of the steps leading to the proof of Theorem II.1.2).

II.2.3 Filtration of Fractional Brownian Motion

A last crucial property of the fractional Brownian B that we will use in order to to prove Theorem II.1.2, is that the natural filtration associated with B is Brownian. We mean by this that there exists a standard Brownian motion $(W_u)_{u\geq 0}$ defined on the same probability space than B such that its filtration satisfies, for all $t > 0$,

$$\sigma\{B_u : u \leq t\} \subset \sigma\{W_u : u \leq t\}. \tag{II.2.9}$$

Property (II.2.9) is an immediate consequence of the Volterra representation of B (see, e.g., [15]). It will be exploited together with the Blumenthal's $0-1$ law, in the end of the proof of Proposition II.3.1.

II.3 Proof of Theorem II.1.2

II.3.1 Upper bound for $\mathrm{Dim}_H\mathcal{L}_x$

By a theorem in [87], for every $\gamma \in (0, H)$, a.s.

$$\mathrm{Dim}_H E_\gamma = 1 - H.$$

On the other hand, observe that for a fixed $\gamma > 0$ and $x \in \mathbb{R}$, the level set $\mathcal{L}_x$ is ultimately included in E_γ. Indeed,

$$\mathcal{L}_x \cap [|x|^{1/\gamma}, +\infty) \subset E_\gamma.$$

We have recalled in Section II.2.1 that the macroscopic Haussdorff dimension is insensitive to the suppression of any bounded subset. As a result, a.s. for every $x \in \mathbb{R}$,

$$\mathrm{Dim}_H\mathcal{L}_x = \mathrm{Dim}_H\left(\mathcal{L}_x \cap [|x|^{1/\gamma}, +\infty)\right) \leq \mathrm{Dim}_H E_\gamma = 1 - H.$$

II.3.2 Lower bound for $\text{Dim}_H \mathcal{L}_x$

Recall S_n from Section II.2.1, and let us introduce the random variables

$$Z_n^x = \frac{L^x(S_n)}{2^{n(1-H)}} \quad \text{and} \quad F_N^x = \sum_{n=1}^{N} Z_n^x. \tag{II.3.1}$$

The random variables $(Z_n^x)_{n\geq -1}$ are positive, so $(F_N^x)_{N\geq 1}$ is non-decreasing. We denote by F_∞^x its limit, i.e. $F_\infty^x = \sum_{n=-1}^{\infty} Z_n^x \in [0, +\infty]$.

Using (II.2.7), we have for all $n \geq 0$

$$Z_n^x \stackrel{d}{=} Z_0^{2^{-nH}x}. \tag{II.3.2}$$

We note that similar random variables $Y_n^x = \dfrac{L^{2^n x}(S_n)}{2^{n(1-H)}}$ were introduced in [87, Section 5.3]. However, the fact that we are dealing with other space variables compared to [87] induce several differences in our proofs. Although its statement is exactly the same than [87, Lemma 5], the meaning and the context of our proof are different. This is why we provide all the details, for the convenience of the reader.
Our aim now is to link the random variable Z_n^x to the microscopic Hausdorff dimension. To this end, let us introduce the random variables

$$A_n := \sup_{0\leq t\leq 2^n} \sup_{0\leq h\leq 2^{n-1}} \sup_{y\in\mathbb{R}} \frac{L^y([t,t+h])}{h^{1-H}(n-\log_2 h)^H}, \tag{II.3.3}$$

where $\log_2$ stands for the binary logarithm (base 2). By (II.2.7), we have

$$\begin{aligned} A_n &= \sup_{0\leq t\leq 1} \sup_{0\leq h\leq 1/2} \sup_{y\in\mathbb{R}} \frac{L^y([2^n t, 2^n(t+h)])}{(2^n h)^{1-H}(-\log_2 h)^H} \\ &\stackrel{d}{=} \sup_{0\leq t\leq 1} \sup_{0\leq h\leq 1/2} \sup_{y\in\mathbb{R}} \frac{L^y([t,t+h])}{h^{1-H}(-\log_2 h)^H}. \end{aligned} \tag{II.3.4}$$

First, let us prove that A_n is finite almost surely. We start by making use of a result of Xiao [128, Theorem 1.2] that describes the scaling behavior of local times of Gaussian processes with stationary increments; in particular, this applies to the fractional Brownian motion and we have, with probability one:

$$M := \lim_{r\downarrow 0} \sup_{0\leq t\leq 1} \sup_{0\leq h\leq r} \sup_{y\in\mathbb{R}} \frac{L^y([t,t+h])}{h^{1-H}(-\log_2 h)^H} < \infty.$$

By the very definition of a limit, we deduce the existence of a (random) real number $0 < r < 1/2$ such that, almost surely,

$$\sup_{0\leq t\leq 1} \sup_{0\leq h\leq r} \sup_{y\in\mathbb{R}} \frac{L^y([t,t+h])}{h^{1-H}(-\log_2 h)^H} \leq 2M. \tag{II.3.5}$$

Now for $r \le h \le 1/2$, we have $h^{1-H}(-\log_2 h)^H \ge r^{1-H}$ and $L^y([t,t+h]) \le L^y([0,3/2])$ for all $0 \le t \le 1$ and $y \in \mathbb{R}$. Moreover by [131, Theorem 4.1], B has a jointly continuous local time $(t,x) \mapsto L^x(t)$ on $[0,3/2] \times \mathbb{R}$. Then, the (random) function $x \mapsto L^x(t)$ is continuous on $\mathbb{R}$ and has a compact support (the occupation measure defined in Section II.2.2 is compactly supported as $B([0,3/2])$ is compact). Hence, $\sup_{y\in\mathbb{R}} L^y([0,3/2])$ is finite and so one gets, almost surely,

$$\sup_{0\le t\le 1}\ \sup_{r\le h\le 1/2}\ \sup_{y\in\mathbb{R}} \frac{L^y([t,t+h])}{h^{1-H}(-\log_2 h)^H} \le r^{H-1} \sup_{0\le t\le 1}\ \sup_{r\le h\le 1/2}\ \sup_{y\in\mathbb{R}} L^y([t,t+h]) \tag{II.3.6}$$
$$\le r^{H-1}\sup_{y\in\mathbb{R}} L^y([0,3/2]) < \infty.$$

Finally, by summing up (II.3.5) and (II.3.6), one has

$$\mathbb{P}\left(\sup_{0\le t\le 1}\ \sup_{0\le h\le 1/2}\ \sup_{y\in\mathbb{R}} \frac{L^y([t,t+h])}{h^{1-H}(-\log_2 h)^H} < \infty \right) = 1.$$

Now for $K > 0$ define the event

$$\Omega_K := \left\{ \sup_{0\le t\le 1}\ \sup_{0\le h\le 1/2}\ \sup_{y\in\mathbb{R}} \frac{L^y([t,t+h])}{h^{1-H}(-\log_2 h)^H} \le K \right\}. \tag{II.3.7}$$

Fix $x \in \mathbb{R}$ and consider the level set $\mathcal{L}_x$ defined by (II.1.2). By recalling Definition II.2.3, we have: if $(I_i = [s_i,t_i])_{i=1}^m \in \mathcal{I}_n(\mathcal{L}_x)$ is a cover minimizing $\widetilde{\nu}^n_{1-H,H}(\mathcal{L}_x)$ then,

$$\widetilde{\nu}^n_{1-H,H}(\mathcal{L}_x) = \sum_{i=1}^m \left(\frac{|t_i - s_i|}{2^n}\right)^{1-H} \left|\log_2 \frac{|t_i - s_i|}{2^n}\right|^H. \tag{II.3.8}$$

Using (II.3.4) and a scaling argument with $t = \frac{s_i}{2^n}$, $h = \frac{t_i - s_i}{2^n}$, and $y = 2^{-nH}x$, we deduce that

$$\left(\frac{|t_i - s_i|}{2^n}\right)^{1-H} \left|\log_2 \frac{|t_i - s_i|}{2^n}\right|^H \ge K^{-1}\frac{L^x(I_i)}{2^{n(1-H)}} \quad \text{on } \Omega_K.$$

Back to (II.3.8), we have

$$\widetilde{\nu}^n_{1-H,H}(\mathcal{L}_x) \ge K^{-1}\sum_{i=1}^m \frac{L^x(I_i)}{2^{n(1-H)}} \ge K^{-1}\frac{L^x(S_n)}{2^{n(1-H)}} = K^{-1}Z_n^x, \quad \text{on } \Omega_K, \tag{II.3.9}$$

where the last inequality holds because the local time $L^x_\cdot$ increases only on the set I_i (whose union covers $\mathcal{L}_x \bigcap S_n$). Finally, one gets

$$\Omega_K \subset \left\{\forall x \in \mathbb{R}, \forall n \ge -1 :\ \widetilde{\nu}^n_{1-H,H}(\mathcal{L}_x) \ge K^{-1}Z_n^x\right\}.$$

Using (II.3.9) for the first inclusion and Lemma II.2.2 for the second one, we can write

$$\begin{aligned}\Omega_K \cap \{\forall x \in \mathbb{R},\, F_\infty^x = +\infty\} \subset &\{\forall x \in \mathbb{R},\, \sum_{n\geq -1} \widetilde{\nu}_{1-H,H}^n(\mathcal{L}_x) = +\infty\} \\ \subset &\{\forall x \in \mathbb{R},\, \mathrm{Dim}_H \mathcal{L}_x \geq 1-H\}.\end{aligned} \tag{II.3.10}$$

But by definition of Ω_K we have

$$\mathbb{P}(\Omega_K) \xrightarrow[K\to+\infty]{} \mathbb{P}\left(\sup_{0\leq t\leq 1} \sup_{0\leq h\leq 1/2} \sup_{y\in\mathbb{R}} \frac{L^y([t,t+h])}{h^{1-H}(-\log_2 h)^H} < \infty \right) = 1. \tag{II.3.11}$$

As a consequence, in order to conclude the proof of Theorem II.1.2, it remains to check that $\mathbb{P}(\forall x \in \mathbb{R},\, F_\infty^x = +\infty) = 1$. Then, using II.3.10, by letting $K \uparrow \infty$ an a.s. uniform lower bound of $\mathrm{Dim}_H \mathcal{L}_x$ is attained. The object of the next proposition is prove that $F_\infty^x = +\infty$ almost surely for all $x \in \mathbb{R}$.

Proposition II.3.1. *We have*

$$\mathbb{P}(\forall x \in \mathbb{R},\, F_\infty^x = +\infty) = 1 \tag{II.3.12}$$

Note that the following stronger statement of Proposition II.3.1 was shown in [87]: for all $x \in \mathbb{R}$, $\mathbb{P}(F_\infty^x = +\infty) = 1$. Our main contribution in the present note is precisely to prove the strongest version stated in Proposition II.3.1.

II.3.3 Proof of Proposition II.3.1

For every $a > 0$, define

$$\widetilde{Z}_n^a = \inf_{x\in[-a,a]} Z_n^x \quad \text{and} \quad \widetilde{F}_\infty^a = \sum_{n\geq 1} \widetilde{Z}_n^a. \tag{II.3.13}$$

Recalling (II.2.7), we get for all $n \geq 0$

$$\widetilde{Z}_n^a = \inf_{x\in[-a,a]} Z_n^x \overset{d}{=} \inf_{x\in[-a,a]} Z_0^{2^{-nH}x} = \inf_{x\in[-2^{-nH}a,2^{-nH}a]} Z_0^x = \widetilde{Z}_0^{2^{-nH}a}. \tag{II.3.14}$$

In the three forthcoming lemmas, the following three facts are established:

(i) the existence of $\varepsilon > 0$ such that $\mathbb{P}(Z_0^0 > 4\varepsilon) > 0$ (Lemma II.3.2),

(ii) the existence of $a > 0$ such that $\mathbb{P}(Z_0^0 > 4\varepsilon) \leq 2\mathbb{P}(\widetilde{Z}_0^a > 0)$ (Lemma II.3.3),

(iii) that $\mathbb{P}\left(\widetilde{F}_\infty^b = \infty\right) \geq \mathbb{P}\left(\widetilde{Z}_0^a > 0\right)$ for all $b > 0$ (Lemma II.3.4).

Combining the results obtained in (i) to (iii), we deduce that

$$\mathbb{P}\left(\widetilde{F}_{\infty}^{b}=\infty\right)>0 \quad \text{for all } b>0. \tag{II.3.15}$$

Set $\widehat{B}_u = u^{2H} B_{1/u}$, $u>0$. By the time inversion property of the fractional Brownian motion, $\widehat{B}$ is a fractional Brownian motion of Hurst index H as well. We can write

$$L^x(S_n)=\frac{1}{2\pi}\int_{\mathbb{R}} dy\, e^{-iyx}\int_{2^{n-1}}^{2^n} du e^{iyu^{2H}\widehat{B}_{1/u}}.$$

As a result, we get that $x \mapsto L^x(S_n)$ is $\sigma\left\{\widehat{B}_u : u \leq 2^{-(n-1)}\right\}$-measurable, implying in turn that

$$\sigma\left\{\widetilde{Z}_n^b \,:\, n\geq M\right\}\subset\sigma\left\{\widehat{B}_u : u\leq 2^{-(M-1)}\right\} \tag{II.3.16}$$

for every $M \geq 1$. Consequently,

$$\left\{\widetilde{F}_{\infty}^{b}=\infty\right\}\in\bigcap_{M\geq 1}\sigma\left\{\widehat{B}_u : u\leq 2^{-(M-1)}\right\}.$$

Using (II.2.9), there exists a standard Brownian motion $(W_u)_{u\geq 0}$ defined on the same probability space such that

$$\left\{\widetilde{F}_{\infty}^{b}=\infty\right\}\in\bigcap_{M\geq 1}\sigma\left\{W_u \,:\, u\leq 2^{-(M-1)}\right\}. \tag{II.3.17}$$

By the Blumenthal's 0-1 law, the probability $\mathbb{P}\left(\widetilde{F}_{\infty}^{b}=\infty\right)$ is either 0 or 1. But by (II.3.15), this probability is strictly positive; hence we conclude that

$$\mathbb{P}\left(\widetilde{F}_{\infty}^{b}=\infty\right)=1 \quad \text{for all } b>0. \tag{II.3.18}$$

For every $b>0$, one has

$$\begin{aligned}&\mathbb{P}\left(\forall x\in[-b,b] :\, F_{\infty}^{x}=\infty\right)=\mathbb{P}\left(\inf_{x\in[-b,b]} F_{\infty}^{x}=\infty\right)=\mathbb{P}\left(\inf_{x\in[-b,b]}\sum_{N\geq 1} Z_N^x=\infty\right)\\ \geq\;&\mathbb{P}\left(\sum_{N\geq 1}\inf_{x\in[-b,b]} Z_N^x=\infty\right)=\mathbb{P}\left(\widetilde{F}_{\infty}^{b}=\infty\right)=1.\end{aligned}$$

We finally conclude that

$$\mathbb{P}\left(\forall x\in\mathbb{R},\, F_{\infty}^{x}=\infty\right)=\lim_{b\to\infty}\mathbb{P}\left(\forall x\in[-b,b],\, F_{\infty}^{x}=\infty\right)=1,$$

which is the desired conclusion of Proposition II.3.1.

To conclude, it remains to state and prove the three lemmas mentioned in points (i) to (iii).

Lemma II.3.2. *For all $\varepsilon > 0$ small enough such that $\mathbb{P}(Z_0^0 > 4\varepsilon) > 0$.*

Proof. Using that $L^0\left([\frac{1}{2}, 1]\right) = \frac{1}{2\pi}\int_{\mathbb{R}} dy \int_{\frac{1}{2}}^{1} du\, e^{iyB_u}$, we have

$$\mathbb{E}\left(L^0\left(\left[\frac{1}{2}, 1\right]\right)\right) = \frac{1}{2\pi}\int_{\frac{1}{2}}^{1} u^{-H}\, du \int_{\mathbb{R}} e^{-\frac{z^2}{2}}\, dz = \frac{1}{\sqrt{2\pi}}\int_{\frac{1}{2}}^{1} u^{-H}\, du > 0.$$

As a result, $\mathbb{P}\left(Z_0^0 > 0\right) = \mathbb{P}\left(L^0\left(\left[\frac{1}{2}, 1\right]\right) > 0\right) > 0$, and the desired conclusion follows. □

Lemma II.3.3. *For every $\varepsilon > 0$ small enough, there exists a real number $a > 0$ such that*

$$0 < \mathbb{P}(Z_0^0 > 4\varepsilon) \le 2\mathbb{P}(\widetilde{Z}_0^a > 0).$$

Proof. Let $\beta < \frac{1}{2}\left(\frac{1}{H} - 1\right)$, $K = [-1, 1]$ and $J = [\frac{1}{2}, 1]$. Set

$$c = c(\omega) := \sup_{x \in K\setminus\{0\}} \frac{\left|L^0(J)(\omega) - L^x(J)(\omega)\right|}{|x|^\beta}.$$

By Lemma II.2.3, we have that $\mathbb{P}(c < \infty) = 1$.

Set $\eta_\varepsilon = \eta_\varepsilon(\omega) := \min\left\{\left(\frac{\varepsilon}{c(\omega)}\right)^{1/\beta}, 1\right\}$. As $[-\eta_\varepsilon, \eta_\varepsilon] \subset [-1, 1]$, one has

$$\forall |x| \le \eta_\varepsilon(\omega),\ \left|\left(L^0(1)(\omega) - L^x(1)(\omega)\right) - \left(L^0\left(\frac{1}{2}\right)(\omega) - L^x\left(\frac{1}{2}\right)(\omega)\right)\right| \le \varepsilon. \tag{II.3.19}$$

By triangle inequality,

$$\left|L^x(1) - L^x\left(\frac{1}{2}\right)\right| \ge \left|L^0(1) - L^0\left(\frac{1}{2}\right)\right| - \left|\left(L^0(1) - L^x(1)\right) - \left(L^0\left(\frac{1}{2}\right) - L^x\left(\frac{1}{2}\right)\right)\right|. \tag{II.3.20}$$

Using (II.3.19) and (II.3.20), we have

$$\left\{Z_0^0 = L^0(1) - L^0(\frac{1}{2}) > 4\varepsilon\right\} \subset \left\{\forall |x| \le \eta_\varepsilon(\omega),\ |L^x(1) - L^x(\frac{1}{2})| \ge 3\varepsilon\right\}. \tag{II.3.21}$$

But $\left\{\forall |x| \le \eta_\varepsilon(\omega),\ \left|L^x(1) - L^x\left(\frac{1}{2}\right)\right| \ge 3\varepsilon)\right\} = \left\{\inf_{x \in [-\eta_\varepsilon, \eta_\varepsilon]}\left|L^x(1) - L^x\left(\frac{1}{2}\right)\right| \ge 3\varepsilon\right\}$. Recalling the definition of $\widetilde{Z}_0^{\eta_\varepsilon}$, we deduce that

$$\mathbb{P}\left(\widetilde{Z}_0^{\eta_\varepsilon} > 0\right) \ge \mathbb{P}\left(\widetilde{Z}_0^{\eta_\varepsilon} > 3\varepsilon\right) \ge \mathbb{P}\left(Z_0^0 > 4\varepsilon\right) > 0. \tag{II.3.22}$$

Now for all $a > 0$, we have

$$\left\{\widetilde{Z}_0^{\eta_\varepsilon} > 0\right\} \subset \left\{\widetilde{Z}_0^a > 0\right\} \cup \left\{\eta_\varepsilon \le a\right\}. \tag{II.3.23}$$

Since $c < \infty$ a.s., one has that $\mathbb{P}(c \geq M) \to 0$ as $M \to \infty$. We can then choose $a > 0$ small enough such that

$$\mathbb{P}(\eta_\varepsilon \leq a) = \mathbb{P}\left(c \geq \frac{\varepsilon}{2a\beta}\right) \leq \frac{1}{2}\mathbb{P}\left(Z_0^0 > 4\varepsilon\right). \qquad \text{(II.3.24)}$$

Using (II.3.22), (II.3.23) and (II.3.24) we deduce that

$$\mathbb{P}\left(Z_0^0 > 4\varepsilon\right) \leq \mathbb{P}\left(\widetilde{Z}_0^{\eta_\varepsilon} > 0\right) \leq \mathbb{P}\left(\widetilde{Z}_0^{a} > 0\right) + \mathbb{P}(\eta_\varepsilon \leq a) \leq \mathbb{P}\left(\widetilde{Z}_0^{a} > 0\right) + \frac{1}{2}\mathbb{P}\left(Z_0^0 > 4\varepsilon\right).$$

Finally, by Lemma II.3.2

$$0 < \mathbb{P}\left(Z_0^0 > 4\varepsilon\right) \leq 2\mathbb{P}\left(\widetilde{Z}_0^{a} > 0\right),$$

which is the desired conclusion. □

Lemma II.3.4. *For any $a, b > 0$, we have*

$$\mathbb{P}\left(\widetilde{F}_\infty^b = \infty\right) \geq \mathbb{P}\left(\widetilde{Z}_0^a > 0\right).$$

Proof. Fix $\gamma > 0$ and $a, b > 0$, consider the event $A_{\gamma,b} = \left\{\widetilde{F}_\infty^b \leq \gamma\right\}$. By Fubini's theorem,

$$\gamma \geq \mathbb{E}\left(\mathbb{1}_{A_{\gamma,b}}\widetilde{F}_\infty^b\right) = \sum_{n\geq -1}\mathbb{E}\left(\mathbb{1}_{A_{\gamma,b}}\widetilde{Z}_n^b\right) = \sum_{n\geq -1}\int_0^\infty \mathbb{P}\left(A_{\gamma,b}\cap\{\widetilde{Z}_n^b > u\}\right)du.$$

Using $\mathbb{P}(A\cap B) \geq (\mathbb{P}(A) - \mathbb{P}(B^c))_+$ where B^c denotes the complement of B, and recalling (II.3.14), we deduce that

$$\gamma \geq \sum_{n\geq 0}\int_0^\infty \left(\mathbb{P}(A_{\gamma,b}) - \mathbb{P}\left(\widetilde{Z}_n^b \leq u\right)\right)_+ du = \sum_{n\geq 0}\int_0^\infty \left(\mathbb{P}(A_{\gamma,b}) - \mathbb{P}\left(\widetilde{Z}_0^{2^{-nH}b} \leq u\right)\right)_+ du.$$

There exists $M \geq 1$ such that $2^{-nH}b \leq a$ for all $n \geq M$. Then, for all $n \geq M$,

$$\mathbb{P}\left(\widetilde{Z}_0^{2^{-nH}b} \leq u\right) \leq \mathbb{P}\left(\widetilde{Z}_0^a \leq u\right)$$

and

$$\gamma \geq \sum_{n\geq M}\int_0^\infty \left(\mathbb{P}(A_{\gamma,b}) - \mathbb{P}\left(\widetilde{Z}_0^a \leq u\right)\right)_+ du.$$

Since the summand does not depend on n and the series is bounded by γ and thus finite, one has necessarily

$$\int_0^\infty \left(\mathbb{P}(A_{\gamma,b}) - \mathbb{P}\left(\widetilde{Z}_0^a \leq u\right)\right)_+ du = 0.$$

Hence, for almost every $u \geq 0$ and every $\gamma \geq 0$,

$$\mathbb{P}\left(\widetilde{F}^b_\infty \leq \gamma\right) = \mathbb{P}\left(A_{\gamma,b}\right) \leq \mathbb{P}\left(\widetilde{Z}^a_0 \leq u\right). \tag{II.3.25}$$

We know that $\mathbb{P}(\widetilde{Z}^a_0 \leq u)$ is increasing as a function of u. Hence, (II.3.25) is actually true for *every* $u \geq 0$ and $\gamma \geq 0$. Hence $\mathbb{P}\left(\widetilde{F}^b_\infty > n\right) \geq \mathbb{P}\left(\widetilde{Z}^a_0 > \frac{1}{n}\right)$ for all $n \in \mathbb{N}$. One conclude that

$$\mathbb{P}\left(\widetilde{F}^b_\infty = \infty\right) \geq \mathbb{P}\left(\widetilde{Z}^a_0 > 0\right).$$

□

Chapter III

Fractal dimensions of the Rosenblatt process

The content of this chapter is a copy of the paper entitled “Fractal dimensions of the Rosenblatt process”, written with “George Kerchev”, and submitted to “Stochastic Processes and their Applications”.

III.1 Introduction

The Rosenblatt process $Z = (Z_t)_{t\geq 0}$ is a stochastic process that is a limit of normalized sums of long-range dependent random variables. It belongs to the class of Hermite processes and is the simplest member that is non-Gaussian. It has continuous but nowhere differentiable paths and is selfsimilar of order $H \in (1/2, 1)$ with stationary increments.

The process Z, due to its self-similarity, can find applications across a multitude of fields like internet traffic [23], hydrology, and turbulence [99, 64]. We refer the reader to [35] and [101] for a detailed review of the properties associated with self-similarity. In particular, the Rosenblatt process is used in finance [118, 109, 41] and statistical inference [69, 30, 88].

From a mathematical standpoint the process has received a lot of interest since its inception in [98]. Its distribution is not known in explicit form but was studied first in [2] and more recently in [72] and [123]. There are three integral representations: in terms of time, the spectrum and on finite intervals, see [113]. There is also a wavelet representation [95] (see also the recent article [7] for the wavelet representation of the generalized Rosenblatt process and its rate of convergence). From a statistical point of view, the value of the Hurst index H is important for practical applications and various estimators exist, see [10, 121].

In the present paper, we focus on the fractal properties of the random sets and measures determined by the sample paths of Z, i.e., if the underlying probability space is $(\Omega, \mathcal{F}, \mathbb{P})$, we study the function $Z(t) = Z_t(\omega)$, for a fixed $\omega \in \Omega$. Some (random) sets of interest are then:

$$\text{Image set: } Z(E) \coloneqq \{Z(t) : t \in E\}\,; \tag{III.1.1}$$
$$\text{Graph set: } Gr_Z(E) \coloneqq \{(t, Z(t)) \in E \times \mathbb{R} : t \in E\}\,; \tag{III.1.2}$$
$$\text{Level set: } \mathcal{L}_Z(x) \coloneqq \{t \in \mathbb{R}_+ : Z(t) = x\}\,, x \in \mathbb{R}; \tag{III.1.3}$$
$$\text{Sojourn set: } E_Z(\gamma) \coloneqq \{t \in \mathbb{R}_+ : |Z(t)| \leq t^\gamma\}\,, \gamma > 0; \tag{III.1.4}$$
$$\text{Inverse image: } Z^{-1}(E') \coloneqq \{t \in \mathbb{R}_+ : Z(t) \in E'\}\,, \tag{III.1.5}$$

where $E \subset \mathbb{R}_+$ and $E' \subset \mathbb{R}$ are Borel sets. These sets , due to self-similarity property of Z, may look like a fractal, see, e.g., Figure III.1, for the sojourn set of the Rosenblatt process. In order to describe such sets quantitatively one can use a type of fractal dimension.

Figure III.1: Simulation of a Rosenblatt process of Hurst index $H = 0.6$. In red - the sojourn set $E_Z(\gamma)$ for $\gamma = 0.6$.

Fractal dimensions give you an intuition about the geometry of a set. Having identified some interesting random sets and possible ways to measure them, we note that such studies can be traced to the pioneering work of Lévy [68] and Taylor [114, 115, 117] on the sample path properties of the Brownian motion. We refer the reader to [103] and [130] for surveys of such results for Lévy and Markov processes respectively.

An important class of such dimensions reflects local properties of the set. One important example is the *classical Hausdorff dimension*, which can be defined as follows using the Hausdorff content, see [38, Section 3.2]. For $E \subset \mathbb{R}$,

$$\dim_H(E) \coloneqq \inf\left\{ s \geq 0 \,:\, \forall \varepsilon > 0, \exists \text{ cover } \{U_i\}_{i=1}^{\infty} \text{ of } E, \text{ s.t. } \sum_{i=1}^{\infty} |U_i|^s \leq \varepsilon \right\}, \tag{III.1.6}$$

where $|F|$ denotes the *diameter* of the set F. Moreover by imposing further restrictions on the sets in the cover $\{U_i\}$ one can recover the definitions of *box dimension*. In particular, for $E \subset \mathbb{R}$, the *lower box dimension* is given by:

$$\underline{\dim}_B (E) := \inf \left\{ s \geq 0 : \begin{array}{c} \forall \varepsilon > 0, \exists \text{ cover } \{U_i\}_{i=1}^{\infty} \text{ of } E, \text{ s.t.} \\ |U_i| = |U_j| \ \forall i,j \text{ and } \sum_{i=1}^{\infty} |U_i|^s \leq \varepsilon \end{array} \right\}. \tag{III.1.7}$$

Similarly, we define the *upper box dimension*:

$$\overline{\dim}_B (E) := \inf \left\{ s \geq 0 : \begin{array}{c} \forall \varepsilon > 0, \exists \delta > 0, \forall \text{ cover } \{U_i\}_{i=1}^{\infty} \text{ of } E, \text{ s t.} \\ |U_i| \leq \delta, \ |U_i| = |U_j| \ \forall i,j \text{ and } \sum_{i=1}^{\infty} |U_i|^s \leq \varepsilon \end{array} \right\}. \tag{III.1.8}$$

The box dimension $\dim_B (E)$ is then given by the common value (if it exists) of $\underline{\dim}_B (E)$ and $\overline{\dim}_B (E)$. Next for $\theta \in [0,1]$, the *θ-intermediate dimensions* $\dim_\theta (E)$ is a dimension that interpolate between the *Hausdorff* and *box dimensions* by increasing restriction on the relative sizes of covering sets as θ increases ($\delta^{1/\theta} \leq |U_i| \leq \delta$ for all i). In particular, one defines $\underline{\dim}_\theta (E)$ and $\overline{\dim}_\theta (E)$ similarly to $\underline{\dim}_B (E)$ and $\overline{\dim}_B (E)$. Then $\dim_\theta (E)$ is the common value if it exists of $\underline{\dim}_\theta (E)$ and $\overline{\dim}_\theta (E)$.

One need not consider only covers for the set E. For example, $\dim_B (E)$ can be defined alternatively using coverings by small balls of equal radius (corresponding to $\underline{\dim}_B (E)$) or using packings by disjoint balls of equal radius that are as dense as possible (corresponding to $\overline{\dim}_B (E)$), see [38, Section 3.4]. If the radii are allowed to differ the covering procedure corresponds to the classical Hausdorff dimension while the packing one is associated to the *packing dimension* $\dim_P (E)$. In linear programming the packing and covering problems are dual of each other and thus the packing dimension can be considered as the *dual analogue* to the classical Hausdorff dimension. The precise definitions are delayed to Section III.3.

Other definitions of the packing and intermediate dimensions are possible by employing methods from potential theory. Thus, $\dim_P (E)$, $\underline{\dim}_\theta (E)$ and $\overline{\dim}_\theta (E)$ can be expressed via capacities with respect to certain kernels, see [40, 22]. This gives rise to *packing and intermediate dimension profiles* - $\dim_{P,\alpha} (E)$, $\underline{\dim}_{\theta,\alpha} (E)$ and $\overline{\dim}_{\theta,\alpha} (E)$ respectively. See III.3.2 for the precise definitions.

All the dimensions in the discussion above pertain to local properties of the set. It is often the case, for instance in statistical physics, that one needs to quantify global properties of an infinite set. The simplest way of assessing the size of such a set is given by its (Lebesgue) density at infinity. In particular, we utilize the *logarithmic density* $\mathrm{Den}_{\log} (E)$ and the *pixel density* $\mathrm{Den}_{pix} (E)$ (the latter corresponding to the "pixelated" image). Alternatively, one can use the *macroscopic Hausdorff dimension* $\mathrm{Dim}_H (E)$ introduced in [11, 12] for the study of the macroscopic properties of random walks. More recent applications can be found in the study of high peaks of solutions of the stochastic heat equation [59, 61]. Definitions of these concepts are provided in Section III.4. A brief summary of all dimensions discussed can be seen in Table III.1.

We also mention a few relations between the dimensions mentioned so far to give the reader some intuition:

$$\dim_H (E) \leq \underline{\dim}_B (E) \leq \overline{\dim}_B (E); \quad \dim_H (E) \leq \underline{\dim}_\theta (E) \leq \overline{\dim}_\theta (E) \leq \overline{\dim}_B (E);$$
$$\dim_P (E) \leq \overline{\dim}_B (E); \quad \mathrm{Den}_{\log} (E) \leq \mathrm{Den}_{pix} (E).$$

Dimension	Name	Cover	Size	Values	Limit
$\dim_H(\cdot)$	Classical Hausdorff	Covering	$(0,\delta]$	$[0,1]$	$\delta\to 0$
$\dim_B(\cdot)$	Box	Upper - Covering Lower - Packing	δ	$[0,1]$	$\delta\to 0$
$\dim_P(\cdot)$	Packing	Packing	$(0,\delta]$	$[0,1]$	$\delta\to 0$
$\dim_\theta(\cdot)$	Intermediate		$\in(\delta^{1/\theta},\delta)$	$[0,1]$	$\delta\to 0$
$\mathrm{Den}_{\log}(\cdot)$	Logarithmic density	Interval	$[1,2^n]$	$[0,1]$	$n\to\infty$
$\mathrm{Den}_{pix}(\cdot)$	Pixel density	Interval*	$[1,2^n]$	$[0,1]$	$n\to\infty$
$\mathrm{Dim}_H(\cdot)$	Macroscopic Hausdorff	Collections of sets in $[2^{n-1},2^n)$	$(0,\delta]$	$[0,1]$	$n\to\infty$

Table III.1: Overview of the types of fractal dimensions. For the pixel density the cover consists of the integer points in the interval at distance less than 1 from E.

Before we list our main results, we outline what is known regarding fractal properties of sample paths of a Hermite process of rank 1, i.e., the fractional Brownian motion. The fractional Brownian motion $X=(X_t)_{t\geq 0}$, like Z, is a selfsimilar stochastic process with stationary increments. Both processes, X and Z, share the same covariance structure and are governed by a parameter H (called Hurst parameter in both cases). Unlike the Rosenblatt process, the process X is Gaussian and $H\in(0,1)$. See Table III.2 for an overview of some fractal properties of sets associated with the sample paths of the fractional Brownian motion.

	$X(E)$	$\mathcal{L}_X(x)$	$E_X(\gamma)$
$\dim_P(\cdot)$	$\frac{1}{H}\dim_{P,H}(E)$ [129]		1
$\underline{\dim}_\theta(\cdot)$	$\frac{1}{H}\underline{\dim}_{\theta,H}(E)$ [21]		1
$\dim_H(\cdot)$	$\min\left(1,\frac{1}{H}\dim_H(E)\right)$ [54]	$1-H$ [38]	1
$\mathrm{Dim}_H(\cdot)$		$1-H$ [28]	$1-H$ [87]
$\mathrm{Den}_{pix}(\cdot)$			$\gamma+1-H$ [87]
$\mathrm{Den}_{\log}(\cdot)$			$\gamma+1-H$ [87]

Table III.2: Table of fractal dimensions and densities of random sets associated with the fractional Brownian motion with $\gamma\in[0,H)$.

For completeness we mention also some results regarding the graph and the inverse sets. If $X:\mathbb{R}^N\mapsto\mathbb{R}^d$ is a fractional Brownian sheet, it has been proved in [1] that, almost surely, $\dim_H\left(Gr_X([0,1]^N)\right)=\min\{N/H,N+(1-H)d\}$. The box dimension of the graph of the fractional Brownian sheet over a non degenerate cube Q of $\mathbb{R}^N$ was determined in [55].

Moreover, with probability 1, $\dim_B (Gr_X (Q)) = N + 1 - H$. Regarding the inverse set, the following holds: for E a closed subset of $\mathbb{R}^d$, $\dim_H (X^{-1} (E)) = N - Hd + \dim_H (E)$ (see [82]). We believe that analogous results can be established for the Rosenblatt process, but the sets in question are not the subject of the current paper.

Many of the results listed above rely on Hölder regularity conditions for the sample paths, and more precisely, for the local time of the process. Such properties have been established for stationary Gaussian processes, like the fractional Brownian motion, by Berman in [16]. His analytic approach, which is based on properties of the Fourier transform of the underlying process, has been adapted to the Rosenblatt setting in [106] where existence of the local time of Z was first established. Hölder regularity was then recovered in the recent paper [57]. These new results now allow to generalize some of the results in Table III.2 for the Rosenblatt case. See Table III.3.

	$Z(E)$	$\mathcal{L}_Z(x)$	$E_Z(\gamma)$
$\dim_P(\cdot)$	$\frac{1}{H}\dim_{P,H}(E)$ [107]	$1-H$	1
$\underline{\dim}_\theta(\cdot)$	$\frac{1}{H}\underline{\dim}_{\theta,H}(E)$	$1-H$	1
$\dim_H(\cdot)$	$\min\left(1, \frac{1}{H}\dim_H(E)\right)$ [107]	$1-H$	1
$\mathrm{Dim}_H(\cdot)$		$1-H$	$1-H$
$\mathrm{Den}_{pix}(\cdot)$			$\gamma+1-H$
$\mathrm{Den}_{\log}(\cdot)$			$\gamma+1-H$

Table III.3: Table of fractal dimensions and densities of random sets associated with the Rosenblatt process with $\gamma \in [0, H)$.

All results in Table III.3 but the ones for the dimensions of the image of the process $Z(E)$ are new. Our findings are collected in the following three propositions. First, for the image set we extend the results of [107] to the intermediate dimensions setting, as in [21]:

Theorem III.1.1. *Let $\theta \in (0,1]$ and $E \subset \mathbb{R}^+$ be compact. Then almost surely:*

$$\underline{dim}_\theta (Z(E)) = \frac{1}{H}\underline{dim}_{\theta,H} (E), \tag{III.1.9}$$

and

$$\overline{dim}_\theta (Z(E)) = \frac{1}{H}\overline{dim}_{\theta,H} (E), \tag{III.1.10}$$

where $\underline{dim}_{\theta,H}(\cdot)$ and $\underline{dim}_{\theta,H}(\cdot)$ are the lower and upper θ-intermediate dimension profiles respectively. For the precise techinical definitions of these two objects see (III.3.7) *and* (III.3.8) *in Section III.3.2.*

Then, we study the proportion of time spent by a stochastic process in a given region. We describe the size of the level sets $\mathcal{L}_Z(x)$ in terms of intermediate dimensions and macroscopic Hausdorff dimension. The following holds:

Theorem III.1.2. *For $E \subset \mathbb{R}$ and $\theta \in [0,1]$, let $dim_\theta(E)$ and $Dim_H(E)$ denote the θ-intermediate and macroscopic Hausdorff dimensions of E. Then, for any $x \in \mathbb{R}$ and $0 < \varepsilon < 1$,*

$$\forall x \in \mathbb{R}, \mathbb{P}\left(dim_\theta\left(\mathcal{L}_Z(x) \cap [\varepsilon, 1]\right) = 1 - H\right) = 1. \tag{III.1.11}$$

And,

$$\forall x \in \mathbb{R}, \mathbb{P}(dim_P\left(\mathcal{L}_Z(x) \cap [\varepsilon, 1]\right) = 1 - H) = 1. \tag{III.1.12}$$

Moreover,

$$\mathbb{P}(\forall x \in \mathbb{R} : \, Dim_H\left(\mathcal{L}_Z(x)\right) = 1 - H) = 1. \tag{III.1.13}$$

We believe that the same uniform result holds for classical Hausdorff dimension but we only prove the pointwise one. Finally, we establish the results for the sojourn times $E_Z(\gamma)$:

Theorem III.1.3. *For $E \subset \mathbb{R}$, let $Den_{pix}(E)$ and $Den_{\log}(E)$ denote the pixel and logarithmic densities of E. Then, for all $\gamma \in [0, H]$,*

$$Den_{pix}\left(E_Z(\gamma)\right) = Den_{\log}\left(E_Z(\gamma)\right) = \gamma + 1 - H, \quad a.s. \tag{III.1.14}$$

Moreover,

$$Dim_H\left(E_Z(\gamma)\right) = 1 - H \quad a.s. \tag{III.1.15}$$

To fill the missing entries in Table III.3 one needs new techniques. In particular, the macroscopic Hausdorff dimension and the two densities of the image set $Z(E)$ should depend on the fractional properties of E (in particular should be 0 if E is bounded). However, intuition regarding this relation is missing. Regarding, the level set $\mathcal{L}_Z(x)$, the approach for the macroscopic Hausdorff dimension does not translate since the key result (Lemma III.4.2) is an artifact of the definition of Dim_H.

The authors believe that many of the results above can be extended to some generalizations of the Rosenblatt process, for instance, when the time and space sets are N and d dimensional, or when the Hurst index is a function of time, as in [106]. To ease the presentation only the case $N = d = 1$ and $H \in (1/2, 1)$ - fixed is considered. However, establishing the results for Hermite processes of rank above 2 requires new techniques and is beyond the scope of the current paper. In particular, the Berman analytic approach relies on a "good" representation for the Fourier transform of the process and this is not known for Hermite processes of higher rank.

The structure of the paper is as follows. The three main results listed above are established in Sections III.3-III.5. Some necessary technical properties of the Rosenblatt process are reviewed and proved in Section III.2.

III.2 Properties of the Rosenblatt process

The Rosenblatt process is formally defined, for $t \geq 0$ and $H \in (1/2, 1)$, as

$$Z_t^H := \lim_{n\to\infty} \frac{\sigma}{n^H} \sum_{k=1}^{\lfloor nt \rfloor} (Y_k^2 - 1), \tag{III.2.1}$$

where $(Y_k)_{k\geq 0}$ is a Gaussian sequence of mean zero, unit variance and covariance $\mathbb{E}[Y_0 Y_k] = (1+k^2)^{-(1-H)/2}$. The series converges in terms of finite dimensional distributions but also as weak convergence of probability measures (see [113] for further details). The parameter σ is an arbitrary constant and is taken such that the limit Z_t^H has unit variance. Letting $t = 1$, one recovers the example, constructed by Rosenblatt in [98], to highlight the limitations of a cental limit theorem for stronly mixing sequences also stated in [98].

The Rosenblatt process can also be defined in terms of Wiener-Itô stochastic integrals.

Following [113], $(Z_t^H)_{t\geq 0}$ is defined as the double Wiener integral with respect to a standard Brownian motion $\{B(x)\}_{x\in\mathbb{R}}$:

$$Z_t^H = \int_{\mathbb{R}^2}' K_H(t, x_1, x_2)\, dB(x_1) dB(x_2), \tag{III.2.2}$$

where $\int_{\mathbb{R}^2}'$ denotes integration over $\mathbb{R}^2$ excluding the diagonal and the kernel function K_H given, for all (t, x_1, x_2) on $\mathbb{R}_+ \times \mathbb{R}^2$, by

$$K_H(t, x_1, x_2) = c(H) \int_0^t (s - x_1)_+^{H-\frac{3}{2}} (s - x_2)_+^{H-\frac{3}{2}}\, ds,$$

with $x^+ = max(x, 0)$. The constant $c(H)$ is a positive normalizing constant and it is chosen such that $\mathbb{E}((Z_1^H)^2) = 1$. More precisely,

$$c(H)^2 = \frac{(H - 1/2)(4H - 3)}{\beta(H - 1/2, 2 - 2H)}$$

This definition is also known as the *time* representation of the Rosenblatt process. Another closely related representation is the *spectral* representation of Z (see [113] and [33]):

$$Z_t^H = C(H) \int_{\mathbb{R}^2} \frac{e^{i(x+y)t} - 1}{i(x + y)} Z_G(dx) Z_G(dy), \tag{III.2.3}$$

where the double Wiener-Ito integral is taken over $x \neq \pm y$ and $Z_G(dx)$ is a complex-valued random white noise with control measure G satisfying $G(tA) = t^{1-H} G(A)$ for all $t \in \mathbb{R}$ and $G(dx) = |x|^{-H} dx$. The constant $C(H)$ in (III.2.3) is such that

$$\mathbb{E}[Z_t^2] = t^{2H} \quad \text{and} \quad \mathbb{E}[Z_t Z_s] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t - s|^{2H}\right),$$

for all $s, t \geq 0$.

Remark 9. Note that in the notation of [113], $Z_G(dx) = |x|^{-H/2} d\hat{B}(x)$, with $(B(t))_{t\in\mathbb{R}}$ the Brownian motion and $d\hat{B}(x)$ is viewed as the complex-valued Fourier transform of $dB(x)$. For more details, see [111].

It is known (see [119]) that the Rosenblatt process has the following properties:

(1) self-similarity: Z is H-self-similar; that is, the processes $\{Z_{ct},\, t \geq 0\}$ and $\{c^H Z_t,\, t \geq 0\}$ have the same distribution.

(2) stationary increments: Z has stationary increments; that is, the distribution of the process $\{Z_{t+s} - Z_s,\, t \geq 0\}$ does not depend on $s \geq 0$.

(3) continuity: the trajectories of the Rosenblatt process Z are δ-Hölder continuous for every $\delta < H$.

We will mention one more property that will be needed in our proofs, and is a consequence of the finite time interval representation [113, Section 7.3] of the Rosenblatt process. The natural filtration associated to a Rosenblatt process is Brownian, i.e., there is a Brownian motion $(B_t)_{t\geq 0}$ defined on the same probability space than Z such that its filtration satisfies

$$\sigma\{Z_s \,:\, s \leq t\} \subset \sigma\{B_s \,:\, s \leq t\}, \tag{III.2.4}$$

for all $t > 0$.
Moreover, by [72, Theorem 1.1], for any $d \geq 1$ and $t_1, \ldots, t_d \geq 0$,

$$(Z_{t_1}, \ldots, Z_{t_d}) \overset{(d)}{=} \left(\sum_{n=1}^{\infty} \lambda_n(t_1)(\varepsilon_n^2 - 1), \ldots, \sum_{n=1}^{\infty} \lambda_n(t_d)(\varepsilon_n^2 - 1) \right), \tag{III.2.5}$$

where $(\varepsilon_n)_{n\geq 1}$ are i.i.d $\mathcal{N}(0,1)$ random variables and $(\lambda_n(t))_{n\geq 1}$ are the (real) eigenvalues of a self-adjoint Hilbert-Schmidt operator associated with the process Z (see [33]).

For our analysis a few properties of the density for the joint process (Z_{t_1}, Z_{t_2}) are needed. Using techniques from [57] we can establish the following:

Proposition III.2.1.

(i) The probability density function $f : \mathbb{R} \to \mathbb{R}_+$ of Z_1 is continuous and $f(x) > 0$ for $x \geq 0$.

(ii) For every $t_1, \ldots, t_n \geq 0$, the vector $(Z_{t_1}, \ldots, Z_{t_n})$ has a continuous density.

Proof of Proposition III.2.1. (i) The density f of Z_1 is continuous (see [123, Corollary 4.3]) and unimodal (see [72]). Therefore $f(0) > 0$ since $\mathbb{E}[Z_1] = 0$. To see that $f(y) > 0$ for all $y > 0$, recall [123, Corrolary 4.5]: for $\alpha > 0$,

$$\lim_{u\to\infty} \frac{\mathbb{P}(Z_1 > u + \alpha)}{\mathbb{P}(Z_1 > u)} = c_H,$$

for a deterministic constant $c_H > 0$. In particular, this shows that for every $y \in \mathbb{R}_+$, there is $x > y$, such that $f(x) > 0$. Combined with the fact that f is continuous and unimodal, this implies that $f(x) > 0$ for every $x \in \mathbb{R}_+$.

(ii) If the characteristic function $\hat{\mu}(z)$ of a probability measure μ in $\mathbb{R}^d$ is integrable, then μ has a continuous density $g(x)$ that tends to 0 as $|x| \to \infty$ (see [103, Proposition 2.5(xii)]). Therefore, it is enough to show that for all $t \in \mathbb{R}_+^n$:

$$\int_{\mathbb{R}^d} \left| \mathbb{E} \exp\left(i \sum_{j=1}^n \xi_j Z_{t_j} \right) \right| d\xi < \infty.$$

At this point we recall [57, Lemma 2.1, 2.2].

Lemma III.2.2. *Let $L^2_G(\mathbb{R})$ be a weighted space with norm $\|f\|^2_{L^2_G} := \int_{\mathbb{R}} |f(x)|^2 G(x) dx$. For $t \in \mathbb{R}_+^n$, $\xi \in \mathbb{R}^n$, let $A_{t,\xi} : L^2_G(\mathbb{R}) \to L^2_G(\mathbb{R})$ be the operator given by*

$$(A_{t,\xi} f)(x) = \int_{\mathbb{R}} \sum_{j=1}^n \xi_j \frac{e^{it_j(x-y)} - 1}{i(x-y)} f(y) |y|^{-H/2} dy.$$

Let $(\lambda_k(t,\xi))_{k\geq 1}$ be the set of eigenvalues of $A_{t,\xi}$. Then,

$$\left| \mathbb{E} \exp\left(i \sum_{j=1}^n \xi_j Z_{t_j} \right) \right| = \prod_{k\geq 1} \frac{1}{(1 + 4\lambda_k(t,\xi))^{1/4}}.$$

Moreover, if $t_0 = 0 < t_1 < \cdots < t_n \leq 1$, for every $k \geq 1$,

$$\lambda_k(t,\xi) \geq C(H) (\max_{1\leq j\leq n} |\xi_j - \xi_{j-1}| |t_j - t_{j-1}|^H)^2 \tilde{\lambda}_k^4, \tag{III.2.6}$$

where $\tilde{\lambda}_k \sim k^{-H/2}$ (independent of t and ξ), $\xi_0 = 0$ and $C(H) > 0$ is a constant that only depends on H.

