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The irreversible pollution game

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ABSTRACT

We investigate the extent to which the irreversibility of pollution shapes the free-riding problems inherent in pollution (differential) games. To this end, we use two-country differential pollution games. Irreversibility is of a *hard* type: While strictly positive and concave below a certain threshold level of pollution, pollution decay drops to zero above this threshold. Assuming that the pollution damage function and preferences are quadratic, we first examine both the cooperative and non-cooperative versions of the game. We innovate in analytically demonstrating the existence of Markov perfect equilibria (MPE) and characterizing these. Second, we demonstrate that when players face the same pollution costs (symmetry), irreversible pollution regimes are more frequently reached than under cooperation, and we evaluate the irreversibility penalty stemming from the absence of cooperation. Incidentally, we prove that open-loop Nash equilibria lead to reach more frequently the irreversible regime than the MPE under our setting. Third, we study the implications of asymmetry in the pollution cost. We find that for equal total pollution costs, asymmetric equilibria produce a lower emission rate than the symmetric under some mild conditions, thereby driving the system to irreversibility less frequently than the latter. Finally, we prove that provided the irreversible regime is reached in both the symmetric and asymmetric cases, long-term pollution is greater in the symmetric case, reflecting more intensive free-riding under symmetry.

1. Introduction

Pollution control has been among the most discussed topics in several disciplines over the last five decades. A key conceptual and modeling aspect is the inherent externality problem: The emission of pollutants due to the actions of a given individual in a given place also affects other individuals in neighboring areas through various diffusion channels (wind and currents, among others), often giving rise to substantial free-riding problems. In this paper, we tackle an important issue at the core of the current debate: **irreversible** pollution. Several researchers claim that irreversible climate regimes have already occurred, and indeed, there is now growing evidence that the buffering capacity of oceans (the most important carbon sink) is near saturation. Accordingly, it is believed that the assimilation capacity of terrestrial ecosystems will peak by mid-century and then decline, with these becoming a net source of carbon by the end of the century. The potential collapse of the North Atlantic meridional overturning circulation is also drawing much attention, as it is projected to occur at a CO_2 concentration of 450 ppm and we have already reached 390 ppm (Yohe et al., 2006; Boucekkine et al., 2013a; Lenton and Ciscarm, 2013). Other examples of irreversible pollution found in the literature include the so-called shallow lake problem (see a very recent exploration in Wagener and de Zeeuw (2021)). This is the

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typical working example for dynamic systems with tipping points (see also van der Ploeg and de Zeeuw (2018). In the lake system, the tipping points are those at which a small increase in phosphorus loading shifts the lake to a poor state, with a significant loss of ecosystem services. Irreversibility has also been studied in the context of the management of natural resources, as in Sakamoto (2014).

This paper is a theoretical contribution to the literature on irreversible pollution, and in particular to the study of the determinants of free-riding under the threat of irreversibility. Our core research questions are: To what extent does the irreversibility of pollution shape the free-riding problems inherent in pollution (differential) games? Under what conditions is irreversible pollution reached in the context of Nash competition? Could cooperation prevent this outcome? Do asymmetries between players (countries) trigger more or less free-riding, and hence, more or less irreversibility and/or long-term pollution with respect to the symmetric case? We shall essentially differentiate between the players in terms of pollution cost.

We build on the seminal contribution of Tahvonen and Withagen (1996), TW hereafter, to build our game-theoretic framework. In TW, there is a single player (say, a country) that faces a standard pollution control problem with the additional complication that pollution is irreversible: Pollution decay may decrease with the level of pollution and eventually drops to zero above a certain threshold. This non-concave feature leads to a sophisticated problem potentially yielding multiple steady-state equilibria and discontinuities, among other non-standard properties. Despite this additional difficulty, the model has been used in several non-game-theoretic contexts (see, for example, Prieur (2009), Boucekkine et al. (2013a,b)). We study dynamic game extensions of TW and analytically characterize the associated Markov perfect equilibria (MPE) in a variety of settings.

From the seminal work of TW, we know that a central planner can only avoid the irreversible pollution regime under some stringent parametric conditions. Accordingly, cooperation between the two players will not rule out the emergence of irreversible regimes. Our dynamic game extension allows the visualization of the irreversibility penalty due to free-riding in non-cooperative games: We accurately characterize the extent to which free-riding will aggravate the global pollution problem by making an irreversible pollution regime significantly easier to reach. In addition, we rigorously derive the implications of asymmetry in pollution costs in terms of the frequency of reaching irreversibility and long-term pollution levels. Indeed, a key aspect already identified in the literature on pollution games and international environmental agreements is heterogeneity across players (see, for example, Hoel (1993), or Xepapadeas (1995)). We study this aspect in the presence of an irreversibility threshold.

In addition to the important methodological contribution, we are able to extract several significant results. We start with the benchmark case of cooperation. As outlined above, the same model is analyzed in depth in TW. However, we employ a different mathematical approach: We use dynamic programming, which is the natural method to characterize MPE, whereas TW use multi-stage optimal control. Thus, it is useful to check that the two methods lead to the same results concerning the reachability of the irreversible pollution regime in the central planner (one-player) case. We indeed corroborate TW's results and find that cooperation between players (implemented through the central planner counterpart of the game) does not always prevent the emergence of irreversible pollution regimes. This is true in particular when pollution costs are low enough.

More importantly, considering that players face the same pollution cost (symmetry), we evaluate the scope for irreversibility in this case, compared to cooperation, by identifying the range of parameters leading to crossing the irreversibility threshold in the two respective institutional settings. This in turn allows us to characterize the extent of free-riding under symmetry. Incidentally, we also demonstrate that, under our setting, open-loop Nash equilibria lead to reach more frequently the irreversible regime compared to the MPE. Last but not least, we study the implications of asymmetry in the pollution cost for the reachability of the irreversibility threshold and for long-term pollution under this regime. We obtain two sets of non-trivial and original results: For equal total pollution costs (summing unit pollution costs across countries), we first demonstrate that asymmetric equilibria produce a lower emission rate than the symmetric under some mild conditions, thereby driving the system to irreversibility less frequently than the latter. Second, we prove that provided the irreversible regime is reached in both the symmetric and asymmetric cases, long-term pollution is greater in the symmetric case. This primarily reflects the more intensive free-riding under symmetry. To our knowledge, this is the first paper pointing at this remarkable property in pollution games.

Relation to the literature There exists an abundant economics literature on pollution games—game-theoretic models aiming to characterize equilibrium free-riding in multi-player models of (a common stock of) pollution. See, for example, Dockner and Van Long (1993), Dutta and Radner (2009) or Bertinelli et al. (2014) for recent examples. However, irreversible pollution has been much more intensively studied in the ecological literature than in economics. Besides TW, only a limited number of economics papers have been written about (hard) irreversibility as found in the ecology literature. Among these, Barrett (2013) is an intriguing contribution to the economics of environmental cooperation treaties under irreversibility thresholds and shows how uncertainty can significantly shape the outcomes of cooperation, depending on whether uncertainty concerns the impact of irreversibility or the level of the irreversibility threshold. Barrett's frame is static, however. Indeed, differential game settings dealing with irreversible pollution are much scarcer. Among the very few contributions to this line of research are Wagener and de Zeeuw (2021) and El Ouardighi et al. (2020).[1] El Ouardighi et al. (2020) explore differential games with variable self-cleaning capacity, but not in the sense of TW. By allowing the self-cleaning capacity to be directly controlled so as to get rid of the non-concavity inherent in TW, they end up exploring a different set of differential games with much *softer* irreversibility constraints. Here, we strictly follow TW in modeling irreversibility with all the technical complications involved (and magnified under the differential game frame). In addition, and contrary to El Ouardighi et al. (2020), we do not consider only symmetric linear–quadratic games: While some linear-quadratic and

[1] As mentioned above, Sakamoto (2014) also explores the implications of irreversibility but in another context, the dynamic management of natural resources.

symmetry assumptions are made either for benchmarking or to ease the extraction of analytical results, we stick to the non-concave and general specification of the pollution decay function postulated in TW, and also study departures from symmetry. Last but not least, all our results are analytical—we only use numerical examples for illustration.

