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QUANTIFYING SOME PROPERTIES OF CURVES AND ARCS ON HYPERBOLIC SURFACES

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Summary

Motivated by the ergodicity of geodesic flow on the unit tangent bundle of a closed hyperbolic surface and its applications, this thesis includes three parts:

Part 1. We present a type of quantitative density of closed geodesics and orthogeodesics on complete finite-area hyperbolic surfaces. The main results are upper bounds on the length of the shortest closed geodesic and the shortest doubly truncated orthogeodesic that are ε-dense on a given compact set on the surface. The content of this part is contained in [18].

Part 2. We investigate the terms arising in Luo-Tan's identity, namely showing that they vary monotonically in terms of lengths and that they verify certain convexity properties. Using these properties, we deduce two results. As a first application, we show how to deduce a theorem of Thurston which states, in particular for closed hyperbolic surfaces, that if a simple length spectrum "dominates" another, then in fact the two surfaces are isometric. As a second application, we show how to find upper bounds on the number of pairs of pants of bounded length that only depend on the boundary length and the topology of the surface. This is joint work with Hugo Parlier and Ser Peow Tan [19].

Part 3. Inspired by a number theoretic application of Bridgeman's identity, the combinatorial proof of McShane's identity by Bowditch and its generalized version by Labourie and Tan, we describe a tree structure on the set of orthogeodesics and give a combinatorial proof of Basmajian's identity in the case of surfaces. We also introduce the notion of orthoshapes with associated identity relations and indicate connections to length equivalent orthogeodesics and a type of Cayley-Menger determinant. As another application, dilogarithm identities following from Bridgeman's identity are computed recursively and their terms are indexed by the Farey sequence. This part is contained in [20].

Contents

Introduction

The ergodicity of geodesic flow [21] tells us that if one takes a point in $T^1(X)$ (the unit tangent bundle of a closed hyperbolic surface X) then, with respect to the Liouville measure, under the action of the geodesic flow, almost surely it will visit almost everywhere in $T^1(X)$.

0.1. Quantifying density

The ergodicity of geodesic flow implies the density of closed orbits on the unit tangent bundle of a closed hyperbolic surface. Indeed, if one covers $T^1(X)$ with a set of balls of radius ε, a "random point" in $T^1(X)$ under the geodesic flow will visit almost everywhere. Thus, after a sufficient amount of time, one can stop the movement and close the orbit to form an ε-dense closed orbit (Anosov closing lemma [2]). A type of quantitative density of closed geodesics on closed hyperbolic surfaces and their unit tangent bundles was investigated by Basmajian, Parlier, and Souto in [3]. In particular, for any closed hyperbolic surface X and any positive number ε, they found an upper bound on the length of the shortest closed geodesic that is ε-dense on X, by which it is meant that for every $x \in X$, the distance from x to the geodesic is less than ε. In Chapter 1, we extend their results to the case of complete finite-area hyperbolic surfaces in two directions. The first is that for any complete finite-area hyperbolic surface and any positive number ε less than or equal to 2, we construct a closed geodesic γ_ε so that γ_ε is ε-dense on a given compact set, namely C, of the surface and its length is bounded above by a quantity which depends on the geometry of X and ε. The second is that we construct a doubly truncated orthogeodesic that is ε-dense on C and also of bounded length. These types of orthogeodesics appear for instance in identities [33] related to McShane's identity [31] and Basmajian's identity [5].

Main results. Let $\mathcal{M}_{g,n}$ be the moduli space of complete connected orientable finite area hyperbolic surfaces of genus g and n cusps. For any X in $\mathcal{M}_{g,n}$ and any positive number $\xi \leq 2$, we define X^ξ as a subset of X such that X^ξ is homeomorphic to X and each boundary element of X^ξ is a horocycle of length ξ. A geodesic arc on X is called a doubly truncated orthogeodesic on X^ξ if it is perpendicular to the horocyclic boundary of X^ξ at both of its endpoints. Then

Theorem 1. *For all $X \in \mathcal{M}_{g,n}$ there exists a constant $C_X > 0$ depending on X such that for all $0 < \xi \leq 1$ and all $0 < \varepsilon \leq 2$ there exists a closed geodesic γ_ε that is ε-dense on X^ξ and such that*

$$\ell(\gamma_\varepsilon) \leq C_X \frac{1}{\varepsilon}\left(\log \frac{1}{\varepsilon} + \log \frac{1}{\xi}\right).$$

Theorem 2. *For all $X \in \mathcal{M}_{g,n}$ there exists a constant $D_X > 0$ depending on X such that for all $0 < \xi \leq 1$ and all $0 < \varepsilon \leq \min\{2\log\frac{1}{\xi}, 2\}$ there exists a doubly truncated orthogeodesic $\mathcal{O}_\varepsilon$ that is ε-dense on X^ξ and such that*

$$\ell(\mathcal{O}_\varepsilon) \leq D_X \frac{1}{\varepsilon}\left(\log \frac{1}{\varepsilon} + \log \frac{1}{\xi}\right).$$

Our main ingredient in the proof of Theorem 1 and Theorem 2 is the following result.

Theorem 3. *For all $X \in \mathcal{M}_{g,n}$, there exists a constant K_X depending on X such that the following holds. For all $0 < \varepsilon \leq 1$, $0 < \xi \leq 1$ and any finite collection $\{c_i\}_{i=1}^N$ of geodesic arcs of average length $\bar{c}$ in X^ξ, there exists a closed geodesic γ of length at most*

$$N\left(K_X + \bar{c} + 10\log\frac{1}{\varepsilon} + 8\log\frac{1}{\xi}\right)$$

containing $\{c_i\}_{i=1}^N$ in its 2ε-neighborhood.

0.2. An investigation on the Luo-Tan identity

The ergodicity of geodesic flow gives rise to two astonishing identities: Bridgeman's identity [13] and Luo-Tan's identity [29]. These two identities connect geometric quantities (lengths of simple closed curves or orthogeodesics) and the Euler characteristic of a hyperbolic surface. In Chapter 2, we investigate the measures of the Luo-Tan identity, and a refinement of the identity due to Hu and Tan [23]. The Luo-Tan identity states:

$$\sum_{P \in \mathcal{P}} \varphi(P) + \sum_{T \in \mathcal{T}} \tau(T) = 8\pi^2(g-1).$$

where the sums are taken over so-called properly embedded pairs of pants and one holed tori. The measures, φ and τ, are functions that depend explicitly on the geometries of P or T. The Hu-Tan variation of the identity can be stated as follows:

$$\sum_{P \in \mathcal{P}} \varphi(P) + \sum_{P \in \mathcal{I}} \eta(P) = 8\pi^2(g-1),$$

where the sums are taken over properly and improperly embedded pants, φ is the same as before and η is a different function from φ but which also depends explicitly on the geometry of P.

Main results. Our main results are about analytic properties of the measures. We state the most striking (and useful) properties here, which concern the measures φ and η. As they depend only on the geometry of the pants, they depend only on the boundary lengths of the pants. Hence φ depends on three real parameters and η only two as two of its boundary curves are of equal length. For practical reasons it is useful to consider, instead of length ℓ, the parameter $t := e^{-\ell/2}$. With these parameters, our results can be expressed as follows.

Theorem 4. *The functions φ and η are strictly increasing on $(0,1]^3$ and $(0,1]^2$, respectively, and satisfy*

$$\varphi(x,y,y) \leq \eta(x,y).$$

Furthermore if we set $t := \sqrt[3]{xyz}$ then

$$\varphi(x,y,z) \geq \varphi(t,t,t) > -24t^3\log(t) + 24t^3$$

for all $x,y,z \in (0,1]$.

In particular this says that the measures are strictly decreasing with respect to boundary length. One might expect this as they necessarily converge to 0 as the lengths increase (because there are infinitely many terms in the sum which adds up to something finite), but in fact there is no obvious geometric reason for this to hold infinitesimally and our proof is entirely analytic. As in Bridgeman's identity, the functions involve Rogers' dilogarithm function and are of intrinsic interest, but our original motivation for studying them was for possible applications.

From our result, we are able to deduce a few corollaries. As a first application, we recover a well-known and useful theorem of Thurston's about dominating length spectra [39].

Corollary 1 (Thurston)**.** *If X and Y are marked and closed hyperbolic surfaces of genus g that satisfy $\ell_X(\gamma) \geq \ell_Y(\gamma)$ for all simple closed geodesics γ, then $X = Y$.*

The surfaces X and Y are points in Teichmüller space (the space of marked hyperbolic metrics) and Thurston used this result to deduce a positivity result for his asymmetric metric on Teichmüller space, related to Lipschitz maps between hyperbolic surfaces.

It should be noted that the same result for surfaces with cusps is easily deduced from McShane's identity. Indeed, the summands in the McShane identity are of the form $1/(e^{(\ell(\alpha)+\ell(\beta))/2}+1)$ and thus are obviously strictly decreasing in both $\ell(\alpha)$ and $\ell(\beta)$. A more general observation of this type can be found in the work of Charette and Goldman [16].

As a second application, we count pants, and find an upper bound on the number of pants of total boundary length L a surface of genus g can have.

Corollary 2. *A closed hyperbolic surface X of genus g has strictly less than*

$$\frac{2\pi^2(g-1)e^{L/2}}{L+6}$$

embedded geodesic pants of total boundary length less than L.

This result is related to other results about curve counting. Of course, by the celebrated results of Mirzakhani [32], the number of pairs of pants grows asymptotically like $C_X L^{6g-6}$ where C_X is a constant that depends on the surface, so it is far from optimal for large L. Nonetheless, the result above is an absolute upper bound that does not depend on the geometry of the surface. In particular it holds for all $L > 0$, including relatively small L. A more directly related result is a result of Buser [14] which says that a surface of genus g has at most $(g-1)e^{L+6}$ primitive closed geodesics of length at most L. This result is used, among other things, to find upper bounds on the number of non-isometric surfaces that can have the same length spectra. Also notice that Buser's result can be applied to find an upper bound on the number of pants of total length L, but the result is a lot weaker. In a nutshell, Buser's upper bound and the above corollary are related, but do not follow from one another.

One of the novelties of the Luo-Tan identity is that it *also* holds for closed surfaces, hence for simplicity we've stated our results in this context. However, with the usual caveats, they generalize without difficulty to surfaces with cusps. This can be seen either by applying the same methods, or by considering cusped surfaces as lying in the compactification of the underlying moduli space.

0.3. A tree structure on the set of orthogeodesics

The set of orthogeodesics, introduced by Basmajian in the early 90's, is the set of geodesic arcs perpendicular to the boundary of a hyperbolic manifold at their ends. In [5], he proved an identity, the so-called Basmajian's identity, which in the case of surfaces, involves the ortho length spectrum and the perimeter (total length of boundary). About 20 years later, Bridgeman discovered an identity [13] which relates the ortho length spectrum and the Euler charateristic of a hyperbolic surface. Let X be a hyperbolic surface of totally geodesic boundary, Basmajian's identity and Bridgeman's identity on X can be expressed respectively as follows:

$$\ell(\partial X) = \sum_{\eta} 2\log(\coth(\ell(\eta)/2)),$$

$$-\frac{\pi^2}{2}\chi(X) = \sum_{\eta} \mathcal{L}\left(1/\cosh^2(\ell(\eta)/2)\right),$$

where both of the sums run over the set of orthogeodesics on the surface and $\mathcal{L}$ is the Roger's dilogarithm function [40], [26].

Recently there have been some results related to the set of orthogeodesics such as the weak rigidity of ortho length spectrum [30], the asymptotic growth of the number of orthogeodesics up to given length [8], and other identities [6], [33]. Among applications of these identities, there is a connection between number theory and Bridgeman's identity in some special surfaces, in particular, it derived classical and infinitely many new dilogarithm identities [12],[25]. Influenced by these results, our initial purpose in studying the set of orthogeodesics was to give a precise description of dilogarithm identities derived from Bridgeman's identity on a pair of pants. This journey leads to a tree structure on the set of oriented orthogeodesics and identity relations, which lead us to a combinatorial proof of Basmajian's identity by using the approach in Bowditch's paper [10]. Bowditch's method was originally used to give a combinatorial proof of McShane's identity [31], then later on was applied to different contexts to obtain many other descendants and generalizations. Furthermore, in [27], Labourie and Tan generalized the idea of Bowditch to a more sophisticated viewpoint and gave a planar tree coding of oriented simple orthogeodesics on hyperbolic surfaces together with a probabilistic explanation of McShane's identity for higher genus surfaces.

Main results. Let S be an orientable hyperbolic surface with boundary ∂S consisting of simple closed geodesics. Let $\eta_{\frac{0}{1}}$ and $\eta_{\frac{1}{1}}$ be two oriented orthogeodesics starting from a simple closed geodesic α at the boundary of S. The starting points of these two orthogeodesics divides α into two open subsegments, namely α_1 and α_2. Suppose that $\alpha_1 \neq \emptyset$. Denote by $\mathcal{O}_{\alpha_1}$ the set of oriented orthogeodesics starting from α_1. Let T_1 be a planar rooted trivalent tree whose the first vertex is of valence 1, and all other vertices are of valence 3. Let $E(T_1)$ be the set of edges of T_1. Each edge of the tree has two sides associated to two neighboring complementary regions of the tree (see Figure 3.6 for an illustration). Let $\Omega(T_1)$ be the set of complementary regions of the tree. Then

Theorem 5. *If $\eta_{\frac{0}{1}}$ and $\eta_{\frac{1}{1}}$ are distinct, there is an order-preserving bijection between $\mathcal{O}_{\alpha_1} \cup \{\eta_{\frac{0}{1}}, \eta_{\frac{1}{1}}\}$ and $\Omega(T_1)$.*

In order to present a combinatorial proof of Basmajian's identity, we define a weight map Φ on the set of edges and complementary regions of the tree T_1. This map satisfies conditions coming from the fact that we want them to correspond to the cosh length function $\cosh(\ell(.))$. In particular, $\Phi : E(T_1) \sqcup \Omega(T_1) \to (1, \infty)$ which has a harmonic relation at any vertex except at the root of the tree. Basmajian's identity for the tree T_1 can be expressed in the following form:

Theorem 6. *(Basmajian's identity for T_1) If $\sup\{\Phi(x) | x \in E(T_1)\} < \infty$, then*

$$\log\left(\frac{x_0 + Y_0 Z_0 + \sqrt{x_0^2 + Y_0^2 + Z_0^2 + 2x_0 Y_0 Z_0 - 1}}{(Y_0 - 1)(Z_0 - 1)}\right) = \sum_{X \in \Omega(T_1)} \log\left(\frac{X+1}{X-1}\right)$$

where (x_0, Y_0, Z_0) is the initial edge region triple at the root of the tree. Note that x_0, X, Y_0, Z_0 are the abbreviations of $\Phi(x_0), \Phi(X), \Phi(Y_0), \Phi(Z_0)$ respectively.

The following corollary is a combinatorial form of Basmajian's identity for the set of oriented orthogeodesics starting from a simple closed geodesic, say o_1, on the boundary of a hyperbolic surface. Suppose that o_1 is divided into n subsegments by a hexagonal decomposition.

Corollary 3. *(Basmajian's identity for T_n) Let T_n be a rooted trivalent tree with n edges starting from the root. If $\sup\{\Phi(x) | x \in E(T_n)\} < \infty$, then*

$$\sum_{k=1}^{n} \operatorname{arccosh}\left(\frac{x_{0,k} + Y_{0,k} Z_{0,k}}{\sqrt{(Y_{0,k}^2 - 1)(Z_{0,k}^2 - 1)}}\right) = \sum_{X \in \Omega(T_n)} \log\left(\frac{X+1}{X-1}\right),$$

where $(x_{0,k}, Y_{0,k}, Z_{0,k})$'s are edge region triples at the root of the tree. Note that $Z_{0,k} = Y_{0,k+1}$ for $k = \overline{1,n}$, in which $Y_{0,n+1} := Y_{0,1}$.

Besides that, we introduce several identity relations of the distances between geodesics and horocycles on the hyperbolic plane: Harmonic relations, Ptolemy relations of geodesics, mixed Ptolemy relations, relations of quintet of geodesics/horocycles, and their special cases so-called orthoshapes (ortho-isosceles trapezoid, orthorectangle, orthokite, orthoparallelogram relation). Some of these relations are restricted to the tree of orthogeodesics on hyperbolic surfaces in the form of recursive formulae, isosceles trapezoid, rectangle, kite, parallelogram, edge relations. As we will see in what follows, these relations are related to a type of Cayley-Menger determinant.

Let U and V be two arbitrarily disjoint geodesics/horocycles in $\mathbb{H}$. Denote κ_U and κ_V to be the geodesic curvatures of U and V respectively. We define a weight function between U and V which is a generalized version of the half-trace and the half of Penner's lambda length of the distance between U and V as follows.

$$\overline{UV} := \frac{e^{\frac{1}{2}d_{\mathbb{H}}(U,V)} + (1-\kappa_U)(1-\kappa_V)e^{-\frac{1}{2}d_{\mathbb{H}}(U,V)}}{2}$$

We obtain the following relation in the form of Cayley-Menger determinant:

Theorem 7. *Let $\{A_1, A_2, A_3, A_4\}$ be the set of four disjoint geodesics/horocycles in $\mathbb{H}$, each of them divides $\mathbb{H}$ into two domains such that the other three lie in the same domain. Then*

$$\det\begin{bmatrix} 2 & 1-\kappa_{A_1} & 1-\kappa_{A_2} & 1-\kappa_{A_3} & 1-\kappa_{A_4} \\ 1-\kappa_{A_1} & 0 & \overline{A_1A_2}^2 & \overline{A_1A_3}^2 & \overline{A_1A_4}^2 \\ 1-\kappa_{A_2} & \overline{A_2A_1}^2 & 0 & \overline{A_2A_3}^2 & \overline{A_2A_4}^2 \\ 1-\kappa_{A_3} & \overline{A_3A_1}^2 & \overline{A_3A_2}^2 & 0 & \overline{A_3A_4}^2 \\ 1-\kappa_{A_4} & \overline{A_4A_1}^2 & \overline{A_4A_2}^2 & \overline{A_4A_3}^2 & 0 \end{bmatrix} = 0.$$

Note that in the special case when $\kappa_{A_i} = 1$ for all $i \in \{1,2,3,4\}$, the equation gives us the Penner's Ptolemy relation [35]. We suspect this theorem can be generalized to hyperbolic spaces of higher dimensions.

Related to the orthoshapes, we also investigate some types of r-orthoshapes (see Definition 6). Generally, an r-orthoshape is a set of (finite) orthogeodesics satisfying some conditions which hold for any hyperbolic structure on a surface. In this thesis, we are interested in r-orthoshapes which are related to length equivalent orthogeodesics. Let S be an arbitrary hyperbolic surface with totally geodesic boundary, we show that:

Theorem 8. *The involution reflections on immersed pair of pants yield infinitely many r-ortho-isosceles-trapezoids, r-orthorectangles and r-orthokites on S. However, there are no r-orthosquares on S.*

One can conjecture that all of these shapes arise from the reflection involutions on immersed pair of pants. We hope, by studying these r-orthoshapes, one may shed a light on the length equivalent problem studied in [1] and [28].

An orthobasis on a hyperbolic surface is a set of pairwise disjoint simple orthogeodesics which decomposes the surface into orthotriangles (see Definition 1). A hyperbolic surface is ortho-integral if the hyperbolic cosine of all ortholengths are integers. Denote by $\mathcal{O}_S$ the set of orthogeodesics on a hyperbolic surface S. Using the recursive formulae and/or edge relations, one can give conditions on pairs of pants and one-holed tori such that they are ortho-integral.

Theorem 9. *Let P be a pair of pants and T a one-holed torus. Then*

- *P is ortho-integral if there is an orthobasis $\{a, b, c\}$ on P such that* $\cosh \ell(a) = \cosh \ell(b) = \cosh \ell(c) \in \{2, 3\}$.
- *T is ortho-integral if there is an orthobasis $\{a, b, c\}$ on T such that one of the following happens*
  - $\cosh \ell(a) = \cosh \ell(b) = \cosh \ell(c) \in \{2, 3\}$
  - $\cosh \ell(a) = 3, \cosh \ell(b) = 17, \cosh \ell(c) = 21$
  - $\cosh \ell(a) = 2, \cosh \ell(b) = 7, \cosh \ell(c) = 10$
  - $\cosh \ell(a) = 17, \cosh \ell(b) = 19, \cosh \ell(c) = 37$
  - $\cosh \ell(a) = 7, \cosh \ell(b) = 17, \cosh \ell(c) = 25.$

Thank to the recursive formulae and/or edge relations, we can compute the hyperbolic cosine of ortholengths of a hyperbolic surface and decribe the terms in Basmajian's identity and Bridgeman's identity recursively. We list here some examples of these identities involving the orthospectrum (with multiplicity) of some ortho-integral surfaces:

- $\phi^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 = 2\left(\frac{10}{9}\right)\left(\frac{32}{31}\right)^2\left(\frac{90}{89}\right)^2\left(\frac{122}{121}\right)^2\left(\frac{242}{241}\right)^2\left(\frac{362}{361}\right)^4\left(\frac{450}{449}\right)^2 \dots$
- $\frac{\pi^2}{6} = \mathcal{L}\left(\frac{2}{3}\right)+\mathcal{L}\left(\frac{2}{18}\right)+2\mathcal{L}\left(\frac{2}{75}\right)+2\mathcal{L}\left(\frac{2}{288}\right)+2\mathcal{L}\left(\frac{2}{363}\right)+2\mathcal{L}\left(\frac{2}{1083}\right)+4\mathcal{L}\left(\frac{2}{1443}\right)+\dots$
- $\frac{\pi^2}{6} = \mathcal{L}\left(\frac{1}{2}\right)+\mathcal{L}\left(\frac{1}{10}\right)+2\mathcal{L}\left(\frac{1}{32}\right)+2\mathcal{L}\left(\frac{1}{90}\right)+2\mathcal{L}\left(\frac{1}{122}\right)+2\mathcal{L}\left(\frac{1}{242}\right)+4\mathcal{L}\left(\frac{1}{362}\right)+\dots$

- $\dfrac{\pi^2}{2} = \mathcal{L}\left(\dfrac{1}{2}\right) + \mathcal{L}\left(\dfrac{1}{9}\right) + 2\mathcal{L}\left(\dfrac{1}{10}\right) + 2\mathcal{L}\left(\dfrac{1}{11}\right) + 2\mathcal{L}\left(\dfrac{1}{19}\right) + 2\mathcal{L}\left(\dfrac{1}{32}\right) + 2\mathcal{L}\left(\dfrac{1}{41}\right) + \dots$
- $\dfrac{\pi^2}{2} = \mathcal{L}\left(\dfrac{1}{9}\right) + \mathcal{L}\left(\dfrac{1}{10}\right) + 2\mathcal{L}\left(\dfrac{1}{19}\right) + 2\mathcal{L}\left(\dfrac{1}{72}\right) + 2\mathcal{L}\left(\dfrac{1}{82}\right) + 2\mathcal{L}\left(\dfrac{1}{90}\right) + 2\mathcal{L}\left(\dfrac{1}{99}\right) + \dots$

One can find more details of these examples in Section 3.5.3.

Chapter 1

Quantifying density

This chapter will give upper bounds on the length of the shortest closed geodesic and the shortest doubly truncated orthogeodesic that are ε-dense on a given compact set of a complete connected orientable finite area hyperbolic surface of genus g and n cusps.

1.1. Geodesics and horocycles in $\mathbb{H}$ and on surfaces

In this section, we introduce some elementary properties of geodesics traveling through subsurfaces which we will use to prove Theorem 10. Let P_n be a hyperbolic subsurface with a single polygonal boundary of n concatenated geodesic edges such that all angles are less than π. Figure 1.1 shows an example of P_1.

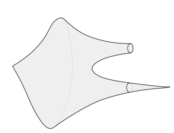

Figure 1.1

The following lemma is an extended version of Lemma 1 in [3].

Lemma 1. *There exists $\theta_P > 0$ such that any geodesic arc c lying inside P_n with endpoints on edges of the n-gons forms an acute angle of at least θ_P in one of its endpoints. Furthermore, the length of c is at most a constant ℓ_P if one of the angles has value less than or equal θ_P.*

Proof. We first label the vertices of the n-gonal boundary of P_n by $A_1, A_2, ..., A_n$ consecutively. For each $i \in \{1, 2, ..., n\}$, we can connect A_i to A_{i+2} by a shortest geodesic arc lying inside the interior of P_n (in which $A_{n+1} := A_1$ and $A_{n+2} := A_2$) such that there is no cusp or geodesic boundary component in the resulting triangle $A_i A_{i+1} A_{i+2}$. We call each such resulting triangle to be an ear of P_n. In the set of inner angles of the ears in P_n, we denote by θ_P their minimum value. Also, in the set of sides of the ears in P_n, we denote by ℓ_P their maximum value.

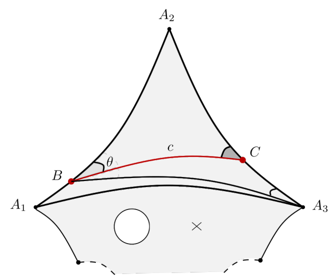

Figure 1.2

Without loss of generality, we can assume that the geodesic arc c leaving from the point B on a side A_1A_2 of the n-gons forms an angle θ of at most θ_P as in Figure 1.2. Since the triangle BA_2A_3 is contained in the ear $A_1A_2A_3$, $\text{Area}(BA_2A_3) \leq \text{Area}(A_1A_2A_3)$. As these two triangles are sharing an angle A_2, by Gauss-Bonnet, the sum of the two remaining angles of BA_2A_3 is greater than or equal the sum of the angles A_1 and A_3 of the ear. Since the angle at A_3 of its is less than the one of the ear, the angle at B of its is greater than the one at A_1 of the ear, and is thus greater than θ_P and θ. Hence c lies inside BA_2A_3, and this implies that c also lies inside the ear and the triangle BA_2C is contained in the ear. By the same argument, we can show that the angle at C (i.e. the remaining angle formed by c and an edge of the n-gons) is greater than θ_P. The fact c lies inside the ear also tell us that the length of c has to be less than or equal at least one of three sides of the ear, hence $\ell(c) \leq \ell_P$. □

In this chapter, we only need to focus on the case of once-punctured polygons. We also refer the reader to [14] (chapter 2) and [7] (chapter 7) for all trigonometric fomulas. The following lemma will give us an upper bound on the length of the geodesic arc that traverses inside the polygon with endpoints lying on the boundary of the polygon.

Lemma 2. *Let P be a once-punctured polygon and $\psi \in [0, \frac{\pi}{2})$ a constant. Let h be a closed horocycle lying inside P. Let d be the maximal distance from a point on ∂P to h. Then for any geodesic arc c in P with two end points on ∂P and $\theta := \measuredangle(c, h) \leq \psi$, we have*

$$\ell(c) \leq \operatorname{arccosh}\left(\frac{2e^{2d}}{\cos^2\psi} - 1\right).$$

Proof. We lift to $\mathbb{H}$ as in Figure 1.3.

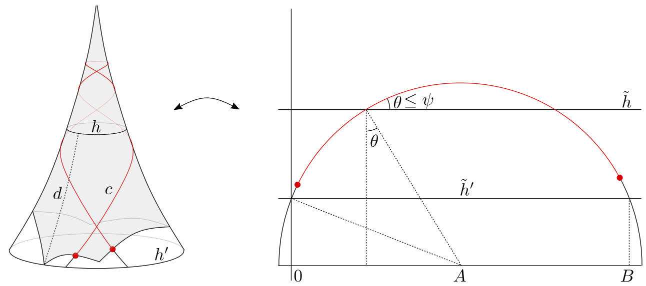

Figure 1.3

The length of c is upper bounded by the length of the geodesic segment with endpoints i and $2A+i$, where

$$A^2+1=\left(\frac{e^d}{\cos\theta}\right)^2.$$

Hence

$$\cosh \ell(c) \leq \cosh(d_{\mathbb{H}}(2A+i,i)) = 1+\frac{|2A|^2}{2} = \frac{2e^{2d}}{\cos^2\theta}-1 \leq \frac{2e^{2d}}{\cos^2\psi}-1.$$

□

The next lemma describes some propeties in a certain type of quadrilateral. Let h_1 and h_2 be two disjoint horocycles in $\mathbb{H}$. Let A_1A_2 be the common orthogonal between h_1 and h_2. For $i=1,2$, let B_i, C_i be points on h_i so that $B_1B_2C_2C_1$ becomes a quadrilateral with two horocyclic edges $\{B_1C_1, B_2C_2\}$ and two geodesic edges $\{B_1B_2, C_1C_2\}$.

Lemma 3. *Suppose the inner acute angles of the quadrilateral $B_1B_2C_2C_1$ are of the same value ψ. Then*

$$\ell(B_1C_1)=\ell(B_2C_2)=2\sqrt{\tan^2\psi+e^{-\ell(A_1A_2)}+1}-2\tan\psi.$$

Furthermore, every geodesic segment which only meets h_1 and h_2 at its endpoints, lies totally inside the quadrilateral if and only if each of two acute angles at the endpoints are of value at least ψ.

Proof. Denote by u the length of A_1A_2. Let h_1 be the horizontal line $y=ie^u$, h_2 be the horocycle centered at 0 and going through i as in Figure 1.4. We can also suppose that $A_1=ie^u$, $A_2=i$, hence $C_1=\ell e^u+ie^u$ where ℓ is defined by the length of the horocyclic segment A_1C_1. By symmetry of the quadrilateral, we can find an involution f which is a non-orientation-preserving isometry sending A_1 to A_2, B_1 to B_2 and C_1 to C_2. By a standard

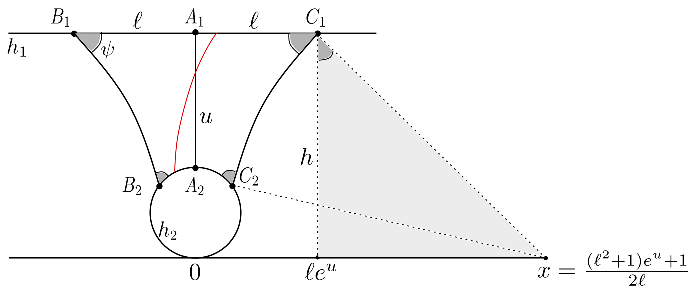

Figure 1.4

computation, $f(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$ where $a = d = 0, b = e^{\frac{u}{2}}$, and $c = e^{\frac{-u}{2}}$. As a consequence,

$$C_2 = f(C_1) = f(\ell e^u + ie^u) = \frac{\ell}{\ell^2+1} + \frac{i}{\ell^2+1}.$$

Let x be a point on the real line of $\mathbb{H}$ such that the Euclidean distances from C_1 and C_2 to x are the same. By computation, $x = \frac{(\ell^2+1)e^u+1}{2\ell}$. Now applying the Euclidean trigonometric formula for the shaded Euclidean right triangle in figure 1.4, noting that value of the angle at C_1 of this triangle is exactly ψ, we get

$$\tan\psi = \frac{\frac{(\ell^2+1)e^u+1}{2\ell} - \ell e^u}{e^u} = \frac{-\ell^2+1+e^{-u}}{2\ell}.$$

From this, we obtain the value of ℓ in terms of ψ and u.

For the second part, we fix an angle ϕ of value between ψ and $\frac{\pi}{2}$ in one endpoint of c, and observe what happens to the acute angle at the other endpoint of c while moving c along the horocycles and keeping the value of the angle ϕ. The behavior of the values of the remaining acute angle is exactly that of a concave function. By symmetry of the quadrilateral, c lies inside the quadrilateral if and only if both acute angles at the endpoints are of value at least ψ. □

Next we recall Lemmas 2.2 and 2.3 from [3].

Lemma 4. [3] *Let* $\frac{\pi}{2} \geq \theta_0 > 0$*, and set* $m(\theta_0) := 2\log\left(\frac{1}{\sin\theta_0}\right) + 2\log(1+\cos\theta_0)$*. If c is an oriented geodesic segment in $\mathbb{H}$ of length at least $m(\theta_0)$ between two (complete) geodesics γ_1, γ_2 such that the starting (resp. end) point of c lies on γ_1 (resp. γ_2) and $\measuredangle(c, \gamma_i) \geq \theta_0$ for $i = 1, 2$, then γ_1 and γ_2 are disjoint.*

Lemma 5. [3] *Let $\frac{\pi}{2} \geq \theta_0 > 0$ be a fixed constant. Let c be a geodesic segment in $\mathbb{H}$ and γ the complete geodesic containing c. Fix $\varepsilon \in (0, 2]$ and let γ_1 and γ_2 be geodesics that intersect γ such that intersection points p_1, p_2 lie on different sides of c. Suppose for $i = 1, 2$*

that $\measuredangle(\gamma_i, \gamma) \geq \theta_0$ and

$$d(c, p_i) \geq \log\left(\frac{1}{\varepsilon}\right) + \log\left(\frac{2e}{\sin\theta_0}\right).$$

Then γ_1 and γ_2 are disjoint. Furthermore, for any geodesic γ intersecting both γ_1 and γ_2, we have the following properties:

(P1) $c \subset B_\varepsilon(\gamma)$.

(P2) The image of the orthogonal projection of c on γ is contained in the middle part of γ (i.e. it lies between γ_1 and γ_2).

Proof. We set

$$r_\varepsilon := \log\left(\frac{1}{\varepsilon}\right) + \log\left(\frac{2e}{\sin\theta_0}\right).$$

Note that the properties: "γ_1 and γ_2 are disjoint" and (P1) are in Lemma 2.3 in [3], here we will fix an incorrect formula in their proof and hence obtain a different value of r_ε under the requirement that $0 < \varepsilon \leq 2$.

