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PROBABILISTIC LIMIT THEOREMS
AND THE GEOMETRY OF RANDOM FIELDS

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Abstract

The topics presented in this thesis lie at the interface of probability theory and stochastic geometry, with emphasis on the asymptotic study of geometric objects associated with Gaussian random fields defined on manifolds. Such a research stream has been rapidly growing in past years, resulting in a number of developments focussing on local and global geometric quantities. Our principal aim is to discuss probabilistic methods allowing one to deal with the asymptotic fluctuations of volumes of zero sets (also called *nodal sets*) of Gaussian Laplace eigenfunctions as the eigenvalues diverge to infinity, with a particular focus on the models of *Arithmetic Random Waves* on the three-dimensional torus and *Berry's Random Plane Wave* on the two-dimensional Euclidean space. We prove universal variance estimates and non-Gaussian limit theorems for the zero sets of multiple independent Arithmetic Random Waves, complementing several related works in the literature. Our analysis for this builds on an abstract cancellation result applicable to the setting of Gaussian Laplace eigenfunctions on manifolds, yielding in particular a formal description of the so-called *Berry's Cancellation Phenomenon* observed in various models of random eigenfunctions. For the Berry Random Plane Wave, we prove spatial functional limit theorems for discretized and truncated versions of the nodal length indexed by rectangular domains. Such a contribution yields a basis for proving a fully general functional limit theorem for the nodal length and opens doors to a number of novel probabilistic limit theorems involving semi-local functionals of these nodal length processes. A common technique lying at the core of our arguments for dealing with these tasks is the asymptotic analysis of the *Wiener-Itô chaos expansion* in Hermite polynomials of such geometric quantities, often allowing one to reduce investigations on Wiener chaoses of lower order. In this context, we discuss properties of generalized Hermite polynomials with matrix arguments, appearing in multivariate statistics and the theory of zonal polynomials. We argue that this family of orthogonal polynomials is particularly effective for deducing chaotic expansions of random variables that are symmetric functionals in the eigenvalues of underlying Gaussian random matrices, notably appearing in different applications dealing with the geometry of random fields. We furthermore present a new characterization of matrix-Hermite polynomials as the eigenfunctions of a generalized *Ornstein-Uhlenbeck* semigroup on matrix spaces. The above mentioned probabilistic limit theorems originate from the systematic use of the *Malliavin-Stein approach* on Gaussian spaces, a collection of analytic statements allowing one to deduce probabilistic limit theorems by means of variational techniques. Such a series of results typically emerges from the combination of *Stein's method* for probabilistic approximations and Malliavin's infinite-dimensional differential calculus. At the end of this thesis, we present preliminary computations yielding variance estimates and Central Limit Theorems for certain non-linear functionals associated with the d-dimensional Berry Random field. Also, we discuss several aspects around optimal convergence rates within Gamma approximations of functionals of Gaussian fields.

Keywords: Gaussian Laplace eigenfunctions, Arithmetic Random Wave, Berry Random Wave, Berry's Cancellation phenomenon, Wiener-Itô chaos expansions, Hermite polynomials, Gaussian analysis, Stein-Malliavin Calculus, Fourth Moment Theorems, Gamma approximations.

Preface

The present manuscript has been submitted for the fulfillment of my doctoral studies at the University of Luxembourg and collects my main achievements of the last four years.

The content of this thesis is divided in six chapters I-VI. Apart from the introductory Chapter I, the content of Chapters II-VI is based on the following works:

- *Fluctuations of nodal sets on the three-torus and general cancellation phenomena*, [Not21]. This work has been accepted for scientific publication in the Latin American Journal of Probability and Mathematical Statistics (*ALEA*).
- *Matrix-Hermite polynomials, random determinants and the geometry of Gaussian fields*, to be submitted in the near future.
- *Some functional convergence result related to Berry's nodal lengths on the plane*. Such a work is at an advanced stage and is to be submitted in the near future.
- *On the d-dimensional Berry Random Wave Model*, based on preliminary computations.
- *Optimality of convergence rates for Gamma approximations*, based on preliminary computations.

Here below we give a global outline of each of the chapters.

Chapter I: *Introduction*

In Chapter I, we present the necessary theoretical background, that is needed for the remaining chapters. Such an introduction is divided into two parts: Section I.1.1 collects properties of (Gaussian) random fields, as well as *Rice formulae* for the geometric measure associated with level sets of Gaussian random fields. In Section I.1.2, we present a thorough introduction to modern Gaussian analysis and Malliavin's infinite-dimensional variational calculus, which is one of the staples of our approach.

Chapter II: *Nodal Sets of Arithmetic Random Waves*

Chapter II deals with the study of Arithmetic Random Waves (ARWs) on the three-torus, a special case of the class of Gaussian Laplace eigenfunctions introduced by Oravecz, Rudnick, and Wigman in [ORW08, RW08]. We consider the ℓ-dimensional Gaussian random field $\mathbf{T}_n^{(\ell)}, \ell = 1, 2, 3$ formed by vectors of respectively one, two and three independent ARWs. Our primary aim is to study the high-energy (that is, as $n \to \infty$) probabilistic fluctuations of the geometric measure $L_n^{(\ell)} = \mathcal{H}_{3-\ell}((\mathbf{T}_n^{(\ell)})^{-1}(0))$ (where $\mathcal{H}_k$ indicates the k-dimensional Hausdorff measure) associated with the nodal set of $\mathbf{T}_n^{(\ell)}$. In Theorem II.1.1, we derive its expected value, asymptotic variance and a universal non-Gaussian limit theorem. Our results for $\ell = 2, 3$ substantially complement prior works by Benatar and Maffucci [BM19] and Cammarota [Cam19], corresponding to the case $\ell = 1$. The proof of Theorem II.1.1 is based on a detailed preliminary study of the Wiener chaos expansion of abstract random variables admitting an integral representation involving multiple Dirac masses and generalized Gramian determinants, allowing us to prove an abstract cancellation result (see Theorem II.2.5). Specifying the content of Theorem II.2.5 to the setting of Gaussian Laplace eigenfunctions on manifolds without boundary, such as ARWs and spherical harmonics on the sphere, yields in particular a neat description of the so-called *Berry Cancellation* observed in different models (see for instance [Wig10, KKW13, DNPR19, MPRW16, NPR19, Cam19]). Such a cancellation typically results in lower order variance estimates and is partially explained by the exact disappearance of the second chaotic projection of the nodal volumes. In Section II.3.1, we present an exhaustive analysis of the fourth-order chaotic projection of $L_n^{(\ell)}$. A subsequent study of higher-order Wiener chaoses allows us to prove that the Wiener chaos expansion of $L_n^{(\ell)}$ is asymptotically dominated by its projection on the fourth Wiener chaos, from which we deduce its non-Gaussian fluctuations. A number of intrinsic number-theoretic estimates available in the literature are also used along our development. In Theorem II.D.3, we prove a deterministic continuity result for nodal volumes associated with vector-valued functions, that is needed in our analysis.

Chapter III: ***Matrix-Hermite Polynomials and Random Determinants***

In Chapter III, we study generalized Hermite polynomials with rectangular matrix argument. Such a family of polynomials is indexed by partitions of integers and is orthogonal with respect to the law of Gaussian matrices. Matrix-Hermite polynomials can be expressed in terms of *zonal polynomials* appearing in multivariate statistics. In Theorem III.3.2, we prove that matrix-Hermite polynomials are particularly tailored for the Wiener chaos decomposition of *spectral random variables*, i.e random variables depending on the eigenvalues of XX^T, where X is a Gaussian matrix. In particular, we obtain explicit formulae for the projection on Wiener chaoses of any order of such random variables, involving integrations of generalized Laguerre polynomials with matrix argument. Such a collection of formulae turns out to be particularly useful when directly dealing with chaos expansions of functionals associated with Gaussian matrices with large dimensions. In Theorem III.3.5, we apply these findings to the case of random determinants of the form $F(X) = \sqrt{\det(XX^T)}$, where X is a rectangular Gaussian matrix whose rows are i.i.d vectors with a non-trivial covariance matrix. In Theorem III.3.6, we show that these projection coefficients admit a geometric interpretation in terms of intrinsic and mixed volumes of ellipsoids. Such a result extends a similar formula for the mean of $F(X)$ by Kabluchko and Zaporozhets [ZK12] to arbitrary projection coefficients associated with the Wiener chaos expansion of F. In a second part of this chapter, we introduce a generalized *Ornstein-Uhlenbeck semigroup* on matrix spaces via a *Mehler-type* formula. In Theorem III.3.10, we prove that matrix-Hermite polynomials are eigenfunctions of these operators, allowing us to deduce a useful orthogonality relation for matrix-Hermite polynomials when these are evaluated in correlated Gaussian matrices (see Theorem III.3.12). In Section III.3.4, we apply our findings to the asymptotic study of the *generalized total variation* of multiple independent Arithmetic Random Waves on the three-dimensional torus: more precisely, by studying its Wiener chaos expansion into matrix-Hermite polynomials, we are able to show that, in the high-energy limit, the total variation is dominated by its projection on the second Wiener chaos, yielding in particular a Gaussian limit theorem (see Theorem III.3.17).

Chapter IV: ***Weak convergence results for Berry's nodal length process***

In Chapter IV, we consider Berry's Random Plane Wave $B_E = \left\{B_E(x) : x \in \mathbb{R}^2\right\}$ with parameter $E > 0$, a stationary and isotropic Gaussian random field which is an eigenfunction of the Laplace operator. Our principal object of interest is the high-energy behaviour (that is, as $E \to \infty$) of the *nodal length process* indexed by rectangles of the type $[0, t_1] \times [0, t_2]$ in the unit square. In [NPR19], Nourdin, Peccati and Rossi prove a one-dimensional Central Limit Theorem for the normalized version of the nodal length restricted to a planar domain. Subsequently, Peccati and Vidotto [PV20] establish multivariate Central limit theorems for vectors of nodal lengths restricted to a collection of domains, proving that the nodal length process converges towards a standard Wiener sheet in the sense of finite-dimensional distributions. Such a series of results suggests a weak convergence result in function spaces for the nodal length process, but were not obtained, due to some intrinsic difficulties when dealing with second-order chaotic projections associated with the chaos expansion of the nodal length. The goal of this chapter is to present some progress towards such a weak convergence result: we prove functional limit theorems for a *discretized* version of the nodal length associated with refining partitions of the unit square (see Corollary IV.1.9), and *truncated* nodal length of increasing degree, formed by chaos projections of large order (see Corollary IV.1.15). In order to prove our results, we study the second order and the higher-order chaotic projections associated with the Wiener chaos expansion of the nodal length. For the second chaotic projections of the nodal length, we prove asymptotic variance estimates and deduce a multi-dimensional Central Limit Theorem (see Theorem IV.1.4) in the high-energy regime, leading in particular to an appealing connection with a *total disorder process* (see Corollary IV.1.5). Our arguments are based on a thorough preliminary investigation of a certain residual boundary term appearing in the projection of the nodal length on the second Wiener chaos (see Section IV.1.2). Combining moment estimates for suprema of stationary Gaussian random fields with a useful criterion by Davydov and Zitikis [DZ08] for proving weak convergence of multivariate processes, we are able to prove that these projections converge weakly to zero (Theorem IV.1.4). For the residue term formed by higher-order chaotic projections, we present a *chaining argument* similar in spirit to Dehling and Taqqu [DT89] and Marinucci and Wigman [MW11]. Such a study allows us to formulate a weak convergence result for the *discretized nodal length process* obtained by refining partitions of the unit square (see Corollary IV.1.9). As a by-product of our results, we deduce a number of novel limit theorems of semi-local type, involving suprema of discretized nodal lengths. In Corollary IV.1.15, we present a weak convergence result for the truncated nodal length process. Our arguments for this are based on the hypercontractivity of Wiener chaoses.

Chapter V: ***Non-linear functionals of d-dimensional Berry's random fields***

In Chapter V, we consider non-linear functionals associated with the d-dimensional Berry Random Wave model B_E (for $d \geq 2$). More precisely, we study random variables of the form

$$Z_E(d, q; \mathcal{D}) = \int_{\mathcal{D}} H_q(B_E(x))dx$$

where H_q is the q-th Hermite polynomial and $\mathcal{D} \subset \mathbb{R}^d$ is a convex domain. Such a random variable typically emerges in the projection on the q-th Wiener chaos of the nodal length associated with the zero set of B_E. In Theorem V.1.1, we prove asymptotic laws for the variance of $Z_E(d, q; \mathcal{D})$. As expected, our results show that the case $(d, q) = (2, 4)$ is the only one in which the variance exhibits logarithmic fluctuations. Such an observation is consistent with the main findings of Nourdin, Peccati and Rossi [NPR19] and conjecturally hint to the fact that, for $d \geq 3$, the chaotic projections of order q of the nodal length are *all* of the same order. In Theorem V.1.2, we prove quantitative Central Limit Theorems for normalized versions of $Z_E(d, q; \mathcal{D})$. We finish this chapter by some comments on *reduction principles* on Wiener chaoses and variance estimates of the nodal length associated with d-dimensional Berry random fields. Our preliminary results are to to be compared with [MR15] by Marinucci and Rossi, where the authors present a similar study for random spherical harmonics on the d-dimensional sphere, see also [MW14] for the earlier study in dimension two.

Chapter VI: ***Optimality of convergence rates in Gamma Approximations***

Chapter VI deals with the task of detecting optimal convergence rates (associated with some probability metric d) for Gamma approximations on a Gaussian space. Formally, for a sequence of chaotic random variables $\{F_n : n \geq 1\}$ converging in distribution to a centred Gamma random variable $G(\nu)$ with parameter ν, and a numerical sequence $\{\phi(n) : n \geq 1\}$ verifying $\phi(n) \to 0$ and $d(F_n, G(\nu)) \leq \phi(n)$, such an optimality is observed as soon as $c_1\phi(n) \leq d(F_n, G(\nu)) \leq c_2\phi(n)$ for some finite constants $0 < c_1 < c_2$ and large enough n. This task can be achieved by assessing exact asymptotics (as $n \to \infty$) for ratios of the form $\phi(n)^{-1}\mathbb{E}\,[h(F_n) - h(G(\nu))]$, where h is some test function related to the probability metric d. Following the lines of Nourdin and Peccati in [NP09b] on optimal rates for normal approximations, our strategy involves a characterization of the joint limiting distribution of the bivariate vector $(F_n, F_n^{(\nu)})$, where $F_n^{(\nu)} := \phi(n)^{-1}(2(F_n + \nu) - \langle DF_n, -DL^{-1}F_n\rangle_H)$ (where D and L^{-1} denote certain Malliavin operators and H is a separable Hilbert space). Our main findings, formulated in Theorem VI.2.5 and Theorem VI.2.6, provide the asymptotic fluctuations of the random variable $F_n^{(\nu)}$ in the case when $F_n = I_2(f_n)$ is an element of the second Wiener chaos, distinguishing between specific cases of finite and infinite rank. In particular, our results allow us to prove that, for a large subclass of sequences living in the second Wiener chaos, the numerical sequence $\{\phi(n) : n \geq 1\}$ leads to a *sub-optimality* phenomenon (see Corollary VI.2.8). Such an observation is in contrast with the setting of normal approximations studied in [NP09b], where a set of sufficient conditions implying optimality can be formulated on the second Wiener chaos. Whether this sub-optimality phenomenon on the second Wiener chaos extends to higher-order Wiener chaoses is partially addressed at the end of the chapter (see in particular Conjecture VI.2.10).

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Massimo, June 11, 2021

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Chapter I

Introduction

I.1 Background and preliminaries

In this first chapter of the thesis, we provide a concise overview of the main theoretical tools that will appear throughout the manuscript. Our exposition will be divided into two parts:

- In Section I.1.1, we introduce random fields in general and discuss a number of key properties such as *stationarity* and *isotropy*, focussing in particular on random fields on Euclidean spaces. Section I.1.1.3 is dedicated to the so-called *Rice formulae*, a collection of formulae playing a pivotal role in the study of geometric measures associated with level sets of Gaussian random fields.

- In Section, I.1.2 we expose the necessary background from Gaussian analysis and Malliavin Calculus, with a particular focus on *Wiener-Itô chaos expansions* and *Gaussian integration by part formulae*. This section constitutes one of the large-scale building blocks of our thesis as the tools presented therein will be used intensively along our work.

Our main bibliographic sources serving as guiding inspiration for this introduction are the books by Adler and Taylor [AT07] and Azaïs and Wshebor [AW09] for Section I.1.1, and the monographs by Nourdin and Peccati [NP12a] and David Nualart [Nua95] for Section I.1.2. One of our principal aims for this expository part is to present the necessary material in both a compact and self-contained way, hopefully allowing the reader to follow it easily without further referencing. For this reason, our exposition also includes the proofs of a number of classical results, the arguments of which we believe are instructive to be presented at this preliminary stage of the dissertation.

I.1.1 Geometry of random fields

I.1.1.1 Generalities on random fields

Our first definition is that of a random field, defined on a certain probability space. Although in this thesis we shall mainly deal with random fields taking values in Euclidean spaces, we formulate our definition for random fields with values in a generic topological space $\mathcal{E}$. For two sets S and T, we denote by S^T the class of functions from T to S.

Definition I.1.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, T a topological space and $\mathcal{E}$ some generic space. An *$\mathcal{E}$-valued random field F* on T is a collection $\{F(t) : t \in T\}$ such that $F(t)$ is an $\mathcal{E}$-valued random variable for every $t \in T$. If $\mathcal{E} = \mathbb{R}$ ($\mathcal{E} = \mathbb{R}^d$, $d \geq 2$), we say that F is a real-valued (d-dimensional) random field[1].

[1] Sometimes, we shall consider the case $\mathcal{E} = \mathbb{C}$, that is when F is a complex-valued random field. In this case, we can decompose $F = F_1 + iF_2$, where both F_1 and F_2 are real-valued random fields and $i = \sqrt{-1} \in \mathbb{C}$.

Unless stated differently, our random fields will be measurable mappings $F : \Omega \to \mathcal{E}^T$, and we will use the notation $F(t,\omega) = F(t)(\omega) = F(\omega)(t), t \in T, \omega \in \Omega$ interchangeably to indicate the value of $F(t)$ and drop the dependence on ω. In this section, we will focus on a rich class of random fields, known as Gaussian fields. We recall the probability density function of Gaussian random variables for completeness and notational reasons.

Definition I.1.2. (i) A real-valued random variable N is said to have the Gaussian distribution with mean μ and variance σ^2 (written $N \sim \mathcal{N}(\mu, \sigma^2)$) if its characteristic function is

$$\mathbb{E}\left[\exp(itN)\right] = \exp(i\mu t - \sigma^2 t^2/2), \quad t \in \mathbb{R}.$$

In the case where $N \sim \mathcal{N}(0,1)$, we say that N is a standard Gaussian random variable.

(ii) For an integer $d \geq 2$, an $\mathbb{R}^d$-valued random variable $N = (N_1, \ldots, N_d) \in \mathbb{R}^d$ is said to have the d-dimensional Gaussian distribution with mean $\mu = (\mu_1, \ldots, \mu_d) \in \mathbb{R}^d$ and covariance function $\Sigma \in \mathcal{M}_{d\times d}(\mathbb{R})$ (written $N \sim \mathcal{N}_d(\mu, \Sigma)$) if its characterisitc function is given by

$$\mathbb{E}\left[\exp(i\langle t, N\rangle)\right] = \exp\left(i\langle \mu, t\rangle - \frac{1}{2}\langle t, \Sigma t\rangle\right), \quad t \in \mathbb{R}^d,$$

where $\langle \bullet, \bullet \rangle$ indicates the canonical scalar product in $\mathbb{R}^d$. If $N \sim \mathcal{N}_d(0, \mathbf{I}_d)$ with $0 \in \mathbb{R}^d$ denoting the zero vector and $\mathbf{I}_d$ the identity matrix, we say that N follows the standard d-dimensional Gaussian distribution.

It is sometimes convenient to consider complex-valued Gaussian random variables. In the notations of the above definition, a d-dimensional complex-valued random variable $N \in \mathbb{C}^d$ is said to have the complex-normal distribution if $N = N_R + iN_I$, where (N_R, N_I) is a $2d$-dimensional real Gaussian vector and $i = \sqrt{-1} \in \mathbb{C}$. We are now in position to define Gaussian random fields taking values in Euclidean spaces.

Definition I.1.3. (i) Let $\{F(t) : t \in T\}$ be a random field in the sense of Definition I.1.1 with $\mathcal{E} = \mathbb{R}$. Then, F is said to be a *Gaussian field* if for every integer $k \geq 1$ and every collection $t_1, \ldots, t_k \in T$, the random vector $(F(t_1), \ldots, F(t_k))$ follows a d-dimensional Gaussian distribution.

(ii) Let $\{F(t) : t \in T\}$ be a random field in the sense of Definition I.1.1 with $\mathcal{E} = \mathbb{R}^d, d \geq 2$. Then F is said to be a d-dimensional Gaussian field if for every $a \in \mathbb{R}^d$, the random function $\langle a, F(\cdot)\rangle$ is a real-valued Gaussian field, where $\langle \cdot, \cdot \rangle$ stands for the canonical inner product in $\mathbb{R}^d$.

For a real-valued Gaussian field F as above, we define its *mean function* and *covariance function* by

$$m_F(t) := \mathbb{E}\left[F(t)\right], \quad \Gamma_F(t,s) := \mathbf{Cov}[F(t), F(s)], \quad t, s \in T.$$

By Kolmogorov's existence Theorem (see for instance [Bil99, Chapter 7]), it is a known fact that the reverse direction actually holds true. More precisely, given a function $m : T \to \mathbb{R}$ and a non-negative definite function[2] $\Gamma : T \times T \to \mathbb{R}$, there exists a real-valued Gaussian process defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with mean function m_F and covariance function Γ_F and such that $m_F(t) = m(t)$ and $\Gamma_F(t,s) = \Gamma(t,s)$ for every $t, s \in T$. In particular, this shows that Gaussian fields are entirely characterized by the knowledge of the sole functions m_F and Γ_F.

[2]Recall that this means $\sum_{i,j=1}^n a_i a_j \Gamma(t_i, t_j) \geq 0$ for every integer $n \geq 1, a \in \mathbb{R}^n, t \in T^n$.