Wagener and de Zeeuw (2021) paper is definitely much closer to ours: It is anchored in the tipping games literature, of which the so-called shallow lake problem is the typical working example. Indeed, in contrast to El Ouardighi et al. (2020), Wagener and de Zeeuw (2021) deal precisely with the hard irreversibility constraint studied in TW. However, their methods – and therefore their results – are valid only for open-loop strategies and involve no time discount. Indeed, when applying their method to our model, the result for the central planner case is the same as ours with no discount rate and a piecewise linear pollution decay function. Besides the generality of our decay function, we compute MPE (closed-loop strategies), which are admittedly more satisfactory from a rationality standpoint as they allow for feedback by construction, in contrast to open-loop strategies which build on strong commitment assumptions. Nonetheless, MPE are technically much more demanding technically speaking. This in no way diminishes the merits of Wagener and de Zeeuw (2021), one of the very few in-depth papers on differential games with tipping points. We come back to this point in detail below when comparing the irreversible regime reachability conditions under open-loop strategies compared to the MPE. As for the role of the asymmetry of pollution costs in the reachability and long-term outcomes of irreversible pollution regimes, to the best of our knowledge this remains unexplored in the literature so far.

We conclude this literature review with a few methodological remarks. First of all, because we rely on TW's non-concave framework, the optimization work needed is far from non-trivial—not to mention the difficulty of computing MPE. More concretely, the individual optimization program is an optimal regime-switching problem. As always, the tricky part of the optimization work lies at the junction of the two regimes, that is, at the switching point (if any). Most of the existing literature on regime switching focuses on optimal individual (for example, central planner) problems, and thus, it has naturally relied on versions of the Pontryagin optimality conditions, including some specific continuity (or transversality) conditions on the maximized Hamiltonians and, in some cases, also on the co-state variables at the switching points (see, for example, Tomiyama (1985), Boucekkine et al. (2013a,b, 2020)). The same technique has been used to characterize open-loop equilibria in dynamic game models with regime switching (see Boucekkine et al. (2011), for a two-country game without pollution irreversibility, in addition to Wagener and de Zeeuw (2021), which does deal with irreversibility). However, in this paper we aim to characterize the MPE (if any). This requires moving to dynamic programming and designing a more adequate analytical approach.

Generally, it is very difficult to explicitly solve optimal control and multi-period differential game problems, even in a linear–quadratic framework. Most of the results in the economics literature rely on numerical solution techniques (see Dawid and Gezer (2022), El Ouardighi et al. (2020), to mention just a few). The main reason is that the continuity conditions at the switching point between different periods (or modes) make it very difficult, if not impossible, to guess the functional form of the Bellman value function and, thus, the strategies. Even with linear–quadratic functions under autonomous settings, the commonly used linear–quadratic functions fail to satisfy the transversality conditions. We show that despite the non-concavity resulting from the irreversibility ingredient, we can analytically characterize the MPE of the different games considered (existence of the irreversible regime and the asymptotics, and other related properties). This is true for any concave, strictly positive pollution decay function in the reversible regime. That being said, we show in each game how the irreversible-regime-crossing conditions "degenerate" in the linear decay case. Our numerical illustrations also consider this the linear decay function for simplicity and for comparison with TW.

The paper is organized as follows. Section 2 presents the basic differential game extension of TW. Section 3 briefly presents the cooperative case, mostly to enable a close comparison with the one-country case studied in TW. Sections 4 and 5 are devoted to non-cooperative games under symmetric versus asymmetric pollution costs. Section 5 provides a central result comparing the equilibrium emission rates of both settings, with the subsequent implications for the crossing of the irreversibility threshold. Section 6 ranks pollution outcomes across strategic settings (central planner, Nash symmetric, and Nash asymmetric) in the irreversible regime and clarifies some aspects of free-riding behavior in this regime. A comparison of the reachability conditions of the irreversible regime between strategic settings is also provided. Section 7 concludes.

2. The game

We briefly present our game-theoretic extension of TW.

In contrast to TW, there are two players, referred to as player $i = 1, 2$, both producing final consumption goods with pollution as a by-product. Ignoring differences in production, we can use their pollution emissions, $y_i(t)$, to measure their output level. Player i's objective is to maximize her social welfare, taking into account transboundary pollution:

$$\max_y W_i = \int_0^{+\infty} (U_i(y_i) - D_i(z))e^{-rt}dt, \tag{1}$$

where r is the time preference, $U_i(y_i)$ is the utility from enjoying the final output generated with pollution $y_i(t)$, and $D_i(z)$ is the damage function from the aggregate pollution stock z. The pollution stock $z(t)$ may decay at rate $\alpha(z)$ if the pollution level is below a threshold $\bar{z}$. In this regime, referred to as the reversible regime in TW, pollution accumulates as

$$\dot{z} = y_1 + y_2 - \alpha(z), \quad z(0) = z_0 \text{ given}, \tag{2}$$

where $\alpha(z)$ is the pollution decay function in the reversible pollution regime. This captures Nature's self-cleaning capacity. If the threshold is attained and crossed, the economy falls into the irreversible regime, where pollution decay drops to zero. Following

TW, we assume that the decay function satisfies the following properties: $\alpha(0) = 0$, $\alpha(z) > 0$ when $z \in [0, \ \bar{z})$, $\alpha(z) = 0$, $\forall z \geq \bar{z}$, and $\alpha''(z) \leq 0$ when $z \in [0, \ \bar{z})$. Though TW do not assume that $\alpha(z)$ is decreasing in the reversible regime, they do work with the affine specification $\alpha(z) = \alpha - \beta z$, with α and β being positive in their numerical examples (though, of course, the specification is only valid locally due to the positivity constraint). We go in the same direction, we only use the affine specification for numerical exercises or to obtain some explicit results.

Due to the differential game framework, we need to posit quadratic utility and damage functions in order to get analytical results—for the existence of MPE, in particular. More specifically, we suppose $U_i(y_i) = a_i y_i - y_i^2$ and $D_i(z) = c_i z^2$, with all coefficients, a_i, c_i, being positive constants. Accordingly, player i's optimal control problems under the reversible and irreversible regimes – called Period I and II hereafter – read as follows:

Period I

$$\max_{y_i} W_i^I = \int_0^T (U_i(y_i) - D_i(z))e^{-rt}dt = \int_0^T (a_i y_i - y_i^2 - c_i z^2)e^{-rt}dt,$$

subject to

$$\dot{z} = y_1 + y_2 - \alpha(z), \quad z(0) = z_0,$$

and

$$\begin{cases} z(T) = \bar{z}, \quad \text{if} \quad T < +\infty, \\ \lim_{t\to\infty} z(t) \leq \bar{z}, \quad \text{if} \quad T = +\infty. \end{cases}$$

Period II

$$\max_{y_i} W_i^{II} = \int_T^{+\infty} (U_i(y_i) - D_i(z))e^{-rt}dt = \int_T^{+\infty} (a_i y_i - y_i^2 - c_i z^2)e^{-rt}dt,$$

subject to

$$\dot{z} = y_1 + y_2, \quad z(T) = \bar{z}.$$

From now on, we define c by $c_1 + c_2 = 2c > 0$, the case $c = c_1 = c_2$ covering the case of pollution cost symmetry. We start with the benchmark cooperative game before exploring the equilibrium properties of Nash games. We shall use dynamic programming along the paper.[2]

3. Cooperative equilibria: the central planner problem

We start with the cooperative game where a benevolent central planner (for example, a credible international institution or a state's federal government) enforces cooperation between the two players. We do this to compare our results with TW's. In both cases, there is a unique optimizing authority, potentially leading to similar outcomes. We indeed refine a few of the results obtained by TW in some way.