(P1) Keeping all notations introduced in Lemma 2.3 in [3], from the proof we already had:

$$\cosh\frac{\ell(\mu)}{2} = \cosh\left(r_\varepsilon + \frac{\ell(c)}{2}\right)\sin(\theta_0);$$

$$\sinh d' = \frac{1}{\sinh\frac{\ell(\mu)}{2}};$$

$$\sinh h' = \sinh d' \cosh\frac{\ell(c)}{2}$$

from which we deduce

$$\sinh^2(h') = \sinh^2(d')\cosh^2\left(\frac{\ell(c)}{2}\right) = \frac{\cosh^2\left(\frac{\ell(c)}{2}\right)}{\cosh^2\left(\frac{\ell(\mu)}{2}\right) - 1} = \frac{\cosh^2\left(\frac{\ell(c)}{2}\right)}{\cosh^2\left(r_\varepsilon + \frac{\ell(c)}{2}\right)\sin^2(\theta_0) - 1}.$$

We want to show that $h' \leq \frac{\varepsilon}{2}$ thus that

$$\frac{\cosh^2\left(\frac{\ell(c)}{2}\right)}{\cosh^2\left(r_\varepsilon + \frac{\ell(c)}{2}\right)\sin^2(\theta_0) - 1} \leq \sinh^2\left(\frac{\varepsilon}{2}\right) \tag{1.1}$$

and Inequality 1.1 is equivalent to the following:

$$\cosh^2\left(r_\varepsilon + \frac{\ell(c)}{2}\right) \geq \frac{\cosh^2\left(\frac{\ell(c)}{2}\right) + \sinh^2\left(\frac{\varepsilon}{2}\right)}{\sinh^2\left(\frac{\varepsilon}{2}\right)\sin^2(\theta_0)}.$$

By using the identities $\cosh(2x) = 2\cosh^2(x) - 1 = 2\sinh^2(x) + 1$, the last inequality can be expressed differently as follows:

$$2r_\varepsilon \geq \operatorname{arccosh}\left(\frac{\cosh \ell(c) + \cosh \varepsilon}{\sinh^2\left(\frac{\varepsilon}{2}\right)\sin^2(\theta_0)} - 1\right) - \ell(c). \tag{1.2}$$

The right hand of Inequality 1.2 can be considered as a function:

$$f(x) = \operatorname{arccosh}(ax + b) - \operatorname{arccosh} x$$

on the domain $[1, \infty)$, in which $a > 0$ and $b > 1$. This function reaches its maximum $x = 1$. Hence (2) will hold if

$$2r_\varepsilon \geq \operatorname{arccosh}\left(\frac{1 + \cosh \varepsilon}{\sinh^2\left(\frac{\varepsilon}{2}\right)\sin^2(\theta_0)} - 1\right). \tag{1.3}$$

For simplicity, we set $A := \dfrac{1 + \cosh \varepsilon}{\sinh^2\left(\frac{\varepsilon}{2}\right)\sin^2(\theta_0)}$. Note that

$$\operatorname{arccosh}(A - 1) < \operatorname{arccosh} A = \log(A + \sqrt{A^2 - 1}) < \log(2A)$$

and

$$\log(2A) = \log\left(\frac{2 + 2\cosh \varepsilon}{\sinh^2\left(\frac{\varepsilon}{2}\right)\sin^2(\theta_0)}\right) = 2\log\left(\frac{2}{\sin\theta_0}\right) + 2\log\left(\frac{e^\varepsilon + 1}{e^\varepsilon - 1}\right)$$

in which

$$\log\left(\frac{e^\varepsilon + 1}{e^\varepsilon - 1}\right) < \log\left(\frac{1}{\varepsilon}\right) + 1$$

for all $\varepsilon \in (0, 2]$.

Thus 1.3 certainly holds provided

$$r_\varepsilon \geq \log\left(\frac{2}{\sin\theta_0}\right) + \log\left(\frac{1}{\varepsilon}\right) + 1.$$

(P2) We consider the worst case scenario: c is a complementary θ_0-transversal of γ_1 and γ_2 (see Figure 1.5). Consider the limit case which is when γ and γ_2 are ultra-parallel. Now orient c from γ_1 to γ_2. Denote by ψ the angle between γ and the geodesic segment connecting the endpoint of γ on γ_1 and the starting point of the oriented geodesic segment c. Denote by θ the angle between the extended part of c toward γ_2 and the geodesic ray starting at the endpoint of c and ending at the endpoint of γ at infinity. Notice that the image of the orthogonal projection of c on γ lies between γ_1 and γ_2 if and only if the angle ψ is acute. Since the sum of four inner angles in a quadrilateral is always less than 2π, $\psi \leq \frac{\pi}{2}$ holds if

$\theta < \frac{\pi}{4}$. By using the same formula as in the proof of Lemma 4, we have:

$$\cos\theta = \frac{\tanh r_\varepsilon - \cos\theta_0}{1 - \tanh r_\varepsilon \cos\theta_0}.$$

Hence $\theta < \frac{\pi}{4}$ holds provided

$$\frac{\tanh r_\varepsilon - \cos\theta_0}{1 - \tanh r_\varepsilon \cos\theta_0} > \frac{1}{\sqrt{2}}. \tag{1.4}$$

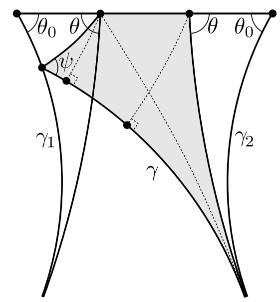

Figure 1.5: The worst case scenario.

By a small manipulation, Inequality 1.4 is equivalent to the following:

$$r_\varepsilon > \frac{1}{2}\log\left(\frac{1}{\sin\theta_0}\right) + \log(1+\sqrt{2}) + \log(1+\cos\theta_0).$$

And this last inequality holds by definition of r_ε. □

1.2. Main tools

Moduli space $\mathcal{M}_{g,n}$ we think of as the space of complete hyperbolic structures up to isometry on a punctured orientable topological surface $\Sigma_{g,n}$ of genus g with n punctures (with $2g + n \geq 3$). For any X in $\mathcal{M}_{g,n}$ and any positive number $\xi \leq 2$, we can define

$$X^\xi := \mathrm{cl}(X \setminus \{\text{all cusp regions of area } \xi\}).$$

In which a cusp region of area ξ is a part of the surface isometric to $\{z : \mathrm{Im} z \geq 1\}/z \mapsto z + \xi$. Thus, X^ξ is a surface of genus g with n boundary components and each connected component of its boundary is a horocycle of length ξ. The following theorem is the main technical part in this chapter:

Theorem 10. *For any $X \in \mathcal{M}_{g,n}$, there exists a constant K_X such that the following holds. For all $0 < \varepsilon \leq 1$, $0 < \xi \leq 1$ and any finite collection $\{c_i\}_{i=1}^N$ of geodesic arcs of average length $\bar{c}$ in X^ξ, there exists a closed geodesic γ of length at most*

$$N\left(K_X + \bar{c} + 10\log\frac{1}{\varepsilon} + 8\log\frac{1}{\xi}\right)$$

which contains $\{c_i\}_{i=1}^N$ *in its* 2ε*-neighborhood.*

Proof. **Part 1: Setup**

Let γ_0 be a closed geodesic on X such that $X \setminus \gamma_0$ consists of a finite collection of ordinary polygons $\{P_i\}_{i\in I}$ and once-punctured polygons $\{P_i\}_{i\in J}$ (I and J are two disjoint finite index sets). Recall that, for each polygon P_i ($i \in I \cup J$) we have the constants θ_{P_i} and ℓ_{P_i} as mentioned in Lemma 1. Also in each ordinary polygon P_i, we denote by D_{P_i} the value of its intrinsic diameter. Note that there is no intrinsic diameter in once-punctured polygons. We define:

$$\theta_0 := \min_{i\in I\cup J}\{\theta_{P_i}\}$$

and

$$D := \max_{i\in I, j\in J}\{D_{P_i}, \ell_{P_j}\}.$$

In this part, we aim to define a classification for geodesics traveling inside polygons in the following way.

We begin by defining a closed horocycle which lies inside a once-punctured polygon (hence γ_0 and this horocycle have no intersection) such that the distance between this horocycle and the horocycle of length ξ (namely h_ξ) is at least

$$r_\varepsilon := \log\left(\frac{1}{\varepsilon}\right) + \log\left(\frac{2e}{\sin\theta_0}\right).$$

Since γ_0 wraps around each cusp at most once, it will not cross transversely the horocycle of length 1 in each cusp region. We also note that ξ is less than 1. Hence, one option which satisfies the above condition is the horocycle which is at distance r_ε from h_ξ. We denote this horocycle by h. Since the decay of length of horocycle in a cusp is e,

$$\ell(h) = \frac{\xi}{e^{r_\varepsilon}} = \frac{\varepsilon\xi}{2e}\sin\theta_0.$$

Now, let c be an arbitrary geodesic arc on X, we extend c by r_ε in one direction to get a new arc c' and the new endpoint p'. Then we continue to extend c' from p'. In the process of extending, the geodesic can intersect γ_0 several times and form angles. An intersection is called a **good intersection** if the acute angle at it is at least θ_0, and otherwise, it will be called a **bad intersection**. The extension will stop at the first good intersection from p'. By Lemma 1, the extensions can be divided into 5 cases as follows:

1. From p', the previous intersection is bad and the next intersection is good.

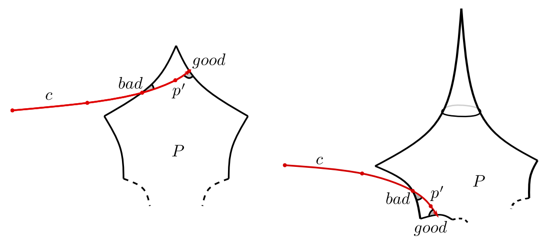

Figure 1.6: Case 1.

2. From p', the previous intersection is good, and the next intersection is bad.

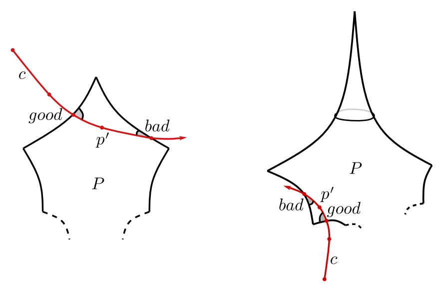

Figure 1.7: Case 2.

3. p' lies inside an ordinary polygon, the previous intersection and the next intersection are both good (see Figure 1.8).

4. p' lies inside a once-punctured polygon P, the previous intersection and the next intersection are both good (see Figure 1.8) and so that the geodesic arc, namely c'', between these two intersections is not too long, more precisely, this arc either intersects the horocycle h at an angle less than a given angle ψ or does not intersect h.

5. p' lies inside a once-punctured polygon and if we continue to extend c' from p', it will intersect the horocycle h at an angle at least ψ (see Figure 1.8).

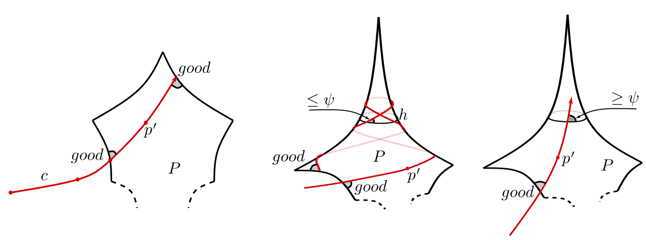

Figure 1.8: Cases 3,4 and 5 respectively.

Now in order to stop the extension, by Lemma 1 and the definition of D above, the distance we need to extend from p' is at most D (in cases 1 and 3), and $2D$ (in case 2). In case 4, let

$s_0 \in \partial P$ such that

$$d_X(s_0, h) = \max_{s \in \partial P}\{d_X(s, h)\}.$$

Let d_P be the distance from s_0 to the closed horocycle of length 1 of the same cusp. Note that the distance between this horocycle and h is $\log\left(\frac{2e}{\varepsilon\xi \sin\theta_0}\right)$. Thus

$$d_X(s_0, h) = d_P + \log\left(\frac{2e}{\varepsilon\xi \sin\theta_0}\right).$$

Then applying Lemma 2 and the inequality $\operatorname{arccosh}(x-1) < \log(2x)$ we have

$$\ell(c'') \leq \operatorname{arccosh}\left(\frac{2e^{2d_P + 2\log\left(\frac{2e}{\varepsilon\xi\sin\theta_0}\right)}}{\cos^2\psi} - 1\right) < 2d_P + 2\log\left(\frac{2e}{\varepsilon\xi\sin\theta_0}\right) + 2\log\left(\frac{2}{\cos\psi}\right). \tag{1.5}$$

Note that, in part 2, we will define ψ as the angle formed by $\tilde{h}$ and $\tilde{\eta}_1$ (see Figure 1.9). By simple computations, we obtain

$$\psi = \arccos\left(\frac{\frac{\varepsilon\xi}{e}\sin\theta_0}{1 + \frac{\varepsilon^2\xi^2}{4e^2}\sin^2(\theta_0)}\right).$$

From there we have

$$2\log\left(\frac{2}{\cos\psi}\right) = 2\log\left(1 + \frac{\varepsilon^2\xi^2}{4e^2}\sin^2(\theta_0)\right) + 2\log\left(\frac{2e}{\varepsilon\xi\sin\theta_0}\right) < 2 + 2\log\left(\frac{2e}{\varepsilon\xi\sin\theta_0}\right). \tag{1.6}$$

Combining 1.5 and 1.6, one has

$$\ell(c'') < 2d_P + 4\log\left(\frac{2e}{\varepsilon\xi\sin\theta_0}\right) + 2.$$

Moreover, if one denotes by d_{γ_0} the maximal value of $\{d_{P_i}\}_{i \in J}$, then

$$m_A := 2D + 2d_{\gamma_0} + 2 + 4\log\left(\frac{2e}{\varepsilon\xi\sin\theta_0}\right)$$

is an upper bound on the length of the extension in **class A** (i.e., cases 1, 2, 3 and 4). Note that, in **class B** (i.e., case 5), the length of the extension is unbounded when $\measuredangle(c', h)$ goes to $\frac{\pi}{2}$.

Part 2: Replacements and estimates

Now, let c be an arbitrary geodesic arc in the collection $\{c_i\}_{i=1}^N$. Denote the two endpoints of c by p and q. We extend c by r_ε in both directions to get a new geodesic arc c'. We now look at different cases.

Case A: The extensions in both directions are in class A.

In order to get good intersections in both directions, we need to extend c' by at most m_A for each of its directions. Hence an upper bound on the length of c after being extended is

$$\ell(c) + 2r_\varepsilon + 2m_A$$

or more precisely,

$$\ell(c) + 10\log\frac{1}{\varepsilon} + 8\log\frac{1}{\xi} + 4D + 4d_{\gamma_0} + 4 + 10\log\left(\frac{2e}{\sin\theta_0}\right). \tag{1.7}$$

<u>Case B</u>: There is a direction where the extension is in class B.

Let p' be the endpoint of c' in this direction, we can suppose p' lies inside a once-punctured polygon, namely P.

What we aim to do is to replace c by another geodesic arc, which is very close to c and controlled in both directions (i.e. the extension in each direction is in class A). Denote the complete geodesic containing c by γ_c. We assume that the horizontal line $y = \frac{2e}{\varepsilon\xi\sin\theta_0}i$, namely $\tilde{h}$, is a lift of the closed horocycle h of length $\frac{\varepsilon\xi}{2e}\sin\theta_0$ in P. From there we can suppose the complete geodesic $\tilde{\gamma_c}$ with an endpoint at 0, forming an angle at least ψ with $\tilde{h}$, is a lift of γ_c. Hence $\tilde{\gamma_c}$ lies between $\tilde{\eta}_1$ and $\tilde{\eta}_2$, in which $\tilde{\eta}_1$ and $\tilde{\eta}_2$ are the complete geodesics with a common endpoint at 0, containing $\frac{2e}{\varepsilon\xi\sin\theta_0}i + 1$ and $\frac{2e}{\varepsilon\xi\sin\theta_0}i - 1$, respectively.

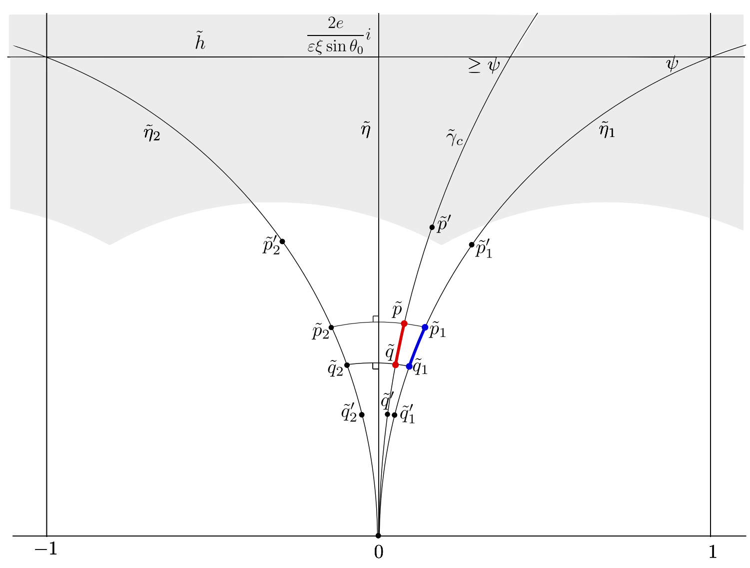

Figure 1.9: Lifting to $\mathbb{H}$ in Case B. The shaded part is a lift of polygon P.

Now we construct lifts of other points from there. Let $\tilde{p}$ and $\tilde{q}$ be lifts of p and q, respectively (see Figure 1.9). Let $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ be geodesics going though $\tilde{p}$ and $\tilde{q}$, respectively, and orthogonal to the axis $x = 0$. Let $\tilde{p}_i := \tilde{\gamma}_1 \cap \tilde{\eta}_i$ and $\tilde{q}_i := \tilde{\gamma}_2 \cap \tilde{\eta}_i$, for $i = 1, 2$.

For $i = 1, 2$, we denote by $p_i q_i$ the projection of $\tilde{p}_i \tilde{q}_i$ to the surface X. Extend the geodesic arc $p_i q_i$ by r_ε in both directions to get a new geodesic arc $p'_i q'_i$. If $p'_i \notin P$, we only need to extend by an extra at most $\frac{\varepsilon\xi}{e} \sin\theta_0 + 2\ell(\gamma_0)$ to get into P. This can be proved by using the inequality in the triangle formed by $\tilde{\eta}_i$, the geodesic segment $\tilde{p}_i{}'\tilde{p}'$ and a lift of γ_0 (one of the boundary components of the shaded part in Figure 1.9).

There are two different sub-cases of case B:

Sub-case BA: The extension in the direction of either q_1 or q_2 is in class A.

Without loss of generality, we assume that the extension in the direction of q_1 is in class A. Note that the geodesic segments $\tilde{p}\tilde{p}_1$ and $\tilde{q}\tilde{q}_1$ are of length at most $\frac{\varepsilon\xi}{e}\sin\theta_0$. Since $\xi \leq 1$ and $\sin\theta_0 \leq 1$, $\frac{\varepsilon\xi}{e}\sin\theta_0 < \frac{\varepsilon}{2}$. Thus any geodesic containing p_1q_1 in its ε-neighborhood contains c in its $\left(\frac{\varepsilon}{2} + \varepsilon\right)$-neighborhood. Hence in this case, we will replace c by p_1q_1.

Recall that we extended p_1q_1 by r_ε in both directions to obtain the geodesic arc $p'_1q'_1$. In order to get good intersections in both directions, we continue to extend $p'_1q'_1$ by at most

$$2m_A + \frac{\varepsilon\xi}{e}\sin\theta_0 + 2\ell(\gamma_0).$$

Hence an upper bound on the length of p_1q_1 after being extended is

$$\ell(c) + 2r_\varepsilon + 2m_A + \frac{\varepsilon\xi}{e}\sin\theta_0 + 2\ell(\gamma_0)$$

which is less than or equal to

$$\ell(c) + 10\log\frac{1}{\varepsilon} + 8\log\frac{1}{\xi} + 4D + 4d_{\gamma_0} + 4 + 10\log\left(\frac{2e}{\sin\theta_0}\right) + \frac{\sin\theta_0}{e} + 2\ell(\gamma_0). \qquad (1.8)$$

Sub-case BB: The extensions in the directions of q_1 and q_2 are both in class B.

Since η, η_1, and η_2 asymptotic in the direction of q, q_1 and q_2, length of the geodesic arc orthogonal to η and connecting q'_1 and q'_2 is very small, roughly less than $\frac{\varepsilon\xi}{2e}\sin\theta_0$. Thus we can suppose that q'_1 and q'_2 lie in the same once-punctured polygon, denoted by P'. Let h' be the closed horocycle of length $\frac{\varepsilon\xi}{2e}\sin\theta_0$ in P'. From there we construct a lift of h', denoted by $\tilde{h}'$. We keep all the notations $A_1, A_2, B_1, B_2, C_1, C_2$ and u as introduced in Lemma 3 (see Figure 1.10).

Now we would like to apply Lemma 3 to the two horocycles $\tilde{h}$ and $\tilde{h}'$ with the angle ψ. Recall that $u = \ell(A_1A_2)$ is the distance from $\tilde{h}$ to $\tilde{h}'$. One can estimate a lower bound and an upper bound on u as follows:

$$2\log\frac{4e}{\varepsilon\xi\sin\theta_0} \leq u \leq \ell(c) + 2r_\varepsilon + \frac{\varepsilon\xi}{e}\sin\theta_0 + 2\ell(\gamma_0) + 2\left(d_{\gamma_0} + \log\frac{2e}{\varepsilon\xi\sin\theta_0}\right).$$

This implies that:

$$0 < u \leq \ell(c) + 4\log\frac{1}{\varepsilon} + 2\log\frac{1}{\xi} + k_X$$

in which $k_X := 2d_{\gamma_0} + 4\log\frac{2e}{\sin\theta_0} + \frac{\sin\theta_0}{e} + 2\ell(\gamma_0)$.

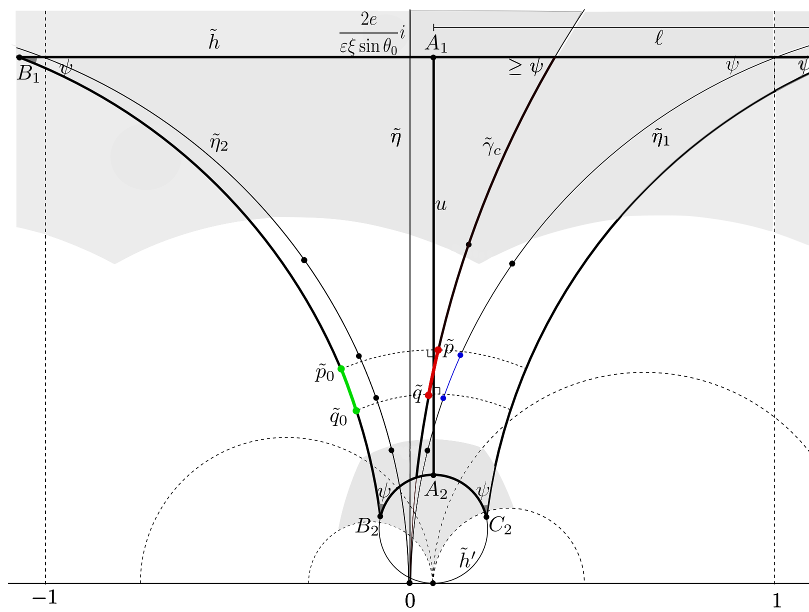

Figure 1.10: Lifting to $\mathbb{H}$ in sub-case BB.

Recall that $\ell := \frac{\ell(B_1C_1)}{2}$. Then by Lemma 3, we have

$$\ell = \sqrt{\tan^2\psi + e^{-u} + 1} - \tan\psi = \frac{e^{-u}+1}{\sqrt{\tan^2\psi + e^{-u} + 1} + \tan\psi} < \frac{2}{\sqrt{\tan^2\psi + 1} + \tan\psi}.$$

Thus since $\psi = \arccos\left(\dfrac{\frac{\varepsilon\xi}{e}\sin\theta_0}{1 + \frac{\varepsilon^2\xi^2}{4e^2}\sin^2(\theta_0)}\right)$, $0 < \varepsilon \leq 1$, $0 < \xi \leq 1$ and $\frac{2e}{\sin\theta_0} > 2e$ we have

$$\ell < \frac{2}{\sqrt{\tan^2\psi + 1} + \tan\psi} = \frac{2\varepsilon\xi}{\frac{2e}{\sin\theta_0}} < 2e^{-1},$$

hence

$$\ell(B_1B_2) = \ell(C_1C_2) \leq 2\ell + u < 4e^{-1} + \ell(c) + 4\log\frac{1}{\varepsilon} + 2\log\frac{1}{\xi} + k_X.$$

Now we draw a geodesic going through $\tilde{p}$, resp. $\tilde{q}$, orthogonal to A_1A_2, meeting B_1B_2 at a point, denoted by $\tilde{p}_0$, resp. $\tilde{q}_0$ (see the Figure 1.10). We also denote by $\tilde{\zeta}$ the geodesic segment $\tilde{p}_0\tilde{q}_0$. By projecting the geodesic segment $\tilde{\zeta}$ to X, we get a geodesic arc on X, denoted by ζ. Note that the geodesic segments $\tilde{p}\tilde{p}_0$ and $\tilde{q}\tilde{q}_0$ are of length at most $2\ell < \frac{4\varepsilon\xi \sin\theta_0}{2e} < \varepsilon$. Thus any geodesic containing ζ in its ε-neighborhood contains c in its 2ε-neighborhood. In this case, we will replace c by ζ.

Since B_1B_2 contains $\tilde{\zeta}$, we will extend B_1B_2 instead of $\tilde{\zeta}$. By Lemma 2, in order to get good intersections in both directions, we need to extend B_1B_2 in each direction by a distance v, where v satisfies:

$$\log\left(\frac{2e}{\varepsilon\xi\sin\theta_0}\right) + \log\frac{1+\sin\psi}{1-\sin\psi} \le v \le \frac{1}{2}\operatorname{arccosh}\left(\frac{2e^{2d}}{\cos^2\psi} - 1\right) + \frac{1}{2}\log\frac{1+\sin\psi}{1-\sin\psi}$$

where $d := d_{\gamma_0} + \log\left(\frac{2e}{\varepsilon\xi\sin\theta_0}\right)$. Similarly to 1.5 and 1.6, one can show that:

$$\log\frac{1}{\varepsilon} + \log\left(\frac{2e}{\xi\sin\theta_0}\right) + \log\frac{1+\sin\psi}{1-\sin\psi} \le v < d_{\gamma_0} + 1 + 3\log\frac{1}{\xi} + 3\log\frac{1}{\varepsilon} + 3\log\left(\frac{2e}{\sin\theta_0}\right).$$

Since the lower bound of v is greater than r_ε, we do not need to extend the segment B_1B_2 in two steps as in the previous cases.

In this way, we obtain an upper bound on the length of B_1B_2 after the extension:

$$4e^{-1} + \ell(c) + 4\log\frac{1}{\varepsilon} + 2\log\frac{1}{\xi} + k_X + 2d_{\gamma_0} + 2 + 6\log\frac{1}{\xi} + 6\log\frac{1}{\varepsilon} + 6\log\left(\frac{2e}{\sin\theta_0}\right)$$

or

$$\ell(c) + 10\log\frac{1}{\varepsilon} + 8\log\frac{1}{\xi} + 4d_{\gamma_0} + 10\log\left(\frac{2e}{\sin\theta_0}\right) + \frac{\sin\theta_0}{e} + 2\ell(\gamma_0) + 4e^{-1} + 2. \quad (1.9)$$

Finally, after comparing upper bounds 1.7, 1.8 and 1.9 in cases A, BA and BB respectively, we set

$$M(c,\varepsilon,\xi,X) := \ell(c) + 10\log\frac{1}{\varepsilon} + 8\log\frac{1}{\xi} + k'_X$$

the upper bound of all cases, in which $k'_X := 4D + 4d_{\gamma_0} + 4 + 10\log\left(\frac{2e}{\sin\theta_0}\right) + \frac{\sin\theta_0}{e} + 2\ell(\gamma_0)$ is a quantity that depends only on X.

Conclusion of parts 1 and 2: We replaced the collection $\{c_i\}_{i=1}^N$ by a new collection, denoted by $\{\zeta_i\}_{i=1}^N$. We also defined a collection of the extended geodesic arcs of $\{\zeta_i\}_{i=1}^N$, denoted by $\{\zeta'_i\}_{i=1}^N$. In this collection, each element ζ'_i, is of length at most $M(c_i,\varepsilon,\xi,X)$, has endpoints lying on γ_0 and forms good angles ($\ge \theta_0$) with γ_0 , and is an extension of ζ_i by at least r_ε in each direction. In short, each ζ_i is an example of the geodesic segment c in Lemma 5. Furthermore, we showed that any geodesic containing ζ_i in its ε-neighborhood contains c_i in its 2ε-neighborhood.

With the new collection $\{\zeta_i\}_{i=1}^N$ and its extension $\{\zeta'_i\}_{i=1}^N$ in hand, one can apply the connecting algorithm in pages 8 and 9 in the proof of Theorem 2.4 in [3] directly without being concerned about the difference between closed surfaces and surfaces with cusps. □

A geodesic arc on X is called a doubly truncated orthogeodesic on X^ξ if it is perpendicular to the horocyclic boundary of X^ξ at its endpoints. As a consequence of Theorem 10, we can also construct a doubly truncated orthogeodesic $\mathcal{O}$ with the same properties:

Theorem 11. *For any $X \in \mathcal{M}_{g,n}$, there exists a constant K_X such that the following holds. For all $0 < \xi \leq 1$, $0 < \varepsilon \leq \min\{\log \frac{1}{\xi}, 1\}$, and any finite collection $\{c_i\}_{i=1}^N$ of geodesic arcs of average length $\bar{c}$ in X^ξ, there exist a doubly truncated orthogedesic $\mathcal{O}$ of length at most*

$$(N+1)(K_X + \bar{c} + 10 \log \frac{1}{\varepsilon} + 8 \log \frac{1}{\xi})$$

containing $\{c_i\}_{i=1}^N$ in its 2ε-neighborhood.

Proof. Let P_0 and P_1 be two arbitrary once-punctured polygons of the partition by γ_0 on X. Firstly, we will construct a doubly truncated orthogeodesic $\mathcal{O}_1$ with endpoints on the horocycles of length 1 associated to the two polygons so that $\mathcal{O}_1$ contains $\{\zeta_i\}_{i=1}^N$ in its ε-neighborhood. We take a shortest one-sided orthogeodesic arc, denoted by ζ_0', oriented with the starting point on the horocycle of length 1 of P_0 and the endpoint on γ_0. We take another shortest one-sided orthogeodesic arc, denoted by ζ_{N+1}', oriented with the starting point on γ_0 and the endpoint on the horocycle of length 1 of P_1 . For each $i \in \{1, 2, ..., N\}$, we orient ζ_i' arbitrarily. The new sequence $\{\zeta_i'\}_{i=0}^{N+1}$ is ordered linearly by its index. We apply the connecting algorithm to this new sequence. Noting that ζ_0' is in the first step and ζ_{N+1}' is in the last step of the algorithm, one will obtain a doubly truncated orthogeodesic $\mathcal{O}_1$ as desired (see Figure 1.11).

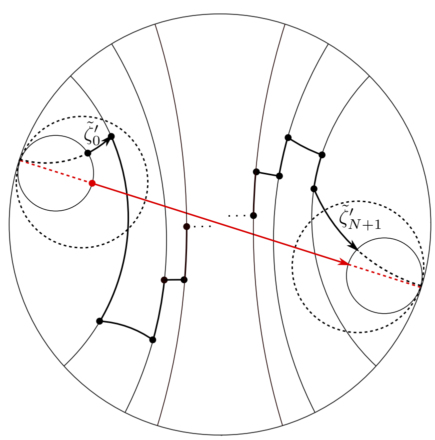

Figure 1.11: Lifting to $\mathbb{H}$.

Since $\mathcal{O}_1$ is an arc, it may not contain entirely either c_1 or c_N in its 2ε-neighborhood. In this case, by applying Lemma 5 (P2), we only need to extend $\mathcal{O}_1$ by an extra segment of length at most ε in both directions. Note that, $\varepsilon \leq \log \frac{1}{\xi}$, and the distance between the horocycle of length 1 and the horocyle of length ξ is $\log \frac{1}{\xi}$, by extending $\mathcal{O}_1$ in both directions until it hits the boundary of X^{ξ}, we obtain $\mathcal{O}$ as desired. □

1.3. Quantitative density on surface

In this section we will prove results about quasi-dense geodesics by applying Theorem 10.

Theorem 12. *For all $X \in \mathcal{M}_{g,n}$ there exists a constant $C_X > 0$ such that for all $0 < \xi \leq 1$ and all $0 < \varepsilon \leq 2$ there exists a closed geodesic γ_ε that is ε-dense on X^{ξ} and such that*

$$\ell(\gamma_\varepsilon) \leq C_X \frac{1}{\varepsilon}\left(\log \frac{1}{\varepsilon} + \log \frac{1}{\xi}\right).$$

Proof. On $\mathbb{H}$, there is a fundamental polygon F whose boundary consists $4g + 2n$ paired geodesic segments (or rays) which, when glued in pairs, turn the polygon into X. This polygon has n ideal vertices and $4g + n$ ordinary vertices.

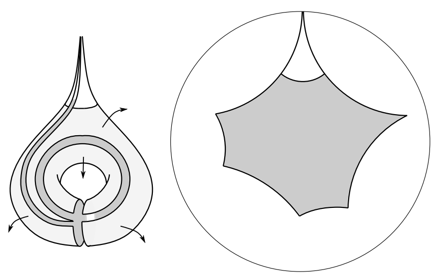

Figure 1.12: An example when g=1, n=1.

Since $X^2 \subset X^{\xi} \subset X$, there is a fundamental polygon of X^{ξ} in F, say F^{ξ}, and a fundamental polygon of X^2 in F^{ξ}, say F^2. We note that the boundary of F^2 consists of n horocyclic segments of length 2 and $4g + 2n$ geodesic segments. By replacing each horocyclic segment by a geodesic segment of length $2 \operatorname{arcsinh} 1$ with the same endpoints, we obtain the convex hull of F^2, denoted by $CH(F^2)$. We denote by P_X the perimeter of $CH(F^2)$, and note that this value depends only on X. On an edge of $CH(F^2)$, we choose the first point at a vertex, then choose the next points such that the segment on the boundary connecting two consecutive points is of length ε. If the length of the segment connecting the last point and the remaining vertex of the same edge is less than ε, that vertex will be chose as the first point of the next edge, we then continue the choosing process. Eventually, we have chosen at most

$$\frac{P_X}{\varepsilon} + 4g + 2n$$

points on the boundary of $CH(F^2)$.