Now, we follow a similar procedure to the one employed for the proof of [57, Proposition 1.3]. Let $f_0 : \mathbb{R}_+^n \times \mathbb{R}^n \to \mathbb{R}_+$ be given by

$$f_0(t,y) := t_1^H |y_1| \vee t_2^H |y_2| \vee \cdots \vee t_n^H |y_n|. \tag{III.2.7}$$

Further, let $\xi' = (\xi_1 - \xi_0, \xi_2 - \xi_1, \ldots, \xi_n - \xi_{n-1})$ and $t' = (t_1 - t_0, t_2 - t_1, \ldots, t_n - t_{n-1})$. Then

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left| \mathbb{E} \exp\left(i \sum_{j=1}^n \xi_j Z_{t_j} \right) \right| d\xi \\
&= \int_{\mathbb{R}^d} \prod_{k\geq 1} (1 + 4\lambda_k(\xi,t))^{-1/4} d\xi \\
&\leq \int_{\mathbb{R}^d} \prod_{k\geq 1} \left(1 + 4C(H) (\max_{1\leq j\leq n} |\xi_j - \xi_{j-1}| |t_j - t_{j-1}|^H)^2 \tilde{\lambda}_k^4 \right)^{-1/4} d\xi \\
&= \int_{\mathbb{R}^d} \prod_{k\geq 1} \left(1 + 4C(H) f_0^2(t',\xi') \tilde{\lambda}_k^4 \right)^{-1/4} d\xi'.
\end{aligned} \tag{III.2.8}$$

Let

$$G(s) := \prod_{k\geq 1}(1+4s^2\tilde{\lambda}_k^4)^{-1/4}.$$

We can now switch to polar coordinates in (III.2.8) via $|\xi'| = r'$, $\xi'/r' = w'$:

$$\begin{aligned}
&\int_{\mathbb{R}^d}\prod_{k\geq 1}\left(1+4C(H)f_0^2(t',\xi')\tilde{\lambda}_k^4\right)^{-1/4}d\xi' \\
\leq &C\int_{|w'|=1}\int_0^\infty (r')^{n-1}G(\sqrt{C(H)}r'f_0(t',w'))dr'\mathcal{H}^{n-1}(dw') \\
= &C\left(\int_0^\infty R^{n-1}G(R)dR\right)\left(\int_{|w'|=1}(f_0(t',w'))^{-n}\mathcal{H}^{n-1}(dw')\right),
\end{aligned}$$

where $\mathcal{H}^{n-1}(dw')$ is the $(n-1)$-dimensional Hausdorff measure on the unit sphere, $C>0$ is a constant that depends on H and the last equality follows with the change of variables $R = \sqrt{C(H)}r'f(t',w')$.

Next, recall [57, Lemma 2.3] that $G(s)$ is finite and positive for any $s>0$ and moreover there are constants $c_1, c_2 > 0$ such that for all $\beta \geq 1$,

$$\int_0^\infty s^{\beta-1}G(s)ds \leq c_2 H c_1^{-\beta H}\Gamma(\beta H),$$

where Γ is the Gamma function.

Finally, since $t_1, \ldots, t_n > 0$ are fixed, by the definition (III.2.7) of $f_0(t',w')$,

$$\begin{aligned}
\int_{|w'|=1}(f_0(t',w'))^{-n}\mathcal{H}^{n-1}(dw') \leq &C(t')\int_{|w'|=1}(|w_1|\vee\cdots\vee|w_n|)^{-n}\mathcal{H}^{n-1}(dw') \\
\leq &C(t')\int_{|w'|=1}(n^{-1/2})^{-n}\mathcal{H}^{n-1}(dw') < \infty,
\end{aligned}$$

where $C(t') := \inf\{t_1'^{Hn}, \ldots, t_n'^{Hn}\}$ is a positive constant.

Therefore, the characteristic function of $(Z_{t_1}, \ldots, Z_{t_n})$ is integrable and thus the joint distribution has a continuous density.

□

Next we establish a time inversion property for the Rosenblatt process:

Proposition III.2.3. *The inverse time process*

$$t \mapsto \tilde{Z}_t := t^{2H}Z_{1/t}, \tag{III.2.9}$$

is also a Rosenblatt process.

Proof. First, using the spectral representation of a Rosenblatt process (III.2.3),

$$\begin{aligned} t^{2H} Z_{1/t} &\stackrel{(d)}{=} C(H) t^{2H} \int_{\mathbb{R}^2} \frac{e^{i(x+y)/t} - 1}{i(x+y)} Z_G(dx) Z_G(dy) \\ &= C(H) t^{2H} \int_{\mathbb{R}^2} \frac{e^{i(x'+y')t} - 1}{i(x'+y')t^2} Z_G(t^2 dx') Z_G(t^2 dy'), \end{aligned}$$

with the change of variables $x = x't^2$ and $y = y't^2$. Now recall the change of variables formula for the Itô integral [32, Proposition 4.2]:

Proposition III.2.4. *Let G and G' be two non-atomic spectral measures such that G is absolutely continuous with respect to G', and let $g(x)$ be a complex valued function such that*

$$\begin{aligned} g(x) =& \overline{g(-x)}, \\ |g^2(x)| =& \frac{d(G(x))}{d(G'(x))}. \end{aligned}$$

Let $f : \mathbb{R}^2 \to \mathbb{C}$ be a measurable function such that:

1. $f(-x_1, -x_2) = \overline{f(x_1, x_2)}$, *and*
2. $||f||^2 = \int |f(x_1, x_2)|^2 G(dx_1) G(dx_n) < \infty$.

Then, for $f'(x_1, x_2) = f(x_1, x_2) g(x_1) g(x_2)$,

$$\int f(x_1, x_2) Z_G(dx_1) Z_G(dx_2) \stackrel{(d)}{=} \int f'(x_1, x_2) Z_{G'}(dx_1) Z_{G'}(dx_2).$$

Let $G_{t^2}(A) := G(At^2) = t^{2(1-H)} G(A)$ for every measurable A. We apply Proposition III.2.4 with G and G_{t^2}, i.e., with $|g(x)|^2 = t^{2(1-H)}$ a constant depending on t. Then,

$$\begin{aligned} & C(H) t^{2H} \int_{\mathbb{R}^2} \frac{e^{i(x'+y')t} - 1}{i(x'+y')t^2} Z_G(t^2 dx') Z_G(t^2 dy') \\ = \; & C(H) t^{2H} \int_{\mathbb{R}^2} \frac{e^{i(x'+y')t} - 1}{i(x'+y')t^2} Z_{G_{t^2}}(dx') Z_{G_{t^2}}(dy') \\ \stackrel{(d)}{=} \; & C(H) t^{2H} \int_{\mathbb{R}^2} \frac{e^{i(x'+y')t} - 1}{i(x'+y')t^2} t^{2(1-H)} Z_G(dx') Z_G(dy') \\ = \; & C(H) \int_{\mathbb{R}^2} \frac{e^{i(x'+y')t} - 1}{i(x'+y')} Z_G(dx') Z_G(dy'), \end{aligned}$$

and we recover the spectral representation of Z_t as desired.

□

Remark 10. For the fractional Brownian motion B_t^H of Hurst index $H \in (1/2, 1)$, the same fact is established using that the process is Gaussian and by comparing covariance functions. However, this property can also be recovered using the approach above. Indeed, we have the following spectral representation:

$$B_t^H \stackrel{(d)}{=} C(H) \int_{\mathbb{R}} \frac{e^{i\lambda t} - 1}{i\lambda} \frac{1}{|\lambda|^{H-1/2}} d\hat{B}(\lambda).$$

The same change of variables yields the desired conclusion.

We also recall a result [57, Proposition 4.2] regarding oscillations:

Proposition III.2.5. *Let $(Z_t)_{t\geq 0}$ be the Rosenblatt process. Then for any $s > 0$ and $h \in (0, s)$,*

$$\mathbb{P}\left(\sup_{t\in[s-h,s+h]} |Z_t - Z_s| \geq u \right) \leq C \exp\left(-\frac{u}{c_1 h^H} \right),$$

where c_1 and C are constants that depend only on H.

We need the following properties of the local time of the Rosenblatt process. Its existence was shown in [106] and one has the representation:

$$L(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_0^t e^{i\xi(x-Z_s)} ds d\xi. \tag{III.2.10}$$

As we mentioned in the beginning of this section, Z is selfsimilar of index H, then its local time at level x also has some selfsimilarity properties in time with index $1 - H$, but with a different level as stated below. More precisely, one has, for every $c > 0$:

$$(L(x, ct))_{t\geq 0, x\in\mathbb{R}} \stackrel{(d)}{=} c^{1-H} (L(c^{-H} x, t))_{t\geq 0, x\in\mathbb{R}}. \tag{III.2.11}$$

Indeed, for every $c > 0$, $t \geq 0$ and $x \in \mathbb{R}$, one has

$$\begin{aligned} L(x, ct) &= \frac{1}{2\pi} \int_{\mathbb{R}} \int_0^{ct} e^{i\xi(x-Z_s)} ds d\xi = c \frac{1}{2\pi} \int_{\mathbb{R}} \int_0^t e^{i\xi(x-Z_{cs})} ds d\xi \\ &\stackrel{(d)}{=} c \frac{1}{2\pi} \int_{\mathbb{R}} \int_0^t e^{i\xi(x-c^H Z_s)} ds d\xi = c^{1-H} \frac{1}{2\pi} \int_{\mathbb{R}} \int_0^t e^{i\xi(c^{-H}x-Z_s)} ds d\xi = L(c^{-H} x, t). \end{aligned}$$

Moreover, a recent result [57, Theorem 1.4] describes the scaling behavior of the local time of Z:

Proposition III.2.6. *The local time $L(x, [0, t])$ is jointly continuous with respect to (x, t) and has finite moments. For a finite closed interval $I \subset (0, \infty)$, let $L^*(I) = \sup_{x\in\mathbb{R}} L(x, I)$. There exist positive constants C_1 and C_2 such that, almost surely, for any $s \in I$,*

$$\limsup_{r\to 0} \frac{L^*([s - r, s + r])}{r^{1-H} (\log\log r^{-1})^{2H}} \leq C_1, \tag{III.2.12}$$

and

$$\limsup_{r\to 0} \sup_{s\in I} \frac{L^*([s - r, s + r])}{r^{1-H} (\log r^{-1})^{2H}} \leq C_2. \tag{III.2.13}$$

Furthermore, we can establish the following property which is key in the study of the classical Hausdorff dimension of the level sets.

Proposition III.2.7. *For* $\beta \in \left(0, \frac{1}{2}\left(\frac{1}{H} - 1\right)\right)$,

$$\mathbb{P}\left(\sup_{x\in[-1,1]\setminus\{0\}} \frac{\left|L\left(0,\left[\frac{1}{2},1\right]\right) - L\left(x,\left[\frac{1}{2},1\right]\right)\right|}{|x|^{\beta}} < \infty\right) = 1.$$

Proof. The result relies on a celebrated lemma due to Garsia, Rodemich and Rumsey [43], as well as on the moment estimates for the local time in [57]. First, let us recall the lemma from [43]:

Lemma III.2.8. *Let $\Psi(u)$ be a non-negative even function on $(-\infty, \infty)$ and $p(u)$ be a non-negative even function on $[-1, 1]$. Assume both $p(u)$ and $\Psi(u)$ are non decreasing for $u \geq 0$. Let f be continuous on $[0, 1]$ and suppose that*

$$\int_0^1 \int_0^1 \Psi\left(\frac{f(u) - f(v)}{p(u-v)}\right) dudv \leq B < \infty.$$

Then, for all $x, y \in [0, 1]$,

$$|f(x) - f(y)| \leq 8 \int_0^{|x-y|} \Psi^{-1}\left(\frac{4B}{u^2}\right) dp(u).$$

where Ψ^{-1} denotes the generalized inverse of Ψ.

Let $\Psi(u) = |u|^p$ and $p(u) = |u|^{\alpha+1/p}$ where $\alpha \geq 1/p$ and $p \geq 1$. Then for any continuous f and $x \in [0, 1]$,

$$|f(x) - f(y)|^p \leq C_{\alpha,p}|x-y|^{\alpha p-1} \int_{[0,1]^2} |f(r) - f(v)|^p |r-v|^{-\alpha p-1} drdv.$$

Here the constant $C_{\alpha,p}$ is given by $C_{\alpha,p} = 4 \cdot 8^p \left(\alpha + p^{-1}\right)^p \left(\alpha - p^{-1}\right)^{-p}$. Thus, for fixed α and large enough p, we have $C_{\alpha,p} \leq C(\alpha)^p$, where $C(\alpha) > 0$ is a constant that depends on the chosen α. We apply this to $f(x) = L\left(2x-1, \left[\frac{1}{2}, 1\right]\right)$:

$$\begin{aligned} &\sup_{x\in[-1,1]\setminus\{0\}} \frac{\left|L\left(x,\left[\frac{1}{2},1\right]\right) - L\left(0,\left[\frac{1}{2},1\right]\right)\right|^p}{\left(\frac{x}{2}\right)^{\alpha p-1}} \\ \leq\ & C'^p \int_{[0,1]^2} \left|L\left(r,\left[\frac{1}{2},1\right]\right) - L\left(v,\left[\frac{1}{2},1\right]\right)\right|^p |r-v|^{-\alpha p-1} drdv. \end{aligned}$$

Using the moment bounds for the occupation density established in [57, Theorem 3.1], one has

$$\begin{aligned} &\mathbb{E}\left[\sup_{x\in[-1,1]\setminus\{0\}} \frac{\left|L\left(x,\left[\frac{1}{2},1\right]\right) - L\left(0,\left[\frac{1}{2},1\right]\right)\right|^p}{\left(\frac{x}{2}\right)^{\alpha p-1}}\right] \\ \leq\ & C'^p \int_{[0,1]^2} \frac{c(\gamma,H)^p p^{p2H(1+\gamma)} 2^{\gamma p}|r-v|^{\gamma p}}{2^{(1-H-\gamma H)}} |r-v|^{-\alpha p-1} drdv, \end{aligned}$$

where $\gamma \in \left[0, \frac{1-H}{2H}\right)$ and $c(\gamma, H) > 0$ is a constant depending only on γ and H. Let $\alpha = \gamma/2$ and $p > 4/\gamma$. Then,

$$\mathbb{E}\left[\sup_{x\in[-1,1]\setminus\{0\}} \frac{\left|L\left(x,\left[\frac{1}{2},1\right]\right) - L\left(0,\left[\frac{1}{2},1\right]\right)\right|^p}{\left(\frac{x}{2}\right)^{\alpha p-1}}\right] \leq C(\gamma, H, p),$$

where $C(\gamma, H, p) > 0$ is a constant that depends on γ, H and p.

Fatou's lemma implies that

$$\mathbb{P}\left(\sup_{x\in[-1,1]/\{0\}} \frac{\left|L\left(0,\left[\frac{1}{2},1\right]\right) - L\left(x,\left[\frac{1}{2},1\right]\right)\right|}{|x|^\beta} < \infty\right) = 1,$$

as desired.

□

Finally, the local time is Hölder continuous in both time and space [57, Corollary 3.2]. In particular:

Proposition III.2.9. *For every $x \in \mathbb{R}$, almost surely, the local time $L(x,t)$ is Hölder continuous in t of order α for every $\alpha \in [0, 1-H)$.*

III.3 Image sets

The present section is dedicated to the study of intermediate dimensions and profiles. To make a comparison, we recall the more popular packing dimensions and profiles.

III.3.1 Packing dimensions

First, recall the definition of the packing dimension. For any $\alpha > 0$, the $\alpha-$dimensional packing measure of $E \subset \mathbb{R}^N$ is

$$\mathcal{P}^s(E) := \inf\left\{\sum_n \mathcal{P}_0^s(E_n) : E \subseteq \bigcup_n E_n\right\},$$

where for $E \subset \mathbb{R}$,

$$\mathcal{P}_0^s(E) := \lim_{\varepsilon\to 0}\sup\left\{\sum_i (2r_i)^s : \overline{B}(x_i, r_i) \text{ are disjoint }, x_i \in E, r_i < \varepsilon\right\}.$$

The packing dimension of E is

$$\dim_P(E) := \inf\{s > 0 : \mathcal{P}^s(E) = 0\} \tag{III.3.1}$$

and the packing dimension of a Borel measure μ on $\mathbb{R}^N$ is defined by

$$\dim_P(\mu) := \inf\{\dim_P(E) \,:\, \mu(E) > 0 \text{ and } E \subset \mathbb{R}^N \text{ is a Borel set}\}.$$

Next, we recall the concept of packing dimension profiles first conceived by Falconer and Howroyd in [40] and [48]. For finite Borel measures μ on $\mathbb{R}^N$ and for any $s > 0$, let

$$F_s^{\mu}(x,r) = \int_{\mathbb{R}} \psi_s\left(\frac{x-y}{r}\right) d\mu(y),$$

be the potential with respect to the kernal $\psi_s(x) = \min\{1, \|x\|^{-s}\}, \forall x \in \mathbb{R}^N$.
The packing dimension profile of μ is defined as follows

$$\dim_{P,s}(\mu) = \sup\left\{\beta \geq 0 : \liminf_{r\to 0} \frac{F_s^{\mu}(x,r)}{r^{\beta}} = 0 \text{ for } \mu - a.e.\, x \in \mathbb{R}^N\right\}.$$

Now for any Borel set $E \subset \mathbb{R}^N$, we define $\mathcal{M}_c^+(E)$ to be the family of finite Borel measures on E with compact support in E. Then

$$\dim_P(E) = \sup\left\{\dim_P(\mu) \,:\, \mu \in \mathcal{M}_c^+(E)\right\}.$$

Motivated by this, Falconer and Howroyd [40] define s-dimensional packing dimension profile of $E \subset \mathbb{R}^N$ by

$$\dim_{P,s}(E) = \sup\left\{\dim_{P,s}(\mu) \,:\, \mu \in \mathcal{M}_c^+(E)\right\}.$$

It is easy to see that $0 \leq \dim_{P,s}(E) \leq s$ and for any $s \geq N$, $\dim_{P,s}(E) = \dim_P(E)$.

III.3.2 Intermediate dimensions

For a bounded and non-empty set $E \subset \mathbb{R}^N$, $\theta \in (0,1]$ and $s \in [0,N]$, define

$$H_{r,\theta}^s(E) = \inf\left\{\sum_i |U_i|^s \,:\, \{U_i\}_i \text{ is a cover of } E \text{ such that } r \leq |U_i| \leq r^{\theta} \text{ for all } i\right\}. \tag{III.3.2}$$

In particular, for $\theta = 0$, $H_{r,0}^s(E)$ is the s-dimensional Hausdorff measure of E. Now, the intermediate dimensions are defined as in [38]:

Definition III.3.1. Let $E \subset \mathbb{R}^N$ be bounded. For $0 \leq \theta \leq 1$, the lower θ-intermediate dimension is

$$\underline{\dim}_{\theta}(E) = \text{ the unique } s \in [0,N] \text{ such that } \liminf_{r\to 0} \frac{\log H_{r,\theta}^s(E)}{-\log r} = 0. \tag{III.3.3}$$

Similarly, the upper θ-intermediate dimension of E is defined by

$$\overline{\dim}_{\theta}(E) = \text{ the unique } s \in [0,N] \text{ such that } \limsup_{r\to 0} \frac{\log H_{r,\theta}^s(E)}{-\log r} = 0. \tag{III.3.4}$$

When $\underline{\dim}_{\theta}(E) = \overline{\dim}_{\theta}(E)$, we refer to the θ-intermediate dimension $\dim_{\theta}(E) = \underline{\dim}_{\theta}(E) = \overline{\dim}_{\theta}(E)$.

Thus, the classical Hausdorff (III.1.6) and box dimensions (III.1.7), (III.1.8) can be viewed as the extremes of a continuum of dimensions with increasing restrictions on the relative sizes of covering sets. Indeed, for every bounded $E \subset \mathbb{R}$,

$$\underline{\dim}_0 E = \overline{\dim}_0 E = \dim_H(E), \quad \underline{\dim}_1 E = \underline{\dim}_B(E) \quad \text{and} \quad \overline{\dim}_1 E = \overline{\dim}_B(E).$$

Moreover, the intermediate dimensions can be defined in terms of capacities with respect to an appropriate kernel denoted by $\phi_{r,\theta}^{s,m}$ (see [22]). For each collection of parameters $\theta \in (0,1]$, $0 < m \le 1$, $0 \le s \le m$ and $0 < r < 1$, let $\phi_{r,\theta}^{s,m} : \mathbb{R}^N \to \mathbb{R}$ be the function

$$\phi_{r,\theta}^{s,m}(x) := \begin{cases} 1 & 0 \le |x| < r, \\ \left(\frac{r}{|x|}\right)^s & r \le |x| < r^\theta, \\ \frac{r^{\theta(m-s)+s}}{|x|^m} & r^\theta \le |x|. \end{cases} \tag{III.3.5}$$

Using this kernel we define the *capacity* of a compact set $E \subset \mathbb{R}^N$ as

$$C_{r,\theta}^{s,m}(E) := \left(\inf_{\mu \in \mathcal{M}(E)} \int\int \phi_{r,\theta}^{s,m}(x-y) d\mu(x) d\mu(y) \right)^{-1}, \tag{III.3.6}$$

where $\mathcal{M}(E)$ is the set of probability measures supported in E.

Now for $0 < m \le N$, the *lower intermediate dimension profiles* of $E \subset \mathbb{R}^N$ are

$$\underline{\dim}_{\theta,m}(E) = \left(\text{the unique } s \in [0,m] \text{ such that } \liminf_{r \to 0} \frac{\log C_{r,\theta}^{s,m}(E)}{-\log r} = s \right), \tag{III.3.7}$$

and the *upper intermediate dimension profiles* are

$$\overline{\dim}_{\theta,m}(E) = \left(\text{the unique } s \in [0,m] \text{ such that } \limsup_{r \to 0} \frac{\log C_{r,\theta}^{s,m}(E)}{-\log r} = s \right). \tag{III.3.8}$$

The intermediate dimension profiles are increasing in m and for $E \subset \mathbb{R}^N$,

$$\underline{\dim}_{\theta,N}(E) = \underline{\dim}_\theta(E) \quad \text{and} \quad \overline{\dim}_{\theta,N}(E) = \overline{\dim}_\theta(E).$$

We note that originally the definitions of capacities and profiles above were established for $E \subset \mathbb{R}^N$ and integers $m \in (0, N]$. However, the recent result [21, Lemma 2.1], allows one to work with the version stated above. In fact, our first main result Theorem III.1.1 is an extension of a similar result in [21] obtained for the index-α fractional Brownian motion. We proceed with the proof of Theorem III.1.1

III.3.3 Proof of Theorem III.1.1

Let $\theta \in (0,1]$. We first state two results due to Burrell [21]. The first one establishes an upper bound for the intermediate dimensions of Hölder images using dimension profiles:

Lemma III.3.2. *[21, Theorem 3.1] Let $E \subset \mathbb{R}$ be a compact, $\theta \in (0,1]$, $m \in \{1,...,n\}$ and $f : E \to \mathbb{R}$. If there exist $c > 0$ and $0 < \alpha \leq 1$ such that*

$$|f(x) - f(y)| \leq c|x - y|^{\alpha},$$

for all $x, y \in E$, then

$$\underline{dim}_{\theta} (f(E)) \leq \frac{1}{\alpha} \underline{dim}_{\theta,\alpha} (E) \quad \text{ and } \quad \overline{dim}_{\theta} (f(E)) \leq \frac{1}{\alpha} \overline{dim}_{\theta,\alpha} (E).$$

The second result gives a lower bound for the intermediate dimensions of image of a compact set E under measurable functions satisfying certain properties:

Lemma III.3.3. *[21, Theorem 3.3] Let $E \subset \mathbb{R}$ be a compact, $\theta \in (0,1]$, $\gamma > 1$ and $s \in [0,1)$. If $f : \Omega \times E \to \mathbb{R}$ is a random function such that for each $\omega \in \Omega$, $f(\omega,.)$ is a continuous measurable function and there exists $c > 0$ satisfying*

$$\mathbb{P}\left(\{\omega \in \Omega \,:\, |f(\omega,x) - f(\omega,y)| \leq r\}\right) \leq c\phi_{r^{\gamma},\theta}^{1/\gamma,1/\gamma}(x-y),$$

for all $x, y \in E$ and $r > 0$, then

$$\underline{dim}_{\theta} (f(\omega,E)) \geq \gamma \underline{dim}_{\theta,\alpha} (E) \quad \text{ and } \quad \overline{dim}_{\theta} (f(\omega,E)) \geq \gamma \overline{dim}_{\theta,\alpha} (E),$$

for almost all $\omega \in \Omega$.

Now let $0 < \varepsilon < H < 1$. The Rosenblatt process Z has Hölder continuous paths in time of order $H - \varepsilon$, see [120, Propostion 3.5], and so there exists, almost surely, $M > 0$ such that

$$|Z_s - Z_t| \leq M|s - t|^{H-\varepsilon},$$

for all $s, t \in E$. In addition by Proposition III.2.1(i), the density function f of Z_1 is continuous and $f(0) > 0$. Then for all $s, t \in E$ and $r > 0$, one has

$$\mathbb{P}\left(|Z_s - Z_t| \leq r\right) = \mathbb{P}\left(|Z_1| \leq \frac{r}{|s-t|^H}\right) \leq 4f(0)\frac{r}{|s-t|^H} = 4f(0)\,\phi_{r^{1/H},\theta}^{H,H}(s-t).$$

Now since the profiles are monotonically increasing, by Lemmas III.3.2 and III.3.3, one has almost surely

$$\frac{1}{H}\underline{\dim}_{\theta,H}(E) \leq \underline{\dim}_{\theta}(Z(E)) \leq \frac{1}{H-\varepsilon}\underline{\dim}_{\theta,H-\varepsilon}(E) \leq \frac{1}{H-\varepsilon}\underline{\dim}_{\theta,H}(E),$$

and

$$\frac{1}{H}\overline{\dim}_{\theta,H}(E) \leq \overline{\dim}_{\theta}(Z(E)) \leq \frac{1}{H-\varepsilon}\overline{\dim}_{\theta,H-\varepsilon}(E) \leq \frac{1}{H-\varepsilon}\overline{\dim}_{\theta,H}(E).$$

Letting $\varepsilon \to 0$ establishes the result.

III.4 Level sets

The present section is devoted to the proof of Theorem III.1.2. First, we establish (III.1.11) and (III.1.12) - the result regarding the θ-intermediate dimensions and the packing dimension. Recall that the definition of $\dim_\theta(E)$ and $\dim_P(E)$ for $E \subset \mathbb{R}$ are given in definition III.3.1 and III.3.1 respectively.
Note that the techniques employed in this section apply for the fractional Brownian motion case. As mentioned earlier, [9, Theorem 5] establishes $\dim_B(\mathcal{L}_X(x) \cap [\varepsilon, 1]) \leq 1 - H$ and $\dim_H(\mathcal{L}_X(x) \cap [\varepsilon, 1]) = 1 - H$ was shown in [38]. Thus from the defintion of the θ-intermediate dimensions (see III.3.2) $\dim_\theta(\mathcal{L}_X(x) \cap [\varepsilon, 1]) = 1 - H$, as well. Relevant results about the local time can be found in [128], which allows us to establish $\dim_P(\mathcal{L}_X(x)) = 1 - H$.

Proof of (III.1.11) *and* (III.1.12). Let $\theta \in [0, 1]$. Recall that for any set $E \subseteq \mathbb{R}$, one has

$$\begin{aligned}
&\dim_H(E) \leq \overline{\dim}_\theta(E) \leq \underline{\dim}_\theta(E) \leq \overline{\dim}_B(E), \text{ and} \\
&\dim_H(E) \leq \dim_P(E) \leq \overline{\dim}_B(E).
\end{aligned}$$

It is enough to show that $\overline{\dim}_B(\mathcal{L}_Z(x) \cap [\varepsilon, 1]) \leq 1 - H$ and $\dim_H(\mathcal{L}_Z(x) \cap [\varepsilon, 1]) \geq 1 - H$ with probability one. Starting with the upper bound, we follow the technique used for [9, Theorem 5] - an upper bound result for the classical Hausdorff dimension of level sets associated to fractional Brownian sheet. But in fact, the covers used are of equal length and so this technique gives an upper bound for the Box dimension.

For $n \geq 1$ we cover $[\varepsilon, 1]$ by $\lceil n^{1/H} \rceil$ subintervals $R_{n,\ell}$ of length $n^{-1/H}$, with $\ell \in \{1, 2, \ldots, \lceil n^{1/H} \rceil\}$. Let $0 < \delta < 1$ be fixed and $\tau_{n,\ell}$ be the left endpoint of the interval $R_{n,\ell}$. We first bound the probability $\mathbb{P}(x \in Z(R_{n,\ell}))$:

$$\begin{aligned}
\mathbb{P}(x \in Z(R_{n,\ell})) \leq \quad & \mathbb{P}(\sup_{t \in R_{n,\ell}} |Z_t - Z_{\tau_{n,\ell}}| \leq n^{-(1-\delta)}, x \in Z(R_{n,\ell})) \\
& + \mathbb{P}(\sup_{t \in R_{n,\ell}} |Z_t - Z_{\tau_{n,\ell}}| \geq n^{-(1-\delta)}) \\
\leq \quad & \mathbb{P}(|Z_{\tau_{n,\ell}} - x| \leq n^{-(1-\delta)}) + C_1 \exp(-c_1 n^{-(1-\delta)}/n^{-1}) \\
\leq \quad & C_2 n^{-(1-\delta)} + C_1 \exp(-c_1 n^\delta) = O(n^{-(1-\delta)}), \qquad \text{(III.4.1)}
\end{aligned}$$

where we have used Proposition III.2.5, and the fact that the density of Z_t is continuous.

We can cover the set $\mathcal{L}_Z(x) \cap [\varepsilon, 1]$ by a sequence of intervals $R'_{n,\ell}$ with $R'_{n,\ell} = R_{n,\ell}$ if $x \in Z(R_{n,\ell})$ and $R'_{n,\ell} = \emptyset$, otherwise, for $\ell \in \{1, 2, \ldots, \lceil n^{1/H} \rceil\}$. We need to show that

$$\mathbb{E}\left[\sum_{\ell=1}^{\lceil n^{1/H} \rceil} |R'_{n,\ell}|^\eta\right] < \infty, \qquad \text{(III.4.2)}$$

for $\eta = 1 - H(1 - \delta)$ and arbitrary $\delta > 0$. In turn this would imply by Fatou's lemma that $\overline{\dim}_B(\mathcal{L}_Z(x) \cap [\varepsilon, 1]) \leq \eta$ almost surely. Then, letting $\delta \to 0$ yields the upper bound on the upper Box dimension.

We establish (III.4.2):

$$\begin{aligned}\mathbb{E}\left[\sum_{\ell=1}^{\lceil n^{1/H}\rceil}|R'_{n,\ell}|^{\eta}\right] &\leq \mathbb{E}\left[\sum_{\ell=1}^{\lceil n^{1/H}\rceil}\left(n^{-1/H}\right)^{\eta}\mathbf{1}_{x\in Z(R_{n,\ell})}\right]\\ \leq \quad & cn^{1/H-1/H(1-H(1-\delta))-(1-\delta)} = c,\end{aligned}$$

where the last inequality follows from the bound (III.4.1) on $\mathbb{P}(x \in Z(R_{n,\ell}))$.

For the lower bound we first recall a relation between the Hölder regularity and the Hausdorff dimension.

Proposition III.4.1 (Theorem 27 in [34])**.** *Let $[u,v] \subset \mathbb{R}$ be a finite interval and $f : [u,v] \to \mathbb{R}$ be a continuous function with occupation density denoted by L. Suppose that L satisfies a Hölder condition of order $\gamma \in (0,1)$ (in the set variable). Then $dim_H \left(f^{-1}_{[u,v]}(x)\right) \geq \gamma$ for all $x \in \mathbb{R}$ such that $L(x,[u,v]) \neq 0$.*

A Hölder regularity condition for the local time of the Rosenblatt process was recently obtained in [57]. In particular, see Proposition III.2.6, for a finite closed interval $I \subset (0,\infty)$, there exists a constant $C > 0$ such that almost surely,

$$\limsup_{r\to 0}\sup_{s\in I}\frac{\sup_{x\in\mathbb{R}} L(x,[s-r,s+r])}{r^{1-H}|\log r|^{2H}} \leq C.$$

Therefore, the occupation density of the Rosenblatt process satisfies a Hölder condition in the set variable of order γ for all $\gamma < 1-H$, and thus $\dim_H(\mathcal{L}_Z(x) \cap [\varepsilon,1]) \geq 1-H$. □

Before we establish the second part (III.1.13) of Theorem III.1.2 we recall some definitions and properties regarding the macroscopic Hausdorff dimension. Of special interest is a relation between $\mathrm{Dim}_H(E_Z(\gamma))$ and $\mathrm{Dim}_H(\mathcal{L}_Z(x))$ which eases the proofs of both (III.1.13) and (III.1.15).

III.4.1 Macroscopic Hausdorff dimension

To set up the notation as in [61, 59], consider the intervals $S_{-1} = [0,1/2)$ and $S_n = [2^{n-1},2^n)$ for $n \geq 0$. For $E \subset \mathbb{R}^+$, we define the set of *proper covers* of E restricted to S_n by

$$\mathcal{I}_n(E) = \left\{\begin{matrix}\{I_i\}_{i=1}^m : & I_i = [x_i,y_i]\,\text{with}\,x_i,y_i \in \mathbb{N},\ y_i > x_i,\\ & I_i \subset S_n \ \text{and}\ E\cap S_n \subset \bigcup_{i=1}^m I_i.\end{matrix}\right\}.$$

For any set $E \subset \mathbb{R}^+$, $\rho \geq 0$ and $n \geq -1$, define

$$\nu_\rho^n(E) := \inf\left\{\sum_{i=1}^m\left(\frac{\mathrm{diam} I_i}{2^n}\right)^{\rho} : \{I_i\}_{i=1}^m \in \mathcal{I}_n(E)\right\},$$

where $\mathrm{diam}[a,b] = b-a$.
The macroscopic Hausdorff dimension of $E \subset \mathbb{R}_+$ is defined as:

$$\mathrm{Dim}_H\,(E) \coloneqq \inf\left\{\rho \geq 0 : \sum_{n\geq 0} \nu_\rho^n(E) < \infty\right\}.$$

Next we establish a relation between (III.1.13) of Theorem III.1.2 and (III.1.15) of Theorem III.1.3 .

Recalling Definitions III.1.3 and III.1.4, for a fixed $\gamma > 0$ and any $x \in \mathbb{R}$, the level set $\mathcal{L}_Z(x)$ is ultimately included in $E_Z(\gamma)$:

$$\mathcal{L}_Z(x) \cap \left\{t \geq |x|^{\frac{1}{\gamma}}\right\} \subset E_Z(\gamma).$$

The macroscopic Hausdorff dimension is left unchanged after the removal of any bounded subset. Then, almost surely, for every $x \in \mathbb{R}$,

$$\mathrm{Dim}_H\,(\mathcal{L}_Z(x)) = \mathrm{Dim}_H\left(\mathcal{L}_Z(x) \cap \left\{t \geq |x|^{\frac{1}{\gamma}}\right\}\right) \leq \mathrm{Dim}_H\,(E_Z(\gamma))\,. \tag{III.4.3}$$

Therefore, to prove (III.1.13) and (III.1.15) it suffices to show that the following two statements hold almost surely:

$$\text{For any } x \in \mathbb{R}, \mathrm{Dim}_H\,(\mathcal{L}_Z(x)) \geq 1-H, \tag{III.4.4}$$

$$\mathrm{Dim}_H\,(E_Z(\gamma)) \leq 1-H. \tag{III.4.5}$$

The proof of (III.4.4) follows in the next subsection while (III.4.5) is established in Section III.5.3.

III.4.2 Lower bound for $\mathrm{Dim}_H\,(\mathcal{L}_Z(x))$

In this section we aim to find a lower bound for $\mathrm{Dim}_H\,(\mathcal{L}_Z(x))$. We first establish a result regarding macroscopic Hausdorff dimension in general.

Lemma III.4.2. *Let $E \subset \mathbb{R}_+$ and suppose that there exist $M > 0$ and $s \in [0,1]$ such that there exists a family of finite measures $\{\mu_n\}_{n\geq -1}$ on S_n such that for all intervals $I \subset S_n$, we have $\mu_n(I) \leq M(\mathrm{diam}I)^s$. If $Dim_H\,(E) = t$ for some $0 \leq t < s$, then $\sum_{n\geq -1} \dfrac{\mu_n\,(E\cap S_n)}{2^{ns}} < +\infty$.*

Proof. As $t < s$ and using the definition of macroscopic Hausdorff dimension we have $\nu^s(A) < +\infty$.
Let $\{I_i\}_{i=1}^m \in \mathcal{I}_n(E)$, then

$$\mu_n(E\cap S_n) \leq \sum_{i=1}^m \mu_n(I_i) \leq \sum_{i=1}^m M\mathrm{diam}{I_i}^s = 2^{ns}M\sum_{i=1}^m\left(\frac{\mathrm{diam}I_i}{2^n}\right)^s.$$

Then $\dfrac{\mu_n(E\cap S_n)}{2^{ns}} \leq M\nu_n^s(E\cap S_n)$ and so $\sum_{n\geq -1} \dfrac{\mu_n(A\cap S_n)}{2^{ns}} < +\infty$. □

By Proposition III.2.9, the local time is Hölder continuous in t of order α for every $\alpha \in [0, 1-H)$. Now we will be using this property and the preceding lemma in order to get a lower bound for $\mathrm{Dim}_H(\mathcal{L}_Z(x))$. To this end, fix $\alpha \in [0, 1-H)$ and introduce the following random variables

$$Y_n^x = \frac{L(x, S_n)}{2^{n\alpha}} \quad \text{and} \quad F_N^x = \sum_{n=1}^{N} Y_n^x. \tag{III.4.6}$$

The random variables $(Y_n^x)_{n \geq -1}$ are positive, so $(F_N^x)_{N \geq 1}$ is non-decreasing. We denote by F_∞^x its limit, i.e. $F_\infty^x = \sum_{n=-1}^{\infty} Y_n^x \in [0, +\infty]$.

As a direct consequence of Lemma III.4.2, there is a connection between $\mathrm{Dim}_H(\mathcal{L}_x)$ and the r.v. Y_n^x. Indeed, for $n \geq -1$ consider the sequence of measures

$$\mu_n(I) := L(x, I), \text{ for all } I \subset S_n,$$

By Proposition III.2.9, there exists $M > 0$ such that for all $n \geq -1$ a.s.

$$\mu_n(I) \leq M \mathrm{diam} I^\alpha, \text{ for all } I \subset S_n.$$

Now by Lemma III.4.2, a.s. for every $x \in \mathbb{R}$, $\mathrm{Dim}_H(\mathcal{L}_x) \geq \alpha$ if

$$\sum_{n \geq -1} \frac{\mu_n(\mathcal{L}_x \cap S_n)}{2^{n\alpha}} = F_\infty^x = +\infty.$$

As a consequence, we see that $\mathrm{Dim}_H(\mathcal{L}_x) \geq \alpha$ for all $x \in \mathbb{R}$ such that $F_\infty^x = +\infty$. Moreover in order to conclude the proof of Theorem III.1.2, it is enough to prove that for all $\alpha \in [0, 1-H)$, a.s. for all $x \in \mathbb{R}$, $\mathrm{Dim}_H(\mathcal{L}_x) \geq \alpha$. Letting $\alpha \uparrow 1-H$ gives that a.s. for all $x \in \mathbb{R}$, $\mathrm{Dim}_H(\mathcal{L}_x) \geq 1-H$. Finally it remains to check that $\mathbb{P}(\forall x \in \mathbb{R},\, F_\infty^x = +\infty) = 1$, for all $\alpha \in [0, 1-H)$. This is the object of the next proposition.

Proposition III.4.3. *Let $\alpha \in [0, 1-H)$ and*

$$Y_n^x = \frac{L(x, S_n)}{2^{n\alpha}}, \text{ for } n \geq -1, \quad \text{and } F_\infty^x = \sum_{n \geq -1} Y_n^x.$$

Then,

$$\mathbb{P}(\forall x \in \mathbb{R},\, F_\infty^x = +\infty) = 1. \tag{III.4.7}$$

Proof. We follow the technique in [28]. For every $a > 0$, let

$$\widetilde{Y}_n^a = \inf_{x \in [-a,a]} Y_n^x, \text{ for } n \geq 1, \quad \text{and} \quad \widetilde{F}_\infty^a = \sum_{n \geq 1} \widetilde{Y}_n^a.$$

Using the self-similarity property of the local time (III.2.11), for all $n \geq 0$,

$$\widetilde{Y}_n^a = \inf_{x \in [-a,a]} Y_n^x \overset{(d)}{=} \inf_{x \in [-a,a]} Y_0^{2^{-nH}x} = \inf_{x \in [-2^{-nH}a, 2^{-nH}a]} Y_0^x = \widetilde{Y}_0^{2^{-nH}a}.$$

The proof now relies on the following technical result:

Lemma III.4.4. *For any $b > 0$, one has*

$$\mathbb{P}(\widetilde{F}^b_\infty = \infty) > 0.$$

Proof of Lemma III.4.4. We first show that there exists $\varepsilon > 0$ such that $\mathbb{P}(Y^0_0 > \varepsilon) > 0$. Recall that $Y^0_0 = L(0, [1/2, 1])$ and is non-negative. Thus, it is enough to show that $\mathbb{E}[L(0, [1/2, 1])] > 0$. Using the following representation of the local time, see [100, Chapter 10], one gets

$$L(0, [1/2, 1]) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{1/2}^{1} \mathbf{1}_{-\varepsilon,\varepsilon}(Z_t)dt.$$

Then using self-similarity of Z and then Proposition III.2.1(i) with some constant $c_1 > 0$, one gets

$$\begin{aligned}
\mathbb{E}[L(0, [1/2, 1])] =& \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{1/2}^{1} \mathbb{P}(Z_t \in [-\varepsilon, \varepsilon])dt \\
=& \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{1/2}^{1} \mathbb{P}(Z_1 \in [-\varepsilon t^{-H}, \varepsilon t^{-H}])dt \\
\geq& \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{1/2}^{1} 2c_1 \varepsilon t^{-H} dt \\
=& \frac{c_1}{1-H}(1 - (1/2)^{1-H}).
\end{aligned}$$

Therefore, $\mathbb{E}[L(0, [1/2, 1])] > 0$ and thus $\mathbb{P}(Z^0_0 > \varepsilon) > 0$ for some $\varepsilon > 0$.

The rest of the proof is based on the following two facts:

1. For every $\varepsilon > 0$ small enough, there exists $a \in \mathbb{R}^+$, such that:

$$0 < \mathbb{P}(Y^0_0 > \varepsilon) \leq 2\mathbb{P}(\widetilde{Y}^a_0 > 0).$$

2. For any $a, b > 0$, we have

$$\mathbb{P}(\widetilde{F}^b_\infty = \infty) \geq \mathbb{P}(\widetilde{Y}^a_0).$$

The statements above correspond to Lemmas II.3.3 and II.3.4 in Chapter II and the proofs are identical as long as the following holds:

$$\mathbb{P}\left(\sup_{x \in [-1,1]/\{0\}} \frac{\left|L\left(0, \left[\frac{1}{2}, 1\right]\right) - L\left(x, \left[\frac{1}{2}, 1\right]\right)\right|}{|x|^\beta} < \infty \right) = 1,$$

where $\beta \in \left(0, \frac{1}{2}\left(\frac{1}{H} - 1\right)\right)$. In [28], this property corresponds to Lemma 5 which is originally due to Geman in Horowitz [45, Theorem 26.1]. For the Rosenblatt case, the above is established in Proposition III.2.7. □

Using the result of Lemma III.4.4 we can establish that $\mathbb{P}(\widetilde{F}^b_\infty = \infty) = 1$ if we can apply Blumental's 0-1 law. This is possible since

$$\left\{\widetilde{F}^b_\infty = \infty\right\} \in \bigcap_{M\geq 1} \sigma\left\{W_u : u < 2^{-(M-1)}\right\}, \tag{III.4.8}$$

where $(W_t)_{t\geq 0}$ is the standard Brownian motion. Indeed, the time inverted process $\widetilde{Z}_t = t^{2H} Z_{1/t}, t > 0$ is distributed as the Rosenblatt process (see Proposition III.2.3). Then, using the representation (III.2.10), the local time $L^x(S_n)$ is $\sigma\left\{\widetilde{Z}_u : u \leq 2^{-(n-1)}\right\}$-measurable. Moreover,

$$\sigma\left\{\widetilde{Y}^b_n : n \geq M\right\} \subset \sigma\left\{\widetilde{Z}_u : u \leq 2^{-(M-1)}\right\},$$

for $M \geq 1$ and thus

$$\left\{\widetilde{F}^b_\infty = \infty\right\} \in \bigcap_{M\geq 1} \sigma\left\{\widetilde{Z}_u : u < 2^{-(M-1)}\right\}. \tag{III.4.9}$$

At this point by (III.2.4) (with $\tilde{Z}$ instead of Z), there exists standard Brownian motion $(W_t)_{t\geq 0}$ such that $\sigma\left\{\tilde{Z}_u : u \leq t\right\} \subset \sigma\left\{W_u : u \leq t\right\}$. This fact combined with (III.4.9) establishes (III.4.8). Then using (III.4.8) and the fact that $\mathbb{P}(\widetilde{F}^b_\infty = \infty) > 0$ for all $b > 0$, one can apply Blumental's 0-1 law and thus gets that $\mathbb{P}(\widetilde{F}^b_\infty = \infty) = 1$, for all $b > 0$.

Finally, for every $b > 0$,

$$\begin{aligned}\mathbb{P}(\forall x \in [-b, b] : F^x_\infty = \infty) &= \mathbb{P}\left(\inf_{x\in[-b,b]} F^x_\infty = \infty\right) = \mathbb{P}\left(\inf_{x\in[-b,b]} \sum_{n\geq 1} Y^x_n = \infty\right) \\ \geq\ & \mathbb{P}\left(\sum_{n\geq 1} \inf_{x\in[-b,b]} Y^x_n = \infty\right) = \mathbb{P}(\widetilde{F}^b_\infty = \infty) = 1.\end{aligned}$$

Therefore,

$$\mathbb{P}(\forall x \in \mathbb{R} : F^x_\infty = \infty) = \lim_{b\to\infty} \mathbb{P}(\forall x \in [-b, b], F^x_\infty = \infty) = 1,$$

and (III.4.7) is established.

□

Next, we establish (III.1.12)- the result regarding Packing dimension. Recall that $\dim_P(\mathcal{L}_Z(x)) \leq \overline{\dim}_B(\mathcal{L}_Z(x)) = 1 - H$. It is enough to show that $\dim_P(\mathcal{L}_Z(x)) \geq 1 - H$, which is the aim of the next section.

III.5 Sojourn times

This section is dedicated to the proof of Theorem III.1.3. We first establish (III.1.14). Recall the definitions of logarithmic and pixel densities. For $E \subset \mathbb{R}^+$, the logarithmic density of E is given by

$$\mathrm{Den}_{\log}(E) := \limsup_{n\to\infty} \frac{\log_2 \mathrm{Leb}(E\cap[1,2^n])}{n},$$

where 'Leb' is the one-dimensional Lebesgue measure.

Let $\mathrm{pix}(E) := \{n \in N : \mathrm{dist}(n,E) \le 1\}$. Then, the pixel density of E is

$$\mathrm{Den}_{pix}(E) := \limsup_{n\to\infty} \frac{\log_2 \#\mathrm{pix}(E\cap[1,2^n])}{n}.$$

The two quantities are closely related, see [61]:

$$\mathrm{Den}_{\log}(E) \le \mathrm{Den}_{pix}(E). \tag{III.5.1}$$

We want to show that for $\gamma \in [0,H)$, $\mathrm{Den}_{pix}(E_Z(\gamma)) = \mathrm{Den}_{\log}(E_Z(\gamma)) = \gamma+1-H$, almost surely. Our strategy is then to establish that $\mathrm{Den}_{pix}(E_Z(\gamma)) \le \gamma+1-H$ and $\mathrm{Den}_{\log}(E_Z(\gamma)) \ge \gamma+1-H$, almost surely.

III.5.1 Upper bound for $\mathrm{Den}_{pix}(E_Z(\gamma))$

Our goal is to obtain an upper bound for $\#\mathrm{pix}(E_Z(\gamma))\cap[1,2^n]$ that holds with probability 1 for all large n. We first study the expectation

$$\begin{aligned}
\mathbb{E}[\mathrm{pix}(E_Z(\gamma)\cap[1,2^n])] &= \sum_{m=1}^{2^n} \mathbb{P}\left(\exists s\in[m-1,m+1], |Z_s| \le s^\gamma\right)\\
&= \sum_{m=1}^{2^n} \mathbb{P}\left(\exists s\in\left[1-\frac{1}{m},1+\frac{1}{m}\right], |Z_s| \le s^\gamma m^{\gamma-H}\right)\\
&= \sum_{m=1}^{2^n} \mathbb{P}\left(\exists s\in\left[1-\frac{1}{m},1\right], |Z_s| \le s^\gamma m^{\gamma-H}\right)\\
&\quad + \mathbb{P}\left(\exists s\in\left[1,1+\frac{1}{m}\right], |Z_s| \le s^\gamma m^{\gamma-H}\right)\\
&\le \sum_{m=1}^{2^n} (A^-_{1/m} + A^+_{1/m}),
\end{aligned} \tag{III.5.2}$$

where

$$\begin{aligned}
A^-_\varepsilon :=& \mathbb{P}(\exists s\in[1-\varepsilon,1], |Z_s| \le \varepsilon^{H-\gamma}),\\
A^+_\varepsilon :=& \mathbb{P}(\exists s\in[1,1+\varepsilon], |Z_s| \le 2\varepsilon^{H-\gamma}).
\end{aligned}$$

Lemma III.5.1. *There is a universal constant $c > 0$, such that, for every ε small enough,*

$$\max(A_\varepsilon^-, A_\varepsilon^+) \le c\varepsilon^{H-\gamma}. \tag{III.5.3}$$

Proof. Consider A_ε^- first. We have

$$\begin{aligned} A_\varepsilon^- \le \quad & \mathbb{P}(\exists s \in [1-\varepsilon, 1], |Z_s| \le \varepsilon^{H-\gamma}, |Z_1| \le 2\varepsilon^{H-\gamma}) \\ & + \mathbb{P}(\exists s \in [1-\varepsilon, 1], |Z_s| \le \varepsilon^{H-\gamma}, |Z_1| \ge 2\varepsilon^{H-\gamma}) \\ \le \quad & \mathbb{P}(|Z_1| \le 2\varepsilon^{H-\gamma}) + \mathbb{P}(\exists s \in [1-\varepsilon, 1], |Z_s - Z_1| \ge \varepsilon^{H-\gamma}). \end{aligned} \tag{III.5.4}$$

To bound the first term on the right-hand side above, we use Proposition III.2.1(i), i.e, the density function f of Z_1 is continuous and $f(0) > 0$. Then one can show, for instance, that for $\varepsilon > 0$ small enough,

$$\mathbb{P}(|Z_1| \le 2\varepsilon^{H-\gamma}) \le 4f(0)\varepsilon^{H-\gamma}. \tag{III.5.5}$$

We are left to study the term $\mathbb{P}(\exists s \in [1-\varepsilon, 1], |Z_s - Z_1| \ge \varepsilon^{H-\gamma})$. Write

$$\begin{aligned} & \mathbb{P}(\exists s \in [1-\varepsilon, 1], |Z_s - Z_1| \ge \varepsilon^{H-\gamma}) \\ \le \quad & \mathbb{P}(\sup_{s \in [1-\varepsilon, 1+\varepsilon]} |Z_s - Z_1| \ge \varepsilon^{H-\gamma}) \\ \le \quad & C \exp\left(-c_1 \varepsilon^{-\gamma}\right), \end{aligned} \tag{III.5.6}$$

where the last inequality follows from Proposition III.2.5 and $C, c_1 > 0$ are constants depending only on H. Note that $\exp(-c_1\varepsilon^{-\gamma}) = O(\varepsilon^\delta)$, for any $\delta > 0$ if any ε is small enough.

