FROM GRAVITY TO STRING TOPOLOGY

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ABSTRACT. The chain gravity properad introduced in [Me2] acts on the cyclic Hochschild of any cyclic A_∞ algebra equipped with a scalar product of degree $-d$. In particular, it acts on the cyclic Hochschild complex of any Poincare duality algebra of degree d, and that action factors through a quotient dg properad $\mathcal{ST}_{3-d}$ of ribbon graphs which is in focus of this paper. We show that its cohomology properad $H^\bullet(\mathcal{ST}_{3-d})$ is highly non-trivial and that it acts canonically on the reduced equivariant homology $\bar{H}^{S^1}_\bullet(LM)$ of the loop space of any simply connected d-dimensional closed manifold M. By its very construction, the string topology properad $H^\bullet(\mathcal{ST}_{3-d})$ comes equipped with a morphism from the gravity properad $\mathcal{G}rav_{3-d}$ which is fully determined by the compactly supported cohomology of the moduli spaces $\mathcal{M}_{g,n}$ of stable algebraic curves of genus g with marked points. This result gives rise to new universal operations in string topology as well as reproduces in a unified way several known constructions: we show that (i) $H^\bullet(\mathcal{ST}_{3-d})$ is also a properad under the properad of involutive Lie bialgebras $\mathcal{L}ieb^\diamond_{3-d}$ whose induced (via $H^\bullet(\mathcal{ST}_{3-d})$) action on $\bar{H}^{S^1}_\bullet(LM)$ agrees precisely with the famous purely geometric construction of M. Chas and D. Sullivan [CS1, CS2], (ii) $H^\bullet(\mathcal{ST}_{3-d})$ is a properad under the properad of *homotopy* involutive Lie bialgebras $\mathcal{H}olieb^\diamond_{2-d}$ which controls (via $H^\bullet(\mathcal{ST}_{3-d})$) four universal string topology operations introduced in [Me1], (iii) E. Getzler's gravity operad *injects* into $H^\bullet(\mathcal{ST}_{3-d})$ implying a purely algebraic counterpart of the geometric construction of C. Westerland [We] establishing an action of the gravity operad on $\bar{H}^{S^1}_\bullet(LM)$.

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CONTENTS

1. Introduction

This paper aims to show new results in a part of string topology which deals with universal operations on the (reduced) equivariant equivariant homology $H^{S^1}_\bullet(LM)$ of the free loop space LM of a connected and simply-connected closed d-dimensional manifold M. A large family of operations on $H^{S^1}_\bullet(LM)$ has been found already in the fundamental work [CS1] by M. Chas and D. Sullivan; later it was shown by C. Westerland in

[We] that these operations provide us with a representation of the gravity operad $grav_{3-d}$ introduced by E. Getzler in [Ge].

M. Chas and D. Sullivan found a geometric construction [CS2] of the involutive Lie bialgebra structure on the reduced equivariant equivariant homology $\bar{H}^{S^1}_\bullet(LM)$ of the free loop space a closed d-dimensional manifold; in the properadic language adopted in this paper, their construction gives us a representation of the properad $\mathcal{L}ieb^\diamond_{3-d}$ on $\bar{H}^{S^1}_\bullet(LM)$ (see §2 below for details). In the case $d=2$ their construction reproduces the famous Goldman-Turaev Lie bialgebra structure [Go, T] on the reduced equivariant homology of free loop spaces of Riemann surfaces; the latter Lie bialgebra structure can be lifted to the full homology (with constant loops including) and leads us to the beautiful formality theory of the geometric and purely algebraic counterparts of this structure which has been developed by A. Alekseev, N. Kawazumi, Y. Kuno, G. Massuyeau and F. Naef in [AKKN1, AKKN2, AN, Ma].

A nice purely algebraic construction of the $\mathcal{L}ieb^\diamond_{3-d}$-algebra structure on $\bar{H}^{S^1}_\bullet(LM)$ for simply connected closed manifolds M has been found by X. Chen, F. Eshmatov and W. L. Gan in [CEG]. It was further studied by F. Naef and T. Willwacher in [NW] who showed, in particular, that this purely algebraic construction gives us the same operations as the geometric one; moreover, they extended it to not necessarily simply connected manifolds M using a partition function Z_M introduced earlier in [CW]. Props of directed ribbon graphs of various types have been introduced and studied in [TZ] with the purpose to get new string topology operations from higher homotopy products on Hochschild (co)homologies of cyclic $\mathcal{A}ss_\infty$ algebras.

In this paper we study string topology applications of the gravity chain properad $\mathcal{C}h\mathcal{G}rav_d$ introduced by the author in [Me2]; it is generated by ribbon graphs with vertices of two types, white vertices which are labelled and black vertices which are unlabelled and are assigned the cohomological degree d; the differential in $\mathcal{C}h\mathcal{G}rav_d$ creates a new black vertex (see §2 below for full details). As explained in §3, this dg properad acts (almost by its very construction) on the Hochschild complex $Cyc(A^*[-1])$ of any cyclic $\mathcal{A}ss_\infty$ algebra A equipped with a degree $-d$ scalar product. The properad $\mathcal{C}h\mathcal{G}rav_d$ has interesting cohomology,

$$(1)\qquad \mathcal{G}rav_d := H^\bullet(\mathcal{C}h\mathcal{G}rav_d) \simeq \left\{ \prod_{\substack{g\geq 0\\ 2g+m+n\geq 3}} H_c^{\bullet - m + d(2g-2+m+n)}(\mathcal{M}_{g,m+n}) \otimes sgn_m \right\}_{m\geq 1, n\geq 0}$$

which is called the *gravity properad* and which is fully determined by the compactly supported cohomology of moduli spaces $\mathcal{M}_{g,m+n}$ of algebraic curves with $m+n$ marked points (m of them are called *out*-points and n of them are called *in* points). Moreover, the gravity properad $\mathcal{G}rav_d$ contains E. Getzler's gravity operad

$$(2)\qquad grav_d := \{H_c^{\bullet+(3-d)(n-1)-1}(\mathcal{M}_{0,1+n})\}_{n\geq 2}$$

as a sub-properad, and the latter is known to play a role in string topology [We]. Our main purpose in this paper is to study string topology applications of its properadic extension $\mathcal{G}rav_d$, more precisely, of its chain version $\mathcal{C}h\mathcal{G}rav_d$ which is more informative and useful in this respect.

Given a Poincaré duality algebra A in degree d, there is a canonical representation of the chain gravity properad (see §3 for full details),

$$\rho_A : \mathcal{C}h\mathcal{G}rav_{3-d} \longrightarrow \mathcal{E}nd_{\overline{Cyc}(\bar{A}^*[-1])}.$$

in the reduced cyclic Hochschild of A,

$$\overline{Cyc}(\bar{A}^*[-1]) \simeq \bigoplus_{k\geq 1} \left(\otimes^k(\bar{A}^*[-1])\right)_{\mathbb{Z}_k^*}.$$

Our interest in this class of representations stems from the fact that if A is a Poincaré duality model of a connected d-dimensional closed manifold M, then there is a linear map

$$\bar{H}^{S^1}_\bullet(LM) \longrightarrow H^\bullet(\overline{Cyc}(\bar{A}^*[-1])$$

from the reduced equivariant equivariant homology $\bar{H}^{S^1}_\bullet(LM)$ of the free loop space LM of M into the cyclic Hochschild cohomology. Moreover, if M is simply connected, this map is an isomorphism (moreover, in this case a Poincaré duality model A for M always exists [LS]) so that the gravity properad $\mathcal{G}rav_d$ acts canonically on $\bar{H}^{S^1}_\bullet(LM)$.

