$\mathbb{Z}_2^n$-Generalization of the Schwarz-Voronov embedding
Local $\mathbb{Z}_2^n$-functor of points

Eduardo Ibargüengoytia
University of Luxembourg
eduardoo@ciencias.unam.mx

Abstract

Fully faithful functors give rise to categorical embeddings. In this poster, I present the construction of fully faithful functors from certain locally small categories, $\mathfrak{C}$ into particular subcategories of **Set**. Such construction can be realized for the categories of *super-manifolds* and *$\mathbb{Z}_2^n$-manifolds*, and the target categories are surprisingly familiar and more workable.

Introduction

Start with a locally small category $\mathfrak{C}$, and consider, for a given object $M \in \mathfrak{C}$ the (contravariant) Hom functor:

$$\begin{aligned} \mathfrak{C} &\longrightarrow \mathbf{Set} \\ A &\longmapsto \mathrm{Hom}(A, M) \\ (\varphi : A \to B) &\longmapsto (- \circ \varphi : \mathrm{Hom}(B, M) \to \mathrm{Hom}(A, M)) \end{aligned}$$

This defines a functor from $\mathfrak{C}$ to the category of functors between the categories $\mathfrak{C}^{op}$ and **Set**, $[\mathfrak{C}^{op}, \mathbf{Set}]$, given by:

$$\begin{aligned} \mathfrak{C} &\longrightarrow [\mathfrak{C}^{op}, \mathbf{Set}] \\ M &\longmapsto \mathrm{Hom}(-, M) \\ (\varphi : M \to N) &\longmapsto \mathrm{Hom}(-, \varphi) \in \mathbf{Nat}(\mathrm{Hom}(-, M), \mathrm{Hom}(-, N)). \end{aligned}$$

The latter is referred to as *The functor of points of M*, and provides the *Yoneda embedding* of the category $\mathfrak{C}$.

Now, let $\mathfrak{S} = \{S_i \in \mathrm{Obj}(\mathfrak{C}) | \ i \in I\}$, be a generating set of $\mathfrak{C}$. This is equivalent to say that we consider a subset of the objects of $\mathfrak{C}$ for which the restricted functor of points

$$\mathfrak{C} \longrightarrow [\mathfrak{S}^{op}, \mathbf{Set}]$$

is still faithful.

Examples

- The category of $\mathbb{R}$-algebras has as generating set the object $\mathbb{R}$. Then, the restricted Hom functor,

  $$A \mapsto \mathrm{Hom}(\mathbb{R}, A) \cong A$$

  has as a target only objects inside the category of $\mathbb{R}$-algebras. The bijection above is in principle, just a bijection as sets, though, the topological and algebraic structures are induced.

  N.b. *The restricted functor of points,*

  $$\begin{aligned} \mathbb{R}\mathrm{Alg} &\longrightarrow [\mathbb{R}^{op}, \boldsymbol{Set}] \\ A &\longmapsto \mathrm{Hom}(-, A) \end{aligned}$$

  *Is faithful, and, in this particular case, it is possible to consider the target category to be $\mathbb{R}$**Alg**, (i.e. one chooses natural transformations to be algebra morphisms), so to get a fully faithful functor, and eventually an equivalence of categories.*

  $$\begin{aligned} \mathbb{R}\mathrm{Alg} &\longrightarrow [|\mathbb{R}^{op}, \mathbb{R}\boldsymbol{Alg}|] \\ A &\longmapsto \mathrm{Hom}(-, A). \end{aligned}$$

- The category of topological spaces, **Top**, has as generating object the point $\{*\}$. The Hom functor,

  $$X \mapsto \mathrm{Hom}(*, X) \cong X$$

  Restricting the possible natural transformations, from being set theoretical maps, to topological homomorphisms, one gets:

  $$\begin{aligned} \mathbf{Top} &\longrightarrow [|*^{op}, \mathbf{Top}|] \\ X &\longmapsto \mathrm{Hom}(-, X). \end{aligned}$$

  is an equivalence of categories.

- (Milnor exercise) The last two examples point out that, in the category of manifolds (Thought of as Locally ringed spaces), the functor,

  $$\begin{aligned} \mathbf{Man} &\longrightarrow [|(*^{op}, \mathbb{R}), \mathbf{Man}|] \\ (M, \mathcal{O}_M) &\longmapsto \mathrm{Hom}(-, M). \end{aligned}$$

  Is an equivalence of categories.

Question:

Does there always exists a subcategory of **Set**, $\mathfrak{D}$, for which the restricted functor of points

$$\mathfrak{C} \longrightarrow [|\mathfrak{S}^{op}, \mathfrak{D}|]$$

is fully faithful.