Finally, for ε small enough, combining (III.5.6) and (III.5.5) in (III.5.4) yields the bound of (III.5.3) for A_ε^-.

Same arguments as above can be applied to A_ε^+ to get an equivalent bound and establish (III.5.3). ☐

Next, applying Lemma III.5.1 in (III.5.2) yields, for some absolute constant $C > 0$,

$$\mathbb{E}[\#\text{pix}(E_Z(\gamma) \cap [1, 2^n])] \le 2C \sum_{m=1}^{2^n} m^{\gamma - H} = O\left(2^{n(\gamma+1-H)}\right).$$

Choose $\rho > \gamma + 1 - H$. Then,

$$\sum_{n \ge 1} \mathbb{P}\left(\#\text{pix}(E_Z(\gamma) \cap [1, 2^n]) > 2^{n\rho}\right) \le C \sum_{n \ge 1} \frac{2^{n(1+\gamma-h)}}{2^{n\rho}} < \infty.$$

By the Borel-Cantelli lemma, with probability one,

$$\#\text{pix}(E_Z(\gamma) \cap [1, 2^n]) \le 2^{n\rho},$$

for every large enough n. Hence, $\text{Den}_{pix}(E_Z(\gamma)) \le \rho$. Letting $\rho \downarrow \gamma + 1 - H$ yields $\text{Den}_{pix}(E_Z(\gamma)) \le \gamma + 1 - H$.

III.5.2 Lower bound for $\mathbf{Den}_{\log}(E_Z(\gamma))$

Introduce

$$S_\gamma([t_1,t_2]) = \mathrm{Leb}(\{t_1 \le s \le t_2 : |Z_s| \le s^\gamma\}), \text{ for all } 0 \le t_1 \le t_2.$$

We will prove that for infinitely many integers n, $S_\gamma([0,2^n]) \ge \frac{c}{2}2^{n(\gamma+1-H)}$, for any $c \in (0,1)$. This implies that $\mathrm{Den}_{\log}(E_Z(\gamma)) \ge \gamma+1-H$ almost surely. Then using (III.5.1), we also obtain $\mathrm{Den}_{pix}(E_Z(\gamma)) \le \gamma+1-H$ and the proof of (III.1.14) is completed.

First we show that for any $c \in (0,1)$, there is a constant $c' > 0$ such that

$$\mathbb{P}(S_\gamma([0,2^n]) \ge c2^{n(1+\gamma-H)}) \ge c'. \tag{III.5.7}$$

By Paley-Zygmund inequality, for any $c \in (0,1)$, we have

$$\mathbb{P}(S_\gamma([0,2^n]) \ge c2^{n(1+\gamma-H)}) \ge (1-c)\frac{\mathbb{E}[S_\gamma([0,2^n])]^2}{\mathbb{E}[S_\gamma([0,2^n])^2]}. \tag{III.5.8}$$

The numerator can be rewritten as:

$$\mathbb{E}[S_\gamma([0,t])] = \int_0^t \mathbb{P}(|Z_s| \le s^\gamma)ds = \int_0^t \mathbb{P}(|Z_1| \le s^{\gamma-H})ds.$$

Now, we establish a lower bound for $\mathbb{P}(|Z_1| \le s^{\gamma-H})$. Apply Proposition III.2.1(i) there is a constant $\alpha > 0$ such that for s large enough, the density function of Z_1 is bounded below by α in $[-s^{\gamma-H}, s^{\gamma-H}]$. Therefore,

$$\mathbb{P}(|Z_1| \le s^{\gamma-H}) \ge 2\alpha s^{\gamma-H} \quad \text{and thus} \quad \mathbb{E}[S_\gamma([0,t])] \ge 2\alpha t^{1+\gamma-H}. \tag{III.5.9}$$

We bound the second moment from above:

$$\begin{aligned}\mathbb{E}[S_\gamma([0,t])^2] &= \int\int_{[0,t]^2} \mathbb{P}(|Z_u| \le u^\gamma, |Z_v| \le v^\gamma)dudv \\ &= t^2 \int\int_{[0,1]^2} \mathbb{P}\left(|Z_u| \le u^\gamma t^{\gamma-H}, |Z_v| \le v^\gamma t^{\gamma-H}\right)dudv.\end{aligned}$$

By Proposition III.2.1(ii), the density function $g_{u,v}$ of (Z_u, Z_v) is continuous and tends to 0 as $|x| \to \infty$. Therefore,

$$\mathbb{E}[S_\gamma([0,t])^2] \le t^2 \int\int_{[0,1]^2} dudv \int\int_{\mathbb{R}^2} g_{u,v}(x,y)\, \mathbb{1}_{\begin{pmatrix}|x| \le u^\gamma t^{\gamma-H} \\ |y| \le v^\gamma t^{\gamma-H}\end{pmatrix}} dxdy \tag{III.5.10}$$

$$\le Ct^{2+2\gamma-2H}. \tag{III.5.11}$$

Applying (III.5.9) and (III.5.10) in (III.5.8) yields (III.5.7). Now, define the event

$$A_{n,\gamma} := \left\{ S_\gamma \left(\left[\frac{c}{2} 2^{n(1+\gamma-H)}, 2^n \right] \right) \geq \frac{c}{2} 2^{n(1+\gamma-H)} \right\}.$$

By (III.5.7), it is easy to see that $\mathbb{P}(A_{n,\gamma}) \geq c' > 0$. Moreover, by the definition of $A_{n,\gamma}$, one has $A_{n,\gamma} \subset \left\{ S_\gamma([0, 2^n]) \geq \frac{c}{2} 2^{n(1+\gamma-H)} \right\}$. Then it is enough to prove that $A_{n,\gamma}$ happens infinitely often which give us that $S_\gamma([0, 2^n]) \geq \frac{c}{2} 2^{n(1+\gamma-H)}$ for infinitly many n. To this end, let A_γ be the event that $A_{n,\gamma}$ happens infinitely often. Recall that for any sequence of events $(A_i)_{i \geq 1}$, one has $\lim_{n\to\infty} \mathbb{P}(\cup_{i\geq n} A_i) = \mathbb{P}(A_i \text{ i. o })$. In other words, one has

$$A_\gamma = \bigcap_{M \geq 1} \bigcup_{n \geq M} A_{n,\gamma}. \tag{III.5.12}$$

We know that $\mathbb{P}(A_\gamma)\ (\geq c')$ is strictly positive. It remains to prove that it is in fact equal to 1. As in Section III.4.2, such a conclusion will follow by using that the time inverted process $\tilde{Z}_t = t^{2H} Z_{1/t}$ is distributed as the Rosenblatt process (see Proposition III.2.3). Now let $\tilde{S}_\gamma$ (resp. $\tilde{A}_{n,\gamma}$, $\tilde{A}_\gamma$) be the event analogous to S_γ (resp. $A_{n,\gamma}$, A_γ), but associated to $\tilde{Z}$ instead of Z. So for any fixed integer $n \geq 0$, we have

$$\tilde{S}_\gamma \left(\left[\frac{c}{2} 2^{n(1+\gamma-H)}, 2^n \right] \right) = \text{Leb} \left(\left\{ \frac{c}{2} 2^{n(1+\gamma-H)} \leq s \leq 2^n : |t^{2H} Z_{1/s}| \leq s^\gamma \right\} \right),$$

which implies in return that $\tilde{A}_{n,\gamma} \in \sigma \left\{ Z_u : u \leq 2^{-n(1+\gamma-H)} \right\}$. As a consequence, for all $M \geq 0$, one has

$$\left\{ \tilde{A}_{n,\gamma} : n \geq M \right\} \in \sigma \left\{ Z_u : u \leq 2^{-M(1+\gamma-H)} \right\}.$$

Recalling definition III.5.12 of A_γ, we obtain that

$$\tilde{A}_\gamma \in \bigcap_{M \geq 1} \sigma \left\{ _{n,\gamma} : n \geq M \right\}.$$

Using (III.2.4), we deduce that

$$\tilde{A}_\gamma \in \bigcap_{M \geq 1} \sigma(B_u : u \leq 2^{-M(1+\gamma-H)}),$$

where $(B_t)_{t\geq 0}$ is the Brownian motion. Therefore, $\tilde{A}_\gamma$ is a tail event and $\mathbb{P}(\tilde{A}_\gamma) = 0$ or 1 by the Blumenthal $0-1$ law. Obviously, as Z and $\tilde{Z}$ have the same distribution, then $\mathbb{P}(\tilde{A}_\gamma) = \mathbb{P}(A_\gamma) \geq c' > 0$ and then $\mathbb{P}(\tilde{A}_\gamma) = \mathbb{P}(A_\gamma) = 1$ as desired.

III.5.3 Upper bound for $\mathrm{Dim}_H(E_Z(\gamma))$

We now turn to the proof of (III.1.15). Following our discussion in Section III.4.1, and in particular the relation (III.4.3) between $\mathrm{Dim}_H(E_Z(\gamma))$ and $\mathrm{Dim}_H(\mathcal{L}_Z(x))$, it is enough to show (III.4.5), i.e., for every $0 \leq \gamma < H$,

$$\mathrm{Dim}_H(E_Z(\gamma)) \leq 1 - H, \text{ a.s.}$$

We follow the technique in [87]. Let us fix $0 \leq \gamma < H$, as well as $\eta > 0$ (as small as necessary). We are going to prove that $\mathrm{Dim}_H(E_Z(\gamma)) \leq 1 - H + \eta$. Letting η tend to zero will then give the result. Fix $\rho > 1 - H + \eta$, our aim is to prove that $\mathrm{Dim}_H(E_Z(\gamma)) \leq \rho$. To this end, consider for every integer $n \geq 1$ and $i \in \left\{0, ..., \left\lfloor \frac{2^{n-1}}{2^{n\frac{\gamma}{H}}} \right\rfloor \right\}$ the intervals

$$I_{n,i} = [t_{n,i}, t_{n,i+1}) \text{ with } t_{n,i} = 2^{n-1} + i2^{n\frac{\gamma}{H}}.$$

And the associated event

$$\mathcal{E}_{n,i} = \{\exists t \in I_{n,i} \,:\, |Z_t| \leq t^\gamma\}.$$

Denote $\varepsilon_{n,i} = 2^{n\frac{\gamma}{H}}/t_{n,i}$, so that $I_{n,i} = [t_{n,i}, t_{n,i}(1+\varepsilon_{n,i}))$, and observe that the ratio between any two of the quantities $2^{n\left(\frac{\gamma}{H}-1\right)}$, $\varepsilon_{n,i}$, and $t_{n,i}^{\frac{\gamma}{H}-1}$ are bounded uniformly with respect to n and i. By self-similarity, we have that, when n becomes large,

$$\begin{aligned}
\mathbb{P}(\mathcal{E}_{n,i}) &= \mathbb{P}(\exists\, t \in I_{n,i} \,:\, |Z_t| \leq t^\gamma) \\
&= \mathbb{P}\left(\exists\, s \in [1, 1+\varepsilon_{n,i}] : |Z_{s.t_{n,i}}| \leq (s.t_{n,i})^\gamma\right) \\
&= \mathbb{P}\left(\exists\, s \in [1, 1+\varepsilon_{n,i}] : |Z_s| \leq t_{n,i}^{\gamma-H}.s^\gamma\right) \\
&= \mathbb{P}\left(\exists\, s \in [1, 1+\varepsilon_{n,i}] : |Z_s| \leq 2t_{n,i}^{\gamma-H}\right) \\
&= \mathbb{P}\left(\exists\, s \in [1, 1+\varepsilon_{n,i}] : |Z_s| \leq c\varepsilon_{n,i}^{H}\right) \\
&= \mathbb{P}\left(\exists\, s \in [1, 1+\varepsilon_{n,i}] : |Z_s| \leq \varepsilon_{n,i}^{H-\eta}\right).
\end{aligned}$$

The last estimate holds because η is a small positive real number and $\varepsilon_{n,i}$ tends to zero when n becomes large. By Lemma III.5.1, we deduce that $\mathbb{P}(\mathcal{E}_{n,i}) \leq c\varepsilon_{n,i}^{H-\eta}$ and so

$$\mathbb{P}(\mathcal{E}_{n,i}) \leq c2^{n(\gamma-H)\frac{H-\eta}{H}}.$$

Now observe that $\mathcal{E}_{n,i}$ is realized if and only if $E_Z(\gamma) \cap I_{n,i} \neq \emptyset$. So, using the intervals $I_{n,i}$ as a covering of $E_Z(\gamma) \cap S_n$, we obtain that

$$\begin{aligned}
\mathbb{E}\left[\nu_\rho^n(E_Z(\gamma))\right] \leq& \mathbb{E}\left(\sum_{0}^{\lfloor 2^{n-1-n\frac{\gamma}{H}} \rfloor} \left(\frac{\mathrm{Leb}(I_{n,i})}{2^n}\right)^\rho \mathbb{1}_{\mathcal{E}_{n,i}}\right) \\
\leq& 2^{\rho n\left(\frac{\gamma}{H}-1\right)} \sum_{0}^{\lfloor 2^{n-1-n\frac{\gamma}{H}} \rfloor} \mathbb{P}(\mathcal{E}_{n,i}) \\
\leq& c2^{n\frac{H-\gamma}{H}(1-H+\eta-\rho)}.
\end{aligned}$$

Thus, the Fubini Theorem entails $\mathbb{E}\left[\sum_{n=1}^\infty \nu_\rho^n(E_Z(\gamma))\right] < +\infty$ as soon as $\rho > 1 - H + \eta$. This implies that for such ρ's, the sum $\sum_{n=1}^\infty \nu_\rho^n(E_Z(\gamma))$ is finite almost surely. In particular, $\mathrm{Dim}_H(E_Z(\gamma)) \leq \rho$, for every $\rho > 1 - H + \eta$. Since such a relation holds for an arbitrary (small) $\rho > 0$, we deduce (III.4.5) as desired.

Chapter IV

Wavelet methods to study the pointwise regularity of the generalized Rosenblatt process

The content of this chapter is a copy of the paper entitled "Wavelet methods to study the pointwise regularity of the generalized Rosenblatt process", written with "Laurent Looseveldt", and to be submitted soon.

IV.1 Introduction

Precise study of path behaviour, and in particular regularity, of stochastic processes is a classical research field, initiated in the 1920s by the works of Wiener [126]. It lies in between probability and (harmonic) analysis and a common strategy is to mix probabilistic arguments with analytical tools. Pioneer works concerned Brownian motion. Among them, one can cite Paley and Wiener's expansion [127] using Fourier series, Lévy's representation [67] obtained with some techniques of interpolation theory or, more recently, Kahane's expansion [54] in the Schauder basis.

In the last decades, the emergence of wavelet analysis allowed to obtain series expansions for many stochastic processes. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying the admissibility condition [79]

$$\int_{\mathbb{R}} \frac{|\widehat{\psi}(\xi)|}{|\xi|} \, d\xi < \infty, \tag{IV.1.1}$$

where $\widehat{\psi}$ is the Fourier transform of ψ. As such it generates an orthonormal basis of $L^2(\mathbb{R})$. More precisely, any function $f \in L^2(\mathbb{R})$ can be decomposed as

$$f = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} c_{j,k} \psi(2^j \cdot -k), \tag{IV.1.2}$$

where

$$c_{j,k} = 2^j \int_{\mathbb{R}} f(x)\psi(2^j x - k)\, dx.$$

It is noteworthy that the expansion (IV.1.2) holds true in many function spaces. We refer to the seminal books [27, 79, 73] for more details and proofs of these facts. Multifractal analysis has demonstrated the efficiency of wavelet methods to study uniform and pointwise Hölder regularity of functions both from a theoretical [13, 14, 25, 49, 50, 52] and a practical points of view [4, 3, 20, 31, 42, 51, 85, 124, 125].

Now, let us consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a real-valued stochastic process X defined on it. If X is smooth enough, for all $\omega \in \Omega$, one can apply expansion (IV.1.2) to the simple path $t \mapsto X(t, \omega)$. This way, one defines a sequence of random wavelet coefficients $(c_{j,k}(\omega))_{j,k\in\mathbb{Z}}$. For instance, if $X = B_H$ is the fractional Brownian motion of Hurst index $H \in (0,1)$ and if ψ is a sufficiently regular wavelet, one has [81, 51]

$$B_H = \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{Z}} 2^{-Hj} \xi_{j,k} \psi_{H+1/2}(2^j \cdot - k) + R, \tag{IV.1.3}$$

where $(R(t,\cdot))_{t\in\mathbb{R}^+}$ is a process with almost surely C^∞ sample paths, $(\xi_{j,k})_{j\in\mathbb{N},k\in\mathbb{Z}}$ is a sequence of independent $\mathcal{N}(0,1)$ random variables and $\psi_{H+1/2}$ is a fractional antiderivative of ψ, see Section IV.2 for a precise definition.

In [36], Esser and Loosveldt undertook a systematic study of Gaussian wavelet series. Thanks to (IV.1.3), it applies in particular to the fractional Brownian motion and leads to the following theorem.

Theorem IV.1.1. *For all $H \in (0,1)$, there exists an event Ω_H of probability 1 satisfying the following assertions for all $\omega \in \Omega_H$ and every non-empty interval I of $\mathbb{R}$.*

- *For almost every $t \in I$,*

$$0 < \limsup_{s\to t} \frac{|B_H(t,\omega) - B_H(s,\omega)|}{|t-s|^H \sqrt{\log\log |t-s|^{-1}}} < +\infty.$$

  Such points are called ordinary points.

- *There exists a dense set of points $t \in I$ such that*

$$0 < \limsup_{s\to t} \frac{|B_H(t,\omega) - B_H(s,\omega)|}{|t-s|^H \sqrt{\log |t-s|^{-1}}} < +\infty.$$

  Such points are called rapid points.

- *There exists a dense set of points $t \in I$ such that*

$$0 < \limsup_{s\to t} \frac{|B_H(t,\omega) - B_H(s,\omega)|}{|t-s|^H} < +\infty.$$

  Such points are called slow points.

Note that Theorem IV.1.1 extends some well-known results of Kahane concerning the Brownian motion [54]. The "ordinary", "rapid" and "slow" terminology is inspired by them. Let us justify it. In a measure-theoretical point of view, the modulus of continuity $x \mapsto |x|^H \sqrt{\log\log|x|^{-1}}$ is the most frequent among the points of singles paths. Thus, it is natural to refer it as ordinary. Now, $|x|^H \sqrt{\log\log|x|^{-1}} = o(|x|^H \sqrt{\log|x|^{-1}})$ if $x \to 0^+$ and thus points for which $x \mapsto |x|^H \sqrt{\log|x|^{-1}}$ is the pointwise modulus of continuity are refereed as rapid. On the other side, points for which $x \mapsto |x|^H$ is the pointwise modulus of continuity are referred as slow because $|x|^H = o(|x|^H \sqrt{\log\log|x|^{-1}})$ if $x \to 0^+$.

Now, let us turn to the stochastic process we will deal with in this paper. The Rosenblatt process appears naturally as a limit of normalized sums of long-range dependent random variables [33]. Like the fractional Brownian motion, it belongs to the class of Hermite processes, fractional Brownian motion being of order 1 while Rosenblatt process is of order 2. Both are selfsimilar stochastic processes with stationary increments and are characterized by a parameter H, called the Hurst exponent. However, unlike the fractional Brownian motion, the Rosenblatt process is not Gaussian. Does it make a big difference regarding ordinary, rapid and slow points? In other words, can Theorem IV.1.1 be extended to cover the non Gaussian Rosenblatt process?

For the last fifteen years the Rosenblatt process has received a significantly increasing interest in both theoretical and practical lines of research. Due to its self-similarity, its applications are numerous across a multitude of fields, including internet traffic [23] and turbulence [99, 64]. From a statistical point of view, estimating the value of the Hurst index H is important for practical applications and various estimators exist, see [10, 121]. Also, from a mathematical point of view the Rosenblatt process has received a lot of interest since its inception in [98]. Its distribution, still not known in explicit form, was studied first in [2] and more recently in [72] and [123].

In this paper, we even consider a generalization of the Rosenblatt process, as defined and studied in [71]. It depends on two parameters $H_1, H_2 \in (\frac{1}{2}, 1)$ which are such that $H_1 + H_2 > \frac{3}{2}$. The generalized Rosenblatt process $\{R_{H_1,H_2}(t,\cdot)\}_{t\in\mathbb{R}_+}$ is defined as a double Wiener-Itô integral of a kernel function K_{H_1,H_2} with respect to a given Brownian motion. More precisely, consider a standard two-sided Brownian motion B, and set

$$R_{H_1,H_2}(t,\cdot) = \int_{\mathbb{R}^2}' K_{H_1,H_2}(t,x_1,x_2)\, dB(x_1)dB(x_2), \tag{IV.1.4}$$

where $\int_{\mathbb{R}^2}'$ denotes integration over $\mathbb{R}^2$ excluding the diagonal. The kernel function in (IV.1.4) is expressed, for all (t,x_1,x_2) on $\mathbb{R}_+ \times \mathbb{R}^2$, by

$$K_{H_1,H_2}(t,x_1,x_2) = \frac{1}{\Gamma\left(H_1 - \frac{1}{2}\right)\Gamma\left(H_2 - \frac{1}{2}\right)} \int_0^t (s-x_1)_+^{H_1-\frac{3}{2}} (s-x_2)_+^{H_2-\frac{3}{2}}\, ds,$$

where Γ stands for the usual Gamma Euler function, and where for $(x,\alpha) \in \mathbb{R}^2$

$$x_+^\alpha = \begin{cases} x^\alpha & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Note that the (standard) Rosenblatt process is the process $\{R_{H,H}(t,\cdot)\}_{t\in\mathbb{R}_+}$ for $H \in (3/4,1)$. The generalized Rosenblatt process $\{R_{H_1,H_2}(t,\cdot)\}_{t\in\mathbb{R}_+}$ is non-Gaussian, belongs to the second Wiener chaos, and has the following basic properties:

(1) Continuity: the trajectories of the Rosenblatt process R_{H_1,H_2} are continuous.

(2) Stationary increments: R_{H_1,H_2} has stationary increments; that is, the distribution of the process $\{R_{H_1,H_2}(t+s,\cdot) - R_{H_1,H_2}(s,\cdot)\}_{t\in\mathbb{R}_+}$ does not depend on $s \geq 0$.

(3) Self-similarity: R_{H_1,H_2} is self-similar with exponent H_1+H_2-1; that is, the processes $\{R_{H_1,H_2}(ct,\cdot)\}_{t\in\mathbb{R}_+}$ and $\left\{c^{H_1+H_2-1}R_{H_1,H_2}(t,\cdot)\right\}_{t\in\mathbb{R}_+}$ have the same distribution for all $c>0$.

In [7], Ayache and Esmili presented a wavelet-type representation of the generalized Rosenblatt process, very similar to the one given in [81] for fractional Brownian motion, excepted for the use of integrals of two-dimensional wavelet bases. This representation is the starting point of this paper. It is one of our key tools to prove the following Theorem IV.1.2 which is the main result of this paper.

Theorem IV.1.2. *For all $H_1, H_2 \in (\frac{1}{2},1)$ such that $H_1+H_2 > \frac{3}{2}$, there exists an event Ω_{H_1,H_2} of probability 1 satisfying the following assertions for all $\omega \in \Omega_{H_1,H_2}$ and every non-empty interval I of $\mathbb{R}$.*

- *For almost every $t \in I$,*

$$0 < \limsup_{s\to t} \frac{|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)|}{|t-s|^{H_1+H_2-1}\log\log|t-s|^{-1}} < +\infty. \tag{IV.1.5}$$

  Such points are called ordinary points.

- *There exists a dense set of points $t \in I$ such that*

$$0 < \limsup_{s\to t} \frac{|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)|}{|t-s|^{H_1+H_2-1}\log|t-s|^{-1}} < +\infty. \tag{IV.1.6}$$

  Such points are called rapid points.

- *There exists a dense set of points $t \in I$ such that*

$$\limsup_{s\to t} \frac{|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)|}{|t-s|^{H_1+H_2-1}} < +\infty. \tag{IV.1.7}$$

  Such points are called slow points.

Theorem IV.1.2 shows in particular that slow, ordinary and rapid points are not specific to Gaussian processes.

Remark 11. Let us compare Theorems IV.1.1 and IV.1.2. Each type of points is defined in the same way when considering their pointwise moduli of continuity. Indeed, if X denotes both the fractional Brownian motion or the generalized Rosenblatt process, we see that the asymptotic behaviour of $|X(t,\omega)-X(s,\omega)|$ is always compared to a modulus of continuity of the form $|t-s|^{\alpha}\theta(|t-s|)$, with α corresponding to the self-similarity exponent of X and θ a potential logarithmic correction. For the ordinary points, θ is an iterated logarithm. More precisely, for the fractional Brownian motion, we have $\theta(|t-s|) = \sqrt{\log\log|t-s|^{-1}}$ while, for the generalized Rosenblatt process, $\theta(|t-s|) = \log\log|t-s|^{-1}$. The same feature appears for the rapid points: in the case of the fractional Brownian motion we have $\theta(|t-s|) = \sqrt{\log|t-s|^{-1}}$ and for the generalized Rosenblatt process we have $\theta(|t-s|) = \log|t-s|^{-1}$. Therefore, the only difference between the corresponding logarithmic corrections is the square root that is used for the fractional Brownian motion and not for the generalized Rosenblatt process. It comes from the estimates that can be done on the tails of the distribution of random variables in the first order Wiener chaos, for the fractional Brownian motion, or the second order, for the generalized Rosenblatt process, see Theorems IV.3.11 and IV.3.12 below. Concerning the slow points, there is no logarithmic correction, $\theta = 1$ in both case. Unfortunately, contrary to the fractional Brownian motion, we did not manage to show the positiveness of the limit in (IV.1.7). In fact, for that, we would need to find an almost-sure uniform lower modulus of continuity for the generalized Rosenblatt process and to be able to judge its optimality, which seems to be a difficult task. This is discussed in details in Remark 20 below, where we give an almost-sure uniform lower modulus of continuity using the techniques we use to prove the positiveness of the limits in (IV.1.5) and (IV.1.6).

Our strategy to prove Theorem IV.1.2 is as follows. First, in Section IV.3 we derive upper-bounds for the oscillations $|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)|$ that are sharp enough to imply the finiteness of the limits (IV.1.5), (IV.1.6) and (IV.1.7). This is done by means of the wavelet-type expansion given in [7], see Theorem IV.3.2 below. Then, in Section IV.4, we give lower bounds for the so-called wavelet-leaders, see Section IV.2, of the generalized Rosenblatt process on a given compactly supported wavelet basis. This will prove the positiveness of the limits (IV.1.5), (IV.1.6). In particular, we use different bases depending on whether we deal with the finiteness of the limits in Theorem IV.1.2 or with their strict positiveness. This is very different from [36] where the authors always work with the same wavelet. The reason is that the expression (IV.3.3) in Theorem IV.3.2 below is not a wavelet series: it involves additional quantities. Therefore, standard arguments linking wavelet coefficients and regularity of the associated functions can no longer be used.

There are a priori no obstacles to extend our results in Section IV.4 to any Hermite process. On the contrary, extending the results of Section IV.3 does not seem obvious at all. This is because a wavelet-type expansion of arbitrary Hermite process is still missing but also because our strategy relies on arguments which are specific to the two-dimensional feature of the Rosenblatt process, see Lemma IV.2.1 for instance.

Notations used through this paper are rather standard except, maybe, that if s,t are two real numbers, $\int_{[s,t]}$ stands for $\int_s^t$ if $s \le t$ and $-\int_s^t = \int_t^s$ otherwise.

IV.2 Some important facts involving wavelets

In this section, we gather all the facts concerning wavelets that we will strongly use all along this article. First, an immediate but important consequence of the admissibility condition (IV.1.1) is that, if the wavelet $\psi \in L^1(\mathbb{R})$, its first moment always vanishes, i.e.

$$\int_{\mathbb{R}} \psi(x)dx = 0.$$

This condition is met for all the wavelets we consider in this paper.

First, while dealing with the upper bounds for the limits in Theorem IV.1.2, we will use a wavelet-type expansion of the generalized Rosenblatt process. It is given in [7] by the mean of the Meyer's wavelet: ψ belongs to the Schwartz class $\mathcal{S}(\mathbb{R})$, and its Fourier transform is compactly supported, see [66]. In particular, for all $H \in (1/2, 1)$, ψ_H, the fractional antiderivative $\widehat{\psi_H}$ of order $H - 1/2$ of ψ is well-defined by means of its Fourier transform as

$$\widehat{\psi_H}(0) = 0 \quad \text{and} \quad \widehat{\psi_H}(\xi) = (i\xi)^{-(H-\frac{1}{2})}\widehat{\psi}(\xi),\ \forall\, \xi \neq 0. \tag{IV.2.1}$$

It also belongs to the Schwartz class $\mathcal{S}(\mathbb{R})$, see [5, 7, 104] for instance. Moreover, some standard facts from distribution theory [104, 5] give us the explicit formula

$$\psi_H(t) = \frac{1}{\Gamma\left(H - \frac{1}{2}\right)} \int_{\mathbb{R}} (t-x)_+^{H-\frac{3}{2}} \psi(x)\, dx.$$

From (IV.2.1), we see that $\operatorname{supp}(\widehat{\psi_H}) = \operatorname{supp}(\widehat{\psi})$ which is the key fact to establish the following lemma, gathering facts already proved in [7].

Lemma IV.2.1. *Let $H_1, H_2 \in (\frac{1}{2}, 1)$. If $(j_1, j_2, k_1, k_2) \in \mathbb{Z}^4$ are such that $|j_1 - j_2| > 1$, then the integral*

$$I_{j_1,j_2}^{k_1,k_2} := \int_{\mathbb{R}} \psi_{H_1}(2^{j_1}x - k_1)\psi_{H_2}(2^{j_2}x - k_2)$$

vanishes. Moreover, for all $(j, k_1, k_2) \in \mathbb{Z}^3$, we have

$$I_{j+1,j}^{k_1,k_2} = 2^{-j} \int_{\mathbb{R}} e^{-i(k_1-2k_2)\xi}\widehat{\psi_{H_1}}(\xi)\overline{\widehat{\psi_{H_2}}(2\xi)}\, d\xi, \tag{IV.2.2}$$

$$I_{j,j}^{k_1,k_2} = 2^{-j} \int_{\mathbb{R}} e^{-i(k_1-k_2)\xi}\widehat{\psi_{H_1}}(\xi)\overline{\widehat{\psi_{H_2}}(\xi)}\, d\xi, \tag{IV.2.3}$$

$$I_{j,j+1}^{k_1,k_2} = 2^{-j} \int_{\mathbb{R}} e^{-i(2k_1-k_2)\xi}\widehat{\psi_{H_1}}(2\xi)\overline{\widehat{\psi_{H_2}}(\xi)}\, d\xi. \tag{IV.2.4}$$

In addition, for all $L > 0$, there exists a constant $C_L > 0$ such that for all $(j, k_1, k_2) \in$

$\mathbb{Z}^3$,

$$|I_{j+1,j}^{k_1,k_2}| \leq C_L \frac{2^{-j}}{(3+|k_1-2k_2|)^L},$$
$$|I_{j,j}^{k_1,k_2}| \leq C_L \frac{2^{-j}}{(3+|k_1-k_2|)^L},$$
$$|I_{j,j+1}^{k_1,k_2}| \leq C_L \frac{2^{-j}}{(3+|2k_1-k_2|)^L}.$$

When dealing with the the lower bounds for the limits in Theorem IV.1.2, we use Daubechies compactly supported wavelets [26]. Note that, if $\mathrm{supp}(\Psi) \subseteq [-N,N]$, for a positive integer N, then, using the first vanishing moment, for all $(j,k) \in \mathbb{N}\times\mathbb{Z}$ and $t \in \mathbb{R}$, one can write

$$c_{j,k} = \int_{-N}^{N} \left(f\left(\frac{x+k}{2^j}\right) - f(t) \right) \Psi(x)\,dx \tag{IV.2.5}$$

Since Ψ is compactly supported, $\Psi(2^j \cdot -k)$ is localized around the dyadic interval

$$\lambda_{j,k} := \left[\frac{k}{2^j}, \frac{k+1}{2^j}\right)$$

and it is therefore common to index wavelets these intervals. For simplicity, we sometimes omit any references to the indices j and k for such intervals by writing $\lambda = \lambda_{j,k}$, and $k = s(\lambda)$. Similarly, c_λ refers to the quantity $c_{j,k}$. The notation Λ_j stands for the set of dyadic intervals λ of $\mathbb{R}$ with side length 2^{-j}. The unique dyadic interval from Λ_j containing the point $t \in \mathbb{R}$ is denoted $\lambda_j(t)$. The set of dyadic intervals is $\Lambda := \cup_{j\in\mathbb{N}}\Lambda_j$. Two dyadic intervals λ and λ' are adjacent if there exist $j \in \mathbb{N}$ such that $\lambda, \lambda' \in \Lambda_j$ and $\mathrm{dist}(\lambda,\lambda') = 0$. The set of dyadic intervals adjacent to λ is denoted by 3λ. In this setting, one defines the *wavelet leader* [50] of f at t and of scale j by

$$d_j(t_0) = \max_{\lambda \in 3\lambda_j(t_0)} \sup_{\lambda' \subseteq \lambda} |c'_\lambda|. \tag{IV.2.6}$$

Then, if $\mathrm{supp}(\Psi) \subseteq [-N,N]$, from (IV.2.5), one can write

$$d_j(t) \leq 2N \sup_{s \in (t_0 - 2^{-j}(N+2), t_0 + 2^{-j}(N+2))} |f(s) - f(t)| \|\Psi\|_{L^\infty}. \tag{IV.2.7}$$

When we study stochastic processes, the wavelet leaders are random variables $d_j(t,\omega)$. Inequality (IV.2.7) with some easy computations implies that in order to obtain the positiveness of the limit (IV.1.5), it suffices to show that for all $\omega \in \Omega_{H_1,H_2}$ and all open intervals $I \subseteq \mathbb{R}^+$, for almost every $t \in I$,

$$0 < \limsup_{j\to+\infty} \frac{d_j(t,\omega)}{2^{-j(H_1+H_2-1)}\log(j)}. \tag{IV.2.8}$$

Similarly, to prove the positiveness of the limit (IV.1.6), we just have to show that for all $\omega \in \Omega_{H_1,H_2}$ and all open intervals $I \subseteq \mathbb{R}^+$, there exists a dense set of points $t \in I$ such that

$$0 < \limsup_{j\to+\infty} \frac{d_j(t,\omega)}{2^{-j(H_1+H_2-1)}j}. \tag{IV.2.9}$$

Remark 12. Let us mention that wavelet leaders can not be used to prove the finiteness of the limits in Theorem IV.1.2 because they do not precisely characterize the pointwise regularity, see for instance [63, 70] for more details.

IV.3 Upper bounds for oscillations

Starting from now and until the end of the paper, we fix $H_1, H_2 \in (\frac{1}{2}, 1)$ such that $H_1+H_2 > \frac{3}{2}$. In this section, we show the finiteness of the limits (IV.1.5), (IV.1.6) and (IV.1.7). Concerning the rapid points, we will in fact show a stronger result, obtaining an almost sure *uniform* modulus of continuity for the generalized Rosenblatt process.

We use a wavelet-type expansion of the generalized Rosenblatt process. It relies on the following random variables.

Definition IV.3.1. For all $(j_1, j_2, k_1, k_2) \in \mathbb{Z}^4$, let $\varepsilon_{j_1,j_2}^{k_1,k_2}$ be the second order Wiener chaos random variable defined by

$$2^{\frac{j_1+j_2}{2}} \int_{\mathbb{R}^2}' \psi(2^{j_1}x_1 - k_1)\psi(2^{j_2}x_2 - k_2)\, dB(x_1)dB(x_2).$$

Remark 13. For all $(j_1, j_2, k_1, k_2) \in \mathbb{Z}^4$, we have ([7, Proposition 2.3])

$$\varepsilon_{j_1,j_2}^{k_1,k_2} = \left(2^{\frac{j_1}{2}} \int_{\mathbb{R}} \psi(2^{j_1}x - k_1)dB(x)\right)\left(2^{\frac{j_2}{2}} \int_{\mathbb{R}} \psi(2^{j_2}x - k_2)dB(x)\right) \tag{IV.3.1}$$

for $j_1 \neq j_2$ or $k_1 \neq k_2$, and

$$\varepsilon_{j_1,j_1}^{k_1,k_1} = \left(2^{\frac{j_1}{2}} \int_{\mathbb{R}} \psi(2^{j_1}x - k_1)dB(x)\right)^2 - 1 \tag{IV.3.2}$$

for $j_1 = j_2$ and $k_1 = k_2$. Using the fact that $(2^{j/2}\psi(2^j\cdot -k))_{(j,k)\in\mathbb{Z}^2}$ forms an orthonormal basis of $L^2(\mathbb{R})$, and elementary properties of Wiener integral, we know that $(2^{j/2}\int_{\mathbb{R}}\psi(2^jx - k)\, dB(x))_{(j,k)\in\mathbb{Z}^2}$ is a family of iid $\mathcal{N}(0,1)$ random variables. So the random variables $\varepsilon_{j_1,j_2}^{k_1,k_2}$ and $\varepsilon_{j_1',j_2'}^{k_1',k_2'}$ are independent as soon as

$$\{(j_1,k_1),(j_2,k_2)\} \cap \{(j_1',k_1'),(j_2',k_2')\} = \emptyset.$$

The following theorem, proved in [7], gives the wavelet-type expansion we use in this section.

Theorem IV.3.2. *Let ψ be the Meyer wavelet and I be any compact interval of $\mathbb{R}_+$. Almost surely, the random series*

$$\sum_{(j_1,j_2,k_1,k_2)\in\mathbb{Z}^4} 2^{j_1(1-H_1)+j_2(1-H_2)}\varepsilon_{j_1,j_2}^{k_1,k_2}\int_0^t \psi_{H_1}(2^{j_1}x-k_1)\psi_{H_2}(2^{j_2}x-k_2)\,dx \tag{IV.3.3}$$

converges uniformly to R_{H_1,H_2} on the interval I.

Remark 14. Any open interval in $\mathbb{R}$ can be written as a countable union of dyadic intervals $(\lambda_{j,k})_{j\in\mathbb{N},k\in\mathbb{Z}}$. Then, to prove Theorem IV.1.2, it is sufficient to show that, for all $j\in\mathbb{N}, k\in\mathbb{Z}$, there exist an event $\Omega_{j,k}$ of probability 1 such that, for all $\omega\in\Omega_{j,k}$, almost every $t\in\lambda_{j,k}$ is ordinary and there exist $t_r\in\lambda_{j,k}$ which is rapid and $t_s\in\lambda_{j,k}$ which is slow. For the sake of simpleness in notation, we will only do the proofs in full details for $\lambda_{0,0}=[0,1)$. In fact, after dilatation and translation, our proofs hold true for any arbitrary dyadic interval.

IV.3.1 Rapid points

Let us first focus on rapid points. We prove that $x\mapsto |x|^{H_1+H_2-1}\log|x|^{-1}$ is almost surely a uniform modulus of continuity for R_{H_1,H_2}.

Proposition IV.3.3. *There exists an event Ω_{rap} of probability 1 such that for all $\omega\in\Omega_{rap}$ there exists $C_R(\omega)>0$ such that, for all $t,s\in(0,1)$, we have*

$$|R_{H_1,H_2}(t,\omega)-R_{H_1,H_2}(s,\omega)|\leq C_R(\omega)|t-s|^{H_1+H_2-1}\log|t-s|^{-1}. \tag{IV.3.4}$$

Let us set, for all $s,t\in(0,1)$ and $(j_1,j_2,k_1,k_2)\in\mathbb{Z}^4$,

$$I_{j_1,j_2}^{k_1,k_2}[t,s]=\int_{[t,s]}\psi_{H_1}(2^{j_1}x-k_1)\psi_{H_2}(2^{j_2}x-k_2)\,dx.$$

All along this section, if $s,t\in(0,1)$ are given, n always refers to the unique positive integer such that

$$2^{-n-1}<|t-s|\leq 2^{-n}. \tag{IV.3.5}$$

Our proof consists in writing

$$|R_{H_1,H_2}(t,\cdot))-R_{H_1,H_2}(s,\cdot)|=\left|\sum_{(j_1,j_2,k_1,k_2)\in\mathbb{Z}^4}2^{j_1(1-H_1)}2^{j_2(1-H_2)}\varepsilon_{j_1,j_2}^{k_1,k_2}I_{j_1,j_2}^{k_1,k_2}[t,s]\right| \tag{IV.3.6}$$

and to split the sum in the right-hand side in subsums determined according to the position of j_1 and j_2 with respect to n. To bound from above some of these subsums the following lemma is key.

Lemma IV.3.4. *[7, Lemma 2.4.] There exist an event Ω^* of probability 1 and a positive random variable C_1 with finite moment of any order, such that, for all $\omega\in\Omega^*$ and for each $(j_1,j_2,k_1,k_2)\in\mathbb{Z}^4$,*

$$|\varepsilon_{j_1,j_2}^{k_1,k_2}(\omega)|\leq C_1(\omega)\sqrt{\log(3+|j_1|+|k_1|)}\sqrt{\log(3+|j_2|+|k_2|)}. \tag{IV.3.7}$$

In view of Lemma IV.3.4, we set

$$L_{j_1,j_2}^{k_1,k_2} = \sqrt{\log(3+|j_1|+|k_1|)}\sqrt{\log(3+|j_2|+|k_2|)}.$$

As a first step, Lemmata IV.3.5 to IV.3.9 are devoted to bound some deterministic series whose general term is

$$2^{j_1(1-H_1)}2^{j_2(1-H_2)}L_{j_1,j_2}^{k_1,k_2}|I_{j_1,j_2}^{k_1,k_2}[t,s]|.$$

This first lemma will be useful to bound the subsums in the right-hand side of (IV.3.6) for $j_1 < n$ and $j_2 < n$.

Lemma IV.3.5. *There exists a deterministic constant $C > 0$ such that, for all $t, s \in (0,1)$, we have*

$$\sum_{j_1<n}\sum_{j_2<n}\sum_{(k_1,k_2)\in\mathbb{Z}^2} 2^{j_1(1-H_1)}2^{j_2(1-H_2)}L_{j_1,j_2}^{k_1,k_2}\left|I_{j_1,j_2}^{k_1,k_2}[t,s]\right| \le C|t-s|^{H_1+H_2-1}\log|t-s|^{-1}.$$

Proof. Let us start by considering, for all $(j_1,j_2)\in\mathbb{Z}^2$, the series

$$R_{j_1,j_2} \,:\, t \mapsto \sum_{(k_1,k_2)\in\mathbb{Z}^2} L_{j_1,j_2}^{k_1,k_2}\int_0^t |\psi_{H_1}(2^{j_1}x-k_1)\psi_{H_2}(2^{j_2}x-k_2)|\,dx \text{ and}$$

$$R'_{j_1,j_2} \,:\, t \mapsto \sum_{(k_1,k_2)\in\mathbb{Z}^2} L_{j_1,j_2}^{k_1,k_2}|\psi_{H_1}(2^{j_1}t-k_1)\psi_{H_2}(2^{j_2}t-k_2)|.$$

The fast decay of the fractional antiderivatives of ψ allows us to write, for all $H\in\{H_1,H_2\}$ and for all $x\in\mathbb{R}$

$$|\psi_H(x)| \le C(1+|x|)^{-4}. \tag{IV.3.8}$$

Moreover, according to [7, Lemma 4.2] for all $L>1$ there exists $C>0$ such that, for all $j\in\mathbb{Z}$ and $x\in\mathbb{R}$

$$\sum_{k\in\mathbb{Z}} \frac{\sqrt{\log(3+|j|+k)}}{(3+|2^jx-k|)^L} \le C\sqrt{\log(3+|j|+2^j|x|)}. \tag{IV.3.9}$$

Therefore, if K is any compact set of $\mathbb{R}_+$, if $s=\sup_K$, for all $t\in K$, we have

$$\begin{aligned}|R_{j_1,j_2}(t)| &\le C\int_0^t \sqrt{\log(3+|j_1|+2^{j_1}|x|)}\sqrt{\log(3+|j_2|+2^{j_2}|x|)}\,dx\\ &\le Cs\sqrt{\log(3+|j_1|+2^{j_1}s)}\sqrt{\log(3+|j_2|+2^{j_2}s)}.\end{aligned}$$

The same arguments can be applied to ${R'}_{j_1,j_2}$, which means that both series converge uniformly on any compact set of $\mathbb{R}_+$. From this, we can use mean value theorem: for all $(j_1,j_2)\in\mathbb{Z}^2$ there is $\xi(j_1,j_2)\in[s,t]$ such that

$$\begin{aligned}&\sum_{(k_1,k_2)\in\mathbb{Z}^2} 2^{j_1(1-H_1)}2^{j_2(1-H_2)}L_{j_1,j_2}^{k_1,k_2}\left|I_{j_1,j_2}^{k_1,k_2}[t,s]\right|\\ &\le |t-s|\sum_{(k_1,k_2)\in\mathbb{Z}^2} L_{j_1,j_2}^{k_1,k_2}|\psi_{H_1}(2^{j_1}\xi-k_1)\psi_{H_2}(2^{j_2}\xi-k_2)|.\end{aligned} \tag{IV.3.10}$$

Now, we use the fast decay of the fractional antiderivatives of ψ (IV.3.8) and inequality (IV.3.9) to bound (IV.3.10) from above: for all $j_1, j_2 < n$,

$$\begin{aligned}
\sum_{(k_1,k_2)\in\mathbb{Z}^2} & L_{j_1,j_2}^{k_1,k_2} |\psi_{H_1}(2^{j_1}\xi - k_1)\psi_{H_2}(2^{j_2}\xi - k_2)| \\
&\leq C \left(\sum_{k_1\in\mathbb{Z}} \frac{\sqrt{\log(3+|j_1|+|k_1|)}}{(3+|2^{j_1}\xi - k_1|)^4} \right) \left(\sum_{k_2\in\mathbb{Z}} \frac{\sqrt{\log(3+|j_2|+|k_2|)}}{(3+|2^{j_2}\xi - k_2|)^4} \right) \\
&\leq C\sqrt{\log(3+|j_1|+2^{j_1}|\xi|)}\sqrt{\log(3+|j_2|+2^{j_2}|\xi|)} \\
&\leq C\sqrt{\log(3+|j_1|+2^{j_1})}\sqrt{\log(3+|j_2|+2^{j_2})},
\end{aligned}$$

as $\xi \in (0,1)$. Let us then remark that

$$\begin{aligned}
\sum_{j_1<n} & 2^{j_1(1-H_1)}\sqrt{\log(3+|j_1|+2^{j_1})} \\
&= \sum_{j_1\leq 0} 2^{j_1(1-H_1)}\sqrt{\log(3+|j_1|+2^{j_1})} + \sum_{j_1=0}^{n-1} 2^{j_1(1-H_1)}\sqrt{\log(3+|j_1|+2^{j_1})} \\
&\leq C + \sum_{j_1=0}^{n-1} 2^{j_1(1-H_1)}\sqrt{\log(3+|j_1|+2^{j_1})} \\
&\leq C 2^{n(1-H_1)}\sqrt{n}, \qquad \text{(IV.3.11)}
\end{aligned}$$

as $1 - H_1 > 0$. The same can be applied to the sum over j_2 and we finally get

$$\begin{aligned}
&\sum_{j_1<n}\sum_{j_2<n}\sum_{(k_1,k_2)\in\mathbb{Z}^2} 2^{j_1(1-H_1)}2^{j_2(1-H_2)} L_{j_1,j_2}^{k_1,k_2} \left| I_{j_1,j_2}^{k_1,k_2}[t,s] \right| \\
&\leq C|t-s| \sum_{j_1<n}\sum_{j_2<n} 2^{j_1(1-H_1)}2^{j_2(1-H_2)}\sqrt{\log(3+|j_1|+2^{j_1})}\sqrt{\log(3+|j_2|+2^{j_2})} \\
&\leq C|t-s|2^{n(2-H_1-H_2)}n \\
&\leq C|t-s|^{H_1+H_2-1}\log|t-s|^{-1}.
\end{aligned}$$

□

Lemmata IV.3.7 and IV.3.8 will help finding an upper bound for the subsums in the right-hand side of (IV.3.6) with $j_1 < n \leq j_2$ or $j_2 < n \leq j_1$ as well as the ones where $n \leq j_1 \leq j_2$ and $n \leq j_2 \leq j_1$. Let us define the following partition of $\mathbb{Z}$, which determines the relative positions of $[k_2 2^{-j_2}, (k_2+1)2^{-j_2})$ and $[s,t]$.