We assume that the central planner maximizes the sum of the utilities of the two players, namely,

$$\max_{y_1,y_2} W_c^I = \int_0^T \sum_{i=1,2} (U_i(y_i) - D_i(z))e^{-rt}dt = \int_0^T \sum_{i=1,2} (a_i y_i - y_i^2 - c_i z^2)e^{-rt}dt,$$

subject to

$$\dot{z} = y_1 + y_2 - \alpha(z), \quad z(0) = z_0,$$

and

$$\begin{cases} z(T) = \bar{z}, \quad \text{if} \quad T < +\infty, \\ \lim_{t\to\infty} z(t) \leq \bar{z}, \quad \text{if} \quad T = +\infty. \end{cases}$$

Furthermore, if $T < \infty$, the system enters the situation with no decay:

$$\max_{y_1,y_2} W_c^{II} = \int_T^\infty \sum_{i=1,2} (U_i(y_i) - D_i(z))e^{-rt}dt = \int_T^\infty \sum_{i=1,2} (a_i y_i - y_i^2 - c_i z^2)e^{-rt}dt,$$

subject to

$$\dot{z} = y_1 + y_2, \quad z(T) = \bar{z}.$$

The central planner's stationary Bellman value function $V(z)$ is given by the following HJB equation:

$$rV_c(z) = \max_{y_1,y_2\geq 0} \left\{ a_1 y_1 + a_2 y_2 - y_1^2 - y_2^2 - 2cz^2 + V_c'(z)\left[y_1 + y_2 - \delta(z)\right] \right\}, \tag{3}$$

where $V_c'(z)$ is the marginal value function and

[2] Of course, the central planner problem can be solved either by dynamic programming or by Pontryagin's principle. Since we essentially characterize feedback controls in the differential games, which requires dynamic programming, we choose the same approach in Section 3 to ease exposition.

$$\delta(z) = \begin{cases} \alpha(z), & z \leq \bar{z}, \\ 0, & z > \bar{z}. \end{cases}$$

In Period II, when $z > \bar{z}$, set $\delta(z) = 0$. Then for the infinite time horizon autonomous optimal control, we can assume that $V_c(z)$ is a quadratic function of z in the form

$$V_c(z) = A_c + B_c z + \frac{C_c}{2} z^2,$$

where A_c, B_c, and C_c are unknown coefficients. Substituting the affine-quadratic value function into the HJB equation and comparing the coefficients, we can obtain the explicit values of coefficients A_c, B_c, and C_c. Furthermore, at $z = \bar{z}$,

$$V_c(\bar{z}) = A_c + B_c \bar{z} + \frac{C_c}{2} \bar{z}^2, \tag{4}$$

which is the initial condition for Period II and the terminal condition for Period I. In Period I, when $z < \bar{z}$ HJB equation (3) takes the form

$$rV_c = \frac{1}{4}\left[\left(a_1 + V_c'\right)^2 + \left(a_2 + V_c'\right)^2\right] - 2cz^2 - V_c' \delta(z). \tag{5}$$

It is worth pointing out that in this period, though the optimization system is still autonomous with free ending time and the state equation and objective functions are still affine or quadratic, the linear–quadratic guess of the value function no longer works (contrary to the solution of Period II) due to the transversality condition (4). This problem arises either in the central planner or the differential games models we will explore. Thus, a different method is needed to tackle this kind of affine-quadratic optimal control problems. We do provide it in this paper.

Based on the above HJB equations, the online Appendix A.1 indeed demonstrates the following proposition.

Proposition 1. *Let $V_c(z), V_c'(z)$ be, respectively, the Bellman value and marginal value functions of the central planner. Then, under the central player's optimal choice $y_i = \frac{a_i + V_c'(z)}{2}$ for $i = 1, 2$, the following properties are true:*

(1.1) If inequality

$$\left(r + \delta'(\bar{z})\right)(a - \delta(\bar{z})) - 4c\bar{z} > 0 \tag{6}$$

holds, then threshold $\bar{z}$ is reached in finite time from some $z_0 < \bar{z}$.

(1.2) After crossing the threshold $\bar{z}$, the dynamic system converges asymptotically to its long-run steady state:

$$\lim_{t\to\infty} z(t) = z_c^* = -\frac{a + B_c}{C_c} = \frac{ar}{4c} (> \bar{z}), \tag{7}$$

where

$$a = \frac{a_1 + a_2}{2}, \qquad C_c = \frac{1}{2}\left[r - \sqrt{r^2 + 16c}\right], \qquad B_c = \frac{aC_c}{r - C_c}. \tag{8}$$

(1.3) If in addition to (6), the special linear decay function $\delta(z) = \alpha - \beta z$ with positive α and β and

$$4cz_0 < (r - \beta)\left(a - \delta\left(z_0\right)\right) \tag{9}$$

holds (in particular, if $a \geq \alpha$), then $\bar{z}$ is reached in finite time for any $0 \leq z_0 \leq \bar{z}$.

(1.4) If the reversed inequality in (6) holds, then $\bar{z}$ is never reached.

As will be made clearer below, the economic interpretation of the results is perfectly in line with the general analysis provided by TW. In particular, after rewriting condition (6) as

$$4c\bar{z} < \left(r + \delta'(\bar{z})\right)(a - \delta(\bar{z})),$$

one can observe that the larger the pollution cost as captured by parameter c, the less likely condition (6) is to hold and the more likely a permanent reversible regime will set in. The reverse occurs with the utility parameter a. That is to say, condition (6) simply compares the cost of pollution and its welfare benefit.

Last but not least, it should be noted that if $\bar{z}$ is reached in finite time, $\bar{T}$, depending on whether $\bar{z} < z_c^*$ or $\bar{z} \geq z_c^*$, the state variable exhibits a different behavior for $t > \bar{T}$. In the first case, the state enters the irreversible regime and $z(t) \to z_c^*$ as $t \to \infty$. In the second case, $z(t) = \bar{z}$ for all $t > \bar{T}$. Therefore, the process stops at $\bar{T}$.

Example 1. Clearly, our cooperative setting need not be different from TW's optimal control model. However, our more specific linear–quadratic (LQ) assumptions regarding the utility and damage functions do allow us to get more clear-cut results. While several key results in TW are formulated as sufficient conditions, we can provide a full and global picture of optimal trajectories, as depicted in Proposition 1 above. Here, we complement our theoretical analysis with a numerical example. In TW, a numerical example is given for the case of one player with

$$U(y) = ay - by^2, \qquad D(z) = cz^2, \qquad \delta(z) = \alpha - \beta z$$

and

$$\dot{z} = y - \delta(z) \qquad \text{for } z < \bar{z}$$

using the parameter values

$$a = 18, \qquad b = 0.5, \qquad c = 0.004, \qquad r = 0.2, \qquad \alpha = 20, \qquad \beta = 0.1.$$

Their model is equivalent to our central planner model with $a_1 = a_2 = a$, $c_1 = c_2 = c/2$, and $y_1 = y_2 = y/2$. We use the values of parameters. The equivalent value of c in our case is $0.004/2 = 0.002$. In TW, $\bar{z} = \alpha/\beta = 200$. At this value,

$$\left(r + \delta'(\bar{z})\right)(a - \delta(\bar{z})) = (r - \beta)\, a = 1.8 > 1.6 = 4c\bar{z}.$$

Hence, (6) holds, and therefore, $\bar{z} = 200$ is reached in finite time. In fact, for any

$$c_1 = c_2 \geq \frac{a(r-\beta)}{4\bar{z}} = 0.00225,$$

the reversed inequality in (6) is satisfied for $\bar{z} = 200$. Therefore, by Proposition 1 $\bar{z}$ is not reachable.