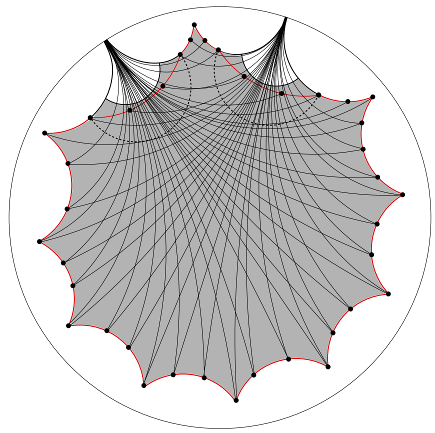

Figure 1.13: An example when g=2, n=2. Note that $CH(F^2)$ is the polygon with red edges.

Now we connect each ideal vertex to the points on the boundary of $CH(F^2)$. Since $CH(F^2)$ is convex, the parts of those geodesic rays in F^ξ are exactly one-sided orthogeodesic segments. By gluing back paired geodesic segments of F in pairs, these segments become one-sided orthogeodesic arcs on X and we have at most

$$n\left(\frac{P_X}{\varepsilon} + 4g + 2n\right)$$

one-sided orthogeodesic arcs on X. By construction, each segment is of length at most

$$\frac{P_X}{2} + \log\left(\frac{2}{\xi}\right).$$

Moreover, the collection of the one-sided orthogeodesic arcs is $\frac{\varepsilon}{2}$-dense on X^ξ. Thus by applying Theorem 10 to this collection, we obtain the closed geodesic γ_ε containing every arcs in its $\frac{\varepsilon}{2}$ neighborhood where length satisfies

$$\ell(\gamma_\varepsilon) \leq n\left(\frac{P_X}{\varepsilon} + 4g + 2n\right)\left(K_X + \frac{P_X}{2} + \log\frac{2}{\xi} + 10\log\frac{2}{\varepsilon} + 8\log\frac{1}{\xi}\right). \tag{1.10}$$

Then by manipulating the right hand of Inequality 1.10, we obtain a constant C_X depending only on X so that:

$$\ell(\gamma_\varepsilon) \leq C_X \frac{1}{\varepsilon}\left(\log\frac{1}{\varepsilon} + \log\frac{1}{\xi}\right).$$

□

By using the same collection of geodesic segments as in Theorem 12, we also obtain the following result:

Theorem 13. *Let $X \in \mathcal{M}_{g,n}$, there exists a constant $D_X > 0$ such that for all $0 < \xi \leq 1$ and all $0 < \varepsilon \leq \min\{2\log\frac{1}{\xi}, 2\}$ there exists a doubly truncated orthogeodesic $\mathcal{O}_\varepsilon$ that is ε-dense on X^ξ and such that*

$$\ell(\mathcal{O}_\varepsilon) \leq D_X \frac{1}{\varepsilon}\left(\log\frac{1}{\varepsilon} + \log\frac{1}{\xi}\right).$$

We end this section with a corollary of Theorem 12 where we apply Theorem 1.2 in [4] to obtain an upper bound on the number of self-intersections of the ε-dense closed geodesic constructed in Theorem 12.

Corollary 4. *Let $X \in \mathcal{M}_{g,n}$, there exists a constant $C_X > 0$ such that for all $0 < \xi \leq 1$ and all $0 < \varepsilon \leq 2$ there exists a closed geodesic γ_ε that is ε-dense on X^ξ and such that*

$$2i(\gamma_\varepsilon, \gamma_\varepsilon) \leq C_X^{\frac{1}{\varepsilon}\left(\log\frac{1}{\varepsilon} + \log\frac{1}{\xi}\right)}.$$

Chapter 2

Measuring pants

2.1. The Luo-Tan identity and variations

In this section we recall a precise formulation of the Luo-Tan identity and rewrite it in a slightly different form more convenient for our purposes.

Let X be a closed, orientable hyperbolic surface of genus $g \geq 2$. We shall be investigating different small complexity subsurfaces of X. The subsurfaces we consider are all either considered up to isotopy or equivalently, we consider their geodesic realizations (their boundary curves are simple closed geodesics). An (geodesic) embedded three-holed sphere $P \subset X$ (or pair of pants) is said to be properly embedded if its closure is embedded. (In other words, all three of its boundary curves are non-isotopic.) Otherwise its closure is an embedded one-holed torus and it is said to be improperly embedded.

With that in hand, the Luo-Tan identity [29] states as following:

$$\sum_{P \in \mathcal{P}} \varphi(P) + \sum_{T \in \mathcal{T}} \tau(T) = 8\pi^2(g-1). \tag{2.1}$$

The right hand side of the identity is the volume of unit tangent bundle. The left hand side has two index sets. The first ($\mathcal{P}$) is the set of properly embedded geodesic pants on X whereas the second ($\mathcal{T}$) is the set of embedded geodesic one holed tori.

The functions depend explicitly on the geometries of the pants and tori. Hence both can be made to depend on three real variables. We think of these functions as being measures on the set of pants and tori, where the measures sum up to full volume.

By cutting a one holed torus along a simple closed geodesic, one obtains a pair of pants. Hence tori contain infinitely many distinct geodesic pants with embedded interior but, because of their boundary geodesics, their closures fail to be embedded. Extending the function for embedded pants to these *improperly* embedded pants leads to under counting and hence an inequality. Nonetheless, Hu and Tan [23] found a way of decomposing the measure associated to a one holed torus as an infinite sum of measures associated to improperly embedded pants. Putting together the results leads to a new identity where the second summand set is on improperly embedded pants (the set of which we denote by $\mathcal{I}$):

$$\sum_{P\in\mathcal{P}} \varphi(P) + \sum_{P\in\mathcal{I}} \eta(P) = 8\pi^2(g-1), \tag{2.2}$$

in which φ and η are functions that depends on the geometry of P.

The function φ on embedded pairs of pants

We now describe the functions explicitly. Let P be a pair of pants with geodesic boundaries $\gamma_1, \gamma_2, \gamma_3$ of lengths ℓ_1, ℓ_2, ℓ_3. For $\{i, j, k\} = \{1, 2, 3\}$, let m_i be the length of the shortest geodesic arc between γ_j and γ_k for $\{i, j, k\} = \{1, 2, 3\}$.

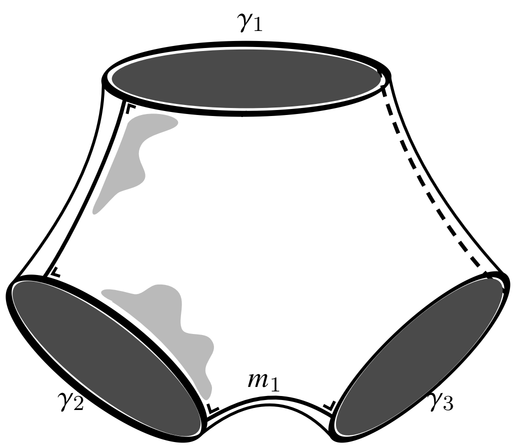

Figure 2.1: A properly embedded pair of pants

The function φ applied to P can now be expressed as:

$$\varphi(P) := 4\sum_{i\neq j}\left[2\mathcal{L}\left(\frac{1-x_i^2}{1-x_i^2 y_j}\right) - 2\mathcal{L}\left(\frac{1-y_j}{1-x_i^2 y_j}\right) - \mathcal{L}(y_j) - \mathcal{L}\left(\frac{(1-y_j)^2 x_i^2}{(1-x_i^2)^2 y_j}\right)\right],$$

where $x_i = e^{-\ell_i/2}$, $y_i = \tanh^2(m_i/2)$, and the Roger's dilogarithm function $\mathcal{L}$ is defined for $x < 1$ by

$$\mathcal{L}(x) = -\int_0^x \frac{\log(1-z)}{z}dz + \frac{1}{2}\log(|x|)\log(1-x).$$

Observe that x_i is monotonic decreasing in ℓ_i.

One of our goals will be to study the variation of this function in terms of the lengths ℓ_i. For that purpose, we shall express the function solely in terms of the x_i. By our definition of y_1:

$$y_1 = \tanh^2(m_1/2) = \frac{\sinh^2(m_1/2)}{\cosh^2(m_1/2)} = \frac{(\cosh(m_1)-1)/2}{(\cosh(m_1)+1)/2} = \frac{\cosh(m_1)-1}{\cosh(m_1)+1}.$$

Using standard hyperbolic trigonometry we have

$$\cosh(m_1) = \frac{\cosh(\ell_1/2)+\cosh(\ell_2/2)\cosh(\ell_3/2)}{\sinh(\ell_2/2)\sinh(\ell_3/2)} = \frac{(x_1+\frac{1}{x_1})/2+(x_2+\frac{1}{x_2})(x_3+\frac{1}{x_3})/4}{(\frac{1}{x_2}-x_2)(\frac{1}{x_3}-x_3)/4}.$$

Then

$$y_1 = \frac{2(x_1+\frac{1}{x_1})+(x_2+\frac{1}{x_2})(x_3+\frac{1}{x_3})-(\frac{1}{x_2}-x_2)(\frac{1}{x_3}-x_3)}{2(x_1+\frac{1}{x_1})+(x_2+\frac{1}{x_2})(x_3+\frac{1}{x_3})+(\frac{1}{x_2}-x_2)(\frac{1}{x_3}-x_3)} = \frac{(x_1x_3+x_2)(x_1x_2+x_3)}{(x_2x_3+x_1)(1+x_1x_2x_3)}.$$

More generally, for $\{i,j,k\}=\{1,2,3\}$, we obtain:

$$y_j = \frac{(x_jx_k+x_i)(x_jx_i+x_k)}{(x_ix_k+x_j)(1+x_1x_2x_3)}.$$

Hence

$$\frac{1-x_i^2}{1-x_i^2y_j} = \frac{1-x_i^2}{1-x_i^2\frac{(x_jx_k+x_i)(x_jx_i+x_k)}{(x_ix_k+x_j)(1+x_1x_2x_3)}} = \frac{(x_ix_k+x_j)(1+x_1x_2x_3)}{x_j+x_i^2x_j+x_ix_k+x_ix_j^2x_k}.$$

Similarly:

$$\frac{1-y_j}{1-x_i^2y_j} = \frac{x_j(1-x_k^2)}{x_j+x_i^2x_j+x_ix_k+x_ix_j^2x_k},$$

and

$$\frac{(1-y_j)^2x_i^2}{(1-x_i^2)^2y_j} = \frac{x_i^2x_j^2(1-x_k^2)^2}{(x_k+x_ix_j)(x_i+x_jx_k)(x_j+x_ix_k)(1+x_1x_2x_3)}.$$

In terms of x_1, x_2 and x_3 we obtain:

$$\begin{aligned}\varphi(x_1,x_2,x_3) := 4\sum_{\{i,j,k\}=\{1,2,3\}}\Bigg[&2\mathcal{L}\left(\frac{(x_ix_k+x_j)(1+x_1x_2x_3)}{x_j+x_i^2x_j+x_ix_k+x_ix_j^2x_k}\right)\\ &-2\mathcal{L}\left(\frac{x_j(1-x_k^2)}{x_j+x_i^2x_j+x_ix_k+x_ix_j^2x_k}\right)-\mathcal{L}\left(\frac{(x_jx_k+x_i)(x_jx_i+x_k)}{(x_ix_k+x_j)(1+x_1x_2x_3)}\right)\\ &-\mathcal{L}\left(\frac{x_i^2x_j^2(1-x_k^2)^2}{(x_k+x_ix_j)(x_i+x_jx_k)(x_j+x_ix_k)(1+x_1x_2x_3)}\right)\Bigg].\end{aligned}$$

The function η on improperly embedded pants

Now let T be a hyperbolic one-holed torus with boundary geodesic β and let α be a non-peripheral simple closed geodesic of T. Let h_α be the length of the shortest simple orthogeodesic from β to itself which is disjoint from α. Let p_α denote the length of the pair of shortest simple orthogeodesics from α to β. Finally let q_α be the length of the shortest simple orthogeodesic from α to itself. Let P be the improperly embedded pair of pants

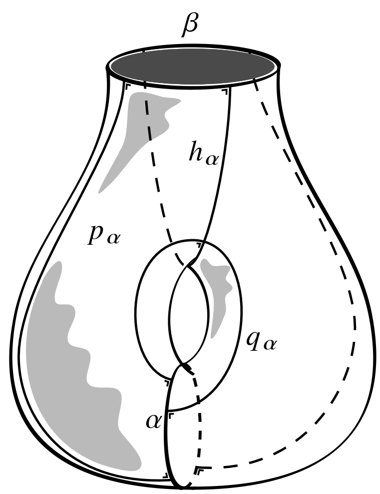

Figure 2.2: An improperly embedded pair of pants

associated to T by cutting T along α. Then the function η is defined as:

$$\eta(P) := 8\Bigg[\mathcal{L}\left(\tanh^2\left(\frac{q_\alpha}{2}\right)\right) + 2\mathcal{L}\left(\tanh^2\left(\frac{p_\alpha}{2}\right)\right) - \mathcal{L}\left(\operatorname{sech}^2\left(\frac{h_\alpha}{2}\right)\right)$$

$$-2\mathrm{La}\left(e^{-\ell(\alpha)}, \tanh^2\left(\frac{p_\alpha}{2}\right)\right) - 2\mathrm{La}\left(e^{-\frac{\ell(\beta)}{2}}, \tanh^2\left(\frac{p_\alpha}{2}\right)\right)\Bigg],$$

where following [29], the lasso function La is defined as follows:

$$\mathrm{La}(a,b) := \mathcal{L}(b) + \mathcal{L}\left(\frac{1-b}{1-ab}\right) - \mathcal{L}\left(\frac{1-a}{1-ab}\right),$$

for $a, b \in (0,1)$.
Let $x := e^{\frac{-\ell(\beta)}{2}}$ and $y := e^{\frac{-\ell(\alpha)}{2}}$, we will express each term of $\eta(P)$ in term of x and y. Using standard hyperbolic trigonometry we have

$$\cosh(q_\alpha) = \frac{\cosh(\ell(\beta)/2) + \cosh(\ell(\alpha)/2)\cosh(\ell(\alpha)/2)}{\sinh(\ell(\alpha)/2)\sinh(\ell(\alpha)/2)} = \frac{(x+\frac{1}{x})/2 + (y+\frac{1}{y})^2/4}{(\frac{1}{y}-y)^2/4},$$

$$\cosh(p_\alpha) = \frac{\cosh(\ell(\alpha)/2) + \cosh(\ell(\alpha)/2)\cosh(\ell(\beta)/2)}{\sinh(\ell(\alpha)/2)\sinh(\ell(\beta)/2)} = \frac{(y+\frac{1}{y})/2 + (y+\frac{1}{y})(x+\frac{1}{x})/4}{(\frac{1}{y}-y)(\frac{1}{x}-x)/4},$$

$$\cosh\left(\frac{h_\alpha}{2}\right) = \sinh\left(\frac{\ell(\alpha)}{2}\right)\sinh(p_\alpha).$$

Hence,

$$\tanh^2\left(\frac{q_\alpha}{2}\right) = \frac{\cosh(q_\alpha)-1}{\cosh(q_\alpha)+1} = \frac{2(x+\frac{1}{x})+(y+\frac{1}{y})^2-(\frac{1}{y}-y)^2}{2(x+\frac{1}{x})+(y+\frac{1}{y})^2+(\frac{1}{y}-y)^2} = \frac{(x+1)^2y^2}{(x+y^2)(xy^2+1)},$$

$$\tanh^2\left(\frac{p_\alpha}{2}\right) = \frac{\cosh(p_\alpha)-1}{\cosh(p_\alpha)+1} = \frac{2(y+\frac{1}{y})+(y+\frac{1}{y})(x+\frac{1}{x})-(\frac{1}{y}-y)(\frac{1}{x}-x)}{2(y+\frac{1}{y})+(y+\frac{1}{y})(x+\frac{1}{x})+(\frac{1}{y}-y)(\frac{1}{x}-x)} = \frac{x+y^2}{xy^2+1},$$

$$\begin{aligned}\operatorname{sech}^2\left(\frac{h_\alpha}{2}\right) &= \frac{1}{\cosh^2\left(\frac{h_\alpha}{2}\right)} = \frac{1}{\sinh^2(\ell(\alpha)/2)\sinh^2(p_\alpha)} = \frac{1}{\sinh^2(\ell(\alpha)/2)(\cosh^2(p_\alpha)-1)}\\ &= \frac{1}{\left((\frac{1}{y}-y)/2\right)^2\left(\left(\frac{(y+\frac{1}{y})/2+(y+\frac{1}{y})(x+\frac{1}{x})/4}{(\frac{1}{y}-y)(\frac{1}{x}-x)/4}\right)^2-1\right)} = \frac{(1-x)^2y^2}{(x+y^2)(xy^2+1)},\end{aligned}$$

$$\mathrm{La}\left(e^{-\ell(\alpha)}, \tanh^2\left(\frac{p_\alpha}{2}\right)\right) = \mathrm{La}\left(y^2, \frac{x+y^2}{xy^2+1}\right) = \mathcal{L}\left(\frac{x+y^2}{xy^2+1}\right) + \mathcal{L}\left(\frac{1-x}{1+y^2}\right) - \mathcal{L}\left(\frac{xy^2+1}{y^2+1}\right),$$

and

$$\mathrm{La}\left(e^{-\frac{\ell(\beta)}{2}}, \tanh^2\left(\frac{p_\alpha}{2}\right)\right) = \mathrm{La}\left(x, \frac{x+y^2}{xy^2+1}\right)\Bigg) = \mathcal{L}\left(\frac{x+y^2}{xy^2+1}\right) + \mathcal{L}\left(\frac{1-y^2}{1+x}\right) - \mathcal{L}\left(\frac{xy^2+1}{x+1}\right).$$

In terms of x and y we obtain:

$$\begin{aligned}\eta(x,y) = 8\mathcal{L}\left(\frac{(x+1)^2y^2}{(x+y^2)(xy^2+1)}\right) - 8\mathcal{L}\left(\frac{(1-x)^2y^2}{(x+y^2)(xy^2+1)}\right) - 16\mathcal{L}\left(\frac{1-x}{1+y^2}\right)\\ +16\mathcal{L}\left(\frac{xy^2+1}{1+y^2}\right) - 16\mathcal{L}\left(\frac{x+y^2}{xy^2+1}\right) - 16\mathcal{L}\left(\frac{1-y^2}{1+x}\right) + 16\mathcal{L}\left(\frac{xy^2+1}{1+x}\right).\end{aligned}$$

Now that we have properly defined the functions φ and η, we refer the reader back to our main result as stated in the introduction (Theorem 4). Recall that if we express $\varphi(P)$ and $\eta(P)$ in terms of the boundary lengths, this theorem tells us that the functions are strictly *decreasing* in terms of these lengths. We defer the proof of Theorem 4 to the final section, and now concentrate on certain of its implications.

2.2. Dominating simple length spectra

For any closed, orientable hyperbolic surface X of genus $g \geq 2$. We will generally be thinking of X as *marked*, hence as a point in Teichmüller space Teich_g, or if the marking is not essential, in moduli space $\mathcal{M}_g$. (Marked in this setting can be thought of as knowing the names of all simple closed geodesics.). In this section we show how to deduce a theorem of Thurston's (Theorem 3.1 in [39]) from the monotonicity properties of the measures.

Theorem 14. *If* $X, Y \in \mathrm{Teich}_g$ *satisfy* $\ell_X(\gamma) \geq \ell_Y(\gamma)$ *for all simple closed geodesics* γ*, then* $X = Y$.

Proof. Just for the purpose of this proof we think of the functions φ and η as being functions of the boundary lengths. In order not to introduce too much notation, we continue to call them φ and η.

Now if $\ell_X(\gamma) \geq \ell_Y(\gamma)$ for all γ, then in particular, by monotonicity of the function φ, for any embedded pair of pants with boundary curves γ_1, γ_2 and γ_3:

$$\varphi(\ell_X(\gamma_1), \ell_X(\gamma_2), \ell_X(\gamma_3)) \leq \varphi(\ell_Y(\gamma_1), \ell_Y(\gamma_2), \ell_Y(\gamma_3))$$

with equality if and only if the lengths are all equal. Similarly, for an improperly embedded pair of pants with boundary curve β and interior simple closed curve α, we have, by monotonicity of the function η:

$$\eta(\ell_X(\beta), \ell_X(\alpha)) \leq \eta(\ell_Y(\beta), \ell_Y(\alpha))$$

with equality if and only if the lengths are equal. As, by Equation 2.2, the sums of these functions, summed over all possible properly and improperly embedded pants, are equal for both X and Y, it follows that each summand is equal.

Now as every simple closed geodesic belongs to certain pairs of pants, either properly or improperly embedded, the result follows by rigidity of the marked simple length spectrum. □

2.3. Counting pants

We now focus on counting the number of pants (embedded or improperly embedded) of boundary length less than L on any surface (of genus g with n cusps).

Fix a hyperbolic closed surface X and let $\mathcal{Y}(X)$ be the set of isotopy classes of geodesic pants on X (so the union of properly $\mathcal{P}(X)$ and improperly embedded pants $\mathcal{I}(X)$). The Hu-Tan variation of the identity (Equation 2.2) allows us to associate to each pair of pants P a measure (either φ or η depending on whether it is properly or improperly embedded), the sum of which adds up to the volume of the unit tangent bundle. The inequalities from Theorem 4 will then allow us to bound the number of pants.

Theorem 15. *For a surface X we set*

$$\mathrm{NP}_X(L) = \sharp\{Y \in \mathcal{Y}(X) | \ell(\partial Y) \leq L\}$$

to be the number of pants of total boundary length less than L. Then any $X \in \mathcal{M}_g$ satisfies

$$\mathrm{NP}_X(L) < \frac{2\pi^2(g-1)e^{L/2}}{L+6}.$$

Proof. Let $Y \in \mathcal{Y}(X)$. Suppose Y is properly embedded, and let ℓ_1, ℓ_2 and ℓ_3 be its boundary lengths and L their sum. Notice that

$$e^{-L/6} = \sqrt[3]{e^{-\frac{\ell_1}{2}} e^{-\frac{\ell_2}{2}} e^{-\frac{\ell_3}{2}}}$$

and thus by Theorem 4 we have

$$\varphi(Y) = \varphi\left(e^{-\ell_1/2}, e^{-\ell_2/2}, e^{-\ell_3/2}\right) \geq \varphi\left(e^{-\frac{L}{6}}, e^{-\frac{L}{6}}, e^{-\frac{L}{6}}\right).$$

Similarly, if Y is improperly embedded with boundary lengths ℓ_1, ℓ_2 and ℓ_2, and L their sum, we have

$$\eta(Y) = \eta\left(e^{-\ell_1/2}, e^{-\ell_2/2}\right) \geq \varphi\left(e^{-\frac{L}{6}}, e^{-\frac{L}{6}}, e^{-\frac{L}{6}}\right)$$

where the last inequality is again from Theorem 4. Now by Theorem 4 again, the measure associated to any Y of total boundary length L is greater than

$$\varphi\left(e^{-\frac{L}{6}}, e^{-\frac{L}{6}}, e^{-\frac{L}{6}}\right) > 24\, e^{-L/2}\frac{L}{6} + 24\, e^{-L/2} = 4\frac{L+6}{e^{L/2}}.$$

Now as the total sum of all the measures is equal to $8\pi^2(g-1)$, we have

$$\mathrm{NP}_X(L) < 8\pi^2(g-1)\frac{1}{4\frac{L+6}{e^{-L/2}}} = \frac{2\pi^2(g-1)e^{L/2}}{L+6}$$

as desired. □

2.4. Behavior of the measures

This is the main technical part of the chapter, where we show the measures satisfy the properties we previously claimed. Several of the intermediate claims, although they are ultimately purely calculus, are in fact quite technical. They can (and have been) checked by formal computational software. In the proofs, the choices of how the terms are grouped might seem somewhat arbitrary at first. In fact, they are the result of playing around with the obtained expressions multiple times so that we can apply standard inequalities for logarithms.

In this section, several of elementary algebraic manipulations will be omitted.

Notation. We will write $\overset{\mathcal{M}}{=}$ to indicate that the equality was checked by Mathematica, in many cases using the *Simplify[]* command.

The following lemma is part of Theorem 4.

Lemma 6. $\varphi(x,x,x) > -24x^3\log(x) + 24x^3$ *for all* $x \in (0,1]$.

Proof. Consider the function $f(x) := \varphi(x,x,x) + 24x^3\log(x) - 24x^3$. This function is continuous on $(0,1]$, so it is enough to prove that f is strictly increasing on $(0,1)$. The derivative of f can be obtained by using the formula of $(\varphi(x,x,x))'$ in Lemma 13

$$\begin{aligned}
f'(x) = {} & (\varphi(x,x,x))' - 48x^2 + 72x^2\log(x) \\
= {} & \frac{24}{x}\log(1+x) + \frac{24(1-2x)}{(1-x)x}\log(1-x+x^2) - \frac{72x^2}{x^3+1}\log(x) + \frac{24(1-2x)}{x^2-x+1}\log(1-x) \\
& - 48x^2 + 72x^2\log(x) \\
\overset{\mathcal{M}}{=} {} & \frac{24}{x}\log(1+x) + \left(\frac{24}{x} - \frac{24}{(1-x)(1-x+x^2)} + \frac{24x}{1-x+x^2}\right)\log(1-x+x^2) \\
& + \frac{72x^5}{x^3+1}\log(x) + \left(\frac{24(1-x)}{x^2-x+1} - \frac{24x}{x^2-x+1}\right)\log(1-x) - 48x^2 \\
= {} & \frac{24}{x}(\log(1+x) + \log(1-x+x^2)) - \frac{24}{(1-x)(1-x+x^2)}\log(1-x+x^2) \\
& + \frac{24x}{1-x+x^2}(\log(1-x+x^2) - \log(1-x)) + \frac{72x^5}{x^3+1}\log(x) + \frac{24(1-x)}{x^2-x+1}\log(1-x) - 48x^2 \\
= {} & \frac{24}{x}\log(1+x^3) - \frac{24}{(1-x)(1-x+x^2)}\log(1-x+x^2) + \frac{24x}{1-x+x^2}\log\left(\frac{1-x+x^2}{1-x}\right) \\
& + \frac{72x^5}{x^3+1}\log(x) + \frac{24(1-x)}{x^2-x+1}\log(1-x) - 48x^2.
\end{aligned}$$

Note that the following Taylor series for $\log(t)$ around 1 is valid for $t \in (0,2]$:

$$\log(t) = (t-1) - \frac{(t-1)^2}{2} + \frac{(t-1)^3}{3} - \ldots = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}(t-1)^k}{k}.$$

We observe that for all $t \in (0,1]$, the terms of the Taylor series are negative which implies that:

$$\log(t) \le (t-1) - \frac{(t-1)^2}{2}.$$

We also note that, $\log(t) \ge 1 - \frac{1}{t}$ for all $t > 0$. Therefore:

$$\begin{aligned}
f'(x) \ge {} & \frac{24}{x}\left(1 - \frac{1}{1+x^3}\right) - \frac{24}{(1-x)(1-x+x^2)}\left((1-x+x^2-1) - \frac{(1-x+x^2-1)^2}{2}\right) \\
& + \frac{24x}{1-x+x^2}\left(1 - \frac{1-x}{1-x+x^2}\right) + \frac{72x^5}{x^3+1}\left(1 - \frac{1}{x}\right) + \frac{24(1-x)}{x^2-x+1}\log(1-x) - 48x^2
\end{aligned}$$

$$\overset{\mathcal{M}}{=} \frac{24(1-x)}{1-x+x^2}\left(\frac{x(2-x+3x^2-4x^3+9x^4-9x^5+2x^6)}{2(1+x)(1-x)(1-x+x^2)}+\log(1-x)\right).$$

We now set

$$g(x) := \frac{x(2-x+3x^2-4x^3+9x^4-9x^5+2x^6)}{2(1+x)(1-x)(1-x+x^2)}+\log(1-x)$$

and so

$$\begin{aligned} g'(x) &\overset{\mathcal{M}}{=} \frac{x^2(5-15x+34x^2-46x^3+28x^4+4x^5-18x^6+13x^7-3x^8)}{(1-x)^2(1+x)^2(1-x+x^2)^2} \\ &\overset{\mathcal{M}}{=} \frac{x^2(2x^8+10x^7(1-x)+(1-x)^2(5-5x+19x^2-3x^3+3x^4+13x^5+5x^6))}{(1-x)^2(1+x)^2(1-x+x^2)^2} \\ &> 0,\ \text{for all } x \in (0,1). \end{aligned}$$

Therefore, $g(x) > g(0) = 0$, for all $x \in (0,1)$. In particular, $f'(x) > 0$, for all $x \in (0,1)$ and thus

$$f(x) > \lim_{x\to 0}\left(\varphi(x,x,x)+24x^3\log(x)-24x^3\right) = 0$$

which completes the proof. □

We now prove the monotonicity of φ. Since φ is a symmetric function, it suffices to show:

Lemma 7.

$$\partial_x\varphi(x,y,z) > 0$$

for all $x, y, z \in (0,1)$.