Definition IV.3.6. For all $j_2 \in \mathbb{N}$, we set

$$\begin{aligned}
\mathbb{Z}_{j_2}^{<}(t,s) &= \{k_2 \in \mathbb{Z} \,:\, k_2 2^{-j_2} < \min\{t,s\}\}, \\
\mathbb{Z}_{j_2}^{>}(t,s) &= \{k_2 \in \mathbb{Z} \,:\, k_2 2^{-j_2} > \max\{t,s\}\}, \\
\text{and } \mathbb{Z}_{j_2}[t,s] &= \mathbb{Z} \setminus (\mathbb{Z}_{j_2}^{<}(t,s) \cup \mathbb{Z}_{j_2}^{>}(t,s)).
\end{aligned}$$

Remark 15. Note that we have $\#\mathbb{Z}_{j_2}[t,s] \leq 2^{j_2-n}+1$.

Let us also observe that for all $a, b > 0$,

$$\log(3+a+b) \leq \log(3+a)\log(3+b). \tag{IV.3.12}$$

Lemma IV.3.7. *There exists a deterministic constant $C>0$ such that, for all $t, s \in (0,1)$ and $j_1 \leq j_2$, the quantities*

$$\sum_{k_1\in\mathbb{Z}} \sum_{k_2\in\mathbb{Z}^{<}_{j_2}(t,s)} L^{k_1,k_2}_{j_1,j_2} \left| I^{k_1,k_2}_{j_1,j_2}[t,s] \right| \tag{IV.3.13}$$

$$\sum_{k_1\in\mathbb{Z}} \sum_{k_2\in\mathbb{Z}^{>}_{j_2}(t,s)} L^{k_1,k_2}_{j_1,j_2} \left| I^{k_1,k_2}_{j_1,j_2}[t,s] \right| \tag{IV.3.14}$$

are bounded from above by

$$C\sqrt{\log(3+|j_1|+2^{j_1})}\sqrt{\log(3+|j_2|+2^{j_2})}2^{-j_2}.$$

Proof. Let us bound (IV.3.13), the proof for (IV.3.14) being similar. From the fast decay of the fractional antiderivatives of ψ (IV.3.8), inequalities (IV.3.9) and (IV.3.12) for $j_1 \leq j_2$, we have

$$\begin{aligned}
\text{(IV.3.13)} &\leq C\int_{[s,t]} \left(\sum_{k_1\in\mathbb{Z}} \frac{\sqrt{\log(3+|j_1|+|k_1|)}}{(3+|2^{j_1}x-k_1|)^4}\right)\left(\sum_{k_2\in\mathbb{Z}^{<}_{j_2}(t,s)} \frac{\sqrt{\log(3+|j_2|+|k_2|)}}{(3+|2^{j_2}x-k_2|)^4}\right) dx \\
&\leq C\sqrt{\log(3+|j_1|+2^{j_1})}\sqrt{\log(3+|j_2|+2^{j_2})} \\
&\quad \int_{[s,t]} \sum_{k_2\in\mathbb{Z}^{<}_{j_2}(t,s)} \frac{\sqrt{\log(3+|2^{j_2}x-k_2|)}}{(3+|2^{j_2}x-k_2|)^4}\, dx.
\end{aligned}$$

For all $x\in[s,t]$ the mapping $y \mapsto (2+2^{j_2}x-2^{j_2}\min\{s,t\}+y)^{-3}$ is decreasing and thus

$$\begin{aligned}
&\int_{[s,t]} \sum_{k_2\in\mathbb{Z}^{<}_{j_2}(t,s)} \frac{\sqrt{\log(3+|2^{j_2}x-k_2|)}}{(3+|2^{j_2}x-k_2|)^4}\, dx \\
&\leq \int_{[s,t]} \sum_{k_2\in\mathbb{Z}^{<}_{j_2}(t,s)} \frac{dx}{(3+2^{j_2}x-k_2)^3} \\
&\leq \int_{[s,t]} \sum_{m=0}^{+\infty} \frac{dx}{(3+2^{j_2}x-2^{j_2}\min\{s,t\}+m)^3} \\
&\leq \int_{[s,t]} \int_0^{+\infty} \frac{dxdy}{(2+2^{j_2}x-2^{j_2}\min\{s,t\}+y)^3} \\
&\leq C2^{-j_2}.
\end{aligned} \tag{IV.3.15}$$

This bound leads to

$$(\text{IV.3.13}) \leq C\sqrt{\log(3+|j_1|+2^{j_1})}\sqrt{\log(3+|j_2|+2^{j_2})}2^{-j_2}. \qquad \text{(IV.3.16)}$$

□

Lemma IV.3.8. *There exists a deterministic constant $C>0$ such that, for all $t,s\in(0,1)$ and $j_1\leq j_2$, the quantities*

$$\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}_{j_2}[t,s]} L_{j_1,j_2}^{k_1,k_2}\left|\int_{-\infty}^{\min\{s,t\}}\psi_{H_1}(2^{j_1}x-k_1)\psi_{H_2}(2^{j_2}x-k_2)\,dx\right|$$

$$\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}_{j_2}[t,s]} L_{j_1,j_2}^{k_1,k_2}\left|\int_{\max\{s,t\}}^{+\infty}\psi_{H_1}(2^{j_1}x-k_1)\psi_{H_2}(2^{j_2}x-k_2)\,dx\right|$$

are bounded from above by

$$C\sqrt{\log(3+|j_1|+2^{j_2})}\sqrt{\log(3+|j_2|+2^{j_2})}2^{-j_2}.$$

Proof. Let us assume that $s\leq t$, the argument for $t<s$ being similar. As $j_2\geq j_1$, we have, by inequality (IV.3.9),

$$\begin{aligned}
&\int_{-\infty}^{s}\left(\sum_{k_1\in\mathbb{Z}}\frac{\sqrt{\log(3+|j_1|+|k_1|)}}{(3+|2^{j_1}x-k_1|)^4}\right)\left(\sum_{k_2\in\mathbb{Z}_{j_2}[t,s]}\frac{\sqrt{\log(3+|j_2|+|k_2|)}}{(3+|2^{j_2}x-k_2|)^4}\right)dx\\
&\leq C_L\int_{-\infty}^{s}\sqrt{\log(3+|j_1|+2^{j_1}|x|)}\left(\sum_{k_2\in\mathbb{Z}_{j_2}[t,s]}\frac{\sqrt{\log(3+|j_2|+|k_2|)}}{(3+|2^{j_2}x-k_2|)^4}\right)dx\\
&\leq C_L\int_{-\infty}^{s}\sqrt{\log(3+|j_1|+2^{j_2}|x|)}\left(\sum_{k_2\in\mathbb{Z}_{j_2}[t,s]}\frac{\sqrt{\log(3+|j_2|+|k_2|)}}{(3+|2^{j_2}x-k_2|)^4}\right)dx.
\end{aligned}$$

For all $k_2\in\mathbb{Z}_{j_2}[t,s]$, $|k_2|\leq 2^{j_2}$, we have, using (IV.3.12),

$$\begin{aligned}
&\log(3+|j_1|+2^{j_2}|x|)\leq\log(3+|j_1|+2^{j_2})\log(3+|2^{j_2}x-k_2|) \text{ and}\\
&\log(3+|j_2|+|k_2|)\leq\log(3+j_2+2^{j_2})\log(3+|2^{j_2}x-k_2|).
\end{aligned}$$

Thus, it only remains us to deal with

$$\int_{-\infty}^{s}\sum_{k_2\in\mathbb{Z}_{j_2}[t,s]}\frac{dx}{(3+|2^{j_2}x-k_2|)^3}.$$

But, for all $x\leq s$ and $k_2\in\mathbb{Z}_{j_2}[t,s]$, $|2^{j_2}x-k_2|=k_2-2^{j_2}x$ and then, using the same method as in (IV.3.15), we get

$$\int_{-\infty}^{s}\sum_{k_2\in\mathbb{Z}_{j_2}[t,s]}\frac{dx}{(3+k_2-2^{j_2}x)^3}\leq C2^{-j_2} \qquad \text{(IV.3.17)}$$

which finally leads to

$$\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}_{j_2}[t,s]} L_{j_1,j_2}^{k_1,k_2}\left|\int_{-\infty}^{s}\psi_{H_1}(2^{j_1}x-k_1)\psi_{H_2}(2^{j_2}x-k_2)\,dx\right|$$
$$\leq C\sqrt{\log(3+|j_1|+2^{j_2})}\sqrt{\log(3+j_2+2^{j_2})}2^{-j_2}. \tag{IV.3.18}$$

We get in the same way,

$$\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}_{j_2}[t,s]} L_{j_1,j_2}^{k_1,k_2}\left|\int_{t}^{+\infty}\psi_{H_1}(2^{j_1}x-k_1)\psi_{H_2}(2^{j_2}x-k_2)\,dx\right|$$
$$\leq C\sqrt{\log(3+|j_1|+2^{j_2})}\sqrt{\log(3+j_2+2^{j_2})}2^{-j_2}.$$

□

Next Lemma will be used to bound the subsums of (IV.3.6) with $j_1 < n \leq j_2$ or $j_2 < n \leq j_1$.

Lemma IV.3.9. *There exists a deterministic constant $C > 0$ such that, for all $t, s \in (0,1)$, the quantities*

$$R^{<\geq n}[t,s] := \sum_{j_1<n}\sum_{j_2\geq n}\sum_{(k_1,k_2)\in\mathbb{Z}^2} 2^{j_1(1-H_1)}2^{j_2(1-H_2)}L_{j_1,j_2}^{k_1,k_2}|I_{j_1,j_2}^{k_1,k_2}[t,s]|$$

$$R^{\geq < n}[t,s] := \sum_{j_1\geq n}\sum_{j_2< n}\sum_{(k_1,k_2)\in\mathbb{Z}^2} 2^{j_1(1-H_1)}2^{j_2(1-H_2)}L_{j_1,j_2}^{k_1,k_2}|I_{j_1,j_2}^{k_1,k_2}[t,s]|$$

are bounded from above by

$$C|t-s|^{H_1+H_2-1}\log|t-s|^{-1}.$$

Proof. As $R^{<\geq n}[t,s]$ and $R^{\geq<n}[t,s]$ can clearly be treated symmetrically, we restrict our attention to $R^{<\geq n}[t,s]$. One sees that

$$\sum_{j_1<n}\sum_{j_2\geq n}\sum_{(k_1,k_2)\in\mathbb{Z}^2} 2^{j_1(1-H_1)}2^{j_2(1-H_2)}L_{j_1,j_2}^{k_1,k_2}I_{j_1,j_2}^{k_1,k_2}[t,s]$$
$$=\sum_{j_1<n}\sum_{j_2\geq n}\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}_{j_2}^{<}(t,s)} 2^{j_1(1-H_1)}2^{j_2(1-H_2)}L_{j_1,j_2}^{k_1,k_2}I_{j_1,j_2}^{k_1,k_2}[t,s] \tag{IV.3.19}$$
$$+\sum_{j_1<n}\sum_{j_2\geq n}\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}_{j_2}^{>}(t,s)} 2^{j_1(1-H_1)}2^{j_2(1-H_2)}L_{j_1,j_2}^{k_1,k_2}I_{j_1,j_2}^{k_1,k_2}[t,s] \tag{IV.3.20}$$
$$+\sum_{j_1<n}\sum_{j_2\geq n}\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}_{j_2}[t,s]} 2^{j_1(1-H_1)}2^{j_2(1-H_2)}L_{j_1,j_2}^{k_1,k_2}I_{j_1,j_2}^{k_1,k_2}[t,s]. \tag{IV.3.21}$$

For (IV.3.19), we use Lemma IV.3.7 to get

$$|(\text{IV.3.19})| \leq C\sum_{j_1<n}2^{j_1(1-H_1)}\sqrt{\log(3+|j_1|+2^{j_1})}\sum_{j_2\geq n}2^{-j_2H_2}\sqrt{\log(3+|j_2|+2^{j_2})}.$$

The sum over over j_1 is bounded just as in (IV.3.11) while, for the sum over j_2, we have

$$\sum_{j_2 \geq n} 2^{-j_2 H_2} \sqrt{\log(3+|j_2|+2^{j_2})} \leq \sum_{j_2 \geq n} 2^{-j_2 H_2} \sqrt{\log(3+2^{j_2+1})}$$
$$\leq C 2^{-nH_2} \sqrt{n}. \tag{IV.3.22}$$

We bound (IV.3.20) in exactly the same way.

For (IV.3.21), let us again assume $s \leq t$, then we write

$$\begin{aligned} I_{j_1,j_2}^{k_1,k_2}[t,s] &= \int_{\mathbb{R}} \psi_{H_1}(2^{j_1}x - k_1)\psi_{H_2}(2^{j_2}x - k_2)\, dx \\ &- \int_{-\infty}^{s} \psi_{H_1}(2^{j_1}x - k_1)\psi_{H_2}(2^{j_2}x - k_2)\, dx \\ &- \int_{t}^{+\infty} \psi_{H_1}(2^{j_1}x - k_1)\psi_{H_2}(2^{j_2}x - k_2)\, dx. \end{aligned} \tag{IV.3.23}$$

Since $j_1 < n$ and $j_2 \geq n$, recalling Lemma IV.2.1, the sum

$$\sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}_{j_2}[t,s]} L_{j_1,j_2}^{k_1,k_2} \int_{\mathbb{R}} \psi_{H_1}(2^{j_1}x - k_1)\psi_{H_2}(2^{j_2}x - k_2)\, dx,$$

vanishes except maybe when $(j_1, j_2) = (n-1, n)$. In this case, note that $\#\mathbb{Z}_n[t,s] \leq 2$ and, for all $k_2 \in \mathbb{Z}_n[t,s]$, $|k_2| \leq 2^n$.Then, by Lemma IV.2.1 and inequality (IV.3.9), we get

$$\begin{aligned} &\sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}_n[t,s]} L_{n-1,n}^{k_1,k_2} \left| I_{n-1,n}^{k_1,k_2} \right| \\ &\leq C 2^{-n} \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}_n[t,s]} \frac{\sqrt{\log(3+n-1+|k_1|)}\sqrt{\log(3+n+|k_2|)}}{(3+|2k_1-k_2|)^4} \\ &\leq C 2^{-n} \sum_{k_2 \in \mathbb{Z}_n[t,s]} \sqrt{\log(3+n-1+|\frac{k_2}{2}|)}\sqrt{\log(3+n+|k_2|)} \\ &\leq C 2^{-n} n \end{aligned}$$

Now, using Lemma IV.3.8, we also get

$$\begin{aligned} &\left| \sum_{j_1 < n} \sum_{j_2 \geq n} \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}_{j_2}[t,s]} 2^{j_1(1-H_1)} 2^{j_2(1-H_2)} L_{j_1,j_2}^{k_1,k_2} \int_{-\infty}^{s} \psi_{H_1}(2^{j_1}x - k_1)\psi_{H_2}(2^{j_2}x - k_2)\, dx \right| \\ &\leq \sum_{j_1 < n} \sum_{j_2 \geq n} 2^{j_1(1-H_1)} 2^{-j_2 H_2} \sqrt{\log(3+|j_1|+2^{j_2})}\sqrt{\log(3+|j_2|+2^{j_2})} \\ &\leq \sum_{j_1 < 0} \sum_{j_2 \geq n} 2^{j_1(1-H_1)} 2^{-j_2 H_2} \sqrt{\log(3+|j_1|)} \log(3+2^{j_2+1}) \\ &\quad + \sum_{j_1=0}^{n-1} \sum_{j_2 \geq n} 2^{j_1(1-H_1)} 2^{-j_2 H_2} \sqrt{\log(3+2^{j_2+1})}\sqrt{\log(3+2^{j_2+1})} \\ &\leq C 2^{n(1-H_1-H_2)} n \end{aligned} \tag{IV.3.24}$$

The series

$$\left|\sum_{j_1<n}\sum_{j_2\geq n}\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}_{j_2}[t,s]} 2^{j_1(1-H_1)}2^{j_2(1-H_2)}L_{j_1,j_2}^{k_1,k_2}\int_t^{+\infty}\psi_{H_1}(2^{j_1}x-k_1)\psi_{H_2}(2^{j_2}x-k_2)\,dx\right|$$

is bounded in exactly the same way and the conclusion follows. □

It remains us to bound the subsums of (IV.3.6) with $j_1\geq n$ and $j_2\geq n$. For this, let us define some random variables associated with dyadic intervals.

Definition IV.3.10. If λ is a dyadic interval of scale n, we define, for all $j\geq n$, the indexation sets

$$S_j^0(\lambda):=\{(k^{(1)},K^{(1)},k^{(2)},K^{(2)})\in\mathbb{Z}^4\ :\ \frac{k^{(1)}}{2^j},\frac{K^{(1)}}{2^j},\frac{k^{(2)}}{2^j},\frac{K^{(2)}}{2^j}\in\lambda\},$$
$$S_j^1(\lambda):=\{(k^{(1)},K^{(1)},k^{(2)},K^{(2)})\in\mathbb{Z}^4\ :\ \frac{k^{(1)}}{2^{j+1}},\frac{K^{(1)}}{2^{j+1}},\frac{k^{(2)}}{2^j},\frac{K^{(2)}}{2^j}\in\lambda\},$$
$$S_j^2(\lambda):=\{(k^{(1)},K^{(1)},k^{(2)},K^{(2)})\in\mathbb{Z}^4\ :\ \frac{k^{(1)}}{2^j},\frac{K^{(1)}}{2^j},\frac{k^{(2)}}{2^{j+1}},\frac{K^{(2)}}{2^{j+1}}\in\lambda\}$$

and consider the random variables, for $(k^{(1)},K^{(1)},k^{(2)},K^{(2)})\in S_j^0(\lambda)$,

$${}^0_j\sum\nolimits_{k^{(1)},K^{(1)}}^{k^{(2)},K^{(2)}}:=\sum_{k^{(1)}\leq k_1\leq K^{(1)}}\ \sum_{k^{(2)}\leq k_2\leq K^{(2)}}\varepsilon_{j,j}^{k_1,k_2}I_{j,j}^{k_1,k_2}\tag{IV.3.25}$$

, for $(k^{(1)},K^{(1)},k^{(2)},K^{(2)})\in S_j^1(\lambda)$,

$${}^1_j\sum\nolimits_{k^{(1)},K^{(1)}}^{k^{(2)},K^{(2)}}:=\sum_{k^{(1)}\leq k_1\leq K^{(1)}}\ \sum_{k^{(2)}\leq k_2\leq K^{(2)}}\varepsilon_{j+1,j}^{k_1,k_2}I_{j+1,j}^{k_1,k_2}\tag{IV.3.26}$$

and, for $(k^{(1)},K^{(1)},k^{(2)},K^{(2)})\in S_j^2(\lambda)$,

$${}^2_j\sum\nolimits_{k^{(1)},K^{(1)}}^{k^{(2)},K^{(2)}}:=\sum_{k^{(1)}\leq k_1\leq K^{(1)}}\ \sum_{k^{(2)}\leq k_2\leq K^{(2)}}\varepsilon_{j,j+1}^{k_1,k_2}I_{j,j+1}^{k_1,k_2}.\tag{IV.3.27}$$

The idea behind the definition of these random variables is, as $|t-s|\leq 2^{-n}$, $s\in 3\lambda_n(t)$ and thus any sum of the form

$$\sum_{k_1\in\mathbb{Z}_j[t,s]}\ \sum_{k_2\in\mathbb{Z}_\ell[t,s]}\varepsilon_{j,\ell}^{k_1,k_2}I_{j,\ell}^{k_1,k_2}\tag{IV.3.28}$$

for $\ell\in\{j,j+1\}$ can be written as the sum of random variables (IV.3.25), (IV.3.26) or (IV.3.27) for some $(k^{(1)},K^{(1)},k^{(2)},K^{(2)})$ belonging to at most two $S_j^\ell(\lambda)$ $(\ell\in\{0,1,2\})$ with $\lambda\in\lambda_n(t)$. Indeed,

- if t and s both belong to $\lambda_n(t)$ then we only need to rewrite (IV.3.28) in the form (IV.3.25), (IV.3.26) or (IV.3.27) for $(k^{(1)}, K^{(1)}, k^{(2)}, K^{(2)}) \in S_j^{\ell}(\lambda_n(t))$;
- if $s \in \lambda$ with $\lambda \in 3\lambda_n(t) \setminus \lambda_n(t)$ then we need to consider a first sum indexed by a quadruple of $S_j^{\ell}(\lambda_n(t))$ and a second indexed by a quadruple of $S_j^{\ell}(\lambda)$.

The reason why we decide to put λ instead of 3λ in the definition of the sets $S_j^{\ell}(\lambda)$ is that if, for all $n \in \mathbb{N}$ and for all $\lambda \in \Lambda_n$ and $j \geq n$, we define the random variable

$$\Xi_j(\lambda) = \max_{\ell \in \{0,1,2\}} \sup_{(k^{(1)},K^{(1)},k^{(2)},K^{(2)}) \in S_j^{\ell}(\lambda))} \frac{\left| {}^{\ell}_{j}\sum_{k^{(1)},K^{(1)}}^{k^{(2)},K^{(2)}} \right|}{\left\| {}^{\ell}_{j}\sum_{k^{(1)},K^{(1)}}^{k^{(2)},K^{(2)}} \right\|_{L^2(\Omega)}}, \tag{IV.3.29}$$

we want $\Xi_j(\lambda)$ to be independent of $\Xi_j(\lambda')$ as soon as $\lambda \cap \lambda' = \emptyset$. Moreover, from the definitions of the random variables (IV.3.26), (IV.3.25) and (IV.3.27), the remarks below Theorem IV.3.2 and the explicit expressions (IV.2.2), (IV.2.3) and (IV.2.4), the law of $\Xi_j(\lambda)$ does not depend on $\lambda \in \Lambda_n$but only on $j - n$.

The key results to estimate the random variables Ξ_j are [53, Theorem 6.7 and Theorem 6.12] that we recall here.

Theorem IV.3.11. *There exists a strictly positive universal deterministic constant $\overset{\star}{C}$ such that, for every random variable X belonging to the second order Wiener chaos and for each real number $y \geq 2$, one has*

$$\mathbb{P}(|X| \geq y \|X\|_{L^2(\Omega)}) \leq \exp(-\overset{\star}{C} y).$$

Theorem IV.3.12. *If X is a random variable belonging to the second order Wiener chaos, there exist $a, b, y_0 > 0$ such that, for all $y \geq y_0$,*

$$\exp(-ay) \leq \mathbb{P}(|X| \geq y) \leq \exp(-by).$$

Remark 16. As stated in [53], the constants a, b in Theorem IV.3.12 are not universal and depend on the law of X. Note that b can be recovered from Theorem IV.3.11 and thus is universal on the unit sphere in $L^2(\Omega)$.

Lemma IV.3.13. *There exists a deterministic constant $C > 0$ such that, for all $n \in \mathbb{N}$, $\lambda \in \Lambda_n$, $j \geq n$, $\ell \in \{0, 1, 2\}$ and $(k^{(1)}, K^{(1)}, k^{(2)}, K^{(2)}) \in S_j^{\ell}(\lambda)$, we have*

$$\left\| {}^{\ell}_{j}\sum_{k^{(1)},K^{(1)}}^{k^{(2)},K^{(2)}} \right\|_{L^2(\Omega)} \leq C 2^{\frac{-j-n}{2}}.$$

Proof. Following an idea from [7, Lemma 2.21], we write

$$\left\| {}^{\ell}_{j}\sum_{k^{(1)},K^{(1)}}^{k^{(2)},K^{(2)}} \right\|_{L^2(\Omega)} \leq \sum_{\mathcal{R} \in \{<,>,=\}} \left\| {}^{\ell}_{j}\sum_{\substack{k^{(1)},K^{(1)} \\ \mathcal{R}}}^{k^{(2)},K^{(2)}} \right\|_{L^2(\Omega)}$$

where

$$ {}_{j}^{\ell}\underset{\mathcal{R}}{\sum\nolimits}_{k^{(1)},K^{(1)}}^{k^{(2)},K^{(2)}} $$

is the subsum of (IV.3.26), (IV.3.25) or (IV.3.27) in which $k_1 \mathcal{R} k_2$. By doing so, we make sure that two random variables $\varepsilon_{j_1,j_2}^{k_1,k_2}$ and $\varepsilon_{j_1',j_2'}^{k_1',k_2'}$ appearing in this subsum are uncorrelated except when $(k_1,k_2)=(k_1',k_2')$. Then from Lemma IV.2.1, we have for $\ell = 0$ (the argument being the same for $\ell = 1$ or $\ell = 2$), for all $\mathcal{R} \in \{<,>,=\}$,

$$
\begin{aligned}
\left\| {}_{j}^{\ell}\underset{\mathcal{R}}{\sum\nolimits}_{k^{(1)},K^{(1)}}^{k^{(2)},K^{(2)}} \right\|_{L^2(\Omega)}^2 &= \sum_{k^{(1)} \le k_1 \le K^{(1)}} \; \sum_{k^{(2)} \le k_2 \le K^{(2)}, k_1 \mathcal{R} k_2} \mathbb{E}[(\varepsilon_{j,j}^{k_1,k_2})^2](I_{j,j}^{k_1,k_2})^2 \\
&\le \sum_{k^{(1)} \le k_1 \le K^{(1)}} \; \sum_{k_2 \in \mathbb{Z}} \frac{2^{-2j}}{(3+|k_1-k_2|)^8}.
\end{aligned}
$$

Since $\#\{k_1 \in \mathbb{Z} \,:\, k^{(1)} \le k_1 \le K^{(1)}\} \le 2^{j-n}$, we conclude that

$$ \left\| {}_{j}^{\ell}{\sum\nolimits}_{k^{(1)},K^{(1)}}^{k^{(2)},K^{(2)}} \right\|_{L^2(\Omega)} \le C 2^{\frac{-j-n}{2}}. \tag{IV.3.30} $$

□

Lemma IV.3.14. *There exist an event $\widetilde{\Omega}$ of probability 1 and a positive random variable C_2 with finite moment of any order such that, on $\widetilde{\Omega}$*

$$ \forall n \in \mathbb{N},\ \forall \lambda \subseteq [0,1],\ \lambda \in \Lambda_n,\ \forall j \ge n,\ \Xi_j(\lambda) \le C_2\,(j-n+1)n. \tag{IV.3.31} $$

Proof. Let us take $\theta > 0$ and consider, for all $n \in \mathbb{N}$ the event

$$ A_n := \{\forall \lambda \subseteq [0,1], \lambda \in \Lambda_n, \forall j \ge n, \Xi_j(\lambda) \le \theta(j-n+1)n\}. $$

If A_n^c stands for the complementary set of A_n in Ω, we have, of course,

$$ \mathbb{P}(A_n^c) = \mathbb{P}(\exists \lambda \subseteq [0,1], \lambda \in \Lambda_n \,:\, \exists j \ge n \text{ s. t. } \Xi_j(\lambda) \ge \theta(j-n+1)n). $$

But, for all $\lambda \subseteq [0,1], \lambda \in \Lambda_n$, $j \ge n$, $\ell \in \{0,1,2\}$ and $(k^{(1)},K^{(1)},k^{(2)},K^{(2)}) \in S_j^\ell(\lambda)$ we have, by Theorem IV.3.11, if $\theta \ge 2$,

$$ \mathbb{P}\left(\frac{\left| {}_{j}^{\ell}{\sum\nolimits}_{k^{(1)},K^{(1)}}^{k^{(2)},K^{(2)}} \right|}{\left\| {}_{j}^{\ell}{\sum\nolimits}_{k^{(1)},K^{(1)}}^{k^{(2)},K^{(2)}} \right\|_{L^2(\Omega)}} \ge \theta(j-n+1)n \right) \le \exp(-\overset{\star}{C}\theta(j-n+1)n). $$

As, for all $j \ge n$, $\#S_j^\ell(\lambda) \le 2^{4(j-n)}$ and $\#\{\lambda \subseteq [0,1] \,:\, \lambda \in \Lambda_n\} = 2^n$, we get

$$
\begin{aligned}
\mathbb{P}(A_n^c) &\le C 2^n \sum_{j \ge n} 2^{4(j-n)} \exp(-\overset{\star}{C}\theta(j-n+1)n) \\
&\le C 2^n \exp(-\overset{\star}{C}\theta n) \sum_{j \ge n} 2^{4(j-n)} \exp(-\overset{\star}{C}\theta(j-n))
\end{aligned}
$$

for a deterministic constant $C > 0$. Therefore, if we take $\theta > 4\log(2)/\overset{\star}{C}$, the conclusion follows from Borel-Cantelli Lemma. □

Lemma IV.3.15. *Let Ω^* and $\widetilde{\Omega}$ be the events of probability 1 given by Lemmata IV.3.4 and IV.3.14 respectively. There exists a positive random variable C_3 with finite moment of any order such that, on $\Omega^* \cap \widetilde{\Omega}$, for all $t, s \in (0,1)$ the random variable*

$$\left|\sum_{j_1\geq n}\sum_{j_2>n}\sum_{(k_1,k_2)\in\mathbb{Z}^2} 2^{j_1(1-H_1)}2^{j_2(1-H_2)}\varepsilon_{j_1,j_2}^{k_1,k_2} I_{j_1,j_2}^{k_1,k_2}[t,s]\right| \tag{IV.3.32}$$

is bounded from above by

$$C_3|t-s|^{H_1+H_2-1}\log|t-s|^{-1}.$$

Proof. We start by splitting the sums in (IV.3.32) in two parts:

$$\sum_{j_1\geq n}\sum_{j_2\geq j_1}\sum_{(k_1,k_2)\in\mathbb{Z}^2} 2^{j_1(1-H_1)}2^{j_2(1-H_2)}\varepsilon_{j_1,j_2}^{k_1,k_2} I_{j_1,j_2}^{k_1,k_2}[t,s] \text{ and}$$
$$\sum_{j_2\geq n}\sum_{j_1>j_2}\sum_{(k_1,k_2)\in\mathbb{Z}^2} 2^{j_1(1-H_1)}2^{j_2(1-H_2)}\varepsilon_{j_1,j_2}^{k_1,k_2} I_{j_1,j_2}^{k_1,k_2}[t,s]. \tag{IV.3.33}$$

We only focus on the first sums, as the argument is symmetric in j_1 and j_2. As in Lemma IV.3.9 we write

$$\sum_{j_1\geq n}\sum_{j_2\geq j_1}\sum_{(k_1,k_2)\in\mathbb{Z}^2} 2^{j_1(1-H_1)}2^{j_2(1-H_2)}\varepsilon_{j_1,j_2}^{k_1,k_2} I_{j_1,j_2}^{k_1,k_2}[t,s]$$
$$= \sum_{j_1\geq n}\sum_{j_2\geq j_1}\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}_{j_2}^{<}(t,s)} 2^{j_1(1-H_1)}2^{j_2(1-H_2)}\varepsilon_{j_1,j_2}^{k_1,k_2} I_{j_1,j_2}^{k_1,k_2}[t,s] \tag{IV.3.34}$$
$$+ \sum_{j_1\geq n}\sum_{j_2\geq j_1}\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}_{j_2}^{>}(t,s)} 2^{j_1(1-H_1)}2^{j_2(1-H_2)}\varepsilon_{j_1,j_2}^{k_1,k_2} I_{j_1,j_2}^{k_1,k_2}[t,s] \tag{IV.3.35}$$
$$+ \sum_{j_1\geq n}\sum_{j_2\geq j_1}\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}_{j_2}[t,s]} 2^{j_1(1-H_1)}2^{j_2(1-H_2)}\varepsilon_{j_1,j_2}^{k_1,k_2} I_{j_1,j_2}^{k_1,k_2}[t,s]. \tag{IV.3.36}$$

To bound (IV.3.34), we use inequality (IV.3.7) and Lemma IV.3.7 to get

$$\begin{aligned}|(\text{IV.3.34})| &\leq CC_1\sum_{j_1\geq n}\sum_{j_2\geq j_1} 2^{j_1(1-H_1)}2^{-j_2H_2}\sqrt{\log(3+|j_1|+2^{j_1})}\sqrt{\log(3+|j_2|+2^{j_2})}\\ &\leq CC_1\sum_{j_1\geq n} 2^{j_1(1-H_1-H_2)}\sqrt{j_1}\\ &\leq CC_1 2^{n(1-H_1-H_2)}n,\end{aligned}$$

by applying twice inequality (IV.3.22). The sum (IV.3.35) is bounded in exactly the same way.

To bound (IV.3.36), we use once again the equality (IV.3.23). First we have, by inequality (IV.3.7) and Lemma IV.3.8,

$$\begin{aligned}
&\left|\sum_{j_1\geq n}\sum_{j_2\geq j_1}\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}_{j_2}[t,s]} 2^{j_1(1-H_1)}2^{j_2(1-H_2)}\varepsilon_{j_1,j_2}^{k_1,k_2}\int_{-\infty}^{s}\psi_{H_1}(2^{j_1}x-k_1)\psi_{H_2}(2^{j_2}x-k_2)\,dx\right| \\
&\leq CC_1\sum_{j_1\geq n}\sum_{j_2\geq j_1} 2^{j_1(1-H_1)}2^{-j_2H_2}\sqrt{\log(3+|j_1|+2^{j_2})}\sqrt{\log(3+|j_2|+2^{j_2})} \\
&\leq CC_1 2^{n(1-H_1-H_2)}n. \qquad \text{(IV.3.37)}
\end{aligned}$$

We bound

$$\left|\sum_{j_1\geq n}\sum_{j_2\geq j_1}\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}_{j_2}[t,s]} 2^{j_1(1-H_1)}2^{j_2(1-H_2)}\varepsilon_{j_1,j_2}^{k_1,k_2}\int_{t}^{+\infty}\psi_{H_1}(2^{j_1}x-k_1)\psi_{H_2}(2^{j_2}x-k_2)\,dx\right|$$

in the same way.

It only remains us to find an estimate for

$$\sum_{j_1\geq n}\sum_{j_2\geq j_1}\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}_{j_2}[t,s]} 2^{j_1(1-H_1)}2^{j_2(1-H_2)}\varepsilon_{j_1,j_2}^{k_1,k_2}I_{j_1,j_2}^{k_1,k_2}$$

and thus, recalling Lemma IV.2.1, we reduce the problem to first bound, for $j\geq n$ and $\ell\in\{j,j+1\}$, the sums

$$\sum_{k_1\in\mathbb{Z}_j^{<}(t,s)}\sum_{k_2\in\mathbb{Z}_\ell[t,s]}\varepsilon_{j,\ell}^{k_1,k_2}I_{j,\ell}^{k_1,k_2}, \qquad \text{(IV.3.38)}$$

$$\sum_{k_1\in\mathbb{Z}_j^{>}(t,s)}\sum_{k_2\in\mathbb{Z}_\ell[t,s]}\varepsilon_{j,\ell}^{k_1,k_2}I_{j,\ell}^{k_1,k_2}, \qquad \text{(IV.3.39)}$$

$$\sum_{k_1\in\mathbb{Z}_j[t,s]}\sum_{k_2\in\mathbb{Z}_\ell[t,s]}\varepsilon_{j,\ell}^{k_1,k_2}I_{j,\ell}^{k_1,k_2} \qquad \text{(IV.3.40)}$$

on $\Omega^*\cap\widetilde{\Omega}$. Let us consider (IV.3.38) with $\ell=j$, the argument for $\ell=j+1$ and (IV.3.39) being similar. Using again Lemmata IV.2.1 and IV.3.4, we have on $\Omega^*\cap\widetilde{\Omega}$, since for all

$k_2 \in \mathbb{Z}_j[t,s]$, $|k_2| \le 2^j$, for $j \ge n$,

$$
\begin{aligned}
|(\text{IV.3.38})| &\le CC_1 \sum_{k_1 \in \mathbb{Z}_j^<(t,s)} \sum_{k_2 \in \mathbb{Z}_j[t,s]} \frac{2^{-j}}{(3+|k_1-k_2|)^4} \sqrt{\log(3+j+|k_1|)}\sqrt{\log(3+j+|k_2|)} \\
&\le CC_1 \sum_{k_1 \in \mathbb{Z}_j^<(t,s)} \sum_{k_2 \in \mathbb{Z}_j[t,s]} \frac{2^{-j}\sqrt{j}}{(3+k_2-k_1)^4} \sqrt{\log(3+j+|k_1|)} \\
&\le CC_1 \sum_{k_1 \in \mathbb{Z}_j^<(t,s)} \sum_{m=0}^{+\infty} \frac{2^{-j}\sqrt{j}}{(3+2^j\min\{s,t\}+m-k_1)^4} \sqrt{\log(3+j+|k_1|)} \\
&\le CC_1 \sum_{k_1 \in \mathbb{Z}_j^<(t,s)} 2^{-j}\sqrt{j} \int_0^{+\infty} \frac{dy}{(2+2^j\min\{s,t\}+y-k_1)^4} \sqrt{\log(3+j+|k_1|)} \\
&\le CC_1 2^{-j}\sqrt{j} \sum_{k_1 \in \mathbb{Z}_j^<(t,s)} \frac{\sqrt{\log(3+j+|k_1|)}}{(2+2^j\min\{s,t\}-k_1)^3} \\
&\le CC_1 2^{-j}\sqrt{j}\sqrt{\log(3+j+2^j\min\{s,t\})} \\
&\le CC_1 2^{-j} j. \qquad \text{(IV.3.41)}
\end{aligned}
$$

It follows that

$$
\left| \sum_{j\ge n} 2^{j(2-H_1-H_2)} \sum_{\ell=j}^{j+1} \sum_{k_1 \in \mathbb{Z}_j^<(t,s)} \sum_{k_2 \in \mathbb{Z}_\ell[t,s]} \varepsilon_{j,\ell}^{k_1,k_2} I_{j,\ell}^{k_1,k_2} \right| \le CC_1 2^{n(1-H_1-H_2)} n
$$

and, similarly,

$$
\left| \sum_{j\ge n} 2^{j(2-H_1-H_2)} \sum_{\ell=j}^{j+1} \sum_{k_1 \in \mathbb{Z}_j^>(t,s)} \sum_{k_2 \in \mathbb{Z}_\ell[t,s]} \varepsilon_{j,\ell}^{k_1,k_2} I_{j,\ell}^{k_1,k_2} \right| \le CC_1 2^{n(1-H_1-H_2)} n.
$$

The bound for (IV.3.40) is obtained using (IV.3.31) and (IV.3.30) which lead to

$$
\begin{aligned}
&\left| \sum_{j\ge n} 2^{j(2-H_1-H_2)} \sum_{\ell=j}^{j+1} \sum_{k_1 \in \mathbb{Z}_j[t,s]} \sum_{k_2 \in \mathbb{Z}_\ell[t,s]} \varepsilon_{j,\ell}^{k_1,k_2} I_{j,\ell}^{k_1,k_2} \right| \\
&\le CC_2 \sum_{j\ge n} 2^{j(\frac{3}{2}-H_1-H_2)} 2^{-\frac{n}{2}} (j-n+1)n \\
&\le CC_2 2^{n(\frac{3}{2}-H_1-H_2)} 2^{-\frac{n}{2}} n \\
&= CC_2 2^{n(1-H_1-H_2)} n,
\end{aligned}
$$

as $\frac{3}{2} < H_1 + H_2$.

Putting all of these together we get that (IV.3.32) is bounded from above by

$$
C\max\{C_1, C_2\}|t-s|^{H_1+H_2-1}\log|t-s|^{-1}
$$

on $\Omega^* \cap \widetilde{\Omega}$. □

We now prove the main result of this subsection.

Proof of Proposition IV.3.3. Let us consider ω in the event $\Omega^* \cap \widetilde{\Omega}$ of probability 1, where Ω^* and $\widetilde{\Omega}$ are given by Lemmata IV.3.4 and IV.3.14 respectively.

If $t, s \in (0,1)$, we write

$$\begin{aligned}
&|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)| \\
&\quad \leq \left| \sum_{j_1<n} \sum_{j_2<n} \sum_{(k_1,k_2)\in\mathbb{Z}^2} 2^{j_1(1-H_1)} 2^{j_2(1-H_2)} \varepsilon_{j_1,j_2}^{k_1,k_2}(\omega) I_{j_1,j_2}^{k_1,k_2}[t,s] \right| \\
&\quad + \left| \sum_{j_1<n} \sum_{j_2\geq n} \sum_{(k_1,k_2)\in\mathbb{Z}^2} 2^{j_1(1-H_1)} 2^{j_2(1-H_2)} \varepsilon_{j_1,j_2}^{k_1,k_2}(\omega) I_{j_1,j_2}^{k_1,k_2}[t,s] \right| \\
&\quad + \left| \sum_{j_1\geq n} \sum_{j_2<n} \sum_{(k_1,k_2)\in\mathbb{Z}^2} 2^{j_1(1-H_1)} 2^{j_2(1-H_2)} \varepsilon_{j_1,j_2}^{k_1,k_2}(\omega) I_{j_1,j_2}^{k_1,k_2}[t,s] \right| \\
&\quad + \left| \sum_{j_1\geq n} \sum_{j_2\geq n} \sum_{(k_1,k_2)\in\mathbb{Z}^2} 2^{j_1(1-H_1)} 2^{j_2(1-H_2)} \varepsilon_{j_1,j_2}^{k_1,k_2}(\omega) I_{j_1,j_2}^{k_1,k_2}[t,s] \right|.
\end{aligned}$$

The first sum is bounded from above by Lemmata IV.3.4 and IV.3.5, the second and the third one are bounded from above by Lemmata IV.3.4 and IV.3.9 and the last one is bounded from above by Lemma IV.3.15. □

Remark 17. Starting from now and until the end of this section, one can reduce our attention to the process

$$\left\{ R'_{H_1,H_2}(t) = \sum_{j_1=0}^{+\infty} \sum_{j_2=0}^{+\infty} \sum_{(k_1,k_2)\in\mathbb{Z}^2} 2^{j_1(1-H_1)} 2^{j_2(1-H_2)} \varepsilon_{j_1,j_2}^{k_1,k_2} I_{j_1,j_2}^{k_1,k_2}[0,\cdot] \right\}$$

because almost surely, it is the most irregular part of R_{H_1,H_2}. Indeed, using different estimates obtained in this subsection, one can see that, almost surely, there exists a constant $C > 0$ such that, for all $s, t \in (0,1)$,

$$\left| \sum_{j_1<0} \sum_{j_2<0} \sum_{(k_1,k_2)\in\mathbb{Z}^2} 2^{j_1(1-H_1)} 2^{j_2(1-H_2)} \varepsilon_{j_1,j_2}^{k_1,k_2} I_{j_1,j_2}^{k_1,k_2}[t,s] \right| \leq C|t-s|,$$

$$\left| \sum_{j_1<0} \sum_{j_2>0} \sum_{(k_1,k_2)\in\mathbb{Z}^2} 2^{j_1(1-H_1)} 2^{j_2(1-H_2)} \varepsilon_{j_1,j_2}^{k_1,k_2} I_{j_1,j_2}^{k_1,k_2}[t,s] \right| \leq C|t-s|^{-H_2} \log|t-s|^{-1}$$

$$\left| \sum_{j_1>0} \sum_{j_2<0} \sum_{(k_1,k_2)\in\mathbb{Z}^2} 2^{j_1(1-H_1)} 2^{j_2(1-H_2)} \varepsilon_{j_1,j_2}^{k_1,k_2} I_{j_1,j_2}^{k_1,k_2}[t,s] \right| \leq C|t-s|^{-H_1} \log|t-s|^{-1}$$

and we conclude because $H_1 + H_2 - 1 < \min\{H_1, H_2\} < 1$.

IV.3.2 Ordinary points

Let us now go to the almost sure finiteness of the limit (IV.1.6) for almost every point. The main idea behind our method is that wavelets which contribute the most in $|R_{H_1,H_2}(t,\cdot) - R_{H_1,H_2}(s,\cdot)|$ are the ones with associated dyadic intervals "close" to the interval $[t,s]$. Thus, we aim at proving the following Proposition.

Proposition IV.3.16. *There exists an event Ω_{ord} of probability 1 such that for all $\omega \in \Omega_{ord}$, for almost every $t \in (0,1)$,*

$$\limsup_{s\to t} \frac{|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)|}{|t-s|^{H_1+H_2-1}\log\log|t-s|^{-1}} < +\infty.$$

As in [36], for all $j \in \mathbb{N}$, we denote by $k_j(t)$ the unique integer such that $t \in [k_j(t)2^{-j}, (k_j(t)+1)2^{-j})$. In other words, $k_j(t) = s(\lambda_j(t))$. If $t \in (0,1)$ is fixed, applying Lemma IV.3.4 to the sequence of random variables $(\xi_{j_1,j_2}^{k_1',k_2'})_{(j_1,j_2,k_1',k_2')\in\mathbb{Z}^4}$ defined by

$$\xi_{j_1,j_2}^{k_1',k_2'} = \varepsilon_{j_1,j_2}^{k_1'+k_{j_1(t)},k_2'+k_{j_2(t)}}$$

we deduce the existence of Ω_t^*, an event of probability 1, and $C_{t,1}$, a positive random variable with finite moment of any order, such that, for all $\omega \in \Omega_t^*$ and for each $(j_1,j_2,k_1,k_2) \in \mathbb{Z}^4$, one has

$$|\varepsilon_{j_1,j_2}^{k_1,k_2}(\omega)| \leq C_{t,1}(\omega)\sqrt{\log(3+|j_1|+|k_1-k_{j_1}(t)|)}\sqrt{\log(3+|j_2|+|k_2-k_{j_2}(t)|)}. \quad \text{(IV.3.42)}$$

In view of this fact, let us set, for $t \in (0,1)$ and $(j_1,j_2,k_1,k_2) \in \mathbb{N}^2 \times \mathbb{Z}^2$

$$L_{j_1,j_2}^{k_1,k_2}(t) = \sqrt{\log(3+j_1+|k_1-k_{j_1}(t)|)}\sqrt{\log(3+j_2+|k_2-k_{j_2}(t)|)}.$$

In what follows, we show how to modify Lemmata IV.3.5 to IV.3.15 from the previous subsection, using $L_{j_1,j_2}^{k_1,k_2}(t)$ instead of $L_{j_1,j_2}^{k_1,k_2}$. Before all, we need the following Lemma which is inspired by results from [36] that can be extended in our case.

Lemma IV.3.17. *For all $L > 2$ there exists a constant $C_L > 0$ such that, for all $n \in \mathbb{N}$ and $t,s \in (0,1)$ such that $2^{-n-1} < |t-s| \leq 2^{-n}$, for all $x \in [s,t]$*

1. *For all $0 \leq j < n$*

$$\sum_{k\in\mathbb{Z}} \frac{\sqrt{\log(3+j+|k-k_j(t)|)}}{(3+|2^jx-k|)^L} \leq C_L\sqrt{\log(3+j)}.$$

2. *For all $j \geq n$*

$$\sum_{k\in\mathbb{Z}} \frac{\sqrt{\log(3+j+|k-k_j(t)|)}}{(3+|2^jx-k|)^L} \leq C_L\sqrt{j-n+1}\sqrt{\log(3+j)}.$$

Proof. For all $j \in \mathbb{N}$, $k \in \mathbb{Z}$ and $x \in [s,t]$, observe that

$$|k - k_j(t)| \le |k - 2^j x| + |2^j x - 2^j t| + |2^j t - k_j(t)| \le |k - 2^j x| + 2^{j-n} + 1. \qquad \text{(IV.3.43)}$$

If $0 \le j < n$, then it follows from (IV.3.43) that $|k - k_j(t)| \le |2^j x - k| + 2$ which allow us to write, thanks to inequality (IV.3.12),

$$\begin{aligned} \frac{\sqrt{\log(3 + j + |k - k_j(t)|)}}{(3 + |2^j x - k|)} &\le \sqrt{\log(3+j)} \frac{\sqrt{\log(5 + |2^j x - k|)}}{|2^j x - k| + 3} \\ &\le C\sqrt{\log(3+j)}. \end{aligned}$$

where $C := \sup_{x \ge 0} \left(\frac{\sqrt{\log(5+x)}}{x+3} \right)$ and we conclude using the boundedness of the function

$$\xi \mapsto \sum_{k \in \mathbb{Z}} \frac{1}{(1 + |\xi - k|)^M} \qquad \text{(IV.3.44)}$$

for all $M > 1$.