We study in §4 the joint kernel of the above class of representation maps ρ_A for all possible A which leads us to an observation that any such representation ρ_A factors through the canonical epimorphism

$$\mathcal{C}h\mathcal{G}rav_d \longrightarrow \mathcal{ST}_d$$

to a *chain string topology properad* $\mathcal{ST}_d$ which is defined as the quotient of $\mathcal{C}h\mathcal{G}rav_d$ by the differential closure of the ideal generated by ribbon graphs with at least one black vertex of valency ≥ 4 or with at least one boundary made of black vertices only. The cohomology properad $H^\bullet(\mathcal{ST}_{3-d})$ acts therefore on $\bar{H}^{S^1}_\bullet(LM)$ for any simply-connected closed d-dimensional manifold M. Our next purpose in this paper is to show that the string topology properad $H^\bullet(\mathcal{ST}_d)$ is very non-trivial, and to establish its connection to known results in string topology.

In §4 we prove that there is

(i) a non-trivial morphism of properads

$$\mathcal{L}ieb^\diamond_d \longrightarrow H^\bullet(\mathcal{ST}_d)$$

which induces, via the above action of $H^\bullet(\mathcal{ST}_{3-d})$ on $\bar{H}^{S^1}_\bullet(LM)$, the Chas-Sullivan involutive Lie bialgebra structure;

(ii) an injection of the gravity operad

$$grav_d \longrightarrow H^\bullet(\mathcal{ST}_d)$$

which gives a purely algebraic counterpart of the geometric construction in [We] establishing an action of $grav_{3-d}$ of $\bar{H}^{S^1}_\bullet(LM)$;

(iii) a morphism of dg properads

$$\mathcal{H}olieb^\diamond_{d-1} \longrightarrow H^\bullet(\mathcal{ST}_d)$$

which is non-trivial on the quartette of *homotopy* involutive Lie bialgebra bialgebra operations (22) found earlier in [Me2] using a very different technique of ribbon *hyper*graphs.

Thus the string topology properad $H^\bullet(\mathcal{ST}_d)$ gives us both well- and less known universal string topology operations on the equivariant cohomology $\bar{H}^{S^1}_\bullet(LM)$ of free loop spaces of simply connected closed manifolds in a unified way. It is an open problem to compute all the cohomology $H^\bullet(\mathcal{ST}_d)$ explicitly, as well as the cohomology of the deformation complex of the morphism (i) which is a Lie algebra under the cohomology Lie algebra of the mysterious even M. Kontsevich graph complex [K, Wi].

1.1. Notation. The set $\{1, 2, \ldots, n\}$ is abbreviated to $[n]$; its group of automorphisms is denoted by $\mathbb{S}_n$. The cyclic subgroup of $\mathbb{S}_n$ generated by the permutation $(12...n)$ is denoted by $\mathbb{Z}^*_n$. The trivial (resp., sign) one-dimensional representation of $\mathbb{S}_n$ is denoted by $\mathbb{1}_n$ (resp. sgn_n). The cardinality of a finite set I is denoted by $\#I$. A linear span of a set I over a field $\mathbb{K}$ is denoted by $\mathbb{K}\langle I\rangle$.

We work throughout in the category of $\mathbb{Z}$-graded vector spaces over a field $\mathbb{K}$ of characteristic zero. If $V = \oplus_{i\in\mathbb{Z}} V^i$ is a graded vector space, then $V[k]$ stands for the graded vector space with $V[k]^i := V^{i+k}$ and and s^k for the associated isomorphism $V \to V[k]$; for $v \in V^i$ we set $|v| := i$. The endomorphism properad[1] of V is denoted by $\mathcal{E}nd_V$.

All our complexes have differential of degree +1. In particular, the cohomology group, $H^\bullet(M)$, of a topological space M is non-negatively graded as usual, while its homology group $H_\bullet(M)$ is *non-positively* graded (so that both spaces are dual to each other as $\mathbb{Z}$-graded spaces).

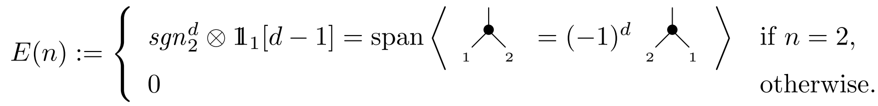
[1] For a nice introduction into the theory of props and properads we refer to the paper [V] by B. Vallette.

2. A brief introduction into the (chain) gravity properad

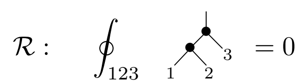
2.1. T. Willwacher's twisting endofunctor. For any $d \in \mathbb{Z}$ the operad of degree d shifted Lie algebras is, by definition, the quotient

$$\mathcal{L}ie_d := \mathcal{F}ree\langle E\rangle/\langle \mathcal{R}\rangle,$$

of the free operad generated by an $\mathbb{S}$-module $E = \{E(n)\}_{n\geq 2}$



modulo the ideal generated by the following relation[2]



This operad controls dg Lie algebras (V, d) with the Lie bracket $[\ ,\ ]$ of degree $1 - d$. Hence the case $d = 1$ corresponds to the ordinary operad of Lie algebras. As usual, Maurer-Cartan elements of a $\mathcal{L}ie_d$-algebra V are defined as degree d elements $\gamma \in V$ satisfying the equation

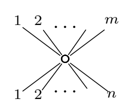
$$d\gamma + \frac{1}{2}[\gamma, \gamma] = 0. \tag{3}$$

Given a dg properad $(\mathcal{P} = \{\mathcal{P}(m, n)\}_{m,n\geq 0}, \partial)$ under $\mathcal{L}ie_d$, that is, a dg properad equipped with a morphism

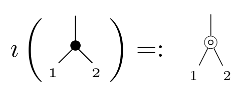
$$\imath : \mathcal{L}ie_d \longrightarrow \mathcal{P}, \tag{4}$$

there is [Wi] an associated dg properad $(\mathsf{tw}\mathcal{P}, \delta)$ which is generated by $\mathcal{P}$ and one extra generator with no inputs and one output; we represent this special element pictorially as a corolla ↓ with black vertex and assign to it the cohomological degree d. Generic elements a of $\mathcal{P}(m, n)$ can also identified with (m, n)-corollas

(5)

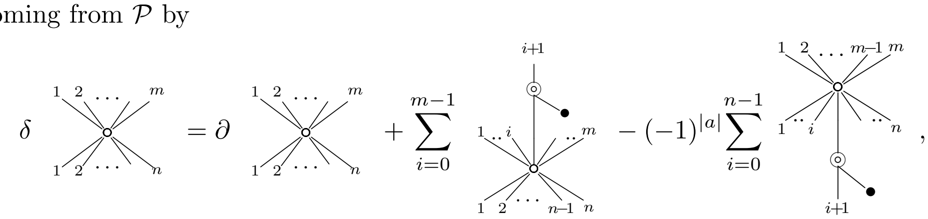


whose (say, white) vertex is decorated by a; then properadic compositions in $\mathcal{P}$ can be represented pictorially by gluing out-legs of such decorated corollas to in-legs of another decorated corollas. Thus the properad $\mathsf{tw}\mathcal{P}$ is freely generated by white corollas (5) and one distinguished $(1, 0)$ corolla ↓. The map (4) sends the generator of $\mathcal{L}ie_d$ into a distinguished degree $1 - d$ element of $\mathcal{P}(1, 2)$,



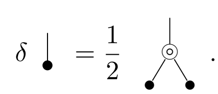
which is represented as a $(1, 2)$-corolla with the vertex denoted by $\circledcirc$; it is a cycle in the complex $(\mathcal{P}, \partial)$. Following [Wi] one equips the extended properad $\mathsf{tw}\mathcal{P}$ with a twisted differential δ which is defined on the generators coming from $\mathcal{P}$ by

(6)



and on the extra generator by

(7)



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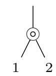
[2] When representing elements of operads and props as decorated graphs we tacitly assume that all edges and legs are *directed* along the flow going from the bottom of the graph to the top.