I.1.1.2 Stationarity and Isotropy of random fields

In this section, we discuss two important notions enjoyed by a large class of random fields arising in numerous applications, namely stationarity and isotropy.

Stationary random fields. For what follows, we endow our parameter space T with a group structure $(+, 0)$. In this setting, we then write $t - s := t + (-s)$, where $-s$ is the inverse of s. Furthermore, we assume that the operation $+$ is commutative[3]. For our purpose, we can always think of T as a subset of an Euclidean space.

Definition I.1.4. Let F be a random field in the sense of Definition I.1.1. Then, we call F *stationary* (or *homogeneous*) if for every integer $k \geq 1$ and every choice of $t_1, \ldots, t_k, s \in T$, we have

$$(F(t_1), \ldots, F(t_k)) \overset{d}{=} (F(t_1 + s), \ldots, F(t_k + s)), \tag{I.1.1}$$

where $\overset{d}{=}$ indicates equality in distribution.

The distributional identity (I.1.1) merely tells that the finite-dimensional distributions of a stationary random field is invariant under translations. As such it becomes clear that whenever F is a stationary random field with mean function m_F and covariance function Γ_F, we necessarily have that m_F is a constant function of t and that Γ_F depends on one variable only, that is, for every $t, s \in T$, $\Gamma_F(t, s)$ only depends on $t - s$. By a slight abuse of notation, we shall always write $\Gamma_F(t, s) = \Gamma_F(t - s)$ in this case. A further particularly nice property for random fields is known as *isotropy*, which we will discuss later in Defintion I.1.7.

We will now discuss several approaches to generate examples of stationary random fields on Euclidean spaces. From now on, we assume that $F = \{F(t) : t \in T\}$ is a centred complex-valued[4] stationary random field on $T = \mathbb{R}^d, d \geq 1$. In this setting, the covariance function Γ_F of F only depends on one single variable and is given by $\Gamma_F(t - s) = \mathbb{E}\left[F(t)\overline{F(s)}\right]$, where $\overline{z} \in \mathbb{C}$ indicates the complex-conjugate of $z \in \mathbb{C}$. The following result is known as *Bochner's Theorem* (see [Boc33]). Heuristically such a result tells that, among the continuous functions on $\mathbb{R}^d$, only those functions representable as the Fourier transform of a finite measure are covariance functions.

Theorem I.1.5. *A continuous function $\Gamma : \mathbb{R}^d \to \mathbb{C}$ is a non-negative definite function if and only if there exists a finite measure ρ on $\mathbb{R}^d$ such that*

$$\Gamma(t) = \int_{\mathbb{R}^d} \exp(i\langle t, \lambda\rangle)\rho(d\lambda) \tag{I.1.2}$$

for every $t \in \mathbb{R}^d$.

The measure ρ in the statement is referred to as *spectral measure* associated with Γ. In view of (I.1.2), it follows that the covariance function Γ_F of a complex-valued centred stationary random field F admits the representation

$$\Gamma_F(t) = \int_{\mathbb{R}^d} \exp(i\langle t, \lambda\rangle)\rho(d\lambda), \quad t \in \mathbb{R}^d$$

for some finite measure $\rho = \rho_F$ on $\mathbb{R}^d$. In this case, we sometimes call ρ the *spectral measure associated with F*. Furthermore, we have that $\mathbf{Var}[F(t)] = \Gamma_F(0) = \rho(\mathbb{R}^d)$.

[3] We remark that if T is a non-abelian group, then one should distinguish between right and left stationarity: F is called right-stationary if for every integer $k \geq 1$ and every choice of $t_1, \ldots, t_k, s \in T$, (I.1.1) holds, and F is called left-stationary if the random field $\widetilde{F} = \left\{\widetilde{F}(t) := F(-t) : t \in T\right\}$ is right-stationary.

[4] Recall that this means that F is written as $F = F_1 + iF_2 \in \mathbb{C}$, where both F_1 and F_2 are real-valued random fields

Our goal is now to show how one can construct stationary random fields from random measures. In order to do this, we introduce complex-valued random measures. Let ρ be a finite measure on $\mathbb{R}^d$ and write Z to indicate a random measure with intensity ρ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, that is, $Z = \left\{Z(B) : B \in \mathcal{B}(\mathbb{R}^d)\right\}$ is a collection of complex-valued random variables verifying the conditions $\mathbb{E}[Z(B_1)] = 0, \mathbb{E}\left[Z(B_1)\overline{Z(B_1)}\right] = \rho(B_1)$ and

$$B_1 \cap B_2 = \emptyset \Longrightarrow Z(B_1 \cup B_2) = Z(B_1) + Z(B_2), \quad \mathbb{P} - a.s.,$$
$$B_1 \cap B_2 = \emptyset \Longrightarrow \mathbb{E}\left[Z(B_1)\overline{Z(B_2)}\right] = 0$$

for every $B_1, B_2 \in \mathcal{B}(\mathbb{R}^d)$. Then, for a deterministic function

$$f \in L^2(\rho) := \left\{g : \mathbb{R}^d \to \mathbb{C} : \|g\|^2_{L^2(\rho)} := \int_{\mathbb{R}^d} g^2 d\rho < \infty\right\},$$

standard techniques allow to define the stochastic integral with respect to Z in $L^2(\mathbb{P})$

$$Z(f) := \int_{\mathbb{R}^d} f(\lambda) Z(d\lambda) \tag{I.1.3}$$

by first defining it for elementary step functions f and then passing to the limit by a density argument. We omit these details here. In particular, such a construction yields the isometry formula

$$\mathbb{E}\left[Z(f)\overline{Z(g)}\right] = \int_{\mathbb{R}^d} f(\lambda)\overline{g(\lambda)}\rho(d\lambda) = \langle f, g\rangle_{L^2(\rho)}$$

valid for every $f, g \in L^2(\rho)$. By construction, it follows that if Z is a Gaussian measure[5] with intensity ρ, then $Z(f)$ in (I.1.3) is also Gaussian.

We can now state the spectral representation theorem, asserting that, if one replaces the deterministic kernel f in (I.1.3) with the function $\lambda \mapsto f_t(\lambda) = \exp(i\langle t, \lambda\rangle)$ for some fixed $t \in \mathbb{R}^d$, then the resulting random field $\left\{Z(f_t) : t \in \mathbb{R}^d\right\}$ is stationary.

Theorem I.1.6. *Let ρ be a finite measure on $\mathbb{R}^d$ and Z be a complex-valued random measure with intensity ρ. Then the complex-valued random field $\left\{F(t) : t \in \mathbb{R}^d\right\}$ defined by*

$$F(t) = \int_{\mathbb{R}^d} \exp(i\langle t, \lambda\rangle) Z(d\lambda), \quad t \in \mathbb{R}^d \tag{I.1.4}$$

is stationary with covariance function

$$\Gamma_F(t - s) = \mathbb{E}\left[F(t)\overline{F(s)}\right] = \int_{\mathbb{R}^d} \exp(i\langle t - s, \lambda\rangle)\rho(d\lambda). \tag{I.1.5}$$

Moreover, for every centred, stationary complex-valued random field $F = \left\{F(t) : t \in \mathbb{R}^d\right\}$ on $\mathbb{R}^d$ with spectral measure ρ, one can associate a complex-valued random measure Z having intensity ρ in such a way that (I.1.4) *holds in $L^2(\mathbb{P})$ for every $t \in \mathbb{R}^d$. The process Z is called the spectral process associated with F.*

[5]In this case, we have that $Z(B)$ is Gaussian for every $B \in \mathcal{B}(\mathbb{R}^d)$. Moreover, in this case, the assumption that for disjoint sets B_1 and B_2, the random variables $Z(B_1)$ and $\overline{Z(B_2)}$ are uncorrelated can be strengthened to independence.

The spectral representation theorem above also has an analog for read-valued random fields, although constructed in a slightly different way. We refer the reader to [AT07, Section 5.4] for more details on this.

Derivatives of random fields. In what follows, we will provide useful formulae to compute covariances between derivatives of random fields. Let us fix some notation first. In order to be as non-technical as possible, we may assume that $T = \mathbb{R}^d$ (or some subset of $\mathbb{R}^d$) and denote its elements by $t = (t_1, \dots, t_d)$. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, we define the differential operator

$$\mathrm{D}_t^\alpha := \frac{\partial^{|\alpha|}}{\partial t_1^{\alpha_1} \dots \partial t_d^{\alpha_d}},$$

where $|\alpha| := \sum_{k=1}^d \alpha_k$. When applied to a random field on $T = \mathbb{R}^d$, such an operator coincides with the L^2-derivative of F of order $|\alpha|$ in the direction of $(e_1, \dots, e_d)$, where e_j stands for the j-th canonical basis element of $\mathbb{R}^d$. We refer the reader to Section 1.4.2 of [AT07] for more details on the existence of such objects. Let us now consider a random field F on $\mathbb{R}^d$ with covariance function $\Gamma_F(t,s) = \mathbb{E}[F(t)F(s)]$. It can be shown by means of the spectral distribution theorem (see Theorem I.1.5) and the spectral representation theorem (see Theorem I.1.6) that the covariance function of derivatives of F is given by

$$\mathbb{E}\left[\mathrm{D}_t^\alpha F(t) \mathrm{D}_s^\beta F(s)\right] = \mathrm{D}_t^\alpha \mathrm{D}_s^\beta \Gamma_F(t,s) = \frac{\partial^{|\alpha|+|\beta|}}{\partial t_1^{\alpha_1} \dots \partial t_d^{\alpha_d} \partial s_1^{\beta_1} \dots \partial s_d^{\beta_d}} \Gamma_F(t,s). \tag{I.1.6}$$

This formula has a particularly nice representation when we assume F to be stationary: indeed in this case, in view of Theorem I.1.5, we can write

$$\Gamma_F(t) = \int_{\mathbb{R}^d} \exp(i\langle t, \lambda\rangle) \rho(d\lambda),$$

where ρ is the spectral measure associated with Γ_F. Then, setting $s = t$ in (I.1.6) yields

$$\begin{aligned}
\mathbb{E}\left[\mathrm{D}_t^\alpha F(t) \mathrm{D}_t^\beta F(t)\right] &= \mathrm{D}_t^\alpha \mathrm{D}_t^\beta \Gamma_F(t-s)_{|t=s} = (-1)^{|\beta|} \mathrm{D}_t^\alpha \mathrm{D}_t^\beta \Gamma_F(t)_{|t=0} \\
&= (-1)^{|\beta|} \mathrm{D}_t^\alpha \mathrm{D}_t^\beta \int_{\mathbb{R}^d} \exp(i\langle t, \lambda\rangle) \rho(d\lambda)_{|t=0} \\
&= (-1)^{|\beta|} i^{|\alpha|+|\beta|} \int_{\mathbb{R}^d} \lambda_1^{\alpha_1} \dots \lambda_d^{\alpha_d} \rho(d\lambda) =: i^{|\alpha|+|\beta|} \Lambda(\alpha).
\end{aligned}$$

The quantity $\Lambda(\alpha)$ in the R.H.S is known as *spectral moment of order* α. A direct consequence of the above discussion is the following: Assume that F is a centred random field on $T = \mathbb{R}^d$, which is not necessarily stationary. For $j = 1, \dots, d$, we let $F_j(t) := \mathrm{D}_t^{e_j} F(t)$, with $e_j \in \mathbb{R}^d$ indicating the j-th canonical basis vector of $\mathbb{R}^d$. Then, in view of the above differentiation rules we compute

$$\mathbb{E}\left[F(t) F_j(t)\right] = \mathrm{D}^{e_j} \Gamma_F(t,s)_{|s=t}, \quad t \in T. \tag{I.1.7}$$

In particular, if F is such that $\mathbf{Var}[F(t)] = \Gamma_F(t,t) = \sigma^2$ is constant for every $t \in T$, then the R.H.S of (I.1.7) is equal to zero, implying that $F(t)$ and $F_j(t)$ are uncorrelated random variables. When F is a Gaussian random field, then this shows that $F(t)$ and $F_j(t)$ are independent. We remark that this argument holds for fixed t and does not extend to the entire processes F and F_j. We shall often consider such a situation in our applications, notably in Chapters II-IV.

Isotropic random fields. As already mentioned at an earlier stage, we will now discuss *isotropic* random fields. For the following definition it is convenient to restrict to the case $T = \mathbb{R}^d$.

Definition I.1.7. Let F be a stationary random field with covariance function $\Gamma_F(t)$. Then, we call F *isotropic* if Γ_F only depends on the Euclidean length $\|t\|$ of $t \in T$.

As for stationarity, we will identify $\Gamma_F(t)$ with $\Gamma_F(\|t\|)$. We remark that in Definition I.1.7, we defined isotropy for stationary F. However, the notion of isotropy can also be defined for non-stationary random fields: in this case, (as is also consistent with Definition I.1.7), we call F isotropic if all its finite-dimensional distributions are invariant under rotation, that is, for every integer $k \geq 1$ and every choice of $t_1, \dots, t_k \in T$, one has that

$$(F(t_1), \dots, F(t_k)) \overset{d}{=} (F(g.t_1), \dots, F(g.t_k)) \tag{I.1.8}$$

for every rotation[6] $g \in SO(d)$. Here for $t \in T$, we indicate by $g.t$ the group action of $g \in SO(d)$ on t. At this stage, it might be instructive to compare (I.1.1) and (I.1.8). Indeed, combining these definitions, yields that stationary and isotropic random fields on Euclidean spaces are invariant under rigid motions, i.e. geometric transformations of the form $\tau(t, s) = g.t + s$, where $g \in SO(d)$ and $t, s \in T$.

We now see how the spectral measure ρ of a stationary random field F is affected under isotropy. Let $g \in SO(d)$ be a rotation. Since $\|t\| = \|g.t\|$ for every $t \in T$, if follows by Definition I.1.7 that $\Gamma_F(t) = \Gamma_F(g.t)$ for every $t \in T$. In particular, by (I.1.2), we can therefore write

$$\Gamma_F(t) = \Gamma_F(g.t) = \int_{\mathbb{R}^d} \exp(i\langle g.t, \lambda\rangle)\rho(d\lambda) = \int_{\mathbb{R}^d} \exp(i\langle t, g.\lambda\rangle)\rho(d\lambda) =: \int_{\mathbb{R}^d} \exp(i\langle t, \lambda\rangle)\rho_g(d\lambda),$$

where ρ_g is the measure on $\mathbb{R}^d$ defined by $\rho_g := \rho \circ g^{-1}$. As this chain of equalities is valid for every $t \in T$, we deduce that $\rho = \rho_g$, yielding that ρ is also invariant under rotations.

In the spirit of Bochner's Theorem I.1.5, the following important result characterizes stationary and isotropic random fields on Eucliedan spaces. As its proof does not require any further technical tools and is only based on notions that we have already introduced, we believe that its proof is instructive.

Theorem I.1.8. *A random field $F = \{F(t) : t \in T\}$ on $T = \mathbb{R}^d$ is stationary and isotropic if and only if its covariance function Γ_F admits the representation*

$$\Gamma_F(t) = \int_{\mathbb{R}_+} \frac{J_{(d-2)/2}(r\|t\|)}{(r\|t\|)^{(d-2)/2}} \Pi(dr), \tag{I.1.9}$$

where Π is a finite measure on $\mathbb{R}_+$ and J_p indicates the Bessel function of the first kind of order p defined by

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+p+1)} \left(\frac{x}{2}\right)^{2k+p}, \quad p \in \mathbb{R}.$$

Proof. Since F is stationary, in view of Bochner's Theorem I.1.5, we can write

$$\Gamma_F(t) = \int_{\mathbb{R}^d} \exp(i\langle t, \lambda\rangle)\rho(d\lambda), \quad t \in \mathbb{R}^d$$

where ρ is the spectral measure associated with Γ_F. Now define a measure $\widetilde{\Pi}$ on $\mathbb{R}_+$ by $\widetilde{\Pi}([0, r]) := \rho(\mathbb{B}_d(r))$, where $\mathbb{B}_d(r)$ indicates the d-dimensional Ball of radius r centred at the origin. Then, passing to polar coordinates $\lambda \equiv (r, u_1, \dots, u_{d-1}) \in \mathbb{R}_+ \times \mathbb{S}^{d-1}$ in the above expression for $\Gamma_F(t)$ applied with the vector $t^\star = (\|t\|, 0, \dots, 0) \in \mathbb{R}^d$, yields

$$\Gamma_F(t^\star) = \int_{\mathbb{R}^d} \exp(i\|t\|\lambda_1)\rho(d\lambda) = \int_{\mathbb{R}_+} \widetilde{\Pi}(dr) \int_{\mathbb{S}^{d-1}} \sigma_d(u_1, \dots, u_{d-1}) \exp(i\|t\|r\theta_1), \tag{I.1.10}$$

[6]Here $SO(d)$ denotes the special orthogonal group of $d \times d$ matrices. A more general version of this particular example is obtained when defining random fields on homogeneous spaces. Recall that whenever G is a group, we say that a set T is a G-homogeneous space if G acts transitively on T. Isotropy in this setting is then defined by (I.1.8), where $g.t \in T$ indicates the action of $g \in G$ on $t \in T$.

where σ_d stands for the surface measure on $\mathbb{S}^{d-1}$. Using hyper-spherical coordinates $(\phi_1, \ldots, \phi_{d-1}) \in [0,\pi] \times [0,2\pi]^{d-2}$ on $\mathbb{S}^{d-1}$ gives $u_1 = \cos\phi_1$, so that the inner integration on $\mathbb{S}^{d-1}$ in (I.1.10) can be computed as

$$\begin{aligned} &\int_{[0,\pi]} d\phi_1 \exp(i\|t\|r\cos\phi_1) \int_{[0,2\pi]^{d-1}} d(\phi_2,\ldots,\phi_{d-2})(\sin\phi_1)^{d-2}(\sin\phi_2)^{d-2}\ldots\sin\phi_{d-2} \\ = \;& C_d \int_{[0,\pi]} d\phi_1 \exp(i\|t\|r\cos\phi_1)(\sin\phi_1)^{d-2}, \qquad \text{(I.1.11)} \end{aligned}$$

where $C_d := \int_{[0,2\pi]^{d-1}} d(\phi_2,\ldots,\phi_{d-2})(\sin\phi_1)^{d-2}(\sin\phi_2)^{d-2}\ldots\sin\phi_{d-2}$. The integral in (I.1.11) can be computed by means of Bessel functions as (see for instance [AS92])

$$\int_{[0,\pi]} d\phi_1 \exp(i\|t\|r\cos\phi_1)(\sin\phi_1)^{d-2} = \frac{J_{(d-2)/2}(r\|t\|)}{(r\|t\|)^{(d-2)/2}},$$

so that the conclusion follows from (I.1.10) once absorbing the normalizing factor C_d into $\widetilde{\Pi}$. The measure Π in the statement is then $C_d\widetilde{\Pi}$. □

Remark I.1.9. In Chapters II and IV, we will focus on the so-called *Berry Random Plane Wave*. Such a random field is a collection $B_E = \left\{B_E(x) : x \in \mathbb{R}^2\right\}$ with parameter $E > 0$, which is the unique centred, isotropic and stationary Gaussian random field on $\mathbb{R}^2$ verifying the Helmholz equation $\Delta B_E + 4\pi^2 E B_E = 0$, where Δ denotes the Laplacian on the plane. Its covariance function is obtained when selecting $\Pi(dr) = \delta_{2\pi\sqrt{E}}(r)$ in (I.1.9), with δ_u denoting the Dirac mass at u, that is $\mathbb{E}\left[B_E(x)B_E(y)\right] = J_0(2\pi\sqrt{E}\|x-y\|)$. We refer the reader to Section II.2.2 of Chapter II and Chapter V for further generalizations on $\mathbb{R}^d$ of this Gaussian random field and Chapter IV for geometric investigations around its nodal length restricted to rectangles.

I.1.1.3 Rice formulae for Gaussian random fields

In this section, we come a step closer to the essence of our applications discussed in this thesis, and consider level sets associated with (Gaussian) random fields. Our scope is to provide explicit formulae allowing one to compute expected values and higher-order moments of geometric measures associated with these level sets. Such a collection of formulae, customarily known as *Rice formulae* (or *Kac-Rice* formulae) will be fruitfully exploited in Chapter II, where we study the geometric measure of *nodal sets* associated with multiple independent *Arithmetic Random Waves*.

In view of what will be studied in this thesis, the statements of this section will be presented for multivariate Gaussian random fields on some Euclidean space, that is, unless otherwise stated, throughout this section, we consider Gaussian random fields $F : \mathbb{R}^d \to \mathbb{R}^{d'}$ for some integers $d, d' \geq 1$. The following definition introduces the principal objects of interest of this section.

Definition I.1.10. Let $u \in \mathbb{R}^{d'}$ be fixed and $\mathcal{R} \subset \mathbb{R}^d$ a Borel set. For a function $f : \mathbb{R}^d \to \mathbb{R}^{d'}$, we define the *level set of f at the level u* as

$$L_{d,d'}(f,\mathcal{R};u) = \{t \in \mathcal{R} : f(t) = u\} = f^{-1}(\{u\}) \cap \mathcal{R}.$$

In the case where $u = 0 \in \mathbb{R}^{d'}$, we sometimes call $L_{d,d'}(f,\mathcal{R};0)$ the *nodal set of f*.

We remark here that the set $L_{d,d'}(f,\mathcal{R};u)$ looks quite different according to whether or not $d = d'$. Indeed, if $d = d'$, the level set will be a collection of isolated points, whereas in the case $d > d'$, in a natural situation, it will be a sufficiently smooth Euclidean manifold of dimension $d - d'$. Indeed, assume that f is C^1-smooth and that its Jacobian matrix $\mathrm{jac}_f(t) \in \mathcal{M}_{d'\times d}(\mathbb{R})$ at $t \in L_{d,d'}(f,\mathcal{R};u)$ has full rank d', so to contain an invertible sub-matrix of dimension $d' \times d'$. In order to avoid technicalities at this stage, we assume that the latter is formed by the last d' columns of $\mathrm{jac}_f(t)$. Writing $t = (\underline{t}, \bar{t}) \in \mathbb{R}^{d-d'} \times \mathbb{R}^{d'}$, it follows by the Implicit Function Theorem that there exist neighborhoods $\underline{V} \subset \mathbb{R}^{d-d'}$ of $\underline{t} \in \mathbb{R}^{d-d'}$ and $\overline{V} \subset \mathbb{R}^{d'}$ of $\bar{t} \in \mathbb{R}^{d'}$ as well as a function $g = g_{\underline{t}} : \underline{V} \to \overline{V}$, such that the set of points $\left\{(\underline{s}, g(\underline{s})) : \underline{s} \in \underline{V}\right\}$ is a local parametrization of $L_{d,d'}(f,\mathcal{R};u)$. This justifies that $L_{d,d'}(f,\mathcal{R};u)$ is a C^1-smooth manifold of dimension $d - d'$.