Returning to the case where $c_1 = c_2 = 0.002$, (6) is satisfied if

$$\bar{z} > \frac{(r-\beta)(\alpha - a)}{\beta(r-\beta) - 4c} = 100.$$

Such values of $\bar{z}$ are all reachable.[3]

4. Non-cooperative equilibria with symmetric pollution costs

We now move to non-cooperative (Nash) games. We first study the case where pollution costs are evenly distributed across players; asymmetric extensions are considered in Section 5. From here onward, we focus on the characterization of Markov perfect equilibria. For simplicity, let $a = \frac{a_1 + a_2}{2}$.

4.1. Characterization of the irreversible regime ($z > \overline{z}$)

Denote the Bellman value function of player i as $V_i(z)$, $\forall z$. Then V_i must satisfy the following Hamilton–Jacobin–Bellman (HJB) equation:

$$rV_i(z) = \max_{y_i} \left[a_i y_i - y_i^2 - c_i z^2 + V_i' \left[y_1 + y_2\right]\right]. \tag{10}$$

The right-hand side's first-order condition (which also satisfies the second-order condition) yields that the optimal choice of player i is

$$y_i = \frac{a_i + V_i'}{2}, \quad i = 1, 2. \tag{11}$$

Thus, the HJB equation (10) becomes

$$rV_i(z) = a_i \frac{a_i + V_i'}{2} - \left(\frac{a_i + V_i'}{2}\right)^2 - c_i z^2 + V_i' \left[\frac{a_1 + V_1'}{2} + \frac{a_2 + V_2'}{2}\right], \quad i = 1, 2. \tag{12}$$

Based on HJB equation (10), in the online Appendix A.2 we obtain that the optimal strategy of player $i = 1, 2$ is

$$y_i^m(z) = \frac{a_i + B^m + C^m z}{2},$$

and the state equation

$$\dot{z} = y_1^m + y_2^m = a + B^m + C^m z, \quad t \geq T$$

yields the explicit solution

$$z^m(t) = (\overline{z} - z_s^*)e^{C^m(t-T)} + z_s^*, \tag{13}$$

where z_s^* is the asymptotically stable long-run steady state and is given by

$$z_s^* = -\frac{a + B^m}{C^m} = \frac{a}{12c}\left(5r + \sqrt{r^2 + 12c}\right). \tag{14}$$

It is straightforward that when $z = \overline{z}$,

$$V_i(\overline{z}) = A_i^m + B^m \overline{z} + \frac{C^m}{2}\overline{z}^2 \equiv \overline{V}_i, \tag{15}$$

which will serve as a terminal condition for the first period under Markovian competition. In other words, this is the transversality condition between Periods I and II. The parameters are given by

[3] Similarly, we can easily show that the model in Wagener and de Zeeuw (2021), when adapted to our cooperative setting, leads to a special case of Proposition 1.

$$C^m = \frac{r - \sqrt{r^2 + 12c}}{3} (< 0) \tag{16}$$

and

$$\begin{cases} B_i^m = B_2^m \equiv B^m = \frac{2aC^m}{2r-3C^m} (< 0), \\ A_1^m = \frac{(a_1+B^m)^2+2B^m(a_2+B^m)}{4r}, \\ A_2^m = \frac{(a_2+B^m)^2+2B^m(a_1+B^m)}{4r}. \end{cases} \tag{17}$$

4.2. Characterization of the reversible regime and threshold-crossing conditions ($z < \bar{z}$)

In Period I, before the pollution threshold is triggered, the accumulation of pollution satisfies

$$\dot{z} = y_1 + y_2 - \delta(z) = y_1 + y_2 - \delta(z), \quad t \leq T,$$

with initial condition $z(0) = z_0$ given. Similarly to Section 3, the value function must satisfy the following HJB equation:

$$rV_i(z) = \max_{y_i} \left[a_i y_i - y_i^2 - c_i z^2 + V_i' \left(y_1 + y_2 - \delta(z) \right) \right]. \tag{18}$$

The right-hand side's first-order condition yields that player i's optimal choice is given by (11). Substituting into (18), one gets

$$rV_i(z) = a_i \frac{a_i + V_i'}{2} - \left(\frac{a_i + V_i'}{2} \right)^2 - c_i z^2 + V_i' \left[\frac{a_1 + V_1'}{2} + \frac{a_2 + V_2'}{2} - \delta(z) \right], \quad i = 1, 2. \tag{19}$$

For simplicity, denote $P_i(z) = V_i'(z)$ for $i = 1, 2$. Taking the derivative of (19) on both sides with respect to state variable z, one gets

$$rP_i = \frac{1}{2} \left\{ P_i' \left[2a + P_1 + P_2 - 2\delta(z) \right] + P_i \left[P_j' - 2\delta'(z) \right] \right\} - 2c_i z. \tag{20}$$

We can now prove the existence of Markovian Nash equilibria in the first period.

Proposition 2 (*Existence Of Stationary Markovian Perfect Nash Equilibria*). *Suppose that $c_1 = c_2 = c$ and that the following equation is solvable in terms of $P_s(\bar{z})$:*

$$3P_s(\bar{z})^2 + 4\left(a - \delta(\bar{z})\right) P_s(\bar{z}) + a_i^2 = 4\left(r\bar{V}_i + c\bar{z}^2 \right). \tag{21}$$

Let $P_s(\bar{z})$ be the root of quadratic equation (21) pertaining to the most concave value function $V_s(z)$. Then, there exist stationary Markovian perfect Nash equilibria, which are given by solutions to the following equation:

$$\left[\frac{3}{2} P_s(z) + a - \delta(z) \right] P_s'(z) = (r + \delta'(z)) P_s(z) + 2cz,$$

with terminal condition $P_s(\bar{z})$.

Furthermore, for the special linear decay function $\delta(z) = \alpha - \beta z$ and α, β as positive constants, the stationary Markovian perfect Nash equilibrium can be more precisely presented as

$$y_i(z) = \frac{a_i}{2} + \frac{1}{3} \left[Q_s(z) - a + \delta(z) \right] \qquad \text{for } z < \bar{z} \ \ i = 1, 2,$$

where $Q_s(z)$ satisfies the following equation:

$$\left| \frac{Q_s(z) - u_s^- \left(z + a_s/b_s \right)}{Q_s(\bar{z}) - u_s^- \left(\bar{z} + a_s/b_s \right)} \right|^{p_s} \left| \frac{Q_s(z) - u_s^+ \left(z + a_s/b_s \right)}{Q_s(\bar{z}) - u_s^+ \left(\bar{z} + a_s/b_s \right)} \right|^{1-p_s} = 1,$$

in which

$$u_s^- = \frac{1}{2} \left[r - \sqrt{r^2 + 4b_s} \right], \quad u_s^+ = \frac{1}{2} \left[r + \sqrt{r^2 + 4b_s} \right]$$

and

$$a_s = (\beta - r)(a - \alpha), \quad b_s = \beta(\beta - r) + 3c.$$

This result does not depend on the explicit form of decay function $\delta(z)$ (see the online Appendix A.2 for a more detailed explanation). We now study the key issue of the reachability of the irreversible regime. We generalize to the game context the intuitive property that such an outcome depends on the position of the steady state of the pollution dynamics induced by the Markovian equilibrium and the irreversibility threshold, $\bar{z}$. To this end, we start by substituting the above Markovian optimal strategies into the dynamic equation for $z < \bar{z}$. This yields

$$\dot{z} = f_s(z) \equiv P_s(z) + a - \delta(z) = \frac{1}{3} \left[2Q_s(z) + a - \delta(z), \right] \tag{22}$$

with initial condition $z(0) = z_0$ given.