The action of the element $\sum_{k=1}^{3}(123)^k \in \mathbb{K}[\mathbb{S}_3]$ on any element a of an $\mathbb{S}_3$-module is denoted by $\oint_{123} a$.

This construction gives us the *twisting* endofunctor

$$\mathsf{tw} : (\mathcal{P}, \partial) \longrightarrow (\mathsf{tw}\mathcal{P}, \delta)$$

in the category of properads under $\mathcal{L}ie_d$ which has been introduced first by Thomas Willwacher in [Wi]. This twisting endofuctor has many nice properties and applications which have been studied in [Wi, DW, DSV].

The distinguished cycle  in $(\mathcal{P}, \partial)$ remains a cycle as an element of $(\mathsf{tw}\mathcal{P}, \delta)$ so that the original morphism (4) extends to the twisted version,

$$\imath : \mathcal{L}ie_d \longrightarrow \mathsf{tw}\mathcal{P},$$

and is given on the generator of $\mathcal{L}ie_d$ by the same formula.

2.1.1. From MC elements to representations of $\mathsf{tw}\mathcal{P}$ [Wi]**.** Assume we have a representation

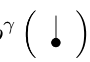
$$\rho : \mathcal{P} \to \mathcal{E}nd_V,$$

of a dg properad $\mathcal{P}$ under $\mathcal{L}ie_d$. Composing ρ with the map (4) one makes the dg vector space (V, d) into a $\mathcal{L}ie_d$-algebra so that it makes sense to talk about the Maurer-Cartan elements (3) in V. Given such an element $\gamma \in V$, there is

(i) an associated twisted differential $d_\gamma = d + [\gamma, \ ]$ in V,
(ii) an associated representation of the twisted properad $\mathsf{tw}\mathcal{P}$ in the dg vector space (V, d_γ),

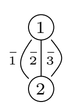
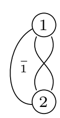
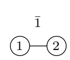
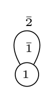
$$\rho^\gamma : \mathsf{tw}\mathcal{P} \longrightarrow \mathcal{E}nd_V$$

which coincides with ρ on the generators coming from $\mathcal{P}$ while on the extra generator one has

ρ^γ  $:= \gamma \in \mathcal{E}nd_V(1,0) \equiv V.$

This construction motivates the main idea of the twisting endofunctor and the terminology.

2.2. Properad of ribbon graphs and involutive Lie bialgebras. Let $\mathcal{RG}ra_d = \{\mathcal{RG}ra_d(m,n)_{m,n\geq 1}\}$ be the properad of ribbon graphs introduced in [MW1]. The $\mathbb{S}_m^{op} \times \mathbb{S}_n$-module $\mathcal{RG}ra_d(m,n)$ is generated by ribbon graphs Γ with n labelled vertices and m labelled boundaries[3] as, for example, the following ones,

 $\in \mathcal{RG}ra_d(3,2)$  $\in \mathcal{RG}ra_d(1,2)$,  $\in \mathcal{RG}ra(1,2)$,  $\in \mathcal{RG}ra_d(2,1)$.

Such graphs are assigned the cohomological degree

$$|\Gamma| := (1-d)\#E(\Gamma)$$

and are equipped with an *orientation* which is defined by

(i) the ordering of edges of Γ (up to the sign action of $\mathbb{S}_{\#E(\Gamma)}$) for d even;
(ii) the choice of a direction on each edge (up to the sign action of $\mathbb{S}_2$) for d odd.

The properadic compositions are given by substituting a vertex v of one ribbon graph into a boundary b of another one and redistributing the edges attached to v (if any) among the vertices belonging to b in all possible ways while respecting the cyclic orderings (see §4 in [MW1] for full details).

A remarkable property of $\mathcal{RG}ra_d$ is that it comes equipped with a canonical morphism of properads

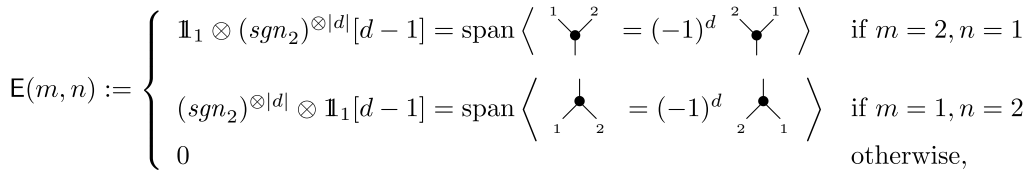
$$i : \mathcal{L}ieb_d \longrightarrow \mathcal{RG}ra_d \tag{8}$$

where $\mathcal{L}ieb_d$ is the properad of (degree shifted) *Lie bialgebras* defined as the quotient of the free properad $\mathcal{F}ree(\mathsf{E})$,

$$\mathcal{L}ieb_d := \mathcal{F}ree\langle \mathsf{E}\rangle / \langle \mathsf{R}\rangle,$$

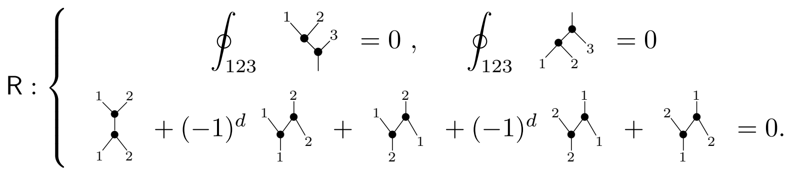
[3] For a ribbon graph Γ we denote by $V(\Gamma)$ its set of vertices, $B(\Gamma)$ its set of boundaries and by $E(\Gamma)$ its set of edges. The genus of Γ is defined by $g = 1 + \frac{1}{2}\left(\#E(\Gamma) - \#V(\Gamma) - \#B(\Gamma)\right)$.

generated by an $\mathbb{S}$-bimodule



by the ideal generated by the following relations

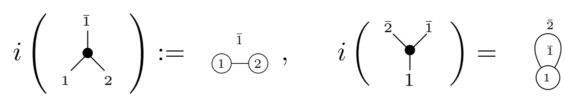
(9)



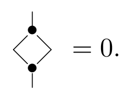
The case $d = 1$ corresponds to the standard notion of Lie bialgebra introduced by V. Drinfeld in [D].