In Definition I.1.10, the function f is deterministic. The idea of this whole part is to replace f with a (Gaussian) random field F. In this setting it becomes evident that the set $L_{d,d'}(F,\mathcal{R};u)$ is random and it is thus natural to study geometric quantities associated with it from a probabilistic point of view. Typically, we are interested in the 'size' of the level set, that is, its geometric measure

$$\mathcal{V}_{d,d'}(F,\mathcal{R};u) := \mathcal{H}_{d-d'}(L_{d,d'}(F,\mathcal{R};u)), \tag{I.1.12}$$

where $\mathcal{H}_{d-d'}$ denotes the $(d-d')$-dimensional Hausdorff measure. Note that in the case where $d = d'$, this reduces to the counting measure. Also, we will only deal with the case $d - d' \geq 0$, as the case $d < d'$ is not interesting from a purely probabilistic view point: indeed, in this case the level set $L_{d,d'}(F,\mathcal{R};u)$, will be almost surely empty with measure zero.

The following deterministic result is a crucial identity known as *Area formula* (in the case $d = d'$) and *Co-Area formula* (in the case $d > d'$), see for instance Proposition 6.1 and Proposition 6.13 in [AW09], respectively. For a matrix $M \in \mathcal{M}_{n\times k}(\mathbb{R})$, we define the function

$$\Phi_{n,k}(M) := \sqrt{\det(MM^T)}.$$

In the case $n = k$, $\Phi_{n,k}(M) = |\det(M)|$ thus reduces to the modulus of the determinant.

Proposition I.1.11. *Let $f : \mathbb{R}^d \to \mathbb{R}^{d'}$ be a C^1 function such that the set of its critical values*[7] *has Lebesgue measure zero. Let $g : \mathbb{R}^{d'} \to \mathbb{R}^d$ be a continuous bounded function. Then, for every Borel set $\mathcal{R} \subset \mathbb{R}^d$,*

$$\int_{\mathbb{R}^{d'}} g(u)\mathcal{V}_{d,d'}(f,\mathcal{R};u)du = \int_{\mathcal{R}} \Phi_{d',d}(\mathrm{jac}_f(t))g(f(t))dt, \tag{I.1.13}$$

where $\mathrm{jac}_f(t) \in \mathcal{M}_{d'\times d}(\mathbb{R})$ denotes the Jacobian matrix $(\frac{\partial}{\partial t_j} f^{(i)}(t))$ evaluated at t.

A more general version of formula (I.1.13) due to Federer (see [Fed59]) is known in the setting where f is a function between Riemannian manifolds. We also point out that the assumption on the critical values of f to have Lebesgue measure zero is automatically verified when f is sufficiently regular in view of Sard's Theorem (see [Sar42]).

An integral representation of geometric measures. For our purpose, the essential role of Proposition I.1.11 is to derive an explicit integral form for the geometric quantities $\mathcal{V}_{d,d'}(f,\mathcal{R};u)$. Indeed, by a suitable approximation argument, it is not more restrictive to directly apply formula (I.1.13) when replacing g with indicator functions. Such a route is efficiently exploited in the literature dealing with the growing study of Gaussian Laplace eigenfunctions, and shall be used at several occasions in this thesis. Here below, we briefly sketch this argument for clarity. Assume that we want to derive an expression for $\mathcal{V}_{d,d'}(f,\mathcal{R};z)$ for some $z \in \mathbb{R}^{d'}$ and that the mapping $z \mapsto \mathcal{V}_{d,d'}(f,\mathcal{R};z)$ of the threshold level z

[7] Critical values of f are the images of its critical points.

is continuous. The idea is to take for g an approximation of the singular Dirac mass δ_z, by setting $g_\varepsilon(u;z) = (2\varepsilon)^{-d'} \prod_{k=1}^{d'} \mathbb{1}_{[z_k-\varepsilon, z_k+\varepsilon]}(u_k), u \in \mathbb{R}^{d'}$ for $\varepsilon > 0$. Using (I.1.13) then leads to

$$\int_{\prod_{k=1}^{d'}[z_k-\varepsilon,z_k+\varepsilon]} (2\varepsilon)^{-d'} \mathcal{V}_{d,d'}(f,\mathcal{R};u)du = \int_{\mathcal{R}} \Phi_{d',d}(\mathrm{jac}_f(t)) g_\varepsilon(f(t);z)dt.$$

Letting $\varepsilon \to 0$ on both sides of the above relation then yields in some natural sense that

$$\mathcal{V}_{d,d'}(f,\mathcal{R};z) = \int_{\mathcal{R}} \Phi_{d',d}(\mathrm{jac}_f(t))\delta_z(f(t))dt,$$

yielding in particular an integral expression for the geometric measure $\mathcal{V}_{d,d'}(f,\mathcal{R};z)$. Such a relation will be the starting point for our analysis led in the forthcoming chapters, when we study the *Wiener-Itô chaos expansion* (that is an orthogonal expansion in Hermite polynomials) of volumes of the type $\mathcal{V}_{d,d'}(f,\mathcal{R};z)$.

We will now come to Rice formulae. The intensive study of Rice formulae for Gaussian processes started in the sixties with the works by Itô [Ito] in 1964, Cramer and Leadbetter [CL65] in 1965 and Belayev [Bel66] in 1966. Loosely speaking, Rice formulae are useful when computing expectations or higher-order moments of the random variables $\mathcal{V}_{d,d'}(F,\mathcal{R};u)$ where F is a random function. At this stage, the reader might wonder why we keep carrying the explicit dependence on d and d' in several occurrences. Although these integers are kept fixed throughout, the reason for this is that the theory of Rice formulae slightly differs whether we consider $d - d' = 0$ or $d - d' > 0$, as is also partially explained by the simpler form of the function $\Phi_{d,d'}$ in the case $d = d'$. In particular, some technical discrepancies show up when looking at the proofs of the respective cases. In order to keep the present part as self-contained as possible, we decide to state Rice formulae in a unified statement, and therefore stick to our notations – our apologies go to the annoyed reader.

In order to give a taste of how Rice formulae come up in general, we present a heuristic argument which we believe is useful to have in mind. In what follows, we replace f by a Gaussian random field $F : \mathbb{R}^d \to \mathbb{R}^d$. Applying Proposition I.1.11 with $f = F$ and a sufficiently regular function g and taking expectations on both sides[8]

$$\begin{aligned}
\int_{\mathbb{R}^{d'}} g(u)\mathbb{E}\left[\mathcal{V}_{d,d'}(F,\mathcal{R};u)\right] du &= \int_{\mathcal{R}} \mathbb{E}\left[\Phi_{d',d}(\mathrm{jac}_F(t))g(F(t))\right] dt \\
= \quad & \int_{\mathcal{R}} dt \int_{\mathbb{R}^{d'}} du \mathbb{E}\left[\Phi_{d',d}(\mathrm{jac}_F(t))g(F(t))|F(t)=u\right] p_{F(t)}(u) \\
= \quad & \int_{\mathbb{R}^{d'}} g(u) \left(\int_{\mathcal{R}} \mathbb{E}\left[\Phi_{d,d'}(\mathrm{jac}_F(t))|F(t)=u\right] p_{F(t)}(u)dt \right) du,
\end{aligned}$$

where $p_{F(t)}(u)$ denotes the probability density function of $F(t)$ computed at u. Since g is arbitrary this shows that for almost every u, we have

$$\mathbb{E}\left[\mathcal{V}_{d,d'}(F,\mathcal{R};u)\right] = \int_{\mathcal{R}} \mathbb{E}\left[\Phi_{d,d'}(\mathrm{jac}_F(t))|F(t)=u\right] p_{F(t)}(u)dt. \tag{I.1.14}$$

This formula is one possible formulation of Rice's formula. In what follows, we introduce some necessary assumptions on F under which the above displayed argument can be made rigorous.

We fix integers $d \geq d' \geq 1$ and $k \geq 1$. Here k will denote the moment-order for later. Also, let $u \in \mathbb{R}^{d'}$ be fixed.

Assumption $A(d,d';k;u)$**.** Assume that $F : \mathbb{R}^d \to \mathbb{R}^{d'}$ is a random field verifying the following four conditions:

[8] freely exchanging integration and expectation operators for the moment, this is why we call it a heuristic argument. This point will be made rigorous by Assumption $A(d,d';k;u)$ later.

1) F is Gaussian

2) the trajectories $t \mapsto F(t)$ are $\mathbb{P}$-almost surely C^1-smooth

3) for every distinct $t_1, \ldots, t_k \in \mathbb{R}^d$, the random vector $(F(t_1), \ldots, F(t_k))$ is non-degenerate in $(\mathbb{R}^{d'})^k$, i.e. its covariance matrix is invertible

4) $\mathbb{P}\left\{\exists t \in \mathbb{R}^d : F(t) = u, \text{rank}(\text{jac}_F(t)) \neq d'\right\} = 0.$

We remark that, in the case $d = d'$, the rank condition stated in **4)** requires that, with probability one, the Jacobian matrix $\text{jac}_F(t)$ is invertible for every t in the level set $L_{d,d'}(F, \mathcal{R}, u)$. In our applications to Gaussian Laplace eigenfunctions presented in Chapters II-III, the assumptions **1)** to **4)** above are all verified. Several sufficient conditions in order for a random field to satisfy the non-degeneracy condition **4)** are studied in the literature (see for instance Propositions 6.5 and 6.12 in [AW09] for further details on this).

We have now presented all the necessary requirements in order to state Kac-Rice formulae (see Theorem 6.3 and Theorem 6.9 in [AW09]). A proof of Rice formula for the mean of the geometric measures was provided by Wschebor [Wsc82] in 1982, and was followed by generalizations to higher moments in [Wsc83] in 1983. In the statement here below, for a vector-valued random variable Z, we indicate by $p_Z(z)$ its probability density function evaluated in z.

Theorem I.1.12. *Let $d \geq d' \geq 1$ and $k \geq 1$ be integers and let $u \in \mathbb{R}^{d'}$. Assume that $F : \mathbb{R}^d \to \mathbb{R}^{d'}$ satisfies Assumption $A(d, d'; k; u)$. Then,*

(i) if $d = d'$, letting $(r)_k := \prod_{j=1}^k (r - j + 1)$ for $r \in \mathbb{R}$,

$$\mathbb{E}\left[(\mathcal{V}_{d,d'}(F, \mathcal{R}; u))_k\right] = \int_{\mathcal{R}^k} \mathbb{E}\left[\prod_{j=1}^k \Phi_{d',d}(\text{jac}_F(t_j)) \middle| F(t_1) = \ldots = F(t_k) = u\right] \cdot p_{(F(t_1),\ldots,F(t_k))}(u, \ldots, u) dt_1 \ldots dt_k$$

(ii) if $d > d'$, then

$$\mathbb{E}\left[\mathcal{V}_{d,d'}(F, \mathcal{R}; u)^k\right] = \int_{\mathcal{R}^k} \mathbb{E}\left[\prod_{j=1}^k \Phi_{d',d}(\text{jac}_F(t_j)) \middle| F(t_1) = \ldots = F(t_k) = u\right] \cdot p_{(F(t_1),\ldots,F(t_k))}(u, \ldots, u) dt_1 \ldots dt_k.$$

A few remarks concerning Theorem I.1.12 are in order.

Remark I.1.13. **(a)** Although in the above formulae the R.H.S of both cases (i) and (ii) look exactly the same (with the only difference appearing in the explicit form of $\Phi_{d',d}$) , there is a crucial difference on the L.H.S. Indeed, whereas in the case $d > d'$, we have a formula for the exact k-th moment of the geometric measure $\mathcal{V}_{d,d'}(F, \mathcal{R}; u)$, in the case $d = d'$ we have a formula for its *falling factorial moments* of order k. This means in particular that, in order to compute exact k-th moments in the case $d = d'$, those have to be established by means of falling factorials: for instance in the case $k = 2$, for a random variable X, one can make use of the identity $X^2 = X(X-1) + X = (X)_2 + (X)_1$, so that upon taking expectations, the second moment of X^2 is obtained as a linear combination of falling factorial moments of X. We shall exploit this in Chapter II in order to deduce variance estimates of certain geometric measures.

(b) Setting $k = 1$ in both formulae above yields the formula for the mean of the geometric measure, as derived heuristically in (I.1.14).

(c) We point out that several extensions of Rice formulae are available. One among them gives similar identities for expected values of products of the form $\prod_{j=1}^{k} \mathcal{V}_{d,d'}(F, \mathcal{R}; u_j)$, where the u_j's are distinct levels. Rice formulae similar to those above with slightly different assumptions are also valid for certain classes of non-Gaussian fields, in particular for random fields that are functionals of an underlying Gaussian field. For both situations, we refer the reader to [AW09, Section 6] for further reading in this directions.

I.1.2 Gaussian analysis and Malliavin Calculus

In the forthcoming sections, we present some background from Gaussian analysis and introduce tools from Paul Malliavin's infinite-dimensional calculus of variations, initiated in the seminal contribution [Mal78]. In a first part, we discuss a number of properties of the classical *Hermite polynomials* on the real line allowing one to formulate the celebrated Wiener-Itô chaos decompositions of square-integrable functionals of a certain underlying Gaussian field. In a second part, we extensively present the so-called *Malliavin operators* and their relations, yielding in particular an intrinsic integration by part formula. We finish this section with a presentation of *Fourth Moment Theorems* on Gaussian Wiener chaoses.

I.1.2.1 Hermite polynomials: the road to Wiener chaos

Let us start with a simple, but important observation. We denote by $\gamma(x)$ the standard Gaussian probability density function. Then, for any sufficiently regular[9] real-valued functions f and g, an application of the classical integration by part formula gives

$$\int_{\mathbb{R}} f'(x)g(x)\gamma(x)dx = -\int_{\mathbb{R}} f(x)\left(g(x)\gamma(x)\right)' dx = -\int_{\mathbb{R}} f(x)\left(g'(x)\gamma(x) - g(x)x\gamma(x)\right) dx.$$

Of course, the above relation can be written as

$$\mathbb{E}\left[f'(N)g(N)\right] = \mathbb{E}\left[f(N)(Ng(N) - g'(N))\right] =: \mathbb{E}\left[f(N)\delta g(N)\right], \tag{I.1.15}$$

where $N \sim \mathcal{N}(0,1)$, and $\delta g(x) := xg(x) - g'(x)$. Relation (I.1.15) is known as a *Gaussian integration by part formula*, and has – among others – many ramifications within the theory of Stein's method[10].

We now define the collection of Hermite polynomials on the real line, which will play a prominent role all over this thesis. There are several equivalent ways to introduce Hermite polynomials.

Definition I.1.14. The collection of *Hermite polynomials* on the real line $\{H_k : k \geq 0\}$ is defined recursively by the relation

$$H_0(x) = 1, H_{k+1}(x) = xH_k(x) - kH_{k-1}(x), \tag{I.1.16}$$

for $k \geq 1$. Equivalenlty, they are defined via *Rodrigues* formula

$$H_k(x) = (-1)^k e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2/2}. \tag{I.1.17}$$

[9] for the moment, by sufficiently regular, we mean such that the constant term in the integration by part formula vanishes.

[10] For instance, taking f equal to a non-zero constant function c yields the relation $0 = \mathbb{E}\left[\delta g(N)\right]$, i.e $\mathbb{E}\left[g'(N)\right] = \mathbb{E}\left[Ng(N)\right]$. This is known as Stein's caracterization for the normal distribution. We refer the reader to Section VI.1.1 of Chapter VI for further reading on Stein's method associated with the normal distribution.

The following proposition gathers some useful properties of Hermite polynomials, when combined with the standard Gaussian distribution. We write $L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \gamma(x)dx) =: L^2(\gamma)$ for brevity and denote by N a standard Gaussian random variable.

Proposition I.1.15. *(i) For every $k \geq 0, x \in \mathbb{R}$, we have that $H_k(x) = \delta^k 1$ (where δ is as in* (I.1.15) *and $\delta^p f := \delta \circ \delta^{p-1} f$ with $\delta^0 f := 1$)*

(ii) For every $p, q \geq 0$, we have that $\mathbb{E}\left[H_p(N)H_q(N)\right] = \mathbb{1}_{p=q}p!$.

(iii) The collection $\left\{(k!)^{-1/2}H_k : k \geq 0\right\}$ is an orthonormal basis of $L^2(\gamma)$.

(iv) For every $x, t \in \mathbb{R}$, we have that $\exp(xt - t^2/2) = \sum_{k\geq 0}(k!)^{-1}H_k(x)t^k$.

Proof. *(i)* Differentiating Rodrigues formula (I.1.17) on both sides gives $H_k'(x) = xH_k(x) - H_{k+1}(x)$. Comparing this with the recurrence relation for Hermite polynomials in (I.1.16) shows that $H_k' = kH_{k-1}$. Substituting this into the recurrence relation, we deduce that $H_{k+1}(x) = xH_k(x) - H_k'(x) = \delta H_k(x) = \delta^2 H_{k-2}(x) = \ldots = \delta^k H_0(x) = \delta^k 1$.

(ii) Combining (i) and the integration by part (I.1.15) yields $\mathbb{E}\left[H_p(N)H_q(N)\right] = \mathbb{E}\left[H_p(N)\delta H_{q-1}(N)\right] = p\mathbb{E}\left[H_{p-1}(N)H_{q-1}(N)\right]$. The claim now follows by induction.

(iii) By (ii), it follows that the family is orthonormal in $L^2(\gamma)$. A standard approximation argument shows that monomials $(x^n : n \geq 0)$ are dense in $L^2(\gamma)$. The desired conclusion then follows when expanding Hermite polynomials in the basis of monomials (see for instance [NP12a, p.19]).

(iv) Since $f_t(x) := e^{xt} \in L^2(\gamma)$ for every t, in view of point (iii), we can expand $e^{xt} = \sum_{k\geq 0} \frac{a_k}{k!} H_k(x)$, where $a_k = \mathbb{E}\left[e^{tN}H_k(N)\right]$. These coefficients can be computed directly via (I.1.15),

$$\mathbb{E}\left[e^{tN}H_k(N)\right] = \mathbb{E}\left[e^{tN}\delta H_{k-1}(N)\right] = t\mathbb{E}\left[e^{tN}H_{k-1}(N)\right] = \ldots = t^k\mathbb{E}\left[e^{tN}\right] = t^k e^{t^2/2},$$

where we used the Gaussian moment generating function and (i). The desired conclusion is then obtained by rearranging the factors accordingly.

□

The next definition introduces a particular class of Gaussian processes, indexed by Hilbert spaces. Throughout, we shall now fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which random objects are defined, as well as a real separable Hilbert space H, endowed with inner product $\langle \cdot, \cdot \rangle_H$ and associated norm $\|\cdot\|_H$.

Definition I.1.16. A real centred Gaussian process $X = \{X(h) : h \in H\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is called *isonormal* if $\mathbb{E}\left[X(h)X(g)\right] = \langle h, g \rangle_H$ for every for every $h, g \in H$.

It can be shown that isonormal Gaussian processes exist given a separable Hilbert space (see for instance [NP12a, Proposition 2.1.1]). Moreover, if X is an isonormal Gaussian process, the associated mappings $h \mapsto X(h)$ are linear in $L^2(\Omega, \mathcal{F}, \mathbb{P})$. We shall from now on assume that $\mathcal{F} = \sigma(X)$ is generated by the isonormal Gaussian process X and use the short-hand notation $L^2(\Omega, \mathcal{F}, \mathbb{P}) =: L^2(\mathbb{P})$.

The following Theorem I.1.18 below is known as *Wiener chaos decomposition* and constitutes a major cornerstone of this thesis. We start by defining the Gaussian Wiener chaoses.

Definition I.1.17. Let X be a isonormal Gaussian process on H. For every integer $k \geq 0$, we define the space C_k^X as the closed linear subspace of $L^2(\mathbb{P})$ generated by random variables of the form $H_k(X(h))$ with $\|h\|_H = 1$. By construction $C_0^X = \mathbb{R}$. We call C_k^X the *k-th Wiener chaos* associated with X.

Theorem I.1.18. *We have $L^2(\mathbb{P}) = \bigoplus_{k\geq 0} C_k^X$, that is, every $F \in L^2(\mathbb{P})$ can be written as a $L^2(\mathbb{P})$-converging series*

$$F = \sum_{k\geq 0} \Pi_k(F), \tag{I.1.18}$$

where $\Pi_k : L^2(\mathbb{P}) \to C_k^X$ stands[11] *for the orthogonal projection operator associated with the k-th Wiener chaos of X.*

Proof. In order to prove the claim, it suffices to show that, if $Y \in L^2(\mathbb{P})$ is such that Y is orthogonal to C_k^X for every $k \geq 0$, then $Y = 0$, $\mathbb{P}$-almost surely. Since Wiener chaoses are spanned by Hermite polynomials, we therefore assume that $\mathbb{E}\left[Y H_k(X(h))\right] = 0$ for every $k \geq 0$ and every $h \in H$ with $\|h\|_H = 1$. Expanding monomials in the basis of Hermite polynomials, it is equivalent to assume that $\mathbb{E}\left[YX(h)^r\right] = 0$ for every $r \geq 0$, and therefore by summation, $\mathbb{E}\left[Y \exp(X(h))\right] = 0$ for every $h \in H, \|h\| = 1$. Applying the latter relation to $h = \sum_{k=1}^n a_k h_k \in H$ and exploiting the linearity of the mapping $h \mapsto X(h)$ shows that $\mathbb{E}\left[Y \exp(\sum_{k=1}^n a_k X(h_k))\right] = 0$ for every $n \geq 1$ and $a_1, \dots, a_n \in \mathbb{R}$. This shows in particular that the Laplace transform on $\mathbb{R}^n$ associated with the measure

$$\mu(A) := \mathbb{E}\left[Y \mathbb{1}[(X(h_1), \dots, X(h_n)) \in A]\right], \quad A \in \mathcal{B}(\mathbb{R}^n)$$

is identically zero on $\mathbb{R}^n$. This implies that $\mu = 0$ on $\mathbb{R}^n$ and therefore $\mathbb{E}\left[Y|\sigma(X)\right] = 0$, which gives the desired conclusion. □

We now introduce the Ornstein-Uhlenbeck semigroup.