Furthermore, pollution accumulation increases or decreases over time depending on the sign of $Q_s(z)$: If $Q_s(z_0) > 0$, then pollution accumulates until the time when $P_s + a - \delta = 0$. Let z'_s denote this root. At this point,

$$Q_s(z'_s) = \frac{1}{2}\left(\delta(z'_s) - a\right).$$

Hence, by Proposition 2, z'_s satisfies

$$S\left(z'_s, \frac{1}{2}\left(\delta(z'_s) - a\right)\right) = S\left(\bar{z}, Q_s(\bar{z})\right). \tag{23}$$

We conclude the above analysis in the following proposition:

Proposition 3. *Suppose $c_1 = c_2 = c$ and that Eq. (21) has a negative real root. Let z'_s satisfy (23). Then, under the Markovian perfect Nash equilibrium given by Proposition 2,*

(a) if $\bar{z} < z'_s$ the pollution decay threshold will be triggered in finite time $\bar{T}_s$, which is given by

$$\bar{T}_s = \int_{z_0}^{\bar{z}} \frac{dz}{P_s(z) + a - \delta(z)}; \tag{24}$$

(b) otherwise, if $\bar{z} \geq z'_s$ the pollution decay threshold will never be reached.

Proposition 3 provides the counterpart to Proposition 1 (cooperative case) for the game-theoretic context under Markovian strategies in the general non-concave decay case. As in Proposition 1, the intuition is clear but obtaining the characterization is highly non-trivial. We go a step further below and express the results in terms of the deep economic parameters of the model.

4.3. Reachability of the threshold and asymptotes

We now uncover the concrete parametric implications of the proposition above to better visualize the economic and ecological determinants of reaching the irreversible regime. We also explore the resulting asymptotes. Particular attention is paid to the comparison of the reachability of the irreversible regime between the cooperative and non-cooperative settings, with concrete numerical examples to support the theoretical arguments. We start with some general reachability conditions.

Proposition 4. *The following are true.*

1. *If $a \leq \delta(\bar{z})$ and*

$$\left(r + \delta'(\bar{z})\right)(a - \delta(\bar{z})) \leq 3c\bar{z}, \tag{25}$$

then $\bar{z}$ is never reached. Furthermore,

$$\lim_{t\to\infty} z(t) = z'_s, \tag{26}$$

where z'_s is given in Proposition 3.

2. *If $a > \delta(\bar{z})$, $\bar{z}$ is never reached if and only if*

$$c\bar{z}^2 \geq \frac{a_i^2 - (a - \delta(\bar{z}))^2}{4} - rV_i(\bar{z}). \tag{27}$$

Case 2 is economically more relevant as the decay is typically very low when we reach the irreversibility threshold (being equal to zero in TW). The intuition behind condition (27) is straightforward. The left-hand side is the direct cost of pollution accumulation, defined in the objective function, at the threshold $\bar{z}$. The right-hand side is the counterpart gain at the threshold and is twofold: The first part is the short-run net gain from emissions net of decay effects while the second part measures the long-run impact on the optimal value function when the threshold is crossed and the ecological system enters the second phase. The above condition provides the rather straightforward information that when the accumulated cost at the threshold is sufficiently high and dominates the gain, more effort will be made by both players to not reach the threshold. Otherwise, when the cost is not high enough, the threshold will be crossed in finite time.

In Proposition 1 for the cooperative case, a permanent reversible regime sets in if and only if

$$4c\bar{z} \geq \left(r + \delta'(\bar{z})\right)(a - \delta(\bar{z})).$$

Here, all the results are derived under condition (25), which is similar to the benchmark condition above. However, the Nash game displays different outcomes even if we restrict ourselves to the TW case discussed in item 2 of Proposition 4 (with $\delta(\bar{z}) = 0$). A second condition is required, namely (27): provided condition (25) holds, we get the permanent reversible case if and only if the addition condition (27) holds. This extra condition is due to the game setting, it does not emerge in the original TW central planner problem. We shall come back to this point in Section 6.1 when we collate all information regarding the equilibrium outcomes obtained in the different game frames considered along this paper. In the meantime, we study below the implications of this intricate additional condition in the linear decay case and prove some additional properties.

Asymptotic behavior. If $\bar{z}$ is reached in finite time, $\bar{T}$, depending on whether $\bar{z} < z_s^*$ or $\bar{z} \geq z_s^*$, the state exhibits different behavior for $t > \bar{T}$. In the first case, the state enters Period II and $z(t) \to z_s^*$ as $t \to \infty$. In the second case, $z(t) = \bar{z}$ for all $t > \bar{T}$.

The following proposition complements Proposition 4, with a particularly clear-cut result in the case of (locally) linear decay functions.

Proposition 5 (***Reachability of*** $\bar{z}$)**.** *Suppose*

$$\left(r + \delta'(\bar{z})\right)(a - \delta(\bar{z})) > 3c\bar{z}. \tag{28}$$

Then $\bar{z}$ is reached in finite time for some z_0 if either $a > \delta(\bar{z})$ or $a \leq \delta(\bar{z})$ and

$$c\bar{z}^2 \geq \frac{a_i^2 - (a - \delta(\bar{z}))^2}{4} - rV_i(\bar{z}) \tag{29}$$

holds. If, in addition, $\delta = \alpha - \beta z$ with positive α and β and $a \geq \alpha$, then $\bar{z}$ is reached in finite time for any $0 \leq z_0 \leq \bar{z}$.

In the case where $a > \delta(\bar{z})$, the two propositions above lead to the following simple necessary and sufficient condition (the detailed proof is given in the online Appendix A.5).

Corollary 1. *Suppose $a > \delta(\bar{z})$. For any fixed $c > 0$, we let $\bar{Z}$ be the largest solution z to the following equation:*

$$4cz^2 = a_i^2 - 4rV_i(z; c) - (a - \delta(z))^2. \tag{30}$$

Then $\bar{z}$ is reached in finite time from some z_0 if and only if $\bar{z} < \bar{Z}$.

Equivalently, for any fixed $\bar{z} > 0$ we let $\bar{C}$ be the largest solution c to the equation

$$4c\bar{z}^2 = a_i^2 - 4rV_i(\bar{z}; c) - (a - \delta(\bar{z}))^2. \tag{31}$$

Then $\bar{z}$ is reached in finite time for some z_0 if and only if $c < \bar{C}$. Furthermore,

$$\bar{C} > \frac{ar}{2\bar{z}}. \tag{32}$$

The case where $a > \delta(\bar{z})$ is quite interesting to get a sense of the implications of the more or less involved conditions displayed in Propositions 4 and 5. In this case, the irreversibility threshold is crossed if

$$c < \frac{1}{4\bar{z}}\left(r + \delta'(\bar{z})\right)(a - \delta(\bar{z}))$$

in the cooperative game, while the Nash counterpart requires $c < \bar{C}$. Since

$$\bar{C} > \frac{1}{4\bar{z}}\left(r + \delta'(\bar{z})\right)(a - \delta(\bar{z}))$$

by (32), the condition for the emergence of the irreversible regime is easier to check in the Nash case. This sounds intuitive: Cooperation generally allows reaching lower pollution levels in pollution games (see the survey by Van Long (2010), and there is no particular reason why the picture should change for irreversibility thresholds. Absence of cooperation will more frequently lead to regimes with more pollution (in this case, irreversibility). Our analytical exploration allows to assess rigorously to which extent free-riding aggravates the irreversibility problem and to closely relate the latter to the deep parameters of the model. We provide here below a further assessment based on open-loop strategies as compared with the MPE studied so far.