The morphism (8) is given on the generators by [MW1]

(10)



The properad $\mathcal{L}ieb_d^\diamond$ of *involutive* Lie bialgebras is the quotient of $\mathcal{L}ieb_d$ by the ideal generated by the following relation



The map (8) factors through the canonical projection

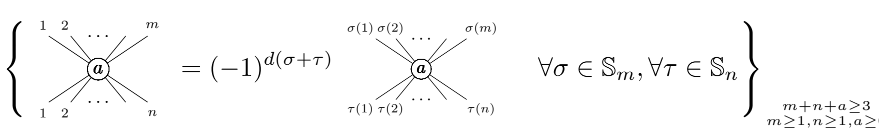
$$\mathcal{L}ieb_d \longrightarrow \mathcal{L}ieb_d^\diamond$$

so that the formulae (10) define in fact a canonical morphism (denoted by the same symbol i) [MW1]

$$i : \mathcal{L}ieb_d^\diamond \longrightarrow \mathcal{RG}ra_d.$$

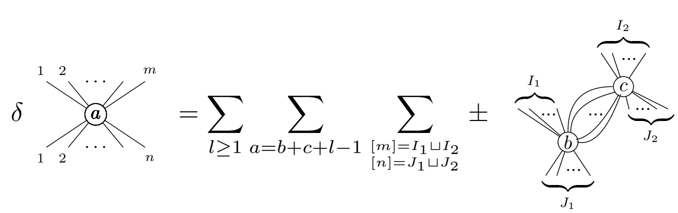
The morphism i was used in [MW1] to built several ribbon graph complexes with interesting cohomology; in this paper we consider its applications in string topology. The minimal resolution of $\mathcal{L}ieb_d^\diamond$ has been constructed in [CMW] and is denoted by $\mathcal{H}olieb_d^\diamond$; this is a free properad generated by the following family of (skew)symmetric corollas of degrees $1 - d(m + n + 2a - 2)$,

(11)



The differential is given by

(12)

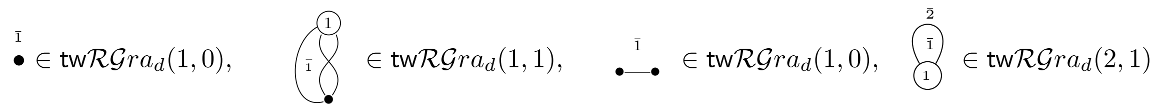


where the summation parameter l counts the number of internal edges connecting the two vertices on the r.h.s.

2.3. Gravity properad. The morphism (8) says, in particular, that $\mathcal{RG}ra_d$ is a properad under $\mathcal{L}ie_d$ and hence can be twisted as explained in §2.1. The twisted properad $\mathsf{tw}\mathcal{RG}ra_d = \{\mathsf{tw}\mathcal{RG}ra_d(m,n)\}_{m\geq 1, n\geq 0}$ is generated by ribbon graphs Γ with all boundaries labelled and with vertices of two types — the white vertices which are labelled, and the black vertices which are unlabelled but assigned the cohomological degree d; their orientation is defined as in the case of ribbon graphs from $\mathcal{RG}ra_d$ except that for d odd one has to choose in addition an ordering (defined up to the sign action of the permutation group) of black vertices. A ribbon graph $\Gamma \in \mathsf{tw}\mathcal{RG}ra_d(m,n)$ has m labelled boundaries, n labelled white vertices and any finite number $k \geq 0$ of black vertices; its cohomological degree is defined by

$$|\Gamma| = (1-d)\#E(\Gamma) + kd$$

For example,



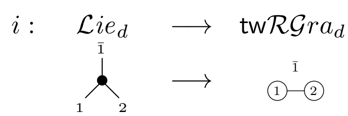
The first graph in this list is precisely the ribbon graph incarnation of the extra $(1,0)$-generator which — in the decorated corolla notation — is represented in §2.1 as $\downarrow$. Substituting the unique boundary of the ribbon graph $\bullet$ into a white vertex of graphs from $\mathcal{RG}ra_d$ one obtains generic elements of $\mathsf{tw}\mathcal{RG}ra_d$ as described above. Given two ribbon graphs $\Gamma_1, \Gamma_2 \in \mathsf{tw}\mathcal{RG}ra_d$, we denote by $\Gamma_1\, {}_i\circ_j\, \Gamma_2$ their partial properadic composition given by substituting the j-th boundary of Γ_2 into the i-th white vertex of Γ_1. The set of black (resp., white) vertices of $\Gamma \in \mathsf{tw}\mathcal{RG}ra_d$ is denoted by $V_\bullet(\Gamma)$ (resp., $V_\circ(\Gamma)$).

As the properad $\mathcal{RG}ra_d$ has trivial differential, the induced differential δ in $\mathsf{tw}\mathcal{RG}ra_d$ is completely determined by the last two summands in (6) which in the ribbon graph incarnation give us the following explicit formula

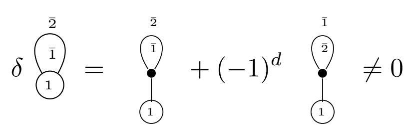
$$\delta\Gamma := \sum_{i=1}^{m} \overset{\bullet}{\underset{①}{|}}\ {}_1\circ_j \Gamma \ - \ (-1)^{|\Gamma|}\sum_{j=1}^{n} \Gamma\ {}_j\circ_1\ \overset{\bullet}{\underset{①}{|}} \ - (-1)^{|\Gamma|}\frac{1}{2}\sum_{v\in V_\bullet(\Gamma)} \Gamma \circ_v (\bullet\!\!-\!\!\bullet) \tag{13}$$

where the symbol $\Gamma \circ_v (\bullet\!\!-\!\!\bullet)$ means substituting the unique boundary of the ribbon graph $\bullet\!\!-\!\!\bullet$ into the black vertex v of the graph Γ and then taking the sum over all possible re-attachments of the half-edges attached earlier to v among the two newly created black vertices in a way which respects the cyclic orderings. Note that if Γ has no univalent black vertices, then $\delta\Gamma$ has no univalent black vertices as well (they all cancel out when applying the above formula).

By the general rule, the twisted properad $\mathsf{tw}\mathcal{RG}ra_d$ comes equipped with a morphism



induced by the above morphism (8) from $\mathcal{L}ie_d$ to $\mathcal{RG}ra_d$. However that morphism does not extend to a morphism from $\mathcal{L}ieb_d$ to $\mathsf{tw}\mathcal{RG}ra_d$ as the image of the second generator of $\mathcal{L}ieb_d$ is no more a cocycle [Me2],

(14) 

Nevertheless the cohomology properad $H^\bullet(\mathsf{tw}\mathcal{RG}ra_d, \delta)$ is proven in [Me2] to be a properad under the properad of degree shifted *quasi* Lie bialgebras; remarkably, the latter properad takes care about *infinitely many* cohomology classes in $H^\bullet(\mathsf{tw}\mathcal{RG}ra_d, \delta)$ (see §3.9 in [Me2]). We do not use this fact in this paper and hence omit any details.