$\mathbb{Z}_2^n$-Geometry

Let $\mathcal{A}$ be a $\mathbb{Z}_2^n$-graded $\mathbb{R}$-algebra $\mathcal{A} = \bigoplus_{\gamma^i \in \mathbb{Z}_2^n} A^{\gamma^i}$, and, define, for homogeneous elements $\mathcal{A}^{\gamma^i} \ni f, g \in \mathcal{A}^{\gamma^j}$ a commutation factor,

$$f \cdot g = (-1)^{\langle \gamma^i, \gamma^j \rangle} g \cdot f$$

This makes of $\mathcal{A}$ a $\mathbb{Z}_2^n$-graded commutative algebra.

A $\mathbb{Z}_2^n$-supermanifold is a locally $\mathbb{Z}_2^n$-supercommutative ringed space, locally modeled on the algebra

$$C^\infty(U)[[\xi^{\mu j}]].$$

Note: For $n = 2$,

$$C^\infty(\mathbb{R}^p)[\xi^1, \xi^2][[y]]$$

is the global algebra of the affine space $\mathbb{R}^{p|\mathbf{1}}$. Notice how non nilpotent, even, formal parameters appear in the $\mathbb{Z}_2^n$ commutative context; for this, it turns out to be necessary to consider **formal power series in such formal generators.**

Functor of points

The category $\mathbb{Z}_2^n$**Man** is locally small. The Yoneda embedding gives rise to the category of generalized $\mathbb{Z}_2^n$-supermanifolds.

Based on the result by $Schwarz$ and $Voronov$, we see how, for **SupMan**, the generating set to consider is the subset of objects composed by supermanifolds with base space being just a point.

$$\boldsymbol{\Lambda} := \{(*, \mathbb{R}[\xi^1, \cdots, \xi^i]) \ | \ i \in \mathbb{N}\}$$

In such context of supergeometry, the result spells:

Theorem 0.1. *The restricted functor of points,*

$$\begin{aligned} \mathbf{SupMan} &\longrightarrow [|\boldsymbol{\Lambda}, \Lambda_0\text{-}\boldsymbol{Man}|] \\ (M, \mathcal{O}_M) &\longmapsto \mathrm{Hom}(-, M). \end{aligned}$$

is fully faithful.

Notes:

1. The field is, in a natural way, extended to Λ_0 *scalars*, so, $\mathbb{R}$-linearity of the differential of a smooth morphism has to be sustitued by Λ_0-linearity.

   $$d_x(a \cdot v) = a \cdot d_x(v) \quad \forall\, x \in M(\Lambda), \ v \in T_x M(\Lambda), \ a \in \Lambda_0$$

2. Nilpotency of the formal generators assures that, if the supermanifold chosen is finite dimensional, then the Λ_0-manifold will be finite dimensional as well.

Our result

The generating set to consider in this setting is, again, all $\mathbb{Z}_2^n$-manifolds with base space made up of a point $(*, \Lambda_{\mathbb{Z}_2^n})$. This set of objects forms a category, labeled by the **global algebras**, which are generalized $\mathbb{Z}_2^n$**-commutative grassmann algebras**

$$\Lambda_{\mathbb{Z}_2^n} = \mathbb{R}[\xi^I][[\psi^J]]$$

Since local sections are now formal power series, the objects in the target category, **Set**, are infinite dimensional manifolds; but a nice class of such, in which calculus is well defined (Fréchet manifolds). Then:

Theorem 0.2. *The restricted functor of points,*

$$\begin{aligned} \mathbb{Z}_2^n\text{-}\mathbf{SupMan} &\longrightarrow [|\boldsymbol{\Lambda}_{\mathbb{Z}_2^n}, \Lambda_{\mathbb{Z}_{2\,0}^n}\text{-}\boldsymbol{NFMan}|] \\ (M, \mathcal{O}_M) &\longmapsto \mathrm{Hom}(-, M). \end{aligned}$$

is fully faithful. And, restricting the objects to representable ones, the functor

$$\begin{aligned} \mathbb{Z}_2^n\text{-}\mathbf{SupMan} &\longrightarrow [|\boldsymbol{\Lambda}_{\mathbb{Z}_2^n}, \Lambda_{\mathbb{Z}_{2\,0}^n}\text{-}\boldsymbol{NFMan}|]_{rep} \\ (M, \mathcal{O}_M) &\longmapsto \mathrm{Hom}(-, M). \end{aligned}$$

is an equivalence of categories

https://arxiv.org/abs/1906.09834

A. Bruce, E. Ibargüengoytia, N. Poncin,
The Schwarz-Voroonov embedding of $\mathbb{Z}_2^n$-manifolds. (To appear in SIGMA)