Proof. Let

$$M := -\frac{16x^4(y^2+z^2+y^2z^2)+8x^3yz(4+x^2+3y^2+3z^2)+8y^2z^2(4x^2-2)-8xyz(1+y^2+z^2)}{x(1-x^2)(x+yz)(y+xz)(z+xy)}.$$

From Lemma 14, for all $x, y, z \in (0,1)$ we have

$$\begin{aligned} \partial_x\varphi(x,y,z) &= -\frac{16(x+yz)\log(x)}{(1-x^2)(1+xyz)} - \frac{16yz\log(y)}{1+xyz} - \frac{16yz\log(z)}{1+xyz} - \frac{8yz(1+x^2+2xyz)\log(1-x^2)}{x(x+yz)(1+xyz)} \\ &+ \frac{8z(1-y^2)\log(1-y^2)}{(y+xz)(1+xyz)} + \frac{8y(1-z^2)\log(1-z^2)}{(z+xy)(1+xyz)} + \frac{(16x+8yz+8x^2yz)\log(x+yz)}{(1-x^2)(1+xyz)} \\ &+ \frac{8yz\log(y+xz)}{1+xyz} + \frac{8yz\log(z+xy)}{1+xyz} + M\log(1+xyz) \\ &= \left(-\frac{16yz}{1+xyz} - \frac{16x}{1-x^2}\right)\log(x) - \frac{16yz\log(y)}{1+xyz} - \frac{16yz\log(z)}{1+xyz} - \frac{8yz(1+x^2+2xyz)\log(1-x^2)}{x(x+yz)(1+xyz)} \\ &+ \frac{8z(1-y^2)\log(1-y^2)}{(y+xz)(1+xyz)} + \frac{8y(1-z^2)\log(1-z^2)}{(z+xy)(1+xyz)} + \left(\frac{8yz}{1+xyz} + \frac{16x}{1-x^2}\right)\log(x+yz) \\ &+ \frac{8yz\log(y+xz)}{1+xyz} + \frac{8yz\log(z+xy)}{1+xyz} + M\log(1+xyz) \end{aligned}$$

$$
\begin{aligned}
&= \frac{8yz}{1+xyz}\left(-2\log(x)-2\log(y)-2\log(z)+\log(x+yz)+\log(y+xz)+\log(z+xy)\right)\\
&\quad -\frac{8yz(1+x^2+2xyz)\log(1-x^2)}{x(x+yz)(1+xyz)}+\frac{8z(1-y^2)\log(1-y^2)}{(y+xz)(1+xyz)}+\frac{8y(1-z^2)\log(1-z^2)}{(z+xy)(1+xyz)}\\
&\quad +\frac{16x}{1-x^2}(-\log(x)+\log(x+yz))+M\log(1+xyz)\\
&= \frac{8yz}{1+xyz}\log\left(\frac{x+yz}{yz}\right)+\frac{8yz}{1+xyz}\log\left(\frac{y+xz}{xz}\right)+\frac{8yz}{1+xyz}\log\left(\frac{z+xy}{xy}\right)\\
&\quad -\frac{8yz(1+x^2+2xyz)\log(1-x^2)}{x(x+yz)(1+xyz)}+\frac{8z(1-y^2)\log(1-y^2)}{(y+xz)(1+xyz)}+\frac{8y(1-z^2)\log(1-z^2)}{(z+xy)(1+xyz)}\\
&\quad +\left(\frac{16x}{1-x^2}\log\left(1+\frac{yz}{x}\right)+M\log(1+xyz)\right)
\end{aligned}
$$

Note that, for all $x, y, z \in (0,1)$,

$$
\frac{8yz}{1+xyz}\log\left(\frac{x+yz}{yz}\right)>0;\quad -\frac{8yz(1+x^2+2xyz)}{x(x+yz)(1+xyz)}\log(1-x^2)>0;
$$

$$
\frac{16x}{1-x^2}\log\left(1+\frac{yz}{x}\right)>\frac{16x}{1-x^2}\log\left(1+xyz\right).
$$

Note again that, $\log(t) \geq 1-\frac{1}{t}$ for all $t > 0$. Therefore,

$$
\begin{aligned}
\partial_x\varphi(x,y,z) >&\ 0+\frac{8yz}{1+xyz}\left(1-\frac{xz}{y+xz}\right)+\frac{8yz}{1+xyz}\left(1-\frac{xy}{z+xy}\right)+0\\
&+\frac{8z(1-y^2)}{(y+xz)(1+xyz)}\left(1-\frac{1}{1-y^2}\right)+\frac{8y(1-z^2)}{(z+xy)(1+xyz)}\left(1-\frac{1}{1-z^2}\right)\\
&+\left(\frac{16x}{1-x^2}\log(1+xyz)+M\log(1+xyz)\right)\\
\overset{\mathcal{M}}{=}&\ 0+\frac{8yz\left(x\left(1-x^2\right)+xy^2+xz^2+2yz\right)}{x(xy+z)(xz+y)(x+yz)}\log(1+xyz)\\
>&\ 0, \text{ for all } x,y,z\in(0,1).
\end{aligned}
$$

□

The next lemma is about the monotonicity of η:

Lemma 8. *The function η satisfies*

$$
\partial\eta_x(x,y)>0 \textit{ and } \partial\eta_y(x,y)>0 \textit{ for all } x,y\in(0,1).
$$

Proof. From Lemma 15, for $x, y \in (0,1)$,

$$
\partial\eta_x(x,y)=-\frac{8\left(y^2+1\right)\log(x)}{(1-x)\left(xy^2+1\right)}-\frac{16y^2\log(y)}{xy^2+1}-\frac{8y^2\left(x^2+2xy^2+1\right)\log(1-x)}{x\left(x+y^2\right)\left(xy^2+1\right)}
$$

$$+\frac{8\left(1-y^4\right)\log(1-y^2)}{(x+y^2)\left(xy^2+1\right)}+\frac{8\left(y^2+1\right)\log(x+y^2)}{(1-x)\left(xy^2+1\right)}-\frac{8\left(x^2+2xy^2-y^2\right)\log(1+xy^2)}{(1-x)x\left(x+y^2\right)}.$$

Note that

$$-\frac{8\left(x^2+2xy^2-y^2\right)}{(1-x)x\left(x+y^2\right)}\overset{\mathcal{M}}{=}\frac{16y^2}{1+xy^2}+\frac{8y^2(1-x^2)}{x(x+y^2)(1+xy^2)}-\frac{8(1+y^2)}{(1-x)(1+xy^2)}$$

Hence,

$$\begin{aligned}
\partial\eta_x(x,y) &= \frac{8\left(y^2+1\right)}{(1-x)\left(xy^2+1\right)}(-\log(x)+\log(x+y^2)-\log(1+xy^2))\\
&+\frac{16y^2}{xy^2+1}(-\log(y)+\log(1+xy^2))+\frac{8y^2\left(x^2+2xy^2+1\right)}{x\left(x+y^2\right)\left(xy^2+1\right)}\log\left(\frac{1}{1-x}\right)\\
&+\frac{8\left(1-y^4\right)\log(1-y^2)}{(x+y^2)\left(xy^2+1\right)}+\frac{8y^2(1-x^2)\log(1+xy^2)}{x(x+y^2)(1+xy^2)}\\
&= \frac{8\left(y^2+1\right)}{(1-x)\left(xy^2+1\right)}\log\left(\frac{x+y^2}{x(1+xy^2)}\right)+\frac{16y^2}{xy^2+1}\log\left(\frac{1+xy^2}{y}\right)\\
&+\frac{8y^2\left(x^2+2xy^2+1\right)}{x\left(x+y^2\right)\left(xy^2+1\right)}\log\left(\frac{1}{1-x}\right)+\frac{8\left(1-y^4\right)\log(1-y^2)}{(x+y^2)\left(xy^2+1\right)}+\frac{8y^2(1-x^2)\log(1+xy^2)}{x(x+y^2)(1+xy^2)}.
\end{aligned}$$

Now as $\log(t)\geq 1-\frac{1}{t}$ for all $t>0$. we deduce for all $x,y\in(0,1)$, that:

$$\begin{aligned}
\partial\eta_x(x,y) \geq\ & \frac{8\left(y^2+1\right)}{(1-x)\left(xy^2+1\right)}\left(1-\frac{x(1+xy^2)}{x+y^2}\right)+\frac{16y^2}{xy^2+1}\left(1-\frac{y}{1+xy^2}\right)\\
&+\frac{8y^2\left(x^2+2xy^2+1\right)}{x\left(x+y^2\right)\left(xy^2+1\right)}(1-(1-x))+\frac{8\left(1-y^4\right)}{(x+y^2)\left(xy^2+1\right)}\left(1-\frac{1}{1-y^2}\right)\\
&+\frac{8y^2(1-x^2)}{x(x+y^2)(1+xy^2)}\left(1-\frac{1}{1+xy^2}\right)\\
&\overset{\mathcal{M}}{=} \frac{8y^2(1+x+x^2+y^2+4xy^2+2x^2y^2+x^3y^2+2xy^4+3x^2y^4+2(1-y)(x+y^2))}{(x+y^2)(1+xy^2)^2}\\
&> 0,\ \text{for all } x,y\in(0,1).
\end{aligned}$$

Now we proceed to the second inequality of this lemma. For $x,y\in(0,1)$,

$$\begin{aligned}
\partial\eta_y(x,y) &= -\frac{16xy\log(x)}{xy^2+1}-\frac{32(x+1)y\log(y)}{(1-y^2)\left(xy^2+1\right)}+\frac{16(1-x^2)y\log(1-x)}{(x+y^2)\left(xy^2+1\right)}\\
&-\frac{16x\left(2xy^2+y^4+1\right)\log(1-y^2)}{y\left(x+y^2\right)\left(xy^2+1\right)}+\frac{16(x+1)y\log(x+y^2)}{(1-y^2)\left(xy^2+1\right)}+\frac{16(x-2xy^2-y^4)\log(1+xy^2)}{y\left(1-y^2\right)\left(x+y^2\right)}.
\end{aligned}$$

Note that

$$\frac{16(x-2xy^2-y^4)}{y\left(1-y^2\right)\left(x+y^2\right)}\overset{\mathcal{M}}{=}\frac{16x(1+2xy^2+y^4)}{y(x+y^2)(1+xy^2)}-\frac{16(1+x)y}{(1-y^2)(1+xy^2)}$$

Hence,

$$\begin{aligned}
\partial\eta_y(x,y) =& -\frac{16xy}{xy^2+1}\log(x) + \frac{16(x+1)y}{(1-y^2)\,(xy^2+1)}(-2\log(y)+\log(x+y^2)-\log(1+xy^2)) \\
&+ \frac{16(1-x^2)y}{(x+y^2)\,(xy^2+1)}\log(1-x) + \frac{16x\left(2xy^2+y^4+1\right)}{y\,(x+y^2)\,(xy^2+1)}(-\log(1-y^2)+\log(1+xy^2)) \\
=& \ \frac{16xy}{xy^2+1}\log\left(\frac{1}{x}\right) + \frac{16(x+1)y}{(1-y^2)\,(xy^2+1)}\log\left(\frac{x+y^2}{y^2(1+xy^2)}\right) \\
&+ \frac{16(1-x^2)y}{(x+y^2)\,(xy^2+1)}\log(1-x) + \frac{16x\left(2xy^2+y^4+1\right)}{y\,(x+y^2)\,(xy^2+1)}\log\left(\frac{1+xy^2}{1-y^2}\right) \\
\geq& \ \frac{16xy}{xy^2+1}(1-x) + \frac{16(x+1)y}{(1-y^2)\,(xy^2+1)}\left(1-\frac{y^2(1+xy^2)}{x+y^2}\right) \\
&+ \frac{16(1-x^2)y}{(x+y^2)\,(xy^2+1)}\left(1-\frac{1}{1-x}\right) + \frac{16x\left(2xy^2+y^4+1\right)}{y\,(x+y^2)\,(xy^2+1)}\left(1-\frac{1-y^2}{1+xy^2}\right) \\
\overset{\mathcal{M}}{=}& \ \frac{16xy(1+2x-x^2+2y^2+2xy^2+3x^2y^2-x^3y^2+y^4+3xy^4)}{(x+y^2)(1+xy^2)^2} \\
>& \ 0, \text{ for all } x,y\in(0,1).
\end{aligned}$$

□

The following lemma is an essential step in our inequalities.

Lemma 9. *The function φ satisfies*

$$\varphi(x,y,z) \geq \varphi(x,\sqrt{yz},\sqrt{yz}),$$

for all $x,y,z\in(0,1]$. Furthermore, the function $\varphi(x,y,z)-\varphi(x,\sqrt{yz},\sqrt{yz})$ is monotone increasing with respect to x.

Proof. By replacing y and z by $\sqrt{yz}$ in the formula of $\partial_x\varphi(x,y,z)$ in Lemma 14, for all $x,y,z\in(0,1)$ we obtain:

$$\begin{aligned}
\partial_x\varphi(x,\sqrt{yz},\sqrt{yz}) =& -\frac{16(x+yz)\log(x)}{(1-x^2)(xyz+1)} - \frac{32yz\log\left(\sqrt{yz}\right)}{xyz+1} - \frac{8yz\left(x^2+2xyz+1\right)\log\left(1-x^2\right)}{x(x+yz)(xyz+1)} \\
&+ \frac{16\sqrt{yz}(1-yz)\log(1-yz)}{(xyz+1)\left(x\sqrt{yz}+\sqrt{yz}\right)} + \frac{8\left(x^2yz+2x+yz\right)\log(x+yz)}{(1-x^2)(xyz+1)} + \frac{16yz\log\left(x\sqrt{yz}+\sqrt{yz}\right)}{xyz+1} \\
&- \frac{8\left(x^5yz+2x^4(yz+yz(yz+1))+x^3yz(6yz+4)+4x^2y^2z^2-xyz(2yz+1)-2y^2z^2\right)\log(xyz+1)}{x(1-x^2)(x+yz)\left(x\sqrt{yz}+\sqrt{yz}\right)^2} \\
=& -\frac{16(x+yz)\log(x)}{(1-x^2)(xyz+1)} - \frac{8yz\log(y)}{xyz+1} - \frac{8yz\log(z)}{xyz+1} - \frac{8yz\left(x^2+2xyz+1\right)\log\left(1-x^2\right)}{x(x+yz)(xyz+1)} \\
&+ \frac{16(1-yz)\log(1-yz)}{(xyz+1)\,(x+1)} + \frac{8\left(x^2yz+2x+yz\right)\log(x+yz)}{(1-x^2)(xyz+1)} + \frac{16yz\log(x+1)}{xyz+1}
\end{aligned}$$

$$-\frac{8\left(x^5+2x^4(yz+2)+x^3(6yz+4)+4x^2yz-x(2yz+1)-2yz\right)\log(xyz+1)}{x(1-x^2)(x+yz)\,(x+1)^2}.$$

Note that

$$-\frac{8\left(x^5+2x^4(yz+2)+x^3(6yz+4)+4x^2yz-x(2yz+1)-2yz\right)}{x(1-x^2)(x+yz)\,(x+1)^2}-\frac{8(1-x)(y-z)^2}{(1+x)(xy+z)(y+xz)}$$
$$\overset{\mathcal{M}}{=}-\frac{16x^4(y^2+z^2+y^2z^2)+8x^3yz(4+x^2+3y^2+3z^2)+8y^2z^2(4x^2-2)-8xyz(1+y^2+z^2)}{x(1-x^2)(x+yz)(y+xz)(z+xy)}.$$

Hence,

$$\begin{aligned}\partial_x\varphi(x,y,z)-\partial_x\varphi(x,\sqrt{yz},\sqrt{yz}) &= -\frac{8yz\log(y)}{1+xyz}-\frac{8yz\log(z)}{1+xyz}-\frac{16yz\log(1+x)}{1+xyz}+\frac{8yz\log(xy+z)}{1+xyz}\\ &+\frac{8yz\log(y+xz)}{1+xyz}+\frac{8(1-y^2)z\log(1-y^2)}{(y+xz)(1+xyz)}+\frac{8y(1-z^2)\log(1-z^2)}{(xy+z)(1+xyz)}-\frac{16(1-yz)\log(1-yz)}{(1+x)(1+xyz)}\\ &-\frac{8(1-x)(y-z)^2\log(1+xyz)}{(1+x)(xy+z)(y+xz)}.\end{aligned}$$

If we can show that $\partial_x\varphi(x,y,z)-\partial_x\varphi(x,\sqrt{yz},\sqrt{yz})\geq 0$ for all $x,y,z\in(0,1)$, then it will imply that:

$$\varphi(x,y,z)-\varphi(x,\sqrt{yz},\sqrt{yz})\geq\varphi(0,y,z)-\varphi(0,\sqrt{yz},\sqrt{yz})=0.$$

Let

$$\begin{aligned}A :=&\frac{(1-y^2)z\log(1-y^2)}{(y+xz)(1+xyz)}+\frac{y(1-z^2)\log(1-z^2)}{(xy+z)(1+xyz)}-\frac{2(1-yz)\log(1-yz)}{(1+x)(1+xyz)},\\ B :=&-\frac{yz\log(y)}{1+xyz}-\frac{yz\log(z)}{1+xyz}-\frac{2yz\log(1+x)}{1+xyz}+\frac{yz\log(xy+z)}{1+xyz}+\frac{yz\log(y+xz)}{1+xyz}\\ &-\frac{(1-x)(y-z)^2\log(1+xyz)}{(1+x)(xy+z)(y+xz)}.\end{aligned}$$

Our aim will be to show that both A and B are non-negative for all $x,y,z\in(0,1)$. As

$$A.(1+xyz)(xy+z)(xz+y)(1+x)\overset{\mathcal{M}}{=}x^2h_2(y,z)+(x-x^2)h_1(y,z)+(1-x^2)h_0(y,z),$$

where

$$\begin{aligned}h_2(y,z) :=&\ 2z(y+z)(1-y^2)\log(1-y^2)+2y(y+z)(1-z^2)\log(1-z^2)-2(y+z)^2(1-yz)\log(1-yz),\\ h_1(y,z) :=&\ z(y+z)(1-y^2)\log(1-y^2)+y(y+z)(1-z^2)\log(1-z^2)-2(y^2+z^2)(1-yz)\log(1-yz),\\ h_0(y,z) :=&\ z^2(1-y^2)\log(1-y^2)+y^2(1-z^2)\log(1-z^2)-2yz(1-yz)\log(1-yz).\end{aligned}$$

The non-negativity of A is implied from the following:

Claim 1.

$$h_0(y,z)\geq 0,\quad h_1(y,z)\geq 0,\ \textit{and}\ h_2(y,z)\geq 0$$

for all $0<y,z<1$.

Proof of claim. Note that:

$$\frac{h_0(y,z)}{y^2z^2} = \frac{1-y^2}{y^2}\log(1-y^2) + \frac{1-z^2}{z^2}\log(1-z^2) - 2\frac{1-yz}{yz}\log(1-yz).$$

We consider the following function:

$$g(t) := \frac{(1-e^t)\log(1-e^t)}{e^t},$$

where $t < 0$. Then

$$g''(t) = \frac{1}{1-e^t} + \frac{\log(1-e^t)}{e^t},$$

which is easily checked to be positive for all $t < 0$. Hence g is convex on its domain. Therefore, for all negative numbers t_1 and t_2, we have:

$$g(t_1) + g(t_2) \geq 2g\left(\frac{t_1+t_2}{2}\right).$$

By substituting t_1, t_2 by $\log(y^2), \log(z^2)$ respectively, we obtain:

$$\frac{1-y^2}{y^2}\log(1-y^2) + \frac{1-z^2}{z^2}\log(1-z^2) \geq 2\frac{1-yz}{yz}\log(1-yz).$$

This implies that $h_0(y,z) \geq 0$ for all $0 < y, z < 1$. Now we prove that $h_2(y,z) \geq 0$. Indeed, for all $y, z \in (0,1)$,

$$\begin{aligned}
\frac{h_2(y,z)}{2(y+z)} &= z(1-y^2)(\log(1-y^2) - \log(1-yz)) + y(1-z^2)(\log(1-z^2) - \log(1-yz)) \\
&= z(1-y^2)\log\left(\frac{1-y^2}{1-yz}\right) + y(1-z^2)\log\left(\frac{1-z^2}{1-yz}\right) \\
&\geq z(1-y^2)\left(1 - \frac{1-yz}{1-y^2}\right) + y(1-z^2)\left(1 - \frac{1-yz}{1-z^2}\right) = 0,
\end{aligned}$$

Lastly, $h_1(y,z)$ is non-negative because of the following:

$$\begin{aligned}
\frac{h_1(y,z)}{y+z} &= \left(z(1-y^2)\log(1-y^2) + y(1-z^2)\log(1-z^2)\right) - 2\frac{(y^2+z^2)(1-yz)}{y+z}\log(1-yz) \\
&= \left(\frac{h_2(y,z)}{2(y+z)} + (z(1-y^2) + y(1-z^2))\log(1-yz)\right) - 2\frac{(y^2+z^2)(1-yz)}{y+z}\log(1-yz) \\
&= \frac{h_2(y,z)}{2(y+z)} - \frac{(y-z)^2(1-yz)}{y+z}\log(1-yz) \geq 0, \quad \text{for all } y, z \in (0,1).
\end{aligned}$$

This completes the proof of Claim 1. □

Finally, we prove the non-negativity of B as follows:

$$B = \frac{yz}{1+xyz}\log\left(\frac{(xy+z)(xz+y)}{yz(1+x)^2}\right) + \frac{(1-x)(y-z)^2}{(1+x)(xy+z)(y+xz)}\log\left(\frac{1}{1+xyz}\right)$$

$$\geq \frac{yz}{1+xyz}\left(1-\frac{yz(1+x)^2}{(xy+z)(xz+y)}\right)+\frac{(1-x)(y-z)^2}{(1+x)(xy+z)(y+xz)}(1-(1+xyz))$$
$$\overset{\mathcal{M}}{=} \frac{x^2y(y-z)^2z(2-yz+xyz)}{(1+x)(xy+z)(y+xz)(1+xyz)} \geq 0, \text{ for all } x,y,z \in (0,1).$$

□

Remark 1. *The previous proof tells us that*

$$\frac{\partial\varphi}{\partial x}(x,y,z) \geq \frac{\partial\varphi}{\partial x}(x,\sqrt{yz},\sqrt{yz}).$$

Hence, Lemma 7 also follows from the following simpler inequality which contains only two variables:

$$\frac{\partial\varphi}{\partial x}(x,y,y) > 0.$$

Lemma 10. *The function φ satisfies*

$$\varphi(x,y,z) \geq \varphi(\sqrt[3]{xyz},\sqrt[3]{xyz},\sqrt[3]{xyz}),$$

for all $x,y,z \in (0,1]$

Proof. By applying Lemma 9 twice, we obtain:

$$\varphi(x,y,z) \geq \varphi(x,\sqrt{yz},\sqrt{yz}) \geq \varphi\left(\sqrt{x\sqrt{yz}},\sqrt{x\sqrt{yz}},\sqrt{yz}\right) \tag{2.3}$$

We define a function f from $(0,1]^3$ to $(0,1]^3$ as follows:

$$f(x,y,z) := (x^{\frac{1}{2}}y^{\frac{1}{4}}z^{\frac{1}{4}}, x^{\frac{1}{2}}y^{\frac{1}{4}}z^{\frac{1}{4}}, y^{\frac{1}{2}}z^{\frac{1}{2}}),$$

Observe that $f^{\circ n}(x,y,z) = (x^{a_n}y^{b_n}z^{b_n}, x^{a_n}y^{b_n}z^{b_n}, x^{c_n}y^{d_n}z^{d_n})$, where $(a_n),(b_n),(c_n)$, and (d_n) are sequences satisfying:

$$a_1 = \frac{1}{2};\ \ b_1 = \frac{1}{4};\ \ c_1 = 0;\ \ d_1 = \frac{1}{2},$$

and

$$a_n = \frac{3a_{n-1}}{4}+\frac{c_{n-1}}{4};\ \ b_n = \frac{3b_{n-1}}{4}+\frac{d_{n-1}}{4};\ \ c_n = \frac{a_{n-1}}{2}+\frac{c_{n-1}}{2};\ \ d_n = \frac{b_{n-1}}{2}+\frac{d_{n-1}}{2},$$

for all integer $n > 1$. We realize that:

$$a_n - c_n = \left(\frac{3a_{n-1}}{4}+\frac{c_{n-1}}{4}\right)-\left(\frac{a_{n-1}}{2}+\frac{c_{n-1}}{2}\right) = \frac{1}{4}(a_{n-1}-c_{n-1}).$$

Hence,

$$a_n - c_n = \frac{1}{4}(a_{n-1}-c_{n-1}) = \ldots = \frac{1}{4^{n-1}}(a_1-c_1) = \frac{1}{4^{n-1}}\left(\frac{1}{2}-0\right) = \frac{2}{4^n}.$$

And then,

$$a_n = \frac{3a_{n-1}}{4} + \frac{c_{n-1}}{4} = \frac{3a_{n-1}}{4} + \frac{a_{n-1} - \frac{2}{4^{n-1}}}{4} = a_{n-1} - \frac{2}{4^n} = \ldots = a_1 - \left(\frac{2}{4^2} + \ldots + \frac{2}{4^n}\right).$$

Therefore $a_n = \frac{1}{3} + \frac{2}{3}\left(\frac{1}{4}\right)^n$, and $c_n = a_n - \frac{2}{4^n} = \frac{1}{3} - \frac{4}{3}\left(\frac{1}{4}\right)^n$. Similarly, we also obtain $b_n = \frac{1}{3} - \frac{1}{3}\left(\frac{1}{4}\right)^n$, and $d_n = \frac{1}{3} + \frac{2}{3}\left(\frac{1}{4}\right)^n$. Hence,

$$\lim_{n\to\infty}(f^{\circ n}(x,y,z)) = \lim_{n\to\infty}(x^{a_n}y^{b_n}z^{b_n}, x^{a_n}y^{b_n}z^{b_n}, x^{c_n}y^{d_n}z^{d_n}) = (x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}, x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}, x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}).$$

Note that, from (2.3), we have a monotonically decreasing sequence:

$$\varphi(x,y,z) \geq \varphi(f(x,y,z)) \geq \varphi((f \circ f)(x,y,z)) \geq \ldots \geq \varphi(f^{\circ n}(x,y,z)) \geq \ldots \tag{2.4}$$

Therefore, from (2.4) and the continuity of the function φ on its domain, we have:

$$\varphi(x,y,z) \geq \lim_{n\to\infty}\varphi(f^{\circ n}(x,y,z)) = \varphi(\lim_{n\to\infty}(f^{\circ n}(x,y,z)) = \varphi(\sqrt[3]{xyz}, \sqrt[3]{xyz}, \sqrt[3]{xyz}).$$

□

The following lemma is a relation between the two functions φ and η:

Lemma 11. *The functions φ and η satisfy:*

$$\eta(x,y) \geq \varphi(x,y,y),$$

for all $x, y, z \in (0,1]$. Furthermore, the function $\eta(x,y) - \varphi(x,y,y)$ is monotone increasing with respect to x.

Proof. Replacing z by y in the formula of $\partial_x\varphi(x,y,z)$ in Lemma 14 and by some standard manipulations, for all $x, y, z \in (0,1)$, we obtain:

$$\begin{aligned}\partial_x\varphi(x,y,y) = &-\frac{16\left(x+y^2\right)\log(x)}{(1-x^2)\left(xy^2+1\right)} - \frac{16y^2\log(y)}{xy^2+1} - \frac{8y^2\left(x^2+2xy^2+1\right)\log(1-x)}{x\left(x+y^2\right)\left(xy^2+1\right)} \\ &+ \frac{16\left(1-y^2\right)\log\left(1-y^2\right)}{(x+1)\left(xy^2+1\right)} - \frac{8\left(1-x^2\right)y^2\log(x+1)}{x\left(x+y^2\right)\left(xy^2+1\right)} + \frac{8\left(x^2y^2+2x+y^2\right)\log\left(x+y^2\right)}{(1-x^2)\left(xy^2+1\right)} \\ &- \frac{8\left(x^5y^2+2x^4\left(\left(y^2+1\right)y^2+y^2\right)+x^3y^2\left(6y^2+4\right)+4x^2y^4-xy^2\left(2y^2+1\right)-2y^4\right)\log\left(xy^2+1\right)}{xy^2\left(1-x^2\right)(x+1)^2\left(x+y^2\right)}\end{aligned}$$

Combine with the formula of $\partial_x\eta(x,y)$ in Lemma 15, after some manipulations, we obtain:

$$\begin{aligned}\partial_x\eta(x,y) - \partial_x\varphi(x,y,y) \overset{\mathcal{M}}{=} &-\frac{8\left(1-y^2\right)\log(x)}{(x+1)\left(xy^2+1\right)} + \frac{8(1-x)\left(1-y^2\right)^2\log(1-y^2)}{(x+1)\left(x+y^2\right)\left(xy^2+1\right)} \\ &+ \frac{8\left(1-x^2\right)y^2\log(x+1)}{x\left(x+y^2\right)\left(xy^2+1\right)} + \frac{8\log(x+y^2)}{x+1} - \frac{8\log(1+xy^2)}{x(x+1)}\end{aligned}$$

Note that,

$$\frac{8}{1+x} \overset{\mathcal{M}}{=} \frac{8y^2}{1+xy^2} + \frac{8(1-y^2)}{(1+x)(1+xy^2)};$$
$$\frac{8}{x(1+x)} \overset{\mathcal{M}}{=} \frac{8(1-x)(1-y^2)^2}{(1+x)(x+y^2)(1+xy^2)} + \frac{8(1-x^2)y^2}{x(x+y^2)(1+xy^2)} + \frac{8y^2}{1+xy^2} + \frac{8(1-y^2)}{(1+x)(1+xy^2)}.$$

Therefore,

$$\begin{aligned}
\partial_x\eta(x,y) - \partial_x\varphi(x,y,y) &= \frac{8\left(1-y^2\right)}{(x+1)\left(xy^2+1\right)}(-\log(x)+\log(x+y^2)-\log(1+xy^2)) \\
&+ \frac{8(1-x)\left(1-y^2\right)^2\log(1-y^2)}{(x+1)\left(x+y^2\right)\left(xy^2+1\right)} + \frac{8\left(1-x^2\right)y^2}{x\left(x+y^2\right)\left(xy^2+1\right)}(\log(x+1)-\log(1+xy^2)) \\
&+ \frac{8y^2}{1+xy^2}(\log(x+y^2)-\log(1+xy^2)) - \frac{8(1-x)(1-y^2)^2}{(1+x)(x+y^2)(1+xy^2)}\log(1+xy^2) \\
&= \frac{8\left(1-y^2\right)}{(x+1)\left(xy^2+1\right)}\log\left(\frac{x+y^2}{x(1+xy^2)}\right) + \frac{8(1-x)\left(1-y^2\right)^2}{(x+1)\left(x+y^2\right)\left(xy^2+1\right)}\log(1-y^2) \\
&+ \frac{8\left(1-x^2\right)y^2}{x\left(x+y^2\right)\left(xy^2+1\right)}\log\left(\frac{x+1}{1+xy^2}\right) + \frac{8y^2}{1+xy^2}\log\left(\frac{x+y^2}{1+xy^2}\right) \\
&+ \frac{8(1-x)(1-y^2)^2}{(1+x)(x+y^2)(1+xy^2)}\log\left(\frac{1}{1+xy^2}\right) \\
&\geq \frac{8\left(1-y^2\right)}{(x+1)\left(xy^2+1\right)}\left(1-\frac{x(1+xy^2)}{x+y^2}\right) + \frac{8(1-x)\left(1-y^2\right)^2}{(x+1)\left(x+y^2\right)\left(xy^2+1\right)}\left(1-\frac{1}{1-y^2}\right) \\
&+ \frac{8\left(1-x^2\right)y^2}{x\left(x+y^2\right)\left(xy^2+1\right)}\left(1-\frac{1+xy^2}{x+1}\right) + \frac{8y^2}{1+xy^2}\left(1-\frac{1+xy^2}{x+y^2}\right) \\
&+ \frac{8(1-x)(1-y^2)^2}{(1+x)(x+y^2)(1+xy^2)}\left(1-(1+xy^2)\right) \\
&\overset{\mathcal{M}}{=} \frac{8(1-x)xy^4(1-y^2)}{(1+x)(x+y^2)(1+xy^2)} \\
&> 0, \text{ for all } x,y,z \in (0,1).
\end{aligned}$$

This implies that:

$$\eta(x,y) - \varphi(x,y,y) \geq \eta(0,y) - \varphi(0,y,y) = 0,$$

for all $x, y \in (0,1]$. □

This completes the proofs of the technical results in this note. We end with an example in a similar vein, which can be obtained by using the same methods, and which illustrates that the function φ has a wealth of yet unexplored properties.

Lemma 12. *The function φ satisfies:*

$$\varphi(x, yz, 1) \geq \varphi(x, y, z),$$

for all $x, y, z \in (0,1]$. Furthermore, the function $\varphi(x, yz, 1) - \varphi(x, y, z)$ is monotone increasing with respect to x.

Sketch of proof. We have $\partial_x\varphi(x, yz, 1) = \lim_{u\to 1}(\partial_x\varphi(x, yz, u))$. Note that $\lim_{t\to 0}(t\log|t|) = 0$. Therefore

$$
\begin{aligned}
&\partial_x\varphi(x, yz, 1) - \partial_x\varphi(x, y, z) \overset{\mathcal{M}}{=} -\frac{8yz\log(xy+z)}{1+xyz} - \frac{8yz\log(y+xz)}{1+xyz} + \frac{8yz\log(x+yz)}{1+xyz} \\
&\quad + \frac{8yz\log(1+xyz)}{1+xyz} - \frac{8\left(1-x^2\right)y\left(1-y^2\right)z\left(1-z^2\right)\log(1+xyz)}{(xy+z)(y+xz)(x+yz)(1+xyz)} \\
&\quad - \frac{8\left(1-y^2\right)z\log\left(1-y^2\right)}{(y+xz)(1+xyz)} - \frac{8y\left(1-z^2\right)\log\left(1-z^2\right)}{(xy+z)(1+xyz)} + \frac{8\left(1-y^2z^2\right)\log\left(1-y^2z^2\right)}{(x+yz)(1+xyz)} \\
&\overset{\mathcal{M}}{=} \frac{8yz}{1+xyz}\log\left(\frac{(x+yz)(1+xyz)}{(xy+z)(y+xz)}\right) + \frac{8\left(1-x^2\right)y\left(1-y^2\right)z\left(1-z^2\right)}{(xy+z)(y+xz)(x+yz)(1+xyz)}\log\left(\frac{1}{1+xyz}\right) \\
&\quad + \frac{8\left(1-y^2\right)z}{(y+xz)(1+xyz)}\log\left(\frac{1-y^2z^2}{1-y^2}\right) + \frac{8y\left(1-z^2\right)}{(xy+z)(1+xyz)}\log\left(\frac{1-y^2z^2}{1-z^2}\right) \\
&\quad + \frac{8\left(1-x^2\right)y\left(1-y^2\right)z\left(1-z^2\right)}{(xy+z)(xz+y)(x+yz)(xyz+1)}\log\left(1-y^2z^2\right) \\
&\geq \frac{8yz}{1+xyz}\left(1-\frac{(xy+z)(y+xz)}{(x+yz)(1+xyz)}\right) + \frac{8\left(1-x^2\right)y\left(1-y^2\right)z\left(1-z^2\right)}{(xy+z)(y+xz)(x+yz)(1+xyz)}\left(1-(1+xyz)\right) \\
&\quad + \frac{8\left(1-y^2\right)z}{(y+xz)(1+xyz)}\left(1-\frac{1-y^2}{1-y^2z^2}\right) + \frac{8y\left(1-z^2\right)}{(xy+z)(1+xyz)}\left(1-\frac{1-z^2}{1-y^2z^2}\right) \\
&\quad + \frac{8\left(1-x^2\right)y\left(1-y^2\right)z\left(1-z^2\right)}{(xy+z)(xz+y)(x+yz)(xyz+1)}\left(1-\frac{1}{1-y^2z^2}\right) \\
&\overset{\mathcal{M}}{=} \frac{8y\left(1-y^2\right)z\left(1-z^2\right)}{\left(1-y^2z^2\right)(xy+z)(xz+y)(x+yz)(xyz+1)^2}\Big(x^4y^2z^2\left(1-y^2z^2\right) + x^3y^3z\left(1-z^2\right) \\
&\quad + x^3yz\left(z^2+2\right) + x^2\left(y^4z^4+2y^2z^2+2y^2+2z^2\right) + xyz\left(y^2z^2+y^2+z^2+2\right) + y^2z^2\Big) \\
&> 0, \text{ for all } x, y, z \in (0,1).
\end{aligned}
$$

□

2.5. Differentiating φ and η

In this section, we prove some of the technical lemmas claimed previously. For the sake of computation, we rewrite the formula of φ in variables x, y, z as follows:

$$
\varphi(x, y, z) = 8\mathcal{L}\left(\frac{(x+yz)(1+xyz)}{x+xy^2+yz+x^2yz}\right) + 8\mathcal{L}\left(\frac{(y+xz)(1+xyz)}{y+x^2y+xz+xy^2z}\right) + 8\mathcal{L}\left(\frac{(xy+z)(1+xyz)}{z+x^2z+xy+xyz^2}\right)
$$

$$
\begin{aligned}
&+8\mathcal{L}\left(\frac{(xy+z)(1+xyz)}{z+y^2z+xy+xyz^2}\right)+8\mathcal{L}\left(\frac{(y+xz)(1+xyz)}{y+xz+xy^2z+yz^2}\right)+8\mathcal{L}\left(\frac{(x+yz)(1+xyz)}{x+yz+x^2yz+xz^2}\right)\\
&-8\mathcal{L}\left(\frac{x\left(1-z^2\right)}{x+xy^2+yz+x^2yz}\right)-8\mathcal{L}\left(\frac{x\left(1-y^2\right)}{x+yz+x^2yz+xz^2}\right)-8\mathcal{L}\left(\frac{y\left(1-z^2\right)}{y+x^2y+xz+xy^2z}\right)\\
&-8\mathcal{L}\left(\frac{y\left(1-x^2\right)}{y+xz+xy^2z+yz^2}\right)-8\mathcal{L}\left(\frac{z\left(1-x^2\right)}{xy+z+y^2z+xyz^2}\right)-8\mathcal{L}\left(\frac{z\left(1-y^2\right)}{z+x^2z+xy+xyz^2}\right)\\
&-8\mathcal{L}\left(\frac{(xy+z)(y+xz)}{(x+yz)(1+xyz)}\right)-8\mathcal{L}\left(\frac{(xy+z)(x+yz)}{(y+xz)(1+xyz)}\right)-8\mathcal{L}\left(\frac{(y+xz)(x+yz)}{(xy+z)(1+xyz)}\right)\\
&-8\mathcal{L}\left(\frac{\left(1-x^2\right)^2y^2z^2}{(xy+z)(y+xz)(x+yz)(1+xyz)}\right)-8\mathcal{L}\left(\frac{x^2\left(1-y^2\right)^2z^2}{(xy+z)(y+xz)(x+yz)(1+xyz)}\right)\\
&-8\mathcal{L}\left(\frac{x^2y^2\left(1-z^2\right)^2}{(xy+z)(y+xz)(x+yz)(1+xyz)}\right).
\end{aligned}
$$