Definition I.1.19. For $F \in L^2(\mathbb{P})$ as in (I.1.18), we define the *Ornstein-Uhlenbeck semigroup* $\{P_t : t \geq 0\}$ by

$$P_t F = \sum_{k\geq 0} e^{-kt} \Pi_k(F) \in L^2(\mathbb{P}), \quad t \geq 0. \tag{I.1.19}$$

It is easy to verify from the above definition that $\{P_t : t \geq 0\}$ indeed satisfies the semigroup property, that is, $P_t \circ P_s = P_{t+s}$ for $t, s \geq 0$, with P_0 being the identity operator on $L^2(\mathbb{P})$. From (I.1.19) we furthermore deduce that the R.H.S actually coincides with the Wiener chaos expansion of $P_t F$, that is, $\Pi_k(P_t F) = e^{-kt}\Pi_k(F)$. We will now derive a different representation of the Ornstein-Uhlenbeck sempigroup. In order to do this, we define an independent copy X' of X on some probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, in such a way that X and X' are both defined on the product space $(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', \mathbb{P} \otimes \mathbb{P}')$. For a parameter $t \geq 0$, we define the auxiliary process $\{X^t(h) : h \in H\}$ given by $X^t(h) := e^{-t}X(h) + \sqrt{1 - e^{-2t}}X'(h)$, $h \in H$. It is straightforward to see that $X^t(h)$ is centred Gaussian and by independence

$$\begin{aligned} \mathbb{E}\left[X^t(h)X^t(g)\right] &= \mathbb{E}\left[\left(e^{-t}X(h) + \sqrt{1 - e^{-2t}}X'(h)\right)\left(e^{-t}X(g) + \sqrt{1 - e^{-2t}}X'(g)\right)\right] \\ &= e^{-2t}\langle h, g\rangle_H + (1 - e^{-2t})\langle h, g\rangle_H = \langle h, g\rangle_H, \end{aligned}$$

for every $g, h \in H$, that is, $X^t \stackrel{d}{=} X$ for every fixed $t \geq 0$. Since $F \in L^2(\mathbb{P})$ is measurable with respect to X, we may write $F = \varphi(X)$ for some measurable $\varphi : \mathbb{R}^H \to \mathbb{R}$. The following result is known as *Mehler's formula* and yields a useful alternative representation of $P_t F$ in (I.1.19).

[11] In the literature, it is customary to find the notation $J_k(F)$ instead of $\Pi_k(F)$. We shall however reserve the symbol J_k for the Bessel function of order k. Throughout this thesis, we will also make use of the notations $\mathrm{proj}(F|C_k^X)$, $\mathrm{proj}_k(F)$ and $F[k]$ to indicate $\Pi_k(F)$.

Proposition I.1.20. *Let $F \in L^2(\mathbb{P})$ be as above. For every $t \geq 0$,*

$$P_t F = \mathbb{E}\left[\varphi(X^t)|X\right] = \mathbb{E}'\left[\varphi(X^t)\right], \tag{I.1.20}$$

where $\mathbb{E}'$ indicates expectation with respect to $\mathbb{P}'$.

Proof. By means of Jensen's inequality, it is easily shown that both definitions of P_t in (I.1.19) and (I.1.20) define linear contractions on $L^2(\mathbb{P})$, that is, $\|P_t F\|_{L^2(\mathbb{P})} \leq \|F\|_{L^2(\mathbb{P})}$. It suffices to prove that both definitions coincide on $L^2(\mathbb{P})$. Since the exponentials $\left\{e^{X(h)-1/2} : h \in H, \|h\| = 1\right\}$ form a dense subset of $L^2(\mathbb{P})$ it is enough to consider the case $F = e^{X(h)-1/2}$ for $h \in H$ with $\|h\|_H = 1$. By (I.1.19) and Proposition I.1.15 (iv) (applied with $t = 1$), we have that $\Pi_k(P_t F) = e^{-kt}\Pi_k(e^{X(h)-1/2}) = e^{-kt}(k!)^{-1}H_k(X(h))$, for every $k \geq 1$. On the other hand, using (I.1.20), yields (bearing in mind the definition of X^t)

$$\begin{aligned} P_t F &= \mathbb{E}'\left[e^{X^t(h)-1/2}\right] = \mathbb{E}'\left[\exp\left(e^{-t}X(h) + \sqrt{1-e^{-2t}}X'(h) - \frac{1}{2}\right)\right] \\ &= \exp\left(e^{-t}X(h) - \frac{1}{2}\right)\mathbb{E}'\left[\exp\left(\sqrt{1-e^{-2t}}X'(h)\right)\right] = \exp\left(e^{-t}X(h) - \frac{e^{-2t}}{2}\right), \end{aligned}$$

where we used the Gaussian moment generating function. Applying again Proposition I.1.15 (iv) yields, $P_t F = \sum_{k\geq 0}(k!)^{-1}H_k(X(h))e^{-tk}$, so that $\Pi_k(P_t F) = (k!)^{-1}H_k(X(h))e^{-tk}$, for every $k \geq 1$. From this we deduce that the projections on every Wiener chaos coincide for both representations of $P_t F$, thus finishing the proof. □

Applying the action of P_t in (I.1.19) on chaotic random variables of the form $H_k(X(h))$ with $\|h\| = 1$, we obtain the relation

$$P_t H_k(X(h)) = e^{-kt}H_k(X(h)). \tag{I.1.21}$$

In particular, this shows that H_k is an eigenfunction of P_t with eigenvalue e^{-kt}. We now show that an appropriate application of Mehler's formula on Hermite polynomials allows one to deduce the following extension of the orthogonality identity in Proposition I.1.15 (ii).

Remark I.1.21. In Chapter III, we study *generalized matrix Hermite polynomials* taking matrices as arguments and introduce generalized Ornstein-Uhlenbeck operators acting on these. Therein, the overall following argument (see Proposition I.1.22) for univariate Hermite polynomials will be mimicked in a suitable fashion in order to derive the counterpart orthogonality relation for matrix-variate Hermite polynomials. Also, we obtain the matrix analog of relation (I.1.21). We refer the reader to Section III.3.3 for more details.

Proposition I.1.22. *Let (X, Y) be a centred Gaussian vector such that $\mathbb{E}\left[X^2\right] = \mathbb{E}\left[Y^2\right] = 1$ and $\mathbb{E}[XY] = \rho$. Then, for every $p, q \geq 0$, we have that*

$$\mathbb{E}\left[H_p(X)H_q(Y)\right] = \mathbb{1}_{p=q}\, p!\,\rho^p.$$

Proof. When $\rho = 0$, X and Y are independent and both sides are trivially zero. Assume therefore that $\rho \neq 0$. By the covariance structure of (X, Y), we have that $(X, Y) \overset{d}{=} (X, \rho X + \sqrt{1-\rho^2}X')$ where X' is independent of X. If $\rho > 0$, by conditioning on X and using (I.1.20), we can write

$$\begin{aligned} \mathbb{E}\left[H_p(X)H_q(Y)\right] &= \mathbb{E}\left[H_p(X)\mathbb{E}\left[H_q\left(\rho X + \sqrt{1-\rho^2}X'\right)\Big|X\right]\right] \\ &= \mathbb{E}\left[H_p(X)P_{\ln(\rho^{-1})}H_q(X)\right] = e^{-q\ln(\rho^{-1})}\mathbb{E}\left[H_p(X)H_q(X)\right] = \rho^p\mathbb{1}_{p=q}\, p!, \end{aligned}$$

where we used (I.1.21) and Proposition I.1.15 (ii). In order to prove the statement for $\rho < 0$, one can proceed similarly and use that fact $X \overset{d}{=} -X$, as well as the symmetry relation $H_q(-X) = (-1)^q H_q(X)$. □

I.1.2.2 Multiple integrals and Malliavin operators

In this section, we introduce the so-called *Mallivin operators* on a Gaussian space. We consider an isonormal Gaussian process $X = \{X(h) : h \in H\}$ indexed by a real separable Hilbert space H. We write $\mathcal{S}$ for the class of smooth random variables, that is, elements of the type $f(X(h_1), \dots, X(h_m))$, $m \geq 1$ for some smooth function $f : \mathbb{R}^m \to \mathbb{R}$ having partial derivatives with polynomial growth. Since $\mathcal{S}$ contains elements of the form $H_k(X(h))$, it is dense in $L^q(\mathbb{P})$ for every $q \geq 1$. We shall start by fixing some more notation, that will be used throughout.

Notation I.1.23. *(On Hilbert spaces)*

- *For a separable Hilbert space H and an integer $r \geq 0$, we write $H^{\otimes r}$ (resp. $H^{\odot r}$) to indicate the r-th (resp. symmetric) tensor product of H, with the convention that $H^{\otimes 1} = H^{\odot 1} = H$ and $H^{\otimes 0} = H^{\odot 0} = \mathbb{R}$.*

- *Let $(e_j : j \geq 1)$ be a orthonormal basis of H. For $h = \sum_{i_1,\dots,i_p \geq 1} u(i_1, \dots, i_p) e_{i_1} \otimes \dots \otimes e_{i_p} \in H^{\otimes p}$, we denote by $\widetilde{h} \in H^{\odot p}$ its symmetrization given by*[12]

$$\widetilde{h} = \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} \sum_{i_1,\dots,i_p \geq 1} u(i_1, \dots, i_p) e_{i_{\sigma(1)}} \otimes \dots \otimes e_{i_{\sigma(p)}},$$

where $\mathfrak{S}_p$ stands for the symmetric group of $\{1, \dots, p\}$. Furthermore, for basis elements $f = e_{i_1} \otimes \dots \otimes e_{i_p} \in H^{\otimes p}$, $g = e_{j_1} \otimes \dots \otimes e_{j_q} \in H^{\otimes q}$ and an integer $r = 0, \dots, p \wedge q$, we define the r-th contraction $f \otimes_r g$ as the element of $H^{\otimes(p+q-2r)}$ defined as $f \otimes_0 g := f \otimes g$ (the usual tensor product) and for $r = 1, \dots, p \wedge q$,

$$f \otimes_r g := \left(\prod_{k=1}^{r} \langle e_{i_k}, e_{j_k} \rangle_H \right) e_{i_{r+1}} \otimes \dots \otimes e_{i_p} \otimes e_{j_{r+1}} \otimes \dots \otimes e_{j_q}. \tag{I.1.22}$$

- *For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a Hilbert space H, the notation $L^2(\mathbb{P}; H)$ stands for random variables Y taking values in H and verifying $\mathbb{E}\left[\|Y\|_H^2\right] < \infty$.*

Definition I.1.24. Let $p \geq 1$ be an integer and $q \geq 1$ be a real number. For $F = f(X(h_1), \dots, X(h_m)) \in \mathcal{S}$, we denote by $D^p F \in L^q(\mathbb{P}; H^{\odot p})$ the *p-th Malliavin derivative of F* given by

$$D^p F = \sum_{i_1,\dots,i_p=1}^{m} \frac{\partial^p f}{\partial x_{i_1} \dots \partial x_{i_p}} (X(h_1), \dots, X(h_m)) h_{i_1} \otimes \dots \otimes h_{i_p}.$$

When $p = 1$, we write $D^1 = D$ and call D the Malliavin derivative.

The operator $D^p : L^q(\mathbb{P}) \to L^q(\mathbb{P}; H^{\odot p})$ is shown to be closable; we write $\mathbb{D}_{p,q}$ to indicate the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{\mathbb{D}_{p,q}} = \left(\sum_{k=1}^{p} \mathbb{E}\left[\|D^k F\|_{H^{\otimes k}}^q\right] \right)^{1/q},$$

[12] It can be shown that, in the case where $H = L^2(A, \mathscr{A}, \mu)$ for some non-atomic measure space $(A, \mathscr{A}, \mu)$, the Hilbert space $H^{\otimes p}$ (resp. $H^{\odot p}$) can be identified with $L^2(A^p, \mathscr{A}^p, \mu^{\otimes p})$ (resp. $L^2_s(A^p, \mathscr{A}^p, \mu^{\otimes p})$). In this case, the symmetrization of h reads $\widetilde{h}(a_1, \dots, a_p) = (p!)^{-1} \sum_{\sigma \in \mathfrak{S}_p} h(a_{i_{\sigma(1)}}, \dots, a_{i_{\sigma(p)}})$ and the contraction kernel in (I.1.22) reduces to $f \otimes_r g(a_1, \dots, a_{p+q-2r}) = \int_{A^r} f(x_1, \dots, x_r, a_1, \dots, a_{p-r}) g(x_1, \dots, x_r, a_{p-r+1}, \dots, a_{p+q-2r}) \mu^{\otimes r}(dx_1, \dots, dx_r)$.

and call $\mathbb{D}_{p,q}$ the domain of D^p in $L^q(\mathbb{P})$. The Malliavin derivative has natural properties: for instance, when $F \in \mathbb{D}_{1,q}$ for some $q \geq 1$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is a sufficiently regular function (for instance C^1-smooth with bounded derivative), then D enjoys the following *chain rule* property ([NP12a, Proposition 2.3.7])

$$D\varphi(F) = \varphi'(F)DF. \tag{I.1.23}$$

We now define the adjoint operator of $D^p : L^2(\mathbb{P}) \to L^2(\mathbb{P}; H^{\odot p})$, called the *divergence operator* and denoted δ^p. We denote by Domδ^p the subset of $L^2(\mathbb{P}; H^{\odot p})$ consisting of those u verifying the condition that there exists a constant $C = C(u) > 0$ such that

$$\left|\mathbb{E}\left[\langle D^p F, u\rangle_{H^{\otimes p}}\right]\right| \leq C\|F\|_{L^2(\mathbb{P})}, \quad \forall F \in \mathcal{S},$$

that is, those u for which the linear functional $\tau_u(\bullet) := \mathbb{E}\left[\langle D^p \bullet, u\rangle_{H^{\otimes p}}\right]$ is continuous from $L^2(\mathbb{P})$ to $\mathbb{R}$. Therefore the following definition is justified by Riesz representation Theorem.

Definition I.1.25. For every $u \in$ Domδ^p, we define $\delta^p(u)$ as the unique element of $L^2(\mathbb{P}; H^{\odot p})$ verifying

$$\mathbb{E}\left[\langle D^p F, u\rangle_{H^{\otimes p}}\right] = \mathbb{E}\left[F\delta^p(u)\right], \quad \forall F \in \mathcal{S}. \tag{I.1.24}$$

When $p = 1$, we simply write $\delta^1 = \delta$ and call it the divergence operator[13].

Now, we are in position to define the so-called multiple integrals: as we will see these objects generalize Hermite polynomials introduced above, thus being indispensable characters in modern Gaussian analysis.

Definition I.1.26. Let $p \geq 1$ be an integer and $f \in H^{\odot p}$. We define $I_p(f) := \delta^p(f)$, and call $I_p(f)$ the *p-th multiple Wiener integral.*

Using the recurrence relation for Hermite polynomials (I.1.16), it can be shown that ([NP12a, Theorem 2.7.7])

$$H_k(X(h)) = I_k(h^{\otimes k}), \quad h \in H, \|h\|_H = 1 \tag{I.1.25}$$

that is, multiple Wiener integrals can be thought of as an infinite-dimensional generalization of Hermite polynomials. Here, $h^{\otimes k}$ stands for the k-fold tensorization of h with itself. In particular, $I_k : H^{\otimes k} \to C_k^X \subset L^2(\mathbb{P})$ defines a linear isometry. Using the duality relation (I.1.24), it is not difficult to prove the following isometry relation ([NP12a, Proposition 2.7.5])

$$\mathbb{E}\left[I_p(f)I_q(g)\right] = \mathbb{1}_{p=q}\, p!\langle f, g\rangle_{H^{\otimes p}}, \quad \forall f \in H^{\otimes p}, g \in H^{\otimes q}. \tag{I.1.26}$$

In particular, we deduce that $\mathbf{Var}\left[I_p(f)\right] = p!\|f\|^2_{H^{\otimes p}}$. By the above discussion, we thus have the representation $C_k^X = I_p(H^{\odot k})$ for the k-th Wiener chaos associated with X. In the parlance of multiple integrals, the content of Theorem I.1.18 asserts that, for every $F \in L^2(\mathbb{P})$, there exist uniquely defined kernels $f_k \in H^{\odot k}$ such that

$$F = \sum_{k\geq 0} I_k(f_k), \tag{I.1.27}$$

with $I_0(f_0) = f_0 = \mathbb{E}[F]$. A useful property enjoyed by multiple Wiener integrals is the *multiplication formula*,

$$I_p(f)I_q(g) = \sum_{r=0}^{p\wedge q} r!\binom{p}{r}\binom{q}{r} I_{p+q-2r}(f\widetilde{\otimes}_r g), \quad \forall f \in H^{\otimes p}, g \in H^{\otimes q}, \tag{I.1.28}$$

[13] It is instructive to compare (I.1.24) with (I.1.15). Indeed, in the one-dimensional setting, the operator δ reduces to $\delta f(x) = f'(x) - xf(x)$, whereas D is the standard differentiation operator of real functions. In the case $p = 1$, the duality relation (I.1.24) reduces to the integration by part formula obtained in (I.1.15).

where $f\widetilde{\otimes}_r g$ indicates the symmetrization of the r-th contraction $f \otimes_r g \in H^{\otimes(p+q-2r)}$. Such a formula implies that the product of two chaotic random variables $I_p(f)$ and $I_q(g)$ is an element living in the orthogonal sum of all the Wiener chaoses of order not exceeding $p + q$.

A further central property possessed by multiple integrals is the following *hypercontractivity* property, [NP12a, Theorem 2.7.2]. For every integer $p \geq 1$ and real number $q \geq 2$,

$$\|I_p(f)\|_{L^q(\mathbb{P})} \leq c(p,q)\|I_p(f)\|_{L^2(\mathbb{P})}, \quad \forall f \in H^{\odot p}, \tag{I.1.29}$$

where $c(p,q) = (q-1)^{p/2}$. Essentially, it means that on a fixed Wiener chaos, all $L^p(\mathbb{P})$ norms are equivalent. We finish this part on multiple integrals by the following remark, justifying their name.

Remark I.1.27. Let $H = L^2([0,T], \mathcal{B}([0,T]), dt)$, and consider a standard Brownian motion $W = \{W(t) : t \geq 0\}$ started from zero. We can realize W as the isonormal Gaussian process X indexed by H via $W(t) \overset{d}{=} X(\mathbb{1}_{[0,t]})$. In this specific framework, an appropriate approximation argument shows that the p-th multiple integral $I_p(f)$, $f \in H^{\odot p}$ with respect to X coincides with the multiple Wiener-Itô integral with respect to W

$$I_p(f) = \int_{[0,T]^p} f(t_1, \ldots, t_p) dW_{t_1} \ldots dW_{t_p}.$$

This representation justifies the name of multiple integrals.

We now define two further Malliavin operators.

Definition I.1.28. Let $F \in L^2(\mathbb{P})$ be as in (I.1.18). We say that $F \in \mathrm{Dom}L$ if $\sum_{k\geq 1} k^2 \mathbb{E}\left[\Pi_k(F)^2\right] < \infty$ and if so, we set

$$LF = \sum_{k\geq 1}(-k)\Pi_k(F), \quad L^{-1}F = \sum_{k\geq 1}(-k)^{-1}\Pi_k(F). \tag{I.1.30}$$

As previously mentioned for the Ornstein-Uhlenbeck semigroup, we see that (I.1.30) implies that $\Pi_k(LF) = -k\Pi_k(F)$ and $\Pi_k(L^{-1}F) = (-k)^{-1}\Pi_k(F)$, respectively. In particular, for $F \in C_k^X$ we observe that $LF = -kF$, that is $C_k^X = \ker(L + k\,\mathbf{I})$, where $\mathbf{I}$ stands for the identity operator on $L^2(\mathbb{P})$. Using the chaotic expansion of P_tF in (I.1.19), it can be shown that L is the infinitesimal generator of $\{P_t : t \geq 0\}$, that is $LF = \lim_{t\to 0} \frac{P_tF - F}{t}$ in $L^2(\mathbb{P})$ (see for instance [Nua95, Proposition 1.4.2]). The operator L^{-1} is called the *pseudo-inverse* of L, in view of the relation

$$LL^{-1}F = \sum_{k\geq 1}(-k)^{-1}L\Pi_k(F) = \sum_{k\geq 1}(-k)^{-1}(-k)\Pi_k(F) = \sum_{k\geq 1}\Pi_k(F) = F - \mathbb{E}[F].$$

The following remarkable result yields an explicit connection between the Malliavin operators D, δ and L (see [NP12a, Proposition 2.8.8]), and has as a by-product a particularly beautiful integration by parts formula (see [NP12a, Theorem 2.9.1]) . Such an intrinsic relation lies at the core of Stein-Malliavin calculus and yields a starting point to deal with a number of quantitative limit theorems that we will present in the next section.

Theorem I.1.29. *(i)* $F \in \mathrm{Dom}L$ *if and and only if* $F \in \mathbb{D}_{1,2}$ *and* $DF \in \mathrm{Dom}\delta$*, and* $LF = -\delta DF$.

(ii) For $F, G \in \mathbb{D}_{1,2}$ *and a* C^1*-smooth function* $g : \mathbb{R} \to \mathbb{R}$ *with bounded derivative, we have*

$$\mathbb{E}[Fg(G)] = \mathbb{E}[F]\,\mathbb{E}[g(G)] + \mathbb{E}\left[g'(G)\langle DG, -DL^{-1}F\rangle_H\right]. \tag{I.1.31}$$

Proof. *(i)* Using the Wiener chaos expansion (I.1.27) and the fact that the spaces C_k^X are dense in $L^2(\mathbb{P})$, it suffices to consider the case where $F = I_k(f)$ for some $f \in H^{\odot k}$ and $k \geq 1$. On the one hand, we then have $LF = -kF$. On the other hand, $-\delta(DF) = -\delta(kI_{k-1}(f)) = -k\delta(I_{k-1}(f)) = -kF$, where we used that $DI_r(f) = rI_{r-1}(f)$ and $\delta I_r(f) = I_{r+1}(f)$ for $r \geq 1$.