One of the main results in [Me2] is the computation of the cohomology properad $H^\bullet(\mathsf{tw}\mathcal{RG}ra_d, \delta)$ using the remarkable geometric K. Costello's theory [C1, C2] of partially compactified moduli spaces $\overline{\mathcal{N}}_{g,m,0,n}$ of connected stable Riemann surfaces Σ of genus g with m marked boundary components and n marked points in the interior $\Sigma \setminus \partial\Sigma$ and with possible nodes on the boundary $\partial\Sigma$. To formulate it precisely we notice that the differential δ in each complex $\mathsf{tw}\mathcal{RG}ra_d(m,n)$ preserves the genus of ribbon graphs so that the latter decomposes into a direct sum,

$$\mathsf{tw}\mathcal{RG}ra_d(m,n) = \bigoplus_{g\geq 0} \mathsf{tw}\mathcal{RG}ra_d(g;m,n)$$

where the subcomplex $\mathsf{tw}\mathcal{RG}ra_d(g;m,n)$ is spanned by ribbon graphs of genus g. Let $\mathcal{M}_{g,N}$ stand for the moduli space of algebraic curves Σ of genus g with $N \geq 1$ marked points. The group $\mathbb{S}_N$ acts on the compactly supported cohomology group $H_c^\bullet(\mathcal{M}_{g,N})$ by relabeling. Assume that the set $S \subset \Sigma$ of N marked points in an algebraic curve Σ is split into the disjoint union of subsets of (so called) "in"- and "out" points,

$$S = S_{in} \sqcup S_{out}$$

which have cardinalities $n := \#S_{in} \geq 0$ and $m := \#S_{out} \geq 1$, and denote the moduli space of such algebraic curves by $\mathcal{M}_{g,m+n}$; its compactly supported cohomology group $H_c^\bullet(\mathcal{M}_{g,m+n})$ is naturally an $\mathbb{S}_m^{op}\times\mathbb{S}_n$-module. The tensor product

$$H_c^\bullet(\mathcal{M}_{g,m+n}) \otimes sgn_m$$

is also an $\mathbb{S}_m^{op} \times \mathbb{S}_n$-module with group $\mathbb{S}_m^{op}$ acting, by definition, diagonally.

2.3.1. Theorem [Me2]**.** (i) *For any $g \geq 0$, $m \geq 1$ and $n \geq 0$ with $2g+m+n \geq 3$ one has an isomorphism of $\mathbb{S}_m^{op} \times \mathbb{S}_n$-modules,*

$$H^\bullet(\mathsf{tw}\mathcal{RG}ra_d(g;m,n)) = H_c^{\bullet-m+d(2g-2+m+n)}(\mathcal{M}_{g,m+n}) \otimes sgn_m \simeq H_c^{\bullet+d(2g-2+m+n)}(\mathcal{M}_{g,m+n}\times\mathbb{R}^m)$$

(ii) *For any $g \geq 0$, $m \geq 1$ and $n \geq 0$ with $2g+m+n < 3$ one has*

$$H^k(\mathsf{tw}\mathcal{RG}ra_d(g;m,n)) = \begin{cases} \mathbb{K} & \text{if } g=0, n=0, m=2, k\geq 1 \ \&\ k \equiv 2d+1 \bmod 4 \\ 0 & \text{otherwise.}\end{cases}$$

where $\mathbb{K}$ is generated by the unique polytope-like ribbon graph with k edges and k bivalent vertices which are all black.

The polytope-like cohomology classes in $H^\bullet(\mathsf{tw}\mathcal{RG}ra_d)$ play no role in this paper so that we can restrict to the dg sub-properad

$$\mathcal{C}h\mathcal{G}rav_d \subset \mathsf{tw}\mathcal{RG}ra_d$$

spanned by ribbon graphs whose black vertices are at least *trivalent*; it is called the *chain gravity properad* in [Me2] and was denoted there simply by $\mathsf{tw}^{\geq 3}\mathcal{RG}ra_d$. Its cohomology properad is called the *gravity properad* and is given by (1); it contains E. Getzler's gravity operad [Ge] as the genus zero sub-operad (2).

3. From gravity to string topology

3.1. A class of representations of $\mathcal{RG}ra_d$. The properadic compositions in $\mathcal{RG}ra_d$ have been designed in [MW1] in such a way that $\mathcal{RG}ra_d$ admits a canonical representation

$$\rho : \mathcal{RG}ra_d \longrightarrow \mathcal{E}nd_{Cyc(W)}, \tag{15}$$

in the space of "cyclic words",

$$Cyc(W) := \bigoplus_{k\geq 0} \left(\otimes^k W\right)_{\mathbb{Z}_k^*},$$

which is generated by a graded vector space W equipped with a scalar product of degree $1-d$,

$$\begin{array}{rccc} \Theta: & W\otimes W & \longrightarrow & \mathbb{K}[1-d] \\ & w_1\otimes w_2 & \longrightarrow & \Theta(w_1,w_2)\end{array}$$

such that

$$\Theta(w_2,w_1) = (-1)^{d+|w_1||w_2|}\Theta(w_1,w_2). \tag{16}$$

We need in this paper a slightly sharpened version of the morphism (15). Note that a differential ∂ in W induces a differential in $Cyc(W)$ and hence in the endomorphism properad $\mathcal{E}nd_{Cyc(W)}$ which is denoted by the same letter ∂.

3.1.1. Proposition. *Let (W, ∂) be as dg vector space equipped with a scalar product $\Theta : W \otimes W \to \mathbb{K}[1-d]$ which satisfies (16) and which is compatible with the differential in the sense that*

$$\Theta(\partial w_1, w_2) + (-1)^{|w_1|}\Theta(w_1, \partial w_2) = 0 \qquad \forall\ w_1, w_2 \in W.$$

Then the morphism (15) extends to a morphism of differential *properads*

$$\rho : (\mathcal{RG}ra_d, 0) \longrightarrow (\mathcal{E}nd_{Cyc(W)}, \partial). \tag{17}$$

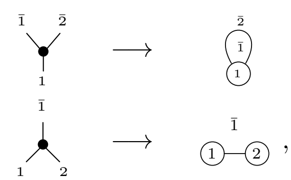
where $\mathcal{RG}ra_d$ is equipped with the trivial differential.

The proposition is equivalent to saying that for any $\Gamma \in \mathcal{RG}ra_d$ its image $\rho(\Gamma)$ is a cycle in the complex $(\mathcal{E}nd_{Cyc(W)}, \partial)$. Which is easy to check using the compatibility of Θ with ∂ and the explicit formula for $\rho(\Gamma)$ given in §4.2.2 of [MW1].

Composing the canonical morphism

$$i : \quad \mathcal{L}ieb_d^{\diamond} \longrightarrow \mathcal{RG}ra_d$$

(18)



with the representation (17) one recovers a wonderful result [Ch] that, given any dg vector space (W, ∂) with the scalar product Θ as above, the associated complex $Cyc(W)$ of cyclic words in W is a dg involutive Lie bialgebra with, e.g., the Lie bracket given explicitly by

$$[(w_1 \otimes ... \otimes w_n)^{\mathbb{Z}_n^*}, (v_1 \otimes ... \otimes v_m)^{\mathbb{Z}_n^*}] :=$$
$$\sum_{i=1}^{n}\sum_{j=1}^{m} \pm\Theta(w_i, v_j)(w_1 \otimes ... \otimes w_{i-1} \otimes v_{j+1} \otimes ... \otimes v_m \otimes v_1 \otimes ... \otimes v_{j-1} \otimes w_{i+1} \otimes \ldots \otimes w_n)^{\mathbb{Z}_{n+m-2}^*}$$

The quotient vector space,

$$\overline{Cyc}(W) := Cyc(W)/\mathbb{K} \simeq \bigoplus_{k\geq 1} \left(\otimes^k W\right)_{\mathbb{Z}_k^*},$$

is also a $\mathcal{L}ieb_d^{\diamond}$-algebra. On the other hand, the subspace

$$Cyc^{\geq 3}(W) := \bigoplus_{k\geq 3} \left(\otimes^k W\right)_{\mathbb{Z}_k^*},$$

is a dg Lie subalgebra of $Cyc(W)$.