Replacing y and z by x, we obtain the formula of $\varphi(x,x,x)$:

$$\varphi(x,x,x)=48\mathcal{L}\left(\frac{1+x^3}{1+x^2}\right)-48\mathcal{L}\left(\frac{1-x}{1+x^2}\right)-24\mathcal{L}\left(\frac{x}{1-x+x^2}\right)-24\mathcal{L}\left(\frac{(1-x)^2x}{(1+x)(1+x^3)}\right).$$

We prove the following lemma

Lemma 13. *For $x\in(0,1)$,*

$$(\varphi(x,x,x))'=\frac{24}{x}\log(1+x)+\frac{24(1-2x)}{(1-x)x}\log(1-x+x^2)-\frac{72x^2}{x^3+1}\log(x)+\frac{24(1-2x)}{x^2-x+1}\log(1-x).$$

Proof. We have

$$(\phi(x,x,x))'=48\mathcal{L}'\left(\frac{1+x^3}{1+x^2}\right)-48\mathcal{L}'\left(\frac{1-x}{1+x^2}\right)-24\mathcal{L}'\left(\frac{x}{1-x+x^2}\right)-24\mathcal{L}'\left(\frac{(1-x)^2x}{(1+x)(1+x^3)}\right).$$

Note that: $\mathcal{L}'(f(x))=-\frac{f'(x)}{2}\left(\frac{\log(f(x))}{1-f(x)}+\frac{\log(1-f(x))}{f(x)}\right)$ for any differentiable function f of range $(0,1)$. Hence

- $\mathcal{L}'\left(\frac{1+x^3}{1+x^2}\right)=-\frac{1}{2}\left(\frac{1+x^3}{1+x^2}\right)'\left(\frac{\log\left(\frac{1+x^3}{1+x^2}\right)}{1-\frac{1+x^3}{1+x^2}}+\frac{\log\left(1-\frac{1+x^3}{1+x^2}\right)}{\frac{1+x^3}{1+x^2}}\right)$
$\overset{\mathcal{M}}{=}\frac{x\left(-x^3-3x+2\right)}{2\left(x^2+1\right)^2}\left(\frac{\left(x^2+1\right)\left(\log(1+x)+\log(x^2-x+1)-\log\left(x^2+1\right)\right)}{x^2(1-x)}+\frac{\left(x^2+1\right)\left(2\log(x)+\log(1-x)-\log\left(x^2+1\right)\right)}{(1+x)(x^2-x+1)}\right)$

- $\mathcal{L}'\left(\frac{1-x}{1+x^2}\right)=-\frac{1}{2}\left(\frac{1-x}{1+x^2}\right)'\left(\frac{\log\left(\frac{1-x}{1+x^2}\right)}{1-\frac{1-x}{1+x^2}}+\frac{\log\left(1-\frac{1-x}{1+x^2}\right)}{\frac{1-x}{1+x^2}}\right)$
$\overset{\mathcal{M}}{=}\frac{-x^2+2x+1}{2\left(x^2+1\right)^2}\left(\frac{\left(x^2+1\right)\left(\log(1-x)-\log\left(x^2+1\right)\right)}{x(1+x)}+\frac{\left(x^2+1\right)\left(\log(x)+\log(x+1)-\log\left(x^2+1\right)\right)}{1-x}\right)$

- $\mathcal{L}'\left(\frac{x}{1-x+x^2}\right)=-\frac{1}{2}\left(\frac{x}{1-x+x^2}\right)'\left(\frac{\log\left(\frac{x}{1-x+x^2}\right)}{1-\frac{x}{1-x+x^2}}+\frac{\log\left(1-\frac{x}{1-x+x^2}\right)}{\frac{x}{1-x+x^2}}\right)$
$\overset{\mathcal{M}}{=}\frac{x^2-1}{2\left(x^2-x+1\right)^2}\left(\frac{\left(x^2-x+1\right)\left(\log(x)-\log\left(x^2-x+1\right)\right)}{(1-x)^2}+\frac{\left(x^2-x+1\right)\left(2\log(1-x)-\log\left(x^2-x+1\right)\right)}{x}\right)$

- $\mathcal{L}'\left(\frac{(1-x)^2x}{(1+x)(1+x^3)}\right)=-\frac{1}{2}\left(\frac{(1-x)^2x}{(1+x)(1+x^3)}\right)'\left(\frac{\log\left(\frac{(1-x)^2x}{(1+x)(1+x^3)}\right)}{1-\frac{(1-x)^2x}{(1+x)(1+x^3)}}+\frac{\log\left(1-\frac{(1-x)^2x}{(1+x)(1+x^3)}\right)}{\frac{(1-x)^2x}{(1+x)(1+x^3)}}\right)$
$\overset{\mathcal{M}}{=}-\frac{(1-x)\left(x^2+1\right)\left(x^2-4x+1\right)}{2(x+1)^3\left(x^2-x+1\right)^2}\left(\frac{\left(x^2-x+1\right)(x+1)^2\left(-\log\left(x^2-x+1\right)+2\log(1-x)+\log(x)-2\log(x+1)\right)}{\left(x^2+1\right)^2}\right.$
$\left.+\frac{\left(x^2-x+1\right)(x+1)^2\left(2\log\left(x^2+1\right)-\log\left(x^2-x+1\right)-2\log(x+1)\right)}{(1-x)^2x}\right).$

Therefore, $(\varphi(x,x,x))'$ can be written in the following form:

$$c_1\log(1+x)+c_2\log(1-x+x^2)+c_3\log(1+x^2)+c_4\log(x)+c_5\log(1-x),$$

where

$$\begin{aligned}
c_1 :=&\frac{24(-x^3-3x+2)}{x(x^2+1)(1-x)}-\frac{24(-x^2+2x+1)}{(x^2+1)(1-x)}-\frac{24(1-x)(x^2-4x+1)}{(x+1)(x^2-x+1)(x^2+1)}\\
&-\frac{24(x^2+1)(x^2-4x+1)}{(x+1)(x^2-x+1)(1-x)x}\overset{\mathcal{M}}{=}\frac{24}{x}\\
c_2 :=&\frac{24(-x^3-3x+2)}{x(x^2+1)(1-x)}-\frac{12(x+1)}{(1-x)\left(x^2-x+1\right)}-\frac{12\left(1-x^2\right)}{x\left(x^2-x+1\right)}-\frac{12(1-x)\left(x^2-4x+1\right)}{(x+1)\left(x^2+1\right)\left(x^2-x+1\right)}\\
&-\frac{12\left(x^2+1\right)\left(x^2-4x+1\right)}{(1-x)x(x+1)\left(x^2-x+1\right)}\overset{\mathcal{M}}{=}\frac{24(1-2x)}{(1-x)x}\\
c_3 :=&-\frac{24x\left(-x^3-3x+2\right)}{(x^2+1)x^2(1-x)}-\frac{24x\left(-x^3-3x+2\right)}{(x+1)\left(x^2+1\right)\left(x^2-x+1\right)}+\frac{24\left(-x^2+2x+1\right)}{(x^2+1)x(x+1)}\\
&+\frac{24\left(-x^2+2x+1\right)}{(1-x)\left(x^2+1\right)}+\frac{24\left(x^2+1\right)\left(x^2-4x+1\right)}{(1-x)x(x+1)\left(x^2-x+1\right)}\overset{\mathcal{M}}{=}0\\
c_4 :=&\frac{48x\left(-x^3-3x+2\right)}{(x+1)\left(x^2+1\right)\left(x^2-x+1\right)}+\frac{24\left(x^2-2x-1\right)}{(1-x)\left(x^2+1\right)}+\frac{12(x+1)}{(1-x)\left(x^2-x+1\right)}\\
&+\frac{12(1-x)\left(x^2-4x+1\right)}{(x+1)\left(x^2+1\right)\left(x^2-x+1\right)}\overset{\mathcal{M}}{=}-\frac{72x^2}{x^3+1}\\
c_5 :=&\frac{24x\left(-x^3-3x+2\right)}{(x+1)\left(x^2+1\right)\left(x^2-x+1\right)}+\frac{24\left(x^2-2x-1\right)}{(x^2+1)x(x+1)}+\frac{24\left(1-x^2\right)}{x\left(x^2-x+1\right)}\\
&+\frac{24(1-x)\left(x^2-4x+1\right)}{(x+1)\left(x^2+1\right)\left(x^2-x+1\right)}\overset{\mathcal{M}}{=}\frac{24(1-2x)}{x^2-x+1}
\end{aligned}$$

□

With the same method, we compute the derivative of φ with respect to x.

Lemma 14. *For $x, y, z \in (0,1)$,*

$$\begin{aligned}\partial_x\varphi(x,y,z) = &-\frac{16(x+yz)\log(x)}{(1-x^2)(1+xyz)} - \frac{16yz\log(y)}{1+xyz} - \frac{16yz\log(z)}{1+xyz} - \frac{8yz(1+x^2+2xyz)\log(1-x^2)}{x(x+yz)(1+xyz)}\\ &+\frac{8z(1-y^2)\log(1-y^2)}{(y+xz)(1+xyz)} + \frac{8y(1-z^2)\log(1-z^2)}{(z+xy)(1+xyz)} + \frac{(16x+8yz+8x^2yz)\log(x+yz)}{(1-x^2)(1+xyz)}\\ &+\frac{8yz\log(y+xz)}{1+xyz} + \frac{8yz\log(z+xy)}{1+xyz} + M\log(1+xyz),\end{aligned}$$

where

$$M := -\frac{16x^4(y^2+z^2+y^2z^2) + 8x^3yz(4+x^2+3y^2+3z^2) + 8y^2z^2(4x^2-2) - 8xyz(1+y^2+z^2)}{x(1-x^2)(x+yz)(y+xz)(z+xy)}.$$

Proof. We have

$$\partial_x\varphi(x,y,z) = 4\sum_{i=1}^{18}\left(\frac{c_i}{b_i}\log(a_i) + \frac{c_i}{a_i}\log(b_i)\right)$$

where

$$a_1 := \frac{(x+yz)(1+xyz)}{x+xy^2+yz+x^2yz};\ \ a_2 := \frac{(y+xz)(1+xyz)}{y+x^2y+xz+xy^2z};\ \ a_3 := \frac{(xy+z)(xyz+1)}{x^2z+xyz^2+xy+z};$$

$$a_4 := \frac{(xy+z)(xyz+1)}{xyz^2+xy+y^2z+z};\ \ a_5 := \frac{(xz+y)(xyz+1)}{xy^2z+xz+yz^2+y};\ \ a_6 := \frac{(x+yz)(xyz+1)}{x^2yz+xz^2+x+yz};$$

$$a_7 := \frac{x\left(1-z^2\right)}{x^2yz+xy^2+x+yz};\ \ a_8 := \frac{x\left(1-y^2\right)}{x^2yz+xz^2+x+yz};\ \ a_9 := \frac{y\left(1-z^2\right)}{x^2y+xy^2z+xz+y};$$

$$a_{10} := \frac{\left(1-x^2\right)y}{xy^2z+xz+yz^2+y};\ \ a_{11} := \frac{\left(1-x^2\right)z}{xyz^2+xy+y^2z+z}; a_{12} := \frac{\left(1-y^2\right)z}{x^2z+xyz^2+xy+z};$$

$$a_{13} := \frac{(xy+z)(xz+y)}{(x+yz)(xyz+1)};\ \ a_{14} := \frac{(xy+z)(x+yz)}{(xz+y)(xyz+1)};\ \ a_{15} := \frac{(xz+y)(x+yz)}{(xy+z)(xyz+1)};$$

$$a_{16} := \frac{\left(1-x^2\right)^2y^2z^2}{(xy+z)(xz+y)(x+yz)(xyz+1)};\ \ a_{17} := \frac{x^2\left(1-y^2\right)^2z^2}{(xy+z)(xz+y)(x+yz)(xyz+1)};$$

$$a_{18} := \frac{x^2y^2\left(1-z^2\right)^2}{(xy+z)(xz+y)(x+yz)(xyz+1)};$$

$$b_1 := 1-a_1 \overset{\mathcal{M}}{=} \frac{xy^2\left(1-z^2\right)}{x+xy^2+yz+x^2yz};\ \ b_2 := 1-a_2 \overset{\mathcal{M}}{=} \frac{x^2y\left(1-z^2\right)}{x^2y+xy^2z+xz+y};$$

$$b_3 := 1-a_3 \overset{\mathcal{M}}{=} \frac{x^2\left(1-y^2\right)z}{x^2z+xyz^2+xy+z};\ \ b_4 := 1-a_4 \overset{\mathcal{M}}{=} \frac{\left(1-x^2\right)y^2z}{xyz^2+xy+y^2z+z};$$

$$b_5 := 1-a_5 \overset{\mathcal{M}}{=} \frac{\left(1-x^2\right)yz^2}{xy^2z+xz+yz^2+y};\ \ b_6 := 1-a_6 \overset{\mathcal{M}}{=} \frac{x\left(1-y^2\right)z^2}{x^2yz+xz^2+x+yz}$$

$$b_7 := 1-a_7 \overset{\mathcal{M}}{=} \frac{(z+xy)(y+xz)}{x^2yz+xy^2+x+yz};\ \ b_8 := 1-a_8 \overset{\mathcal{M}}{=} \frac{(y+xz)(z+xy)}{x^2yz+xz^2+x+yz};$$

$$b_9 := 1-a_9 \overset{\mathcal{M}}{=} \frac{(x+yz)(z+xy)}{x^2y+xy^2z+xz+y};\ \ b_{10} := 1-a_{10} \overset{\mathcal{M}}{=} \frac{(x+yz)(z+xy)}{xy^2z+xz+yz^2+y};$$

$$b_{11} := 1 - a_{11} \overset{\mathcal{M}}{=} \frac{(x+yz)(y+xz)}{xyz^2+xy+y^2z+z}; \quad b_{12} := 1 - a_{12} \overset{\mathcal{M}}{=} \frac{(x+yz)(y+xz)}{x^2z+xyz^2+xy+z};$$

$$b_{13} := 1 - a_{13} \overset{\mathcal{M}}{=} \frac{x\left(1-y^2\right)\left(1-z^2\right)}{(x+yz)(xyz+1)}; \quad b_{14} := 1 - a_{14} \overset{\mathcal{M}}{=} \frac{\left(1-x^2\right)y\left(1-z^2\right)}{(xz+y)(xyz+1)};$$

$$b_{15} := 1 - a_{15} \overset{\mathcal{M}}{=} \frac{\left(1-x^2\right)\left(1-y^2\right)z}{(xy+z)(xyz+1)}; \quad b_{16} := 1 - a_{16} \overset{\mathcal{M}}{=} \frac{x\left(xy^2z+xz+yz^2+y\right)\left(xyz^2+xy+y^2z+z\right)}{(xy+z)(xz+y)(x+yz)(xyz+1)};$$

$$b_{17} := 1 - a_{17} \overset{\mathcal{M}}{=} \frac{y\left(x^2yz+xz^2+x+yz\right)\left(x^2z+xyz^2+xy+z\right)}{(xy+z)(xz+y)(x+yz)(xyz+1)};$$

$$b_{18} := 1 - a_{18} \overset{\mathcal{M}}{=} \frac{z\left(x^2yz+xy^2+x+yz\right)\left(x^2y+xy^2z+xz+y\right)}{(xy+z)(xz+y)(x+yz)(xyz+1)};$$

$$c_1 := -\partial_x a_1 \overset{\mathcal{M}}{=} \frac{\left(1-x^2\right)y^3z\left(1-z^2\right)}{\left(x+xy^2+yz+x^2yz\right)^2}; \quad c_2 := -\partial_x a_2 \overset{\mathcal{M}}{=} \frac{xy\left(1-z^2\right)\left(xy^2z+xz+2y\right)}{\left(x^2y+xy^2z+xz+y\right)^2};$$

$$c_3 := -\partial_x a_3 \overset{\mathcal{M}}{=} \frac{x\left(1-y^2\right)z\left(xyz^2+xy+2z\right)}{\left(x^2z+xyz^2+xy+z\right)^2};$$

$$c_4 := -\partial_x a_4 \overset{\mathcal{M}}{=} -\frac{y^2z\left(\left(x^2+1\right)y\left(z^2+1\right)+2xy^2z+2xz\right)}{\left(xyz^2+xy+y^2z+z\right)^2};$$

$$c_5 := -\partial_x a_5 \overset{\mathcal{M}}{=} -\frac{yz^2\left(x^2\left(y^2+1\right)z+2xy\left(z^2+1\right)+\left(y^2+1\right)z\right)}{\left(xy^2z+xz+yz^2+y\right)^2};$$

$$c_6 := -\partial_x a_6 \overset{\mathcal{M}}{=} \frac{\left(1-x^2\right)y\left(1-y^2\right)z^3}{\left(x^2yz+xz^2+x+yz\right)^2}; \quad c_7 := \partial_x a_7 \overset{\mathcal{M}}{=} \frac{\left(1-x^2\right)yz\left(1-z^2\right)}{\left(x^2yz+xy^2+x+yz\right)^2};$$

$$c_8 := \partial_x a_8 \overset{\mathcal{M}}{=} \frac{\left(1-x^2\right)y\left(1-y^2\right)z}{\left(x^2yz+xz^2+x+yz\right)^2}; \quad c_9 := \partial_x a_9 \overset{\mathcal{M}}{=} -\frac{y\left(1-z^2\right)\left(2xy+y^2z+z\right)}{\left(x^2y+xy^2z+xz+y\right)^2};$$

$$c_{10} := \partial_x a_{10} \overset{\mathcal{M}}{=} -\frac{y\left(x^2\left(y^2+1\right)z+2xy\left(z^2+1\right)+\left(y^2+1\right)z\right)}{\left(xy^2z+xz+yz^2+y\right)^2};$$

$$c_{11} := \partial_x a_{11} \overset{\mathcal{M}}{=} -\frac{z\left(\left(x^2+1\right)y\left(z^2+1\right)+2xy^2z+2xz\right)}{\left(xyz^2+xy+y^2z+z\right)^2};$$

$$c_{12} := \partial_x a_{12} \overset{\mathcal{M}}{=} -\frac{\left(1-y^2\right)z\left(2xz+yz^2+y\right)}{\left(x^2z+xyz^2+xy+z\right)^2}; \quad c_{13} := \partial_x a_{13} \overset{\mathcal{M}}{=} -\frac{\left(1-x^2\right)y\left(1-y^2\right)z\left(1-z^2\right)}{(x+yz)^2(xyz+1)^2};$$

$$c_{14} := \partial_x a_{14} \overset{\mathcal{M}}{=} \frac{y\left(1-z^2\right)\left(x^2\left(y^2+1\right)z+2xy\left(z^2+1\right)+\left(y^2+1\right)z\right)}{(xz+y)^2(xyz+1)^2};$$

$$c_{15} := \partial_x a_{15} \overset{\mathcal{M}}{=} \frac{\left(1-y^2\right)z\left(\left(x^2+1\right)y\left(z^2+1\right)+2xy^2z+2xz\right)}{(xy+z)^2(xyz+1)^2};$$

$$c_{16} := \partial_x a_{16} \overset{\mathcal{M}}{=} -\frac{\left(1-x^2\right)y^2z^2\left(\left(x^2+1\right)\left(y^2+1\right)z+2xy\left(z^2+1\right)\right)\left(\left(x^2+1\right)y\left(z^2+1\right)+2x\left(y^2+1\right)z\right)}{(xy+z)^2(xz+y)^2(x+yz)^2(xyz+1)^2};$$

$$c_{17} := \partial_x a_{17} \overset{\mathcal{M}}{=} \frac{(1-x^2)x(1-y^2)^2yz^3\left(2x^2yz+xy^2z^2+xy^2+xz^2+x+2yz\right)}{(xy+z)^2(xz+y)^2(x+yz)^2(xyz+1)^2};$$

$$c_{18} := \partial_x a_{18} \overset{\mathcal{M}}{=} \frac{x\left(1-x^2\right)y^3z\left(1-z^2\right)^2\left(2x^2yz+x\left(y^2+1\right)\left(z^2+1\right)+2yz\right)}{(xy+z)^2(xz+y)^2(x+yz)^2(xyz+1)^2}.$$

By standard manipulations of logarithm, we can rewrite $\partial_x\varphi(x,y,z)$ in the following form.

$$\begin{aligned}\partial_x\varphi(x,y,z) = 4\big(&f_1\log(x)+f_2\log(y)+f_3\log(z)+f_4\log(1-x^2)+f_5\log(1-y^2)+f_6\log(1-z^2)\\&+f_7\log(x+yz)+f_8\log(y+xz)+f_9\log(z+xy)+g_1\log(x+xy^2+yz+x^2yz)\\&+g_2\log(y+x^2y+xz+xy^2z)+g_3\log(x^2z+xyz^2+xy+z)+g_4\log(xyz^2+xy+y^2z+z)\\&+g_5\log(xy^2z+xz+yz^2+y)+g_6\log(x^2yz+xz^2+x+yz)+m\log(1+xyz)\big),\end{aligned}$$

where

$$f_1 := \frac{c_1}{a_1}+2\frac{c_2}{a_2}+2\frac{c_3}{a_3}+\frac{c_6}{a_6}+\frac{c_7}{b_7}+\frac{c_8}{b_8}+\frac{c_{13}}{a_{13}}+\frac{c_{16}}{a_{16}}+2\frac{c_{17}}{b_{17}}+2\frac{c_{18}}{b_{18}} \overset{\mathcal{M}}{=} -\frac{4(x+yz)}{(1-x^2)\,(xyz+1)};$$

$$f_2 := 2\frac{c_1}{a_1}+\frac{c_2}{a_2}+2\frac{c_4}{a_4}+\frac{c_5}{a_5}+\frac{c_9}{b_9}+\frac{c_{10}}{b_{10}}+\frac{c_{14}}{a_{14}}+2\frac{c_{16}}{b_{16}}+\frac{c_{17}}{a_{17}}+2\frac{c_{18}}{b_{18}} \overset{\mathcal{M}}{=} -\frac{4yz}{xyz+1};$$

$$f_3 := \frac{c_3}{a_3}+\frac{c_4}{a_4}+2\frac{c_5}{a_5}+2\frac{c_6}{a_6}+\frac{c_{11}}{b_{11}}+\frac{c_{12}}{b_{12}}+\frac{c_{15}}{a_{15}}+2\frac{c_{16}}{b_{16}}+2\frac{c_{17}}{b_{17}}+\frac{c_{18}}{a_{18}} \overset{\mathcal{M}}{=} -\frac{4yz}{xyz+1};$$

$$f_4 := \frac{c_4}{a_4}+\frac{c_5}{a_5}+\frac{c_{10}}{b_{10}}+\frac{c_{11}}{b_{11}}+\frac{c_{14}}{a_{14}}+\frac{c_{15}}{a_{15}}+2\frac{c_{16}}{b_{16}} \overset{\mathcal{M}}{=} -\frac{2yz\left(x^2+2xyz+1\right)}{x(x+yz)(xyz+1)};$$

$$f_5 := \frac{c_3}{a_3}+\frac{c_6}{a_6}+\frac{c_8}{b_8}+\frac{c_{12}}{b_{12}}+\frac{c_{13}}{a_{13}}+\frac{c_{15}}{a_{15}}+2\frac{c_{17}}{b_{17}} \overset{\mathcal{M}}{=} \frac{2\left(1-y^2\right)z}{(xz+y)(xyz+1)};$$

$$f_6 := \frac{c_1}{a_1}+\frac{c_2}{a_2}+\frac{c_7}{b_7}+\frac{c_9}{b_9}+\frac{c_{13}}{a_{13}}+\frac{c_{14}}{a_{14}}+2\frac{c_{18}}{b_{18}} \overset{\mathcal{M}}{=} \frac{2\left(1-z^2\right)y}{(xy+z)(xyz+1)};$$

$$\begin{aligned}f_7 :=& \frac{c_1}{b_1}+\frac{c_6}{b_6}+\frac{c_9}{a_9}+\frac{c_{10}}{a_{10}}+\frac{c_{11}}{a_{11}}+\frac{c_{12}}{a_{12}}-\frac{c_{13}}{b_{13}}-\frac{c_{13}}{a_{13}}+\frac{c_{14}}{b_{14}}+\frac{c_{15}}{b_{15}}-\frac{c_{16}}{b_{16}}-\frac{c_{16}}{a_{16}}-\frac{c_{17}}{b_{17}}-\frac{c_{17}}{a_{17}}-\frac{c_{18}}{b_{18}}-\frac{c_{18}}{a_{18}}\\ \overset{\mathcal{M}}{=}& \frac{2\left(x^2yz+2x+yz\right)}{(1-x^2)\,(xyz+1)};\end{aligned}$$

$$\begin{aligned}f_8 :=& \frac{c_2}{b_2}+\frac{c_5}{b_5}+\frac{c_7}{a_7}+\frac{c_8}{a_8}+\frac{c_{11}}{a_{11}}+\frac{c_{12}}{a_{12}}+\frac{c_{13}}{b_{13}}-\frac{c_{14}}{b_{14}}-\frac{c_{14}}{a_{14}}+\frac{c_{15}}{b_{15}}-\frac{c_{16}}{b_{16}}-\frac{c_{16}}{a_{16}}-\frac{c_{17}}{b_{17}}-\frac{c_{17}}{a_{17}}-\frac{c_{18}}{b_{18}}-\frac{c_{18}}{a_{18}}\\ \overset{\mathcal{M}}{=}& \frac{2yz}{xyz+1};\end{aligned}$$

$$\begin{aligned}f_9 :=& \frac{c_3}{b_3}+\frac{c_4}{b_4}+\frac{c_7}{a_7}+\frac{c_8}{a_8}+\frac{c_9}{a_9}+\frac{c_{10}}{a_{10}}+\frac{c_{13}}{b_{13}}+\frac{c_{14}}{b_{14}}-\frac{c_{15}}{b_{15}}-\frac{c_{15}}{a_{15}}-\frac{c_{16}}{b_{16}}-\frac{c_{16}}{a_{16}}-\frac{c_{17}}{b_{17}}-\frac{c_{17}}{a_{17}}-\frac{c_{18}}{b_{18}}-\frac{c_{18}}{a_{18}}\\ \overset{\mathcal{M}}{=}& \frac{2yz}{xyz+1};\end{aligned}$$

$$g_1 := -\frac{c_1}{a_1}-\frac{c_1}{b_1}-\frac{c_7}{a_7}-\frac{c_7}{b_7}+\frac{c_{18}}{a_{18}} \overset{\mathcal{M}}{=} 0;\quad g_2 := -\frac{c_2}{a_2}-\frac{c_2}{b_2}-\frac{c_9}{a_9}-\frac{c_9}{b_9}+\frac{c_{18}}{a_{18}} \overset{\mathcal{M}}{=} 0;$$

$$g_3 := -\frac{c_3}{a_3}-\frac{c_3}{b_3}-\frac{c_{12}}{a_{12}}-\frac{c_{12}}{b_{12}}+\frac{c_{17}}{a_{17}} \overset{\mathcal{M}}{=} 0;\quad g_4 := -\frac{c_4}{a_4}-\frac{c_4}{b_4}-\frac{c_{11}}{a_{11}}-\frac{c_{11}}{b_{11}}+\frac{c_{16}}{a_{16}} \overset{\mathcal{M}}{=} 0;$$

$$g_5 := -\frac{c_5}{a_5}-\frac{c_5}{b_5}-\frac{c_{10}}{a_{10}}-\frac{c_{10}}{b_{10}}+\frac{c_{16}}{a_{16}} \overset{\mathcal{M}}{=} 0;\quad g_6 := -\frac{c_6}{a_6}-\frac{c_6}{b_6}-\frac{c_8}{a_8}-\frac{c_8}{b_8}+\frac{c_{17}}{a_{17}} \overset{\mathcal{M}}{=} 0;$$

$$\begin{aligned}m :=& \frac{c_1}{b_1}+\frac{c_2}{b_2}+\frac{c_3}{b_3}+\frac{c_4}{b_4}+\frac{c_5}{b_5}+\frac{c_6}{b_6}-\frac{c_{13}}{a_{13}}-\frac{c_{13}}{b_{13}}-\frac{c_{14}}{b_{14}}-\frac{c_{14}}{a_{14}}-\frac{c_{15}}{a_{15}}-\frac{c_{15}}{b_{15}}-\frac{c_{16}}{a_{16}}-\frac{c_{16}}{b_{16}}-\frac{c_{17}}{a_{17}}\\&-\frac{c_{17}}{b_{17}}-\frac{c_{18}}{a_{18}}-\frac{c_{18}}{b_{18}}\\ \overset{\mathcal{M}}{=}& -\frac{4x^4(y^2+z^2+y^2z^2)+2x^3yz(4+x^2+3y^2+3z^2)+2y^2z^2(4x^2-2)-2xyz(1+y^2+z^2)}{x(1-x^2)(x+yz)(y+xz)(z+xy)}\end{aligned}$$

□

Next, we differentiate the function η. From Section 2.1, we have:

$$\eta(x,y) = 8\mathcal{L}\left(\frac{(x+1)^2y^2}{(x+y^2)(xy^2+1)}\right) - 8\mathcal{L}\left(\frac{(1-x)^2y^2}{(x+y^2)(xy^2+1)}\right) - 16\mathcal{L}\left(\frac{1-x}{1+y^2}\right) + 16\mathcal{L}\left(\frac{xy^2+1}{1+y^2}\right)$$
$$- 16\mathcal{L}\left(\frac{x+y^2}{xy^2+1}\right) - 16\mathcal{L}\left(\frac{1-y^2}{1+x}\right) + 16\mathcal{L}\left(\frac{xy^2+1}{1+x}\right).$$

Lemma 15. *For $x, y \in (0,1)$,*

$$\partial\eta_x(x,y) = -\frac{8\left(y^2+1\right)\log(x)}{(1-x)\left(xy^2+1\right)} - \frac{16y^2\log(y)}{xy^2+1} - \frac{8y^2\left(x^2+2xy^2+1\right)\log(1-x)}{x\left(x+y^2\right)\left(xy^2+1\right)}$$
$$+ \frac{8\left(1-y^4\right)\log(1-y^2)}{\left(x+y^2\right)\left(xy^2+1\right)} + \frac{8\left(y^2+1\right)\log(x+y^2)}{(1-x)\left(xy^2+1\right)} - \frac{8\left(x^2+2xy^2-y^2\right)\log(1+xy^2)}{(1-x)x\left(x+y^2\right)},$$
$$\partial\eta_y(x,y) = -\frac{16xy\log(x)}{xy^2+1} - \frac{32(x+1)y\log(y)}{\left(1-y^2\right)\left(xy^2+1\right)} + \frac{16(1-x^2)y\log(1-x)}{\left(x+y^2\right)\left(xy^2+1\right)}$$
$$- \frac{16x\left(2xy^2+y^4+1\right)\log(1-y^2)}{y\left(x+y^2\right)\left(xy^2+1\right)} + \frac{16(x+1)y\log(x+y^2)}{\left(1-y^2\right)\left(xy^2+1\right)} + \frac{16(x-2xy^2-y^4)\log(1+xy^2)}{y\left(1-y^2\right)\left(x+y^2\right)}.$$