(ii) Using the fact that $LL^{-1}F = F - \mathbb{E}[F]$, we can write

$$\begin{aligned}\mathbb{E}\left[(F-\mathbb{E}[F])g(G)\right] &= \mathbb{E}\left[L(L^{-1}F)g(G)\right] = \mathbb{E}\left[\delta(-DL^{-1}F)g(G)\right] \\ &= \mathbb{E}\left[\langle Dg(G), -DL^{-1}F\rangle_H\right] = \mathbb{E}\left[g'(G)\langle DG, -DL^{-1}F\rangle_H\right],\end{aligned}$$

where we used the conclusion of (i), the duality relation (I.1.24) and the chain rule property (I.1.23). □

I.1.2.3 Fourth moment theorems

In this section, we present the so-called *Fourth Moment Theorems*, which – roughly speaking – consist in a series of statements asserting that, for sequences of chaotic random variables F_n, their convergence in distribution to a Gaussian random variable N is guaranteed by the sole convergence of the fourth moment of F_n to that of N. In this respect, fourth moment theorems provide a radical simplification of the classical method of moments for proving probabilistic limit theorems. Since its derivation in 2005 by Nualart and Peccati, fourth moment theorems have become a popular and efficient tool to deal with central limit theorems on Wiener chaoses. An updated list with a countless number of works dealing with applications around the Fourth Moment Theorem can be found on the website

`https://sites.google.com/site/malliavinstein/home`

maintained by Ivan Nourdin.

We recall that for a random variable $Y \in L^p(\mathbb{P})$ for every $p \geq 1$, the sequence of cumulants $\left\{\kappa_p(F) : p = 1, 2, \dots\right\}$ is defined by the relation

$$\log \mathbb{E}\left[\exp(itY)\right] = \sum_{p\geq 1} \kappa_p(Y)\frac{(it)^p}{p!}, \quad t \in \mathbb{R}$$

in such a way that $\kappa_1(Y) = \mathbb{E}[Y]$, $\kappa_2(Y) = \mathbf{Var}[Y]$, etc.

The following theorem is due to Nualart and Peccati in 2005 ([NP05]) and Nualart and Ortiz-Latorre in 2008 ([NOL08]). It gives a set of equivalent conditions characterizing convergence in law of a sequence of multiple Wiener integrals to a Gaussian random variable. Their proof is based on the use of the multiplication formulae (I.1.28) for multiple Wiener integrals and the *Dambis-Dubins-Schwarz Theorem* (see for instance [RY99, Chapter 5]) to transform F_n into a suitable time-changed Brownian motion. The equivalence between (i) and (v) was proven by Nualart and Ortiz-Latorre in [NOL08] using techniques from Malliavin calculus. A somewhat different and more general approach was initiated by Ledoux in [Led12], where normal approximations of sequences of eigenfunctions associated with Markov chaoses are studied from a spectral view point. We refer the reader to [ACP14] and [AMMP16] for further works in this direction. We refer the reader to Notation I.1.23 for the definition of contraction operators on Hilbert spaces.

Theorem I.1.30. *Let* $\{F_n : n \geq 1\}$ *be a sequence of random variables* $F_n = I_p(f_n)$ *for some* $p \geq 2$ *and* $f_n \in H^{\odot p}$ *such that* $\mathbf{Var}[F_n] \to \sigma^2$ *as* $n \to \infty$. *Let* $N \sim \mathcal{N}(0, \sigma^2)$. *As* $n \to \infty$, *the following assertions are equivalent:*

(i) $F_n \xrightarrow{d} N$

(ii) $\mathbb{E}\left[F_n^4\right] \to \mathbb{E}\left[N^4\right] = 3\sigma^4$ *(or equivalently* $\kappa_4(F_n) \to \kappa_4(N) = 0$*)*

(iii) $\|f_n \widetilde{\otimes}_r f_n\|_{H^{\otimes(2p-2r)}} \to 0$ *for every* $r = 1, \ldots, p-1$

(iv) $\|f_n \otimes_r f_n\|_{H^{\otimes(2p-2r)}} \to 0$ *for every* $r = 1, \ldots, p-1$

(v) $\mathbf{Var}\left[\|DF_n\|_H^2\right] \to 0.$

The following multi-dimensional generalization of Theorem I.1.30 is due to Peccati and Tudor, see [PT05].

Theorem I.1.31. *Let* $d \geq 2$ *be an integer. Let* $\{\mathbf{F}_n : n \geq 1\}$ *be a sequence of vector-valued random variables* $\mathbf{F}_n = (F_n^1, \ldots, F_n^d)$ *such that for every* $k = 1, \ldots, d$, $F_n^k = I_{p_k}(f_n^k)$ *for some* $p_k \geq 1$ *and* $f_n^k \in H^{\odot p_k}$. *Assume that the covariance matrix* Σ_n *of* F_n *satisfies* $\Sigma_n(i,j) = \mathbb{E}\left[F_n^i F_n^j\right] \to \sigma(i,j)$ *as* $n \to \infty$ *for every* $i, j = 1, \ldots, d$. *Let* $\mathbf{N} = (N^1, \ldots, N^d) \sim \mathcal{N}_d(0, \Sigma)$, *where* $\Sigma(i,j) := \sigma(i,j)$. *As* $n \to \infty$, *we have*

$$\left(\mathbf{F}_n \xrightarrow{d} \mathbf{N}\right) \iff \left(F_n^k \xrightarrow{d} N^k, \quad \forall k = 1, \ldots, d\right).$$

In particular, the above statement allows to deduce that, for a sequence of random vectors with chaotic components, marginal convergence in distribution is equivalent to their joint convergence. By virtue of Theorem I.1.30, the condition in the R.H.S above can equivalently be replaced with any of the statements (ii)-(v) in Theorem I.1.30. Such a structural phenomenon enjoyed by Wiener chaoses constitutes a further motivation of their study.

Quantitative versions. We point out that several quantitative versions of the above statements have been established. In order to state some of these, we introduce three probability metrics. Consider the classes of functions

$$\begin{aligned}
\mathscr{H}_{\mathrm{W}} &:= \{h : \mathbb{R} \to \mathbb{R} : \|h'\|_\infty \leq 1\}, \\
\mathscr{H}_{\mathrm{Kol}} &:= \{h(x) = \mathbb{1}_{(-\infty, z]}(x) : z \in \mathbb{R}\}, \\
\mathscr{H}_{\mathrm{TV}} &:= \{h(x) = \mathbb{1}_B(x) : B \in \mathcal{B}(\mathbb{R})\}.
\end{aligned}$$

For random variables X, Y, we define $d_{\mathrm{U}}(X,Y) := \sup\{|\mathbb{E}[h(X) - h(Y)]| : h \in \mathscr{H}_{\mathrm{U}}\}$, called *Wasserstein* (U = W), *Kolmogorov* (U = Kol), and *total variation* (U = TV) distance, respectively. It is a known fact that the topology of these three distances is strictly stronger than the topology induced by convergence in distribution, that is, whenever $\{X_n : n \geq 1\}$ is a sequence of random variables such that $X_n \xrightarrow{d} X_\infty$ as $n \to \infty$, then $d_{\mathrm{U}}(X_n, X_\infty) \to 0$ for every $\mathrm{U} \in \{\mathrm{W}, \mathrm{Kol}, \mathrm{TV}\}$ (see for instance [NP12a, Proposition C.3.1]). The following theorem (see [NP12a, Theorem 5.2.6]) collects quantitative estimates for the three above distances whenever X is a random variable belonging to a fixed Wiener chaos and the target random variable X_∞ is centred Gaussian.

Theorem I.1.32. *Let* $p \geq 2$ *be an integer and* $F = I_p(f)$ *for* $f \in H^{\odot p}$ *such that* $\mathbf{Var}[F] = \sigma^2 > 0$. *Let* $N \sim \mathcal{N}(0, \sigma^2)$. *Then, for every* $\mathrm{U} \in \{\mathrm{W}, \mathrm{Kol}, \mathrm{TV}\}$, *we have that*

$$d_{\mathrm{U}}(F, N) \leq C_{\mathrm{U}} \sqrt{\frac{p-1}{3p} \kappa_4(F)},$$

where $C_{\mathrm{W}} = \sigma^{-1}\sqrt{2/\pi}$, $C_{\mathrm{Kol}} = \sigma^{-2}$, $C_{\mathrm{TV}} = 2\sigma^{-2}$.

The following theorem (see [NP12a, Theorem 6.1.1]) is quantitative d-dimensional analog in Wasserstein distance when $\mathbf{F}$ is a sufficiently regular random vector. Here, for a matrix $\mathcal{M} \in \mathcal{M}_{d\times d}(\mathbb{R})$, we denote by $\|M\|_{op} := \sup\{\|My\| : \|y\| = 1\}$ its operator norm. Also, d_{W} stands for the Wasserstein distance between vector-valued random variables, defined similarly as above.

Theorem I.1.33. *Let $d \geq 2$ be an integer and $\mathbf{F} = (F^1, \dots, F^d)$ be a centred random vector such that $F^k \in \mathbb{D}_{1,4}$ for every $k = 1, \dots, d$. For some positive definite symmetric matrix $\Sigma \in \mathcal{M}_{d\times d}(\mathbb{R})$, we let $\mathbf{N} \sim \mathcal{N}_d(0, \Sigma)$. Then,*

$$d_{\mathrm{W}}(\mathbf{F}, \mathbf{N}) \leq \sqrt{d}\|\Sigma^{-1}\|_{op}\|\Sigma\|_{op}^{1/2}\sqrt{\sum_{i,j=1}^{d}\mathbb{E}\left[(\Sigma(i,j) - \langle DF^j, -DL^{-1}F^i\rangle)^2\right]}.$$

I.2 Main contributions of the thesis

In this section, we take a closer look to what the main contributions of this thesis are. We shall present some of our main findings in a somewhat informal and simplified way, as further technical background will be needed in order to make formulations more precise. For complete formulations and thorough expositions, we refer the reader to the respective chapters.

Let us briefly motivate and situate our contributions in the literature. Chapters II-V deal with the growing line of research on random Gaussian Laplace eigenfunctions on manifolds. Such a research domain finds its roots in various areas of mathematics and mathematical physics, and roughly originates from an experimental observation due to the musician and physicist Chladni at the end of the 18-th century. He observed that when exciting a vibrating metal plate with sand on it with his violin, produced a number of curious geometric patterns, known as *Chladni figures*. These figures were later observed to correspond to zero sets of Laplace eigenfunctions (see for instance [GK12]). Gaussian Laplace eigenfunctions are typically obtained as random linear combinations of square integrable basis elements of the manifold, when the coefficients are normal random variables. Many questions of interest are assigned to their study, taking vast and diverse directions, ranging from local to global considerations. Typically, local quantities can be investigated in small areas and then glued together to recover their full behaviour. Such quantities include for instance geometric measures of level sets, Euler characteristics, or critical points, see for instance [EF16, EL16, NPR19, KKW13, DNPR19, Wig10, Wig12] and references therein. Global geometric quantities include the study of connected components of zero sets, and require different techniques, see for instance [NS09, NS16, GW11] and references therein for some works in this direction. We also refer the reader to [Ale96, BG17, BM18] for some representative works on percolation theory associated with random eigenfunctions. Several explicit models of Gaussian Laplace eigenfunctions are studied in the literature, a pivotal role being played by *spherical harmonics* on the sphere, *Arithmetic Random Waves* on the torus, and *Berry's random waves*, see for instance [Wig10, MPRW16, MRW20, KKW13, DNPR19, Cam19, NPR19, PV20] for a selected collection of works on these models. In [Zel09], the authors introduce *monochromatic random waves* on smooth compact manifolds, for which it is shown that Berry's random field is the scaling limit, see for instance [CH20] and [DNPR20].

In this thesis, we will mainly focus on local geometric aspects associated with two explicit models of Gaussian Laplace eigenfunctions, namely Arithmetic Random Waves on the torus (Chapter II and partially Chapter III), and Berry's Random Plane Waves (Chapter IV and also Chapter V for Berry's random field in higher dimensions), focussing on the geometric measure of their nodal sets. Our investigations are essentially motivated by the fact that Berry's model of *random* eigenfunctions are conjectured to be a good modeling for *deterministic* eigenfunctions on Riemannian manifolds (see for instance [Ber77] and also

[CH20, DNPR20] for universality results), thus providing a basis for a global understanding of related models of Laplace eigenfunctions.

The last chapter of this thesis discusses some aspects of optimal convergence rates in Gamma approximations, and is related to Stein's method for probabilistic approximations. We refer the reader to Chapter VI (and references therein) for an introduction to Stein's method for normal and Gamma approximations.

Chapter II: *Nodal Sets of Arithmetic Random Waves*

This chapter deals with the study of Arithmetic Random Waves (ARW) on the three-dimensional torus, a model of Laplace eigenfunctions first introduced in [ORW08, RW08] by Oravecz, Rudnick, and Wigman on tori of arbitrary dimension. These can be defined as

$$T_n(x) = \frac{1}{\sqrt{\mathcal{N}_n}} \sum_{\lambda \in \Lambda_n} a_\lambda \exp(2\pi i \langle \lambda, x\rangle) , \quad x \in \mathbb{T}^3 , \quad n \in S_3$$

where the coefficients $\{a_\lambda : \lambda \in \Lambda_n\}$ are complex Gaussian random variables and independent except for the relation $a_\lambda = \overline{a_{-\lambda}}$. Here S_3 denotes the set of integers that are representable as a sum of three integers squares, and the summation is over all vectors $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ such that $\|\lambda\| = n$, with cardinality $\mathcal{N}_n$. It follows from the spectral analysis on the torus that the eigenvalues (or *energies*) of the negative Laplacian $-\Delta$ are written as $E_n := 4\pi^2 n$, where $n \in S_3$ is an integer representable as the sum of three integer squares. The definition of T_n implies that, $\mathbb{P}$-almost surely, T_n verifies the equation $\Delta T_n + 4\pi^2 n T_n = 0$. Equivalently, T_n can be shown to be a smooth centred Gaussian field on the torus. The existing literature on the three-dimensional torus focuses on the asymptotic study (as $n \to \infty$) of the nodal set Z_n of T_n and its associated two-dimensional geometric measure $\mathcal{A}_n := \mathcal{H}_2(Z_n)$, that is the nodal surface of Z_n. In [BM19], Benatar and Maffucci derive an exact *universal* asymptotic behaviour for its variance, namely as $n \to \infty$,

$$\mathbf{Var}[\mathcal{A}_n] = \frac{n}{\mathcal{N}_n^2} \left(\frac{32}{375} + O\left(n^{-1/28+o(1)}\right)\right) .$$

The limiting distribution of the normalized nodal surface was subsequently addressed by Cammarota in [Cam19], where the following non-Gaussian, *universal* limit theorem was established in the high-energy regime:

$$\frac{\mathcal{A}_n - \mathbb{E}\left[\mathcal{A}_n\right]}{\sqrt{\mathbf{Var}[\mathcal{A}_n]}} \xrightarrow{d} \frac{1}{\sqrt{10}} \cdot (5 - \chi^2(5)) ,$$

where $\chi^2(5)$ denotes a chi-squared random variable with 5 degrees of freedom. Our main goal is to extend these findings to the geometric measures associated with zero sets of multiple independent ARW. In order to achieve this task, we consider three independent copies $T_n^{(1)}, T_n^{(2)}, T_n^{(3)}$ of T_n and define the ℓ-dimensional Gaussian field

$$\mathbf{T}_n^{(\ell)} := \left\{ \mathbf{T}_n^{(\ell)}(x) := \left(T_n^{(1)}(x), \dots, T_n^{(\ell)}(x)\right) : x \in \mathbb{T}^3 \right\} , \quad \ell = 1, 2, 3,$$

and its nodal measure

$$L_n^{(\ell)} := \mathcal{H}_{3-\ell}\Big(\bigcap_{i=1}^{\ell} (T_n^{(i)})^{-1}(0) \Big) ,$$

where $\mathcal{H}_k$ indicates the k-dimensional Hausdorff measure. Our main result is Theorem I.2.1 below, providing the exact mean, the asymptotic variance and the subsequent second order probabilistic fluctuations of the properly rescaled nodal volumes. Specified to the case $\ell = 1$, our findings recover the main results of [BM19] and [Cam19]. For integers $1 \le \ell \le k$, we set

$$\alpha(\ell,k) := \frac{(k)_\ell \kappa_k}{(2\pi)^{\ell/2}\kappa_{k-\ell}},$$

where $(k)_\ell := k!/(k-\ell)!$ and $\kappa_k := \frac{\pi^{k/2}}{\Gamma(1+k/2)}$ stands for the volume of the unit ball in $\mathbb{R}^k$. Recall that $E_n := 4\pi^2 n$.

Theorem I.2.1. *(see Theorem II.1.1) Let the above notation prevail. Then the following holds:*

(i) *(Expected nodal volume) For every $n \in S_3$,*

$$\mathbb{E}\left[L_n^{(\ell)}\right] = \left(\frac{E_n}{3}\right)^{\ell/2} \frac{\alpha(\ell,3)}{(2\pi)^{\ell/2}}.$$

(ii) *(Universal asymptotic nodal variance) As $n \to \infty, n \not\equiv 0,4,7 \pmod 8$,*

$$\mathbf{Var}\left[L_n^{(\ell)}\right] \sim (c_n^{(\ell)})^2\left(\ell \cdot \frac{1}{250} + \frac{\ell(\ell-1)}{2}\cdot\frac{76}{375}\right), \quad c_n^{(\ell)} = \left(\frac{E_n}{3}\right)^{\ell/2} \frac{2}{(2\pi)^{\ell/2}} \frac{\alpha(\ell,3)}{\mathcal{N}_n}.$$

(iii) *(Universal asymptotic distribution of the nodal volume) As $n \to \infty, n \not\equiv 0,4,7 \pmod 8$,*

$$\widetilde{L_n^{(\ell)}} := \frac{L_n^{(\ell)} - \mathbb{E}\left[L_n^{(\ell)}\right]}{\sqrt{\mathbf{Var}\left[L_n^{(\ell)}\right]}} \xrightarrow{d} \left(\ell\cdot\frac{1}{250} + \frac{\ell(\ell-1)}{2}\cdot\frac{76}{375}\right)^{-1/2} Y^{(\ell)} M^{(\ell)} (Y^{(\ell)})^T,$$

where $Y^{(\ell)} \sim \mathcal{N}_{\ell(9\ell-4)}(0, \mathbf{I}_{\ell(9\ell-4)})$ is a $\ell(9\ell-4)$-dimensional standard Gaussian vector and $M^{(\ell)} \in \mathcal{M}_{\ell(9\ell-4)\times\ell(9\ell-4)}(\mathbb{R})$ is the deterministic matrix given by

$$M^{(\ell)} = \frac{-1}{50}\mathbf{I}_{5\ell} \oplus \frac{-1}{25}\mathbf{I}_{\frac{5\ell(\ell-1)}{2}} \oplus \frac{1}{25}\mathbf{I}_{\frac{5\ell(\ell-1)}{2}} \oplus \frac{1}{50}\mathbf{I}_{\frac{5\ell(\ell-1)}{2}} \oplus \frac{-1}{6}\mathbf{I}_{\frac{3\ell(\ell-1)}{2}}.$$

Here, for two matrices M_1 and M_2, we indicate by $M_1 \oplus M_2$ their direct sum.

Our analysis for proving Theorem I.2.1 relies on an intrinsic preliminary study of the Wiener-Itô chaos expansion of abstract random elements of the form (see Definition II.2.1)

$$J(\mathbf{G}, W; u^{(\ell)}) := \int_Z \prod_{i=1}^{\ell} \delta_{u_i}(G^{(i)}(z)) \cdot W(z)\, \mu(dz), \quad u^{(\ell)} := (u_1,\dots,u_\ell) \in \mathbb{R}^\ell$$

where $(Z, \mathscr{Z}, \mu)$ is some finite measurable space, $\mathbf{G}$ is a centred ℓ-dimensional Gaussian field with i.i.d coordinates $G^{(i)}, i = 1,\dots,\ell$ and $W = \{W(z) : z \in Z\}$ is some other random field, which is not necessarily Gaussian. As usual, δ_a denotes the Dirac mass at a. Our motivation to study such an object comes from the Area/Co-Area formulae (see Proposition I.1.11), as we shall prove that, both in $L^2(\mathbb{P})$ and $\mathbb{P}$-almost surely, $L_n^{(\ell)} = J(\mathbf{G}, W; u^{(\ell)})$, where $\mathbf{G} = \mathbf{T}_n^{(\ell)}$ and $W(z)$ is the square root of the Gramian determinant of the Jacobian matrix of $\mathbf{T}_n^{(\ell)}$ computed at z. Let us introduce further notation needed in order to state our main result: For every $i = 1,\dots,\ell$, let

$$\mathbf{X}^{(i)} = \left\{\mathbf{X}^{(i)}(z) := (X_0^{(i)}(z), X_1^{(i)}(z), \dots, X_k^{(i)}(z)) : z \in Z\right\}$$

be a $(k+1)$-dimensional standard Gaussian field. For $z \in Z$, we write $\mathbf{X}_\star^{(i)}(z)$ to indicate the vector $(X_1^{(i)}(z),\dots,X_k^{(i)}(z))$ and write

$$\mathbf{X}_\star(z) := \left\{X_j^{(i)}(z) : (i,j) \in [\ell]\times[k]\right\} \in \mathcal{M}_{\ell\times k}(\mathbb{R})$$

for the $\ell \times k$ matrix whose i-th row is given by $\mathbf{X}_\star^{(i)}(z)$. If $\ell \geq 2$, for every $i_1 \neq i_2 \in [\ell]$, we assume that the random fields $\mathbf{X}^{(i_1)}$ and $\mathbf{X}^{(i_2)}$ are stochastically independent.