3.2. Chain gravity properad and cyclic $\mathcal{A}ss_\infty$ algebras. Let (V, ∂) be a dg vector space[4] equipped with a degree $-d$ non-degenerate scalar product

$$\begin{array}{rccc} \langle\ ,\ \rangle : & V \odot V & \longrightarrow & \mathbb{K}[-d] \\ & v_1 \odot v_2 & \longrightarrow & \langle v_1, v_2\rangle \end{array}$$

which is compatible with the differential ∂. Hence there is an isomorphism of complexes

$$V \longrightarrow V^*[-d]$$

so that the vector space $V^*[-1] \simeq V[d-1]$ comes equipped with a a degree $d-2$ pairing

[4]We work in this section in the category of possibly infinite-dimensional vector spaces V which are direct limits, $V = \varinjlim V_p$, of finite-dimensional ones. Their dual vector spaces are defined as projective limits, $V^* = \varprojlim V_p^*$.

$$\begin{array}{rccl}\Theta: & V^*[-1]\otimes V^*[-1] & \longrightarrow & \mathbb{K}[d-2]] = \mathbb{K}[1-(3-d)]] \\ & \mathfrak{s}^{d-1}v_1\otimes \mathfrak{s}^{d-1}v_2 & \longrightarrow & (-1)^{(d-1)|v_1|}\mathfrak{s}^{2d-2}\langle v_1, v_2\rangle\end{array}$$

such that

$$\begin{aligned}\Theta(\mathfrak{s}^{d-1}v_2\otimes\mathfrak{s}^{d-1}v_1) &= (-1)^{(d-1)|v_2|}\mathfrak{s}^{2d-2}\langle v_2, v_1\rangle \\ &= (-1)^{(d-1)|v_2|+|v_1||v_2|}\mathfrak{s}^{2d-2}\langle v_1, v_2\rangle \\ &= (-1)^{(d-1)(|v_1|+|v_2|)+|v_1||v_2|}\mathfrak{s}^{2d-2}\Theta(\mathfrak{s}^{d-1}v_1, \mathfrak{s}^{d-1}v_2) \\ &= (-1)^{3-d+|\mathfrak{s}^{d-1}v_1||\mathfrak{s}^{d-1}v_1|}\Theta(\mathfrak{s}^{d-1}v_1, \mathfrak{s}^{d-1}v_2).\end{aligned}$$

Here we used the fact that

$$(-1)^{3-d+|\mathfrak{s}^{d-1}v_1||\mathfrak{s}^{d-1}v_1|} = (-1)^{3-d+(d-1+|v_1|)(d-1+|v_1|)} = (-1)^{(d-1)(|v_1|+|v_2|)+|v_1||v_2|}$$

Therefore, Proposition **3.1.1** says that the properad $\mathcal{RG}ra_{3-d}$ acts canonically on the graded vector space $\overline{Cyc}(V^*[-1])$; in particular this space is a $\mathcal{L}ieb^{\diamond}_{3-d}$-algebra and its subspace $Cyc^{\geq 3}(V^*[-1])\subset\overline{Cyc}(V^*[-1])$ a dg $\mathcal{L}ie_{3-d}$-subalgebra with Lie bracket $[\ ,\ ]$ of degree $d-2$.

3.2.1. Lemma. *There is a one-to-one correspondence between cyclic $\mathcal{A}ss_\infty$ structures in $(V, \langle\ ,\ \rangle)$ and Maurer-Cartan elements of the dg Lie algebra $(Cyc^{\geq 3}(V^*[-1]), [\ ,\ ], \partial)$.*

Proof. This is a well-known and easy observation; hence we show only a sketch. A degree $3-d$ element $\gamma\in Cyc^{\geq 3}(V^*[-1])$ gives naturally rise to a degree 1 element γ' in $\mathrm{Hom}(\otimes^{\geq 2}V[1], V[1])$. Then the Maurer-Cartan equation

$$\partial\gamma + \frac{1}{2}[\gamma,\gamma] = 0$$

says that sum

$$\partial + \gamma' \in \mathrm{Hom}\left(\otimes^{\geq 1}V[1], V[1]\right)$$

is a codifferential of the tensor coalgebra $\otimes^{\geq 1}(V[1])$, i.e. an $\mathcal{A}ss_\infty$ structure in V. If $\gamma\in(\otimes^3V^*[-1])_{\mathbb{Z}_3}$, then it makes V into dg associative algebra. □

Given a cyclic $\mathcal{A}ss_\infty$ structure in a dg vector space V, the associated Maurer-Cartan element $\gamma\in Cyc^{\geq 3}(V^*[-1])$ makes the vector space of "cyclic words" $\overline{Cyc}(V^*[-1])$ into a complex with the differential

$$\partial_\gamma := \partial + [\gamma,\ ]$$

It is called the cyclic Hochschild complex of the cyclic $\mathcal{A}ss_\infty$-algebra V. The general property of T. Willwacher's twisting endofunctor tw (see §2.1) implies the following result.

3.2.2. Proposition. *The chain gravity properad $\mathcal{C}h\mathcal{G}rav_{3-d}$ admits a canonical representation,*

$$\rho_\gamma : (\mathcal{C}h\mathcal{G}rav_{3-d}, \delta) \longrightarrow \left(\mathcal{E}nd_{\overline{Cyc}(V^*[-1])}, \partial_\gamma\right)$$

in the cyclic Hochschild complex of any cyclic $\mathcal{A}ss_\infty$ structure in a dg vector space V equipped with a non-degenerate scalar product $\odot^2V\to\mathbb{K}[-d]$.

We are interested in this paper in applications of this general statement in string topology of connected and simply connected closed manifolds M via associated Poincaré models.

3.3. An application to Poincaré duality algebras. Let (A, ∂) be a finite-dimensional non-negatively graded complex equipped with a degree zero multiplication

$$\begin{array}{rcl} A\odot A & \longrightarrow & A \\ a\otimes b & \longrightarrow & a\cdot b \end{array}$$

making A into a dg commutative associative algebra. It is called a *dg Poincaré duality algebra of degree d* if it comes equipped with a degree $-d$ *orientation map*

$$\mathfrak{o}: A \to \mathbb{K}[-d]$$

such that the induced graded symmetric scalar product,

$$\begin{array}{rrcl} \langle\ ,\ \rangle: & A\odot A & \longrightarrow & \mathbb{K}[-d] \\ & a\odot b & \longrightarrow & \mathfrak{o}(a\cdot b) \end{array} \tag{19}$$

is non-degenerate and, moreover, $\mathfrak{o}(\partial a) = 0$ for any $a \in A$. We also assume that A is connected and augmented,

$$A = \mathbb{K} \oplus \bar{A}$$

with $\bar{A}$ being a positively graded vector space. It is shown in [LS] that any connected and simply connected closed d-dimensional manifold M admits a Poincaré duality model A in degree d which is quasi-isomorphic to the de Rham algebra $\Omega^\bullet_M$ of M as a homotopy commutative associative algebra.

The scalar product in A induces an isomorphism

$$\iota: A \longrightarrow A^*[-d] := \mathrm{Hom}(A, \mathbb{K})[-d]$$

such that for any $a, b \in A$ a one has $\imath(a)(b) = \mathfrak{o}(a\cdot b)$. This isomorphism combined with the dualization of the multiplication map induces in turn a degree d diagonal on A,

$$\begin{array}{rrcl} \Delta: & A & \longrightarrow & A\odot A[d] \\ & a & \longrightarrow & \Delta(a) =: \sum a'\otimes a'' \end{array}$$

which satisfies

$$\Delta(a\cdot b) = \sum (a\cdot b')\otimes b'' = \sum (-1)^{|a||b'|} b'\otimes (a\cdot b'') = \sum (-1)^{|a''||b|} (a'\cdot b)\otimes a'' = \sum a'\otimes (a''\cdot b)$$

for any $a, b \in A$, and hence makes A into a *Frobenius algebra*.