Proof. We have

$$\partial\eta_x(x,y) = \sum_{i=1}^{7}\left(\frac{p_i}{n_i}\log(m_i) + \frac{p_i}{m_i}\log(n_i)\right); \quad \partial\eta_y(x,y) = \sum_{i=1}^{7}\left(\frac{q_i}{n_i}\log(m_i) + \frac{q_i}{m_i}\log(n_i)\right)$$

where

$$m_1 := \frac{(x+1)^2y^2}{(x+y^2)(xy^2+1)}; \quad n_1 := 1-m_1 \overset{\mathcal{M}}{=} \frac{x\left(1-y^2\right)^2}{\left(x+y^2\right)\left(xy^2+1\right)};$$
$$p_1 := -4\partial_x m_1 \overset{\mathcal{M}}{=} \frac{4\left(1-x^2\right)y^2\left(1-y^2\right)^2}{\left(x+y^2\right)^2\left(xy^2+1\right)^2}; \quad q_1 := -4\partial_y m_1 \overset{\mathcal{M}}{=} \frac{8x(x+1)^2y\left(y^4-1\right)}{\left(x+y^2\right)^2\left(xy^2+1\right)^2};$$
$$m_2 := \frac{(1-x)^2y^2}{(x+y^2)(xy^2+1)}; \quad n_2 := 1-m_2 \overset{\mathcal{M}}{=} \frac{x\left(y^2+1\right)^2}{\left(x+y^2\right)\left(xy^2+1\right)};$$
$$p_2 := 4\partial_x m_2 \overset{\mathcal{M}}{=} -\frac{4\left(1-x^2\right)y^2\left(y^2+1\right)^2}{\left(x+y^2\right)^2\left(xy^2+1\right)^2}; \quad q_2 := 4\partial_y m_2 \overset{\mathcal{M}}{=} -\frac{8(x-1)^2xy\left(y^4-1\right)}{\left(x+y^2\right)^2\left(xy^2+1\right)^2};$$
$$m_3 := \frac{1-x}{1+y^2}; \quad n_3 := 1-m_3 \overset{\mathcal{M}}{=} \frac{x+y^2}{y^2+1}; \quad p_3 := 8\partial_x m_3 \overset{\mathcal{M}}{=} -\frac{8}{y^2+1}; \quad q_3 := 8\partial_y m_3 \overset{\mathcal{M}}{=} \frac{16(x-1)y}{\left(y^2+1\right)^2};$$
$$m_4 := \frac{xy^2+1}{1+y^2}; \quad n_4 := 1-m_4 \overset{\mathcal{M}}{=} \frac{(1-x)y^2}{y^2+1}; \quad p_4 := -8\partial_x m_4 \overset{\mathcal{M}}{=} -\frac{8y^2}{y^2+1};$$
$$q_4 := -8\partial_y m_4 \overset{\mathcal{M}}{=} -\frac{16(x-1)y}{\left(y^2+1\right)^2};$$
$$m_5 := \frac{x+y^2}{xy^2+1}; \quad n_5 := 1-m_5 \overset{\mathcal{M}}{=} \frac{(1-x)\left(1-y^2\right)}{xy^2+1}; \quad p_5 := 8\partial_x m_5 \overset{\mathcal{M}}{=} \frac{8\left(1-y^4\right)}{\left(xy^2+1\right)^2};$$

$$q_5 := 8\partial_y m_5 \overset{\mathcal{M}}{=} -\frac{16\left(x^2-1\right)y}{\left(xy^2+1\right)^2};$$

$$m_6 := \frac{1-y^2}{1+x};\ \ n_6 := 1-m_6 \overset{\mathcal{M}}{=} \frac{x+y^2}{x+1};\ \ p_6 := 8\partial_x m_6 \overset{\mathcal{M}}{=} -\frac{8\left(1-y^2\right)}{(x+1)^2};\ \ q_6 := 8\partial_y m_6 \overset{\mathcal{M}}{=} -\frac{16y}{x+1};$$

$$m_7 := \frac{xy^2+1}{1+x};\ \ n_7 := 1-m_7 \overset{\mathcal{M}}{=} \frac{x\left(1-y^2\right)}{x+1};\ \ p_7 := -8\partial_x m_7 \overset{\mathcal{M}}{=} \frac{8\left(1-y^2\right)}{(x+1)^2};$$

$$q_7 := -8\partial_y m_7 \overset{\mathcal{M}}{=} -\frac{16xy}{x+1}.$$

By standard manipulations of the logarithm function, we rewrite $\partial\eta_x(x,y)$ and $\partial\eta_x(x,y)$ in the following form:

$$\begin{aligned}\partial\eta_x(x,y) =& d_1\log(x)+d_2\log(y)+d_3\log(1-x)+d_4\log(1-y^2)+d_5\log(1+x)+d_6\log(1+y^2)\\&+d_7\log(x+y^2)+d_8\log(1+xy^2),\\ \partial\eta_y(x,y) =& e_1\log(x)+e_2\log(y)+e_3\log(1-x)+e_4\log(1-y^2)+e_5\log(1+x)+e_6\log(1+y^2)\\&+e_7\log(x+y^2)+e_8\log(1+xy^2),\end{aligned}$$

where

$$d_1 := \frac{p_1}{m_1}+\frac{p_2}{m_2}+\frac{p_7}{m_7} \overset{\mathcal{M}}{=} -\frac{8\left(y^2+1\right)}{(1-x)\left(xy^2+1\right)};\ \ d_2 := 2\frac{p_1}{n_1}+2\frac{p_2}{n_2}+2\frac{p_4}{m_4} \overset{\mathcal{M}}{=} -\frac{16y^2}{xy^2+1};$$

$$d_3 := 2\frac{p_2}{n_2}+\frac{p_3}{n_3}+\frac{p_4}{m_4}+\frac{p_5}{m_5} \overset{\mathcal{M}}{=} -\frac{8y^2\left(x^2+2xy^2+1\right)}{x\left(x+y^2\right)\left(xy^2+1\right)};$$

$$d_4 := 2\frac{p_1}{m_1}+\frac{p_5}{m_5}+\frac{p_6}{n_6}+\frac{p_7}{m_7} \overset{\mathcal{M}}{=} \frac{8\left(1-y^4\right)}{\left(x+y^2\right)\left(xy^2+1\right)};$$

$$d_5 := 2\frac{p_1}{n_1}-\frac{p_6}{m_6}-\frac{p_6}{n_6}-\frac{p_7}{n_7}-\frac{p_7}{m_7} \overset{\mathcal{M}}{=} 0;\ \ d_6 := 2\frac{p_2}{m_2}-\frac{p_3}{m_3}-\frac{p_3}{n_3}-\frac{p_4}{m_4}-\frac{p_4}{n_4} \overset{\mathcal{M}}{=} 0;$$

$$d_7 := -\frac{p_1}{n_1}-\frac{p_1}{m_1}-\frac{p_2}{m_2}-\frac{p_2}{n_2}+\frac{p_3}{m_3}+\frac{p_5}{n_5}+\frac{p_6}{m_6} \overset{\mathcal{M}}{=} \frac{8\left(y^2+1\right)}{(1-x)\left(xy^2+1\right)};$$

$$d_8 := -\frac{p_1}{m_1}-\frac{p_1}{n_1}-\frac{p_2}{m_2}-\frac{p_2}{n_2}+\frac{p_4}{n_4}-\frac{p_5}{n_5}-\frac{p_5}{m_5}+\frac{p_7}{n_7} \overset{\mathcal{M}}{=} -\frac{8\left(x^2+2xy^2-y^2\right)}{(1-x)x\left(x+y^2\right)};$$

$$e_1 := \frac{q_1}{m_1}+\frac{q_2}{m_2}+\frac{q_7}{m_7} \overset{\mathcal{M}}{=} -\frac{16xy}{xy^2+1};\ \ e_2 := 2\frac{q_1}{n_1}+2\frac{q_2}{n_2}+2\frac{q_4}{m_4} \overset{\mathcal{M}}{=} -\frac{32(x+1)y}{\left(1-y^2\right)\left(xy^2+1\right)};$$

$$e_3 := 2\frac{q_2}{n_2}+\frac{q_3}{n_3}+\frac{q_4}{m_4}+\frac{q_5}{m_5} \overset{\mathcal{M}}{=} \frac{16(1-x^2)y}{\left(x+y^2\right)\left(xy^2+1\right)};$$

$$e_4 := 2\frac{q_1}{m_1}+\frac{q_5}{m_5}+\frac{q_6}{n_6}+\frac{q_7}{m_7} \overset{\mathcal{M}}{=} -\frac{16x\left(2xy^2+y^4+1\right)}{y\left(x+y^2\right)\left(xy^2+1\right)};$$

$$e_5 := 2\frac{q_1}{n_1}-\frac{q_6}{m_6}-\frac{q_6}{n_6}-\frac{q_7}{n_7}-\frac{q_7}{m_7} \overset{\mathcal{M}}{=} 0;\ \ e_6 := 2\frac{q_2}{m_2}-\frac{q_3}{m_3}-\frac{q_3}{n_3}-\frac{q_4}{m_4}-\frac{q_4}{n_4} \overset{\mathcal{M}}{=} 0;$$

$$e_7 := -\frac{q_1}{n_1}-\frac{q_1}{m_1}-\frac{q_2}{m_2}-\frac{q_2}{n_2}+\frac{q_3}{m_3}+\frac{q_5}{n_5}+\frac{q_6}{m_6} \overset{\mathcal{M}}{=} \frac{16(x+1)y}{\left(1-y^2\right)\left(xy^2+1\right)};$$

$$e_8 := -\frac{q_1}{m_1}-\frac{q_1}{n_1}-\frac{q_2}{m_2}-\frac{q_2}{n_2}+\frac{q_4}{n_4}-\frac{q_5}{n_5}-\frac{q_5}{m_5}+\frac{q_7}{n_7} \overset{\mathcal{M}}{=} \frac{16(x-2xy^2-y^4)}{y\left(1-y^2\right)\left(x+y^2\right)}.$$

□

Chapter 3

Orthotree, orthoshapes and ortho-integral surfaces

Structure: Section 3.1 and 3.2 contain necessary notations used throughout this chapter. Section 3.3 introduces several identity relations of geodesics and horocycles on hyperbolic plane. Section 3.4 describes a tree structure on a subset of oriented orthogeodesics, related identity relations and the construction of some types of r-orthoshapes. Section 3.5 will be about some applications: A combinatorial proof of Basmajian's identity, ortho-integral surfaces, infinitely (dilogarithm) identities. The last section will be about some remarks and further questions.

3.1. Preliminaries

Let Σ be a surface of negative Euler characteristic obtained from removing points from a closed oriented surface. Denote by $\partial\Sigma$ the set of these removing points. We also denote the Teichmüller space $\mathcal{T}(\Sigma)$ to be the set of marked hyperbolic structures on Σ up to isotopy such that each point in $\mathcal{T}(\Sigma)$ is represented by a hyperbolic surface with its boundary consisting of cusps and/or simple closed geodesics.

Let $S \in \mathcal{T}(\Sigma)$, we define a **truncated surface** on S, say $\hat{S}$, to be a hyperbolic surface with its boundary $\partial\hat{S}$ consisting of simple closed geodesics and/or simple closed horocycles obtained from cutting off all the cusp regions of S. Denote by $\hat{S}(2)$ the **natural truncated surface** of S where all removing cusp regions are of the same area 2. Note that $\hat{S} = \hat{S}(2) = S$ if there are no cusps on S. We define $\hat{\mathcal{T}}(\Sigma)$ to be the set of all pairs $(S, \hat{S})$ with $S \in \mathcal{T}(\Sigma)$. Similarly, a **natural concave core** (or a concave core of grade 1 as defined in [6]), denoted by $S^\star$, is the surface obtained by cutting off all natural collars of cusps and simple closed geodesics on ∂S of S. Note that a **natural collar** of a boundary component $\beta \in \partial S$ is

- a cusp region of area 2 surrounding β if β is a cusp,

- a set of points at distance less than $\operatorname{arcsinh}(1/\sinh(\ell(\beta)/2))$ from β if β is a simple closed geodesic.

Let η be a homotopy class of arcs relative to $\partial\Sigma$. The geometric realization of η with respect to an $S \in \mathcal{T}(\Sigma)$ is an **orthogeodesic** which is a geodesic arc perpendicular to ∂S at both ends, with a convention that any geodesic with an endpoint at a cusp is said to be perpendicular to that cusp. We also note that an orthogeodesic is of infinite length if one of its endpoint is at a cusp of S.

Let $\bar{S}$ be either $\hat{S}$ or $S^\star$ defined as above. Each orthogeodesic η on S will be truncated by $\partial\bar{S}$ and associated to a truncated orthogeodesic which is a subarc of η perpendicular to $\partial\bar{S}$ at both endpoints. Denote by $\ell_{\hat{S}}(\eta)$ and $\ell_{S^\star}(\eta)$ the **truncated lengths** of η with respect to $\hat{S}$ and $S^\star$ on S. We also denote

$$\ell_{\bar{S}}(\eta) := \begin{cases} \ell_{\hat{S}}(\eta) & \text{if} \quad \bar{S} = \hat{S} \\ \ell_{S^\star}(\eta) & \text{if} \quad \bar{S} = S^\star \end{cases}$$

Two homotopy classes of arcs relative to $\partial\Sigma$ are length equivalent in $\mathcal{T}(\Sigma)$ if their orthogeodesics are of the same truncated length with respect to any pair $(S, \hat{S}) \in \hat{\mathcal{T}}(\Sigma)$. One can show that if two orthogeodesics are length equivalent, then their endpoints are on the same pair of elements in $\partial\Sigma$. Thus two orthogeodesics, say η_1 and η_2, are **length equivalent** if and only if $\ell_{S^\star}(\eta_1) = \ell_{S^\star}(\eta_2)$ for all $S \in \mathcal{T}(\Sigma)$.

There is an equivalent way to define length equivalent orthogeodesics based on length equivalent closed geodesics. Indeed, each orthogeodesic η is roughly a seam between two boundary components, say β_1 and β_2, of a collection of immersed pairs of pants on S. Among these immersed pair of pants, there is a unique maximal one, say P^*, which contains all of the rest. One can associate η to the closed geodesic, say β_3, at the remaining boundary component of P^* (see the precise definition in [6] or [8]). Then, two orthogeodesics are **length equivalent** if their associated closed geodesics are of the same length for all $S \in \mathcal{T}(\Sigma)$. Here is the relation between $\ell_{S^\star}(\eta)$ and β_1, β_2 and β_3:

$$\ell_{S^\star}(\eta) = \log\left(\frac{a_1 a_2 + a_3 + \sqrt{a_1^2 + a_2^2 + a_3^2 + 2a_1 a_2 a_3 - 1}}{(a_1+1)(a_2+1)}\right),$$

in which a_i denotes the half-trace of β_i, i.e. $\cosh(\ell(\beta_i)/2)$, for any $i \in \{1, 2, 3\}$. We also observe that $e^{\ell_{S^\star}(\eta)}$ is a root of the following equation:

$$(a_1+1)(a_2+1)x^2 - 2(a_1 a_2 + a_3)x + (a_1 - 1)(a_2 - 1) = 0.$$

Now we introduce the notions of orthotriangle, orthobasis, orthotriangulation.

Definition 1. *An* ***orthotriangle*** *is a geometric realization of an immersed disc with three punctures on its boundary. With respect to an $S \in \mathcal{T}(\Sigma)$, an orthotriangle is a polygon with* *its boundary consisting of three orthogeodesics and at most three geodesic subsegments of ∂S (i.e. it is an n-gons with $(6-n)$ ideal vertices, where $3 \leq n \leq 6$).*

Note that by this definition, we realize that orthotriangles are related to the so-called generalized triangles in Buser's book [14].

Definition 2. *An **orthobasis** is a geometric realization of a collection of pairwise disjoint simple arcs which decomposes Σ into a collection of interior disjoint discs. Each of these discs is geometrically realized as an embedded orthotriangle. The set of all orthotriangles coming from an orthobasis is called an **orthotriangulation**.*

Definition 3. *An orthotriangulation Δ is **standard** if for any orthotriangle in Δ, the three orthogeodesics at its boundary are pairwise distinct.*

Similarly to the notion of orthotriangle, we introduce the notions of orthoquadrilateral, ortho-isosceles-trapezoid, orthorectangle, orthokite, orthoparallelogram on Σ.

Definition 4. *An **orthoquadrilateral** is the geometric realization of an immersed disc with four punctures on its boundary. With respect to an $S \in \mathfrak{T}(\Sigma)$, an orthoquadrilateral $\mathbb{O} := XYZT$ is a polygon with its boundary consisting of four orthogeodesics, namely (XY, YZ, ZT, TX) in a cyclic order, and at most four geodesic segments of ∂S (i.e. it is an n-gons with $(8-n)$ ideal vertices, where $4 \leq n \leq 8$). Let XZ and YT be diagonal orthogeodesics of $\mathbb{O}$. Let $\bar{S}$ be either the natural concave core $S^\star$ or a truncated surface $\hat{S}$ on S. Then with respect to $\bar{S}$,*

- *$\mathbb{O}$ is an ortho-isosceles-trapezoid if $\ell_{\bar{S}}(XT) = \ell_{\bar{S}}(YZ)$ and $\ell_{\bar{S}}(XZ) = \ell_{\bar{S}}(YT)$,*
- *$\mathbb{O}$ is an orthorectangle if $\ell_{\bar{S}}(XT) = \ell_{\bar{S}}(YZ)$, $\ell_{\bar{S}}(XY) = \ell_{\bar{S}}(ZT)$ and $\ell_{\bar{S}}(XZ) = \ell_{\bar{S}}(YT)$,*
- *$\mathbb{O}$ is an orthokite if $\ell_{\bar{S}}(XY) = \ell_{\bar{S}}(XT)$ and $\ell_{\bar{S}}(ZY) = \ell_{\bar{S}}(ZT)$,*
- *$\mathbb{O}$ is an orthoparallelogram if $\ell_{\bar{S}}(XY) = \ell_{\bar{S}}(ZT)$ and $\ell_{\bar{S}}(XT) = \ell_{\bar{S}}(YZ)$,*
- *$\mathbb{O}$ is a orthorhombus if $\ell_{\bar{S}}(XT) = \ell_{\bar{S}}(YZ) = \ell_{\bar{S}}(XY) = \ell_{\bar{S}}(ZT)$,*
- *$\mathbb{O}$ is an orthosquare if $\ell_{\bar{S}}(XT) = \ell_{\bar{S}}(YZ) = \ell_{\bar{S}}(XY) = \ell_{\bar{S}}(ZT)$ and $\ell_{\bar{S}}(XZ) = \ell_{\bar{S}}(YT)$.*

An **r-orthoshape** is a set of orthogeodesics satisfying some equality conditions on their truncated lengths which hold for any pair $(S, \hat{S}) \in \hat{\mathfrak{T}}(\Sigma)$. By this definition, a pair of length equivalent orthogeodesics is an example of an r-orthoshape. We are interested in some types of r-orthoshapes which closely related to length equivalent orthogeodesics.

Definition 5. *$\mathbb{O}$ is an **r-ortho**-isosceles-trapezoid/rectangle/kite/parallelogram/rhombus/square if it is an ortho-isosceles-trapezoid/rectangle/kite/parallelogram/rhombus/square with respect to any pair $(S, \hat{S}) \in \hat{\mathfrak{T}}(\Sigma)$.*

Due to the definition of length equivalent orthogeodesics, Definition 5 is equivalent to the following one:

Definition 6. *$\mathbb{O}$ is an **r-ortho**-isosceles-trapezoid/rectangle/kite/parallelogram/rhombus/square if it is an ortho-isosceles-trapezoid/rectangle/kite/parallelogram/rhombus/square with respect to $S^{\star}$ for any $S \in \mathcal{T}(\Sigma)$.*

3.2. Notation

Since the following notations will be used throughout this chapter, we put them into a seperated section so that readers can revisit whenever they get confused. Let X be the shortest geodesic arc from A to B. To avoid long expressions in several formulae in this chapter, if A is a geodesic and B is a geodesic/point in the hyperbolic plane, we use

- X (or AB) for $\cosh(\ell_{\mathbb{H}}(X))$,
- $\overline{X}$ (or $\overline{AB}$) for $\cosh(\ell_{\mathbb{H}}(X)/2)$ (The half "trace" $\frac{1}{2}tr(X)$ of X),
- $\widetilde{X}$ (or $\widetilde{AB}$) for $\sinh(\ell_{\mathbb{H}}(X))$.

If A is a horocycle/geodesic/point and B is a horocycle in the hyperbolic plane, we use

- X (or AB) for $\frac{1}{2}e^{d_{\mathbb{H}}(A,B)}$,
- $\lambda(X)$ (or $\lambda(A,B)$) for $e^{\frac{1}{2}d_{\mathbb{H}}(A,B)}$ (Penner's lambda length),
- $\overline{X}$ (or $\overline{AB}$) for $\frac{1}{2}e^{\frac{1}{2}d_{\mathbb{H}}(A,B)}$ (The half Penner's lambda length).

If A and B are two points in the hyperbolic plane, we use

- $\overline{X}$ (or $\overline{AB}$) for $\sinh(\ell_{\mathbb{H}}(X)/2)$.

3.3. The basics

In this section, we will present several relations between the distances from a geodesic or a horocycle to a finite set of geodesics and/or horocycles in $\mathbb{H}$. It turns out some of the resulting relations are similar to the relations of a point to a finite set of points in the Euclidean plane. For example: there are Ptolemy relations and in particular, the orthorectangle and ortho-isosceles trapezoid relations are exactly identical to the relation of the distances from a point to four vertices of a rectangle and an isosceles trapezoid in the Euclidean plane. These relations can also be translated to relations in term of cross ratios. We refer the reader to [14] (chapter 2) and [7] (chapter 7) for hyperbolic trigonometry formulae used in this section.

3.3.1 Penner's Ptolemy relation

Let A, B, C, P be four disjoint horocycles in $\mathbb{H}$. Each of them divides $\mathbb{H}$ into two domains such that the other three horocycles lie in the same domain.

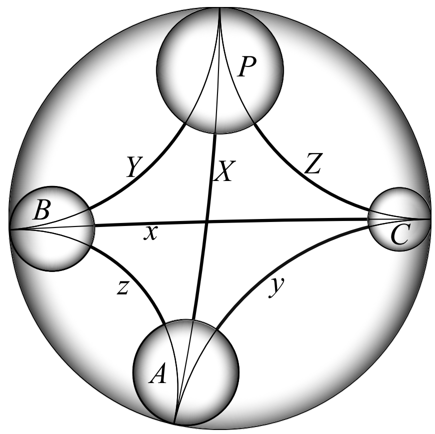

Figure 3.1: Penner's Ptolemy relation.

If P, B, A, C are in a cyclic order as in Figure 3.1, then the Penner's Ptolemy relation (see [35] and [36]) is

$$\overline{X} = \frac{\overline{y}\overline{Y} + \overline{z}\overline{Z}}{\overline{x}}. \tag{3.1}$$

This equation is equivalent to a harmonic relation:

$$\frac{\overline{x}}{\overline{Y}.\overline{Z}} = \frac{\overline{y}}{\overline{X}.\overline{Z}} + \frac{\overline{z}}{\overline{X}.\overline{Y}},$$

in which $\frac{\overline{x}}{\overline{Y}.\overline{Z}}; \frac{\overline{y}}{\overline{X}.\overline{Z}}; \frac{\overline{z}}{\overline{X}.\overline{Y}}$ are respectively the lengths of segments on P between PB and PC; PA and PC; PA and PB. Now we consider the case where all horocycles are replaced by geodesics.

3.3.2 Ptolemy relation of geodesics

Let A, B, C, P be four disjoint geodesics in $\mathbb{H}$. Each of them divides $\mathbb{H}$ into two domains such that the other three geodesics lie in the same domain. Denote $X := \cosh d_{\mathbb{H}}(P, A)$, $Y := \cosh d_{\mathbb{H}}(P, B)$, $Z := \cosh d_{\mathbb{H}}(P, C)$, $x := \cosh d_{\mathbb{H}}(B, C)$, $y := \cosh d_{\mathbb{H}}(A, C)$, $z := \cosh d_{\mathbb{H}}(A, B)$.

Lemma 16. *If B, A, C, P are in a cyclic order as in Figure 3.2 then we have a harmonic* *relation as follows:*

$$\operatorname{arccosh}\frac{x+YZ}{\sqrt{(Y^2-1)(Z^2-1)}}=\operatorname{arccosh}\frac{z+XY}{\sqrt{(X^2-1)(Y^2-1)}}+\operatorname{arccosh}\frac{y+XZ}{\sqrt{(X^2-1)(Z^2-1)}}.$$

Proof. Let m,n be two segments respectively between X,Y and X,Z on P. The harmonic relation follows from computing the length of m,n and $m+n$ by using hyperbolic trigonometric formula for right-angled hexagons. □

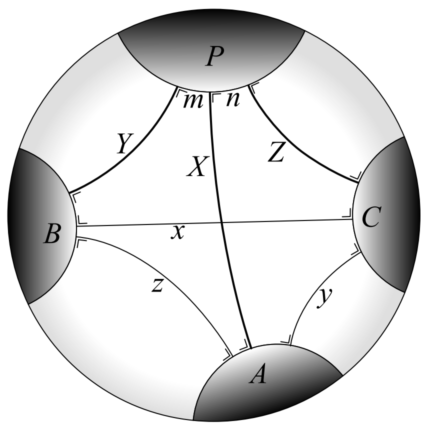

Figure 3.2: Ptolemy relation of geodesics.

Lemma 17. *(Ptolemy relation of geodesics) If B,A,C,P are in a cyclic order as in Figure 3.2, then*

$$X=\frac{(xy+z)Y+(xz+y)Z+\mathcal{P}(x,y,z)\mathcal{P}(x,Y,Z)}{x^2-1},$$

where $\mathcal{P}(a,b,c):=\sqrt{a^2+b^2+c^2+2abc-1}$ for any a,b,c.

Proof. Since $\cosh(m+n)=\cosh(m)\cosh(n)+\sinh(m)\sinh(n)$,

$$\frac{x+YZ}{\sqrt{(Y^2-1)(Z^2-1)}}=\frac{z+XY}{\sqrt{(X^2-1)(Y^2-1)}}\cdot\frac{y+XZ}{\sqrt{(X^2-1)(Z^2-1)}}$$

$$+\sqrt{\left(\left(\frac{z+XY}{\sqrt{(X^2-1)(Y^2-1)}}\right)^2-1\right)\left(\left(\frac{y+XZ}{\sqrt{(X^2-1)(Z^2-1)}}\right)^2-1\right)}.$$

Equivalently,

$$(x+YZ)(X^2-1)-(z+XY)(y+XZ)=\mathcal{P}(X,Y,z)\mathcal{P}(X,y,Z). \tag{3.2}$$

Under the condition $X, Y, Z, x, y, z > 1$, Equation 3.2 can be solved to obtain the formula of X in term of x, y, z, Y, Z as desired. □

Corollary 5. *(Quadruplet of geodesics) The following equation holds for any order of P, A, B, C:*

$$(x^2-1)X^2 + (y^2-1)Y^2 + (z^2-1)Z^2 - 2(xy+z)XY - 2(yz+x)YZ - 2(xz+y)XZ$$
$$= x^2 + y^2 + z^2 + 2xyz - 1.$$

Proof. If B, A, C, P are in a cyclic order as in Figure 3.2, one has Equation 3.2. Taking the square of both sides of Equation 3.2 and simplifying the result, one obtains the equation as desired. Since this equation is symmetric in term of X, Y, Z and x, y, z, it holds for any order of A, B, C, P. □

Remark. This equation has two roots in the variable X. If B, A, C, P are in a cyclic order,

$$X = \frac{(xy+z)Y + (xz+y)Z + \mathcal{P}(x,y,z)\mathcal{P}(x,Y,Z)}{x^2-1}.$$

Otherwise,

$$X = \frac{(xy+z)Y + (xz+y)Z - \mathcal{P}(x,y,z)\mathcal{P}(x,Y,Z)}{x^2-1}.$$

3.3.3 Mixed Ptolemy relations

Let P, A, B be three disjoint geodesics and horocycles in $\mathbb{H}$. Each of them divides $\mathbb{H}$ into two domains such that the other two lie in the same domain. Let m be the length of the segment between PA, PB on P. With the notations in Section 3.2, using standard calculations in the upper half-plane model, one can compute m in terms of PA, PB and AB.

Lemma 18.
- *If P, A, B are horocycles, then from* [35]*:* $m = \sqrt{\dfrac{AB}{2.PA.PB}}$.
- *If P, A are horocycles, and B is a geodesic, then:* $m = \sqrt{\dfrac{1}{4.PB^2} + \dfrac{AB}{2.PA.PB}}$.
- *If P is a horocycle, and A, B are geodesics, then:* $m = \sqrt{\dfrac{1}{4.PA^2} + \dfrac{1}{4.PB^2} + \dfrac{AB}{2.PA.PB}}$.
- *If P is a geodesic, and A, B are horocycles, then:* $m = \operatorname{arccosh}\left(1 + \dfrac{AB}{PA.PB}\right)$.
- *If P, A are geodesics, and B is a horocycle, then:* $m = \operatorname{arccosh}\left(\dfrac{PA.PB + AB}{PB\sqrt{PA^2-1}}\right)$.

Remark. Using these formulae for m and the argument in Lemma 16, one can establish different forms of harmonic relation depending on whether P, A, B, C are horocycles and/or geodesics. In total, there are 10 different forms of the harmonic relation including also the two cases where P, A, B, C are all geodesics/horocycles discussed before.

Let A, B, C, P be four disjoint geodesics and horocycles in $\mathbb{H}$. Each of them divides $\mathbb{H}$ into two domains such that the other three lie in the same domain. Assume that P, B, A, C are in a cyclic order. By combining Lemma 18 and the argument in Lemma 17, one can express $X := PA$ in terms of $Y := PB, Z := PC, x := AB, y := AC, z := BC$ in several different cases

Lemma 19. *(Mixed Ptolemy and quadruplet relations)*

- *If P, A, C are geodesics, and B is a horocycle, then*

$$X = \frac{xyY + xzZ + zY + \sqrt{(x^2 + z^2 + 2xyz)(x^2 + Y^2 + 2xYZ)}}{x^2}.$$

$$x^2X^2 + (y^2 - 1)Y^2 + z^2Z^2 - 2XY(xy + z) - 2YZ(yz + x) - 2xzXZ = x^2 + z^2 + 2xyz.$$

- *If P, B, C are geodesics, and A is a horocycle, then*

$$X = \frac{xyY + xzZ + zY + yZ + \sqrt{(y^2 + z^2 + 2xyz)(x^2 + Y^2 + Z^2 + 2xYZ - 1)}}{x^2 - 1}.$$

$$(x^2 - 1)X^2 + y^2Y^2 + z^2Z^2 - 2XY(xy + z) - 2yzYZ - 2(xz + y)XZ = y^2 + z^2 + 2xyz.$$

- *If P, B are geodesics, and A, C are horocycles, then*

$$X = \frac{xyY + xzZ + yZ + \sqrt{(y^2 + 2xyz)(x^2 + Z^2 + 2xYZ)}}{x^2}.$$

$$x^2X^2 + y^2Y^2 + z^2Z^2 - 2xyXY - 2yzYZ - 2(xz + y)XZ = y^2 + 2xyz.$$

- *If P, A are geodesics, and B, C are horocycles, then*

$$X = \frac{yY + zZ + \sqrt{(x + 2yz)(x + 2YZ)}}{x}.$$

$$x^2X^2 + y^2Y^2 + z^2Z^2 - 2xyXY - 2(yz + x)YZ - 2xzXZ = x^2 + 2xyz.$$

- *If P is a geodesic, and A, B, C are horocycles, then*

$$X = \frac{yY + zZ + \sqrt{2yz(x + 2YZ)}}{x}.$$

$$x^2X^2 + y^2Y^2 + z^2Z^2 - 2xyXY - 2yzYZ - 2xzXZ = 2xyz.$$

- *If P is a horocycle, and A, B, C are geodesics, then*

$$X = \frac{xyY + xzZ + zY + yZ + \sqrt{(x^2 + y^2 + z^2 + 2xyz - 1)(Y^2 + Z^2 + 2xYZ)}}{x^2 - 1}.$$

$$(x^2-1)X^2+(y^2-1)Y^2+(z^2-1)Z^2-2(xy+z)XY-2(yz+x)YZ-2(xz+y)XZ = 0.$$

- *If P, A are horocycles, and B, C are geodesics, then*

$$X = \frac{xyY + xzZ + zY + yZ + \sqrt{(y^2 + z^2 + 2xyz)(Y^2 + Z^2 + 2xYZ)}}{x^2 - 1}.$$

$$(x^2 - 1)X^2 + y^2Y^2 + z^2Z^2 - 2(xy + z)XY - 2yzYZ - 2(xz + y)XZ = 0.$$

- *If P, A, B are horocycles, and C is a geodesic, then*

$$X = \frac{xyY + xzZ + zY + \sqrt{(z^2 + 2xyz)(Y^2 + 2xYZ)}}{x^2}.$$

$$(x^2 - 1)X^2 + y^2Y^2 + z^2Z^2 - 2(xy + z)XY - 2yzYZ - 2xzXZ = 0.$$

- *If P, A, B, C are horocycles, then from Equation 3.1*

$$X = \frac{(\sqrt{yY} + \sqrt{zZ})^2}{x}.$$

$$x^2X^2 + y^2Y^2 + z^2Z^2 - 2xyXY - 2yzYZ - 2xzXZ = 0.$$

Remark. Since the curvatures of geodesic and horocycle are 0 and 1 respectively, one can try to get a unique formula applied in all cases of curves of constant curvatures in $[-1, 1]$. One may also try to generalize the notion of sign distance between a horocycle and a geodesic mentioned in [38]. In section 3.3.6 we will present a unique formula in form of Cayley-Menger determinant.

3.3.4 A quintet of geodesics

Let A, P, B, C, D be five disjoint geodesics in $\mathbb{H}$ with a cyclic order. Each of them divides $\mathbb{H}$ into two domains such that the other four geodesics lie in the same domain. Let O be the intersection between AC and BD. With the notations in Section 3.2, we have the following relations:

Lemma 20. *(Orthoquadrilateral relation)*

$$\frac{PA.OC + PC.OA}{PB.OD + PD.OB} = \frac{\widetilde{AC}}{\widetilde{BD}}.$$

Proof. Let α and β be the angles between OA, OP and OB, OP as in Figure 3.3.

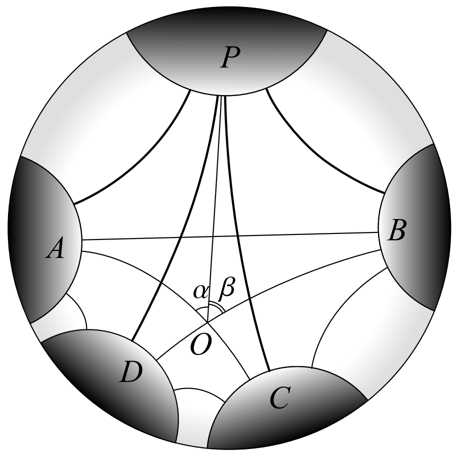

Figure 3.3: A quintet of geodesics.

Using the trigonometry formulae for hyperbolic pentagons with four right-angles ([14], [7]), we have:

$$PA = -OP.OA.\cos(\alpha) + \widetilde{OP}.\widetilde{OA}; \qquad PB = -OP.OB.\cos(\beta) + \widetilde{OP}.\widetilde{OB};$$

$$PC = -OP.OC.\cos(\pi - \alpha) + \widetilde{OP}.\widetilde{OC}; \qquad PD = -OP.OD.\cos(\pi - \beta) + \widetilde{OP}.\widetilde{OD}.$$

These equations imply that:

$$PA.OC + PC.OA = \widetilde{OP}(\widetilde{OA}.OC + \widetilde{OC}.OA),$$

$$PB.OD + PD.OB = \widetilde{OP}(\widetilde{OD}.OB + \widetilde{OB}.OD).$$

And so

$$\frac{PA.OC + PC.OA}{PB.OD + PD.OB} = \frac{\widetilde{OA}.OC + \widetilde{OC}.OA}{\widetilde{OD}.OB + \widetilde{OB}.OD} = \frac{\widetilde{AC}}{\widetilde{BD}}.$$

□

The following are special cases of the orthoquadrilateral relation.

Corollary 6. *(Ortho-isosceles trapezoid, orthorectangle, orthoparallelogram and orthokite relation of geodesics)*

1. *If $AD = BC$ and $AC = BD$ then:* $\dfrac{\overline{PA}^2 - \overline{PB}^2}{\overline{AB}} = \dfrac{\overline{PD}^2 - \overline{PC}^2}{\overline{CD}}.$

2. *If $AB = CD$, $AD = BC$ and $AC = BD$ then:* $\overline{PA}^2 + \overline{PC}^2 = \overline{PB}^2 + \overline{PD}^2.$

3. *If* $AB = CD,\ AD = BC$ *then:* $\dfrac{PA+PC}{PB+PD} = \sqrt{\dfrac{AC-1}{BD-1}}$.