Now, if $j \ge n$, from (IV.3.43) we get $|k - k_j(t)| \le |2^j x - k| + 2^{j-n+1}$ and thus, again by inequality (IV.3.12),

$$\begin{aligned} \frac{\sqrt{\log(3 + j + |k - k_j(t)|)}}{(3 + |2^j x - k|)} &\le \sqrt{\log(3 + 2^{j-n+1})} \sqrt{\log(3+j)} \frac{\sqrt{\log(3 + |2^j x - k|)}}{|2^j x - k| + 3} \\ &\le C' \sqrt{j - n + 1} \sqrt{\log(3+j)}. \end{aligned}$$

where $C' := \sqrt{3} \sup_{x \ge 0} \left(\frac{\sqrt{\log(3+x)}}{x+3} \right)$ and the conclusion comes again from the boundedness of the function in (IV.3.44) for all $M > 1$ □

Lemma IV.3.18. *There exists a deterministic constant $C > 0$ such that, for all $t, s \in (0,1)$ we have*

$$\begin{aligned} \sum_{0 \le j_1 < n} \sum_{0 \le j_2 < n} \sum_{(k_1,k_2) \in \mathbb{Z}^2} & 2^{j_1(1-H_1)} 2^{j_2(1-H_2)} L_{j_1,j_2}^{k_1,k_2}(t) \left| I_{j_1,j_2}^{k_1,k_2}[t,s] \right| \\ & \le C|t-s|^{H_1+H_2-1} \log\log|t-s|^{-1}. \end{aligned}$$

Proof. If $\xi \in [s,t]$, we get from the fast decay of the fractional antiderivatives of ψ (IV.3.8) and inequality (IV.3.42), for $0 \le j_1, j_2 < n$,

$$\begin{aligned} &\sum_{(k_1,k_2) \in \mathbb{Z}^2} L_{j_1,j_2}^{k_1,k_2}(t) |\psi_{H_1}(2^{j_1}\xi - k_1) \psi_{H_2}(2^{j_2}\xi - k_2)| \\ &\qquad \le CC_1 \left(\sum_{k_1 \in \mathbb{Z}} \frac{\sqrt{\log(3 + j_1 + |k_1 - k_{j_1}(t)|)}}{(3 + |2^{j_1}\xi - k_1|)^4} \right) \\ &\qquad\qquad \left(\sum_{k_2 \in \mathbb{Z}} \frac{\sqrt{\log(3 + j_2 + |k_2 - k_{j_2}(t)|)}}{(3 + |2^{j_2}\xi - k_2|)^4} \right). \end{aligned}$$

These last two sums are bounded by the first point of Lemma IV.3.17. Using

$$\sum_{j_1=0}^{n-1} 2^{j_1(1-H_1)}\sqrt{\log(3+j_1)} \leq C2^{n(1-H_1)}\sqrt{\log(n)} \tag{IV.3.45}$$

instead of (IV.3.11), we conclude, just as in Lemma IV.3.5, that the desired inequality holds. □

Lemma IV.3.19. *There exists a deterministic constant $C>0$ such that, for all $t,s\in(0,1)$ and $0\leq j_1<n\leq j_2$, the quantities*

$$\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}^<_{j_2}(t,s)} L^{k_1,k_2}_{j_1,j_2}(t)\left|I^{k_1,k_2}_{j_1,j_2}[t,s]\right| \tag{IV.3.46}$$

$$\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}^>_{j_2}(t,s)} L^{k_1,k_2}_{j_1,j_2}(t)\left|I^{k_1,k_2}_{j_1,j_2}[t,s]\right| \tag{IV.3.47}$$

are bounded from above by

$$C\sqrt{j_2-n+1}\sqrt{\log(3+j_1)}\sqrt{\log(3+j_2)}2^{-j_2}.$$

Proof. Let us prove the bound for (IV.3.46), the argument for (IV.3.47) being similar. We have, by the first part of Lemma IV.3.17, for $0\leq j_1<n\leq j_2$,

$$\text{(IV.3.46)} \leq C\sqrt{\log(3+j_1)}\int_{[s,t]}\sum_{k_2\in\mathbb{Z}^<_{j_2}(t,s)}\frac{\sqrt{\log(3+j_2+|k_2-k_{j_2}(t)|)}}{(3+|2^{j_2}x-k_2|)^4}\,dx$$

and, as for all $k_2\in\mathbb{Z}^<_{j_2}(t,s)$ and $x\in[s,t]$ we have

$$|k_2-k_{j_2}(t)|\leq|2^{j_2}x-k_2|+|k_{j_2}(t)-2^{j_2}x|\leq|2^{j_2}x-k_2|+2^{j_2-n}+1$$

and, by inequality (IV.3.12),

$$\sqrt{\log(3+j_2+|k_2-k_{j_2}(t)|)}\leq C\sqrt{j_2-n+1}\sqrt{\log(3+j_2)}\sqrt{\log(3+|2^{j_2}x-k_2|)}$$

it just remains us to use the bound (IV.3.15) to write

$$\text{(IV.3.46)}\leq C\sqrt{j_2-n+1}\sqrt{\log(3+j_1)}\sqrt{\log(3+j_2)}2^{-j_2}. \tag{IV.3.48}$$

□

Lemma IV.3.20. *There exists a deterministic constant $C>0$ such that, for all $t,s\in(0,1)$ and $0\leq j_1<n\leq j_2$, the quantities*

$$\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}_{j_2}[t,s]} L^{k_1,k_2}_{j_1,j_2}(t)\left|\int_{-\infty}^{\min\{s,t\}}\psi_{H_1}(2^{j_1}x-k_1)\psi_{H_2}(2^{j_2}x-k_2)\,dx\right| \tag{IV.3.49}$$

$$\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}_{j_2}[t,s]} L_{j_1,j_2}^{k_1,k_2}(t)\left|\int_{\max\{s,t\}}^{+\infty}\psi_{H_1}(2^{j_1}x-k_1)\psi_{H_2}(2^{j_2}x-k_2)\,dx\right| \tag{IV.3.50}$$

are bounded from above by

$$C\sqrt{\log(3+j_1)}\sqrt{\log(3+j_2)}\sqrt{j_2-n+1}2^{-j_2}.$$

Proof. Again we assume $s\le t$. First, using the fast decay of the fractional antiderivatives of ψ (IV.3.8), (IV.3.49) is bounded from above by

$$\int_{-\infty}^{s}\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}_{j_2}[t,s]}\frac{\sqrt{\log(3+j_1+|k_1-k_{j_1}(t)|)}}{(3+|2^{j_1}x-k_1|)^4}\frac{\sqrt{\log(3+j_2+|k_2-k_{j_2}(t)|)}}{(3+|2^{j_2}x-k_2|)^4}\,dx. \tag{IV.3.51}$$

Observe that, for all $k_1\in\mathbb{Z}$, $k_2\in\mathbb{Z}_{j_2}[t,s]$ and $x\in(-\infty,s]$, we have, as $j_1<n\le j_2$,

$$\begin{aligned}|2^{j_1}x-k_{j_1}(t)| &\le |2^{j_1}x-2^{j_1-j_2}k_2|+|2^{j_1-j_2}k_2-2^{j_1}t|+|2^{j_1}t-k_{j_1}(t)|\\ &\le |2^{j_2}x-k_2|+2\end{aligned}$$

and therefore

$$|k_1-k_{j_1}(t)|\le|2^{j_1}x-k_1|+|2^{j_2}x-k_2|+2$$

while

$$|k_2-k_{j_2}(t)|\le|k_2-2^{j_2}t|+|2^{j_2}t-k_{j_2}(t)|\le 2^{j_2-n}+1.$$

It allows to write, thanks to inequality (IV.3.12), the boundedness of the function (IV.3.44) and inequality (IV.3.17)

$$\begin{aligned}|(\text{IV.3.51})| &\le C\sqrt{\log(3+j_1)}\sqrt{\log(3+j_2)}\sqrt{j_2-n+1}\int_{-\infty}^{s}\sum_{k_2\in\mathbb{Z}_{j_2}[t,s]}\frac{dx}{(3+|2^{j_2}x-k_2|)^3}\\ &\le C\sqrt{\log(3+j_1)}\sqrt{\log(3+j_2)}\sqrt{j_2-n+1}2^{-j_2}.\end{aligned}$$

We bound the second sums in the same way. □

Lemma IV.3.21. *There exists a deterministic constant $C>0$ such that, for all $t,s\in(0,1)$, the quantities*

$$\sum_{0\le j_1<n}\sum_{j_2\ge n}\sum_{(k_1,k_2)\in\mathbb{Z}^2}2^{j_1(1-H_1)}2^{j_2(1-H_2)}L_{j_1,j_2}^{k_1,k_2}(t)|I_{j_1,j_2}^{k_1,k_2}[t,s]|$$

$$\sum_{j_1\ge n}\sum_{0\le j_2<n}\sum_{(k_1,k_2)\in\mathbb{Z}^2}2^{j_1(1-H_1)}2^{j_2(1-H_2)}L_{j_1,j_2}^{k_1,k_2}(t)|I_{j_1,j_2}^{k_1,k_2}[t,s]|$$

are bounded from above by

$$C|t-s|^{H_1+H_2-1}\log\log|t-s|^{-1}.$$

Proof. The proof is exactly the same as the one of Lemma IV.3.9 excepted that we use Lemmata IV.3.19 and IV.3.20 instead of Lemmata IV.3.7 and IV.3.8 respectively and that we conclude using again (IV.3.45) instead of (IV.3.11) and

$$\sum_{j_2=n}^{+\infty} 2^{-j_2 H_2} \sqrt{j_2 - n + 1}\sqrt{\log(3 + j_2)} \leq C' 2^{-nH_2} \sqrt{\log(n)}. \tag{IV.3.52}$$

instead of (IV.3.22). □

Lemma IV.3.22. *There exists a deterministic constant $C > 0$ such that, for all $t, s \in (0, 1)$ and $n \leq j_1 \leq j_2$, the quantities* (IV.3.46) *and* (IV.3.47) *are bounded from above by*

$$C\sqrt{j_2 - n + 1}\sqrt{j_1 - n + 1}\sqrt{\log(3 + j_1)}\sqrt{\log(3 + j_2)}2^{-j_2}.$$

Proof. The proof is exactly the same as for Lemma IV.3.19 except that, here, we use the second part of Lemma IV.3.17 instead of the first one. □

Lemma IV.3.23. *There exists a deterministic constant $C > 0$ such that, for all $t, s \in (0, 1)$ and $n \leq j_1 \leq j_2$ the quantities* (IV.3.49) *and* (IV.3.50) *are bounded from above by*

$$C\sqrt{j_1 - n + 1}\sqrt{j_2 - n + 1}\sqrt{\log(3 + j_1)}\sqrt{\log(3 + j_2)}2^{-j_2}.$$

Proof. The proof is exactly the same as for Lemma IV.3.20 except that, here, we use the second part of Lemma IV.3.17 instead of the first one. □

Just as we did for the rapid points, it remains us to bound the random variables $\Xi_j(\lambda)$ (IV.3.29). Here, we don't want anymore to show the existence of an uniform modulus but only a pointwise modulus of continuity at a fixed point of interest t. Therefore, we just have to bound, for all $n \in \mathbb{N}$ the random variables $\Xi_j(\lambda)$ for $j \geq n$ and $\lambda \in 3\lambda_n(t)$. We thus have the following result.

Lemma IV.3.24. *For all $t \in (0, 1)$, there exist an event $\widetilde{\Omega_t}$ of probability 1 and a positive random variable $C_{t,2}$ with finite moment of any order such that, on $\widetilde{\Omega_t}$,*

$$\forall n \in \mathbb{N},\ \forall \lambda \in 3\lambda_n(t),\ \forall j \geq n,\ \Xi_j(\lambda) \leq C_{t,2}\,(j - n + 1)\log(n). \tag{IV.3.53}$$

Proof. If $t \in (0, 1)$ is fixed and $\theta > 0$, let us define the event

$$A_n(t) = \{\forall \lambda \in 3\lambda_n(t)\ \forall j \geq n,\ \Xi_j(\lambda) \leq \theta(j - n + 1)\log(n)\}.$$

Similarly to Lemma IV.3.14, we get

$$\begin{aligned}\mathbb{P}(A_n(t)^c) &\leq C \sum_{j\geq n} 2^{4(j-n)} \exp(-\overset{\star}{C}\theta(j - n + 1)\log(n)) \\ &\leq C \exp(-\overset{\star}{C}\theta\log(n)) \sum_{j\geq n} 2^{4(j-n)} \exp(-\overset{\star}{C}\theta(j - n)),\end{aligned}$$

for a determistic constant $C > 0$. Therefore, if we take again $\theta > 4\log(2)/\overset{\star}{C}$ then Borel-Cantelli Lemma implies the existence of an event $\widetilde{\Omega_t}$ of probability 1 and $C_{t,2}$ a positive random variable of finite moment of any order such that, on $\widetilde{\Omega_t}$, assertion (IV.3.53) holds. □

Lemma IV.3.25. *If $t \in (0,1)$, let Ω_t^* be the event of probability 1 where inequality* (IV.3.42) *holds and $\widetilde{\Omega_t}$ be the event of probability 1 given by Lemma IV.3.24. There exists a positive random variable $C_{t,3}$ with finite moment of any order such that, on $\Omega_t^* \cap \widetilde{\Omega_t}$, for all $s \in (0,1)$ the random variable*

$$\left| \sum_{j_1 \geq n} \sum_{j_2 > n} \sum_{(k_1,k_2) \in \mathbb{Z}^2} 2^{j_1(1-H_1)} 2^{j_2(1-H_2)} \varepsilon_{j_1,j_2}^{k_1,k_2} I_{j_1,j_2}^{k_1,k_2}[t,s] \right| \tag{IV.3.54}$$

is bounded from above by

$$C_{t,3} |t-s|^{H_1+H_2-1} \log |t-s|^{-1}.$$

Proof. Again, we use the split (IV.3.33) and we only do the details for the first sum. We deal with the series (IV.3.34) and (IV.3.35) in the same way that in Lemma IV.3.15 but using inequality (IV.3.42) and Lemmata IV.3.22 and IV.3.23 and finally inequality (IV.3.52).

For (IV.3.36), first, by Lemma IV.3.23 and inequality (IV.3.42), we have, on $\Omega_t^* \cap \widetilde{\Omega_t}$

$$\begin{aligned} &\left| \sum_{j_1 \geq n} \sum_{j_2 \geq j_1} \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}_{j_2}[t,s]} 2^{j_1(1-H_1)} 2^{j_2(1-H_2)} \varepsilon_{j_1,j_2}^{k_1,k_2} \int_{-\infty}^{s} \psi_{H_1}(2^{j_1}x - k_1) \psi_{H_2}(2^{j_2}x - k_2)\, dx \right| \\ &\leq C C_{t,1} \sum_{j_1 \geq n} \sum_{j_2 \geq j_1} 2^{j_1(1-H_1)} 2^{-j_2 H_2} \sqrt{j_1 - n + 1} \sqrt{j_2 - n + 1} \sqrt{\log(3+j_1)} \sqrt{\log(3+j_2)} \\ &\leq C C_1 2^{n(1-H_1-H_2)} \log(n). \end{aligned} \tag{IV.3.55}$$

We bound

$$\left| \sum_{j_1 \geq n} \sum_{j_2 \geq j_1} \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}_{j_2}[t,s]} 2^{j_1(1-H_1)} 2^{j_2(1-H_2)} \varepsilon_{j_1,j_2}^{k_1,k_2} \int_{-\infty}^{s} \psi_{H_1}(2^{j_1}x - k_1) \psi_{H_2}(2^{j_2}x - k_2)\, dx \right|$$

on $\Omega_t^* \cap \widetilde{\Omega_t}$ exactly in the same way.

To finish the proof, again, we have to bound (IV.3.38), (IV.3.39) and (IV.3.40) for $\ell \in \{j, j+1\}$ (with $j \geq n$) on $\Omega_t^* \cap \widetilde{\Omega_t}$. For (IV.3.38), in the case $\ell = j$, one can note that, for all $k_2 \in \mathbb{Z}_j[t,s]$, $|k_2 - k_j(t)| \leq 2^{j-n} + 1$ and, for all $k_1 \in \mathbb{Z}_j^<(t,s)$, $|k_1 - k_j(t)| \leq |2^j \min\{t,s\} - k_1| + 2^{j-n} + 1$. Using the same tricks as in (IV.3.41), we get, on $\Omega_t^* \cap \widetilde{\Omega_t}$

$$\begin{aligned} |(\text{IV.3.38})| &\leq C C_{t,1} (j-n+1) \log(3+j) 2^{-j} \sum_{k_1 \in \mathbb{Z}_j^<(t,s)} \frac{1}{(2 + |2^j \min\{s,t\} - k_1|)^3} \\ &\leq C(j-n+1) \log(3+j) 2^{-j}. \end{aligned}$$

The bounds for (IV.3.39) and in the case $\ell = j+1$ are obtained in the same way. Finally to bound (IV.3.40), we use (IV.3.53) and (IV.3.30) and get on $\Omega_t^* \cap \widetilde{\Omega_t}$

$$\begin{aligned}
&\left| \sum_{j \geq n} 2^{j(2-H_1-H_2)} \sum_{\ell=j}^{j+1} \sum_{k_1 \in \mathbb{Z}_j[t,s]} \sum_{k_2 \in \mathbb{Z}_\ell[t,s]} \varepsilon_{j,\ell}^{k_1,k_2} I_{j,\ell}^{k_1,k_2} \right| \\
&\leq C C_{t,2} \sum_{j \geq n} 2^{j(\frac{3}{2}-H_1-H_2)} 2^{-\frac{n}{2}} (j-n+1) \log(n) \\
&\leq C C_{t,2} 2^{n(1-H_1-H_2)} \log(n).
\end{aligned}$$

We conclude that (IV.3.54) is bounded from above by

$$C \max\{C_{t,1}, C_{t,2}\} |t-s|^{H_1+H_2-1} \log |t-s|^{-1}$$

on $\Omega_t^* \cap \widetilde{\Omega_t}$. □

We can now prove Proposition IV.3.16.

Proof of Proposition IV.3.16. Let us fix $t \in (0,1)$ and consider $\omega \in \Omega_t^* \cap \widetilde{\Omega_t}$. For all $s \in (0,1)$, we write[1]

$$\begin{aligned}
&|R'_{H_1,H_2}(t,\omega) - R'_{H_1,H_2}(s,\omega)| \\
&\quad \leq \left| \sum_{0 \leq j_1 < n} \sum_{0 \leq j_2 < n} \sum_{(k_1,k_2) \in \mathbb{Z}^2} 2^{j_1(1-H_1)} 2^{j_2(1-H_2)} \varepsilon_{j_1,j_2}^{k_1,k_2}(\omega) I_{j_1,j_2}^{k_1,k_2}[t,s] \right| \\
&\quad + \left| \sum_{0 \leq j_1 < n} \sum_{j_2 \geq n} \sum_{(k_1,k_2) \in \mathbb{Z}^2} 2^{j_1(1-H_1)} 2^{j_2(1-H_2)} \varepsilon_{j_1,j_2}^{k_1,k_2}(\omega) I_{j_1,j_2}^{k_1,k_2}[t,s] \right| \\
&\quad + \left| \sum_{j_1 \geq n} \sum_{0 \leq j_2 < n} \sum_{(k_1,k_2) \in \mathbb{Z}^2} 2^{j_1(1-H_1)} 2^{j_2(1-H_2)} \varepsilon_{j_1,j_2}^{k_1,k_2}(\omega) I_{j_1,j_2}^{k_1,k_2}[t,s] \right| \\
&\quad + \left| \sum_{j_1 \geq n} \sum_{j_2 \geq n} \sum_{(k_1,k_2) \in \mathbb{Z}^2} 2^{j_1(1-H_1)} 2^{j_2(1-H_2)} \varepsilon_{j_1,j_2}^{k_1,k_2}(\omega) I_{j_1,j_2}^{k_1,k_2}[t,s] \right|.
\end{aligned}$$

We bound from above the first sum by inequality (IV.3.42) and Lemma IV.3.18, the second and the third sums by inequality (IV.3.42) and Lemma IV.3.21 and the last sum by Lemma IV.3.25.

Using inequalities (IV.3.52) and (IV.3.55) and Remark 17, one can finally write that for all $t \in (0,1)$, for all ω in the event of probability 1 $\Omega_t^* \cap \widetilde{\Omega_t}$

$$\limsup_{s \to t} \frac{|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)|}{|t-s|^{H_1+H_2-1} \log\log |t-s|^{-1}} < +\infty$$

and we conclude by Fubini Theorem. □

[1]We recall that R'_{H_1,H_2} is defined in Remark 17.

IV.3.3 Slow points

In this section, we aim at showing that the generalized Rosenblatt process admits slow points: we prove the following Proposition.

Proposition IV.3.26. *There exists an event Ω_{slo} of probability 1 such that for all $\omega \in \Omega_{slo}$ there exist $t \in (0,1)$ such that*

$$\limsup_{s\to t} \frac{|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)|}{|t-s|^{H_1+H_2-1}} < +\infty. \tag{IV.3.56}$$

In [54], Kahane described a procedure to insure the existence of slow points for the Brownian motion. This procedure was then generalized in [36] to fit for any arbitrary fractional Brownian motion. It consists in showing that for any $m > 0$, almost surely, there exist $\mu > 0$ and $t \in (0,1)$ such that, if one sets

$$\Lambda_j^0(t) = \{\lambda \in \Lambda_j : |s(\lambda(t)) - s(\lambda)| \leq 1\} \tag{IV.3.57}$$

and, for all $1 \leq l$

$$\Lambda_j^l(t) = \{\lambda \in \Lambda_j, : 2^{m(l-1)} < |s(\lambda(t)) - s(\lambda)| \leq 2^{ml}\}, \tag{IV.3.58}$$

then, for all $\lambda \in \Lambda_j^l(t)$ we have

$$|\varepsilon_\lambda| \leq 2^l \mu, \tag{IV.3.59}$$

where ε_λ is the random variable

$$2^{\frac{j}{2}} \int_{\mathbb{R}} \psi_\lambda(x)\, dB(x).$$

In this procedure, if $\mu \in \mathbb{N}$,for all $j, l \in \mathbb{N}_0$ and $\lambda \in \Lambda_j$, $\lambda \subseteq [0,1]$, we define

$$\Lambda_{j,l}(\lambda) = \{\lambda' \in \Lambda_j, : |s(\lambda) - s(\lambda')| \leq 2^{ml}\}$$

and the random set

$$S_{j,l}^\mu = \{\lambda' \in \Lambda_j, : 2^l \mu < |\varepsilon_{\lambda'}| \leq 2^{l+1}\mu\}.$$

Finally we consider the random set

$$I_j^\mu = \{\lambda \in \Lambda_j, \lambda \subseteq [0,1] : \forall l \in \mathbb{N}_0,\, \Lambda_{j,l}(\lambda) \cap S_{j,l}^\mu = \emptyset\},$$

and show that almost surely, there exists $\mu \in \mathbb{N}$ such that

$$S_{\text{low}}^\mu = \bigcap_{j \in \mathbb{N}_0} \bigcup_{\lambda \in I_j^\mu} \overline{\lambda} \neq \emptyset$$

which is equivalent to the fact that, for any J

$$S_{\text{low},J}^\mu = \bigcap_{j \leq J} \bigcup_{\lambda \in I_j^\mu} \overline{\lambda} \neq \emptyset$$

as $(S^\mu_{\text{low},J})_J$ is a decreasing sequence of compact sets. To do so, let us denote by $2S^\mu_{\text{low},J}$ the sets of dyadic intervals of scale $J+1$ obtained by cutting in two the remaining intervals[2] in $S^\mu_{\text{low},J}$ and remark that $S^\mu_{\text{low},J+1}$ is obtained from $2S^\mu_{\text{low},J}$ by removing the dyadic intervals λ such that $\Lambda_{J+1,l}(\lambda) \cap S^\mu_{J+1,l} \neq \emptyset$ for a $l \in \mathbb{N}_0$. But now, if $\xi \sim \mathcal{N}(0,1)$, we set, for all such a l

$$p_l(\mu) = \mathbb{P}(2^l\mu < |\xi| \leq 2^{l+1}\mu).$$

and note that, if N is the number of intervals of $S^\mu_{\text{low},J}$, counting the number of intervals in $2S^\mu_{\text{low},J} \cap S^\mu_{J+1,l}$ is a binomial random variable of parameter $2N$ and $p_l(\mu)$ and this number is thus bounded by

$$2N(p_l(\mu) + (l+1)\sqrt{p_l(\mu)(1-p_l(\mu))})$$

on an event of probability $1-(l+1)^{-2}N^{-1}$. Therefore, to pass from $S^\mu_{\text{low},J}$ to $S^\mu_{\text{low},J+1}$ we remove at most

$$2N\sum_{l=0}^{+\infty}(2^{ml+1}+1)(p_l(\mu) + (l+1)\sqrt{p_l(\mu)(1-p_l(\mu))})$$

intervals with probability greater than $1-N^{-1}$. But if μ is large enough, as $p_l(\mu)$ is of order $\frac{e^{-(2^l\mu)^2}}{2^l\mu}$, one can make sure that this last term is bounded by $\frac{N}{2}$. So, if N^μ_J is the random variable counting the number of subintervals of $S^\mu_{\text{low},J}$, we have

$$\mathbb{P}(N^\mu_{J+1} \geq \frac{3}{2}N^\mu_J | N^\mu_J = N) \geq 1-N^{-1}$$

which leads to the recursive formula

$$\mathbb{P}(N^\mu_{J+1} \geq (\frac{3}{2})^{J+1}) \geq (1-(\frac{2}{3})^J)\mathbb{P}(N^\mu_J \geq (\frac{3}{2})^J), \quad \forall J \in \mathbb{N}_0,$$

see [36, Lemma 3.6 and Theorem 3.7.]. Finally, we deduce

$$\mathbb{P}\big(\bigcup_\mu \bigcap_{J\in\mathbb{N}_0}(N^\mu_J \geq 1)\big) = 1. \tag{IV.3.60}$$

Moreover, we can show that, in this case, $S^\mu_{\text{low}} \cap (0,1) \neq \emptyset$. If $\alpha > 0$, applying this procedure with $\frac{1}{m} < \alpha$ gives us that any point $t \in S^\mu_{\text{low}} \cap (0,1)$ is a slow point of the fractional Brownian motion of exponent α.

From formulas (IV.3.1) and (IV.3.2), we see that this procedure is also useful to bound the random variables appearing in the expansion (IV.3.3) of the generalized Rosenblatt process. But, from the proofs of Propositions IV.3.3 and IV.3.16 we know that this is not sufficient and we also need to give a bound for the random variables $\Xi_j(\lambda)$, for $\lambda \in 3\lambda_n(t), n \in \mathbb{N}$ and $j \geq n$. Such dyadic intervals are precisely the ones in the set $\Lambda_{n,0}(\lambda_n(t))$ and this fact forces us to consider the following modification of the procedure. For all

[2]The interval $[k2^{-j},(k+1)2^{-j}]$ is cut into $[(2k)2^{-(j+1)},(2k+1)2^{-(j+1)}]$ and $[(2k+1)2^{-(j+1)},(2k+2)2^{-(j+1)}]$.

$j \in \mathbb{N}$, if $l \neq 0$, the sets $S^{\mu}_{j,l}$ remain untouched as well as its associated probability $p_l(\mu)$ while for $l = 0$ we set

$$S^{\mu}_{j,0} = \{\lambda' \in \Lambda_j, \lambda' \subseteq [0,1] \,:\, \exists j' \geq j\, \Xi_{j'}(\lambda') > (j' - j + 1)\mu\},$$

with associated probability (which only depends on μ)

$$p_0(\mu) = \mathbb{P}(\exists j' \geq j\, \Xi_{j'}(\lambda) > (j' - j + 1)\mu).$$

As $\Xi_{j'}(\lambda_1)$ is independent of $\Xi_{j'}(\lambda_2)$ as soon as $\lambda_1 \cap \lambda_2 = \emptyset$, for all $J \in \mathbb{N}$, if N is again the number of dyadic intervals of $S^{\mu}_{\text{low},J}$, the number of such intervals in $2S^{\mu}_{\text{low},J} \cap S^{\mu}_{J+1,0}$ is still a binomial random variable of parameter $2N$ and $p_0(\mu)$. Therefore if μ is large enough, using Theorems IV.3.11 and IV.3.12, one can still affirm

$$2N \sum_{l=0}^{+\infty} (2^{ml+1} + 1)(p_l(\mu) + (l+1)\sqrt{p_l(\mu)(1 - p_l(\mu))}) \leq \frac{N}{2}$$

and the end of the procedure is saved: equality (IV.3.60) still holds. Now, if $t \in S^{\mu}_{\text{low}} \cap (0,1)$ we know that

$$\forall\, n \in \mathbb{N},\, \forall\, \lambda \in 3\lambda_n(t)\,, \forall j \geq n\,, \Xi_j(\lambda) \leq (j - n + 1)\mu. \tag{IV.3.61}$$

Let us remark that, as for all $\lambda \in \Lambda_n$, $|\varepsilon^2_{\lambda}| \leq 2\Xi_n(\lambda)+1$, we still have, in this case, for all $\lambda \in 3\lambda_n(t)$, $|\varepsilon_{\lambda}| \leq C\mu$, for a deterministic constant $C > 0$. Starting from now we take m such that $1/m < \min\{H_1, H_2\}$ and $2/m < 1 - H_1 - H_2$.

In order to use notations (IV.3.57) and (IV.3.58), here after λ_1 (resp. λ_2) will always stand for the dyadic interval $[k_1 2^{-j_1}, (k_1{+}1)2^{-j_1})$ (resp. $[k_2 2^{-j_2}, (k_2{+}1)2^{-j_2})$) and ψ_{λ_1} (resp. ψ_{λ_2}) will be the associated antiderivative of wavelet $\psi_{H_1}(2^{j_1} \cdot - k_1)$ (resp. $\psi_{H_2}(2^{j_2} \cdot - k_2)$) and $I_{\lambda_1,\lambda_2}[t,s]$ will stand for $I^{k_1,k_2}_{j_1,j_2}[t,s]$. Finally, $\varepsilon_{\lambda_1,\lambda_2}$ will stand for $\varepsilon^{k_1,k_2}_{j_1,j_2}$. If $t \in (0,1)$, let $(y_{\lambda}(t))_{\lambda\Lambda}$ be the sequence defined by

$$y_{\lambda}(t) = 2^l \text{ if } \lambda \in \Lambda^l_j(t).$$

Note that, if we apply the preceding procedure, we find Ω_{slo} an event of probability 1 such that, for all $\omega \in \Omega_{\text{slo}}$, there exists μ for which $S^{\mu}_{\text{low}} \cap (0,1) \neq \emptyset$. Then, if t belong to this set, we have, thanks to inequality (IV.3.59) and equalities (IV.3.1) and (IV.3.2)

$$|\varepsilon_{\lambda_1,\lambda_2}(\omega)| \leq C\mu^2 y_{\lambda_1}(t) y_{\lambda_2}(t), \tag{IV.3.62}$$

for a deterministic constant $C > 0$. Again, we need to adapt the Lemmata from previous sections with this alternative upper bound.

Lemma IV.3.27. *There exists a deterministic constant $C > 0$ such that, for all $t, s \in (0,1)$ we have*

$$\sum_{0 \leq j_1 < n} \sum_{0 \leq j_2 < n} \sum_{\lambda_1 \in \Lambda_{j_1}, \lambda_2 \in \Lambda_{j_2}} 2^{j_1(1-H_1)} 2^{j_2(1-H_2)} y_{\lambda_1}(t) y_{\lambda_2}(t)\, |I_{\lambda_1,\lambda_2}[t,s]| \\ \leq C|t-s|^{H_1+H_2-1}.$$

Proof. If $\xi \in [s,t]$ and $\lambda \in \lambda_j^l(t)$, for $0 \leq j < n$ and $l \geq 1$,

$$|2^j\xi - s(\lambda)| \geq |s(\lambda(t)) - s(\lambda)| - 2 > 2^{m(l-1)} - 2$$

and so, using the fast decay of the fractional antiderivatives of ψ (IV.3.8) and the definition of $(y_\lambda)_{\lambda\in\Lambda}$, we get for $0 \leq j_1, j_2 < n$

$$\begin{aligned}
&\sum_{\lambda_1\in\Lambda_{j_1},\lambda_2\in\Lambda_{j_2}} y_{\lambda_1}(t)y_{\lambda_2}(t)|\psi_{\lambda_1}(\xi)\psi_{\lambda_2}(\xi)| \\
&\quad= \sum_{(l_1,l_2)\in\mathbb{N}_0^2} \sum_{\lambda_1\Lambda_{j_1}^{l_1}(t)} \sum_{\lambda_2\Lambda_{j_2}^{l_2}(t)} y_{\lambda_1}(t)y_{\lambda_2}(t)|\psi_{\lambda_1}(\xi)\psi_{\lambda_2}(\xi)| \\
&\quad\leq C \sum_{(l_1,l_2)\in\mathbb{N}_0} \sum_{\lambda_1\Lambda_{j_1}^{l_1}(t)} \sum_{\lambda_2\Lambda_{j_2}^{l_2}(t)} \frac{2^{l_1+l_2}}{(3+|2^{j_1}\xi-k_1|)^4(3+|2^{j_2}\xi-k_2|)^4} \\
&\quad\leq C \sum_{(l_1,l_2)\in\mathbb{N}_0} \sum_{\lambda_1\Lambda_{j_1}^{l_1}(t)} \sum_{\lambda_2\Lambda_{j_2}^{l_2}(t)} \frac{2^{l_1+l_2}2^{-m(l_1+l_2)}}{(3+|2^{j_1}\xi-k_1|)^3(3+|2^{j_2}\xi-k_2|)^3} \\
&\leq C \sum_{k_1\in\mathbb{Z}} \frac{1}{(3+|2^{j_1}\xi-k_1|)^3} \sum_{k_2\in\mathbb{Z}} \frac{1}{(3+|2^{j_2}\xi-k_2|)^3} \\
&\leq C.
\end{aligned} \tag{IV.3.63}$$

It leads, just as in Lemmata IV.3.5 and IV.3.18, to the desired estimate. □

In what follows, we use these notations instead of the one given in Definition IV.3.6:

$$\Lambda_{j_2}^{<}(t,s) = \{\lambda_2 \in \Lambda_{j_2} \; : \; s(\lambda_2) \in \mathbb{Z}_{j_2}^{<}(t,s)\},$$

$$\Lambda_{j_2}^{>}(t,s) = \{\lambda_2 \in \Lambda_{j_2} \; : \; s(\lambda_2) \in \mathbb{Z}_{j_2}^{>}(t,s)\},$$

$$\Lambda_{j_2}[t,s] = \{\lambda_2 \in \Lambda_{j_2} \; : \; s(\lambda_2) \in \mathbb{Z}_{j_2}[t,s]\}.$$

Lemma IV.3.28. *There exists a deterministic constant $C > 0$ such that, for all $t, s \in (0,1)$ and $0 \leq j_1 < n \leq j_2$, the quantities*

$$\sum_{\lambda_1\in\Lambda_{j_1}} \sum_{\lambda_2\in\Lambda_{j_2}^{<}(t,s)} y_{\lambda_1}(t)y_{\lambda_2}(t)\,|I_{\lambda_1,\lambda_2}[t,s]| \tag{IV.3.64}$$

$$\sum_{\lambda_1\in\Lambda_{j_1}} \sum_{\lambda_2\in\Lambda_{j_2}^{>}(t,s)} y_{\lambda_1}(t)y_{\lambda_2}(t)\,|I_{\lambda_1,\lambda_2}[t,s]| \tag{IV.3.65}$$

are bounded by

$$C2^{\frac{1}{m}(j_2-n)}2^{-j_2}.$$

Proof. Again, we prove the bound for (IV.3.64), the reasoning for (IV.3.65) being similar. Let us remark that, if $j_2 \geq n$ $x \in [s,t]$ and $\lambda_{j_2}(x) \in \Lambda_{j_2}^l(t)$ then, the construction and the definition of $(y_\lambda(t))_{\lambda\in\Lambda}$ gives that

- $l \leq \frac{1}{m}(j_2 - n)$, as $|s-t| \leq 2^{-n}$,
- if $\lambda \in \Lambda_{j_2}^{l_2}(x)$ then $|y_\lambda| \leq 2^{l_2} 2^{l+1}\mu$ while, by definition, if $l_2 \geq 1$

$$3 + |2^{j_2}x - s(\lambda)| \geq 2 + 2^{m(l_2-1)}.$$

Therefore, if we set

$$D_{j_2}^l(t) = \bigcup_{\lambda \in \Lambda_{j_2}^l(t)} \lambda,$$

we have

$$\begin{aligned}
\text{(IV.3.64)} &\leq \sum_{\lambda_1 \in \Lambda_{j_1}} \sum_{\lambda_2 \in \Lambda_{j_2}^{<}(t,s)} y_{\lambda_1}(t) y_{\lambda_2}(t) \int_{[s,t]} |\psi_{\lambda_1}(x)\psi_{\lambda_2}(x)|\, dx \\
&\leq \sum_{0 \leq l \leq \frac{1}{m}(j_2-n)} \sum_{\lambda_1 \in \Lambda_{j_1}} \sum_{\lambda_2 \in \Lambda_{j_2}^{<}(t,s)} y_{\lambda_1}(t) y_{\lambda_2}(t) \int_{D_{j_2}^l(t)} |\psi_{\lambda_1}(x)\psi_{\lambda_2}(x)|\, dx.
\end{aligned} \tag{IV.3.66}$$

But, for all $x \in D_{j_2}^l$, using the same method as in (IV.3.63), but splitting the sums according to the set $\Lambda_{j_1}^{l_1}(x)$ and $\Lambda_{j_2}^{l_2}(x)$ on which $y_{\lambda_1}(t) y_{\lambda_2}(t) \leq 2^{l+l_1+l_2+1}$ we get

$$\begin{aligned}
&\sum_{\lambda_1 \in \Lambda_{j_1}} \sum_{\lambda_2 \in \Lambda_{j_2}^{<}(t,s)} |\varepsilon_{\lambda_1,\lambda_2}||\psi_{\lambda_1}(x)\psi_{\lambda_2}(x)| \\
&\qquad \leq C 2^{l+1} \sum_{\lambda_1 \in \Lambda_{j_1}} \frac{1}{(3 + |2^{j_1}x - k_1|)^3} \sum_{\lambda_2 \in \Lambda_{j_2}^{<}(t,s)} \frac{1}{(3 + |2^{j_2}x - k_2|)^3} \\
&\qquad \leq C 2^{l+1} \sum_{\lambda_2 \in \Lambda_{j_2}^{<}(t,s)} \frac{1}{(3 + |2^{j_2}x - k_2|)^3}.
\end{aligned} \tag{IV.3.67}$$

Finally, using the techniques in (IV.3.15), we get

$$\text{(IV.3.64)} \leq C 2^{\frac{1}{m}(j_2-n)} 2^{-j_2}.$$

□

Lemma IV.3.29. *There exists a deterministic constant $C > 0$ such that, for all $t, s \in (0,1)$ and $0 \leq j_1 < n \leq j_2$, the quantities*

$$\sum_{\lambda_1 \in \Lambda_{j_1}} \sum_{\lambda_2 \in \Lambda_{j_2}[t,s]} y_{\lambda_1}(t) y_{\lambda_2}(t) \left| \int_{-\infty}^{\min\{s,t\}} \psi_{H_1}(2^{j_1}x - k_1)\psi_{H_2}(2^{j_2}x - k_2)\, dx \right| \tag{IV.3.68}$$

$$\sum_{\lambda_1 \in \Lambda_{j_1}} \sum_{\lambda_2 \in \Lambda_{j_2}[t,s]} y_{\lambda_1}(t) y_{\lambda_2}(t) \left| \int_{\max\{s,t\}}^{+\infty} \psi_{H_1}(2^{j_1}x - k_1)\psi_{H_2}(2^{j_2}x - k_2)\, dx \right| \tag{IV.3.69}$$

are bounded by

$$C 2^{\frac{1}{m}(j_2-n)} 2^{-j_2}.$$

Proof. Again, we assume $s \le t$. If $x \in (-\infty, s]$ is such that $\lambda_{j_1}(x) \in \Lambda^l_{j_1}(s)$, we have, for all $\lambda_1 \in \Lambda^{l_1}_{j_1}(x)$ and $\lambda_2 \in \Lambda_{j_2}[t,s] \cap \Lambda^{l_2}_{j_2}(s)$ (with $j_1 < n \le j_2$),

$$\begin{aligned}
\frac{y_{\lambda_1}(t) y_{\lambda_2}(t)}{(3+|2^{j_1}x-k_1|)^4(3+|2^{j_2}x-k_2|)^4} &\le C \frac{2^{\frac{1}{m}(j_2-n)+l+l_1+l_2+1}\mu^2}{(3+|2^{j_1}x-k_1|)^4(3+|2^{j_2}x-k_2|)^5} \\
&\le C \frac{2^{\frac{1}{m}(j_2-n)+l+1}}{(3+|2^{j_1}x-k_1|)^3(3+|2^{j_2}x-k_2|)^4} \qquad \text{(IV.3.70)} \\
&\le C \frac{2^{\frac{1}{m}(j_2-n)}}{(3+|2^{j_1}x-k_1|)^3(3+|2^{j_2}x-k_2|)^3}
\end{aligned}$$

because

$$3 + |2^{j_2}x - k_2| = 3 + k_2 - 2^{j_2}x \ge 2 + 2^{j_1}(s-x) \ge 2^{m(l-1)}.$$

Thus we get, using the fast decay of the fractional antiderivatives of the wavelet before splitting the integral over $(-\infty, s]$ into the integral over the sets $(-\infty, s] \cap D^l_{j_1}(s)$, in the same way as in (IV.3.66), using (IV.3.70) and finally the boundedness of the function (IV.3.44) for $M = 3$ and inequality (IV.3.17)

$$\begin{aligned}
&\sum_{\lambda_1 \in \Lambda_{j_1}} \sum_{\lambda_2 \in \Lambda_{j_2}[t,s]} \int_{-\infty}^{s} |\psi_{\lambda_1}(x)\psi_{\lambda_2}(x)|\, dx \\
&\quad \le C 2^{\frac{1}{m}(j_2-n)} \int_{-\infty}^{s} \sum_{\lambda_1 \in \Lambda_{j_1}} \sum_{\lambda_2 \in \Lambda_{j_2}[t,s]} \frac{dx}{(3+|2^{j_1}x-k_1|)^3(3+|2^{j_2}x-k_2|)^3} \\
&\quad \le C 2^{\frac{1}{m}(j_2-n)} \int_{-\infty}^{s} \sum_{\lambda_2 \in \Lambda_{j_2}[t,s]} \frac{dx}{(3+|2^{j_2}x-k_2|)^3} \\
&\quad \le C 2^{\frac{1}{m}(j_2-n)} 2^{-j_2}.
\end{aligned}$$

In the same way we get

$$\sum_{\lambda_1 \in \Lambda_{j_1}} \sum_{\lambda_2 \in \Lambda_{j_2}[t,s]} y_{\lambda_1}(t) y_{\lambda_2}(t) \left| \int_{\max\{t,s\}}^{+\infty} \psi_{H_1}(2^{j_1}x - k_1)\psi_{H_2}(2^{j_2}x-k_2)\, dx \right| \le C 2^{-j_2}. \quad \text{(IV.3.71)}$$

□

Lemma IV.3.30. *There exists a deterministic constant $C > 0$ such that, for all $t, s \in (0,1)$, the quantities*

$$\sum_{0 \le j_1 < n} \sum_{j_2 \ge n} \sum_{\lambda_1 \in \Lambda_{j_1}, \lambda_2 \in \Lambda_{j_2}} 2^{j_1(1-H_1)} 2^{j_2(1-H_2)} y_{\lambda_1}(t) y_{\lambda_2}(t) |I_{\lambda_1,\lambda_2}[t,s]|$$

$$\sum_{j_1 \ge n} \sum_{0 \le j_2 < n} \sum_{\lambda_1 \in \Lambda_{j_1}, \lambda_2 \in \Lambda_{j_2}} 2^{j_1(1-H_1)} 2^{j_2(1-H_2)} y_{\lambda_1}(t) y_{\lambda_2}(t) |I_{\lambda_1,\lambda_2}[t,s]|$$

are bounded by

$$C|t-s|^{H_1+H_2-1}.$$

Proof. The proof is exactly the same as the one of Lemma IV.3.9 excepted that we use Lemmata IV.3.28 and IV.3.29 instead of Lemmata IV.3.7 and IV.3.8 respectively. It leads on one side us to consider the sums

$$\left(\left(\sum_{j_1=0}^{n-1} 2^{j_1(1-H_1)} \sum_{j_2=n}^{+\infty} 2^{\frac{1}{m}(j_2-n)} 2^{-j_2 H_2}\right) + 2^{n(1-H_1-H_2)}\right)$$

which are bounded by

$$C 2^{n(1-H_1-H_2)} \leq C|t-s|^{H_1+H_2-1}$$

because $\frac{1}{m} < H_2$. On the other side, if we write I_{λ_1,λ_2} for $I_{j_1,j_2}^{k_1,k_2}$ in Lemma IV.2.1, we have, from it,

$$\begin{aligned}
&\left|\sum_{\lambda_1\in\Lambda_{n-1}} \sum_{\lambda_2\in\Lambda_n[t,s]} y_{\lambda_1}(t) y_{\lambda_2}(t) I_{\lambda_1,\lambda_2}\right| \\
&\leq C 2^{-n} \left|\sum_{l_1=0}^{+\infty} \sum_{\lambda_1\in\Lambda_{n-1}^{l_1}(t)} \sum_{\lambda_2\in\Lambda_n[t,s]} \frac{2^{l_1}}{(3+|2k_1-k_2|)^4}\right| \\
&\leq C 2^{-n} \left|\sum_{\lambda_1\in\Lambda_{n-1}} \sum_{\lambda_2\in\Lambda_n[t,s]} \frac{1}{(3+|2k_1-k_2|)^3}\right| \\
&\leq C 2^{-n}.
\end{aligned} \tag{IV.3.72}$$

□

Lemma IV.3.31. *There exists a deterministic constant $C > 0$ such that, for all $t, s \in (0,1)$ and $n \leq j_1 \leq j_2$ the quantities* (IV.3.64) *and* (IV.3.65) *are bounded by*

$$C 2^{\frac{1}{m}(j_1-n)} 2^{\frac{1}{m}(j_2-n)} 2^{-j_2}.$$

Proof. The proof is essentially the same as for Lemma IV.3.28 excepted that, now, as $n \leq j_1 \leq j_2$, we remark that if $x \in D_{j_2}^{l}(t)$ for a $0 \leq l \leq \frac{1}{m}(j_2-n)$ then $x \in D_{j_1}^{l'}(t)$ for a $0 \leq l' \leq \frac{1}{m}(j_1-n)$. □

Lemma IV.3.32. *There exists a deterministic constant $C > 0$ such that, for all $t, s \in (0,1)$ and $n \leq j_1 \leq j_2$ the quantities* (IV.3.68) *and* (IV.3.69) *are bounded by*

$$C 2^{\frac{1}{m}(j_1-n)} 2^{\frac{1}{m}(j_2-n)} 2^{-j_2}.$$

Proof. The proof is essentially the same as for Lemma IV.3.29 and the only modification is the same as in the proof of Lemma IV.3.31. □

This time, the bound for the random variables $\Xi_j(\lambda)$ is already considered in the construction and we can directly go to the proof of the main Proposition of this subsection.

Proof of Proposition IV.3.26. If we apply the procedure with m such that $1/m < \min\{H_1, H_2\}$ and $2/m < 1 - H_1 - H_2$, we find an event Ω_{slo} of probability 1 such that, for all $\omega \in \Omega_{\text{slo}}$, there is $\mu \in \mathbb{N}$ for which $S^\mu_{\text{low}} \cap (0,1) \neq \emptyset$. Then, if $\omega \in \Omega_{\text{slo}}$ and $t \in S^\mu_{\text{low}}(\omega) \cap (0,1)$ and $s \in (0,1)$, we write

$$\begin{aligned}
&|R'_{H_1,H_2}(t,\omega) - R'_{H_1,H_2}(s,\omega)| \\
&\qquad \leq \left| \sum_{0 \leq j_1 < n} \sum_{0 \leq j_2 < n} \sum_{\lambda_1 \in \Lambda_{j_1}, \lambda_2 \in \Lambda_{j_2}} 2^{j_1(1-H_1)} 2^{j_2(1-H_2)} \varepsilon_{\lambda_1,\lambda_2}(\omega) I_{\lambda_1,\lambda_2}[t,s] \right| \\
&\qquad + \left| \sum_{0 \leq j_1 < n} \sum_{j_2 \geq n} \sum_{\lambda_1 \in \Lambda_{j_1}, \lambda_2 \in \Lambda_{j_2}} 2^{j_1(1-H_1)} 2^{j_2(1-H_2)} \varepsilon_{\lambda_1,\lambda_2}(\omega) I_{\lambda_1,\lambda_2}[t,s] \right| \\
&\qquad + \left| \sum_{j_1 \geq n} \sum_{0 \leq j_2 < n} \sum_{\lambda_1 \in \Lambda_{j_1}, \lambda_2 \in \Lambda_{j_2}} 2^{j_1(1-H_1)} 2^{j_2(1-H_2)} \varepsilon^{k_1,k_2}_{j_1,j_2}(\omega) I_{\lambda_1,\lambda_2}[t,s] \right| \\
&\qquad + \left| \sum_{j_1 \geq n} \sum_{j_2 \geq n} \sum_{\lambda_1 \in \Lambda_{j_1}, \lambda_2 \in \Lambda_{j_2}} 2^{j_1(1-H_1)} 2^{j_2(1-H_2)} \varepsilon_{\lambda_1,\lambda_2}(\omega) I_{\lambda_1,\lambda_2}[t,s] \right|.
\end{aligned} \tag{IV.3.73}$$

As inequality (IV.3.62) holds, we use Lemma IV.3.27 to bound the first sum, and Lemma IV.3.30 to bound the second and the third one. For the last sum, from inequality (IV.3.62) and Lemmata IV.3.31 and IV.3.32, it just remains us to find bound for the random variables (IV.3.38), (IV.3.39) and (IV.3.40)with $\ell \in \{j, j+1\}$ on Ω_{slo}. For (IV.3.38) with $\ell = j$, we have, as in (IV.3.72) and then (IV.3.41)

$$\begin{aligned}
&\left| \sum_{\lambda_1 \in \Lambda^<_j(t,s)} \sum_{\lambda_2 \in \Lambda_j[t,s]} \varepsilon_{\lambda_1,\lambda_2}(\omega) I_{\lambda_1,\lambda_2} \right| \\
&\qquad \leq C 2^{-j} 2^{\frac{2}{m}(j-n)} \mu^2 \left| \sum_{\lambda_1 \in \Lambda^<_j(t,s)} \sum_{\lambda_2 \in \Lambda_j[t,s]} \frac{1}{(3 + |2k_1 - k_2|)^3} \right| \\
&\qquad \leq C 2^{-j} 2^{\frac{2}{m}(j-n)} \mu^2.
\end{aligned}$$

The same bound holds when we consider the sums over $\lambda_1 \in \Lambda^>_j(t,s)$ or $\lambda_2 \in \Lambda_{j+1}[t,s]$, i.e. for(IV.3.38) and (IV.3.39). Finally the construction and especially (IV.3.61) insures us that

$$\left| \sum_{\lambda_1 \in \Lambda_j[t,s]} \sum_{\lambda_2 \in \Lambda_j[t,s]} \varepsilon_{\lambda_1,\lambda_2}(\omega) I_{\lambda_1,\lambda_2} \right| \leq C(j - n + 1) 2^{\frac{-j-n}{2}} \mu.$$

Therefore, the last term in (IV.3.73) is bounded from above by

$$
\begin{aligned}
&C\mu^2\left(\sum_{j_1\geq n}2^{j_1(1-H_1)}2^{\frac{1}{m}(j_1-n)}\sum_{j_2\geq j_1}2^{-j_2H_2}2^{\frac{1}{m}(j_2-n)}+\sum_{j\geq n}2^{j(\frac{3}{2}-H_1-H_2)}(j-n+1)2^{-\frac{n}{2}}\right)\\
&\leq C\mu^2\left(\sum_{j_1\geq n}2^{j_1(1-H_1-H_2)}2^{\frac{2}{m}(j_1-n)}+2^{n(\frac{3}{2}-H_1-H_2)}2^{-\frac{n}{2}}\right)\\
&\leq C\mu^2 2^{n(1-H_1-H_2)}\\
&\leq C\mu^2|t-s|^{H_1+H_2-1}
\end{aligned}
$$

and thus inequality (IV.3.56) holds. □

IV.4 Lower bounds for wavelet leaders

In this section, we show that the limits (IV.1.5) and (IV.1.6) are strictly positive. In [8], the authors used the independence of the increments of the Brownian motion to bound from below its wavelet leaders. But, for the (generalized) Rosenblatt process this nice feature is not met anymore. Nevertheless, following an idea by Ayache in a close but different context[3] [6], we decompose the wavelet coefficients of the generalized Rosenblatt process in two parts. We gain some independence properties in the first part while the second is, in some sense, negligible compared to the first, see Proposition IV.4.5 below. All along this section, in order to ease notations we set

$$
C_{H_1,H_2}:=\frac{1}{\Gamma\left(H_1-\frac{1}{2}\right)\Gamma\left(H_2-\frac{1}{2}\right)}
$$

and for $s,x_1,x_2\in\mathbb{R}$

$$
f_{H_1,H_2}(s,x_1,x_2)=(s-x_1)_+^{H_1-3/2}\,(s-x_2)_+^{H_2-3/2}
$$

Let Ψ be a wavelet with compact support included in $[-N,N]$. Using formula (IV.2.5) at $t=k/2^j$, the wavelet coefficient $c_{j,k}$ of the generalized Rosenblatt process is given by

$$
\begin{aligned}
c_{j,k}&=\int_{-N}^{N}\left[R_{H_1,H_2}\left(\frac{x+k}{2^j}\right)-R_{H_1,H_2}\left(\frac{k}{2^j}\right)\right]\Psi(x)dx\\
&=c_{H_1,H_2}\int_{-N}^{N}\Psi(x)\int_{\mathbb{R}^2}'\int_{\frac{k}{2^j}}^{\frac{x+k}{2^j}}f_{H_1,H_2}(s,x_1,x_2)\,ds\,dB(x_1)\,dB(x_2)\,dx\\
&=c_{H_1,H_2}\int_{\mathbb{R}^2}'\int_{-N}^{N}\Psi(x)\int_{\frac{k}{2^j}}^{\frac{x+k}{2^j}}f_{H_1,H_2}(s,x_1,x_2)\,ds\,dx\,dB(x_1)\,dB(x_2)\\
&=c_{H_1,H_2}\int_{A}'\int_{-N}^{N}\Psi(x)\int_{\frac{k}{2^j}}^{\frac{x+k}{2^j}}f_{H_1,H_2}(s,x_1,x_2)\,ds\,dx\,dB(x_1)\,dB(x_2)
\end{aligned}
$$

[3] In [6], Ayache does not consider wavelets at all but directly work on Wiener-Itô integrals

where $A := \left]-\infty, \frac{k+N}{2^j}\right]^2$, because, as soon as $x \in [-N, N]$ and $s \in [k2^{-J}, (k+N)2^{-j}]$, $f(s, x_1, x_2)$ vanishes for all x_1, x_2 outside of A.