The following theorem provides explicit expressions of chaotic projections onto Wiener chaos of order 0, 1 and 2 (denoted by $\mathrm{proj}_q(\cdot)$) associated with $\left\{\mathbf{X}^{(1)}, \dots, \mathbf{X}^{(\ell)}\right\}$ of the random variable $J := J(\mathbf{G}, W; u^{(\ell)})$ defined above in the setting where

$$\mathbf{G} = \left\{(X_0^{(1)}(z), \dots, X_0^{(\ell)}(z)) : z \in Z\right\}, \quad W = \{\Phi_{\ell,k}(\mathbf{X}_\star(z)) : z \in Z\},$$

and $\Phi_{\ell,k} : \mathbb{R}^{\ell \times k} \to \mathbb{R}_+$ is a certain matrix-variate function verifying a number of appropriate conditions (see Assumption A in Definition II.2.2 for precise requirements[14]). We set $\mathbb{E}\left[\Phi_{\ell,k}(\mathbf{X}_\star(z))\right] =: \alpha_{\ell,k}$ and define the numerical quantities for $i = 1, \dots, \ell$

$$D^{(i)} := \frac{1}{k}\sum_{j=1}^{k} \int_Z X_j^{(i)}(z)^2 \,\mu(dz) - \int_Z X_0^{(i)}(z)^2 \,\mu(dz), \qquad m^{(i)} := \int_Z X_0^{(i)}(z)\,\mu(dz).$$

Theorem I.2.2. *(see Theorem II.2.5) Let the above setting prevail. Then, writing $\gamma(z) := \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$, we have*

(i) *(Projection formulae)*

$$\mathrm{proj}_0(J) = \mathbb{E}[J] = \alpha_{\ell,k} \cdot \prod_{i=1}^{\ell} \gamma(u_i), \quad \mathrm{proj}_1(J) = \alpha_{\ell,k} \cdot \prod_{i=1}^{\ell} \gamma(u_i) \cdot \sum_{i=1}^{\ell} m^{(i)} u_i,$$

$$\mathrm{proj}_2(J) = \frac{\alpha_{\ell,k}}{2} \cdot \prod_{i=1}^{\ell} \gamma(u_i) \cdot \sum_{i=1}^{\ell} \left(u_i^2 \int_Z (X_0^{(i)}(z)^2 - 1)\,\mu(dz) + D^{(i)} \right).$$

(ii) *(Abstract cancellation) If $u_i = D^{(i)} = 0$ for every $i \in [\ell]$, then*

$$\mathrm{proj}_0(J) = \mathbb{E}[J] = \frac{\alpha_{\ell,k}}{(2\pi)^{\ell/2}}, \quad \mathrm{proj}_{2q+1}(J) = \mathrm{proj}_2(J) = 0, \quad q \geq 0.$$

Specializing the content of part (ii) of Theorem I.2.2 to the setting of Gaussian Laplace eigenfunctions on manifolds without boundary yields an abstract reproduction of *Berry's cancellation phenomenon* first observed in [Ber02] and widely discussed in the literature, see for instance [Wig10, MR21, KKW13, DNPR19, MPRW16, NPR19, Cam19] for works on related models.

Applying the above to our study of nodal sets of multiple Arithmetic Random Waves, we are able to prove that $D^{(i)}$ vanishes for every $i = 1, \dots, \ell$, showing that $\mathrm{proj}_2(L_n^{(\ell)}) = \mathrm{proj}_{2q+1}(L_n^{(\ell)}) = 0$ for every $q \geq 0$. A direct investigation of the fourth-Wiener chaos as well as higher-order chaotic projections associated with $L_n^{(\ell)}$ allows us to show that as $n \to \infty$,

$$\widetilde{L_n^{(\ell)}} = \frac{\mathrm{proj}_4\,(L_n^{(\ell)})}{\sqrt{\mathbf{Var}\left[\mathrm{proj}_4(L_n^{(\ell)})\right]}} + o_{\mathbb{P}}(1)$$

where $o_{\mathbb{P}}(1)$ is a sequence of random variables converging to zero in probability. The above displayed relation implies that the probabilistic behaviour of $L_n^{(\ell)}$ is characterized by the one of its projection onto the fourth Wiener chaos. A subsequent explicit expression of the fourth-order chaotic projection in terms of T_n and its derivatives then allows to establish the exact form of its limiting distribution.

[14]Such a set of assumptions on W is formulated in such a way that it best features the properties of Gramian determinants.

An important by-product of our analysis is the following deterministic continuity result for nodal volumes associated with vector-valued functions on the torus. Such a result generalizes the findings of [APP18], where the authors consider the case of real-valued functions. Here $E = C^1(\mathbb{T}^d, \mathbb{R}^k)$ denotes the set of C^1 real vector-valued functions on the d-dimensional torus $\mathbb{T}^d$, $Z_K(F_n)$ denotes the zeros of F_n lying in a compact subset $K \subset \mathbb{T}^d$ and vol stands for the $(d-k)$-dimensional Hausdorff measure of $Z_K(F_n)$.

Theorem I.2.3. *(see Theorem II.D.3) Let $(F_n)_{n\geq 1} \subset E$ and $F \in E$ be such that F is non-degenerate on a compact $K \subset \mathbb{T}^d$ and $F_n \to F$ in the C^1 topology on K as $n \to \infty$. Then, as $n \to \infty$,*

$$\mathrm{vol}(Z_K(F_n)) \to \mathrm{vol}(Z_K(F)) .$$

Chapter III: *Matrix-Hermite Polynomials and Random Determinants*

In this chapter, we consider matrix-Hermite polynomials defined in terms of zonal polynomials, a collection of symmetric polynomials in the eigenvalues of the matrix variable. Matrix-Hermite polynomials, introduced in [Hay69], are indexed by partitions of integers and can be defined in a number of equivalent ways, one of them being for instance Rodrigues formula

$$H_\kappa^{(\ell,n)}(X)\gamma^{(\ell,n)}(X) = 4^{-k}\left(\frac{n}{2}\right)_\kappa^{-1} C_\kappa(\partial X \partial X^T)\gamma^{(\ell,n)}(X), \quad \kappa \vdash k$$

where $\gamma^{(\ell,n)}$ indicates the Gaussian density of dimension $\ell \times n$, C_κ denotes the *zonal polynomial* corresponding to the partition κ and ∂X is a matrix of differentials. Here we use the notation $\kappa \vdash k$ to indicate that κ is a partition of an integer $k \geq 0$. By fruitfully exploiting the one-dimensional Rodrigues formula in (I.1.17) for classical Hermite polynomials, we can prove the following result, thus providing explicit formulae for the projection coefficients associated with the Wiener chaos expansion of random variables depending on the eigenvalues of XX^T. Here μ_X denotes the *spectral measure* of XX^T associated with a Gaussian matrix $X \sim \mathcal{N}_{\ell\times n}(0, \mathbf{I}_\ell \otimes \mathbf{I}_n)$ and $L^2(\mu_X) := L^2(\Omega, \sigma(\mu_X), \mathbb{P})$ indicates the closed subspace of $L^2(\mathbb{P})$ of those random variables that are measurable with respect to μ_X. For a random variable F, we denote by the $\mathrm{proj}(F|C_k^X)$ the projection of F on the k-th Wiener chaos associated with X.

Theorem I.2.4. *(see Theorem III.3.2) For integers $1 \leq \ell \leq n$, let $X \sim \mathcal{N}_{\ell\times n}(0, \mathbf{I}_\ell \otimes \mathbf{I}_n)$ and write $\mathbf{X} = \mathrm{Vec}(X)$ for its vectorization. Then, for every integer $k \geq 0$ and every partition $\kappa \vdash k$, we have that $H_\kappa^{(\ell,n)}(X)$ is an element of the $2k$-th Wiener chaos $C_{2k}^{\mathbf{X}}$ associated with $\mathbf{X}$ and for every $F \in L^2(\mu_X)$,*

$$\mathrm{proj}(F|C_{2k}^{\mathbf{X}}) = \sum_{\kappa\vdash k} \widehat{F}(\kappa) H_\kappa^{(\ell,n)}(X),$$

where $\widehat{F}(\kappa)$ is given by

$$\widehat{F}(\kappa) := \left(4^{-k}\left(\frac{n}{2}\right)_\kappa^{-1} k! C_\kappa(\mathbf{I}_\ell)\right)^{-1} \int_{\mathbb{R}^{\ell\times n}} F(X) H_\kappa^{(\ell,n)}(X)\gamma^{(\ell,n)}(X)(dX).$$

In particular, we have that $\mathrm{proj}(F|C_{2k+1}^{\mathbf{X}}) = 0$. *Moreover, if $F(X) = f_0(XX^T)$ is a radial function, then*

$$\widehat{F}(\kappa) = \frac{1}{2^{n\ell/2}\Gamma_\ell(\frac{n}{2})} \frac{(-2)^k}{k! C_\kappa(\mathbf{I}_\ell)} \int_{\mathcal{P}_\ell(\mathbb{R})} f_0(R) L_\kappa^{(\frac{n-\ell-1}{2})}(2^{-1}R)\mathrm{etr}\left(-2^{-1}R\right) \det(R)^{\frac{n-\ell-1}{2}} \nu(dR), \qquad \text{(I.2.1)}$$

where $\mathrm{etr}\,(A) := \exp(\mathrm{tr}(A))$, *$L_\kappa^{(\gamma)}$ denotes the generalized Laguerre polynomial of order $\gamma > -1$ associated with the partition κ and $\nu(dR)$ is the Lebesgue measure on the space of $\ell \times \ell$ positive definite matrices $\mathcal{P}_\ell(\mathbb{R})$.*

Formula (I.2.1) is particularly useful when the integral in the R.H.S is explicitly computable and is based on the use of the instrinsic relation

$$\gamma_\kappa \cdot L_\kappa^{(\frac{n-\ell-1}{2})}(XX^T) = H_\kappa^{(\ell,n)}(\sqrt{2}X), \quad \gamma_\kappa := (-2)^{-k}\left(\frac{n}{2}\right)_\kappa^{-1},$$

linking matrix-variate Hermite polynomials and generalized Laguerre polynomials. In view of our applications dealing with certain geometric functionals of random fields, our particular interest of spectral random variables are random *Gramian determinants* of the type $\det(XX^T)^{1/2}$. Such a quantity notably appears in the Area/Co-Area formula (see Proposition I.1.11) for integral representations of geometric measures of level sets. In the following theorem, we provide explicit formulae for projection coefficients $\widehat{F}(\kappa;\Sigma)$ associated with such radial determinants in the case where the rows of the underlying Gaussian matrix are i.i.d Gaussian vectors with covariance matrix Σ, and show that they admit a geometric interpretation in terms of intrinsic volumes.

Theorem I.2.5. *(see Theorem III.3.6) For integers $1 \le \ell \le n$ and $\Sigma \in \mathbb{R}^{n\times n}$ positive-definite symmetric, let $X \sim \mathcal{N}_{\ell\times n}(0, \mathbf{I}_\ell \otimes \Sigma)$. Then, for $F(X) = \det(XX^T)^{1/2}$, we have*

$$\widehat{F}(\kappa;\Sigma) = \frac{(-2)^k}{\det(\Sigma)^{\ell k} k!}\left(\frac{n}{2}\right)_\kappa \sum_{s=0}^{k}\sum_{\sigma\vdash s}\binom{\kappa}{\sigma}(-1)^s \frac{(\frac{n+1}{2})_\sigma}{(\frac{n}{2})_\sigma}\cdot\frac{(n)_\ell}{(2\pi)^{\ell/2}}\binom{n}{\ell}^{-1} V_\ell(\mathcal{E}_\Sigma)$$

where $V_\ell(\mathcal{E}_\Sigma)$ stand for the ℓ-th intrinsic volume of the ellipsoid $\mathcal{E}_\Sigma$ associated with the covariance matrix Σ. In particular, for $\kappa = (0)$,

$$\mathbb{E}\left[\det(XX^T)^{1/2}\right] = \frac{(n)_\ell}{(2\pi)^{\ell/2}}\binom{n}{\ell}^{-1} V_\ell(\mathcal{E}_\Sigma). \tag{I.2.2}$$

We point out that formula (I.2.2) has been established by Kabluchko and Zaporozhets [ZK12]. Therefore, the content of Theorem I.2.5 can be seen as a generalization of their results to *arbitrary* projection coefficients associated with matrix-variate Hermite polynomials. As a by-product of such a result, we are able to establish the following new identity for computing intrinsic volumes of ellipsoids

$$V_\ell(\mathcal{E}_\Sigma) = \binom{n}{\ell}\frac{\kappa_n}{\kappa_{n-\ell}}\det(\Sigma)^{-\ell/2}\int_{O(n,\ell)}\det(U\Sigma^{-1}U^T)^{-(n+1)/2}\tilde{\mu}(dU), \quad 1 \le \ell \le n,$$

where κ_k is the volume of the k-dimensional unit ball and the integration in the R.H.S is with respect to the Haar probability measure on the *Stiefel manifold* $O(n,\ell)$ of ℓ-frames in $\mathbb{R}^n$.

In a second part of this chapter, we introduce a suitable collection of operators on matrix spaces via a *generalized Mehler-type formula*, whose definition is amenable to that of the classical Ornstein-Uhlenbeck semigroup on the real line,

$$O_{t;A}^{(\ell,n)} f(X) = \mathbb{E}\left[\int_{O(n)} f(XHe^{-tA} + X_0(\mathbf{I}_n - e^{-2tA})^{1/2})\tilde{\mu}(dH)\middle| X\right], \quad t \ge 0. \tag{I.2.3}$$

Here, the expectation is taken with respect to $X_0 \sim \mathcal{N}_{\ell\times n}(0, \mathbf{I}_\ell \otimes \mathbf{I}_n)$, $A \in \mathbb{R}^{n\times n}$ is a diagonal matrix with non-negative entries, e^{tA} denotes the matrix exponential of A, and $\tilde{\mu}$ is the Haar probability measure on the orthogonal group $O(n)$. Such a relation is analogous to the classical Mehler's formula for the univariate Ornstein-Uhlenbeck semigroup $\{P_t : t \ge 0\}$ (see Proposition I.1.20). The following result characterizes the action of the operator $O_{t;A}^{(\ell,n)}$ on the class of matrix-variate Hermite polynomials, showing in particular that, as for classical Hermite polynomials and the semigroup P_t, matrix-variate Hermite polynomials are eigenfunctions of $O_{t;A}^{(\ell,n)}$. Let $\Pi(\ell,n)$ denote the class of matrix-variate functions that are right-invariant by rotations, that is, $f : \mathbb{R}^{\ell\times n} \to \mathbb{R}$ such that $f(XO) = f(X)$ for every orthogonal matrix $O \in O(n)$.

Theorem I.2.6. *(see Theorem III.3.10) For every diagonal matrix* $A = \mathrm{diag}(a_1, \dots, a_n) \in \mathbb{R}^{n\times n}$ *such that* $a_1, \dots, a_n \geq 0$*, every integer* $k \geq 0$ *and every partition* $\kappa \vdash k$*, we have that*

$$O^{(\ell,n)}_{t;A} H^{(\ell,n)}_{\kappa}(X) = \frac{C_\kappa(e^{-2tA})}{C_\kappa(\mathbf{I}_n)} H^{(\ell,n)}_{\kappa}(X). \tag{I.2.4}$$

In particular, the family $\left\{O^{(\ell,n)}_{t;A} : t \geq 0\right\}$ *is a semigroup on the class* $\Pi(\ell, n)$ *if and only if* $a_1 = \dots = a_n = a$*. More precisely, in this case,* $O^{(\ell,n)}_{t;A}$ *coincides with* $P^{(\ell n)}_{at}$ *on the class* $\Pi(\ell, n)$*.*

Exploiting the above result and the generalized Mehler's formula in (I.2.3), allows us to prove the following extended orthogonality relation for matrix-variate Hermite polynomials.

Theorem I.2.7. *(see Theorem III.3.12) Let* $X, X_0 \sim \mathcal{N}_{\ell\times n}(0, \mathbf{I}_\ell \otimes \mathbf{I}_n)$ *be independent and* R *be a deterministic matrix of dimension* $n \times n$*. Let* $Y \overset{d}{=} XR + X_0(\mathbf{I}_n - R^2)^{1/2}$ *in distribution. Then, for every integers* $k, l \geq 0$ *and every partitions* $\kappa \vdash k, \sigma \vdash l$*, we have*

$$\mathbb{E}\left[H^{(\ell,n)}_{\kappa}(X) H^{(\ell,n)}_{\sigma}(Y)\right] = \mathbb{1}_{\kappa=\sigma} \cdot 4^{-k} \left(\frac{n}{2}\right)^{-1}_{\kappa} k! C_\kappa(R^2) \frac{C_\kappa(\mathbf{I}_\ell)}{C_\kappa(\mathbf{I}_n)}. \tag{I.2.5}$$

We finish this chapter by applying our results to the study of the *total variation* of multiple independent Arithmetic Random Waves $\mathbf{T}^{(\ell)}_n$ on the three-torus (as considered in Chapter II)

$$\mathbf{V}(\mathbf{T}^{(\ell)}_n; \mathbb{T}^3) = \left(\frac{E_n}{3}\right)^{\ell/2} \int_{\mathbb{T}^3} \Phi(\widetilde{\mathrm{jac}}_{\mathbf{T}^{(\ell)}_n}(z)) dz. \tag{I.2.6}$$

Here $\Phi(M) := \sqrt{\det(MM^T)}$ and $\widetilde{\mathrm{jac}}_{\mathbf{T}^{(\ell)}_n}(z)$ denotes the normalized Jacobian matrix of $\mathbf{T}^{(\ell)}_n$. We characterize the asymptotic probabilistic behaviour of the random variable in (I.2.6) by providing its exact mean, an asymptotic law for its variance and subsequent second-order Gaussian fluctuations for the suitably normalized total variation $\widehat{\mathbf{V}}(\mathbf{T}^{(\ell)}_n; \mathbb{T}^3)$.

Theorem I.2.8. *(see Theorem III.3.17) Let the above notation prevail.*

(i) *(Expected total variation) For every* $n \in S_3$*, we have*

$$\mathbb{E}\left[\mathbf{V}(\mathbf{T}^{(\ell)}_n; \mathbb{T}^3)\right] = \left(\frac{E_n}{3}\right)^{\ell/2} 2^{\ell/2} \frac{\Gamma_\ell(2)}{\Gamma_\ell(\frac{3}{2})}$$

(ii) *(Asymptotic variance) As* $n \to \infty, n \not\equiv 0, 4, 7 \pmod 8$*,*

$$\mathbf{Var}\left[\mathbf{V}(\mathbf{T}^{(\ell)}_n; \mathbb{T}^3)\right] = \left(\frac{E_n}{3}\right)^{\ell} 2^{\ell} \frac{\Gamma_\ell(2)^2}{\Gamma_\ell(\frac{3}{2})^2} \frac{\ell}{2\mathcal{N}_n} \left(1 + O(n^{-1/28+o(1)})\right)$$

(iii) *(Central Limit Theorem) As* $n \to \infty, n \not\equiv 0, 4, 7 \pmod 8$*,*

$$\widehat{\mathbf{V}}(\mathbf{T}^{(\ell)}_n; \mathbb{T}^3) := \frac{\mathbf{V}(\mathbf{T}^{(\ell)}_n; \mathbb{T}^3) - \mathbb{E}\left[\mathbf{V}(\mathbf{T}^{(\ell)}_n; \mathbb{T}^3)\right]}{\sqrt{\mathbf{Var}\left[\mathbf{V}(\mathbf{T}^{(\ell)}_n; \mathbb{T}^3)\right]}} \overset{d}{\longrightarrow} \mathcal{N}(0, 1),$$

where $\overset{d}{\longrightarrow}$ *denotes convergence in distribution.*

Our proof of Theorem I.2.8 is based on the expansion of the total variation in matrix-Hermite polynomials by means of Theorem I.2.4. In particular, we are able to prove that its high-energy behaviour is fully characterized by the fluctuations of its second chaotic projection, entailing the asymptotic Gaussianity.

Chapter III: *Weak convergence of Berry's nodal length process*

In this chapter, we study the nodal length associated with zero sets of the two-dimensional Berry random wave B_E with parameter $E > 0$, that is, the unique stationary and isotropic Gaussian field verifying the equation $\Delta B_E + 4\pi^2 E B_E = 0$. Such a model was already studied in seminal works by Berry [Ber02, Ber77] and later by Nourdin, Peccati and Rossi [NPR19] and Peccati, Vidotto [PV20], confirming Berry's observations, see also [KW18, BCW17, CH20, DNPR20] for further works. In [NPR19] and [PV20], the authors prove one-dimensional Central Limit Theorems and multidimensional extensions for the nodal length restricted to planar domains, and in particular deduce that, in the high-energy limit, the normalized nodal length process

$$X_E = \left\{ X_E(t_1,t_2) := \sqrt{\frac{512\pi}{\log E}}\left(\mathscr{L}_E([0,t_1]\times[0,t_2]) - \mathbb{E}\left[\mathscr{L}_E([0,t_1]\times[0,t_2])\right]\right) : (t_1,t_2) \in [0,1]^2 \right\},$$

(where $\mathscr{L}_E([0,t_1]\times[0,t_2])$ indicates the nodal length of B_E restricted to the rectangle $[0,t_1]\times[0,t_2]$) converges in the sense of finite-dimensional distributions to a standard two-parameter Wiener sheet on $[0,1]^2$, thus suggesting a functional limit theorem for X_E. There are multiple motivations for studying such weak convergence result, typically giving access to a number of new limit theorems of semi-local type, involving for instance suprema of normalized nodal length processes, which provide global indicators for the discrepancy between the nodal length and its mean. The installment of a general functional limit theorem for X_E was however not successful in [PV20], motivated by technical difficulties when dealing with second chaotic projections associated with X_E. The goal of this chapter is to overcome these difficulties by studying these components in detail. In the following theorem, we derive variance estimates and a multi-dimensional Central Limit Theorem for the second chaotic projections.