As we discussed in §**3.2** above, the graded vector space $\overline{Cyc}(A^*[-1])$ carries canonically a representation of the properad $\mathcal{RG}ra_{3-d}$; in particular, it is a dg $\mathcal{L}ie_{3-d}$-algebra which, in accordance with Lemma **3.2.1**, admits a Maurer-Cartan element $\gamma \in (\otimes^3 A^*[-1])_{\mathbb{Z}_3}$ encoding the graded commutative associative algebra structure in A. It can be given explicitly as follows [CFL, NW],

$$\gamma = \sum_{\alpha,\beta\gamma=1}^{\dim A} \mathfrak{o}(e_\alpha\cdot e_\beta\cdot e_\gamma) s^{-1}e^\alpha\otimes s^{-1}e^\beta\otimes s^{-1}e^\gamma \tag{20}$$

where $\{e_\alpha\}_{\alpha\in[\dim A]}$ is an arbitrary basis of A and e^α the associated dual basis of A^*, If we choose $e_1 = 1 \in \mathbb{K}$ (resp., $e^1 =: 1^* \in A^*$) to be the basis vector of a basis of A respecting the direct sum decomposition $\mathbb{K}\oplus\bar{A}$ of A and set $\omega^* := \mathfrak{s}^d\iota(1) \in \bar{A}^*$, then

$$\gamma \;=\; \sum_{\alpha,\beta=1}^{\dim A} \mathfrak{o}\left(e_\alpha\cdot e_\beta\right)\left(\mathfrak{s}^{-1}1^*\otimes\mathfrak{s}^{-1}e^\alpha\otimes\mathfrak{s}^{-1}e^\beta\right)^{\mathbb{Z}_3^*} + \sum_{\alpha,\beta,\gamma=2}^{\dim A} \mathfrak{o}(e_\alpha\cdot e_\beta\cdot e_\gamma) s^{-1}e^\alpha\otimes s^{-1}e^\beta\otimes s^{-1}e^\gamma$$

By Proposition **3.2.2**, *the chain gravity properad $\mathcal{C}h\mathcal{G}rav_{3-d}$ acts canonically on the cyclic Hochschild complex $(\overline{Cyc}(A^*[1]), \partial_\gamma)$ of any Poincaré duality algebra A in degree d.*

It is well-known (and easy to check) that the subspace $\overline{Cyc}(\bar{A}^*[1]) \subset \overline{Cyc}(A^*[1])$ is respected by the differential ∂_γ giving us the so called *reduced* cyclic Hochschild complex of A. Moreover, as all black vertices of ribbon graphs from $\mathcal{C}h\mathcal{G}rav_{3-d}$ are at least trivalent, the operations $\rho(\Gamma)$ for any $\Gamma \in \mathcal{C}h\mathcal{G}rav_{3-d}$ can not create cyclic words with letter 1^* when applied to elements of $\overline{Cyc}(\bar{A}^*[1])$. Hence we obtain finally the following

3.3.1. Theorem. *The chain gravity properad $\mathcal{C}h\mathcal{G}rav_{3-d}$ acts canonically and, in general, non-trivially on the reduced Hochschild complex $(\overline{Cyc}(\bar{A}^*[1]), \partial_\gamma)$ of any Poincaré duality algebra A in degree d.*

More details on the non-triviality of this action will be given in the next section where we show that this action factors through a new dg *string topology properad* which has rich cohomology and which explains in a unified way many well-known universal operations in string topology introduced and studied earlier in [CS1, CS2, CEG, We, NW, Me2].

3.4. Gravity and equivariant cohomology of free loop spaces. Let LM stand for the space of free loops in a closed oriented n-dimensional manifold M, and $\bar{C}^{S^1}_\bullet(LM))$ for the reduced equivariant chain complex of LM with respect to the obvious S^1 action on LM; note that we work in the cohomological setting everywhere so that $\bar{C}^{S^1}_\bullet(LM)$ is non-positively graded with the boundary differential of cohomological degree $+1$. Let $\bar{H}^{S^1}_\bullet(LM)$ stand for its homology.
Assume A is a Poincaré model of M, then there is a morphism of cohomology groups

$$\bar{H}^{S^1}_\bullet(LM) \longrightarrow H^\bullet(\overline{Cyc}(\bar{A}^*[-1]), \partial_\gamma), \tag{21}$$

which is an isomorphisms if M is a closed, connected and simply connected closed manifold. This result was first proven in [CEG]; another proof can be found in [NW]. Therefore this isomorphism combined with Theorem **3.3.1** implies immediately the following

3.4.1. Corollary. *The gravity properad $\mathcal{G}rav_{3-d}$ admits a representation on the reduced equivariant equivariant homology $\bar{H}^{S^1}_\bullet(LM)$ of the free loop space LM of any connected and simply-connected closed manifold M.*

We shall discuss other implications of Theorem **3.3.1** as well as non-trivially of the action of $\mathcal{G}rav_{3-d}$ on $\bar{H}^{S^1}_\bullet(LM)$ for generic manifolds in the next section.

4. String topology properad

4.1. On the joint kernel of the representation maps. Given a Poincaré duality algebra A in degree d, by Theorem **3.3.1** there is a canonical morphism of dg properads,

$$\rho_A : (\mathcal{C}h\mathcal{G}rav_{3-d}, \delta) \longrightarrow \left(\mathcal{E}nd_{\overline{Cyc}(\bar{A}^*[-1])}, \partial_\gamma\right).$$

Consider the differential closure $\langle I, \delta I\rangle$ of the ideal I in the non-differential properad $(\mathcal{C}h\mathcal{G}rav_{3-d}, 0)$ which is generated by ribbon graphs Γ such that

(i) every black vertex of Γ (if any) is at least four valent, or
(ii) Γ contains a boundary made of black vertices only.

It is worth noting that the subspace of $(\mathcal{C}h\mathcal{G}rav_{3-d}, 0)$ generated by graphs with property (ii) only do *not* form an ideal as their properadic compositions with other graphs can create graphs Γ' with no boundaries made of black vertices solely; however such graphs Γ' would contain at least one black vertex of valency ≥ 4 so that the subspace $I \subset (\mathcal{C}h\mathcal{G}rav_{3-d}, 0)$ generated by graphs of *both* types (i) and (ii) is a properadic ideal indeed.
It is clear that

$$\langle I, \delta I\rangle \in \mathsf{Ker}\, \rho_A$$

for any Poincaré duality algebra A because

(i) graphs Γ from (i) above do not contribute into ρ_A as the Maurer-Cartan element $\gamma \in \overline{Cyc}(A^*[-1])$ (which decorates black vertices upon the representation) is a linear combination of cyclic words with precisely three letters from $A^*[-1]$ (see (20)),
(ii) graphs Γ from (ii) above do not contribute since we work with the quotient spaces, $\overline{Cyc}(V) = Cyc(V)/\mathbb{K}$, which do not contain the "empty" cyclic word, i.e. the one with no letters from a vector space V.