4. *If* $AB = AD, CB = CD$ *then:* $PD+PB = \dfrac{2(AC.BC+AB)}{AC^2-1}.PA+\dfrac{2(AC.AB+BC)}{AC^2-1}.PC$.

Proof. 1. By using hyperbolic trigonometry for pentagons and right-angled hexagons, from the conditions $AD = BC$ and $AC = BD$, we can show that $OA = OB$, $OC = OD$ and then

$$\frac{PA.OC + PC.OA}{PB.OC + PD.OA} = 1.$$

The last equality implies that: $\dfrac{OA}{OC} = \dfrac{PA - PB}{PD - PC} = \dfrac{\overline{PA}^2 - \overline{PB}^2}{\overline{PD}^2 - \overline{PC}^2}$.

We also have

$$AB = -OA.OB.\cos(\alpha+\beta) + \widetilde{OA}.\widetilde{OB} = -OA^2.\cos(\alpha+\beta) + OA^2 - 1,$$

$$CD = -OC.OD.\cos(\alpha+\beta) + \widetilde{OC}.\widetilde{OD} = -OC^2.\cos(\alpha+\beta) + OC^2 - 1.$$

And so: $\dfrac{\overline{AB}}{\overline{CD}} = \dfrac{OA}{OC}$.

3. By using hyperbolic trigonometry in pentagons and right-angled hexagons, from the conditions $AB = CD$ and $AD = BC$, we can show that $OA = OC = \overline{AC}$, $OC = OD = \overline{CD}$ and hence

$$\frac{PA+PC}{PB+PD} = \frac{\overline{BD}}{\overline{AC}}.\frac{\widetilde{AC}}{\widetilde{BD}} = \sqrt{\frac{\overline{AC}^2 - 1}{\overline{BD}^2 - 1}}.$$

4. Denote $b := AB = AD$, $c := CB = CD$ and $a := AC$. One can show that $PD > PB$, then by applying the relation of quadruplet of geodesics (Corollary 5) for two quadruples $\{P; A, C, D\}$ and $\{P; A, C, B\}$, we have that PB and PD are two different roots of the following quadratic equation of variable t:

$$\begin{aligned} &PA^2(c^2-1) + PC^2(b^2-1) + t^2(a^2-1) \\ &= 2(bc+a).PA.PC + 2(ac+b).PA.t + 2(ab+c).PC.t + a^2 + b^2 + c^2 + 2abc - 1. \end{aligned}$$

Thus we have the formula of the sum of two roots: $PD+PB = \dfrac{2(ac+b)}{a^2-1}.PA+\dfrac{2(ab+c)}{a^2-1}.PC$.

□

Remark. If P is identical to one of the four geodesics, then the orthorectangle relation becomes the Pythagorean relation.

3.3.5 A quintet of horocycles

Let A, P, B, C, D be five disjoint horocycles in $\mathbb{H}$ with a cyclic order. Each of them divides $\mathbb{H}$ into two domains such that the other four horocycles lie in the same domain. Let O be the intersection between AC and BD. Let α and β be the angles between OA, OP and OB, OP respectively as in Figure 3.4.

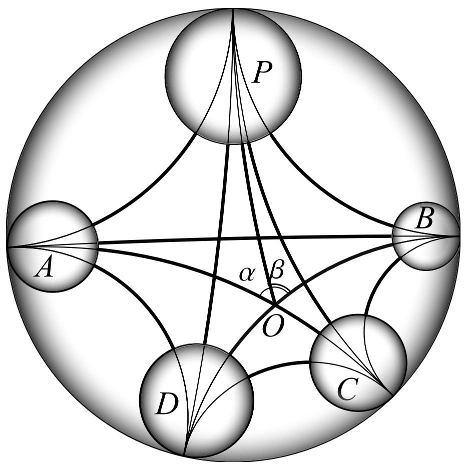

Figure 3.4: A quintet of horocycles.

With the notations in Section 3.2, and using standard calculations in the upper half-plane model of hyperbolic plane, we have the following relations:

Lemma 21. *With P, A, O and α defined above, we have*

$$PA = OP.OA(1 - \cos\alpha).$$

Applying Lemma 21 and the Penner's Ptolemy relation, one can compute OA, OB, OC, OD in terms of AC, BD, AD, AB, CD, CB as follows:

Lemma 22. *With P, A, B, C, D and O defined above, we have*

$$OA = \frac{\overline{AC}}{2}\sqrt{\frac{\overline{AD}.\overline{AB}}{\overline{CD}.\overline{CB}}};\ \ OC = \frac{\overline{AC}}{2}\sqrt{\frac{\overline{CD}.\overline{CB}}{\overline{AD}.\overline{AB}}};\ \ OB = \frac{\overline{BD}}{2}\sqrt{\frac{\overline{BA}.\overline{BC}}{\overline{DA}.\overline{DC}}};\ \ OD = \frac{\overline{BD}}{2}\sqrt{\frac{\overline{DA}.\overline{DC}}{\overline{BA}.\overline{BC}}}.$$

Applying these above formulae and the argument as in the proof of Lemma 20, we obtain the following relation:

Lemma 23. *(Orthoquadrilateral relation of horocycles) With P, A, B, C, D and O defined above, we have*

$$\frac{PA.OC + PC.OA}{PB.OD + PD.OB} = \frac{AC}{BD}.$$

Equivalently,

$$\frac{PA.\overline{CD}.\overline{CB} + PC.\overline{AD}.\overline{AB}}{PB.\overline{DA}.\overline{DC} + PD.\overline{BA}.\overline{BC}} = \frac{\overline{AC}}{\overline{BD}}.$$

Similarly, one also has the following properties of Penner's lambda lengths in special cases:

Corollary 7. *(Ortho-isosceles trapezoid, orthorectangle, orthoparallelogram relation of horocycles) With P, A, B, C, D defined above, we have*

- *If $AD = BC$ and $AC = BD$ then:* $\dfrac{\overline{PA}^2 - \overline{PB}^2}{\overline{AB}} = \dfrac{\overline{PD}^2 - \overline{PC}^2}{\overline{CD}}$.
- *If $AB = CD$, $AD = BC$ and $AC = BD$ then:* $\overline{PA}^2 + \overline{PC}^2 = \overline{PB}^2 + \overline{PD}^2$.
- *If $AB = CD$, $AD = BC$ then:* $\dfrac{\overline{PA}^2 + \overline{PC}^2}{\overline{PB}^2 + \overline{PD}^2} = \dfrac{\overline{AC}}{\overline{BD}}$.

3.3.6 A unique formula in form of Cayley-Menger determinant

In this section one will see that all of the relations in previous sections can be put into a unique formula in form of Cayley-Menger determinant. Let $|UV|$ be the Euclidean distance between two arbitrary points U and V in a Euclidean space. In the field of distance geometry, the Cayley-Menger determinant allows us to compute the volume of an $n-$simplex in a Euclidean space in terms of the squares of all the distances between pairs of its vertices [15]. In a special case, if A, B, C, D are four points in the Euclidean plane, one has a relation as follows:

$$\det \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & |AB|^2 & |AC|^2 & |AD|^2 \\ 1 & |AB|^2 & 0 & |BC|^2 & |BD|^2 \\ 1 & |AC|^2 & |BC|^2 & 0 & |CD|^2 \\ 1 & |AD|^2 & |BD|^2 & |CD|^2 & 0 \end{bmatrix} = 0.$$

If A, B, C, D are four points in the hyperbolic plane, one has a similar formula [9] as follows:

$$\det \begin{bmatrix} -2 & 1 & 1 & 1 & 1 \\ 1 & 0 & \overline{AB}^2 & \overline{AC}^2 & \overline{AD}^2 \\ 1 & \overline{AB}^2 & 0 & \overline{BC}^2 & \overline{BD}^2 \\ 1 & \overline{AC}^2 & \overline{BC}^2 & 0 & \overline{CD}^2 \\ 1 & \overline{AD}^2 & \overline{BD}^2 & \overline{CD}^2 & 0 \end{bmatrix} = 0.$$

Now we consider geodesics and horocycles in the hyperbolic plane. Let X be a curve of constant curvature in $\mathbb{H}$, denote by κ_X the geodesic curvature of X. Thus $\kappa_X = 0$ or 1 if X is a geodesic or a horocycle respectively. The relations of quadruplet of geodesics/horocycles in Lemma 19 can be written in a unique form as follows

Theorem 16. *Let A, B, C, D be four disjoint geodesics/horocycles, each of them divides $\mathbb{H}$ into two domains such that the other three lie in the same domain. Then*

$$\det \begin{bmatrix} 2 & 1-\kappa_A & 1-\kappa_B & 1-\kappa_C & 1-\kappa_D \\ 1-\kappa_A & 0 & \overline{AB}^2 & \overline{AC}^2 & \overline{AD}^2 \\ 1-\kappa_B & \overline{AB}^2 & 0 & \overline{BC}^2 & \overline{BD}^2 \\ 1-\kappa_C & \overline{AC}^2 & \overline{BC}^2 & 0 & \overline{CD}^2 \\ 1-\kappa_D & \overline{AD}^2 & \overline{BD}^2 & \overline{CD}^2 & 0 \end{bmatrix} = 0.$$

Remark. We suspect that the result can be generalized to higher dimensions in the following form: Let U and V be two arbitrarily disjoint hyperbolic spaces or horoballs of co-dimension 1 in $\mathbb{H}^n$. Denote by κ_U and κ_V the geodesic curvatures of U and V respectively. We define a weight function between U and V which is a generalized version of the half-trace and the half of Penner's lambda length of the distance between U and V as follows.

$$\overline{UV} := \frac{e^{\frac{1}{2}d_{\mathbb{H}^n}(U,V)} + (1-\kappa_U)(1-\kappa_V)e^{-\frac{1}{2}d_{\mathbb{H}^n}(U,V)}}{2}$$

Let $\{A_1, A_2, ..., A_k, A_{k+1}, A_{k+2}\}$ be the set of $k+2$ disjoint hypersurfaces of constant geodesic curvatures in n-dimensional hyperbolic space $\mathbb{H}^n$ $(k \geq n)$, each of them divides $\mathbb{H}^n$ into two domains such that the other $k+1$ hypersurfaces lie in the same domain. If $\kappa_{A_i} \in \{0, 1\}$ for all $i \in \{1, 2, ..., k+2\}$, then

$$\det \begin{bmatrix} 2 & 1-\kappa_{A_1} & 1-\kappa_{A_2} & 1-\kappa_{A_3} & \cdots & 1-\kappa_{A_{k+2}} \\ 1-\kappa_{A_1} & 0 & \overline{A_1A_2}^2 & \overline{A_1A_3}^2 & \cdots & \overline{A_1A_{k+2}}^2 \\ 1-\kappa_{A_2} & \overline{A_2A_1}^2 & 0 & \overline{A_2A_3}^2 & \cdots & \overline{A_2A_{k+2}}^2 \\ 1-\kappa_{A_3} & \overline{A_3A_1}^2 & \overline{A_3A_2}^2 & 0 & \cdots & \overline{A_3A_{k+2}}^2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-\kappa_{A_{k+2}} & \overline{A_{k+2}A_1}^2 & \overline{A_{k+2}A_2}^2 & \overline{A_{k+2}A_3}^2 & \cdots & 0 \end{bmatrix} = 0.$$

3.4. Orthotree, identity relations and r-orthoshapes

For simplicity, in this section we only present a tree structure and identity relations of orthogeodesics on a hyperbolic surface with its boundary consisting of simple closed geodesics.

3.4.1 Orthotree

In [27], Labourie and Tan generalized Bowditch's method to give a planar tree coding of oriented simple orthogeodesics on a hyperbolic surface by doing special flips on an orthotriangulation of the surface. In this section, we extend their method to obtain a construction which can be applied for any suitable subset of the set of oriented orthogeodesics starting from a boundary component. This construction may be useful for studying coherent markings introduced by Parlier in [33].

Let S be an orientable hyperbolic surface with boundary ∂S consisting of simple closed geodesics. Let $\eta_{\frac{0}{1}}$ and $\eta_{\frac{1}{1}}$ be two oriented orthogeodesics starting from a simple closed geodesic α at the boundary of S. The starting points of these two orthogeodesics divides α into two open subsegments, namely α_1 and α_2. Suppose that $\alpha_1 \neq \emptyset$. Denote by $\mathcal{O}_{\alpha_1}$ the set of oriented orthogeodesics starting from α_1. The orientation of S gives an order on $\mathcal{O}_{\alpha_1}$ independent of the hyperbolic structure. In particular, the order on $\mathcal{O}_{\alpha_1}$ is defined by the order of the starting points of $\mathcal{O}_{\alpha_1}$ on α_1. By that, if η_x, η_y and η_z are three elements in $\mathcal{O}_{\alpha_1}$ and the starting point of η_y lies between η_x and η_z with respect to the subsegment α_1, we say η_y lies between η_x and η_z.

Labelling elements in a subset of $\mathcal{O}_{\alpha_1}$:

A Farey pair $(\frac{p}{q}, \frac{m}{n})$ is a pair of two reduced fractions $\frac{p}{q}$ and $\frac{m}{n}$ such that $\frac{0}{1} \leq \frac{p}{q} < \frac{m}{n} \leq \frac{1}{1}$ and $qm - pn = 1$. Their Farey sum is defined as $\frac{p}{q} \oplus \frac{m}{n} := \frac{p+m}{q+n}$. Thus $\frac{p}{q} < \frac{p}{q} \oplus \frac{m}{n} < \frac{m}{n}$. Furthermore, $(\frac{p}{q}, \frac{p}{q} \oplus \frac{m}{n})$ and $(\frac{p}{q} \oplus \frac{m}{n}, \frac{m}{n})$ become two other Farey pairs.

Let G be a countable subset of $\mathcal{O}_{\alpha_1}$ such that for any two arbitrary elements of G, there are infinitely many other elements of G lying between them. For example, G can be taken as the set $\mathcal{O}_{\alpha_1}$ itself or the set of simple oriented orthogeodesics starting from α_1 when $\eta_{\frac{0}{1}}, \eta_{\frac{1}{1}}$ and α_1 are chosen suitably. We label elements in the set G by rational numbers between 0 and 1 as follows:

Step 1: Take an arbitrary element in G and name it by $\eta_{\frac{0}{1}\oplus\frac{1}{1}}$ or equivalently $\eta_{\frac{1}{2}}$. The starting point of this element divides α_1 into two disjoint subsegments. By abuse of notation, we denote these two subsegments by $(\eta_{\frac{0}{1}}, \eta_{\frac{1}{2}})$ and $(\eta_{\frac{1}{2}}, \eta_{\frac{1}{1}})$.

Step $k \geq 2$: We have 2^{k-1} disjoint subsegments of the form $(\eta_{\frac{p_1}{q_1}}, \eta_{\frac{p_2}{q_2}})$ where $(\frac{p_1}{q_1}, \frac{p_2}{q_2})$ is a Farey pair. Denote by $G(\eta_{\frac{p_1}{q_1}}, \eta_{\frac{p_2}{q_2}})$ the set of oriented orthogeodesics in G starting from the subsegment $(\eta_{\frac{p_1}{q_1}}, \eta_{\frac{p_2}{q_2}})$. Similarly to step 1, we take an arbitrary element in $G(\eta_{\frac{p_1}{q_1}}, \eta_{\frac{p_2}{q_2}})$ and name it by $\eta_{\frac{p_1}{q_1}\oplus\frac{p_2}{q_2}}$ or equivalently $\eta_{\frac{p_1+p_2}{q_1+q_2}}$. We do it for all other subsegments and obtain 2^k new disjoint subsegments.

The above labelling gives an order-preserving injection

$$\mathring{\Psi}_1 : \mathbb{Q} \cap (0,1) \to G.$$

This map can be extended naturally to a map

$$\Psi_1 : \mathbb{Q} \cap [0,1] \to G \cup \{\eta_{\frac{0}{1}}, \eta_{\frac{1}{1}}\}.$$

Note that if $\eta_{\frac{0}{1}}$ and $\eta_{\frac{1}{1}}$ are distinct oriented orthogeodesics, then Ψ_1 is also an order-preserving injection.

The bijectivity:

For each orthogeodesic X, we denote $h(X) := \frac{1}{2}\log\left(\frac{X+1}{X-1}\right)$ the radius of the stable neighborhood of X. In each step $k \geq 1$, we have 2^{k-1} disjoint subsegments. Let (X,Y) be one of these subsegments and let $\ell(X,Y)$ be the length of this subsegment. Let $m(X,Y) := \ell(X,Y) - h(X) - h(Y)$ the modified length of the segment (X,Y). We denote by m_k the maximum of the modified lengths of the associated 2^{k-1} disjoint subsegments in step k, then

Lemma 24. *The map $\mathring{\Psi}_1$ is an order-preserving bijection if $m_k \to 0$ as $k \to \infty$.*

Proof. Assume that X is an oriented orthogeodesic in G such that $\mathring{\Psi}_1(p/q) \neq X$ for all $p/q \in \mathbb{Q} \cap (0,1)$. Let (Y_k, Z_k) be the subsegment containing X at step k. It is easy to see that $m(Y_k, Z_k) \geq 2h(X)$ for all $k \geq 1$, a contradiction to the fact that $m_k \to 0$ as $k \to \infty$. □

In the example below, we will present another way to define $\mathring{\Psi}_1$ based on a given orthotriangulation of S which is closely related to the construction in [27].

Labelling complementary regions of a tree:

Let T_1 be a planar rooted trivalent tree whose the first vertex is of valence 1, and all other vertices are of valence 3. We visualize it by embedding T_1 to the lower half-plane with the root is located at the origin (see Figure 3.6 for an example). Let $E(T_1)$ be the set of edges of T_1. Each edge of the tree has two sides associated to two distinct complementary regions of the tree. Let $\Omega(T_1)$ be the set of complementary regions of the tree. We label elements in $\Omega(T_1)$ as follows: The two initial regions is labeled by fraction numbers $\frac{0}{1}$ and $\frac{1}{1}$. Since each vertex except the root has three regions surrounding it, so if we know the labels of two of them, we can label the third region by their Farey sum. Thus, one obtains a bijection

$$\Psi_2 : \mathbb{Q} \cap [0,1] \to \Omega(T_1).$$

By this map, the order on $\mathbb{Q} \cap [0,1]$ then descents to an order on $\Omega(T_1)$. Denote by $\mathring{\Omega}(T_1)$ the set $\Psi_2(\mathbb{Q} \cap (0,1))$. Combining with Lemma 24, we obtain the following theorem:

Theorem 17. *If $m_k \to 0$ as $k \to \infty$, the map $\mathring{\Psi} := \mathring{\Psi}_1 \circ \Psi_2^{-1}|_{\mathring{\Omega}(T_1)}$ is an order-preserving bijection from $\mathring{\Omega}(T_1)$ to G. Furthermore if $\eta_{\frac{0}{1}}$ and $\eta_{\frac{1}{1}}$ are distinct, then the map $\Psi := \Psi_1 \circ \Psi_2^{-1}$ is also an order-preserving bijection.*

The map Ψ gives us a label of $\Omega(T_1)$ by the set $G \sqcup \{\eta_{\frac{0}{1}}\} \sqcup \{\eta_{\frac{1}{1}}\}$.

Labelling edges of the tree T_1:

Let Y and Z be two arbitrary complementary regions of T_1. Thus, $\Psi(Y)$ and $\Psi(Z)$ are two oriented orthogeodesics starting from α_1 and define a subsegment $(\Psi(Y), \Psi(Z))$ of α_1. The two orthogeodesics $\Psi(Y)$ and $\Psi(Z)$ then together with the subsegment $(\Psi(Y), \Psi(Z))$ define a unique orthogeodesic denoted by $F(\Psi(Y), \Psi(Z), \alpha_1)$ such that they are four sides of an orthotriangle on S. Note that $F(\Psi(Y), \Psi(Z), \alpha_1)$ and $(\Psi(Y), \Psi(Z))$ are two opposite sides in this orthotriangle. If x is an edge of the tree T_1 with two neighboring complementary regions Y and Z, then we label x by the orthogeodesic $F(\Psi(Y), \Psi(Z), \alpha_1)$.

A simple example:

Let Δ be an orthotriangulation. Let $(\eta_{\frac{0}{1}}, \eta_{\frac{1}{1}}, \alpha_1)$ be three sides of an orthotriangle in Δ. For simplicity, we will take G as the set $\mathcal{O}_{\alpha_1}$ itself. For any element $X \in \mathcal{O}_{\alpha_1}$, we denote by $N(X)$ the number of intersections of X with the orthobasis of Δ. We define the associated map Ψ_1 as follows:

Step 1: Take an element X in $\mathcal{O}_{\alpha_1}$ such that $N(X)$ is minimal. We name it by $\eta_{\frac{0}{1}\oplus\frac{1}{1}}$ or equivalently $\eta_{\frac{1}{2}}$. The starting point of this element divides α_1 into two disjoint subsegments, denoted by $(\eta_{\frac{0}{1}}, \eta_{\frac{1}{2}})$ and $(\eta_{\frac{1}{2}}, \eta_{\frac{1}{1}})$.

Step $k \geq 2$: We have 2^{k-1} disjoint subsegments of the form $(\eta_{\frac{p_1}{q_1}}, \eta_{\frac{p_2}{q_2}})$ where $(\frac{p_1}{q_1}, \frac{p_2}{q_2})$ is a Farey pair. Denote by $G(\eta_{\frac{p_1}{q_1}}, \eta_{\frac{p_2}{q_2}})$ the set of oriented orthogeodesics in G starting from the subsegment $(\eta_{\frac{p_1}{q_1}}, \eta_{\frac{p_2}{q_2}})$. Apply step 1 for each of these subsegments, one obtains 2^k new disjoint subsegments.

By this construction, we can exhaust all elements in $\mathcal{O}_{\alpha_1}$, thus Ψ_1 is an order-preserving bijection from $\mathbb{Q} \cap [0, 1]$ to $\mathcal{O}_{\alpha_1} \cup \{\eta_{\frac{0}{1}}, \eta_{\frac{1}{1}}\}$. Hence $\Psi := \Psi_1 \circ \Psi_2^{-1}$ gives us a label of $\Omega(T_1)$ by the set $\mathcal{O}_{\alpha_1} \cup \{\eta_{\frac{0}{1}}, \eta_{\frac{1}{1}}\}$. By looking at the universal cover, one can also see that if X is a chosen element in step k, then $N(X) = k$.

Furthermore, if Δ be a standard orthotriangulation (see Definition 3), each oriented orthogeodesic in $\mathcal{O}_{\alpha_1}$ can also be encoded by its crossing sequence with the orthobasis of Δ. Note that two oriented orthogeodesics are of the same crossing sequence (word) iff they are two different directions of an orthogeodesic with a symmetric word. This labeling is useful when we only care of the ortholength spectrum. Due to the labeling of edges, each path of edges starting from the root is associated to a continuous sequence of elements in the orthobasis (crossing sequence). We label the associated complementary region of the path by capitalizing the word formed from the crossing sequence (see Figure 3.6 for a tree with labels on edges and complementary regions and see Figure 3.7 for an example of the crossing sequence of an orthogeodesic).

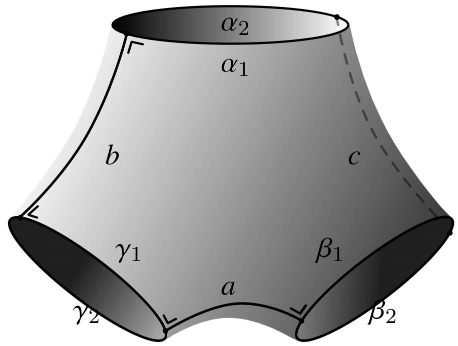

Figure 3.5: A standard orthotriangulation on a pair of pants.

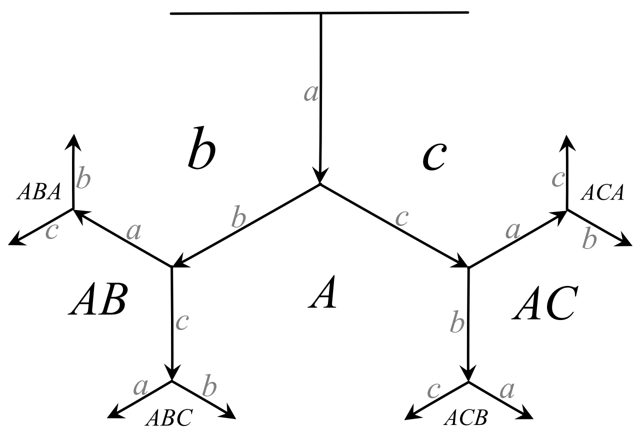

Figure 3.6: Labeling the tree of oriented orthogeodesics starting at α_1.

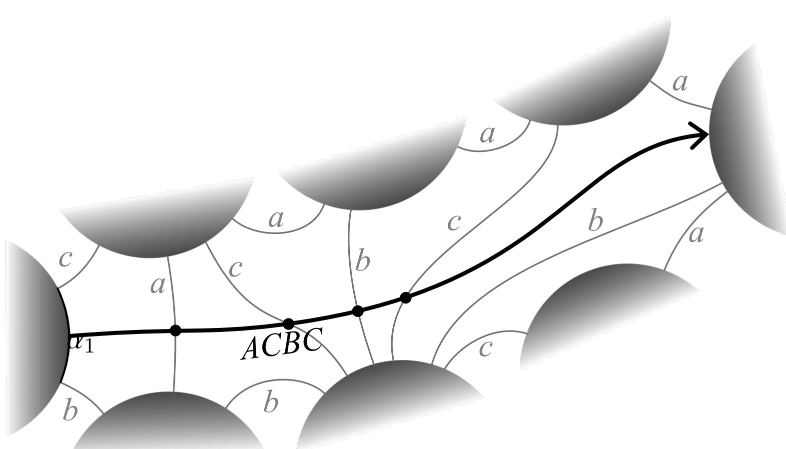

Figure 3.7: A lift of an oriented orthogeodesic ($ACBC$) starting at α_1.

Note that we have only defined the tree of oriented orthogeodesics starting at a subsegment of a simple closed geodesic, say o_1, at the boundary of a surface. Since the orthobasis of Δ divides o_1 into a finite number of disjoint subsegments. One can glue the roots of suitable labeled trees associated to all the subsegments in a cyclic order to get a planar rooted trivalent tree of all oriented orthogeodesics starting from o_1.

3.4.2 Identity relations

Recursive formulae and vertex relations

Let X, Y, Z be three complementary regions surrounding a vertex v (not at the root) of the tree T_1. Let x, y, z be three edges which intersect X, Y, Z respectively at only v. Suppose that the set of edges of T_1 is oriented outwards from the root. If the end point of the oriented edge x and the starting point of the oriented edges y and z coincide at v (see Figure 3.8), by the Ptolemy relation of geodesics in Lemma 17, one can compute X in term of Y, Z, x, y, z:

$$X = \frac{(xy+z)Y + (xz+y)Z + \sqrt{(x^2+y^2+z^2+2xyz-1)(x^2+Y^2+Z^2+2xYZ-1)}}{x^2-1}.$$

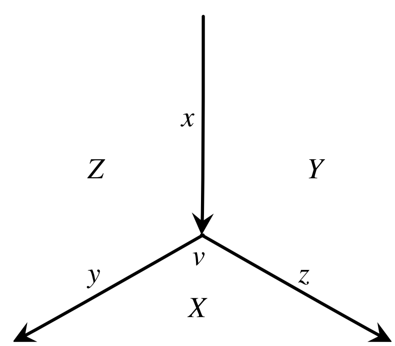

Figure 3.8: Positions of X, Y, Z, x, y, z.

Note that by Corollary 5, we also have a **vertex relation** as follows:

$$(x^2-1)X^2 + (y^2-1)Y^2 + (z^2-1)Z^2 - 2(xy+z)XY - 2(yz+x)YZ - 2(xz+y)XZ$$
$$= x^2 + y^2 + z^2 + 2xyz - 1.$$

Isosceles trapezoid, rectangle, kite, parallelogram and edge relations

We define the weight function $UV :=$ weight(U, V) between two complementary regions U and V to be the hyperbolic cosine of the length of the orthogeodesic $F(\Psi(U), \Psi(V), \alpha_1)$ as defined in previous section 3.4.1. Let X, Y, Z, T be four complementary regions, by Corollary 6 in Section 3.3, we have the following relations:

- If $XT = YZ$, and $XZ = YT$, then: $\dfrac{X-Y}{\sqrt{XY+1}} = \dfrac{Z-T}{\sqrt{ZT+1}}$.

- If $XT = YZ$, $XY = ZT$, and $XZ = YT$, then: $X + T = Y + Z$.

- If $XY = XT$, and $ZY = ZT$, then: $T + Y = \dfrac{2(XZ.YZ + XY)}{(XZ)^2 - 1}X + \dfrac{2(XZ.XY + YZ)}{(XZ)^2 - 1}Z$.

- If $XY = ZT$, and $XT = YZ$, then: $X + Z = (Y + T)\sqrt{\dfrac{XZ-1}{YT-1}}$.

Furthermore, we have edge relations of orthogeodesics on certain special surfaces as in the following examples.

Example 1: Edge relations on a pair of pants. Denote by $\{a, b, c\}$ the standard orthobasis of a pair of pants, that is the set of shortest simple orthogeodesics connecting two distinct boundary components. This orthobasis cuts each boundary geodesic component into two geodesic segments of equal length. The six resulting segments are denoted by $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ as in Figure 3.5. Let T_1 be the tree of oriented orthogeodesics starting from α_1. As in Figure 3.5, b and c are on the left and the right of the segment α_1 and a is opposite to α_1. Each edge of T_1 is labeled by either a or b or c following the grammar of the dual graph of the orthotriangulation. We also use words formed from the capital letters A, B, C to label the complemetary regions of T_1 except for the two initial regions. Figure 3.6 is an illustration of the tree T_1.

Edge relations are special cases of orthokite relations when four edges of the orthokite are elements in the standard orthobasis of a pair of pants. There are four regions surrounding each edge - except for the edge with an end point at the root. We choose arbitrarily an edge $x \in \{a, b, c\}$ with four surrounding regions X_1, Y, Z, X_2 with $Y \cap Z = x$. Due to the standard orthobasis of a pair of pants, $X_1Y = X_2Y =: z$ and $X_1Z = X_2Z =: y$, where y, z are distinct elements in $\{a, b, c\} - \{x\}$. Thus one can denote the labels of $X_1 \cap Y$, $X_1 \cap Z$, $X_2 \cap Y$, $X_2 \cap Z$ to be z, y, z, y respectively (see e.g. Figure 3.9). We have an edge relation of orthogeodesics on a pair of pants:

$$X_1 + X_2 = \frac{2(xy + z)}{x^2 - 1}Y + \frac{2(xz + y)}{x^2 - 1}Z. \tag{3.3}$$

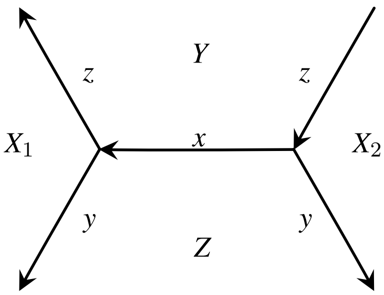

Figure 3.9: Positions of X_1, X_2, Y, Z, x, y, z in the case of a pair of pants.

Example 2: Edge relations on a one-holed torus. We denote $\{a, b, c\}$ to be an arbitrary orthobasis of a one-holed torus, that is the set of three non-crossing simple orthogeodesics. This orthobasis cuts the boundary geodesic component of the one-holed torus into six geodesic segments.

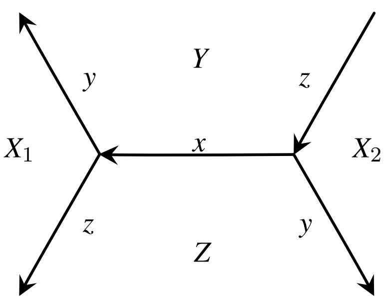

Figure 3.10: Positions of X_1, X_2, Y, Z, x, y, z in the case of a one-holed torus.

Let T_1 be the tree of oriented orthogeodesics starting from one of these six segments. There are four regions surrounding each edge - except for the edge with an end point at the root. We choose arbitrarily an edge $x \in \{a, b, c\}$ with four surrounding regions X_1, Y, Z, X_2 with $Y \cap Z = x$. We have $X_1Y = X_2Z =: y$ and $X_1Z = X_2Y =: z$, where y, z are distinct elements in $\{a, b, c\} - \{x\}$. Thus one can denote the labels of $X_1 \cap Y$, $X_1 \cap Z$, $X_2 \cap Y$, $X_2 \cap Z$ to be y, z, z, y respectively (see Figure 3.10). Then

$$X_1 + X_2 = (Y + Z)\sqrt{\frac{X_1X_2 - 1}{YZ - 1}}.$$

Note that X_1X_2 is the cosh of the length of the simple orthogeodesic with label X crossing the edge x. Thus one can use the recursive formula (Section 3.4.2) to compute X_1X_2 in terms of x, y, z as follows:

$$X_1 X_2 = \frac{(xy+z)y + (xz+y)z + x^2 + y^2 + z^2 + 2xyz - 1}{x^2 - 1} = \frac{(y+z)^2}{x-1} + 1.$$

Note that $YZ = x$, thus we have an edge relation of orthogeodesics on a one-holed torus:

$$X_1 + X_2 = \frac{y+z}{x-1}(Y+Z). \tag{3.4}$$

In the next section, we will use reflection involutions on immersed pairs of pants to construct a family of ortho-isosceles-trapezoids, orthorectangles and orthokites on a general hyperbolic surface.