Definition IV.4.1. Given an integer $M \geq 0$, $c_{j,k}$ can be written as following

$$c_{j,k} = \widetilde{c_{j,k}}^M + \widetilde{\mathcal{C}}c_{j,k}{}^M$$

where

$$\widetilde{c_{j,k}}^M = c_{H_1,H_2} \int'_{\lambda^M_{j,k}} \int_{-N}^{N} \Psi(x) \int_{\frac{k}{2^j}}^{\frac{x+k}{2^j}} f_{H_1,H_2}(s, x_1, x_2)\, ds\, dx\, dB(x_1)\, dB(x_2) \tag{IV.4.1}$$

with

$$\lambda^M_{j,k} := \left] \frac{k - NM}{2^j}, \frac{k+N}{2^j} \right]^2$$

and

$$\widetilde{\mathcal{C}}c_{j,k}{}^M = c_{H_1,H_2} \int'_{A \setminus \lambda^M_{j,k}} \int_{-N}^{N} \Psi(x) \int_{\frac{k}{2^j}}^{\frac{x+k}{2^j}} f_{H_1,H_2}(s, x_1, x_2)\, ds\, dx\, dB(x_1)\, dB(x_2).$$

Remark 18. Let us highlight the fact that using time change of variable for Wiener-Itô integrals [90, Theorem 8.5.7], for all j, k, we have $\widetilde{c_{j,k}}^M$ is equal in law to the random variable

$$c_{H_1,H_2} 2^{-j(H_1+H_2-1)} \int'_{I_M} \int_{-N}^{N} \psi(x) \int_0^x f_{H_1,H_2}(s, x_1, x_2)\, dsdx\, dB(x_1)dB(x_2)$$

with $I_M = (-MN, N]^2$, while $\widetilde{\mathcal{C}}c_{j,k}{}^M$ is equal in law to the random variable

$$c_{H_1,H_2} 2^{-j(H_1+H_2-1)} \int'_{I'_M} \int_{-N}^{N} \psi(x) \int_0^x f_{H_1,H_2}(s, x_1, x_2)\, dsdx\, dB(x_1)dB(x_2)$$

with $I'_M = (-\infty, N]^2 \setminus (-MN, N]^2$.

Definition IV.4.2. For all $(j, k) \in \mathbb{N} \times \mathbb{Z}$ and $M \in \mathbb{N}$ we define the random variables

$$\widetilde{\varepsilon_{j,k}}^M := \frac{\widetilde{c_{j,k}}^M}{2^{-j(H_1+H_2-1)}} \text{ and } \widetilde{\mathcal{C}}\varepsilon_{j,k}{}^M := \frac{\widetilde{\mathcal{C}}c_{j,k}{}^M}{2^{-j(H_1+H_2-1)}}.$$

Remark 19. Note that $\widetilde{\varepsilon_{j,k}}^M$ and $\widetilde{\varepsilon_{j',k'}}^M$ are independent when

$$\lambda^M_{j,k} \cap \lambda^M_{j',k'} = \emptyset. \tag{IV.4.2}$$

Indeed, if $(f_j)_j$ is a sequence of real-valued step functions on $\mathbb{R}^2 \setminus \{(x, x) : x \in \mathbb{R}\}$ which converge to the integrand with respect to $dB(x_1)dB(x_2)$ in (IV.4.1) then $\int'_{\mathbb{R}^2} f_j(x_1, x_2)\, dB(x_1)dB(x_2)$ is a polynomial function of a finite number of increments $B(t_2) - B(t_1)$ of the Brownian motion for some $t_1, t_2 \in \lambda_{j,k}^M$. Thus $\widetilde{\varepsilon_{j,k}}^M$ is measurable with respect to the σ-algebra generated by these increments

$$\sigma_{j,k}^M := \sigma\left(\{B(t_2) - B(t_1) \, : \, t_1, t_2 \in \lambda_{j,k}^M\}\right).$$

Using the independence of the increments of the Brownian motion, one concludes that $\sigma_{j,k}^M$ and $\sigma_{j',k'}^M$ are independent as soon as condition (IV.4.2) is met and so the same holds for $\widetilde{\varepsilon_{j,k}}^M$ and $\widetilde{\varepsilon_{j,k'}}^M$. Moreover, $\widetilde{\varepsilon_{j_1,k_1}}^M, \ldots, \widetilde{\varepsilon_{j_n,k_n}}^M$ are independent when the following condition is satisfied

$$\lambda_{j_i,k_i}^M \cap \lambda_{j_l,k_l}^M = \emptyset \text{ for all } 1 \leq i < l \leq n. \tag{IV.4.3}$$

This leads to defining the following condition.

Definition IV.4.3. Let $n \geq 2$. We say $\lambda_{j_1,k_1}, \ldots, \lambda_{j_n,k_n}$ satisfy condition (C_M) if (IV.4.3) is satisfied.

From Remark 18, we know that $(\widetilde{\varepsilon_{j,k}}^M)_{\lambda\in\Lambda}$ is a family of identically distributed second order Wiener chaos random variables. Moreover, $\widetilde{\varepsilon_{j_1,k_1}}^M, \ldots, \widetilde{\varepsilon_{j_n,k_n}}^M$ are independent as soon as $\lambda_{j_1,k_1}, \ldots, \lambda_{j_n,k_n}$ satisfies (C_M). The following proposition provides a lower bound (independent of M) for the tail behavior of the random variable $\widetilde{\varepsilon_{j,k}}^M$.

Proposition IV.4.4. *Let $M \in \mathbb{N}$ and $y \in \mathbb{R}^+$. If M and y are large enough, then the exists a deterministic constant $c_2 > 0$ (independent of M) such that*

$$\mathbb{P}\left(|\widetilde{\varepsilon_{j,k}}^M| > y\right) \geq \exp\left(-c_2 y\right) \tag{IV.4.4}$$

for all $(j,k) \in \mathbb{N} \times \mathbb{Z}$

Proof. Fix $y \in \mathbb{R}^+$ (large enough). Our aim is to prove the existence of lower bound for $\mathbb{P}\left(|\widetilde{\varepsilon_\lambda}^M| > y\right)$ which is independent of M. To this end, we start by proving the following lemma

Lemma IV.4.5. *There exist three strictly positive deterministic constants C_{Ψ,H_1,H_2}, C'_{Ψ,H_1,H_2} and C^*_{Ψ,H_1,H_2} such that for all $(j,k) \in \mathbb{N} \times \mathbb{Z}$ and $M \geq 2$ one has*

$$\begin{aligned} C_{\Psi,H_1,H_2} 2^{-j(H_1+H_2-1)} \leq \left\| \widetilde{c_{j,k}}^M \right\|_2 &\leq C'_{\Psi,H_1,H_2} 2^{-j(H_1+H_2-1)} \\ \left\| \widetilde{\mathcal{C}c_{j,k}}^M \right\|_2 &\leq C^*_{\Psi,H_1,H_2} 2^{-j(H_1+H_2-1)} M^{\max\{H_1,H_2\}-1} \end{aligned}$$

Proof. Let us assume, w.l.o.g. that $H_1 \geq H_2$. We define the functions

$$\Phi_1 \, : \, (x_1, x_2) \mapsto \int_{-N}^{N} \Psi(x) \int_0^x f_{H_1,H_2}(s, x_1, x_2) ds \, dx,$$

$$\Phi_2 \, : \, (x_1, x_2) \mapsto \int_{-N}^{N} \Psi(x) \int_0^x f_{H_1,H_2}(s, x_2, x_1) ds \, dx,$$

and the symmetric function[4]

$$\Phi = \frac{1}{2}\left(\Phi_1 + \Phi_2\right).$$

By Remark 18 we have, using the "Wiener isometry" [5] [113, Section 5],

$$\left\|\widetilde{c_{j,k}}^M\right\|_2 = \sqrt{2} c_{H_1,H_2} 2^{-j(H_1+H_2-1)} \left\|\Phi\right\|_{L^2(I_M)}$$

and thus it suffices to take

$$C_{\Psi,H_1,H_2} := \sqrt{2} c_{H_1,H_2} \left\|\Phi\right\|_{L^2([-N,N]^2)}$$
$$C'_{\Psi,H_1,H_2} := \sqrt{2} c_{H_1,H_2} \left\|\Phi\right\|_{L^2((-\infty,N]^2)}$$

Now, still using Remark 18 and "Wiener isometry" we have

$$\begin{aligned}\left\|\widetilde{\mathcal{C}c_{j,k}}^M\right\|_2 &= \sqrt{2} c_{H_1,H_2} 2^{-j(H_1+H_2-1)} \left\|\Phi\right\|_{L^2(I'_M)} \\ &\le \sqrt{2} c_{H_1,H_2} 2^{-j(H_1+H_2-1)} \left\|\Phi_1\right\|_{L^2(I'_M)}.\end{aligned}$$

Also as

$$I'_M = (-\infty, N]^2 \setminus (-MN, N]^2 \subset \mathbb{R} \times (-\infty, -MN] \bigcup (-\infty, -MN] \times \mathbb{R},$$

we write

$$\begin{aligned}&\left\|\Phi_1\right\|^2_{L^2(I'_M)} \\ &= \int_{I'_M} \left| \int_{-N}^{N} \Psi(x) \int_0^x (s-x_1)_+^{H_1-\frac{3}{2}} (s-x_2)_+^{H_2-\frac{3}{2}} ds\, dx \right|^2 dx_1\, dx_2 \\ &\le \int_{I'_M} \left(\int_{-N}^{N} |\Psi(x)| \int_{[0,x]} (s-x_1)_+^{H_1-\frac{3}{2}} (s-x_2)_+^{H_2-\frac{3}{2}} ds\, dx \right)^2 dx_1\, dx_2 \\ &\le \int_{\mathbb{R}} \int_{-\infty}^{-MN} \left(\int_{-N}^{N} |\Psi(x)| \int_{[0,x]} (s-x_1)_+^{H_1-\frac{3}{2}} (s-x_2)_+^{H_2-\frac{3}{2}} ds\, dx \right)^2 dx_1\, dx_2 \\ &+ \int_{\mathbb{R}} \int_{-\infty}^{-MN} \left(\int_{-N}^{N} |\Psi(x)| \int_{[0,x]} (s-x_1)_+^{H_1-\frac{3}{2}} (s-x_2)_+^{H_2-\frac{3}{2}} ds\, dx \right)^2 dx_2\, dx_1.\end{aligned}$$

Let us deal with the first term in the last sum, the second one can be treated similarly by permuting the roles of H_1 and H_2 as well as x_1 and x_2. As the function $y \mapsto y^{H_1-3/2}$ is decreasing, one gets

$$\begin{aligned}&\int_{\mathbb{R}} \int_{-\infty}^{-MN} \left(\int_{-N}^{N} |\Psi(x)| \int_{[0,x]} (s-x_1)_+^{H_1-\frac{3}{2}} (s-x_2)_+^{H_2-\frac{3}{2}} ds\, dx \right)^2 dx_1\, dx_2 \\ &\le \left(\int_{-\infty}^{-MN} (-N-x_1)^{2H_1-3}\, dx_1 \right) \times \int_{\mathbb{R}} \left(\int_{-N}^{N} |\Psi(x)| \left| \int_{[0,x]} (s-x_2)_+^{H_2-3/2}\, ds \right| dx \right)^2 dx_2.\end{aligned}$$

[4] The function Φ is in the fact the symmetrization of Φ_1.

[5] For f a symmetric function in $L^2(\mathbb{R}^2)$, and $I_2(f)$ the second order Wiener-Itô integral of f. One has $\mathbb{E}(I_m(f))^2 = 2!||f||_{L^2(\mathbb{R}^2)}$.

Concerning the first integral, we have, as $M \geq 2$

$$\begin{aligned}\int_{-\infty}^{-NM} (-N - x_1)^{2H_1-3}\, dx_1 &= \frac{1}{2-2H_1} (NM - N)_+^{2H_1-2} \\ &= \frac{1}{2-2H_1} N^{2H_1-2}(M-1)^{2H_1-2} \\ &\leq c \cdot M^{2H_1-2}\end{aligned}$$

while, using again the "Wiener isometry",

$$\begin{aligned}&\int_{\mathbb{R}} \left(\int_{-N}^{N} |\Psi(x)| \left| \int_{[0,x]} (s - x_2)_+^{H_2-3/2} \right| dx \right)^2 dx_2 \\ &\leq 2N \left\| \Psi \right\|_\infty \sup_{x \in [-N,N]} \int_{\mathbb{R}} \left| \int_{[0,x]} (s - x_2)_+^{H_2-3/2}\, ds \right|^2 dx_2 \\ &= 2N \left\| \Psi \right\|_\infty \sup_{x \in [-N,N]} \mathbb{E}\left[|B_{H_2}(x) - B_{H_2}(0)|^2 \right] \\ &\leq 2N \left\| \Psi \right\|_\infty \sup_{x \in [-N,N]} C_{H_2} (|x|)^{2H_2} \leq c,\end{aligned}$$

where B_{H_2} denotes the fractional Brownian motion with parameter H_2. As a result, there exists a positive constant C^*_{Ψ,H_1,H_2} such that, as we suppose $H_1 \geq H_2$, one has

$$\left\| \widetilde{\mathcal{C}c_{j,k}}^M \right\|_2 \leq C^*_{\Psi,H_1,H_2} 2^{-j(H_1+H_2-1)} M^{H_1-1}.$$

□

By Lemma IV.4.5, one can remark that as $M \to +\infty$, $(\widetilde{\varepsilon_{j,k}}^M)_M$ converges in $L^2(\Omega)$ to the random variable

$$\varepsilon_{j,k} := \frac{c_{j,k}}{2^{-j(H_1+H_2-1)}}$$

with, for all $M \in \mathbb{N}$,

$$\varepsilon_{j,k} - \widetilde{\varepsilon_{j,k}}^M = \widetilde{\mathcal{C}\varepsilon_{j,k}}^M.$$

By Theorem IV.3.12, there exists a constant $c_1 > 0$ such that, for all $\lambda \in \Lambda$ and y sufficiently large

$$\mathbb{P}(|\varepsilon_{j,k}| \geq y) \geq \exp(-c_1 y).$$

Then, for all $M \in \mathbb{N}$, we have, for all such λ and y

$$\begin{aligned}\mathbb{P}\left(|\widetilde{\varepsilon_{j,k}}^M| \geq y \right) &\geq \mathbb{P}\left(\{ |\widetilde{\varepsilon_{j,k}}^M| \geq y \} \cap \{ |\widetilde{\mathcal{C}\varepsilon_{j,k}}^M| \leq y \} \right) \\ &\geq \mathbb{P}\left(\{ |\varepsilon_{j,k}| - |\widetilde{\mathcal{C}\varepsilon_{j,k}}^M| \geq y \} \cap \{ |\widetilde{\mathcal{C}\varepsilon_{j,k}}^M| \leq y \} \right) \\ &\geq \mathbb{P}\left(\{ |\varepsilon_{j,k}| \geq 2y \} \cap \{ |\widetilde{\mathcal{C}\varepsilon_{j,k}}^M| \leq y \} \right) \\ &\geq \mathbb{P}(|\varepsilon_{j,k}| \geq 2y) - \mathbb{P}\left((|\widetilde{\mathcal{C}\varepsilon_{j,k}}^M| > y \right).\end{aligned}$$

Using Lemma IV.4.5 and Theorem IV.3.11 one has

$$\mathbb{P}\left(|\widetilde{\mathcal{C}}\varepsilon_{j,k}{}^M| > y\right) \leq \mathbb{P}\left((|\widetilde{\mathcal{C}}c_{j,k}{}^M| > y \left\|\widetilde{\mathcal{C}}c_{j,k}{}^M\right\|_2 (C^*_{\Psi,H_1,H_2})^{-1} M^{1-\max\{H_1,H_2\}}\right)$$
$$\leq \exp(-\overset{\star}{C}(C^*_{\Psi,H_1,H_2})^{-1} M^{1-\max\{H_1,H_2\}} y).$$

Thus, if M is large enough, one has, as $1 - \max\{H_1, H_2\} > 0$,

$$\exp(-\overset{\star}{C}(C^*_{\Psi,H_1,H_2})^{-1} M^{1-\max\{H_1,H_2\}} y) \leq \frac{1}{2} \exp\left(-2c_1 y\right)$$

which gives that, for all large enough y, one gets

$$\mathbb{P}\left(|\widetilde{\varepsilon_{j,k}}^M| > y\right) \geq \exp\left(-c_2 y\right) \tag{IV.4.5}$$

with $c_2 := 2c_1$. In the sequel, we will implicitly always consider such large enough M. □

In the following two subsections, Lemmata IV.4.7 and IV.4.10 follow the lines of Lemmata 3.6 and 3.8 in [8] respectively, with some subtle modifications as the authors in [8] deal with $\mathcal{N}(0,1)$ random variables while, here, we focus on random variables in the second order Wiener chaos that depend on the parameter M. For the sake of completeness and clarity, we write the proofs in full details.

IV.4.1 Ordinary Points

In this section our aim is to prove the following proposition.

Proposition IV.4.6. *There exists $\Omega_1^* \subset \Omega$ with probability 1 such that for all $\omega \in \Omega_1^*$ and Lebesgue almost every $t \in (0,1)$ one has*

$$\limsup_{j \to +\infty} \frac{d_j(t,\omega)}{2^{-j(H_1+H_2-1)} \log j} > 0. \tag{IV.4.6}$$

To this end, as a first step, let us state the following lemma concerning the random variable $\widetilde{\varepsilon_\lambda}^M$. If $\lambda = \lambda_{j,k}$ is a dyadic interval and $m \in \mathbb{N}$, $S_{\lambda,m} = S_{j,k,m}$ stands for the finite set of cardinality 2^m whose elements are the dyadic intervals of scale $j+m$ included in $\lambda_{j,k}$, formally speaking

$$S_{j,k,m} := \{\lambda \in \Lambda_{j+m} \, : \, \lambda \subset \lambda_{j,k}\}$$

Lemma IV.4.7. *There is a deterministic constant $C > 0$ such that the following holds: for all $M \in \mathbb{N}$ and for all $t \in (0,1)$, there exists $\Omega_{t,1} \subset \Omega$ with probability 1 such that for all $\omega \in \Omega_{t,1}$ there are infinitely many $j \in \mathbb{N}$ such that*

$$\max_{\substack{\lambda' \in S_{\lambda, \lfloor \log_2(NM) \rfloor + 2} \\ \lambda \in 3\lambda_j(t)}} \left|\widetilde{\varepsilon_{\lambda'}}^M(\omega)\right| \geq C \log j.$$

Proof. Let us fix $t \in (0,1)$ and $j \in \mathbb{N}$. For any $\lambda \in S_{j,k_j(t),m}$, there exists a unique decreasing finite sequence $(I_n)_{0\leq n\leq m}$ of decreasing dyadic intervals in the sense of inclusion such that $I_0 = \lambda_{j,k_j(t)}$, $I_m = \lambda$ and $I_n \in S_{j,k_j(t),n}$. Then, define the sequence $(T_n)_{1\leq n\leq m}$ of unique dyadic intervals such that $I_{n-1} = I_n \cup T_n$. Note that for all $1 \leq n \leq m$, $T_n \in 3I_n$. Moreover, as $(I_n)_{0\leq n\leq m}$ is decreasing, $(T_n)_{1\leq n\leq m}$ are pairwisely disjoint. Furthermore, for every $n \in \{1, ..., m\}$, there exist $T'_n = \lambda_{j_n,k_n} \in S_{T_n,\lfloor \log_2 NM\rfloor+2}$ such that

$$\left(\frac{k_n - NM}{2^{j_n}}, \frac{k_n + N}{2^{j_n}}\right) \subset T_n.$$

As a consequence, the associated random variables $\left(\widetilde{\varepsilon_{T'_n}}^M\right)_{1\leq n\leq m}$ are independent as the dyadic intervals $(T'_n)_{1\leq n\leq m}$ satisfies condition (C_M) in Definition IV.4.3. Next, for a constant $C > 0$ to be chosen later, we set

$$\mathcal{E}_{j,m}(t) = \left\{\omega \in \Omega \,:\, \max_{1\leq n\leq m} \left|\widetilde{\varepsilon_{T'_n}}^M\right| \geq C\log(2m)\right\}.$$

Note that, as the random variables $\left(\widetilde{\varepsilon_{T'_n}}^M\right)_{1\leq n\leq m}$ are independent,

$$\mathbb{P}\left(\mathcal{E}_{j,m}(t)\right) = 1 - \prod_{n=1}^{m} \mathbb{P}\left(\left|\widetilde{\varepsilon_{T'_n}}^M\right| < C\log(2m)\right)$$

Recalling (IV.4.4), and the fact that $\log(1-x) \leq -x$ if $x \in (0,1)$, one gets, for m is large enough,

$$\begin{aligned}
\mathbb{P}\left(\mathcal{E}_{j,m}(t)\right) &\geq 1 - (1 - \exp(-Cc_2\log(2m))^m \\
&= 1 - \left(1 - \left(\frac{1}{2m}\right)^{Cc_2}\right)^m \\
&\geq 1 - \exp\left(\frac{m}{(2m)^{Cc_2}}\right) \\
&= 1 - \exp\left(\frac{m^{1-Cc_2}}{2^{Cc_2}}\right).
\end{aligned}$$

Finally, choosing C such that $0 < Cc_2 < 1$, one obtain that

$$\sum_{p\in\mathbb{N}} \mathbb{P}\left(\mathcal{E}_{2^p,2^p}(t)\right) = +\infty.$$

Knowing that the events $\mathcal{E}_{2^p,2^p}(t)$ are independent for all $p \in \mathbb{N}$, one concludes using Borel-Cantelli Lemma that

$$\mathbb{P}\left(\limsup_{m\to+\infty} \mathcal{E}_{2^m,2^m}(t)\right) = 1$$

It follows that for a fixed $t \in \mathbb{R}$, almost surely, there are infinitely many $j \in \mathbb{N}$ such that

$$\max_{\substack{\lambda' \in S_{\lambda,\lfloor \log_2 NM \rfloor + 2} \\ \lambda \in 3\lambda_j(t)}} \left| \widetilde{\varepsilon_{\lambda'}}^{M}(\omega) \right| \geq C \log j.$$

□

Concerning the "non-independent part" of the wavelet coefficients, one can state the following Lemma.

Lemma IV.4.8. *There is a deterministic constant $C' > 0$ such that, for all $M \in \mathbb{N}$ and for all $t \in (0,1)$, there exists $\Omega_{t,2} \subset \Omega$ with probability 1 such that for all $\omega \in \Omega_{t,2}$ there exists $J \in \mathbb{N}$ such that, for all $j \geq J$,*

$$\max_{\substack{\lambda' \in S_{\lambda,\lfloor \log_2 (NM) \rfloor + 2} \\ \lambda \in 3\lambda_j(t)}} \left| \widetilde{\mathcal{C}\varepsilon_{\lambda'}}^{M}(\omega) \right| \leq C' M^{\max\{H_1,H_2\}-1} \log j.$$

Proof. Let us fix $t \in (0,1)$. For any $C' > 0$, for all j sufficiently large and $\lambda \in 3\lambda_j(t)$, we have, by Theorem IV.3.11,

$$\begin{aligned}
&\mathbb{P}\left(\exists \lambda' \in S_{\lambda,\lfloor \log_2 NM \rfloor + 2} \, : \, \left| \widetilde{\mathcal{C}\varepsilon_{\lambda'}}^{M} \right| \geq C' M^{\max\{H_1,H_2\}-1} \log j \right) \\
&\leq \sum_{\lambda' \in S_{\lambda,\lfloor \log_2 NM \rfloor + 2}} \mathbb{P}\left(\left| \widetilde{\mathcal{C}\varepsilon_{\lambda'}}^{M}) \right| \geq C' M^{\max\{H_1,H_2\}-1} \log j \right) \\
&\leq \sum_{\lambda' \in S_{\lambda,\lfloor \log_2 NM \rfloor + 2}} \mathbb{P}\left(\left| \widetilde{\mathcal{C}\varepsilon_{\lambda'}}^{M} \right| \geq C' (C^*_{\Psi,H_1,H_2})^{-1} \| \widetilde{\mathcal{C}\varepsilon_{\lambda'}}^{M} \|_{L^2(\Omega)} \log j \right) \\
&\leq 4NM \exp(-\overset{\star}{C} C' (C^*_{\Psi,H_1,H_2})^{-1} \log j)
\end{aligned}$$

Thus, for $C' > C^*_{\Psi,H_1,H_2} / \overset{\star}{C}$, the conclusion follows by Borel-Cantelli Lemma. □

Proof of Proposition IV.4.6. The constant C and C' of Lemmata IV.4.7 and IV.4.8 being deterministic and independent of M, on can choose M large enough such that

$$C - C' M^{\max\{H_1,H_2\}-1} > 0.$$

Let us fix $t \in (0,1)$ and consider $\omega \in \Omega_{t,1} \cap \Omega_{t,2}$, where the events, of probability 1, $\Omega_{t,1}$ and $\Omega_{t,2}$ are given by the same Lemmata. For all $J \in \mathbb{N}$, by Lemma IV.4.7, there exist $j \geq J$ and $\lambda'(j) \subseteq 3\lambda_j(t)$ of scale $j' = j + \lfloor \log NM \rfloor + 2$ such that

$$\left| \widetilde{c_{\lambda'(j)}}^{M}(\omega) \right| \geq C 2^{-j'(H_1+H_2-1)} \log j.$$

If J is large enough, we also have, for all such $j \geq J$, by Lemma IV.4.8,

$$\left| \widetilde{\mathcal{C}c_{\lambda'(j)}}^{M}(\omega) \right| \leq C' M^{\max\{H_1,H_2\}-1} 2^{-j'(H_1+H_2-1)} \log j.$$

From this we deduce that

$$\begin{aligned} d_j(t,\omega) \geq & \left|c_{\lambda'(j)}(\omega)\right| \\ \geq & \left|\widetilde{c_{\lambda'(j)}}^M(\omega)\right| - \left|\widetilde{\mathcal{C}}c_{\lambda'(j)}{}^M(\omega)\right| \\ \geq & 2^{-j'(H_1+H_2-1)} \log j \left(C - C' M^{\max\{H_1,H_2\}-1}\right) \\ \geq & 2^{-j(H_1+H_2-1)} (4NM)^{1-H_1-H_2} \log j \left(C - C' M^{\max\{H_1,H_2\}-1}\right) \end{aligned}$$

Therefore, (IV.4.6) holds true for all $t \in (0,1)$ and $\omega \in \Omega_{t,1} \cap \Omega_{t,2}$. The conclusion follows then from Fubini Theorem. □

IV.4.2 Rapid Points

In this section our aim is to prove the following proposition.

Proposition IV.4.9. *There exists $\Omega_2^* \subset \Omega$ with probability 1 such that, for all $\omega \in \Omega_2^*$, there exist $t \in (0,1)$ such that*

$$\limsup_{j\to+\infty} \frac{d_j(t,\omega)}{2^{-j(H_1+H_2-1)}j} > 0. \tag{IV.4.7}$$

As in the previous subsection, we start by working with the random variables $\widetilde{\varepsilon_\lambda}^M$.

Lemma IV.4.10. *There exists a deterministic constant $C > 0$ such that for all M there is $\Omega_2 \subset \Omega$ with probability 1 such that for all $\omega \in \Omega_2$ there exist $t \in (0,1)$ such that*

$$\limsup_{j\to+\infty} \frac{\left|\widetilde{\varepsilon_{\lambda_j(t)}}^M(\omega)\right|}{j} \geq C. \tag{IV.4.8}$$

Proof. Let us fix $a \in (0,1)$ and $C > 0$ to be chosen later on. For every $(j,l) \in \mathbb{N} \times \left\{0,\dots,\lfloor 2^{j(1-a)}\rfloor - 1\right\}$, we set

$$S_{j,l}^M = \left\{l\lfloor 2^{aj}/(2NM)\rfloor, \dots, (l+1)\lfloor 2^{aj}/(2NM)\rfloor - 1\right\}$$

and consider the event

$$\mathcal{E}_{j,l}^M = \left\{\omega \in \Omega \,:\, \max_{k\in S_{j,l}^M} \left|\widetilde{\varepsilon_{j,2kNM}}^M(\omega)\right| \geq Cj\right\}$$

Let j_0 be the smallest integer such that $\lfloor 2^{aj}/(2NM)\rfloor \geq 1$. If we assume that

$$\Omega_2^* = \bigcup_{J\geq j_0} \bigcap_{j\geq J} \bigcap_{l\in\left\{0,\dots,\lfloor 2^{j(1-a)}\rfloor - 1\right\}} \mathcal{E}_{j,l}^M \tag{IV.4.9}$$

is an event of probability 1 and we consider $\omega \in \Omega_2^*$. For every $j \geq j_0$, denote by

$$G_j^M(\omega) := \left(k \in \{0, \dots, 2^j - 1\} \,:\, \left|\widetilde{\varepsilon_{j,k}}^M(\omega)\right| \geq Cj\right). \tag{IV.4.10}$$

Moreover, for every $n \geq j_0$, one considers

$$O_n^M(\omega) := \bigcup_{j \geq n} U_j^M(\omega), \quad \text{where } U_j^M(\omega) := \bigcup_{k \in G_j^M(\omega)} \left(\frac{k}{2^j}, \frac{k+1}{2^j}\right). \tag{IV.4.11}$$

If one proves that $O_n^M(\omega)$ is dense in $(0,1)$, then by Baire's theorem the set $\cap_{n \geq j_0} O_n^M(\omega)$ is non-empty and let t be an element of this set. Then for every $n \geq j_0$, there is $j \geq n$ such that $\left|\widetilde{\varepsilon_{\lambda_j(t)}}^M(\omega)\right| \geq Cj$, and so desired statement (IV.4.8) is true.
We still have to prove two points:

1. $O_n^M(\omega)$ is dense in $(0,1)$.

2. Ω_2^* is an event of probability 1.

Indeed, starting with statement 1, consider $t \in (0,1)$, $j \geq j_0$ and k such that $\lambda_j(t) = \lambda_{j,k}$. Then, we have two cases:

Case 1 : There is $l \in \{0, \dots, \lfloor 2^{j(1-a)} \rfloor - 1\}$ such that

$$k \in \left\{l\lfloor 2^{aj} \rfloor, \dots, (l+1)\lfloor 2^{aj} \rfloor - 1\right\}$$

Using (IV.4.9) and (IV.4.10), there is $k' \in \{l\lfloor 2^{aj}/(2NM)\rfloor, \dots, (l+1)\lfloor 2^{aj}/(2NM)\rfloor - 1\}$ such that $2k'NM \in G_j(\omega)$. Then, by (IV.4.11),

$$\left(\frac{2NMk'}{2^j}, \frac{2NMk'+1}{2^j}\right) \subset O_n^M(\omega).$$

which is at is at most $2^{-j}\left(\lfloor 2^{aj} \rfloor + 2NM\lfloor 2^{aj}/(2NM)\rfloor\right)$ from t. Finally, we get that t is at a distance at most $22^{j(a-1)}$ of $U_j^M(\omega)$.

Case 2 : $k \in \{\lfloor 2^{j(1-a)} \rfloor \lfloor 2^{ja} \rfloor, \dots, 2^j - 1\}$. Again by (IV.4.9) and (IV.4.10), there is $k' \in S_{j,l}^M$ such that $2k'NM \in G_j^M(\omega)$, and similarly, we get that t is at a distance at most $c2^{j(a-1)}$ of $U_j^M(\omega)$, for some constant $c > 0$ depending only on N, M and a.

Finally, in both cases t is at a distance at most $c2^{j(a-1)}$, and so the density follows.
Now for statement 2, in order to prove that Ω_2^* has a probability 1, it is enough to prove that

$$\mathbb{P}\left(\mathcal{C}\left(\bigcap_{l \in \left\{0, \dots, \lfloor 2^{j(1-a)} \rfloor - 1\right\}} \mathcal{E}_{j,l}^M\right)\right) \tag{IV.4.12}$$

is the general term of a convergent series, then the result follows by Borel-Cantelli Lemma. Note that the variables $\widetilde{\varepsilon_{j,2NMk}}^M$, $k \in S_{j,l}^M$ and $l \in \{0, \dots, \lfloor 2^{j(1-a)} \rfloor - 1\}$, are independent because for every $k \neq k'$, $|2NMk - 2NMk'| \geq 2NM$ and so $\lambda_{j,2MNk}^M \cap \lambda_{j,2MNk'}^M = \emptyset$. Consequently, one has

$$\begin{aligned}
&\mathbb{P}\left(\mathcal{C}\left(\bigcap_{l \in \left\{0,\dots,\lfloor 2^{j(1-a)}\rfloor - 1\right\}} \mathcal{E}_{j,l}^M\right)\right) \\
=&1 - \mathbb{P}\left(\bigcap_{l \in \left\{0,\dots,\lfloor 2^{j(1-a)}\rfloor - 1\right\}} \mathcal{C}\left(\mathcal{E}_{j,l}^M\right)\right) \\
=&1 - \prod_{l \in \left\{0,\dots,\lfloor 2^{j(1-a)}\rfloor - 1\right\}} \left(1 - \prod_{k \in S_{j,l}^M} \mathbb{P}\left(\left|\widetilde{\varepsilon_{j,2NMk}}^M\right| < Cj\right)\right) \\
=&1 - \left(1 - \left(1 - \mathbb{P}\left(|\varepsilon| \geq Cj\right)\right)^{\lfloor 2^{aj}/(2NM)\rfloor}\right)^{\lfloor 2^{j(1-a)}\rfloor} \\
\leq&1 - \exp\left(2^{j(1-a)}\log(1 - p_j)\right) \qquad \text{(IV.4.13)}
\end{aligned}$$

where ε is a random variable belonging to the Wiener chaos of order 2 distributed according to the $(\widetilde{\varepsilon_\lambda})_{\lambda\in\Lambda}$ and $p_j = \left(1 - \mathbb{P}\left(|\varepsilon| \geq Cj\right)\right)^{\lfloor 2^{aj}/(2NM)\rfloor}$. Remark that p_j is a positive term that tends to 0 as $j \to +\infty$. Indeed, using the fact that $\log(1-x) \leq -x$ if $x \in (0,1)$ together with (IV.4.4), there exists $J \in \mathbb{N}$ such that for all $j \geq J$,

$$\begin{aligned}
0 \leq p_j &\leq \left(1 - \exp\left(-C\, c_2\, j\right)\right)^{\left\lfloor 2^{aj}/(2NM)\right\rfloor} \\
&\leq \exp\left(-\left\lfloor \frac{2^{aj}}{2NM} \right\rfloor \exp\left(-C\, c_2\, j\right)\right) \\
&\leq \exp\left(-C' \exp\left(\log 2^{aj}\right) \exp\left(-C\, c_2\, j\right)\right) \\
&\leq \exp\left(-C' \exp j(a\log 2 - C\, c_2)\right) \qquad \text{(IV.4.14)}
\end{aligned}$$

where C' depends only on N, M and a and c_2 is the constant given in (IV.4.4). It is enough to choose C such that $a\log 2 - C\, c_2 > 0$ to deduce that and so $p_j \to 0$ as $j \to +\infty$. Similarly, one can get for all $j \geq J$

$$0 \leq 2^{j(1-a)} p_j \leq \exp\left(-C' \exp j(\log 2 - C\, c_2)\right)$$

which indeed shows that $2^{j(1-a)}p_j$ tends to 0 as $j \to +\infty$. Now, using the fact that $\log(1-x) = -x + o(x)$ and $\exp(x) = 1 + x + o(x)$ as $x \to 0$, together with (IV.4.13) we obtain that for all $\delta > 0$

$$\mathbb{P}\left(\mathcal{C}\left(\bigcap_{l \in \left\{0,\dots,\lfloor 2^{j(1-a)}\rfloor - 1\right\}} \mathcal{E}_{j,l}^M\right)\right) \leq 2^{j(1-a)}\left(\delta(p_j + \delta p_j) + p_j + \delta p_j\right)$$

for j large enough. Using the upper bound in (IV.4.14), one can finally conclude that (IV.4.12) is indeed the general term of a convergent series. □

Concerning the random variable $\widetilde{\mathcal{C}\varepsilon_\lambda}^M$, one can give an almost sure upper bound.

Lemma IV.4.11. *There exists a deterministic constant $C' > 0$ such that for all M there is $\Omega_2' \subset \Omega$ with probability 1 such that for all $\omega \in \Omega_2'$ there exist $J \in \mathbb{N}$ such that, for all $j \geq J$, for all $\lambda \in \Lambda_j$, $\lambda \subseteq [0,1]$,*

$$\left|\widetilde{\mathcal{C}\varepsilon_\lambda}^M(\omega)\right| \leq C' M^{\max\{H_1,H_2\}-1} j$$

Proof. If $C' > 0$, for all j sufficiently large, we have, by Theorem IV.3.11

$$\begin{aligned}
&\mathbb{P}\left(\exists \lambda \in \Lambda_j,\, \lambda \subseteq [0,1] \,:\, \left|\widetilde{\mathcal{C}\varepsilon_\lambda}^M(\omega)\right| \geq C' M^{\max\{H_1,H_2\}-1} j\right) \\
&\leq \sum_{\lambda \in \Lambda_j\, \lambda \subseteq [0,1]} \mathbb{P}\left(\left|\widetilde{\mathcal{C}\varepsilon_\lambda}^M(\omega)\right| \geq C' M^{\max\{H_1,H_2\}-1} j\right) \\
&\leq 2^j \exp(-\overset{\star}{C} C' (C^*_{\Psi,H_1,H_2})^{-1} j)
\end{aligned}$$

and thus, if $C' > \log(2) C^*_{\Psi,H_1,H_2} / \overset{\star}{C}$, the conclusion follows by Borel-Cantelli Lemma. □

Proof of Proposition IV.4.9. Again, one can choose M large enough such that

$$C - C' M^{\max\{H_1,H_2\}-1} > 0,$$

where C and C' are the constant given by Lemmata IV.4.10 and IV.4.11 respectively. Let us consider $\omega \in \Omega_2^* := \Omega_2 \cap \Omega_2'$ where the evnets, of probability 1, Ω_2 and Ω_2' are giving by the same Lemmata. We use the same notations as in them. First there exist $t \in (0,1)$ such that for all $J \in \mathbb{N}$ there exist $j \geq n$ such that

$$\left|\widetilde{c_{\lambda_j(t)}}^M(\omega)\right| \geq C\, j 2^{-j(H_1+H_2-1)}. \tag{IV.4.15}$$

Moreover, if J is large enough, for all such j we also have

$$\left|\widetilde{\mathcal{C}c_{\lambda_j(t)}}^M(\omega)\right| \leq C'\, M^{\max\{H_1,H_2\}-1} 2^{-j(H_1+H_2-1)} j. \tag{IV.4.16}$$

In this case, as in IV.4.6 we have that for all J great enough, there is $j \geq J$ such that

$$d_j(t,\omega) \geq 2^{-j(H_1+H_2-1)} j \left(C - C' M^{\max\{H_1,H_2\}-1}\right)$$

and so one can conclude that (IV.4.7) holds true for all $\omega \in \Omega_2^*$. □

IV.5 Proof of the main Theorem

Theorem IV.1.2 is then a straightforward consequence of Propositions IV.3.3, IV.3.16, IV.3.26, IV.4.6 and IV.4.9.

Proof of Theorem IV.1.2. Let us denote by Ω_R the event obtained by taking the intersection of all the events of probability 1 induced by Propositions IV.3.3, IV.3.16, IV.3.26, IV.4.6 and IV.4.9.

If we consider ω belonging to this event of probability 1, first, from Proposition IV.3.3 there exists $C_R > 0$ such that, for all $t, s \in (0,1)$

$$|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)| \leq C_R |t-s|^{H_1+H_2-1} \log|t-s|^{-1} \tag{IV.5.1}$$

while, for almost every $t_o \in (0,1)$, from Propositions IV.3.16 and IV.4.6

$$0 < \limsup_{s \to t_o} \frac{|R_{H_1,H_2}(t_o,\omega) - R_{H_1,H_2}(s,\omega)|}{|t_o - s|^{H_1+H_2-1} \log\log|t_o - s|^{-1}} < +\infty.$$

Nevertheless, from Proposition IV.4.9 we also know that there exists $t_r \in (0,1)$ such that

$$0 < \limsup_{s \to t_r} \frac{|R_{H_1,H_2}(t_r,\omega) - R_{H_1,H_2}(s,\omega)|}{|t_r - s|^{H_1+H_2-1} \log|t_r - s|^{-1}}$$

which, combined with (IV.5.1), gives that, for all such a t_r,

$$0 < \limsup_{s \to t_r} \frac{|R_{H_1,H_2}(t_r,\omega) - R_{H_1,H_2}(s,\omega)|}{|t_r - s|^{H_1+H_2-1} \log|t_r - s|^{-1}} < +\infty.$$

Moreover, from Proposition IV.3.26, we also know that one can find $t_\sigma \in (0,1)$ such that

$$\limsup_{s \to t_\sigma} \frac{|R_{H_1,H_2}(t_\sigma,\omega) - R_{H_1,H_2}(s,\omega)|}{|t_\sigma - s|^{H_1+H_2-1}} < +\infty.$$

The conclusion follows by Remark 14. □

Remark 20. Unfortunately, our method does not allow us to affirm the positiveness of the limit (IV.1.7), at the opposite of limits (IV.1.5) and (IV.1.6). Indeed, as for almost every $\omega \in \Omega$

$$\limsup_{s \to t} \frac{|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)|}{|t - s|^{H_1+H_2-1}}$$

is finite for *some* t, we would need to show its positiveness for *all* t and thus the positiveness of the limit

$$\limsup_{j \to +\infty} \frac{d_j(t,\omega)}{2^{-j(H_1+H_2-1)}} \tag{IV.5.2}$$

for all t.

Concerning the random variables $(\widetilde{\varepsilon_\lambda}^M)_\lambda$, one can obtain a positive result[6]. Indeed, from [53, Theorem 6.9 and Remark 6.10] we know that there exists an universal deterministic constant $\gamma \in [0,1)$ such that, for each random variable X in the Wiener chaos of order 2

$$\mathbb{P}\left(|X| \leq \frac{1}{2}\|X\|_2\right) \leq \gamma.$$

[6]This result is again a generalization of [8, Lemma 3.3.] where most of the modifications comes from the fact that we are working in the Wiener chaos of order 2

As $0 \leq \gamma < 1$, of course, one can find $\ell_0 \in \mathbb{N}$ such that

$$\gamma^{\ell_0} < 2^{-1}. \tag{IV.5.3}$$

Let us go back to the construction starting the proof of Lemma IV.4.7. If the dyadic interval $\lambda_{j,k}$ and $m \in \mathbb{N}$ are fixed and $S \in \mathcal{S}_{j,k,m}$ we define the sequences of dyadic intervals $(I_n)_{0\leq n\leq m}$ and $(T_n)_{1\leq n\leq m}$ in the same way: $I_0 = \lambda_{j,k}$, $I_m = S$ and, for all $1 \leq n \leq m$, $I_{n-1} = I_n \cup T_n$. Now, for any $1 \leq n \leq m$, there are ℓ_0 dyadic intervals $(T_n^\ell = \lambda_{j_n^{(\ell)},k_n^{(\ell)}})_{1\leq \ell\leq \ell_0}$ in $S_{T_n,\lfloor \log_2(\ell_0 NM)\rfloor+2}$ such that, for all $1 \leq \ell \leq \ell_0$

$$\left(\frac{k_n^{(\ell)} - NM}{2^{j_n^{(\ell)}}}, \frac{k_n^{(\ell)} + N}{2^{j_n^{(\ell)}}}\right) \subseteq T_n$$

and, if $\ell \neq \ell'$, $T_n^\ell \cap T_n^{\ell'} = \emptyset$. Therefore, the dyadic intervals $(T_n^\ell)_{1\leq n\leq m,1\leq \ell\leq \ell_0}$ satisfy condition (C_M) in Definition IV.4.3. From this, for all $S \in \mathcal{S}_{j,k,m}$ we define the Bernouilli random variable

$$\mathcal{B}_{j,k,m}(S) = \prod_{1\leq n\leq m, 1\leq \ell\leq \ell_0} \mathbf{1}_{\{|\widetilde{\varepsilon_{T_n^\ell}}^M| < 2^{-1}C_{\Psi,H_1,H_2}\}}$$

for which, by Proposition IV.4.5, we have, using the independence of the random variables $(\widetilde{\varepsilon_{T_n^\ell}}^M)_{1\leq n\leq m,1\leq \ell\leq \ell_0}$, $\mathbb{E}[\mathcal{B}_{j,k,m}(S)] \leq \gamma^{m\ell_0}$. Therefore, if we define the random variable

$$\mathcal{G}_{j,k,m} = \sum_{S\in\mathcal{S}_{j,k,m}} \mathcal{B}_{j,k,m}(S)$$

then $\mathbb{E}[\mathcal{G}_{j,k,m}] \leq (2\gamma^{\ell_0})^m$ and it follows from inequality (IV.5.3) and Fatou Lemma that

$$\mathbb{E}\left[\liminf_{m\to+\infty} \mathcal{G}_{j,k,m}\right] = 0.$$

As a consequence,

$$\Omega_1 = \bigcap_{j\in\mathbb{N}, 0\leq k<2^j} \{\omega \,:\, \liminf_{m\to+\infty} \mathcal{G}_{j,k,m}(\omega) = 0\}$$

is an event of probability 1.