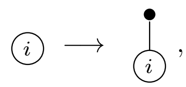
These observations together with Theorem **3.3.1** and isomorphism (21) imply the following

4.1.1. Definition-Theorem. *The quotient dg properad*

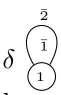
$$\mathcal{ST}_d := \mathcal{C}h\mathcal{G}rav_d/\langle I, \delta I\rangle, \quad d \in \mathbb{Z}.$$

is called the chain string topology *properad. The properad $\mathcal{ST}_{3-d}$ admits a canonical representation on the reduced cyclic Hochschild complex $(\overline{Cyc}(\bar{A}^*[-1]), \partial_\gamma)$ of a Poincaré duality algebra A in degree d. In particular, its cohomology properad $H^\bullet(\mathcal{ST}_{3-d})$ (called the* string topology *properad) acts on the reduced equivariant equivariant homology $\bar{H}^{S^1}_\bullet(LM)$ of the free loop space LM of any connected and simply-connected closed manifold M.*

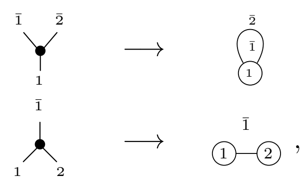
The induced differential δ in $\mathcal{ST}_d$ acts only on white vertices splitting them as follows



and redistributing edges among the new vertices in such a way that the newly created black vertex becomes strictly trivalent. In particular, the differential in $\mathcal{ST}_d$ acts trivially on univalent white vertices. Let us check next the non-triviality of the cohomology properad $H^\bullet(\mathcal{ST}_d)$.

4.2. $H^\bullet(\mathcal{ST}_d)$ and involutive Lie bialgebras. The r.h.s. in the formula (14) for δ  in $\mathcal{C}h\mathcal{G}rav_d$ belongs to the ideal $\langle I, \delta I\rangle$ and hence vanishes in $\mathcal{ST}_d$. Therefore there is a morphism of dg properads (cf. (18))

$$i: \quad (\mathcal{L}ieb^\diamond_d, 0) \longrightarrow (\mathcal{ST}_d, \delta)$$

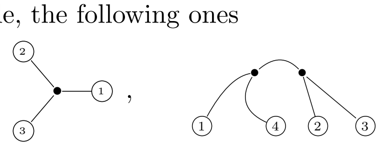


implying that the reduced cyclic Hochschild complex (and hence its cohomology) is always an involutive Lie bialgebra. In particular, this morphism and Theorem **4.1.1** imply a purely algebraic construction of the involutive Lie bialgebra $\mathcal{L}ieb^\diamond_{3-d}$-structure on $\bar{H}^{S^1}_\bullet(LM)$; this construction was first found by X. Chen, F. Eshmatov, and W. L. Gan in [CEG]. It was proven by F. Naef and T. Willwacher in [NW] that this algebraic construction of $\mathcal{L}ieb^\diamond_{3-d}$ structure on $\bar{H}^{S^1}_\bullet(LM)$ agrees precisely with the original geometric construction of M. Chas and D. Sullivan in [CS1, CS2].

4.3. An injection of the gravity operad into $H^\bullet(\mathcal{ST}_d)$. The chain gravity properad $\mathcal{C}h\mathcal{G}rav_d$ contains a sub-operad $\mathsf{tw}\mathcal{RT}ree_d$ spanned by genus zero ribbon graphs with precisely one boundary, i.e. by ribbon trees. It is proven in [Wa] that its cohomology operad

$$H^\bullet(\mathsf{tw}\mathcal{RT}ree_d) = grav_v$$

can be identified with the E. Getzler's gravity operad (2). Moreover, every cohomology class in $H^\bullet(\mathcal{RT}ree_d)$ admits a representation in terms of a linear combination of ribbon trees with all white vertices univalent and black vertices trivalent as, for example, the following ones

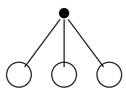
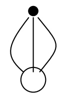
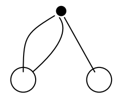
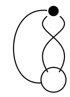


Therefore we can make the following conclusion.

4.3.1. Lemma. *The gravity operad $grav_d$ injects into the string topology properad $H^\bullet(\mathcal{ST}_d)$.*

This Lemma together with Theorem **4.1.1** imply an action of the gravity properad $grav_{3-d}$ on the reduced equivariant equivariant homology $\bar{H}^{S^1}_\bullet(LM)$ of the free loop space LM of any connected and simply-connected closed manifold M. This is a purely algebraic counterpart of the geometric construction by C. Westerland in [We]. Conjecturally, both constructions describe identical string topology operations.

4.4. $H^\bullet(\mathcal{ST}_d)$ and homotopy involutive Lie bialgebras. Consider the following four graphs in $\mathcal{ST}_d$, $\forall d \in \mathbb{Z}$,

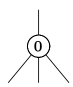
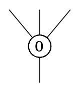
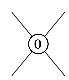
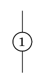
(22)  $\in \mathcal{ST}_d(1,3)$,  $\in \mathcal{ST}_d(3,1)$,  $\in \mathcal{ST}_d(2,2)$,  $\in \mathcal{ST}_d(1,1)$,

whose white vertices and boundaries are symmetrized (resp., skewsymmetrized) for d odd (resp., d even) so that we can omit their numerical labels in pictures. These elements are obviously cycles in $\mathcal{ST}_d$ which — due to the (skew)symmetrizations — can not be coboundaries. Hence they give us four non-trivial cohomology classes in $H^\bullet(\mathcal{ST}_d)$ which, for $d \leq 1$, give us in turn four universal string topology operations on $\bar{H}^{S^1}_\bullet(LM)$ for any closed, connected and simply-connected $(3-d)$-dimensional manifold M. These four operations have been discovered in [Me2] using a completely different technical gadget called the properad of ribbon *hyper*graphs. Moreover, the first half of Proposition 6.3.1 from [Me2] together with with the above observation imply the following result.

4.4.1. Proposition. *There is a morphism of dg properads*

$$j : (\mathcal{H}olieb^\diamond_{d-1}, \delta) \longrightarrow (\mathcal{ST}_d, \delta)$$

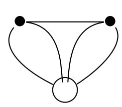
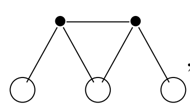
which vanishes on all generators (11) of $\mathcal{H}olieb^\diamond_{d-1}$ except the following quartette,

 ,  ,  , 

which is sent, respectively, by j into the list (22) of cycles in $\mathcal{ST}_d$.

This Proposition together with Theorem **4.1.1** gives us a new surprising proof of the second part of the Proposition 6.3.1 in [Me2] asserting a non-trivial role of $\mathcal{H}olieb^\diamond_{2-d}$ in string topology of generic d-dimensional manifolds.

There are many more non-trivial cohomology classes in $H^\bullet(\mathcal{ST}_d)$ (and hence universal operations in string topology), e.g., the following ones

 ,  , *etc.*

where the first (resp. second) ribbon graph has all boundaries (resp. vertices) skew-symmetrized for d odd/symmetrized for d even. We conclude that the string topology properad $H^\bullet(\mathcal{ST}_d)$ is highly non-trivial and is worth of further study. It is also an interesting open problem to study the cohomology group of the deformation complex of the morphism i discussed above,

$$\mathsf{Def}(\mathcal{L}ieb^\diamond_d \stackrel{i}{\to} \mathcal{ST}_d),$$

as it comes equipped with a morphism from the cohomology group (cf. [MW2])

$$H^\bullet(\mathsf{GC}_{2d}) \longrightarrow H^\bullet\left(\mathsf{Def}(\mathcal{L}ieb^\diamond_d \stackrel{i}{\to} \mathcal{ST}_d)\right),$$

of the mysterious even Kontsevich graph complex GC_{2d} which have been introduced in [K] and studied in [Wi].

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