3.4.3 A construction of r-orthoshapes using reflection involutions

Let P be a pair of pants with boundary components o_1, o_2, o_3. Let a, b, c, A, B, C be simple orthogeodesics connecting two elements in each of pairs (o_2, o_3), (o_1, o_3), (o_1, o_2), (o_1, o_1), (o_2, o_2), (o_3, o_3) respectively. Note that a, b, c are also called the seams of P. Let r be the unique orientation-reversing isometry over P which fixes a, b, c pointwisely. Note that if $\ell(o_1)$, $\ell(o_2)$ and $\ell(o_3)$ are pairwise different, then the isometry group of P is $\{id, r\}$ with $r^2 = id$. Let $\{i, j, k\} = \{1, 2, 3\}$. If $\ell(o_i) = \ell(o_j)$, let r_k be the unique orientation-reserving isometry over P which fixes o_k and interchanges o_i with o_j. The set of fixed points of each of isometries r_1, r_2, r_3 form each of orthogeodesics A, B, C respectively. Note that if $\ell(o_i) = \ell(o_j) \neq \ell(o_k)$, the isometry group of P will be $\{id, r_k, r, r \circ r_k\}$ with $r^2 = r_k^2 = (r \circ r_k)^2 = id$ and $r \circ r_k = r_k \circ r$. A thorough treatment on pairs of pants can be found in chapter 3 of Buser's book [14]. Before going to the construction of r-orthoshapes, we need the following notions:

- P is called an **isosceles pair of pants** if it has two boundary components of the same length. It is called a **regular pair of pants** if its three boundary components are of the same length.
- A **reflection involution** i on P is an orientation-reversing isometry on P which fixes a simple orthogeodesic pointwisely (the "symmetry axis" of i). For example: r is a reflection involution on P and it has three symmetry axes a, b, c. If $\ell(o_i) = \ell(o_j)$, then r_k is another reflection involution on P.

Let Σ be an oriented topological surface with punctures and negative Euler characteristic. A point in $\mathcal{T}(\Sigma)$ can be represented by a hyperbolic surface, namely S, with ∂S consisting of simple closed geodesics and/or cusps such that the interior of S is homeomorphic to Σ. In the following, we construct a class of r-orthoshapes on Σ coming from reflection involutions on immersed pairs of pants.

Lemma 25. *An orthogeodesic γ and its reflection through a reflection involution form infinitely many r-ortho-isosceles-trapezoids and r-orthokites on Σ.*

Proof. Without loss of generality, one can assume that γ is a common orthogeodesic of $S \in \mathcal{T}(\Sigma)$ and P where P is an immersed pair of pants on S. Let r be a reflection involution on P which fixes its seams. Lifting to the universal cover of S. The lift of P has an injective simple connected fundamental domain. Let $\tilde{a}$ be a lift of a symmetry axis of r (i.e. a lift of the seam a of P). A lift of γ and its reflection through $\tilde{a}$ form either an r-orthokite or an r-ortho-isosceles-trapezoid on the universal cover of S depending on the chosen lift of γ. Since there are infinitely many lifts of γ, there are also infinitely many r-ortho-isosceles-trapezoids and r-orthokites on the universal cover of S such that their projection to the surface gives infinitely many different r-ortho-isosceles-trapezoids and r-ortho kites. □

The special case of an r-ortho-isosceles-trapezoid is an r-orthorectangle happened in an immersed isosceles pair of pants when there is a reflection involution between the other pair of the opposite orthogeodesics in the r-ortho-isosceles-trapezoid.

Lemma 26. *The endpoints of orthogeodesics forming an r-orthorectangle are always on the same boundary component of* Σ.

Proof. Let $ABCD$ be an r-orthorectangle in which each pair of orthogeodesics (AB, CD), (BC, DA), (AC, BD) are length equivalent. If the endpoints of AB, CD, BC, DA are on two distinct boundary components, say β_1 and β_2, of Σ. Then both endpoints of AC are on the same boundary component, say β_1 (without loss of generality). Thus both endpoints of BD have to be on β_2. It contradicts the fact that (AC, BD) are length equivalent. □

Lemma 27. *There are infinitely many r-orthorectangles on* Σ.

Proof. Let γ be an orthogeodesic with its endpoints on the same boundary component of $S \in \mathcal{T}(\Sigma)$. Let P be an immersed isosceles pair of pants on S with γ as one of its seams. Thus, there are two geodesic boundary components of P, say o_1 and o_2, such that they are embedded to a single boundary component of S. Let r and r_3 be two reflection involutions on P where r fixes the seams of P and r_3 interchanges o_1 and o_2. Let η be the symmetry axis of r_3, hence η is an arc with its endpoints on the third boundary of P and it is perpendicular to γ at their midpoints. Let α be an orthogeodesic in P with its ends on o_1 such that α wraps n times around o_2 and zero time around o_1, where $n \geq 2$. Then $\beta := r_3(\alpha)$ is an orthogeodesic in P with its ends on o_2 and it wraps n times around o_1 without wraping around o_2. Lifting to the universal cover of S. The lift of P has an injective simple connected fundamental domain. Under these conditions, one is always able to find lifts of γ, η, α and β, namely $\tilde{\gamma}$, $\tilde{\eta}$, $\tilde{\alpha}$ and $\tilde{\beta}$, such that

- $\tilde{\eta}$, $\tilde{\alpha}$ and $\tilde{\beta}$ are disjoint,
- $\tilde{\beta}$ is the reflection of $\tilde{\alpha}$ through $\tilde{\eta}$ on the universal cover of S,
- $\tilde{\gamma}$ is perpendicular to $\tilde{\eta}$ at their midpoints, and perpendicular to $\tilde{\alpha}$ and $\tilde{\beta}$ at the midpoints of $\tilde{\alpha}$ and $\tilde{\beta}$.

Thus $\tilde{\alpha}, \tilde{\beta}$ together with two other associated orthogeodesics form an r-ortho rectangle on the universal cover of S. Projecting it back to S one obtains an r-ortho rectangle on S. Since n can be chosen arbitrarily in $\mathbb{N}_{\geq 2}$ and even more than that γ can be chosen arbitrarily in the set of orthogeodesics with endpoints on the same boundary component of S, there are infinitely many r-orthorectangles on S. □

Lemma 28. *There is no r-orthosquare on* Σ*.*

Proof. Suppose that there exists $\mathbb{O} := XYZT$ an r-orthosquare on Σ. We consider a hyperbolic structure S^* on Σ which satisfies:

- S^* is a cusped surface.
- There is a truncated orthobasis (i.e. a decorated ideal triangulation) on S^* including only orthogeodesics of lambda length 1.

Note that if we lift this decorated ideal triangulation of S^* to $\mathbb{H}$, we obtains the Farey tessellation with Ford circles. Also note that by Penner's Ptolemy relation (see Equation 3.1), the tree of lambda lengths of orthogeodesics on S^* is the tree of Fibonacci function (see [11] for a definition). Thus the lambda length of any orthogeodesics on S^* is always an integer (also see [37] for a direct computation on the Farey decoration).

Again by Penner's Ptolemy relation, $\lambda(X,Z)\lambda(Y,T) = \lambda(X,Y)\lambda(Z,T) + \lambda(X,T)\lambda(Y,Z)$. Thus $\lambda(X,Z) = \lambda(X,Y)\sqrt{2}$ since $\mathbb{O}$ is an orthosquare on S^*. It contradicts the fact that the lambda length of any orthogeodesic on S^* is always an integer. □

3.5. Applications

In this section we present some applications from the study of the tree structure on the set of orthogeodesics.

3.5.1 A combinatorial proof of Basmajian's identity

Let S be a hyperbolic surface of totally geodesic boundary, Basmajian's identity on S is as follows:

$$\ell(\partial S) = \sum_{\eta} 2\log(\coth(\ell(\eta)/2)),$$

where the sum runs over the set of orthogeodesics on the surface. This identity was proved by using elementary hyperbolic geometry and the fact that the limit set of a non-elementary second kind Fuchsian group is of 1- dimensional measure zero. In this section, we will present a combinatorial proof of Basmajian's identity in the case of a hyperbolic surface with totally geodesic boundary.

Let T_1 be a rooted trivalent tree whose first vertex is of valence 1, and all other vertices are of valence 3. We visualize it by embedding T_1 in the lower half-plane with the root is located at the origin (see Figure 3.6). Let $E(T_1)$ be the set of edges of T_1 where each edge is oriented outwards from the root. Let $\Omega(T_1)$ be the set of complementary regions of T_1. Similarly to the definition of the Markoff map in [10], one can define a map $\Phi : E(T_1) \sqcup \Omega(T_1) \to (1, \infty)$ which has a harmonic relation at any vertex except at the root of the tree as follows:

$$\operatorname{arccosh} \frac{x + YZ}{\sqrt{(Y^2-1)(Z^2-1)}} = \operatorname{arccosh} \frac{z + XY}{\sqrt{(X^2-1)(Y^2-1)}} + \operatorname{arccosh} \frac{y + XZ}{\sqrt{(X^2-1)(Z^2-1)}},$$

where x, y, z, X, Y, Z are the abbreviations of $\Phi(x), \Phi(y), \Phi(z), \Phi(X), \Phi(Y), \Phi(Z)$ respectively and they are as in Figure 3.8. Note that the idea of this harmonic relation comes from Lemma 16. We define a function h on $\Omega(T_1)$ as follows:

$$h(X) := \operatorname{arccosh}\left(\frac{X}{\sqrt{X^2-1}}\right) = \frac{1}{2}\log\left(\frac{X+1}{X-1}\right).$$

A triple $(x, Y, Z) \in E(T_1) \times \Omega(T_1) \times \Omega(T_1)$ is called an edge region triple if $x = Y \cap Z$ (i.e. $Y, Z \in \Omega(T_1)$ are two neighboring complementary regions of $x \in E(T_1)$). We define a function f on the set of edge region triples as follows:

$$f(x, Y, Z) := \operatorname{arccosh}\left(\frac{x + YZ}{\sqrt{(Y^2-1)(Z^2-1)}}\right).$$

Note that $\lim_{Y\to\infty} f(y, X, Y) = \lim_{Z\to\infty} f(z, X, Z) = h(X)$. The harmonic relation can be rewritten as:

$$f(x, Y, Z) = f(y, X, Z) + f(z, X, Y),$$

for all X, Y, Z, x, y, z as in Figure 3.8. Thus we have a Green formula:

$$\sum_{x\in C_n} f(x, Y, Z) = f(x_0, Y_0, Z_0),$$

where (x_0, Y_0, Z_0) is the initial edge region triple (i.e., $x_0 \in E(T_1)$ is the initial edge from the root, $Y_0, Z_0 \in \Omega(T_1)$ are two neighboring complementary regions of x_0), and C_n is the set of edges at combinatorial distance $n \in \mathbb{N}$ from the root. We use Bowditch's argument to prove Basmajian's identity:

Theorem 18. *(Basmajian's identity for T_1) If* $\sup\{\Phi(x) | x \in E(T_1)\} < \infty$*, then*

$$\sum_{X\in\Omega(T_1)} 2h(X) = h(Y_0) + h(Z_0) + f(x_0, Y_0, Z_0).$$

or equivalently,

$$\sum_{X\in\Omega(T_1)} \log\left(\frac{X+1}{X-1}\right) = \frac{1}{2}\log\left(\frac{Y_0+1}{Y_0-1}\right) + \frac{1}{2}\log\left(\frac{Z_0+1}{Z_0-1}\right) + \operatorname{arccosh}\left(\frac{x_0+Y_0Z_0}{\sqrt{(Y_0^2-1)(Z_0^2-1)}}\right),$$

where (x_0, Y_0, Z_0) *is the initial edge region triple of the tree.*

Proof. It is easy to show that $f(x,Y,Z) \geq h(Y)+h(Z)$ for any edge region triple (x,Y,Z). Then by using the Green formula we obtain:

$$\sum_{X\in\Omega(T_1)} 2h(X) \leq h(Y_0)+h(Z_0)+f(x_0,Y_0,Z_0). \tag{3.5}$$

Let $\mu := \sup\{\Phi(x) | x \in E(T_1)\}$, then we will show that

$$f(x,Y,Z) \leq h(Y)+x.h(Z) \leq h(Y)+\mu.h(Z), \tag{3.6}$$

for any edge region triple (x,Y,Z). Indeed, we define

$$F(x) := h(Y)+x.h(Z)-f(x,Y,Z),$$

then compute $F'(x)$ and $F''(x)$, in particular

$$F''(x) = \frac{x+YZ}{(x^2+Y^2+Z^2+2xYZ-1)^{3/2}} > 0,$$

for all $x,Y,Z > 1$. It implies that $F'(x) > F'(1)$. Note that

$$F'(x) = \log\left(\frac{Z+1}{\sqrt{Z^2-1}}\right) - \frac{1}{\sqrt{x^2+Y^2+Z^2+2xYZ-1}},$$

so

$$F'(x) > F'(1) = \frac{1}{2}\log\left(\frac{Z+1}{Z-1}\right) - \frac{1}{Y+Z} > \frac{1}{2}\log\left(\frac{Z+1}{Z-1}\right) - \frac{1}{Z+1} > 0,$$

for all $x,Y,Z > 1$. Hence $F(x) > F(1) = 0$.

By combining Inequality 3.6 and the Green formula we have:

$$f(x_0,Y_0,Z_0) = \sum_{x\in C_n} f(x,Y,Z) \leq \sum_{X\in\Omega_n} 2h(X) - h(Y_0) - h(Z_0) + 2\mu \sum_{X\in\Omega_{n+1}-\Omega_n} 2h(X),$$

for all $n \in \mathbb{N}$ where Ω_n is the set of complementary regions at combinatorial distance at most n from the root. Let $n \to \infty$. Note that $\sum_{X\in\Omega_{n+1}-\Omega_n} 2h(X)$ tends to 0 due to Inequality 3.5.

One has

$$f(x_0, Y_0, Z_0) \leq \sum_{X \in \Omega(T_1)} 2h(X) - h(Y_0) - h(Z_0). \tag{3.7}$$

Finally, Basmajian's identity follows from Inequalities 3.5 and 3.7. □

As a consequence, one can express a combinatorial form of Basmajian's identity for the set of oriented orthogeodesics starting from a simple closed geodesic, say o_1, at the boundary of a hyperbolic surface, assuming that o_1 is divided into n subsegments by an orthobasis. Note that the finite sum on the right hand side of Equation 3.8 is a combinatorial form of the length of o_1.

Corollary 8. *(Basmajian's identity for T_n) Let T_n be a rooted trivalent tree with n edges starting from the root. If $\sup\{\Phi(x) | x \in E(T_n)\} < \infty$, then*

$$\sum_{X \in \Omega(T_n)} \log\left(\frac{X+1}{X-1}\right) = \sum_{k=1}^{n} \operatorname{arccosh}\left(\frac{x_{0,k} + Y_{0,k} Z_{0,k}}{\sqrt{(Y_{0,k}^2 - 1)(Z_{0,k}^2 - 1)}}\right). \tag{3.8}$$

where $(x_{0,k}, Y_{0,k}, Z_{0,k})$'s are edge region triples at the root of the tree. Note that $Z_{0,k} = Y_{0,k+1}$ for $k = \overline{1,n}$, in which $Y_{0,n+1} := Y_{0,1}$.

Remarks.

1. By adapting suitable harmonic relations (see the remark after Lemma 18), one can extend this result to the general case in which the boundary of surface consists cusps and at least one simple closed geodesic. We suspect that this method can be generalized to higher dimensions.

2. In the proof of Theorem 18, we need two necessary inequalities: $h(Y) + h(Z) \leq f(x, Y, Z)$ and $f(x, Y, Z) \leq h(Y) + x.h(Z)$. It is not difficult to see the geometric meaning of the first inequality. However, the second one seems unnatural.

3.5.2 Ortho-integral surfaces

A hyperbolic surface is **ortho-integral** if it has an integral ortho cosh-length spectrum, that is $\cosh(\ell(\eta)) \in \mathbb{N}$ for any orthogeodesic η on the surface. Denote by $\mathcal{O}_S$ the set of orthogeodesics on a hyperbolic surface S. Using the recursive formulae and/or edge relations, one can give conditions on pairs of pants and one-holed tori such that they are ortho-integral.

Theorem 19. *Let P be a pair of pants and T a one-holed torus. Then*

- *P is ortho-integral if there is an orthobasis $\{a, b, c\}$ on P such that $\cosh \ell(a) = \cosh \ell(b) = \cosh \ell(c) \in \{2, 3\}$.*

- *T is ortho-integral if there is an orthobasis $\{a, b, c\}$ on T such that one of the following happens*

- $\cosh \ell(a) = \cosh \ell(b) = \cosh \ell(c) \in \{2, 3\}$
- $\cosh \ell(a) = 3, \cosh \ell(b) = 17, \cosh \ell(c) = 21$
- $\cosh \ell(a) = 2, \cosh \ell(b) = 7, \cosh \ell(c) = 10$
- $\cosh \ell(a) = 17, \cosh \ell(b) = 19, \cosh \ell(c) = 37$
- $\cosh \ell(a) = 7, \cosh \ell(b) = 17, \cosh \ell(c) = 25.$

Proof. We will give a proof of the case where a one-holed torus has an orthobasis $\{a = 3, b = 17, c = 21\}$ (other cases are either simpler or similar). Note that this one-holed torus is obtained from identifying two boundary geodesics of a pair of pants with a standard basis $\{a = 19, b = c = 3\}$. The recursive formulae and edge relations will be

$$X = \frac{(xy+z)Y + (xz+y)Z + \sqrt{(x^2+y^2+z^2+2xyz-1)(x^2+Y^2+Z^2+2xYZ-1)}}{x^2-1},$$

$$X_1 + X_2 = \frac{y+z}{x-1}(Y+Z),$$

where $\{x, y, z\} = \{3, 17, 21\}$. Thus, we have three different recursive formulae:

$$X = 9Y + 10Z + 3\sqrt{5(Y^2+Z^2+6YZ+8)},$$

$$X = \frac{Y}{4} + \frac{5Z}{4} + \frac{\sqrt{5(Y^2+Z^2+34YZ+288)}}{12},$$

$$X = \frac{2Y}{11} + \frac{9Z}{11} + \frac{3\sqrt{5(Y^2+Z^2+42YZ+440)}}{55},$$

and three different edge relations:

$$X_1 + X_2 = \frac{3+17}{21-1}(Y+Z) = Y+Z,$$

$$X_1 + X_2 = \frac{3+21}{17-1}(Y+Z) = \frac{3}{2}(Y+Z),$$

$$X_1 + X_2 = \frac{17+21}{3-1}(Y+Z) = 19(Y+Z).$$

Using the recursive formulae, one has $A = 723, B = 37, C = 21$. Let (x, Y, Z) be an arbitrary edge region triple, by induction one can show that

- If $x = 3$ then $Y \equiv Z \pmod 4$.
- If $x \in \{17, 21\}$ then $Y + Z \equiv 0 \pmod 4$.

Combining with the edge relations, one concludes that a one-holed torus with $\{a = 3, b = 17, c = 21\}$ is ortho-integral. ☐

These above results tell us the following:

Corollary 9. *Each of the following Diophantine equations has infinitely many solutions:*

$$X^2 + Y^2 + Z^2 - 3XY - 3YZ - 3XZ = 10$$

$$X^2 + Y^2 + Z^2 - 4XY - 4YZ - 4XZ = 9$$

$$X^2 + 36Y^2 + 55Z^2 - 18XY - 90YZ - 20XZ = 360$$

$$X^2 + 16Y^2 + 33Z^2 - 16XY - 48YZ - 18XZ = 144$$

$$4X^2 + 5Y^2 + 19Z^2 - 10XY - 20YZ - 18XZ = 360$$

$$X^2 + 6Y^2 + 13Z^2 - 6XY - 18YZ - 8XZ = 144$$

Furthermore, there is an algorithm to express a collection of its positive solutions on a tree.

Moreover, by glueing these ortho-integral building blocks together without twisting, one can obtain other ortho-integral surfaces. For example:

Corollary 10. *The four-holed sphere formed by gluing two pairs of pants ($a = b = c = 3$) without twisting is ortho-integral.*

Proof. We choose a standard orthobasis where two arbitrary neighboring hexagons forms an ortho-parallelogram. By this choice, one can observe that this four-holed sphere is iso-orthospectral (up to multiplicity 2) to a one-holed torus with $\{a = 3, b = 17, c = 21\}$. Thus the four-holed sphere is also ortho-integral. □

3.5.3 Infinite (dilogarithm) identities

We now look at a new type of identities due to Bridgeman. Let S be a hyperbolic surface of totally geodesic boundary, the Bridgeman identity [13] on S is as follows:

$$-\frac{\pi^2}{2}\chi(S) = \sum_{\eta} \mathcal{L}\left(\frac{1}{\cosh^2(\ell(\eta)/2)}\right), \tag{3.9}$$

where the sum runs over the set of orthogeodesics on the surface and $\mathcal{L}$ is the Roger's dilogarithm function. Using recursive formula (Section 3.4.2) and/or edge relation (Section 3.4.2), we compute the ortho length spectrum in some special surfaces and then express Basmajian's identity and Bridgeman's identity in each case.

Example 1: We consider a pair of pants with an orthobasis $a = b = c = 3$. The recursive formula in this case is:

$$X = \frac{1}{2}\left(3Y + 3Z + \sqrt{5(Y^2 + Z^2 + 6YZ + 8)}\right).$$

The edge relation will be $X_1 + X_2 = 3(Y + Z)$. This is the case where all the square of the cosh of the half lengths of orthogeodesics are integers $\{2, 10, 32, 90, 122, 242, 362, 450, \dots\}$ with the formula $\overline{X}_1^2 = 3(\overline{Y}^2 + \overline{Z}^2) - \overline{X}_2^2 - 2$. Furthermore, they all are even numbers of the form $10k$ or $10k + 2$. By computing terms of Basmajian's identity (Theorem 6) in this case and manipulating them, we obtain an identity involving the golden ratio:

$$\phi^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 = 2\left(\frac{10}{9}\right)\left(\frac{32}{31}\right)^2\left(\frac{90}{89}\right)^2\left(\frac{122}{121}\right)^2\left(\frac{242}{241}\right)^2\left(\frac{362}{361}\right)^4\left(\frac{450}{449}\right)^2\dots$$

By Bridgeman's identity in Equation 3.9, we have a dilogarithm identity as follows:

$$\frac{\pi^2}{2} = 3\mathcal{L}\left(\frac{1}{2}\right) + 3\mathcal{L}\left(\frac{1}{10}\right) + 6\mathcal{L}\left(\frac{1}{32}\right) + 6\mathcal{L}\left(\frac{1}{90}\right) + 6\mathcal{L}\left(\frac{1}{122}\right) + 6\mathcal{L}\left(\frac{1}{242}\right) + 12\mathcal{L}\left(\frac{1}{362}\right) + \dots$$

Example 2: Similarly, we consider a pair of pants with an orthobasis $a = b = c = 2$. The recursive formula and edge relation in this case will be respectively:

$$X = 2Y + 2Z + \sqrt{3(Y^2 + Z^2 + 4YZ + 3)}; \qquad X_1 + X_2 = 4(Y + Z).$$

This is the case where all the squares of the cosh of the half lengths of orthogeodesics are half integers: $\{\frac{3}{2}, 9, \frac{75}{2}, 144, \frac{363}{2}, \frac{1083}{2}, \dots\}$ with the formula $\overline{X}^2 = 3(\overline{Y}^2 + \overline{Z}^2) - \overline{W}^2 - 2$. By computing terms in Basmajian's identity and manipulating them, we obtain an identity as follows:

$$\left(\frac{1+\sqrt{3}}{2}\right)^2 = \left(\frac{3}{2}\right)\left(\frac{18}{16}\right)\left(\frac{75}{73}\right)^2\left(\frac{288}{286}\right)^2\left(\frac{363}{361}\right)^2\left(\frac{1083}{1081}\right)^2\left(\frac{1443}{1441}\right)^4\left(\frac{1728}{1726}\right)^2\dots$$

By Bridgeman's identity, we have a dilogarithm identity as follows:

$$\frac{\pi^2}{2} = 3\mathcal{L}\left(\frac{2}{3}\right) + 3\mathcal{L}\left(\frac{2}{18}\right) + 6\mathcal{L}\left(\frac{2}{75}\right) + 6\mathcal{L}\left(\frac{2}{288}\right) + 6\mathcal{L}\left(\frac{2}{363}\right) + 6\mathcal{L}\left(\frac{2}{1083}\right) + 12\mathcal{L}\left(\frac{2}{1443}\right) + \dots$$

Note that dilogarithm identities in these above examples differ from those in [12] and [25]. Their terms are arranged over the set of complementary regions of a trivalent tree which can be associated to a Farey sequence as in McShane's identity and other identities [23], [24], [22] involving the set of simple closed geodesics on a once-punctured torus. Figure 3.11 illustrates the two cases.

One can also investigate the set of one-holed tori with a regular orthobasis (see Proposition 4.3 in [34]), then show that there are also two of them having the same ortho length spectra as in the two examples above.

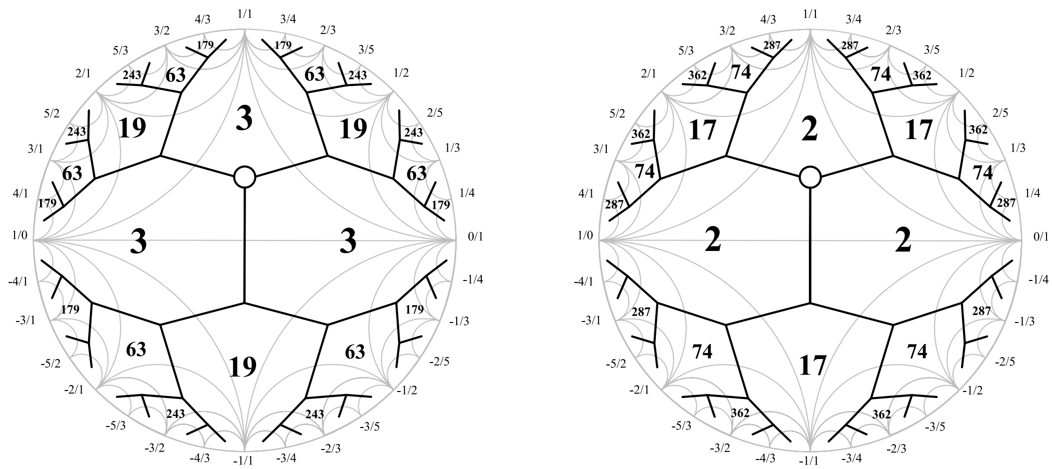

Figure 3.11: $a = b = c \in \{2, 3\}$.

Example 3: Consider a one-holed torus with an orthobasis $a = 3, b = 17, c = 21$. In the proof of Theorem 19, we presented the recursive formulae and edge relations for this case. Therefore, by computing terms in Basmajian's identity and manipulating them, we obtain an identity as follows:

$$\phi^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \left(\frac{3}{2}\right)\left(\frac{10}{9}\right)\left(\frac{11}{10}\right)\left(\frac{19}{18}\right)\left(\frac{32}{31}\right)\left(\frac{41}{40}\right)\cdots$$

By Bridgeman's identity, we have a dilogarithm identity as follows:

$$\frac{\pi^2}{2} = \mathcal{L}\left(\frac{1}{2}\right) + \mathcal{L}\left(\frac{1}{9}\right) + 2\mathcal{L}\left(\frac{1}{10}\right) + 2\mathcal{L}\left(\frac{1}{11}\right) + 2\mathcal{L}\left(\frac{1}{19}\right) + 2\mathcal{L}\left(\frac{1}{32}\right) + 2\mathcal{L}\left(\frac{1}{41}\right) + \ldots$$

Example 4: Description of Bridgeman's identity on a one-holed torus with an orthobasis $a = 37, b = 17, c = 19$:

$$\frac{\pi^2}{2} = \mathcal{L}\left(\frac{1}{9}\right) + \mathcal{L}\left(\frac{1}{10}\right) + 2\mathcal{L}\left(\frac{1}{19}\right) + 2\mathcal{L}\left(\frac{1}{72}\right) + 2\mathcal{L}\left(\frac{1}{82}\right) + 2\mathcal{L}\left(\frac{1}{90}\right) + 2\mathcal{L}\left(\frac{1}{99}\right) + \ldots$$

3.6. Further questions and remarks

3.6.1 Questions following from Section 3.4.3

In the similar vein with theorems in Section 3.4.3, we ask the following question

Question 1. *Is it true that: All r-ortho-isosceles-trapezoids, r-orthokites and r-orthorectangles* *arise from reflection involutions; there is no r-orthorhombus; if S is not a one-holed torus, then any r-orthoparallelogram on S is an r-orthorectangle.*

We come up with a conjecture closely related to the length equivalent problem of closed geodesics on hyperbolic surfaces stated in [1] and [28]:

Conjecture 1. *Two orthogeodesics α and β are length equivalent if and only if there is a finite sequence of reflection involutions $r_1, r_2, ..., r_n$ so that $r_1 \circ r_2 \circ ... \circ r_n(\alpha) = \beta$. In other words, there is a finite sequence of r-ortho-isosceles-trapezoids "connecting" α and β.*

Note that each r-ortho-isosceles-trapezoid consists 6 orthogeodesics which are associated to at most 8 closed geodesics (including also at most two simple closed geodesics at the boundary). Thus the identity relations of orthogeodesics can be translated to trace-type identities of closed geodesics. Also each closed geodesic is a boundary of infinitely many immersed pairs of pants, each of them is associated to an (improper) orthogeodesics (an arc perpendicular to two closed geodesics), thus it would be interesting to study the set of all (improper) orthogeodesics.

3.6.2 Questions following from Section 3.5.2

If a surface is ortho-integral, then each Diophantine equation at each vertex of the orthotree is solvable in integers (X, Y, Z). For example, in the case of a pair of pants with a standard orthobasis $a = b = c = 3$, the vertex relation (see Section 3.4.2) is:

$$X^2 + Y^2 + Z^2 - 3XY - 3YZ - 3XZ = 10.$$

Question 2. *Each orthotree of an ortho-integral surface gives a collection of Diophantine equations ($aX^2 + bY^2 + cZ^2 + dXY + eYZ + fXZ = g$) at its vertices. Is it true that we can find all the solutions (up to signs) of these Diophantine equations on the set of triples of complementary regions surrounding vertices?*

Question 3. *Is it true that surfaces formed by glueing pairs of pants $a = b = c = 3$ (or $a = b = c = 2$) without twisting are ortho-integral. Is this the way to get all ortho-integral surfaces?*

In a combinatorial context, one can ask how to put weights on the set of edges and initial complementary regions of a planar rooted trivalent tree such that by a recursive formula one obtains an integral weight spectrum on the set of complementary regions of the tree?

3.6.3 A relation to the parallelogram rule

It seems there is a connection between the parallelogram rule in the Conway's construction of a topograph of a binary quadratic form (see [17]) and the edge relations. Also, the topograph of the binary quadratic form $B(x,y) = x^2$ is identical to the orthotree of truncated orthogeodesics on a pair of pants with three cusps decorated with horocycles of length 2. Note that the Diophantine equation at each non-root vertex of this orthotree is $X^2+Y^2+Z^2-2XY-2YZ-2XZ=0$. The parallelogram rule also tells us that $B(u+v)$ and $B(u-v)$ are two roots of the equation: $X^2-2(B(u)+B(v))X+B(u+v)B(u-v)=0$. And so $(B(u+v),B(u),B(v))$ and $(B(u-v),B(u),B(v))$ are two roots of the equation:

$$X^2+Y^2+Z^2-2XY-2YZ-2XZ = (B(u)-B(v))^2 - B(u+v)B(u-v).$$

We will show that

Lemma 29. *If $\{u-v,u,v,u+v\}$ is the set of four vectors associated to four regions surrounding an edge in the topograph of $B(x,y)=ax^2+bxy+cy^2$, then*

$$(B(u)-B(v))^2 - B(u+v)B(u-v) = b^2-4ac.$$

Proof. If $u=(x_1,y_1)$ and $v=(x_2,y_2)$ are two vectors associated to two neighboring regions on the topograph, then $|x_1y_2-x_2y_1|=1$. Hence

$$(B(u)-B(v))^2 - B(u+v)B(u-v) = (b^2-4ac)(x_1y_2-x_2y_1)^2 = b^2-4ac.$$

□

By Lemma 29, the values of a binary quadratic form $B(x,y)=ax^2+bxy+cy^2$ at three regions surrounding a vertex in the topograph is a solution of the following equation:

$$X^2+Y^2+Z^2-2XY-2YZ-2XZ = b^2-4ac.$$

It defines a bijection from the set of equivalent classes of binary quadratic forms (two elements are in the same class if they represent the same function on a lattice with two different bases)

$$\{B(x,y)=ax^2+bxy+cy^2 | a,b,c\in\mathbb{R}\}/\sim$$

to the set of equations in the form of Cayley-Menger determinant in Theorem 7:

$$\left\{ \det \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & X/2 \\ 0 & 1/2 & 0 & 1/2 & Y/2 \\ 0 & 1/2 & 1/2 & 0 & Z/2 \\ 0 & X/2 & Y/2 & Z/2 & 0 \end{bmatrix} = \frac{b^2-4ac}{8} \,\middle|\, a,b,c\in\mathbb{R} \right\}. \tag{3.10}$$

Also, by Theorem 7, each orthotree is associated to a subset of equations of the form

$$\left\{ \det \begin{bmatrix} 2 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_1 & 0 & \overline{z}^2 & \overline{y}^2 & \overline{X}^2 \\ \alpha_2 & \overline{z}^2 & 0 & \overline{x}^2 & \overline{Y}^2 \\ \alpha_3 & \overline{y}^2 & \overline{x}^2 & 0 & \overline{Z}^2 \\ \alpha_4 & \overline{X}^2 & \overline{Y}^2 & \overline{Z}^2 & 0 \end{bmatrix} = 0 \,\middle|\, \alpha_i \in \{0,1\} \right\}. \tag{3.11}$$

Thus the equation $X^2 + Y^2 + Z^2 - 2XY - 2YZ - 2XZ = 0$ or equivalently

$$\det \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & X/2 \\ 0 & 1/2 & 0 & 1/2 & Y/2 \\ 0 & 1/2 & 1/2 & 0 & Z/2 \\ 0 & X/2 & Y/2 & Z/2 & 0 \end{bmatrix} = 0,$$

is the bridge connecting the two sets in 3.10 and 3.11.

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