Theorem I.2.9. *(see Theorem IV.1.4) For every* $\mathbf{t} = (t_1,t_2) \in [0,1]^2$*, we set* $\mathcal{D}_{\mathbf{t}} := [0,t_1]\times[0,t_2]$ *with boundary* $\partial\mathcal{D}_{\mathbf{t}}$ *and let*

$$Y_E(\mathbf{t}) := \frac{\mathscr{L}_E[2](\mathcal{D}_{\mathbf{t}})}{\sqrt{\mathbf{Var}[\mathscr{L}_E[2](\mathcal{D}_{\mathbf{t}})]}}.$$

For every integer $d \geq 1$ *and every collection of* $\mathbf{t}_1,\dots,\mathbf{t}_d \in [0,1]^2$*, we have that, as* $E \to \infty$

$$(Y_E(\mathbf{t}_1),\dots,Y_E(\mathbf{t}_d)) \xrightarrow{d} Z \sim \mathcal{N}_d(0,\Sigma)$$

where Z *is a centred* d*-dimensional Gaussian vector with covariance matrix* $\Sigma = \{\Sigma(i,j) : i,j = 1,\dots,d\}$ *given by the relation*

$$\Sigma(i,j) = \frac{\lambda(\partial\mathcal{D}_{\mathbf{t}_i},\partial\mathcal{D}_{\mathbf{t}_j})}{\sqrt{\lambda(\partial\mathcal{D}_{\mathbf{t}_i},\partial\mathcal{D}_{\mathbf{t}_i})\lambda(\partial\mathcal{D}_{\mathbf{t}_j},\partial\mathcal{D}_{\mathbf{t}_j})}},$$

where $\lambda(\partial\mathcal{D}_{\mathbf{t}_i},\partial\mathcal{D}_{\mathbf{t}_j})$ *denotes the signed length of* $\partial\mathcal{D}_{\mathbf{t}_i}\cap\partial\mathcal{D}_{\mathbf{t}_j}$*.*

The proof of Theorem I.2.9 is based on a preliminary study of random variables taking the form (see Definition IV.1.17)

$$\phi_E(C) = \frac{1}{8\pi\sqrt{2E}}\int_C B_E(x)\langle\nabla B_E(x), \mathrm{n}_C(x)\rangle dx,$$

where C is a *polygonal curve*, dx indicates one-dimensional Hausdorff measure on C and n_C denotes the unit normal vector to C computed at x. Such a definition is motivated by the expression of the second chaotic projections associated with the nodal length. For such random variables, we derive the following result, yielding asymptotic dependence structures and subsequent second-order results.

Theorem I.2.10. *(see Theorem IV.1.20 and Theorem IV.1.21) For polygonal curves C_1 and C_2, we have that, as $E \to \infty$,*

$$\mathbf{Cov}[\phi_E(C_1), \phi_E(C_2)] = \frac{\lambda(C_1, C_2)}{16\pi^2\sqrt{E}} + o\left(\frac{1}{\sqrt{E}}\right),$$

where $\lambda(C_1, C_2)$ denotes the signed length of the $C_1 \cap C_2$. Furthermore, writing $\widetilde{\phi}_E(C_1) := 4\pi E^{1/4}\phi_E(C_1)$, we have that, for every integer $d \geq 1$ and every collection of polygonal curves $C_1, \dots, C_d$

$$\left(\widetilde{\phi}_E(C_1), \dots, \widetilde{\phi}_E(C_d)\right) \xrightarrow{d} \mathcal{N}_d(0, \Sigma),$$

where $\Sigma = \{\Sigma(i,j) : i, j = 1, \dots, d\}$ is the $d \times d$ matrix defined by

$$\Sigma(i,j) := \lambda(C_i, C_j), \quad i, j = 1, \dots, d.$$

In particular, specializing the findings of Theorem I.2.9 to the family of concentric squares $R_t := [1/2 - t, 1/2 - t]^2, 0 < t < 1/2$ centred at the point $(1/2, 1/2)$, yields that the limit of Y_E is a total disorder process, that is, a Gaussian process whose linear span contains an uncountable collection of i.i.d standard Gaussian random variables. Total disorder processes appear as the limiting object in a number of works dealing with Random Matrix Theory or Mathematical Physics, see for instance [Leb83, Wie02, DE01, HNY08, Sel92] for such a collection of works.

Corollary I.2.11. *(see Corollary IV.1.5) The limiting process of $\left\{Y_E(\mathbf{t}) : \mathbf{t} \in [0,1]^2\right\}$ is a total disorder process.*

Combining the above results with a suitable tightness criterion by Davydov and Zitikis [DZ08] for proving weak convergence of processes on $[0,1]^d$ with some moment estimates for suprema of Gaussian fields, we are able to prove that the second chaotic projections of X_E converge weakly to zero in the Skorohod space $\mathbf{D}_2 = D([0,1]^2, \mathbb{R})$.

Corollary I.2.12. *(see Corollary IV.1.6) As $E \to \infty$, the process $\left\{X_E[2](\mathbf{t}) : \mathbf{t} \in [0,1]^2\right\}$ converges weakly to zero in $\mathbf{D}_2$.*

In view of the above results for second chaotic projections, in order to prove a weak convergence result, we study the higher-order projections associated with X_E. Based on a chaining argument inspired by works by Dehling, Taqqu [DT89] and Marinucci, Wigman [MW11] in the framework of empirical processes associated with certain underlying Gaussian fields and spherical harmonics on the sphere, we are able to prove a weak convergence result for a *discretized* version of the nodal length. More precisely, for an integer $K \geq 1$ and a partition of the unit square into squares of side length 2^{-K}, we define the *discretized nodal length* by

$$\mathscr{L}_E^K([0,t_1] \times [0,t_2]) := \mathscr{L}_E\left([0, p_{i_{1,K}(t_1)}(K)] \times [0, p_{i_{2,K}(t_2)}(K)]\right)$$

where $p_{i_{1,K}(t_1)}(K)$ and $p_{i_{2,K}(t_2)}(K)$ are the coordinates of the partition point which are closest to t_1 and t_2, respectively. We denote by X_E^K its normalized version

$$X_E^K(\mathbf{t}) = \sqrt{\frac{512\pi}{\log E}}\left(\mathscr{L}_E^K([0,t_1] \times [0,t_2]) - \mathbb{E}\left[\mathscr{L}_E^K([0,t_1] \times [0,t_2])\right]\right).$$

For the residue term formed by higher-order projections associated with X_E^K, we prove the uniform convergence result.

Theorem I.2.13. *(see Theorem IV.1.8) Let* $\{K(E) : E > 0\}$ *be a numerical sequence such that* $K(E) \to \infty$ *and* $K(E) = o((\log E)^{1/10})$ *as* $E \to \infty$*. Then, for every* $\varepsilon > 0$*,*

$$\mathbb{P}\left\{ \sup_{\mathbf{t}\in[0,1]^2} \left| R_E^{K(E)}(\mathbf{t}) \right| > \varepsilon \right\} \to 0.$$

Combining Theorem I.2.13 with Corollary I.2.12 and the fact that the fourth chaotic projections of X_E converge weakly to a standard Wiener sheet (thanks to the main findings of [PV20]), we deduce the following functional limit theorem in the high-energy regime for the discretized nodal length process.

Corollary I.2.14. *(see Corollary IV.1.9) Let* $\{K(E) : E > 0\}$ *be a numerical sequence such that* $K(E) \to \infty$ *and* $K(E) = o((\log E)^{1/10})$ *as* $E \to \infty$*. Then, as* $E \to \infty$*, the normalized process* $X_E^{K(E)}$ *converges weakly to a standard Wiener sheet* $\mathbf{W}$ *on* $[0,1]^2$ *in* $\mathbf{D}_2$*.*

The weak convergence result formulated above allows us to have access to a number of novel limit theorem, dealing for instance with suprema of discretized nodal length processes. The following statement gives an explicit expression for the asymptotic distribution function of the supremum over the boundary of the unit square of the discretized nodal length.

Corollary I.2.15. *(see Corollary IV.1.11) Let* $\{K(E) : E > 0\}$ *be a numerical sequence such that* $K(E) \to \infty$ *and* $K(E) = o((\log E)^{1/10})$ *as* $E \to \infty$*. Then, for every* $z \in \mathbb{R}$*, we have that, as* $E \to \infty$*,*

$$\mathbb{P}\left\{ \sup_{\mathbf{t}\in\partial[0,1]^2} \left| X_E^{K(E)}(\mathbf{t}) \right| \le z \right\} \to \mathbb{P}\left\{ \sup_{\mathbf{t}\in\partial[0,1]^2} |\mathbf{W}(\mathbf{t})| \le z \right\} = 1 - 3\Phi(-z) + e^{4z^2}\Phi(-3z),$$

where $\Phi(z) := \mathbb{P}\{N \le z\}, N \sim \mathcal{N}(0,1)$*.*

Our difficulties to extend such a functional limit theorem from $X_E^{K(E)}$ to X_E emerge in the planar chaining argument, presented in order to prove Theorem I.2.13, the essential difficulty being that the mean of the nodal length grows exponentially faster than the logarithmic normalization factor.

Furthermore, combining our results on the second Wiener chaos with a hypercontractivity argument on Wiener chaoses, allows us to deduce the following weak convergence result for the *truncated* nodal length process of increasing degree,

$$X_E(\mathbf{t}; N) := \sum_{q=1}^{N} X_E[2q](\mathbf{t}).$$

Corollary I.2.16. *(see Corollary IV.1.15) Let* $N(E) = \log_5(\log E)$*. Then, as* $E \to \infty$*, the process* $\left\{X_E(\mathbf{t}; N(E)) : \mathbf{t} \in [0,1]^2\right\}$ *converges weakly to a standard Wiener sheet* $\mathbf{W}$ *on* $[0,1]^2$ *in* $\mathbf{D}_2$*.*

Chapter V: *Non-linear functionals of d-dimensional Berry's random fields*

In this chapter, we consider the d-dimensional Berry Random Wave model and investigate the high-energy behaviour as $E \to \infty$ of non-linear random variables taking the form

$$Z_E(d, q; \mathcal{D}) := \int_{\mathcal{D}} H_q(B_E^{(d)}(x))dx, \quad \mathcal{D} \subset \mathbb{R}^d$$

where q and d are integers, $B_E^{(d)}$ stands for the Berry random field in $\mathbb{R}^d$, and H_q is the q-th Hermite polynomial on the real line. Our motivation for studying such a type of random variables comes from the Wiener-Itô chaos expansion of geometric measures associated with level sets of the $B_E^{(d)}$. Indeed, the random variables $Z_E(d,q;\mathcal{D})$ appear as a typical element in the projection on the q-th Wiener chaos of these geometric measures. In Theorem I.2.17 below, we provide asymptotic variance estimates for the random variable $Z_E(d,q;\mathcal{D})$, whenever q and d are elements of the set

$$S := \{(d,q) : d \geq 2, q \geq 3\} \setminus \{(2,3),(3,3)\}.$$

Our reason for omitting certain pairs from the set S above originates from some technical difficulties arising when trying to optimally bound the contribution of some residual terms and is addressed in Remark V.1.3.

Theorem I.2.17. *(see Theorem V.1.1) Let $(d,q) \in S$. If $(d,q) = (2,4)$, we have that, as $E \to \infty$,*

$$\mathbf{Var}[Z_E(d,q;\mathcal{D})] \sim \frac{9}{\pi^3}\mathrm{vol}_2(\mathcal{D})\frac{\log E}{E},$$

whereas, if $(d,q) \neq (2,4)$, we have that, as $E \to \infty$

$$\mathbf{Var}[Z_E(d,q;\mathcal{D})] \sim \alpha(d;q)\frac{1}{E^{d/2}},$$

where

$$\alpha(d;q) := \mathrm{vol}_d(\mathcal{D})q!d\kappa_d(2\pi)^{-\frac{q}{2}(d-2)}\int_0^\infty d\psi\, J_{\frac{d-2}{2}}(2\pi\psi)^q\psi^{(d-1)(1-\frac{q}{2})+\frac{q}{2}}, \tag{I.2.7}$$

with κ_d denoting the volume of the unit ball in $\mathbb{R}^d$ and $\mathrm{vol}_d(\mathcal{D})$ the d-dimensional volume of $\mathcal{D}$.

Our result shows that $(d,q) = (2,4)$ is the only pair of integers leading to *logarithmic* variance fluctuations, thus recovering the findings of Nourdin, Peccati and Rossi in [NPR19]. Furthermore, it becomes clear that, whenever $(d,q) \neq (2,4)$, the variance is always of lower order $E^{-d/2}$. Such an observation is particularly interesting when $d \geq 3$ is fixed: indeed, in this case, our results conjecturally hint to the fact that the projections of the geometric measure of level sets associated with the d-dimensional Berry random wave model are *all* of same order, thus not resulting in a dominance principle as is observed for instance in dimension two and on the related model of Arithmetic Random Waves in dimensions two and three (see Chapter II). Our arguments for proving Theorem I.2.17 rely on the asymptotic study of moments of Bessel functions.

In view of Theorem I.2.17, we consider the normalized random variables $\widetilde{Z}_E(d,q;\mathcal{D}) := E^{d/4}Z_E(d,q;\mathcal{D})$ for $(d,q) \neq (2,4)$ and prove the following quantitative Central Limit Theorem in Kolmogorov, total variation and Wasserstein distance. Note that, in the statement below, the exponent $d - q\frac{d-1}{2} < 0$, which implies the asymptotic Gaussianity.

Theorem I.2.18. *(see Theorem V.1.2) Assume that $(d,q) \in S$ is such that $(d,q) \neq (2,4)$ and let $N \sim \mathcal{N}(0,\alpha(d;q)^2)$, where $\alpha(d;q)$ is as in* (I.2.7)*. Then, for* $\mathrm{U} \in \{\mathrm{Kol},\mathrm{TV},\mathrm{W}\}$*, we have that*

$$d_{\mathrm{U}}\left(\widetilde{Z}_E(d,q;\mathcal{D}),N\right) \leq c_1(\log E)^{3/2}\sqrt{E}^{d-q\frac{d-1}{2}},$$

where c_1 is some absolute constant that is independent of E. In particular, $\widetilde{Z}_E(d,q;\mathcal{D})$ converges in distribution to N, as $E \to \infty$.

Our findings of this chapter are to be compared with the works [MR15, MW11], where the authors present the analog study for random spherical harmonics on the unit d-sphere. We finish this chapter with several comments on *reduction principles* on Wiener chaoses and argue how our findings might be useful for deducing variance estimates for the nodal length of the d-dimensional Berry random field.

Chapter VI: *Optimality of convergence rates in Gamma Approximations*

The last chapter of this dissertation is generally speaking about characterizations of optimal convergence rates within Stein's method for Gamma approximations. More precisely, given a suitable probability metric d, a sequence of random variables $\{F_n : n \geq 1\}$ converging in law to some target random variable F_∞, and a strictly positive numerical sequence $\{\phi(n) : n \geq 1\}$ such that $d(F_n, F_\infty) \leq \phi(n)$, one is interested in knowing whether or not $\phi(n)$ is an *optimal* rate of convergence for the metric d. Conventionally, such an optimality is detected as soon as there are positive constants $0 < c_1 < c_2$ such that $c_1\phi(n) \leq d(F_n, F_\infty) \leq c_2\phi(n)$ is verified for sufficiently large n (see in particular Definition VI.1.4). Several works on optimality of convergence rates within Steins's method have been published in the literature, see for instance [NP09b, NP15, BBNP12, Cam13] for works around normal approximations and [AEK20] for Gamma approximations.

In this manuscript, we deal with the case where $F_\infty = G(\nu)$ is a *centred Gamma random variable with parameter* ν and d is the Wasserstein distance, that is, $d_W(F, G) := \sup\{|\mathbb{E}[h(F)] - \mathbb{E}[h(G)]| : h\}$, where the supremum is taken over the class of test functions that are Lipschitz continuous with constant less than one. Furthermore we assume that each $F_n = I_2(f_n)$ is an element of the second Wiener chaos associated with some isonormal Gaussian process, and is such that $\mathbb{E}\left[F_n^2\right] = 2\nu$ and $F_n \xrightarrow{d} G(\nu)$ as $n \to \infty$. Our methodological approach is to represent F_n as the series converging both $\mathbb{P}$-almost surely and in $L^2(\mathbb{P})$

$$F_n = I_2(f_n) = \sum_{j=1}^{N(f_n)} \lambda_j(f_n) H_2(N_j) \tag{I.2.8}$$

where $\lambda_j(f_n)$ are the eigenvalues of a certain Hilbert-Schmidt operator associated with the kernel f_n, $N(f_n)$ denotes its rank and $(N_j : j \geq 1)$ is a sequence of i.i.d standard Gaussian random variables. As usual, $H_2(x) = x^2 - 1$ stands for the second Hermite polynomial. Following the route of Nourdin and Peccati in [NP09b], our strategy consists in characterizing the joint convergence of the bivariate vector $(F_n, F_n^{(\nu)})$, where

$$F_n^{(\nu)} = \frac{2(F_n + \nu) - \langle DF_n, -DL^{-1}F_n\rangle_H}{\phi_\nu(n)}, \quad \phi_\nu(n) := \sqrt{\mathbb{E}\left[(2(F_n + \nu) - \langle DF_n, -DL^{-1}F_n\rangle_H)^2\right]},$$

and where D and L^{-1} indicate the Malliavin derivative and the pseudo-inverse of the generator of the Ornstein-Uhlenbeck semigroup, respectively. It is not difficult to prove that, whenever $F_n = I_2(f_n)$ then $F_n^{(\nu)}$ is also an element of the second Wiener chaos (see Proposition VI.2.2). Moreover, if $F_n = I_2(f_n) \xrightarrow{d} G(\nu)$, we necessarily have that ν is an integer, and it holds that $\lambda_j(f_n) \to 1$ for $j = 1, \ldots, \nu$ and $\lambda_j(f_n) \to 0$ for $j \geq \nu + 1$, as $n \to \infty$ (see Proposition VI.2.11). The starting point of our analysis is the Stein bound in Wasserstein distance $d_W(F_n, G(\nu)) \leq \max(1, 2/\nu)\phi_\nu(n)$ derived by Döbler and Peccati in [DP18]. Our main result is a partial characterization of the probabilistic fluctuations of the random variable $F_n^{(\nu)}$ above in the two cases where the rank $N(f_n)$ in (I.2.8) is finite or infinite, and can be compactly formulated as follows. Here, for numerical sequences $(a_n), (b_n)$, we write $a_n \asymp b_n$ if $0 < \liminf_{n\to\infty} |a_n/b_n| \leq \limsup_{n\to\infty} |a_n/b_n| < \infty$.

Theorem I.2.19. *(see Theorem VI.2.5 and Theorem VI.2.6) Let the above framework prevail and define the sequences*

$$\omega(n) := \max\left\{|\lambda_{n,j} - 1| : j = 1, \ldots, \nu\right\}, \quad \vartheta(n) := \sum_{j=\nu+1}^{N} \lambda_{n,j}^2.$$

(i) *Assume* $N(f_n) = N < \infty$ *for every* $n \geq 1$. *Then there exist real numbers* $\left\{\ell_j : j = 1, \ldots, N\right\}$ *such that* $\sum_{j=1}^{\nu} \ell_j = 0$ *and* $F_n^{(\nu)} \xrightarrow{d} \sum_{j=1}^{N} \ell_j H_2(N_j)$, *as* $n \to \infty$.

(ii) *Assume* $N(f_n) \to \infty$ *and* $\vartheta(n) \asymp \omega(n)$ *as* $n \to \infty$. *Then,* $F_n^{(\nu)} \xrightarrow{d} N_0$, *where* N_0 *is a standard Gaussian random variable that is independent of* $(N_j : j \geq 1)$.

We remark that part (ii) of the above statement, does not include the case where $\vartheta(n) = o(\omega(n))$. Preliminary computations show that in this case, the limiting distribution of $F_n^{(\nu)}$ is non Gaussian. We refer the reader to Remark VI.2.7 for more details on this.

By suitably applying the integration by part formula on a Gaussian space, our conclusions derived in the above theorem are sufficient to deduce the following corollary, showing in particular that the sequence $\{\phi_\nu(n) : n \geq 1\}$ provides *non-strongly optimal* rates of convergence (see Definition VI.1.4). For an integer $\nu > 0$, we define the subclass $\Sigma(\nu)$ of the second Wiener chaos to be the collection of all sequences $\{F_n = I_2(f_n), n \geq 1\}$ such that $\mathbf{Var}[F_n] = 2\nu$ for every $n \geq 1$, $F_n \xrightarrow{d} G(\nu)$ and verifying the conditions and (i) $N(f_n) = N < \infty$ for some $N \in \mathbb{N}$ or (ii) $N(f_n) \to \infty$ and $\vartheta(n) \asymp \omega(n)$ as $n \to \infty$.

Corollary I.2.20. *(see Corolloray VI.2.8) Let the above setting prevail. Then, for every* $\{F_n : n \geq 1\} \in \Sigma(\nu)$ *and every* $h \in \mathrm{Lip}(1)$, *we have that, as* $n \to \infty$

$$\frac{\mathbb{E}\left[h(F_n)\right] - \mathbb{E}\left[h(G(\nu))\right]}{\phi_\nu(n)} \to 0.$$

Whether such a phenomenon continues to hold *on every fixed Wiener chaos* of order greater than two is a natural question. We refer the reader to Conjecture VI.2.10 and the subsequent Section VI.2.1 for further considerations towards a more general result. A preliminary specific example on the fourth Wiener chaos, in which we detect the same phenomenon, is exposed in Section VI.2.3.

Chapter II

Fluctuations of nodal sets on the three-torus and abstract cancellation phenomena

In 2017, Benatar and Maffucci [BM19] established an asymptotic law for the variance of the nodal surface of arithmetic random waves on the 3-torus in the high-energy limit. In a subsequent work, Cammarota [Cam19] proved a universal non-Gaussian limit theorem for the nodal surface. In this chapter, we study the nodal intersection length and the number of nodal intersection points associated, respectively, with two and three independent arithmetic random waves of same frequency on the 3-torus. For these quantities, we compute their expected value, asymptotic variance as well as their limiting distribution. Our results are based on Wiener-Itô expansions for the volume and naturally complement the findings of Cammarota [Cam19]. At the core of our analysis lies an abstract cancellation phenomenon applicable to the study of level sets of arbitrary Gaussian random fields, that we believe has independent interest.

II.1 Introduction

II.1.1 Overview

The present chapter deals with the high-energy behaviour of the nodal set associated with arithmetic random waves (ARW) on the 3-torus, $\mathbb{T}^3$. ARWs (first introduced in [ORW08, RW08] for tori of arbitrary dimension) are Gaussian stationary eigenfunctions of the Laplace operator on the torus. In recent years, such a model has been intensively studied, in the framework of a more general program, focussing on the high-energy behaviour of local and non-local functionals of random Laplace eigenfunctions on generic manifolds (see e.g. [CS19, SW19, Roz17, WY19, KKW13, RW08, GW17, MPRW16, DNPR19, Tod20, Tod19, PV20, DEL21]).