4.4. Comparison with open-loop strategies

Open-loop (OL) strategies are an alternative class of Nash strategies, which has been often studied in pollution games. OL players do not observe the state variables or they decide to commit to a given time function, therefore not involving any kind of feedback control from the players. Though MPE may suggest that by construction they leave more room for free-riding, comparison between MPE and OL strategies in terms of social welfare and other aggregate and individual indicators is often tricky, often delivering ambiguous outcomes, see for example Bertinelli et al. (2014, 2018).[4] As outlined in the Introduction, Wagener and de Zeeuw (2021) is the unique paper, to the best of our knowledge, which derives OL strategies under irreversibility thresholds. Rather than replicating their analysis on our model, we focus here on a key economic question: Do OL strategies lead to cross the irreversibility threshold more and less often than the MPE? We can respond to this question with a minimal amount of algebra. Consider our model presented in Section 2, more precisely the model in Period I, in the reversible regime. To solve for OL strategies, we depart by construction from dynamic programming and use the Pontryagin principle. The current value Hamiltonian for player i is[5]

$$\tilde{H}_i = \left(a_i y_i - y_i^2 - c_i z^2\right) + \lambda_i\left(y_1 + y_2 - \delta(z)\right), \qquad i = 1, 2.$$

where λ_i is the associated co-state variable. Using the Pontryagin maximum principle, the optimal control y_i^* satisfies $\frac{\partial H_i}{\partial y_i} = 0$, thus yielding

[4] For more formal definitions, examples and comparisons on MPE and OL strategies, see Dockner et al. (2000), Chapters 3 and 4.

[5] There is no discounting, $r = 0$, in Wagener and de Zeeuw (2021).

$$y_i^* = \frac{a_i + \lambda_i}{2} \qquad \text{for } i = 1, 2.$$

As a result,

$$E^* = y_1^* + y_2^* = a + \frac{\lambda_1 + \lambda_2}{2}.$$

Also using the Pontryagin condition $\partial H_i / \partial z = -\dot{\lambda}_i + r\lambda_i$ at the steady state of the reversible regime (so with $\dot{\lambda}_i = 0$), it follows that

$$-2c_i z^* - \lambda_i \delta'\left(z^*\right) = r\lambda_i.$$

Therefore,

$$4cz^* + 2\left(E^* - a\right)\left(\delta'\left(z^*\right) + r\right) = 0.$$

Combined with the steady state relation, $\dot{z} = 0$, that is

$$E^* = \delta\left(z^*\right),$$

it follows that

$$2cz^* = \left(\delta'\left(z^*\right) + r\right)\left(a - \delta\left(z^*\right)\right).$$

Hence, $\bar{z}$ being reachable if and only if $\bar{z} > z^*$, one gets the following reachability condition

$$2c\bar{z} < \left(\delta'\left(\bar{z}\right) + r\right)\left(a - \delta\left(\bar{z}\right)\right).$$

This condition should be compared with the counterpart under MPE, condition (28):

$$3c\bar{z} < \left(r + \delta'\left(\bar{z}\right)\right)\left(a - \delta\left(\bar{z}\right)\right)$$

It follows that MPE strategies require significantly lower pollution costs to lead to crossing the irreversibility threshold. Said differently, OL strategies require much higher pollution costs to avoid irreversibility. When irreversibility comes to the story, the flexibility of the MPE (based on a larger information set) clearly outperforms the OL strategies in terms of escaping irreversibility.

5. Non-cooperative games with asymmetric pollution costs

We now come to one of the most important contributions of this paper: the role of asymmetries. We relax the assumption of identical pollution costs and study the implications in terms of the frequency of emergence of the irreversible regime compared to the (symmetric) benchmark case. Using the methodology applied successfully above to this general case, we have not been able to uncover economically interpretable conditions for the emergence of the irreversible pollution regime, the obtained conditions being largely implicit. Only extreme asymmetric cases can allow for the derivation of explicit results (see the online Appendix A.8 for detailed computations in this special case, we shall use this case for illustrative purposes in Section 6). We therefore apply a different approach to infer the role of asymmetry. We explore a shortcut through the analysis of the equilibrium state equations

$$\dot{z} = f(z),$$

using a prior investigation of the optimal responses of the players as in Section 4.2 (with $f(.) \equiv f_s(.)$ where $f_s(.)$ is given by Eq. (22)). More specifically, we study the behavior of equilibrium emission rates (as captured by $f(z)$) in the neighborhood of the threshold $\bar{z}$. To this end, some analysis of equilibrium outcomes in the general case is needed.

Characterization of the irreversible regime ($z > \bar{z}$) For $z > \bar{z}$, $\delta(z) = 0$. We seek the value functions in the quadratic form

$$V_i(z) = A_i + B_i z + \frac{C_i}{2} z^2, \qquad i = 1, 2. \tag{33}$$

Substituting the quadratic functions into (12), it follows that

$$r\left[A_i + B_i z + \frac{C_i}{2} z^2\right] = \frac{1}{4}\left(a_i + B_i + C_i z\right)^2 + \frac{1}{2}\left(B_i + C_i z\right)\left(a_j + B_j + C_j z\right) - c_i z^2.$$

Comparing coefficients, we find that

$$\begin{aligned} 4rA_i &= \left(a_i + B_i\right)^2 + 2B_i\left(a_j + B_j\right), \\ 2rB_i &= C_i\left(a_i + B_i\right) + B_i C_j + C_i\left(a_j + B_j\right), \\ 2rC_i &= C_i^2 + 2C_i C_j - 4c_i. \end{aligned} \tag{34}$$

The existence and uniqueness of the solution to the above system with $C_1, C_2 < 0$ is given in Lemma 1 in the online Appendix A.7.

Characterization of the reversible regime ($z < \bar{z}$) and threshold-crossing conditions As derived in Section 4.2, $P_i(z) \equiv V_i'(z)$ satisfies (20). This is a linear system of differential equations for P_1 and P_2. Solving P_1' and P_2' from the system, we can write

$$P_i' = \frac{2\left[2a + P_1 + P_2 - 2\delta(z)\right]\left[\left(r + \delta'(z)\right)P_i + 2c_i z\right] - 2P_i\left[(r-\beta)P_j + 2c_j z\right]}{\left[2a + P_1 + P_2 - 2\delta(z)\right]^2 - P_1 P_2} \tag{35}$$

for $i, j = 1, 2,\ j \neq i$. The terminal values $P_i(\bar{z})$ are obtained by solving (19) at $\bar{z}$, which takes the form

$$r\bar{V}_i = \frac{1}{4}\left(a_i + P_i(\bar{z})\right)^2 + \frac{1}{2}P_i(\bar{z})\left[a_j + P_j(\bar{z}) - 2\delta(\bar{z})\right] - c_i\bar{z}^2 \qquad \text{for } i, j = 1, 2, \quad j \neq i \tag{36}$$

with

$$\bar{V}_i = A_i + B_i\bar{z} + \frac{C_i}{2}\bar{z}^2.$$

With the values $P_i(z)$ for $z < \bar{z}$ solved, we can find the value function $V_i(z)$ for $z < \bar{z}$ by

$$V_i(z) = \frac{1}{4r}\left(a_i + P_i(z)\right)^2 + \frac{1}{2r}P_i(z)\left[a_j + P_j(z) - 2\delta(z)\right] - \frac{c_i}{r}z^2.$$

To solve (36), we write the equation as

$$P_i(\bar{z})^2 + 4(a - \delta(\bar{z}))P_i(\bar{z}) + 2P_1(\bar{z})P_2(\bar{z}) - \Delta_i = 0,$$

where

$$\Delta_i = 4\left(r\bar{V}_i + c_i\bar{z}^2\right) - a_i^2.$$

Let $\lambda = 2P_1(\bar{z})P_2(\bar{z})$. Then, the equation becomes

$$P_i(\bar{z})^2 + 4(a - \delta(\bar{z}))P_i(\bar{z}) + \lambda - \Delta_i = 0.$$

The solution is

$$P_i(\bar{z}) = -2(a - \delta(\bar{z})) \pm \sqrt{4(a - \delta(\bar{z}))^2 + \Delta_i - \lambda} \qquad \text{for } i = 1, 2.$$

We assume that $P_i(\bar{z}) \leq 0$. Then the sign in front of the square root is positive if $a \geq \delta(\bar{z})$ and $\lambda \geq \Delta_i$, and it is negative if $a \geq \delta(\bar{z})$ or if $a < \delta(z)$ and $\lambda < \Delta_i$. Let $\sigma_i = 1$ if the sign is positive and $\sigma_i = -1$ if the sign is negative. We have

$$P_i(\bar{z}) = -2(a - \delta(\bar{z})) + \sigma_i\sqrt{4(a - \delta(\bar{z}))^2 + \Delta_i - \lambda}. \tag{37}$$

Then, λ is a solution to the following equation:

$$\lambda = 2\prod_{i=1}^{2}\left\{-2(a - \delta(\bar{z})) + \sigma_i\sqrt{4(a - \delta(\bar{z}))^2 + \Delta_i - \lambda}\right\}. \tag{38}$$

It ultimately follows that $f(\bar{z})$ has the form

$$f(\bar{z}) = -(a - \delta(\bar{z})) + \frac{1}{2}\sum_{i=1}^{2}\sigma_i\sqrt{4(a - \delta(\bar{z}))^2 + \Delta_i - \lambda}. \tag{39}$$

We can now compare the outcomes of symmetric vs asymmetric equilibria.