Now if $\omega \in \Omega_1$ and $t \in (0,1)$, we take $j \in \mathbb{N}$ and $k = k_j(t)$ and since, for all m, $\mathcal{G}_{j,k_j(t),m}$ has values in $\{0,\ldots,2^m\}$ we conclude that there are infinitely many m for which, for every $S \in \mathcal{S}_{j,k_j(t),m}$, $\mathcal{B}_{j,k,m}(S) = 0$. Considering such a m and $S = \lambda_{j+m}(t)$ then we first remark that, for all $1 \leq n \leq m$, $I_n = \lambda_{j+n}(t)$ and thus $T_n \in 3\lambda_{j+n}(t)$. Now, as $\mathcal{B}_{j,k,m}(\lambda_{j+m}(t)) = 0$, one can find $1 \leq n \leq m$ and $1 \leq \ell \leq \ell_0$ such that

$$|\widetilde{\varepsilon_{T_n^\ell}}^M(\omega)| \geq 2^{-1}C_{\Psi,H_1,H_2}.$$

Thus we have showed that, for all $\omega \in \Omega_1$ and $t \in (0,1)$ there exist infinitely many $j' \in \mathbb{N}$ such that

$$\max_{\substack{\lambda' \in S_{\lambda,\lfloor \log_2(\ell_0 NM)\rfloor+2} \\ \lambda \in 3\lambda_{j'}(t)}} \left|\widetilde{\varepsilon_{\lambda'}}^M(\omega)\right| \geq 2^{-1}C_{\Psi,H_1,H_2}.$$

To pass to the wavelet leaders, in the spirit of Propositions IV.4.6 and IV.4.9, we would need to get from Borel-Cantelli Lemma an upper bound of

$$\max_{\substack{\lambda' \in S_{\lambda,\lfloor \log_2(\ell_0 NM)\rfloor+2} \\ \lambda \in 3\lambda_{j'}(t)}} \left|\widetilde{\mathcal{C}\varepsilon_{\lambda'}}^M(\omega)\right|$$

for *all* j sufficiently large on an event of probability 1 which does not depend on t. Then, as

$$\mathbb{P}\left(\exists \lambda \in \Lambda_j,\, \lambda \subseteq [0,1] \,:\, \max_{\substack{\lambda'' \in S_{\lambda',\lfloor \log_2(\ell_0 NM)\rfloor+2} \\ \lambda' \in 3\lambda}} \left|\widetilde{\mathcal{C}\varepsilon_{\lambda'}}^M(\omega)\right| \geq C' M^{\max\{H_1,H_2\}-1} j\right)$$
$$\leq 2^j 4\ell_0 NM \exp(-\overset{\star}{C} C' (C^*_{\Psi,H_1,H_2})^{-1} j),$$

if $C' > \log(2) C^*_{\Psi,H_1,H_2}/\overset{\star}{C}$ this probability is the general term of some convergent series and in this case one can affirm the existence of an event Ω_1' of probability 1 such that, for all $\omega \in \Omega_1'$ there exist $J \in \mathbb{N}$ such that, for all $j \geq J$, for all $\lambda \in \Lambda_j$, $\lambda \subseteq [0,1]$,

$$\max_{\substack{\lambda'' \in S_{\lambda',\lfloor \log_2(\ell_0 NM)\rfloor+2} \\ \lambda' \in 3\lambda}} \left|\widetilde{\mathcal{C}\varepsilon_{\lambda'}}^M(\omega)\right| \leq C' M^{\max\{H_1,H_2\}-1} j.$$

It seems to be the sharper upper bound that we can hope to find with our constraints and the fact that we don't have any independence property to take advantage of when dealing with the random variables $\widetilde{\mathcal{C}\varepsilon_\lambda}^M$. This is insufficient to consider properly limit (IV.5.2). Nevertheless, if, instead of working with an uniform constant M we make it depends on the scale j by setting $M_j = (4C'C^{-1}_{\Psi,H_1,H_2} j)^{\frac{1}{1-\max\{H_1,H_2\}}}$, where $C' > \log(2) C^*_{\Psi,H_1,H_2}/\overset{\star}{C}$ is the same constant as in Lemma IV.4.11,

$$\lambda_{j,k}^{M_j} := \left]\frac{k - NM_j}{2^j}, \frac{k+N}{2^j}\right]^2,$$

$$\widetilde{c_{j,k}}^{M_j} = c_{H_1,H_2} \int'_{\lambda_{j,k}^{M_j}} \int_{-N}^{N} \Psi(x) \int_{\frac{k}{2^j}}^{\frac{x+k}{2^j}} f_{H_1,H_2}(s,x_1,x_2)\, ds\, dx\, dB(x_1)\, dB(x_2)$$

and

$$\widetilde{\mathcal{C}c_{j,k}}^{M_j} = c_{j,k} - \widetilde{c_{j,k}}^{M_j}$$

then Proposition IV.4.5 stills holds if we replace M by M_j with j sufficiently large and, by directly adapting what precedes one can find on event Ω_1^* of probability 1 such that, for all $\omega \in \Omega_1^*$ and $t \in (0,1)$ there exist infinitely many $j \in \mathbb{N}$ such that[7]

$$\max_{\substack{\lambda' \in S_{\lambda,\lfloor \log_2(\ell_0 NM_j)\rfloor+2} \\ \lambda \in 3\lambda_j(t)}} \left|\widetilde{\varepsilon_{\lambda'}}^{M_j}(\omega)\right| \geq 2^{-1} C_{\Psi,H_1,H_2}.$$

[7]The random variables $\widetilde{\varepsilon_{\lambda'}}^{M_j}$ and $\widetilde{\mathcal{C}\varepsilon_{\lambda'}}^{M_j}$ are defined in an obvious way.

while there exist $J \in \mathbb{N}$ such that, for all $j \geq J$, for all $\lambda \in \Lambda_j$, $\lambda \subseteq [0,1]$,

$$\max_{\substack{\lambda'' \in S_{\lambda', \lfloor \log_2(\ell_0 N M_j) \rfloor + 2} \\ \lambda' \in 3\lambda}} \left| \widetilde{\mathcal{C}\varepsilon_{\lambda'}}^{M_j}(\omega) \right| \leq C'(M_j)^{\max\{H_1,H_2\}-1} j$$

$$\leq 4^{-1} C_{\Psi,H_1,H_2}.$$

As a consequence, as in Proposition IV.4.6, for all $J \in \mathbb{N}$ there exist $j \geq J$ with

$$d_j(t,\omega) \geq 2^{-j(H_1+H_2-1)} (4C' C_{\Psi,H_1,H_2}^{-1} j)^{\frac{1-H_1-H_2}{1-\max\{H_1,H_2\}}} (4\ell_0 N)^{1-H_1-H_2} 4^{-1} C_{\Psi,H_1,H_2}$$

which allows to state that, for all $t \in (0,1)$ and $\omega \in \Omega_1$,

$$\limsup_{j \to +\infty} \frac{d_j(t,\omega)}{2^{-j(H_1+H_2-1)} j^{\frac{1-H_1-H_2}{1-\max\{H_1,H_2\}}}} > 0$$

, and thus, for all $\omega \in \Omega_1$ and for all $t \in (0,1)$,

$$\limsup_{s \to t} \frac{|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)|}{|t-s|^{H_1+H_2-1} (\log |t-s|^{-1})^{\frac{1-H_1-H_2}{1-\max\{H_1,H_2\}}}} > 0.$$

In particular, we find an almost sure uniform lower modulus of continuity for the generalized Rosenblatt process, similar to the one established in [57] for the Rosenblatt process. However, we are not able to judge the optimality of this modulus, which seems to be a difficult problem, as already stated in [6, Remark 1.2].

An interesting corollary of Remark 20 and Proposition IV.3.3 is the fact that, almost surely, the pointwise Hölder exponent of the generalized Rosenblatt process is everywhere $H_1 + H_2 - 1$ and, in particular, it is nowhere differentiable.

Similarly, one can also take $(M_j = (4C' C_{\Psi,H_1,H_2}^{-1} \log(j)^{\frac{1}{1-\max\{H_1,H_2\}}})_j$, where C' is this time the same constant that in Lemma IV.4.8 and show, precisely like in this Lemma, that there exists a deterministic constant $C' > 0$ such that, for all $t \in (0,1)$ there exists $\Omega_{t,2} \subset \Omega$ with probability 1 such that for all $\omega \in \Omega_{t,2}$ there exist $J \in \mathbb{N}$ such that, for all $j \geq J$,

$$\max_{\substack{\lambda' \in S_{\lambda, \lfloor \log_2(\ell_0 N M_j) \rfloor + 2} \\ \lambda \in 3\lambda_j(t)}} \left| \widetilde{\mathcal{C}\varepsilon_{\lambda'}}^{M_j}(\omega) \right| \leq 4^{-1} C_{\Psi,H_1,H_2}.$$

and conclude in the same way that there exists an event of probability 1 such that, for all ω in this event and for almost every $t \in (0,1)$

$$\limsup_{s \to t} \frac{|R_{H_1,H_2}(t,\omega) - R_{H_1,H_2}(s,\omega)|}{|t-s|^{H_1+H_2-1} (\log\log |t-s|^{-1})^{\frac{1-H_1-H_2}{1-\max\{H_1,H_2\}}}} > 0.$$

Chapter V

Potential methods and projection theorems for macroscopic Hausdorff dimension

The content of this chapter is a copy of the paper entitled "Potential methods and projection theorems for macroscopic Hausdorff dimension", written with "Stéphane Seuret", and to be submitted soon.

V.1 Introduction

Fractal geometry provides a general framework for studying sets possessing either irregular or self-reproducing (deterministic or random, self-similar or self-affine) properties. Most definitions of fractal dimensions of sets included in $\mathbb{R}^d$ are based on the local properties (also known as microscopic) of the set. Taking into consideration that many statistical physics models are built on discrete spaces, Barlow and Taylor [12, 11] introduced a new notion of dimension to study unbounded "fractal-like" sets on discrete space. This so-called macroscopic Hausdorff dimension (see Definition V.2.2 below) has proved to be useful in quantifying the behavior at infinity of several objects, beyond the transient range of random walks in $\mathbb{Z}^d$ which was the original motivation of Barlow and Taylor in [12].

Macroscopic Hausdorff dimension is actually defined for every set (not only discrete) in $\mathbb{R}^d$ [12]. It is a discrete analog of Hausdorff dimension, and the word macroscopic comes from the fact that this dimension ignores the local structure of the sets. At the same time, the macroscopic Hausdorff dimension assesses the asymptotic behavior at infinity of the sets, so it is very relevant when one is interested in the description of infinite objects, how they fill the space "at large scale". The macroscopic Hausdorff dimension was a key tool used by Xiao et Zheng [132] in studying the range of a random walk in random environment. It is related to [61] where Khoshnevisan and Xiao are concerned with the macroscopic geometry of other random sets. In [59], Khoshnevisan, Kim and Xiao found out a multifractal behavior for the macroscopic dimension of tall peaks of solutions to stochastic PDEs. Georgiou *et el* [46] solved Barlow and Taylor question [12, Problem, p. 145] by qualifying the range of an arbitrary transient random walk. The macroscopic Hausdorff dimension was also useful for studying the large scale structure of sojourn sets associated to the Brownian motion [105], the fractional Brownian motion [87, 28], and the Rosenblatt process [29].

In this paper we are interested in building various methods for estimating the macroscopic Hausdorff dimension. Recalling the fact that macroscopic Hausdorff dimension is a discrete analog of the Hausdorff dimension, we start by stating the estimating methods used for the Hausdorff dimension. In most cases, when estimating the Hausdorff dimension of a set F, the difficult part consists in finding a suitable lower bound for $\dim_H(F)$. Various methods exist to find lower bounds for the standard Hausdorff dimension, and it is a natural question to ask whether these methods have their counterparts for the macroscopic Hausdorff dimension. The two usual techniques are the mass distribution principle and the potential theoretic method.

The mass distribution principle, see for instance [38, page 67], states that if a set $F \subset \mathbb{R}^d$ and a Borel finite measure μ are such that $\mu(F) = 1$ and $\mu(B(x,r)) \leq Cr^s$ for every $x \in \mathbb{R}^d$ and $r > 0$, then the s-dimensional Hausdorff measure $\mathcal{H}^s(F)$ is larger than $\mu(F)/C$, and so F has at least Hausdorff dimension s.

The potential theoretic method is based on an integral analysis: if for some probability measure μ, $\mu(F) = 1$ and the integral $\displaystyle\iint_{(\mathbb{R}^d)^2} \frac{d\mu(x)d\mu(y)}{\|x-y\|_2^s}$ is finite, then again F has at least Hausdorff dimension s. In addition to bounding the Hausdorff dimension from below, the potential theoretic method plays a key role in proving the projection theorem.

The first aim of this paper is to establish similar results for the macroscopic Hausdorff dimension. This happens to be very easy for the mass distribution principle, and follows essentially from previous works. It is much more challenging for the potential theoretic method, and a careful analysis is needed.

As an application of the new potential theoretic method, we obtain a Marstrand-like projection theorem, describing the dimension of almost all projections on lines of sets $F \in \mathbb{R}^2$. Dealing with the dimensions of projections of Borel sets is a line of research that has a long history. It started with the investigation by Marstrand [77] of the projection theorem associated to the Hausdorff dimension. He dealt with orthogonal projections on linear subspaces and proved that

$$\text{for every Borel set } E \subset \mathbb{R}^2, \quad \dim_H(\text{proj}_V E) = \min\{\dim_H E, 1\}$$

for almost every 1-dimensional subspaces V, where proj_V denotes the orthogonal projection onto V and $\dim_H E$ denotes the Hausdorff dimension of E. Afterwards Marstrand's results was proved by Kaufman but using potential theoretic methods [56]. Subsequently in 1975 Mattila extended these results to Borel sets $E \subset \mathbb{R}^n$ and almost all V in the Grassmannian $G(n,m)$[78]. We prove analog results for the macroscopic Hausdorff dimension, using the potential theory method we developed above.

V.2 Definitions and statements of the results

Here and in the reset of the paper, let $(\mathbb{R}^d, \|.\|_2)$ be the d-dimensional Euclidean space equipped with the L^2- norm.

V.2.1 The macroscopic Hausdorff dimension

For $x \in \mathbb{R}^d$ and $r > 0$, $B(x, r)$ denotes the Euclidean ball with center x and radius r. For $E \subset \mathbb{R}^d$, the diameter of a set E is denoted by $|E|$.

Let us recall the definition of the Barlow-Taylor macroscopic Hausdorff dimension $\mathrm{Dim}_H(E)$ of a set $E \subseteq \mathbb{R}^d$, developed in [11, 12].

Define, for all integer $n \in \mathbb{N}$, the n-th shell of $\mathbb{R}^d$ by

$$S_0 = B(0,1) \quad \text{and} \quad S_n := B(0, 2^n) \setminus B(0, 2^{n-1}) \text{ for all } n \geq 1. \tag{V.2.1}$$

Like the standard Hausdorff dimension, the macroscopic Hausdorff dimension $\mathrm{Dim}_H(E)$ aims at describing how a set E can be efficiently covered by balls. Since Dim_H is concerned only with large scale behaviors, Barlow and Taylor proposed to study the covers of the intersections $E \cap S_n$ by balls, for every $n \in \mathbb{N}$, and the balls used to cover the sets $E \cap S_n$ will all be of diameter at least 1. Again this is justified by the fact that this dimension is supposed to describe discrete sets (so small balls are not relevant).

To this end, let us introduce, for $E \subseteq \mathbb{R}^d$, the set of *covers* of E restricted to S_n defined by

$$\widetilde{\mathcal{C}}_n(E) = \left\{ \{B(x_i, r_i)\}_{i=1}^m : \; m \in \mathbb{N},\, x_i \in S_n,\, r_i \geq 1,\, E \cap S_n \subset \textstyle\bigcup_{i=1}^m B(x_i, r_i) \right\}.$$

Finally, for $s \geq 0$ and $n \in \mathbb{N}$, set

$$\widetilde{\nu}_n^s(E) = \inf \left\{ \sum_{i=1}^m \left(\frac{r_i}{2^n} \right)^s : \{B_i = B(x_i, r_i)\}_{i=1}^m \in \widetilde{\mathcal{C}}_n(E) \right\}. \tag{V.2.2}$$

Observe that $\widetilde{\nu}_n^s$ is sub-additive, i.e. $\widetilde{\nu}_n^s(A \cup B) \leq \widetilde{\nu}_n^s(A) + \widetilde{\nu}_n^s(B)$ for every sets A and B, but is not a measure (because of the constraints on r_i).

Definition V.2.1. When $\widetilde{\nu}_n^s(E) = \sum_{i=1}^m \left(\frac{r_i}{2^n} \right)^s$ and $E \cap S_n \subset \bigcup_{i=1}^m B(x_i, r_i)$, the finite family of balls $\{B_i = B(x_i, r_i)\}_{i=1}^m$ is called an s-optimal cover of $E \cap S_n$.

The existence of optimal covers is not guaranteed. We will deal with this issue in Section V.3.

We are now ready to define the Barlow-Taylor macroscopic Hausdorff dimension.

Definition V.2.2. For every $s \geq 0$ and $E \subset \mathbb{R}^d$, define

$$\widetilde{\nu}^s(E) = \sum_{n\geq 1} \widetilde{\nu}_n^s(E).$$

The macroscopic Hausdorff dimension of $E \subset \mathbb{R}^d$ is defined by

$$\mathrm{Dim}_H\,(E) = \inf\left\{s \geq 0 : \widetilde{\nu}^s(E) < +\infty\right\}. \tag{V.2.3}$$

One easily checks that $\mathrm{Dim}_H\,(E) \in [0,d]$ for all $E \subset \mathbb{R}^d$, that $\mathrm{Dim}_H\,(E) = 0$ when E is bounded, and that an alternative definition for $\mathrm{Dim}_H\,(E)$ is

$$\mathrm{Dim}_H\,(E) = \sup\left\{s \geq 0 : \widetilde{\nu}^s(E) = +\infty\right\},$$

where $\sup \emptyset = 0$ by convention. It is also standard that $\mathrm{Dim}_H\,(f(E)) \leq \mathrm{Dim}_H\,(E)$ for every Lipschitz mapping $f : \mathbb{R}^d \to \mathbb{R}^d$.

A key ingredient when working with the standard Hausdorff dimension is the existence of s-sets, i.e. sets $E \subset \mathbb{R}^d$ with Hausdorff dimension $\dim_H(E) = s$ and such that its s-Hausdorff measure $\mathcal{H}^s(E)$ is finite. We introduce a similar notion for the macroscopic Hausdorff dimension.

Definition V.2.3. Let $s \geq 0$. A set $E \subset \mathbb{R}^d$ is called a macroscopic s-set when $\mathrm{Dim}_H\,(E) = s$ and $\widetilde{\nu}^s(E) < +\infty$.

We prove the existence of macroscopic s-sets.

Theorem V.2.4. *Let $E \subset \mathbb{R}^d$ be such that $\widetilde{\nu}^s(E) = +\infty$. Then there exists a macroscopic s-set $\widetilde{E}$ such that $\widetilde{E} \subset E$.*

This extraction theorem is a key ingredient at various places in our proofs.

V.2.2 Methods to find lower bounds for $\mathrm{Dim}_H\,(E)$

For every set B and every measure μ, $\mu_{|B}$ stands for the restriction of μ on B, i.e. $\mu_{|B}(A) = \mu(A \cap B)$.

As recalled above, the mass distribution principle is a powerful, albeit simple, tool allowing to find a lower bound of the Hausdorff dimension by considering measures supported on the set, see [38, page 67]. We prove a similar result for the macroscopic Hausdorff dimension Dim_H.

Proposition V.2.5 (Macroscopic mass distribution principle)**.** *Let E be a Borel subset of $\mathbb{R}^d$ and $s > 0$. Suppose that there exists a Radon measure μ on $\mathbb{R}^d$ such that $\mu(E) = +\infty$ and a constant $c > 0$ such that for all $n \in \mathbb{N}$, $x \in S_n$ and $1 \leq r \leq 2^n$,*

$$\mu_{|S_n}(B(x,r)) \leq c\left(\frac{r}{2^n}\right)^s.$$

Then, for all $n \in \mathbb{N}$, $\widetilde{\nu}_n^s(E) \geq \dfrac{\mu_{|S_n}(E)}{c}$ and $\mathrm{Dim}_H\,(E) \geq s$.

The proof of the macroscopic mass distribution principle is not complicated. Although it was not exactly stated before as we write it, it essentially follows directly from previous results, and so it is not so innovative.

This is not the case for the potential method below. Let us first introduce the macroscopic s-energy of a measure.

Definition V.2.6. Let $s \geq 0$, and let μ be a finite mass distribution on $\mathbb{R}^d$. The macroscopic (μ, s)-potential at a point x is defined as

$$\phi_\mu^s(x) := \int_{\mathbb{R}^d} \frac{d\mu(y)}{\|x-y\|_2^s \vee 1}. \tag{V.2.4}$$

The macroscopic s-energy of μ is

$$I_s(\mu) := \int_{\mathbb{R}^d} \phi_\mu^s(x) d\mu(x) = \iint_{(\mathbb{R}^d)^2} \frac{d\mu(x)d\mu(y)}{\|x-y\|_2^s \vee 1}. \tag{V.2.5}$$

In the case of standard Hausdorff dimension, in the integrals (V.2.4) and (V.2.5), the quantity $\|x-y\|_2^s \vee 1$ is simply $\|x-y\|_2^s$. This modification is justified by the fact that Dim_H is not concerned with local behavior, so we are not interested in small interactions $\|x-y\|_2 < 1$.

Theorem V.2.7. *Let E be a subset of $\mathbb{R}^d$.*

1. *If there exists a Radon measure μ on $\mathbb{R}^d$ such that $\mu(E) = +\infty$ and if*

$$\sum_{n \geq 0} 2^{ns} I_s(\mu_{|S_n}) < +\infty,$$

then $\widetilde{\nu}^s(E) = +\infty$ and $Dim_H(E) \geq s$.

2. *If $\widetilde{\nu}^s(E) = +\infty$, then for all $0 < \varepsilon < s$ there exists a Radon measure μ^ε on $\mathbb{R}^d$ such that $\mu^\varepsilon(E) = +\infty$ and $\sum_{n \geq 0} 2^{n(s-\varepsilon)} I_{s-\varepsilon}(\mu^\varepsilon_{|S_n}) < +\infty$.*

The potential theoretic methods we demonstrated in Theorem V.2.7 are very comparable to the ones established for the standard Hausdorff dimension [38, Theorem 4.13]. Unlike the standard Hausdorff dimension case, for the macroscopic Hausdorff dimension, we consider the measure μ is define on $\mathbb{R}^d$, and we focus on the restriction of μ on every annulus S_n. For this reason, we deal with sums over n.

V.2.3 Application to projections

Projection theorems for Hausdorff dimensions have recently regained a lot of attention after some breakthroughs by M. Hochman and P. Shmerkin [47] and others, who used these theorems to tackle many longstanding questions in geometric measure theory and dynamical systems. It is quite satisfactory that they have natural counterparts in terms of macroscopic Hausdorff dimensions, as stated in the following theorem.

Theorem V.2.8. *Let $E \subset \mathbb{R}^2$ be a Borel set. Define L_θ as the straight line passing through 0 with angle θ, and $proj_\theta E$ as the orthogonal projection of E onto L_θ.*

(a) If $Dim_H(E) < 1$, then $Dim_H(proj_\theta E) = Dim_H(E)$ for Lebesgue almost every $\theta \in [0,\pi]$.

(b) If $Dim_H(E) \geq 1$, then $Dim_H(proj_\theta E) = 1$ for Lebesgue almost every $\theta \in [0,\pi]$.

As in the standard Hausdorff dimension case, the proof is based on a subtle use of the potential method and Theorem V.2.7.

It can be expected that Theorem V.2.8 can be extended in higher dimensional spaces, and that both Theorem V.2.7 and Theorem V.2.8 are useful in other situations that the one we describe here.

The structure of the paper is as follows. The main three results, Theorems V.2.4, V.2.7 and V.2.8 are established in Sections V.4, V.5, and V.6 respectively. Some necessary technical properties of the macroscopic Hausdorff dimension are proved in Section V.3.

V.3 First properties of Macroscopic Hausdorff Dimension

V.3.1 An alternative definition for the macroscopic Hausdorff dimension

We will use an alternative, easier to handle with, definition for the macroscopic Hausdorff dimension, based on a simple modification of the $\widetilde{\nu}_n^s$ quantities. We restrict ourselves to covers centered on integer points, with integer radii. We show that, up to a constants, this does not modify the values of the quantities involved in the computations, and the value of the macroscopic Hausdorff dimension is left unchanged.

We introduce for $E \subseteq \mathbb{R}^d$ and $n \geq 0$, the set of *proper covers* of E restricted to S_n by

$$\mathcal{C}_n(E) = \left\{ \{B(x_i,r_i)\}_{i=1}^m \ : \ m \in \mathbb{N},\, x_i \in \mathbb{Z}^d \cap S_n,\, r_i \in \mathbb{N}^*,\, E \cap S_n \subset \bigcup_{i=1}^m B(x_i,r_i) \right\}.$$

Definition V.3.1. For every $s \geq 0$, $n \geq 0$ and $E \subset \mathbb{R}^d$, define

$$\nu_n^s(E) = \inf \left\{ \sum_{i=1}^m \left(\frac{r_i}{2^n}\right)^s \,:\, \{B_i = B(x_i,r_i)\}_{i=1}^m \in \mathcal{C}_n(E) \right\} \tag{V.3.1}$$

and

$$\nu^s(E) = \sum_{n \geq 1} \nu_n^s(E). \tag{V.3.2}$$

Due to the fact that the x_i are (multi)-integers, as well as the r_i, the above infimum (V.3.1) in $\nu_n^s(E)$ is reached for some cover $\{B_i = B(x_i, r_i)\}_{i=1}^m \in \mathcal{C}_n(E)$.

Observe that ν_n^s is still sub-additive, i.e. $\nu_n^s(A \cup B) \leq \nu_n^s(A) + \nu_n^s(B)$ for every sets A and B.

Lemma V.3.2. *For every $n \geq 0$, every set $E \subset \mathbb{R}^d$, one has*

$$\widetilde{\nu}_n^s(E) \leq \nu_n^s(E) \leq (2 + \sqrt{d})^s \widetilde{\nu}_n^s(E). \tag{V.3.3}$$

In particular, one still has

$$Dim_H(E) = \inf \{s \geq 0 : \nu^s(E) < +\infty\} = \sup \{s \geq 0 : \nu^s(E) = +\infty\}. \tag{V.3.4}$$

Proof. The fact that $\mathcal{C}_n(E) \subset \widetilde{\mathcal{C}}_n(E)$ implies directly that $\widetilde{\nu}_n^s(E) \leq \nu_n^s(E)$.

Now, let $\{B(\tilde{x}_i, \tilde{r}_i)\}_{i=1}^m \in \widetilde{\mathcal{C}}_n(E)$. Each ball $B(\tilde{x}_i, \tilde{r}_i)$ is included in a ball $B(x_i, \tilde{r}_i + \sqrt{d})$, where $x_i \in \mathbb{Z}^d \cap E_n$. So $\left\{B\left(x_i, \left\lceil \tilde{r}_i + \sqrt{d} \right\rceil\right)\right\}_{i=1}^m \in \mathcal{C}_n(E)$, and using that $\left\lceil \tilde{r}_i + \sqrt{d} \right\rceil \leq \tilde{r}_i + \sqrt{d} + 1 \leq (2 + \sqrt{d})\tilde{r}_i$ (since $\tilde{r}_i \geq 1$), one has

$$\sum_{i=1}^m \left(\frac{\tilde{r}_i + \sqrt{d}}{2^n}\right)^s \leq (2 + \sqrt{d})^s \sum_{i=1}^m \left(\frac{\tilde{r}_i}{2^n}\right)^s.$$

This holds for any cover $\{B(\tilde{x}_i, \tilde{r}_i)\}_{i=1}^m \in \widetilde{\mathcal{C}}_n(E)$, so $\nu_n^s(E) \leq (2 + \sqrt{d})^s \widetilde{\nu}_n^s(E)$. □

Lemma V.3.2 shows in particular that the convergence/divergence properties of $\widetilde{\nu}^s(E)$ and $\nu^s(E)$ are identical.

The main advantage of dealing with $\nu^s(E)$ is the existence of optimal proper s-covers, i.e. covers $\{B_i = B(x_i, r_i)\}_{i=1}^m \in \mathcal{C}_n(E)$ such that $\nu_n^s(E) = \sum_{i=1}^m \left(\frac{r_i}{2^n}\right)^s$. These optimal covers exists because x_i and r_i are positive integers.

In our further analysis, the size of the balls of optimal covers will matter, justifying the following definition.

Definition V.3.3. For $E \subset \mathbb{Z}^d$, $n \in \mathbb{N}$ and $0 < s < d$, define

$$\beta_n^s(E) := \max \left\{ \max_{1 \leq i \leq p} \frac{r_i}{2^n} : (B(x_i, r_i))_{i=1}^p \text{ is an } s\text{-optimal proper cover of } E \cap S_n \right\}.$$

The quantity $\beta_n^s(E)$ will be important, in particular for Theorem V.2.7 about potential methods and for the projection Theorem V.2.8.

V.3.2 Some preliminary results

We first prove two propositions that will be needed later.

Proposition V.3.4. *Let μ_n be a Borel measure on S_n, $E \subset \mathbb{R}^d$ be a Borel set and $0 < c < +\infty$ be a constant.*

a) If $\max_{r\in\mathbb{N}^*} \dfrac{\mu_n\left(B(x,r)\right)}{(r/2^n)^s} \leq c$ *for all* $x \in E \cap S_n$, *then* $\nu_n^s(E) \geq \dfrac{\mu_n(E)}{c2^s}$.

b) If $\max_{r\in\mathbb{N}^*} \dfrac{\mu_n\left(B(x,r)\right)}{(r/2^n)^s} > c$ *for all* $x \in E \cap S_n$, *then* $\nu_n^s(E) \leq \dfrac{(5(1+\sqrt{d}/2))^s}{c}\mu_n(S_n)$.

Proof. a) Let $\{B(x_i,r_i)\}_{i=1}^m \in \mathcal{C}_n(E)$. For each $1 \leq i \leq m$, there exists $y_i \in B(x_i,r_i)\cap E\cap S_n$ such that $B(x_i,r_i) \subset B(y_i,2r_i)$, so

$$\mu_n(B(x_i,r_i)) \leq \mu_n(B(y_i,2r_i)) \leq c\left(\frac{2r_i}{2^n}\right)^s = c2^s\left(\frac{r_i}{2^n}\right)^s.$$

Then,

$$\mu_n(E\cap S_n) \leq \sum_{i=1}^m \mu_n(B(x_i,r_i)) \leq c2^s\sum_{i=1}^m\left(\frac{r_i}{2^n}\right)^s,$$

which is true for all covers $\{B(x_i,r_i)\}_{i=1}^m \in \mathcal{C}_n(E)$. Finally, taking the infimum over all elements of $\mathcal{C}_n(E)$, one gets

$$\mu_n(E) = \mu_n(E\cap S_n) \leq c2^s\nu_n^s(E).$$

b) Consider the family of balls

$$\mathcal{B}_n = \left\{B(x,r) : x \in E\cap S_n,\ r \in \{1,2,...,2^n\} \text{ and } \mu_n(B(x,r)) > c\left(\frac{r}{2^n}\right)^s\right\}.$$

Then

$$E\cap S_n \subset \bigcup_{B(x,r)\in\mathcal{B}_n} B(x,r).$$

Now, we invoke the following $5r$-covering Lemma [37, Lemma 4.8].

Lemma V.3.5. *Let $\mathcal{B}$ be a family of balls in $\mathbb{R}^N$ and suppose that $\sup_{B\in\mathcal{B}} d(B) < \infty$. Then there exists a countable sub-family of disjoint balls $\mathcal{B}_0$ of $\mathcal{B}$ such that*

$$\bigcup_{B\in\mathcal{B}} B \subset \bigcup_{i\in\mathcal{B}_0} 5B_i.$$

Using the previous lemma, there exists a finite family $(B_i = B(x_i, r_i))_{i=1,\ldots,m}$ of disjoint balls, all elements of $\mathcal{B}_n$, such that $\bigcup_{B\in\mathcal{B}_n} B \subset \bigcup_{i=1}^m 5B_i$. The finiteness of the family comes from the boundedness of S_n and the fact that the balls all have a diameter greater than 1. Up to a small translation of each x_i by a vector of length at most $\sqrt{d}/2$, one can assume that $x_i \in \mathbb{Z}^d$ and that

$$\bigcup_{B\in\mathcal{B}_n} B \subset \bigcup_{i=1}^m 5B\left(x_i, \left\lceil r_i + \sqrt{d}/2\right\rceil\right).$$

With the translations that we added, some balls $B \in \mathcal{B}_n$ may intersect, but this does not affect our argument.

Using the definition of $\nu_n^s(E)$, one finally gets

$$\begin{aligned}\nu_n^s(E) &\leq \sum_{i=1}^m \left(\frac{5\left\lceil r_i + \sqrt{d}/2\right\rceil}{2^n}\right)^s \leq (5(2+\sqrt{d}/2))^s \sum_{i=1}^m \left(\frac{r_i}{2^n}\right)^s \\ &\leq \frac{(5(2+\sqrt{d}/2))^s}{c} \sum_{i=1}^m \mu_n(B_i) \leq \frac{(5(2+\sqrt{d}/2))^s}{c}\mu_n(S_n),\end{aligned}$$

where the last equality comes from the disjointness of the B_i's. □

The following proposition guarantees that given a measure μ on a set E, there exists a smaller set $F \subset E$ such that the measure μ has a controlled local scaling behavior on F.

Proposition V.3.6. *Let $E \subset \mathbb{R}^d$ be a Borel set. Then, for every $0 < s \leq d$ there exists a constant $c_s > 0$ (depending only on s) and a set $\emptyset \neq F \subset E$ such that for every $n \geq 1$,*

(a) $\frac{4}{5}\nu_n^s(E) \leq \nu_n^s(F) \leq \nu_n^s(E)$

(b) $\nu_n^s\left(F \cap B\left(x, r\right)\right) \leq c_s \left(\frac{r}{2^n}\right)^s$ *for all $x \in \mathbb{Z}^d \cap S_n$ and $r \geq 1$.*

Proof. Let $E \subset \mathbb{R}^d$ and set for every $n \geq 1$

$$F_n := \left\{x \in E \cap S_n : \max_{r\geq 1} \frac{\nu_n^s\left(E \cap B(x,r)\right)}{\left(r/2^n\right)^s} > 5(5(2+\sqrt{d}/2))^s\right\}.$$

Using Proposition V.3.4 (b) applied to the set F_n and the measure $\mu_n(A) = \nu_n^s(E \cap A)$, one gets

$$\mu_n(F_n) \leq (5(2+\sqrt{d}/2))^s 5^{-1}(2+\sqrt{d}/2))^{-s}\mu_n(S_n) = \frac{1}{5}\mu_n(E).$$

Then $\mu_n(E \setminus F_n) \geq \frac{4}{5}\mu_n(E)$, i.e. as soon as $E \cap S_n$ is not empty, $(E \setminus F_n) \cap S_n \neq \emptyset$. Finally, the set $F = \bigcup_{n\geq 0} E \setminus F_n$ satisfies the two conditions mentioned above, with the constant $c_s = 5(5(2+\sqrt{d}/2))^s$. □

V.3.3 Proof of the mass distribution principle : Proposition V.2.5

For $n \in \mathbb{N}$, let $\{B(x_i, r_i)\}_{i=1}^m \in \widetilde{\mathcal{C}}_n(E)$, then

$$\mu_{|S_n}(E \cap S_n) \leq \mu_{|S_n}\left(\bigcup_{i=1}^m B(x_i, r_i)\right) \leq \sum_{i=1}^m \mu_{|S_n}(B(x_i, r_i)) \leq c \sum_{i=1}^m \left(\frac{r_i}{2^n}\right)^s.$$

Taking infimum over all proper covers $\{B(x_i, r_i)\}_{i=1}^m \in \widetilde{\mathcal{C}}_n(E)$, one gets

$$\frac{\mu_{|S_n}(E \cap S_n)}{c} \leq \nu_n^s(E).$$

Then $\widetilde{\nu}^s(E) \geq \dfrac{\sum_{n\geq 0} \mu_{|S_n}(E)}{c} = \dfrac{\mu(E)}{c} = +\infty$ and so $\mathrm{Dim}_H(E) \geq s$.

Observe that the same proof works if $\widetilde{\mathcal{C}}_n(E)$ and $\widetilde{\nu}_n^s(E)$ are replaced respectively by $\mathcal{C}_n(E)$ and $\nu_n^s(E)$.

V.4 Subsets of finite macroscopic measure

In this section, we prove a stronger version than Theorem V.2.4, more precisely:

Theorem V.4.1. *Let $E \subset \mathbb{R}^d$ such that $\nu^s(E) = +\infty$. Then there exists a macroscopic s-set $\widetilde{E}$ such that $\widetilde{E} \subset E$ and $\lim_{n\to+\infty} \sup_{t\in[0,d]} \beta_n^t(\widetilde{E}) = 0$.*

Observe that we can either work with $\widetilde{\nu}^s$ or ν^s, since $(\widetilde{\nu}^s(E) < +\infty) \Leftrightarrow (\nu^s(E) < +\infty)$. We choose to work with ν^s, and in this case $\beta_n^s(\widetilde{E})$ is defined without ambiguity.

We start with three technical lemmas, that will later help us extract a macroscopic s-set and prove the projection theorem.

Lemma V.4.2. *Let $(a_n)_{n\geq 1}$ be a bounded sequence of positive real numbers, such that $\lim\limits_{n\to+\infty} A_n := \sum\limits_{k=1}^{+\infty} a_k = +\infty$. For every $\varepsilon > 0$, $\sum\limits_{n=1}^{+\infty} \dfrac{a_n}{A_n^{1+\varepsilon}} < +\infty$ and $\sum\limits_{n=1}^{+\infty} \dfrac{a_n}{A_n} = +\infty$.*

This is a standard exercise, we prove it for completness.

Proof. Let $\varepsilon > 0$. For $n \geq 2$ and $\varepsilon > 0$, one has $\displaystyle\int_{A_{n-1}}^{A_n} \frac{dx}{x^{1+\varepsilon}} \geq \int_{A_{n-1}}^{A_n} \frac{dx}{A_n^{1+\varepsilon}} = \frac{a_n}{A_n^{1+\varepsilon}}$. Then, $\displaystyle\frac{1}{\varepsilon}\frac{1}{A_1^\varepsilon} \geq \frac{1}{\varepsilon}\left(\frac{1}{A_1^\varepsilon} - \frac{1}{A_n^\varepsilon}\right) = \int_{A_1}^{A_n} \frac{dx}{x^{1+\varepsilon}} \geq \sum_{k=2}^n \frac{a_k}{A_k^{1+\varepsilon}}$. So the sums $\displaystyle\sum_{k=1}^n \frac{a_n}{A_n^{1+\varepsilon}}$ are uniformly bounded and the series converges. Similarly, $\ln(A_n) - \ln(A_1) = \displaystyle\int_{A_n}^{A_1} \frac{dx}{x} \leq \sum_{k=2}^n \frac{a_k}{A_{k-1}}$.

Since $A_n \to +\infty$ as $n \to +\infty$, the series $\displaystyle\sum_{k=2}^{n} \frac{a_k}{A_{k-1}}$ diverges. Also, since (a_n) is bounded, $A_n \sim A_{n-1}$ and the series $\displaystyle\sum_{k=2}^{n} \frac{a_k}{A_k}$ diverges. □

Lemma V.4.3. *Let $(a_n)_{n\geq 1}$ be a positive sequence converging to zero, $(b_n)_{n\geq 1}$ be a bounded sequence of positive real numbers, such that $\sum_{n\geq 1} a_n b_n = +\infty$. Then, there exists a sequence $(c_n)_{n\geq 1}$ such that:*

1. *either $c_n = b_n$, or $c_n = 0$,*
2. *$\sum_{n\geq 1} a_n c_n = +\infty$,*
3. *$\sum_{n\geq 1} a_n^2 c_n < +\infty$.*

Proof. We assume without loss of generality that $0 \leq a_n, b_n < 1$ for every n, and that $(a_n)_{n\in\mathbb{N}}$ is a non-increasing sequence.

For $j \geq 0$, let us call $D_j = \{n \geq 0 : 2^{-j-1} \leq a_n < 2^{-j}\}$, and $B_j = \sum_{n\in D_j} b_n$. We call $d_j = \max(D_j)$, which is finite since $a_n \to 0$. Observe that the integer sets D_j are arranged in increasing order: $d_j + 1 = \min(D_{j+1})$. Also, one has

$$\frac{1}{2}\sum_{j=0}^{+\infty} 2^{-j} B_j \leq \sum_{n\geq 0} a_n b_n = \sum_{j=0}^{+\infty}\sum_{n\in D_j} a_n b_n \leq \sum_{j=0}^{+\infty} 2^{-j} B_j,$$

so that $\sum_{j=0}^{+\infty} 2^{-j} B_j = +\infty$.

We put $n_1 = 0$, $j_1 = 1$, and $c_n = 0$ for every $n \in D_0 \cup D_1$.

Remark that $\sum_{n\geq d_1+1} a_n b_n \geq 1/2 \sum_{j\geq 2} 2^{-j} B_j = +\infty$.

Let us call n_2 the first integer n such that $\sum_{n=d_1+1}^{n_2} a_n b_n > 1/2$. Observing that for $n \geq d_1 + 1$, $a_n b_n \leq 2^{-1}$, one necessarily has $1/2 < \sum_{n=d_1+1}^{n_2} a_n b_n < 1$.

We call j_2 the unique integer such that $n_2 \in D_{j_2}$, and we put $c_n = b_n$ for every $n \in \{d_1 + 1, ..., n_2\}$, and $c_n = 0$ for every $n \in \{n_2 + 1, ..., d_{j_2}\}$. By construction,

$$1/2 < \sum_{j=j_1+1}^{j_2}\sum_{n\in D_j} a_n c_n < 1.$$

We iterate the construction. Assume that we have built two finite sequences of integers $(n_k)_{k=1,...,p}$ and $(j_k)_{k=1,...,p}$ such that:

1. for $k = 1, ..., p-1$, $j_{k+1} > j_k$, and for $k = 1, ..., p$, $n_k \in D_{j_k}$
2. for $k = 1, ..., p$, $c_n = b_n$ if $n \in \{d_{j_{k-1}} + 1, ..., n_k\}$, and $c_n = 0$ if $n \in \{n_k + 1, ..., d_{j_k}\}$,

3. for $k = 1, ..., p$, one has

$$1/(k+1) < \sum_{j=j_{k-1}+1}^{j_k} \sum_{n \in D_j} a_n c_n < 2/k. \tag{V.4.1}$$

Let us call n_{p+1} the first integer such that $\sum_{n=d_p+1}^{n_{p+1}} a_n b_n > 1/(p+2)$. Observing that for $n \geq d_p + 1$, $a_n b_n \leq 2^{-j_p} \leq 1/(p+1)$ (since $j_p \geq p$), one necessarily has $1/(p+2) < \sum_{n=d_p+1}^{n_{p+1}} a_n b_n < 1/(p+2) + 1/(p+1) \leq 2/(p+1)$.

We call j_{p+1} the unique integer such that $n_{p+1} \in D_{j_{p+1}}$, and we put $c_n = b_n$ for every $n \in \{d_p + 1, ..., n_{p+1}\}$, and $c_n = 0$ for every $n \in \{n_{p+1} + 1, ..., d_{j_{p+1}}\}$. Clearly, these n_{p+1} and j_{p+1} satisfy the recurrence properties.

Now, gathering the information, we deduce by (V.4.1) that

$$\sum_{n \geq 0} a_n c_n = \sum_{k=1}^{+\infty} \sum_{j=j_{k-1}+1}^{j_k} \sum_{n \in D_j} a_n c_n \geq \sum_{k=1}^{+\infty} 1/(k+1) = +\infty$$

and, using that $a_n \leq 2^{-j}$when $n \geq D_j$, and that $j_{k-1} \geq k - 1$,

$$\begin{aligned}\sum_{n \geq 0} a_n^2 c_n = \sum_{k=1}^{+\infty} \sum_{j=j_{k-1}+1}^{j_k} \sum_{n \in D_j} a_n^2 c_n &\leq \sum_{k=1}^{+\infty} \sum_{j=j_{k-1}+1}^{j_k} 2^{-j} \sum_{n \in D_j} a_n c_n \\ &\leq \sum_{k=1}^{+\infty} 2^{-k+1}/(k+1) < +\infty.\end{aligned}$$

This concludes the proof. □

The same lines of computations can certainly be adapted to impose $\sum_{n \geq 0} a_n c_n = +\infty$ and $\sum_{n \geq 0} h(a_n) c_n < +\infty$ for any map $h : \mathbb{R}^+ \to \mathbb{R}^+$ such that $h(x) = o(x)$ when $x \to 0^+$.

As a first step toward Theorem V.4.1, we reduce the problem to sets that can be covered by small sets only.

Proposition V.4.4. *Let $E \subset \mathbb{R}^d$ such that $\nu^s(E) = +\infty$. Then, there exists a set $\widetilde{E} \subset E$ such that $\nu^s(\bar{E}) = +\infty$ and $\lim_{n \to +\infty} \sup_{t \in [0,d]} \beta_n^t(\widetilde{E}) = 0$.*

Proof. It is an application of Lemma V.4.2.

Call $A_n = \sum_{k=1}^n \nu_k^s(E)$ and $\alpha_n = A_n^{-1}$. By assumption, $\alpha_n \to 0$ when $n \to +\infty$.

For every $n \geq 1$, S_n can be covered by at most $2\alpha_n^{-1}$ balls of diameter $2^n \alpha_n^{1/d}$. Call $\mathcal{A}_n$ such a family of sets. One obviously has

$$\nu_n^s(E) \leq \sum_{A \in \mathcal{A}_n} \nu_n^s(E \cap A)$$

Thus there must exist $A_n \in \mathcal{A}_n$ such that $\nu_n^s(E \cap A_n) \geq \alpha_n \nu_n^s(E)$. Then one defines the set $\widetilde{E}$ as

$$\widetilde{E} = \bigcup_{n \geq 1} E \cap A_n.$$

By Lemma V.4.2,

$$\sum_{n \geq 0} \nu_n^s(\widetilde{E}) \geq \sum_{n \geq 0} \nu_n^s(E \cap A_n) \geq \sum_{n \geq 0} \alpha_n \nu_n^s(E) = +\infty.$$

Now, it is clear that for every n, $|\widetilde{E} \cap S_n| \leq 2^n \alpha_n^{1/d}$, so by Definition V.3.3, for every $t > 0$

$$\beta_n^t(\widetilde{E}) \leq \alpha_n^{1/d}.$$

Actually, this implies more: necessarily $\nu_n^s(\widetilde{E}) \leq \alpha_n^{s/d}$. In particular, $\beta_n^t(\widetilde{E}) \to 0$ as $n \to +\infty$ uniformly in t.

□

Finally, we prove Theorem V.4.1.

Proof. Let E be such that $\nu^s(E) = +\infty$. By Proposition V.4.4, one also assumes that $\lim_{n \to +\infty} \sup_{s \in [0,d]} \beta_n^s(\widetilde{E}) = 0$. This fact will not be used in this proof only, but will be key in the next section.

Observe that since for every n $\nu_n^s(E) \leq 1$, then $A_n := \sum_{k=0}^{n} \nu_k^s(E) \leq n$.

The idea consists in replacing E by a set $\widetilde{E}$ such that $\nu_n^s(\widetilde{E}) \sim b_n \nu_n^s(E)$, such that $\sum_{n \geq 1} \nu_n^s(\widetilde{E}) < +\infty$ but b_n is "as large as possible". Lemma V.4.2 helps to build such a sequence.

First, for every $\varepsilon > 0$, denote by

$$B_n^\varepsilon = \sum_{k \geq n} \frac{\nu_k^s(E)}{A_k^{1+\varepsilon}}$$

By Lemma V.4.2, one knows that $B_n^\varepsilon \to 0$ as $n \to \infty$, for every $\varepsilon > 0$.

We build iteratively a non-increasing sequence $(\varepsilon_n)_{n \geq 0} \subset \mathbb{R}^+$, and a sequence of integers $(n_k)_{k \geq 1}$.

Consider n_1 as the smallest positive integer such that $B_{n_1}^{\frac{1}{4}} \leq 1$ and set $\varepsilon_n = \frac{1}{2}$ for all $0 \leq n \leq n_1$.

Next we proceed by induction to build $(\varepsilon_n)_{n \geq 0}$ and $(n_k)_{k \geq 1}$.

Assume that $n_1 < n_2 < ... < n_p$ are defined.

Define n_{p+1} as the smallest integer such that

$$n_p < n_{p+1} \text{ and } B_{n_{p+1}}^{\frac{1}{2^{p+2}}} \leq \frac{1}{2^p}. \tag{V.4.2}$$

Put $\varepsilon_n = \dfrac{1}{2^{p+1}}$ for all $n_p < n \leq n_{p+1}$. Finally, let

$$b_n = \min\left\{1/2, (A_n)^{-(1+\varepsilon_n)}\right\}. \tag{V.4.3}$$

Then by construction of ε_n, one has:

(i) $\varepsilon_n \to 0$ as $n \to +\infty$,

(ii) By (V.4.2), and the fact that $B_{n_k}^{\frac{1}{2^{k+1}}} \leq B_{n_k}^{\frac{1}{2^k}} \leq 2^{-k-1}$,

$$\sum_{n\geq 0} b_n \nu_n^s(E) \leq \sum_{n\geq 0} \frac{\nu_n^s(E)}{A_n^{1+\varepsilon_n}} \leq \sum_{n=0}^{n_1} \frac{\nu_n^s(E)}{A_n^{1+\frac{1}{2}}} + \sum_{k\geq 1} \sum_{n=n_k+1}^{n_{k+1}} \frac{\nu_n^s(E)}{A_n^{1+\frac{1}{2^{k+1}}}} \tag{V.4.4}$$

$$\leq \sum_{n=0}^{n_1} \frac{\nu_n^s(E)}{A_n^{\frac{3}{2}}} + \sum_{k\geq 1} B_{n_k}^{\frac{1}{2^{k+1}}} \leq \sum_{n=0}^{n_1} \frac{\nu_n^s(E)}{(A_n)^{\frac{3}{2}}} + \sum_{k\geq 1} \frac{1}{2^{k-1}} < +\infty. \tag{V.4.5}$$

Next, we construct a set $\widetilde{E} \subset E$ such that for all $n \in \mathbb{N}$, one has

$$|\nu_n^s(\widetilde{E}) - b_n \nu_n^s(E)| \leq 2^{-ns}.$$

To achieve this, observe that by (V.2.1), S_n contains a finite number of lattice points, and denote by $M_{n,d}$ their cardinality. These points are denote by x_i for $i \in \{1, \ldots, M_{n,d}\}$.