The implications of asymmetry for equilibrium free-riding We focus on the equilibrium-state dynamics

$$\dot{z} = f(z)$$

in order to evaluate the sign of the emission rate f in the neighborhood of $\bar{z}$. The emission rate f in Period I is given by

$$f(z) = a + P(z) - \delta(z),$$

where

$$P(z) = \frac{P_1(z) + P_2(z)}{2}.$$

Intuitively, if $f(\overline{z}) > 0$, the threshold $\overline{z}$ is located in a z-interval where pollution keeps on accumulating, which in turn would lead to crossing the threshold in finite time. In contrast, if $f(\overline{z}) < 0$, the dynamic system must already have reached its long-run steady state $\dot{z} = f(z) = 0$, which in turn would mean that the threshold will never be reached. In line with these intuitive arguments, we compare the emission rates $f(\bar{z})$ and $f_s(\bar{z})$ under asymmetry and symmetry, respectively, to infer the role of asymmetry in equilibrium free-riding. The next proposition presents one of the main results obtained.

Proposition 6. *Suppose c_1 and c_2 are both positive. Let $c = \left(c_1 + c_2\right)/2$, and let $P_s(z)$ be defined as in Proposition 2. Then, there is a $\tilde{z} > 0$ and an $\varepsilon > 0$ such that*

$$f(\bar{z}) \leq f_s(\bar{z})$$

whenever $\bar{z} > \tilde{z}$ and $\left|c_1 - c_2\right| < \varepsilon$.

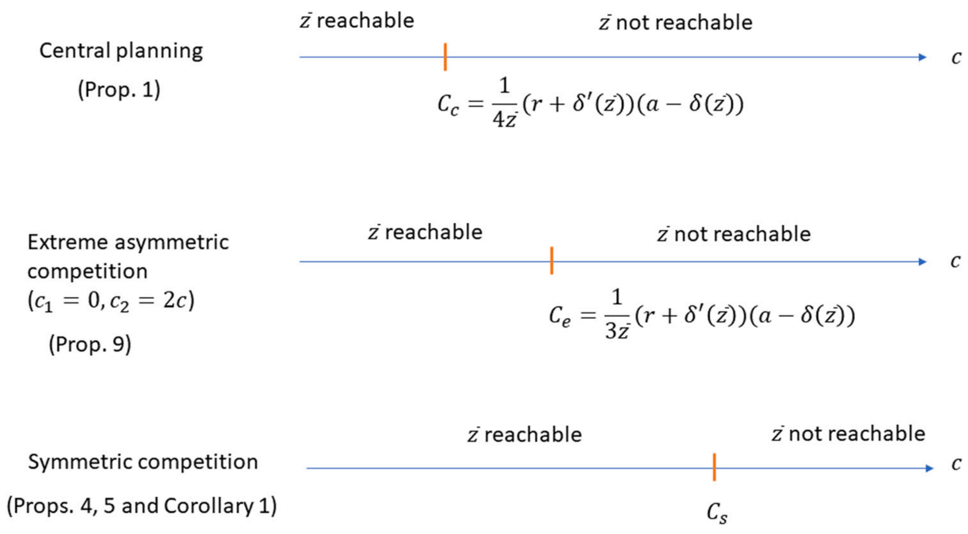

Fig. 1. Comparison of reachability conditions.

A proof is given in the online Appendix A.7. Consequently, if $\bar{z}$ is unreachable in the symmetric case it is also unreachable in the asymmetric case. Conversely, if $\bar{z}$ is reachable in the asymmetric case, it is also reachable in the symmetric case.

Note that the property according to which crossing the irreversibility threshold is more frequent under symmetry comes naturally from the fact that the equilibrium emission rate is greater in the latter, reflecting more intensive free-riding. We also note that the results are obtained for bounded asymmetry ($|c_1 - c_2| < \epsilon$). In fact, we demonstrate in the online Appendix A.8 that this property holds for extreme asymmetry: $c_1 = 0$, $c_2 = 2c > 0$. This limit case is indeed the one that displays the lowest level of free-riding (for the same total cost per unit of pollution, which reinforces the "local" result obtained just above. We dig deeper into the economics below.

6. Reachability, irreversibility, and institutional settings

This section synthesizes the main results of the sections above and provides additional comparative results. The first section focuses on the threshold reachability conditions essentially established in the above Propositions 1, 4, 5 and Corollary 1. The second section compares long-term outcomes once the irreversibility threshold is crossed, under different competition settings.

6.1. Reachability conditions under different competition settings

Fig. 1 summarizes the reachability conditions of threshold $\overline{z}$ in terms of the average cost parameter $\left(c_1 + c_2\right)/2 = c$ under the different settings, including the symmetric and asymmetric competition (including the extreme case, with $c_1 = 0$) described in Section 5 and analyzed in detail in the online Appendix A.8).

Before any further investigation, we must remark that in the last part of Fig. 1, the position of C_s depends on the combination of parameters. In the case where $a > \delta(\bar{z})$,

$$C_s = \bar{C} > \frac{1}{3\bar{z}}\left(r + \delta'(\bar{z})\right)(a - \delta(\bar{z})) = C_e > C_c$$

by Corollary 1, where $\bar{C}$ is the largest solution to Eq. (31). In the case where $a < \delta(\bar{z})$, by Proposition 5 $C_s = C_e$, provided that (29) holds. In particular, in the case where $a > \delta(\bar{z})$, the ranking of $C_s > C_e > C_c$ is unambiguous.

It is straightforward that regardless of the position of $\overline{C}$ (see Corollary 1), the $\overline{z}$-unreachable interval, that is, $[C_c, \infty)$, is the largest. This is not surprising as central planning yields the best scenario. The ranking between symmetric and extreme asymmetric competition is rather complicated as it depends on the location of $\overline{C}$. In the case of the bottom part of Fig. 1, the unreachable cost interval is larger under extreme asymmetric competition than in the symmetric situation: $[\overline{C}, \infty) \subset [C_e, \infty)$. But it may happen that $\overline{C} < C_e$, given the condition in Corollary 1, and then the opposite conclusion is true. The extra condition $\overline{C}$, defined in Corollary 1, comes directly from the competition differences; the intuition behind this condition was explained below Proposition 4. In the case where there is only one decision-maker who cares about the accumulated pollution cost (who suffers the most from the accumulation of pollution) – either the central planner or player 2 in the extreme asymmetric competition – the decision is made unambiguously depending on the cost-gain benefit analysis (explained below Proposition 1). This yields the conditions in terms of the threshold values C_c, C_s, and C_e. However, between these two polar cases – for , under symmetric competition – both players' efforts additionally depend on conditions of type (26), that is, on the cost generated by pollution accumulation at the threshold level compared to their respective net values. To compare the reachability of $\bar{z}$ in the case of asymmetric competition, we take into consideration