Consider the following function:

$$\begin{aligned} g_n : \{0, 1, \ldots, M_{n,d}\} &\longrightarrow \mathbb{R}^+ \\ m &\longmapsto \nu_n^s\left(\bigcup_{i=1}^{m} E \cap B(x_i, 1)\right). \end{aligned}$$

where $g_n(0) = 0$ by convention. It is clear that g_n is non-decreasing, and ranges from 0 to $\nu_n^s(E)$. Moreover, for all $m \in \{1, \ldots, M_{n,d} - 1\}$, if $\{B(y_j, r_j)\}_{j=1}^p$ is an s-optimal cover of $\bigcup_{i=1}^{m} E \cap B(x_i, 1)$, then $\left\{(B(y_j, r_j))_{j=1}^p, B(x_{m+1}, 1)\right\}$ is a proper cover of $\bigcup_{i=1}^{m+1} E \cap B(x_i, 1)$ (not necessarily optimal). Using these two covers, one gets

$$g_n(m+1) - g_n(m) \leq \left(\sum_{j=1}^{p} \left(\frac{r_j}{2^n}\right)^s + \frac{1}{2^{ns}}\right) - \sum_{j=1}^{p} \left(\frac{r_j}{2^n}\right)^s \leq 2^{-ns}.$$

Hence, g_n has only small increments.

Recalling (V.4.3), $0 = g_n(0) \leq b_n \nu_n^s(E) \leq \nu_n^s(E) = g_n(M_{n,d})$, so there must exist an integer $m_n \in \{1, \dots, M_{n,d}\}$ such that

$$b_n \nu_n^s(E) \leq g_n(m_n) \leq b_n \nu_n^s(E) + 2^{-ns}.$$

Put

$$\widetilde{E}_n = \bigcup_{i=1}^{m_n} E \cap B(x_i, 1) \text{ and } \widetilde{E} = \bigcup_{n \geq 0} \widetilde{E}_n. \tag{V.4.6}$$

Then by construction, $\widetilde{E} \subset E$, and for all $n \in \mathbb{N}$ one has

$$b_n \nu_n^s(E) \leq \nu_n^s(\widetilde{E}) \leq b_n \nu_n^s(E) + 2^{-ns}.$$

And so, by (V.4.5),

$$\nu^s(\widetilde{E}) = \sum_{n \geq 0} \nu_n^s(\widetilde{E}) \leq \sum_{n \geq 0} \left(b_n \nu_n^s(E) + 2^{-ns} \right) < +\infty.$$

To complete the proof, it is enough to show that for all $\varepsilon > 0$, $\nu^{s-\varepsilon}(\widetilde{E}) = +\infty$. To this end, fix $\varepsilon > 0$, and let $(B(x_i, r_i))_{i=1}^m$ be an optimal $(s-\varepsilon)$-cover of $\widetilde{E} \cap S_n$, and assume that for this specific cover, $\beta_n^{s-\varepsilon}(\widetilde{E})$ is reached, i.e. there exists $i \in \{1, ..., m\}$ such that $r_i = 2^n \beta_n^{s-\varepsilon}(\widetilde{E})$. In particular, $\nu_n^{s-\varepsilon}(\widetilde{E}) \geq (\beta_n^{s-\varepsilon}(\widetilde{E}))^{s-\varepsilon}$.

One sees that

$$\nu_n^{s-\varepsilon}(\widetilde{E}) = \sum_{i=1}^m \left(\frac{r_i}{2^n} \right)^{s-\varepsilon} \geq \sum_{i=1}^m \left(\frac{r_i}{2^n} \right)^s \cdot (\beta_n^{s-\varepsilon}(\widetilde{E}))^{-\varepsilon} \geq (\beta_n^{s-\varepsilon}(\widetilde{E}))^{-\varepsilon} \cdot \nu_n^s(\widetilde{E}). \tag{V.4.7}$$

Two cases are separated.

On the one hand, if $\beta_n^{s-\varepsilon}(\widetilde{E}) \leq \sqrt[s]{\dfrac{\nu_n^s(E)}{A_n}}$, then (V.4.7) yields

$$\begin{aligned} \nu_n^{s-\varepsilon}(\widetilde{E}) &\geq \left(\frac{A_n}{\nu_n^s(E)} \right)^{\varepsilon/s} \cdot \nu_n^s(\widetilde{E}) \geq \left(\frac{A_n}{\nu_n^s(E)} \right)^{\varepsilon/s} \cdot b_n \cdot \nu_n^s(E) \\ &\geq \frac{(\nu_n^s(E))^{1-\varepsilon/s}}{A_n^{1+\varepsilon_n - \varepsilon/s}} \geq \frac{\nu_n^s(E)}{A_n^{1+\varepsilon_n - \varepsilon/s}}. \end{aligned} \tag{V.4.8}$$

where the fact that $\nu_n^s(E) \leq 1$ has been used in the last step.

On the other hand, if $\beta_n^{s-\varepsilon}(\widetilde{E}) \geq \sqrt[s]{\dfrac{\nu_n^s(E)}{A_n}}$, one has

$$\nu_n^{s-\varepsilon}(\widetilde{E}) \geq (\beta_n^{s-\varepsilon}(\widetilde{E}))^{s-\varepsilon} \geq \frac{(\nu_n^s(E))^{1-\varepsilon/s}}{A_n^{1-\varepsilon/s}} \geq \frac{\nu_n^s(E)}{A_n^{1-\varepsilon/s}}. \tag{V.4.9}$$

Finally, using the fact that $\varepsilon_n \to 0$ together with the lower bounds (V.4.8) and (V.4.9), one gets that for every large n, $\nu_n^{s-\varepsilon}(\widetilde{E}) \geq \dfrac{\nu_n^s(E)}{A_n}$. By Lemma V.4.2, $\displaystyle\sum_{n \geq 0} \frac{\nu_n^s(E)}{A_n} = +\infty$, hence $\nu^{s-\varepsilon}(\widetilde{E}) = \sum_{n \geq 0} \nu_n^{s-\varepsilon}(\widetilde{E}) = +\infty$.

This holds for every $\varepsilon > 0$, so $\mathrm{Dim}_H \left(\widetilde{E} \right) = s$. □

V.5 Potential Methods

V.5.1 First part of Theorem V.2.7

Consider $E \subset \mathbb{R}^d$, and assume that there exists a Radon measure μ on $\mathbb{R}^d$ such that $\mu(E) = +\infty$ and $\sum_{n\geq 0} 2^{ns} I_s(\mu_{|S_n}) < +\infty$. We prove that $\nu^s(E) = +\infty$, which implies that $\widetilde{\nu}^s(E) = +\infty$ and $\mathrm{Dim}_H(E) \geq s$.

For $n \in \mathbb{N}$, we write $\mu_n = \mu_{|S_n}$, and define

$$\phi^s_{\mu_n} := \int_{\mathbb{R}^d} \frac{d\mu_n(y)}{\|x-y\|_2^s \vee 1} \quad \text{ and } E_n = \left\{ x \in E \cap S_n : \max_{r\geq 1} \frac{\mu_n\left(B(x,r)\right)}{\left(\frac{r}{2^n}\right)^s} \leq 1 \right\}$$

For every $x \in E_n^c$, there exists an integer r_x such that $\dfrac{\mu_n\left(B(x,r_x)\right)}{\left(\frac{r_x}{2^n}\right)^s} \geq 1$. One has

$$\phi^s_{\mu_n}(x) = \int_{\mathbb{R}^d} \frac{d\mu_n(y)}{\|x-y\|_2^s \vee 1} \geq \int_{B(x,r_x)} \frac{d\mu_n(y)}{\|x-y\|_2^s \vee 1} \geq \frac{\mu_n\left(B(x,r_x)\right)}{r_x^s} \geq \frac{1}{2^{ns}}.$$

Then $I_s(\mu_n) \geq \displaystyle\int_{E_n^c} \phi^s_{\mu_n}(x) d\mu_n(x) \geq \frac{1}{2^{ns}} \mu_n(E_n^c)$, which implies that

$$\sum_{n\geq 0} \mu_n(E_n^c) \leq \sum_{n\geq 0} 2^{ns} I_s(\mu_n) < +\infty.$$

But as $E \cap S_n = E_n \cup E_n^c$ and $\sum_{n\geq 0} \mu_n(E \cap S_n) = +\infty$, then $\sum_{n\geq 0} \mu_n(E_n) = +\infty$. Moreover, by Proposition V.3.4 a), one has $\nu_n^s(E_n) \geq \frac{\mu_n(E_n)}{2^s}$. Finally, $\nu^s(E) = \sum_{n\geq 0} \nu_n^s(E_n) = +\infty$ which gives that $\mathrm{Dim}_H(E) \geq s$.

V.5.2 Second part of Theorem V.2.7

This is the most delicate part. Assume now that $\widetilde{\nu}^s(E) = +\infty$, and fix $0 < \varepsilon < s$.

Our goal is to build a Radon measure μ^ε on $\mathbb{R}^d$ such that $\mu^\varepsilon(E) = +\infty$ and $\sum_{n\geq 0} 2^{n(s-\varepsilon)} I_{s-\varepsilon}(\mu^\varepsilon_{|S_n}) < +\infty$. We are going to build each measure $\mu_n^\varepsilon = \mu^\varepsilon_{|S_n}$.

For this, we use the results we previously proved.

By Theorem V.4.1, there exists a macroscopic s-set $E_1 \subset E$ such that $\lim_{n\to+\infty} \sup_{t\in[0,d]} \beta_n^t(E_1) = 0$, $\mathrm{Dim}_H(E_1) = s$ and $\nu^s(E_1) = +\infty$.

Consider an optimal $(s-\frac{\varepsilon}{2})$-cover $\{B(x_i,r_i)\}_{i=1}^m$ of $E_1\cap S_n$. One sees that

$$\begin{aligned}\left(\beta_n^{s-\varepsilon/2}(E_1)\right)^{\frac{\varepsilon}{4}}\nu_n^{s-\frac{\varepsilon}{2}}(E_1) &= \left(\beta_n^{s-\varepsilon/2}(E_1)\right)^{\frac{\varepsilon}{4}}\sum_{i=1}^m\left(\frac{r_i}{2^n}\right)^{s-\frac{\varepsilon}{2}}\\ &= \left(\beta_n^{s-\varepsilon/2}(E_1)\right)^{\frac{\varepsilon}{4}}\sum_{i=1}^m\left(\frac{r_i}{2^n}\right)^{s-\frac{\varepsilon}{4}}\left(\frac{r_i}{2^n}\right)^{-\frac{\varepsilon}{4}}\\ &\geq \sum_{i=1}^m\left(\frac{r_i}{2^n}\right)^{s-\frac{\varepsilon}{4}}\geq \nu_n^{s-\frac{\varepsilon}{4}}(E_1),\end{aligned}$$

where we used that $\beta_n^{s-\varepsilon/2}(E_1)\geq \dfrac{r_i}{2^n}$. Recalling that $\operatorname{Dim}_H(E_1)=s$, it follows that $\sum_{n\geq}\left(\beta_n^{s-\varepsilon/2}(E_1)\right)^{\frac{\varepsilon}{4}}\nu_n^{s-\frac{\varepsilon}{2}}(E_1)=+\infty$.

Setting $a_n=\left(\beta_n^{s-\varepsilon/2}(E_1)\right)^{\frac{\varepsilon}{4}}$ and $b_n=\nu_n^{s-\frac{\varepsilon}{2}}(E_1)$, one then sees that the sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ satisfies the assumptions of Lemma V.4.3. Consider the sequence $(c_n)_{n\geq 1}$ given by this Lemma, and define the set $E_2\subset E_1$ as follows: for every $n\geq 1$,

- if $c_n=0$, then $E_2\cap S_n=\emptyset$,
- if $c_n=b_n$, then $E_2\cap S_n=E_1\cap S_n$.

It is immediate from the construction and Lemma V.4.3 that $c_n=\nu_n^{s-\varepsilon/2}(E_2)$ and

$$\begin{aligned}&\sum_{n\geq}\left(\beta_n^{s-\frac{\varepsilon}{2}}(E_1)\right)^{\frac{\varepsilon}{4}}\nu_n^{s-\frac{\varepsilon}{2}}(E_2)=+\infty\\ \text{and } &\sum_{n\geq}\left(\beta_n^{s-\frac{\varepsilon}{2}}(E_1)\right)^{\frac{\varepsilon}{2}}\nu_n^{s-\frac{\varepsilon}{2}}(E_2)<+\infty\end{aligned}\tag{V.5.1}$$

Finally, by Proposition V.3.6, there exists $\emptyset\neq E_3\subset E_2\subset E$ such that for all $n\in\mathbb{N}$,

$$\frac{4}{5}\nu_n^{s-\frac{\varepsilon}{2}}(E_2)\leq \nu_n^{s-\frac{\varepsilon}{2}}(E_3)\leq \nu_n^{s-\frac{\varepsilon}{2}}(E_2)\tag{V.5.2}$$

$$\text{and}\qquad \nu_n^{s-\frac{\varepsilon}{2}}(E_3\cap B(x,r))\leq c_{s-\frac{\varepsilon}{2}}\left(\frac{r}{2^n}\right)^{s-\frac{\varepsilon}{2}}\tag{V.5.3}$$

for all $x\in S_n\cap\mathbb{Z}^d$ and $r\geq 1$.

Define the measures $\mu_n^{\varepsilon}(A):=\left(\beta_n^{s-\frac{\varepsilon}{2}}(E_1)\right)^{\frac{\varepsilon}{4}}\nu_n^{s-\frac{\varepsilon}{2}}(E_3\cap A)$. Then by our construction and (V.5.2), one has

$$\begin{aligned}\sum_{n\geq 0}\mu_n^{\varepsilon}(E\cap S_n) &= \sum_{n\geq 0}\left(\beta_n^{s-\frac{\varepsilon}{2}}(E_1)\right)^{\frac{\varepsilon}{4}}\nu_n^{s-\frac{\varepsilon}{2}}(E_3)\\ &\geq \frac{4}{5}\sum_{n\geq 0}\left(\beta_n^{s-\frac{\varepsilon}{2}}(E_1)\right)^{\frac{\varepsilon}{4}}\nu_n^{s-\frac{\varepsilon}{2}}(E_2)=+\infty.\end{aligned}$$

We are left to prove that

$$\sum_{n\geq 0} 2^{n(s-\varepsilon)} I_{s-\varepsilon}(\mu_n^\varepsilon) = \sum_{n\geq 0} 2^{n(s-\varepsilon)} \int_{\mathbb{R}^d} \phi^{s-\varepsilon}_{\mu_n^\varepsilon}(x) d\mu_n^\varepsilon(x) < +\infty$$

For $x \in S_n$, one can write

$$\phi^{\mu_n^\varepsilon}_{s-\varepsilon}(x) = \int_{S_n} \frac{d\mu_n^\varepsilon(y)}{\|x-y\|_2^{s-\varepsilon} \vee 1}$$

Every $y \in S_n$ belongs to the ball $B(x, 2^{n+1})$. For $1 \leq r \leq 2^{n+1}$, denote by $m_n^\varepsilon(r) = \mu_n^\varepsilon(B(x,r))$. By (V.5.3), one has

$$m_n^\varepsilon(r) = \left(\beta_n^{s-\frac{\varepsilon}{2}}(E_1)\right)^{\frac{\varepsilon}{4}} \nu_n^{s-\frac{\varepsilon}{2}} \left(E_3 \cap B(x,r)\right) \leq c_s \left(\beta_n^{s-\frac{\varepsilon}{2}}(E_1)\right)^{\frac{\varepsilon}{4}} \left(\frac{r}{2^n}\right)^{s-\frac{\varepsilon}{2}}. \qquad \text{(V.5.4)}$$

Using the fact that $B(x, 2^{n+1}) = \bigcup_{r=1}^{2^{n+1}} B(x,r) \setminus B(x, r-1)$, one has

$$\begin{aligned}
\phi^{\mu_n^\varepsilon}_{s-\varepsilon}(x) &\leq \sum_{r=1}^{2^{n+1}} \int_{B(x,r)\setminus B(x,r-1)} \frac{d\mu_n^\varepsilon(y)}{\|x-y\|_2^{s-\varepsilon} \vee 1} \\
&= \mu_n^\varepsilon(B(x,1)) + \sum_{r=2}^{2^{n+1}} \int_{B(x,r)\setminus B(x,r-1)} \frac{d\mu_n^\varepsilon(y)}{\|x-y\|_2^{s-\varepsilon}}.
\end{aligned}$$

One the one hand, by (V.5.2), $\mu_n^\varepsilon(B(x,1)) \leq c_s \left(\beta_n^{s-\frac{\varepsilon}{2}}(E_1)\right)^{\frac{\varepsilon}{4}} 2^{-n(s-\frac{\varepsilon}{2})}$. On the other hand,

$$\begin{aligned}
&\sum_{r=2}^{2^{n+1}} \int_{B(x,r)\setminus B(x,r-1)} \frac{d\mu_n^\varepsilon(y)}{\|x-y\|_2^{s-\varepsilon}} \\
&= \sum_{r=2}^{2^{n+1}} \int_{r-1}^{r} t^{\varepsilon-s} dm_n^\varepsilon(t) \\
&= \sum_{r=2}^{2^{n+1}} \left(\left[t^{\varepsilon-s} m_n^\varepsilon(t)\right]_{r-1}^{r} + (s-\varepsilon) \int_{r-1}^{r} t^{\varepsilon-s-1} m_n^\varepsilon(t) dt \right) \\
&\leq c_{s-\frac{\varepsilon}{2}} \left(\beta_n^{s-\frac{\varepsilon}{2}}(E_1)\right)^{\frac{\varepsilon}{4}} 2^{-n(s-\frac{\varepsilon}{2})} \sum_{r=1}^{2^n} \left(\left[t^{\frac{\varepsilon}{2}}\right]_{r-1}^{r} + (s-\varepsilon) \int_{r-1}^{r} t^{\frac{\varepsilon}{2}-1} dt \right) \\
&\leq c_{s-\frac{\varepsilon}{2}} \left(1 + 2\frac{s-\varepsilon}{\varepsilon}\right) \left(\beta_n^{s-\frac{\varepsilon}{2}}(E_1)\right)^{\frac{\varepsilon}{4}} 2^{-n(s-\frac{\varepsilon}{2})} \sum_{r=1}^{2^{n+1}} \left(r^{\frac{\varepsilon}{2}} - (r-1)^{\frac{\varepsilon}{2}}\right) \\
&\leq C \left(\beta_n^{s-\frac{\varepsilon}{2}}(E_1)\right)^{\frac{\varepsilon}{4}} 2^{-n(s-\varepsilon)}.
\end{aligned}$$

for some constant C. So

$$\phi^{\mu_n^\varepsilon}_{s-\varepsilon}(x) \leq c_s, \left(\beta_n^{s-\frac{\varepsilon}{2}}(E_1)\right)^{\frac{\varepsilon}{4}} 2^{-n(s-\frac{\varepsilon}{2})} + C \left(\beta_n^{s-\frac{\varepsilon}{2}}(E_1)\right)^{\frac{\varepsilon}{4}} 2^{-n(s-\varepsilon)} \leq \widetilde{C} \left(\beta_n^{s-\frac{\varepsilon}{2}}(E_1)\right)^{\frac{\varepsilon}{4}} 2^{-n(s-\varepsilon)}.$$

Moving to the integral, one gets

$$I_{s-\varepsilon}(\mu_n^\varepsilon) = \int_{\mathbb{R}^d} \phi_{s-\varepsilon}^{\mu_n^\varepsilon}(x) d\mu_n^\varepsilon(x) \le C \left(\beta_n^{s-\frac{\varepsilon}{2}}(E_1)\right)^{\frac{\varepsilon}{4}} 2^{-n(s-\varepsilon)} \mu_n^\varepsilon(E_3).$$

Finally, recalling (V.5.1), (V.5.2), (V.5.3) and the definition of μ_n^ε, one has

$$\begin{aligned}\sum_{n\geq 0} 2^{n(s-\varepsilon)} I_{s-\varepsilon}(\mu_n^\varepsilon) &\le C \sum_{n\geq 0} \left(\beta_n^{s-\frac{\varepsilon}{2}}(E_1)\right)^{\frac{\varepsilon}{4}} \mu_n^\varepsilon(E_3) \\ &\le C \sum_{n\geq 0} \left(\beta_n^{s-\frac{\varepsilon}{2}}(E_1)\right)^{\frac{\varepsilon}{2}} \nu_n^{s-\frac{\varepsilon}{2}}(E_3) < +\infty\end{aligned}$$

as desired.

V.6 Projection of a Set

In this section we are considering the orthogonal projection of sets in $\mathbb{R}^2$ and we aim at proving the projection Theorem V.2.8 for the macroscopic Hausdorff dimension.

Let us introduce some notations.

For every $\theta \in [0, 2\pi]$, call $e_\theta = (\cos\theta, \sin\theta)$ the vector with angle θ, and L_θ the straight line in $\mathbb{R}^2$ with angle θ passing through the origin.

Then, recall that $proj_\theta : \mathbb{R}^2 \to L_\theta$ is the orthogonal projection onto L_θ.

V.6.1 Case where $\mathrm{Dim}_H(E) \geq 1$

Let us start by proving item b) of Theorem V.2.8, assuming that item a) is proved.

Consider $E \subset \mathbb{R}^2$ with $\mathrm{Dim}_H(E) \geq 1$.

By Theorem V.4.1, for every $p \geq 2$, there exists $E_p \subset E$ such that $\mathrm{Dim}_H(E_p) = 1-1/p$. For each set E_p, by item a), there exists a set $\Theta_p \subset [0, \pi]$ of full Lebesgue measure such that for every $\theta \in \Theta_p$, $\mathrm{Dim}_H(proj_\theta(E_p)) = 1 - 1/p$. In particular, this implies that $\mathrm{Dim}_H(proj_\theta(E)) \geq 1 - 1/p$.

Consider now the set $\Theta = \bigcap_{p\geq 2} \Theta_p$. The above arguments show that Θ is still of full Lebesgue measure in $[0, \pi]$, and that for every $\theta \in \Theta$, $\mathrm{Dim}_H(proj_\theta(E)) \geq 1$. Since obviously $\mathrm{Dim}_H(proj_\theta(E))$ is always less than 1 (since it is included in L_θ), the result follows.

V.6.2 First extractions when $\mathrm{Dim}_H(E) < 1$

Fix a set $E \subset \mathbb{R}^2$ with $0 < \mathrm{Dim}_H(E) = s < 1$. The rest of the section is devoted to prove that $\mathrm{Dim}_H(\mathrm{proj}_\theta E) = \mathrm{Dim}_H(E)$ for almost every $\theta \in [0, \pi]$.

Writing $L_\theta = \{\lambda e_\theta : \lambda \in \mathbb{R}\}$, we can define the n-th shells inside L_θ as $S_n^\theta = \{v = (x,y) \in L_\theta : \|v\|_2 \in [2^{n-1}, 2^n]\}$. Identifying L_θ with $\mathbb{R}$, the results we obtained before in dimension 1 apply to L_θ and S_n^θ.

We are going to project 2-dimensional measures onto the lines L_θ. For this, let us define for every $n \geq 0$ the cylinders

$$C_n^\theta := \mathrm{proj}_\theta^{-1} S_n^\theta. \tag{V.6.1}$$

We are going to prove that for every $0 < \varepsilon < s$, the set

$$\Theta_{s-\varepsilon} = \{\theta \in [0,\pi] : \mathrm{Dim}_H\,(proj_\theta(E)) \geq s - \varepsilon\} \tag{V.6.2}$$

has full Lebesgue measure. The conclusion then follows using the same argument as the one used to prove item b). More precisely, from the properties above, $\Theta := \bigcap_{p\geq 1} \Theta_{s-1/p}$ has full Lebesgue measure, and for every $\theta \in \Theta$, $\mathrm{Dim}_H\,(proj_\theta(E)) \geq s$. But since $proj_\theta$ is a Lipschitz mapping, $\mathrm{Dim}_H\,(proj_\theta(E)) \leq s = \mathrm{Dim}_H\,(E)$. Finally one gets $\mathrm{Dim}_H\,(\mathrm{proj}_\theta E) = \mathrm{Dim}_H\,(E)$ for almost all $\theta \in [0,\pi]$.

Fix $0 < \varepsilon < s$.

Applying Theorem V.2.7(2), there exists a Borel measure μ^ε supported by E such that

$$\sum_{n\geq 0} \mu_n^\varepsilon(E \cap S_n) = +\infty, \tag{V.6.3}$$

$$\text{and} \quad \sum_{n\geq 0} 2^{n(s-\varepsilon)} I_{s-\varepsilon}(\mu_n^\varepsilon) < +\infty, \tag{V.6.4}$$

where μ_n^ε is a simplified notation for $\mu^\varepsilon_{|S_n}$. Observe that in fact, via the finer Theorem V.4.1 and Proposition V.4.4, we can impose that $\lim_{n\to+\infty} \mu_n^\varepsilon(E \cap S_n) = 0$.

We need to impose an additional condition on μ^ε, namely that

$$\sum_{n\geq 0} 2^{-n} \mu_n^\varepsilon(E \cap S_n) \left(\sum_{k=0}^{n} 2^k \mu_k^\varepsilon(E \cap S_k) \right) < +\infty. \tag{V.6.5}$$

This is achieved thanks to the following lemma.

Lemma V.6.1. *Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two positive sequences converging to zero, such that $\sum_{n\geq 1} a_n = +\infty$ and $\sum_{n\geq 1} a_n b_n = +\infty$. There exists a sequence $(c_n)_{n\geq 1}$ such that:*

1. *either $c_n = a_n$, or $c_n = 0$,*
2. *$\sum_{n\geq 1} c_n = +\infty$,*
3. *$\sum_{n\geq 1} c_n b_n < +\infty$.*

Proof. Again, without loss of generality, we assume that $0 < a_n, b_n < 1$. Let us call $D_j = \{n \geq 0 : 2^{-j-1} \leq b_n < 2^{-j}\}$, for $j \geq 0$.

Put $c_n = 0$ for every $n \in D_0 \cup D_1$, and $n_0 = 0$, $j_0 = 1$.

We know that $\sum_{j\geq 2} \sum_{n\in D_j} a_n b_n = +\infty$. We go through each D_j in increasing order. Consider the first couple (n_1, j_1) such that $n_1 \in D_{j_1}$ and $\sum_{j=2}^{j_1-1} \sum_{n\in D_j} a_n b_n + \sum_{n\in D_{j_1}, n\leq n_1} a_n b_n \geq 1/2$. Put $c_n = a_n$ for all $n \in \bigcup_{j=2}^{j_1-1} D_j \cup \{n \in D_{j_1} : n \leq n_1\}$, and $c_n = 0$ for all $n \in \{n \in D_{j_1} : n > n_1\}$. By our choice,

$$1/2 \leq \sum_{j=0}^{j_1} \sum_{n\in D_j} c_n b_n = \sum_{j=2}^{j_1-1} \sum_{n\in D_j} a_n b_n + \sum_{n\in D_{j_1}, n\leq n_1} a_n b_n < 1.$$

We then iterate the process: assume that we have built two finite sequences of integers $(n_k)_{k=1,\dots,p}$ and $(j_k)_{k=1,\dots,p}$ such that

1. for $k = 1, ..., p-1$, $j_{k+1} > j_k$, and for $k = 1, ..., p$, $n_k \in D_{j_k}$
2. for $k = 1, ..., p$, $c_n = a_n$ if $n \in \bigcup_{j=j_{k-1}}^{j_k-1} D_j \cup \{n \in D_{j_k} : n \leq n_k\}$, and $c_n = 0$ for all $n \in \{n \in D_{j_k} : n > n_k\}$.
3. for $k = 1, ..., p$, one has

$$2^{-k} \leq \sum_{j=j_{k-1}}^{j_k} \sum_{n\in D_j} c_n b_n < 2^{-k+1}. \tag{V.6.6}$$

We know that $\sum_{j\geq j_p+1} \sum_{n\in D_j} a_n b_n = +\infty$. Consider the first couple (n_{p+1}, j_{p+1}) such that $n_{p+1} \in D_{j_{p+1}}$ and $\sum_{j=j_p}^{j_{p+1}-1} \sum_{n\in D_j} a_n b_n + \sum_{n\in D_{j_{p+1}}, n\leq n_{p+1}} a_n b_n \geq 2^{-(p+1)}$. Put $c_n = a_n$ for all $n \in \bigcup_{j=j_p}^{j_{p+1}-1} D_j \cup \{n \in D_{j_{p+1}} : n \leq n_{p+1}\}$, and $c_n = 0$ for all $n \in \{n \in D_{j_{p+1}} : n > n_{p+1}\}$. Then, since for all the selected integers n, $a_n b_n \leq 2^{-j_{p+1}} \leq 2^{-(p+1)}$, (V.6.6) holds true.

Collecting the information, on one hand one has by (V.6.6)

$$\sum_{n\geq 0} c_n b_n = \sum_{k\geq 1} \sum_{j=j_{k-1}}^{j_k} \sum_{n\in D_j} c_n b_n \leq \sum_{k\geq 1} 2^{-k+1} < +\infty.$$

On the other hand, since $j_k \geq k+1$, one sees that for each $n \in D_j$ for $j \in \{j_{k-1}, ... j_k\}$, $b_n \leq 2^{-k}$, so again by (V.6.6),

$$\sum_{n\geq 0} c_n = \sum_{k\geq 1} \sum_{j=j_{k-1}}^{j_k} \sum_{n\in D_j} c_n \geq \sum_{k\geq 1} 2^k \sum_{j=j_{k-1}}^{j_k} \sum_{n\in D_j} c_n b_n \geq \sum_{k\geq 1} 1 = +\infty,$$

hence the result. □

Setting $a_n = \mu_n^\varepsilon(E)$, then $(a_n)_{n\geq 0}$ tends to zero when n tends to infinity. Define then

$$b_n = 2^{-n}\sum_{k=0}^{n} 2^k a_k.$$

Since $\sum_{k=0}^{n} 2^k \sim 2^n$, $(b_n)_{n\geq 0}$ is a generalized Caesaro mean associated with the sequence $(a_n)_{n\geq 0}$, and converges to zero when n tends to infinity.

So either $\sum_{n\geq 1} a_n b_n < +\infty$, and (V.6.5) is true, or $\sum_{n\geq 1} a_n b_n = +\infty$ and we are exactly in the situation of Lemma V.6.1: there exists a sequence $(c_n)_{n\geq 1}$ such that:

1. either $c_n = a_n$, or $c_n = 0$,
2. $\sum_{n\geq 1} c_n = +\infty$,
3. $\sum_{n\geq 1} c_n b_n < +\infty$.

Setting $\widetilde{E} = \bigcup_{n\geq 0: a_n = c_n} E \cap S_n$, by construction one has $\mu^\varepsilon(\widetilde{E}) = \sum_{n\geq 1} c_n = +\infty$, and since $\mu_k^\varepsilon(\widetilde{E}\cap S_k) = c_k \leq a_k = \mu_k^\varepsilon(E\cap S_k)$, one has

$$\sum_{n\geq 0} 2^{-n}\mu_n^\varepsilon(\widetilde{E}\cap S_n)\left(\sum_{k=0}^{n} 2^k \mu_k^\varepsilon(\widetilde{E}\cap S_k)\right) \leq \sum_{n\geq 1} c_n b_n < +\infty,$$

hence (V.6.5) is obtained for $\widetilde{E}$. This property will be used at the very end of the proof of Proposition V.6.4 only. It is obvious that if Theorem V.2.8 is proved for this smaller set E, it is also true for the original set.

Finally, observe that, replacing $\widetilde{E}$ by $\bigcup_{n\geq 0} \widetilde{E}\cap S_{2n}$ or $\bigcup_{n\geq 0} \widetilde{E}\cap S_{2n+1}$, one can assume in addition to (V.6.3), (V.6.4) and (V.6.5) that

$$\text{if } S_n \neq \emptyset, \quad \text{then } S_{n-1} = S_{n+1} = \emptyset. \tag{V.6.7}$$

To resume this section, we have proved that the original set E contains a subset, still denoted by E for simplification, and a measure μ^ε supported by E such that (V.6.3), (V.6.4), (V.6.5) and (V.6.7) simultaneously hold.

V.6.3 Final proof of item a) of Theorem V.2.8

Consider the set E obtained after extraction above. For all $\theta \in [0,\pi]$, $k \geq n$ and $A \subset L_\theta$, we focus on the restriction of μ_k^ε on C_n^θ

$$(\mu_k^\varepsilon)_{|C_n^\theta}(A) := \mu_k^\varepsilon(\{x \in E\cap S_k \,:\, \mathrm{proj}_\theta x \in A \cap S_n^\theta\}),$$

Equivalently for each non-negative function f, one has

$$\int_{-\infty}^{+\infty} f(t)d(\mu_k^\varepsilon)_{|C_n^\theta}(t) \;=\; \int_{C_n^\theta \cap S_k} f(x.e_\theta)d\mu_k^\varepsilon(x).$$

where $x.e_\theta$ denotes the scalar product. Since e_θ is unitary, we identify $x.e_\theta$ with $\mathrm{proj}_\theta x$, the orthogonal projection of x onto L_θ.

Definition V.6.2. The projected measure $\mu^{\varepsilon,\theta}$ is defined as $\mu^{\varepsilon,\theta} = \sum_{n\geq 1} \mu_n^{\varepsilon,\theta}$, where

$$\mu_n^{\varepsilon,\theta} = \sum_{k\geq n} (\mu_k^{\varepsilon})_{|C_n^\theta}. \tag{V.6.8}$$

Note that each $\mu_n^{\varepsilon,\theta}$ is a measure supported on $\text{proj}_\theta E \cap S_n^\theta$.

We are going to prove that for almost all $\theta \in [0,\pi]$,

$$\sum_{n\geq 0} \mu_n^{\varepsilon,\theta}(\text{proj}_\theta E) = +\infty \text{ and } \sum_{n\geq 0} 2^{n(s-\varepsilon)} I_{s-\varepsilon}(\mu_n^{\varepsilon,\theta}) < \infty. \tag{V.6.9}$$

for almost all $\theta \in [0,\pi]$. Then item a) of Theorem V.2.7 will allow us to conclude that the set $\Theta_{s-\varepsilon}$ defined by (V.6.2) has full Lebesgue measure, as announced.

This is the purpose of the next two propositions.

Proposition V.6.3. *For every $\theta \in [0,\pi]$,*

$$\mu^{\varepsilon,\theta}(\mathit{proj}_\theta E) = +\infty. \tag{V.6.10}$$

Proof. This simply follows from the observation that

$$\mu^{\varepsilon,\theta}(\text{proj}_\theta E) = \sum_{n\geq 0} \mu_n^{\varepsilon,\theta}(\text{proj}_\theta E) = \sum_{n\geq 0}\sum_{k\geq n} (\mu_k^{\varepsilon})_{|C_n^\theta}(E) \geq \sum_{n\geq 0} \mu_n^{\varepsilon}(E) = +\infty,$$

since the union of the $(C_n^\theta)_{n\geq 1}$ cover $\mathbb{R}^2$ (there are small overlaps (their borders) between the C_n^θ). Hence the result. $\square$

So the first part of (V.6.9) is proved.

Let us move to the second part. Observe that even if $\mu^{\varepsilon,\theta}(\text{proj}_\theta E) = +\infty$, it is likely that $\text{proj}_\theta E$ has dimension less than $\text{Dim}_H(E)$. A trivial example is when the s-dimensional set E is included in a straight line of angle ϕ passing through 0, and $\theta = \phi + \pi/2$.

Proposition V.6.4. *One has*

$$\mathbb{E}_\theta \left[\sum_{n\geq 0} 2^{n(s-\varepsilon)} I_{s-\varepsilon}(\mu_n^{\varepsilon,\theta}) \right] < +\infty. \tag{V.6.11}$$

Proof. Remark that if (V.6.11) is proved, then $\sum_{n\geq 0} 2^{n(s-\varepsilon)} I_{s-\varepsilon}(\mu_n^{\varepsilon,\theta}) < +\infty$ for Lebesgue almost every $\theta \in [0,\pi]$, so (V.6.9) and item a) of Theorem V.2.8 are proved.

We start with the following lemma.

Lemma V.6.5. *There exists a constant $C_0 > 0$ such that the following holds. Let $x \in S_k$ for some $k \geq 0$. For all $0 \leq n \leq k$, the set $J_{x,n} = \{\theta \in [0,\pi] : x \in C_k^\theta\}$ is an interval modulo π, and $|J_{x,n}| \leq C_0 2^{n-k}$.*

Proof. The fact that $J_{x,k}$ is an interval is obvious.

Let $x = (u, v) \in S_k$. We study the case where $x_1 \geq 0$, the case $x_1 < 0$ being symmetric. Using polar coordinates, one has $x = (r\cos\theta_0, r\sin\theta_0)$ for some $2^{k-1} \leq r \leq 2^k$ and $\theta_0 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Then the projection of x on L_θ is given by:

$$\mathrm{proj}_\theta x = (r\cos(\theta - \theta_0)\cos\theta, r\cos(\theta - \theta_0)\sin\theta).$$

Recall (V.6.1), one sees that for $0 \leq n \leq k$,

$$\begin{aligned} x \in C_n^\theta &\iff 2^{n-1} \leq r\cos(\theta - \theta_0) \leq 2^n \\ &\iff \frac{2^{n-1}}{r} \leq \cos(\theta - \theta_0) \leq \min\left\{1, \frac{2^n}{r}\right\} \\ &\iff \theta \in \left[\theta_0 + \arccos\left(\frac{2^{n-1}}{r}\right), \theta_0 + \arccos\left(\min\left\{1, \frac{2^n}{r}\right\}\right)\right] \quad \mod \pi. \end{aligned}$$

The Taylor development $\arccos(y) = \frac{\pi}{2} - y + o(y)$ together with the fact that $2^{k-1} \leq r \leq 2^k$ yields that $|J_{n,x}| = 2^{n-k}(1 + o(1))$. □

From the proof, it also follows that $|J_{x,n}| \sim C2^{n-k}$ when n/k is quite small.

Let us study (V.6.11). One has

$$\begin{aligned} &\mathbb{E}_\theta\left[\sum_{n\geq 0} 2^{n(s-\varepsilon)} I_{s-\varepsilon}(\mu_n^{\varepsilon,\theta})\right] \\ &= \int_0^\pi \left[\sum_{n\geq 0} 2^{n(s-\varepsilon)} I_{s-\varepsilon}(\mu_n^{\varepsilon,\theta})\right] d\theta \\ &= \int_0^\pi \left[\sum_{n\geq 0} 2^{n(s-\varepsilon)} \int_{S_n^\theta}\int_{S_n^\theta} \frac{d\mu_n^{\varepsilon,\theta}(u)\, d\mu_n^{\varepsilon,\theta}(v)}{|u-v|^{s-\varepsilon} \vee 1}\right] d\theta \\ &= \int_0^\pi \left[\sum_{n\geq 0} 2^{n(s-\varepsilon)} \sum_{j,k\geq n} \int_{E\cap S_j\cap C_n^\theta}\int_{E\cap S_k\cap C_n^\theta} \frac{d\mu_k^\varepsilon(x)\, d\mu_j^\varepsilon(y)}{|x\cdot e_\theta - y\cdot e_\theta|^{s-\varepsilon} \vee 1}\right] d\theta \\ &:= I_1 + 2I_2 \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_0^\pi \left[\sum_{n\geq 0} 2^{n(s-\varepsilon)} \sum_{k\geq n} \iint_{(E\cap S_j\cap C_n^\theta)^2} \frac{d\mu_k^\varepsilon(x)\, d\mu_k^\varepsilon(y)}{|(x-y)\cdot e_\theta|^{s-\varepsilon} \vee 1}\right] d\theta \\ I_2 &= \int_0^\pi \left[\sum_{n\geq 0} 2^{n(s-\varepsilon)} \sum_{k>j\geq n} \int_{E\cap S_j\cap C_n^\theta}\int_{E\cap S_k\cap C_n^\theta} \frac{d\mu_k^\varepsilon(x)\, d\mu_j^\varepsilon(y)}{|(x-y)\cdot e_\theta|^{s-\varepsilon} \vee 1}\right] d\theta. \end{aligned}$$

Starting with I_1, one has

$$
\begin{aligned}
I_1 &= \int_0^\pi \left[\sum_{n\geq 0} 2^{n(s-\varepsilon)} \sum_{k\geq n} \iint_{(E\cap S_k\cap C_n^\theta)^2} \frac{d\mu_k^\varepsilon(x)\, d\mu_k^\varepsilon(y)}{|(x-y)\cdot e_\theta|^{s-\varepsilon} \vee 1} \right] d\theta \\
&= \int_0^\pi \left[\sum_{n\geq 0} 2^{n(s-\varepsilon)} \sum_{k\geq n} \iint_{(E\cap S_k)^2} \frac{\mathbb{1}_{C_n^\theta}(x)\mathbb{1}_{C_n^\theta}(y)}{|(x-y)\cdot e_\theta|^{s-\varepsilon} \vee 1} d\mu_k^\varepsilon(x)\, d\mu_k^\varepsilon(y) \right] d\theta \\
&= \sum_{n\geq 0} 2^{n(s-\varepsilon)} \sum_{k\geq n} \iint_{(E\cap S_k)^2} \int_0^\pi \frac{\mathbb{1}_{C_n^\theta}(x)\mathbb{1}_{C_n^\theta}(y)}{|(x-y)\cdot e_\theta|^{s-\varepsilon} \vee 1} d\theta d\mu_k^\varepsilon(x)\, d\mu_k(y) \\
&\leq \sum_{n\geq 0} 2^{n(s-\varepsilon)} \sum_{k\geq n} \iint_{(E\cap S_k)^2} \left[\int_0^\pi \frac{\mathbb{1}_{x\in C_n^\theta}(\theta)\mathbb{1}_{y\in C_n^\theta}(\theta)}{|\tau_{x-y}\cdot e_\theta|^{s-\varepsilon}} d\theta \right] \frac{d\mu_k^\varepsilon(x)\, d\mu_k^\varepsilon(y)}{\|x-y\|_2^{s-\varepsilon} \vee 1},
\end{aligned}
$$

where τ_{x-y} is the unit vector in the direction of $x-y$. By Lemma V.6.5, when $x \in S_k$ one has $\mathbb{1}_{x\in C_n^\theta}(\theta) = \mathbb{1}_{J_{n,x}}(\theta)$. Then

$$
\int_0^\pi \frac{\mathbb{1}_{x\in C_n^\theta}(\theta)\mathbb{1}_{y\in C_n^\theta}(\theta)}{|\tau_{x-y}\cdot e_\theta|^{s-\varepsilon}} d\theta = \int_{J_{n,x}\cap J_{n,y}} \frac{d\theta}{|\cos(\widehat{\tau_{x-y}, e_\theta})|^{s-\varepsilon}}.
$$

By Lemma V.6.5, the interval $J_{n,x} \cap J_{n,y}$ has length smaller than $C_0 2^{n-k}$. So the integral above is taken over an interval of length at most $C_0 2^{n-k}$. Moreover, as $s < 1$, the integral reaches its largest value when θ close to $\dfrac{\pi}{2}$. Thus

$$
\int_0^\pi \frac{\mathbb{1}_{x\in C_n^\theta}(\theta)\mathbb{1}_{y\in C_n^\theta}(\theta)}{|\tau_{x-y}\cdot e_\theta|^{s-\varepsilon}} d\theta \leq \int_{\frac{\pi}{2}-C_0 2^{n-k}}^{\frac{\pi}{2}+C_0 2^{n-k}} \frac{d\theta}{|\cos(\theta)|^{s-\varepsilon}} \leq \int_{-C_0 2^{n-k}}^{C_0 2^{n-k}} \frac{d\theta}{|\theta|^{s-\varepsilon}} = C 2^{(n-k)(1-s+\varepsilon)}. \tag{V.6.12}
$$

where $C > 0$ is some positive constant. Then going back to I_1 and using V.6.12, one gets

$$
\begin{aligned}
I_1 &\leq C \sum_{n\geq 0} 2^{n(s-\varepsilon)} \sum_{k\geq n} 2^{(n-k)(1-s+\varepsilon)} \iint_{(E\cap S_k)^2} \frac{d\mu_k^\varepsilon(x)\, d\mu_k^\varepsilon(y)}{\|x-y\|_2^\varepsilon \vee 1} \\
&= C \sum_{n\geq 0} \sum_{k\geq n} 2^{n+k(s+\varepsilon-1)} \iint_{(E\cap S_k)^2} \frac{d\mu_k^\varepsilon(x)\, d\mu_k^\varepsilon(y)}{\|x-y\|_2^{s-\varepsilon} \vee 1} \\
&= C \sum_{n\geq 0} 2^{n(s+\varepsilon-1)} \sum_{k=0}^{n} 2^k \iint_{(E\cap S_n)^2} \frac{d\mu_n^\varepsilon(x)\, d\mu_n^\varepsilon(y)}{\|x-y\|_2^{s-\varepsilon} \vee 1} \\
&\leq 2C \sum_{n\geq 0} 2^{n(s-\varepsilon)} I_{s-\varepsilon}(\mu_n^\varepsilon) < +\infty,
\end{aligned}
$$

which is finite by (V.6.4).

Moving to I_2, the same manipulations as above for I_1 yield

$$\begin{aligned}
I_2 &= \int_0^\pi \left[\sum_{n\geq 0} 2^{n(s-\varepsilon)} \sum_{k>j\geq n} \int_{E\cap S_j \cap C_n^\theta} \int_{E\cap S_k\cap C_n^\theta} \frac{d\mu_k^\varepsilon(x)\, d\mu_j^\varepsilon(y)}{|(x-y)\cdot e_\theta|^{s-\varepsilon} \vee 1} \right] d\theta. \\
&= \int_0^\pi \left[\sum_{n\geq 0} 2^{n(s-\varepsilon)} \sum_{k>j\geq n} \int_{E\cap S_j} \int_{E\cap S_k} \frac{\mathbb{1}_{C_n^\theta}(x)\mathbb{1}_{C_n^\theta}(y)}{|(x-y)\cdot e_\theta|^{s-\varepsilon}} d\mu_k^\varepsilon(x)\, d\mu_j^\varepsilon(y) \right] d\theta \\
&= \sum_{n\geq 0} 2^{n(s-\varepsilon)} \sum_{k>j\geq n} \int_{E\cap S_j} \int_{E\cap S_k} \left[\int_0^\pi \frac{\mathbb{1}_{J_{n,x}}(\theta)\mathbb{1}_{J_{n,y}}(\theta)}{|\tau_{x-y}\cdot e_\theta|^{s-\varepsilon}} d\theta \right] \frac{d\mu_k^\varepsilon(x)\, d\mu_j^\varepsilon(y)}{\|x-y\|_2^{s-\varepsilon}}.
\end{aligned}$$

As before, by Lemma V.6.5, $|J_{k,x}| \leq 2^{n-k}$ and $|J_{j,y}| \leq 2^{n-j}$ for all $x \in S_k \cap C_n^\theta$ and $y \in S_j \cap C_n^\theta$). Then, as $k \geq j+1$, the same argument as in (V.6.12) yields

$$\int_0^\pi \frac{\mathbb{1}_{x\in C_n^\theta}(\theta)\mathbb{1}_{y\in C_n^\theta}(\theta)}{|\tau_{x-y}\cdot e_\theta|^{s-\varepsilon}} d\theta \leq C 2^{(n-k)(1-s+\varepsilon)}. \tag{V.6.13}$$

for some $C > 0$.

Next, we make use of equation (V.6.7) : indeed, it is not possible that μ_j^ε and μ_{j+1}^ε are simultaneously non-zero. Hence, for $x \in S_k$ and $y \in S_j$ such that $j < k$ and μ_j^ε and μ_k^ε not both equal to zero, then necessarily $|k-j| \geq 2$ and $2^{k-2} \leq \|x-y\|_2 \leq 2^{k+1}$. This implies in particular that

$$\int_{E\cap S_j} \int_{E\cap S_k} \frac{d\mu_k^\varepsilon(x)\, d\mu_j^\varepsilon(y)}{\|x-y\|_2^{s-\varepsilon}} \leq C 2^{-k(s-\varepsilon)} \mu_k^\varepsilon(E\cap S_k)\, \mu_j^\varepsilon(E\cap S_j), \tag{V.6.14}$$

the inequality being in fact close to be sharp.

Finally, combining (V.6.14) and (V.6.13)), one gets that for some $C' > 0$,

$$\begin{aligned}
I_2 &\leq C' \sum_{n\geq 0} 2^{n(s-\varepsilon)} \sum_{k>j\geq n} 2^{(n-k)(1-s+\varepsilon)} 2^{-k(s-\varepsilon)} \mu_k^\varepsilon(E\cap S_k)\, \mu_j^\varepsilon(E\cap S_j) \\
&= C' \sum_{n\geq 0} 2^n \sum_{k>j\geq n} 2^{-k} \mu_j^\varepsilon(E\cap S_j) \mu_k^\varepsilon(E\cap S_k) \\
&= C' \sum_{j\geq 0} \left(\sum_{n=0}^{j} 2^n \right) \mu_n^\varepsilon(E\cap S_n) \sum_{k\geq n+1} 2^{-k} \mu_k^\varepsilon(E\cap S_k) \\
&\leq C' \sum_{n\geq 0} 2^n \mu_n^\varepsilon(E\cap S_n) \sum_{k\geq n+1} 2^{-k} \mu_k^\varepsilon(E\cap S_k) \\
&\leq C' \sum_{n\geq 0} 2^{-n} \mu_n^\varepsilon(E\cap S_n) \left(\sum_{k=0}^{n} 2^k \mu_k^\varepsilon(E\cap S_k) \right).
\end{aligned}$$

This last double sum is finite, because the set E was chosen so that (V.6.5) holds true. This concludes the proof. □

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