Proposition 6 and the continuity of the emission rates with respect to c_1 and c_2. For a fixed value of $c = \left(c_1 + c_2\right)/2$, the pollution growth rate in the asymmetric case, $f_a(z)$, is close to the rate in the symmetric case, $f_s(z)$, if the gap $\left|c_1 - c_2\right|$ is small, and $f_a(z)$ is close to the rate of the extreme asymmetric case, $f_e(z)$, if the gap $\left|c_1 - c_2\right|$ is large. In view of Proposition 6, $f_a(\bar{z}) \leq f_s(\bar{z})$ if $\bar{z}$ is large and the gap is small. Hence, if such a $\bar{z}$ is unreachable in the symmetric case, it is also unreachable in the asymmetric case, provided that the gap $\left|c_1 - c_2\right|$ is small. On the other hand, if the gap $\left|c_1 - c_2\right|$ is large, $f_a(\bar{z})$–being close to $f_e(\bar{z})$, which is negative for $c > C_e$–is also negative. Hence, $\bar{z}$ is again unreachable. For those c that satisfy

$$C_e < c < C_s,$$

as shown in Example 3 $\bar{z}$ is reachable if $\left|c_1 - c_2\right|$ is sufficiently small and is unreachable if $\left|c_1 - c_2\right|$ is sufficiently large. These facts appear to indicate that in terms of the ease of crossing the threshold, $\bar{z}$, the asymmetric case is between the extreme asymmetric case and the symmetric case and depends on the gap $\left|c_1 - c_2\right|$ in a decreasing way.

6.2. Institutional settings and long-term pollution outcomes

In this section, we provide an ordering of steady-state pollution levels with the four strategic settings so far considered once the irreversible threshold is crossed. Interestingly, we uncover that asymmetric games deliver less pollution in the long run than symmetric ones. We dig into the intuition and the economic interpretation later.

Let the steady states of pollution in the irreversible regime be denoted by z_c^*, z_s^*, z_e^*, and z_a^* for the central planner, symmetric, Nash extreme asymmetric Nash ($c_1 = 0$), and asymmetric Nash cases ($c_1, c_2 \neq 0$), respectively. The first three are given by (7), (14), and the online Appendix (92), respectively, and it can be shown – similar to the derivation of z_s^* – that

$$z_a^* = \frac{2a\left(\lambda + r^2 + r\sqrt{r^2 + 4(2c - \lambda)}\right)}{r(4c - \lambda) + (4c - 3\lambda)\sqrt{r^2 + 4(2c - \lambda)}}, \tag{40}$$

with λ being the unique positive solution to the following equation:

$$\left(r - \sqrt{r^2 + 4\left(c_1 - \lambda\right)}\right)\left(r - \sqrt{r^2 + 4\left(c_2 - \lambda\right)}\right) = 2\lambda. \tag{41}$$

In the online Appendix A.7, we show the following ranking.

Proposition 7. *For any non-negative c_1 and c_2, the steady states of pollution without decay, z_c^*, z_e^*, z_s^*, and z_a^*, given by (7), (92), (14), and (40), respectively, with $c = \left(c_1 + c_2\right)/2$, are ordered as*

$$z_c^* < z_e^* < z_a^* < z_s^*.$$

We first illustrate the finding with a numerical example.

Example 2. Using the same parameter values as in Example 1, TW show that the threshold $\bar{z} = 200$ is reached in finite time. We have shown in Examples 1–3 that the same $\bar{z}$ is reached in all other cases for equivalent parameter values. In addition, the limit of $z(t)$ is $z_c^* = 450$ in the central planner case (with $c = 0.002$), $z_s^* = 939.74$ in the symmetric case (with $c_1 = c_2 = 0.002$), and $z_e^* = 900$ in the extreme asymmetric case (with $c_1 = 0$ and $c_2 = 0.004$). For the asymmetric case with $c_1 = 0.0015$ and $c_2 = 0.0025$, the limit is $z_a^* = 937.14$. Thus, the ranking in the proposition is satisfied.

The intuition behind Proposition 7 is the following. The steady states are comparable only if they lie within the same pollution regime; we focus on the irreversible regime with zero decay, which is the more original exercise in this respect. Obviously, the central planner's optimal choice yields the best outcome with the lowest pollution accumulation. The other three cases are more intricate to compare at once, but we can visualize the results better if we compare first the symmetric and extreme asymmetric configurations, as the general asymmetric setting can be approached as an intermediate case between the two former.

Consider the pollution accumulation dynamics $\dot{z} = y_1 + y_2$ and the efforts of both players $y_i = \frac{a_i}{2} + \frac{B_i + C_i z}{2}$ and $y_i^m = \frac{a_i}{2} + \frac{B^m + C^m z}{2}$ under extreme asymmetric and symmetric competition, respectively. Given $B_i \leq 0, B^m < 0$, and $C_i < 0, C^m < 0$, we can interpret $\frac{B_i + C_i z}{2}$ and $\frac{B^m + C^m z}{2}$ as the effort made by players to reduce pollution accumulation in these two cases. Thus, it is straightforward to see that under the extreme asymmetric competition ($c_1 = 0$), player 1 makes no effort to help reducing pollution (since $C_1 = 0, B_1 = 0$). In contrast, player 2, who bears the highest cost from the accumulation of pollution, will make a substantial effort to reduce pollution. It is easy to show that

$$\frac{C_2}{2} < C^m < 0 \text{ and } \frac{B_2}{2} < B^m < 0.$$

Thus, for any z it follows that at the aggregate level,

$$\frac{B_2 + C_2 z}{2} < B^m + C^m z.$$

In other words, under extreme asymmetric competition, player 2 with $c_2 = 2c$ makes more effort to clean up pollution than the sum of two players in the symmetric case ($c_1 = c_2 = c$). In the symmetric case, the well-known free-riding mechanism is fully at work: Both players wait for the other to make more effort, and neither ends up making enough effort. The general asymmetric case lies in between: The player who faces higher accumulated pollution damage will make more effort to reduce pollution, while the one that is less sensitive to accumulated pollution free-rides on the other's efforts. But the global impact of free-riding is lower than under symmetry.[6]

7. Conclusion

In this paper, we have developed an extension of the TW hard pollution irreversibility problem to differential game settings. As we have kept the inherent original non-concavity feature, the induced mathematical setting is highly non-trivial. However, we have been able to provide a comprehensive theoretical analysis of the considered games. Abstracting from the methodological insight, some original contributions should be stressed. First, considering that pollution costs are evenly distributed across players, we have accurately characterized the extent of the irreversibility penalty in this case relative to cooperation. Moreover, we have compared the open-loop with the MPE strategies regarding the reachability of the irreversible regime, and find that feedback control inherent in MPE might well significantly lower the probability to cross the irreversibility thresholds. Finally, we have studied the implications of asymmetry in the pollution cost. We find that for equal total pollution costs, asymmetric equilibria produce a lower emission rate than the symmetric under some mild conditions, thereby driving the system to irreversibility less frequently than the latter. Finally, we have proven that provided the irreversible regime is reached in both the symmetric and asymmetric cases, long-term pollution is greater in the symmetric case, reflecting more intensive free-riding under symmetry.

Needless to say, our results are worth examining in richer settings. One quite interesting extension – also extending the related analysis of Barrett (2013) from static to dynamic games – would be to consider uncertainty either in the value of irreversibility thresholds ($\bar{z}$) or in the extent of irreversibility (that is, under a random magnitude of the drop in pollution decay for a given threshold). A second even more promising avenue is to exploit our finding regarding free-riding in symmetric versus asymmetric games for the design of international environmental agreements (see, for example, Carraro and Siniscalco (1993)). In the typical case (symmetric game with n players), the key parameter is the size of the coalition (see Wagener and de Zeeuw (2021)). Our results indicate that cost heterogeneity (or perhaps other types of heterogeneity) might also be very relevant. Obviously, it is not granted that we can keep the fully analytical approach when dealing with these natural extensions.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.jeem.2023.102841.

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[6] In the online Appendix A.10, we show that the same ranking is true for the perpetual reversible region with a constant decay rate.

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