Our specific aim is to extend the findings of Benatar and Maffucci [BM19], that first provided an exact asymptotic variance for the nodal surface area of the nodal set of ARW on $\mathbb{T}^3$, and Cammarota [Cam19], that subsequently derived the limiting distribution of the normalized nodal surface area. More precisely, our goal is to study the high-energy behaviour of two further geometric quantities associated with vectors of ARWs, namely: (i) the nodal length of so-called *dislocation lines* of ARWs (see e.g. [Den01]), obtained when intersecting the zero sets of two independent ARWs with the same eigenvalue and (ii) the number of intersection points obtained when intersecting the zero sets of three independent ARWs with the same eigenvalue. For both quantities, we provide the exact expected value, precise variance asymptotics and second-order limit results. Our findings recover and extend the work of [Cam19]. Such a contribution is located in a series of works exploiting Wiener chaos techniques for deriving limit results of geometric functionals associated with Gaussian fields (see e.g. [CMW16b, CMW16a, EL16, DNPR19, MPRW16, NPR19, Cam19, DEL21]). Our main source of arithmetic results, serving as building blocks for the nodal variance asymptotics, is [BM19].

An important contribution of our analysis is a detailed study of the Wiener-Itô chaos expansion associated with non-linear geometric functionals of (possibly multi-dimensional) Gaussian fields admitting an integral representation in terms of generalized Jacobians (see Appendix II.A). In particular, our findings of Section II.2.1 provide a full description of a general cancellation phenomenon that (i) explains all exact cancellations for the nodal length of Gaussian Laplace eigenfunctions on manifolds without boundary encountered so far (see e.g. [DNPR19, MRW20, MPRW16, Cam19]); (ii) contains as special cases the projection formulae (see also Appendix II.B) for nodal length and number of phase singularities of Berry's Random Wave model (see [NPR19]).

Notation. Throughout this chapter, every random object is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by $\mathbb{E}[\cdot]$ and $\mathbf{Var}[\cdot]$ the mathematical expectation and the variance with respect to $\mathbb{P}$, respectively. Also, $\gamma(x) := (2\pi)^{-1/2}e^{-x^2/2}$ denotes the standard Gaussian probability density on the real line.

For sequences $\{A_n : n \geq 1\}, \{B_n : n \geq 1\}$, we will use the notation $A_n \ll B_n$ or $A_n = O(B_n)$ to indicate that $|A_n| \leq C|B_n|$ for some absolute constant C. We write $A_n = o(B_n)$ whenever $A_n/B_n \to 0$ as $n \to \infty$. Also, we write $A_n \sim B_n$ whenever $A_n/B_n \to 1$ as $n \to \infty$. For random variables, the symbols $\overset{d}{=}$ and $\overset{d}{\to}$ denote equality and convergence in distribution, respectively.

For an integer $n \geq 1$, we write $[n] := \{1, \dots, n\}$. For $n \geq 0$, we denote by $\mathbf{I}_n$ the n-dimensional identity matrix with the convention that $\mathbf{I}_0 := 0 \in \mathbb{R}$. For $A \in \mathcal{M}_{p\times q}(\mathbb{R})$ and $B \in \mathcal{M}_{p'\times q'}(\mathbb{R})$, we write

$$A \oplus B := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathcal{M}_{(p+p')\times(q+q')}(\mathbb{R})$$

for the direct sum of A and B with the convention $A \oplus \mathbf{I}_0 := A$ for every $A \in \mathcal{M}_{p\times q}(\mathbb{R})$.

II.1.2 Models of ARW and relevant existing results

Let (M, g) be a smooth compact Riemannian manifold and let Δ be the associated Laplace-Beltrami operator. The spectrum of Δ is purely discrete, that is: (i) there exists a non-decreasing sequence $\left\{\lambda_j : j \geq 0\right\}$ of non-negative eigenvalues of $-\Delta$, customarily called the *energy levels of* M, and (ii) the associated eigenfunctions $\left\{f_j : j \geq 0\right\}$, satisfying

$$\Delta f_j + \lambda_j f_j = 0\,, \quad j \geq 0\,, \tag{II.1.1}$$

form an $L^2(M)$-orthonormal system. The *nodal set* of f_j is its zero set $f_j^{-1}(\{0\})$. In [Che76] it is shown that, except on a closed set of lower dimension, $f_j^{-1}(\{0\}) \subset M$ is a submanifold of codimension one. Of particular interest are quantities associated with the nodal set of f_j, such as the *nodal volume*, in the *high-energy* regime, that is, as $\lambda_j \to \infty$. Yau's conjecture [Yau82, Yau93] asserts that there exist constants $c_M, C_M > 0$, uniquely depending on M, such that

$$c_M \sqrt{\lambda_j} \leq \mathrm{vol}(f_j^{-1}(\{0\})) \leq C_M \sqrt{\lambda_j}\,,$$

with $\mathrm{vol}(\cdot)$ denoting the volume measure on M. This conjecture was proven for real-analytic manifolds M in [DF88], whereas the lower bound is a result by [Log18] in the more general case where M is smooth.

Arithmetic random waves on $\mathbb{T}^d$. Let us specialize the above framework to the setting of the d-dimensional torus. Let $d \geq 1$ be an integer, let $M = \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d = [0,1]^d/_\sim$ denote the d-dimensional flat torus, and let Δ be the Laplace-Beltrami operator on it. One is interested in quantities associated with the nodal sets of real-valued random eigenfunctions of Δ, i.e. random solutions $f : \mathbb{T}^d \to \mathbb{R}$ of (II.1.1) for some appropriate λ_j. It is a known fact that the eigenvalues of $-\Delta$ are positive real numbers of the form $E = E_n = 4\pi^2 n$, where $n \in S_d$, with

$$S_d := \left\{ m \geq 1 : \exists (m_1, \ldots, m_d) \in \mathbb{Z}^d, m = m_1^2 + \ldots + m_d^2 \right\} ,$$

that is, n is an integer expressible as a sum of d integer squares. For $n \in S_d$, we introduce the set of *frequencies*

$$\Lambda_n := \left\{ \lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{Z}^d : \lambda_1^2 + \ldots + \lambda_d^2 = n \right\} ,$$

and write $\mathrm{card}(\Lambda_n) =: \mathcal{N}_n$ (card denoting the cardinality; note that we do not mark the dependency on d) to indicate the number of ways in which n can be represented as a sum of d integer squares. An $L^2(\mathbb{T}^d)$-orthonormal system for the eigenspace $\mathscr{E}(E_n)$ associated with E_n is given by the complex exponentials

$$\{ e_\lambda(\cdot) := \exp(2\pi i \langle \lambda, \cdot \rangle) : \lambda \in \Lambda_n \} ,$$

so that $\dim \mathscr{E}(E_n) = \mathrm{card}(\Lambda_n) = \mathcal{N}_n$. For $n \in S_d$, the *arithmetic random wave of order n*, denoted by T_n, is defined as the following random linear combination of complex exponentials

$$T_n(x) = \frac{1}{\sqrt{\mathcal{N}_n}} \sum_{\lambda \in \Lambda_n} a_\lambda e_\lambda(x) , \quad x \in \mathbb{T}^3 ,$$

where the coefficients $\{a_\lambda : \lambda \in \Lambda_n\}$ are complex $\mathcal{N}(0,1)$-distributed[1] and independent except for the relation $a_\lambda = \overline{a_{-\lambda}}$, which makes T_n real-valued. It is easily seen that the law of T_n is uniquely characterized by the property of being a centred Gaussian field on $\mathbb{T}^d$ with covariance function

$$r_n(x, y) := \mathbb{E}\left[T_n(x) \cdot T_n(y) \right] = \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} e_\lambda(x - y) =: r_n(x - y) . \qquad \text{(II.1.2)}$$

The function r_n depends only on the difference of the arguments, meaning that the field $\left\{ T_n(x) : x \in \mathbb{T}^d \right\}$ is stationary. Note that the normalization factor $\mathcal{N}_n^{-1/2}$ in the definition of $T_n(x)$ does not change the zero set of T_n, and appears purely for computational reasons; indeed, it implies that $r_n(0) = 1$, that is: for every $x \in \mathbb{T}^3$, the variance of $T_n(x)$ is equal to 1.

Equidistribution of lattice points on $\mathbb{S}^{d-1}$. The set of frequencies Λ_n induces a probability measure on the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$, given by

$$\mu_{n,d} := \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \delta_{\lambda/\sqrt{n}} ,$$

where $\delta_{\lambda/\sqrt{n}}$ denotes the Dirac mass at $\lambda/\sqrt{n}$. Since the measure $\mu_{n,d}$ is compactly supported, it is determined by its Fourier coefficients

$$\widehat{\mu_{n,d}}(k) := \int_{\mathbb{S}^{d-1}} z^{-k} \mu_{n,d}(dz) , \quad k \in \mathbb{Z} .$$

[1]Recall that a random variable X has the complex $\mathcal{N}(0,1)$ distribution, if $X = Y + iZ$ where Y, Z are independent real $\mathcal{N}(0,1/2)$ random variables.

Up to rescaling its argument, the measure $\mu_{n,d}$ is the spectral measure of the Gaussian field $\left\{T_n(x) : x \in \mathbb{T}^d\right\}$, as can be seen by rewriting (II.1.2) as

$$r_n(x-y) = \int_{\mathbb{S}^{d-1}} \exp\left(2\pi i \langle \sqrt{n}\xi, x-y\rangle\right) \mu_{n,d}(d\xi) .$$

The problem of angular distribution of the lattice points in dimension d has been investigated by Linnik [Lin68]. A notable difference arises when comparing dimensions $d = 2$ and $d = 3$: indeed, it is known that there exists a density-1 subsequence $\left\{n_j : j \geq 1\right\} \subset S_2$ such that $\mu_{n_j,2}$ converges weakly to the uniform distribution on the unit circle as $\mathcal{N}_{n_j} \to \infty$ [EH99], but there are infinitely many other weak limits of $\{\mu_{n,2} : n \in S_2\}$; such limits are referred to as *attainable measures* [KW17]. Instead, when $d = 3$, subject to the condition $n \to \infty, n \not\equiv 0, 4, 7 \pmod 8$, the probability measures $\{\mu_{n,3} : n \in S_3\}$ converge weakly to the uniform probability measure on $\mathbb{S}^2$ [Duk88], implying asymptotic equidistribution [DSP90]. In this context, the arithmetic condition $n \not\equiv 0, 4, 7 \pmod 8$ arises naturally from the result by Gauss and Legendre asserting that $n \in S_3$ if and only if n is not of the form $4^a(8b+7)$ (see e.g. [Gro85]).

Previous work on this model. ARWs on the d-dimensional torus have been introduced in [ORW08], where the authors consider the *Leray measure* of the nodal set of ARWs and study its asymptotic variance. A quantitative Central Limit Theorem for the Leray measure on the two-dimensional torus (in the high-frequency limit) is provided in [PR18]. In [RW08], the authors take interest in the $(d-1)$-dimensional nodal volume of ARWs. Denoting by Z_n the zero set of T_n and $\mathcal{V}_n := \mathcal{H}_{d-1}(Z_n)$ its $(d-1)$-dimensional Hausdorff measure, the expected nodal volume is shown to be a constant multiple of the square root of the energy level, that is, $\mathbb{E}\left[\mathcal{V}_n\right] = C_d \sqrt{E_n}$, where C_d is an explicit constant depending only on the dimension, which is in particular consistent with Yau's conjecture. Concerning the variance of the nodal volume, the authors derive the asymptotic upper bound

$$\mathbf{Var}[\mathcal{V}_n] \ll \frac{E_n}{\sqrt{\mathcal{N}_n}} , \quad \mathcal{N}_n \to \infty$$

and conjecture the stronger bound $\ll E_n/\mathcal{N}_n$ to hold.

Recent developments on the two and three-dimensional torus concerning exact asymptotic laws for variances and subsequent second-order results for fluctuations of quantities associated with the nodal set of Laplacian eigenfunctions have gained great attention in the literature. We will now briefly discuss these works.

Work on the two-dimensional torus. In [KKW13], for any probability measure μ on the circle, the authors define

$$c(\mu) := \frac{1+\hat{\mu}(4)^2}{512}$$

and derive a precise asymptotic law for the variance of the nodal length $\mathcal{L}_n$ of ARW, namely

$$\mathbf{Var}[\mathcal{L}_n] \sim c(\mu_{n,2}) \cdot \frac{E_n}{\mathcal{N}_n^2} , \quad \mathcal{N}_n \to \infty . \tag{II.1.3}$$

This suggests that, if $\left\{n_j : j \geq 1\right\} \subset S_2$ is a subsequence such that $\mu_{n_j,2}$ converges weakly to some symmetric probability measure μ on $\mathbb{S}^1$, then $c(\mu_{n_j,2}) \to c(\mu)$ as $\mathcal{N}_{n_j} \to \infty$ and hence

$$\mathbf{Var}\left[\mathcal{L}_{n_j}\right] \sim c(\mu) \cdot \frac{E_{n_j}}{\mathcal{N}_{n_j}^2} , \quad \mathcal{N}_{n_j} \to \infty , \tag{II.1.4}$$

yielding an asymptotic variance estimate with non-fluctuating order of magnitude. In particular, the order of magnitude of the variance is $E_n/\mathcal{N}_n^2$, which significantly improves the previously conjectured bound $E_n/\mathcal{N}_n$ in [RW08]. Such a lower order of magnitude is known as *Berry's arithmetic cancellation phenomenon*, which follows from the exact vanishing of the second-order projection of the Wiener-Itô expansion of the nodal length, as pointed out in [MPRW16]; such a cancellation phenomenon is not observed when dealing with non-zero level sets, in which case the variance would be commensurate to $E_n/\mathcal{N}_n$. The asymptotic estimate in (II.1.4) depends on the angular distribution of the lattice points, and is therefore referred to as a *non-universal* result. Second-order results of the normalized nodal length were addressed in [MPRW16], where the authors show that for a subsequence $\left\{n_j : j \geq 1\right\} \subset S_2$ such that $|\widehat{\mu_{n_j,2}}(4)| \to \eta$, for some $\eta \in [0,1]$ and $\mathcal{N}_{n_j} \to \infty$,

$$\frac{\mathcal{L}_{n_j} - \mathbb{E}\left[\mathcal{L}_{n_j}\right]}{\sqrt{\mathbf{Var}\left[\mathcal{L}_{n_j}\right]}} \xrightarrow{d} \frac{1}{2\sqrt{1+\eta^2}}\left(2 - (1+\eta)X_1^2 + (1-\eta)X_2^2\right) ,$$

where (X_1, X_2) is a standard Gaussian vector in dimension two. In particular, this shows that the limiting probability distribution of the normalized nodal length is parametrised by $\eta \in [0,1]$, which depends on the high-energy behaviour of the spectral measures $\mu_{n,2}$ via the fourth Fourier coefficient. This fact emphasizes that, similarly to the asymptotic law for the variance, the limiting distribution of the normalized length is also non-universal. It is easily checked that the above limiting distributions are different for distinct values of η and non-Gaussian. A quantitative version of this limit theorem is proven in [PR18].

Phase singularities of complex ARWs on the 2-torus have been investigated in [DNPR19]; there, the authors consider the number of intersection points of the nodal sets of two independent ARWs of same energy level. More precisely, if T_n and T_n' denote two independent ARWs associated with eigenvalue E_n and $I_n := \operatorname{card}(T_n^{-1}(\{0\}) \cap T_n'^{-1}(\{0\}))$, the authors establish the following non-universal asymptotic law for the variance: as $\mathcal{N}_n \to \infty$,

$$\mathbf{Var}[I_n] \sim c(\mu_{n,2}) \cdot \frac{E_n^2}{\mathcal{N}_n^2}\,, \quad c(\mu_{n,2}) := \frac{3\widehat{\mu_{n,2}}(4)^2 + 5}{128\pi^2}\,.$$

Similar to the asymptotic variance of the nodal length, the variance of I_n fluctuates due to the fact that lattice points are not necessarily asymptotically equidistributed. The following distributional limit result is also provided: for $\left\{n_j : j \geq 1\right\} \subset S_2$ such that $|\widehat{\mu_{n_j,2}}(4)| \to \eta$, for some $\eta \in [0,1]$ and $\mathcal{N}_{n_j} \to \infty$,

$$\frac{I_{n_j} - \mathbb{E}\left[I_{n_j}\right]}{\sqrt{\mathbf{Var}\left[I_{n_j}\right]}} \xrightarrow{d} \frac{1}{2\sqrt{10+6\eta^2}}\left(\frac{1+\eta}{2}A + \frac{1-\eta}{2}B - 2(C-2)\right) ,$$

where A, B, C are independent random variables such that $A \stackrel{d}{=} B \stackrel{d}{=} 2X_1^2 + 2X_2^2 - 4X_3^2$ and $C \stackrel{d}{=} X_1^2 + X_2^2$, and (X_1, X_2, X_3) is a standard Gaussian vector in dimension three.

Related works on the two-dimensional torus include the study of the volume of the nodal set intersected with a fixed reference curve [RW18], or line segment [Maf17]. In [BMW20], the authors restrict the nodal length of ARWs to shrinking balls and prove that the restricted nodal length is asymptotically fully correlated with the total nodal length. In [GW17], Granville and Wigman study the small scale distribution of the L^2-mass of Laplacian eigenfunctions. Finally, in the recent work [CMR20] the authors investigate the probabilistic fluctuations of Lipschitz-Killing curvatures in the high-frequency regime.

Work on the three-dimensional torus. Statements on the three-dimensional torus include the arithmetic relation $n \not\equiv 0, 4, 7 \pmod 8$ and, unlike the two-dimensional case, they do not rely on the spectral measures $\{\mu_{n,3} : n \in S_3\}$ due to equidistribution of lattice points on the unit two-sphere. The existing literature in $d = 3$ considers the nodal set Z_n of T_n and its two-dimensional Hausdorff measure $\mathcal{A}_n := \mathcal{H}_2(Z_n)$, that is the nodal surface of Z_n. In [BM19], an exact asymptotic law for the variance is provided, namely as $n \to \infty, n \not\equiv 0, 4, 7 \pmod 8$,

$$\mathbf{Var}[\mathcal{A}_n] = \frac{n}{\mathcal{N}_n^2}\left(\frac{32}{375} + O\left(n^{-1/28+o(1)}\right)\right) . \tag{II.1.5}$$

Similarly to the two-dimensional case, the order of magnitude of the variance is commensurate to $E_n/\mathcal{N}_n^2$, which originates from the cancellation of the second chaotic projection in the Wiener chaos expansion of the nodal surface. As a consequence of the asymptotic equidistribution of lattice points on $\mathbb{S}^2$, the leading coefficient in front of $n/\mathcal{N}_n^2$ in (II.1.5) does not fluctuate. The limiting distribution of the normalized nodal surface was investigated in [Cam19], where the following non-Gaussian, *universal* result was derived: as $n \to \infty, n \not\equiv 0, 4, 7 \pmod 8$,

$$\frac{\mathcal{A}_n - \mathbb{E}[\mathcal{A}_n]}{\sqrt{\mathbf{Var}[\mathcal{A}_n]}} \xrightarrow{d} \frac{1}{\sqrt{10}} \cdot (5 - \chi^2(5)) ,$$

where $\chi^2(5)$ denotes a chi-squared random variable with 5 degrees of freedom. This distributional limit result is analogous to the case $d = 2$ in the sense that the limiting distribution is a linear combination of independent chi-squared random variables, but does not involve any non-universality phenomenon.

Results on the intersection of nodal sets against a surface can be found in [RWY16, RW16], see also [Maf20] for a study of the intersection length obtained when intersecting nodal sets of ARWs with planes.

II.1.3 Our main results

Let T_n be an arithmetic random wave on $\mathbb{T}^3$ and $T_n^{(1)}, T_n^{(2)}, T_n^{(3)}$ be i.i.d. copies of T_n. Fix $\ell \in [3]$ and consider the centred ℓ-dimensional Gaussian field

$$\mathbf{T}_n^{(\ell)} := \left\{\mathbf{T}_n^{(\ell)}(x) := \left(T_n^{(1)}(x), \dots, T_n^{(\ell)}(x)\right) : x \in \mathbb{T}^3\right\} , \tag{II.1.6}$$

to which we associate the quantity

$$L_n^{(\ell)}(u^{(\ell)}) := \mathcal{H}_{3-\ell}\left(\bigcap_{i=1}^{\ell} (T_n^{(i)})^{-1}(\{u_i\})\right), \quad u^{(\ell)} = (u_1, \dots, u_\ell) \in \mathbb{R}^\ell \tag{II.1.7}$$

where, for a k-dimensional measurable domain $A \subset \mathbb{T}^3$, $\mathcal{H}_k(A)$ denotes the k-dimensional Haussdorff measure of A, that is $(\mathcal{H}_2, \mathcal{H}_1, \mathcal{H}_0) =$ (area, length, card). Whe denote the normalized volume by

$$\widetilde{L_n^{(\ell)}}(u^{(\ell)}) := \frac{L_n^{(\ell)}(u^{(\ell)}) - \mathbb{E}\left[L_n^{(\ell)}(u^{(\ell)})\right]}{\sqrt{\mathbf{Var}\left[L_n^{(\ell)}(u^{(\ell)})\right]}} .$$

The main object of study in this chapter is the nodal volume, obtained when setting $u^{(\ell)} = 0 \in \mathbb{R}^\ell$; we will simply write $L_n^{(\ell)} := L_n^{(\ell)}(0), 0 \in \mathbb{R}^\ell$. Since $T_n^{(1)}, T_n^{(2)}$ and $T_n^{(3)}$ are i.i.d. copies of T_n, we have

$$r_n^{(i)}(x-y) := \mathbb{E}\left[T_n^{(i)}(x) \cdot T_n^{(i)}(y)\right] = r_n(x-y) , \quad i \in [\ell],$$

where r_n is as in (II.1.2).

Our main result, stated in Theorem II.1.1 below, provides exact second order results for the three quantities $L_n^{(1)}, L_n^{(2)}, L_n^{(3)}$, and thus contains the findings of [Cam19] in the special case $\ell = 1$. The statement is divided into three parts: (i) gives the precise expected nodal volume, (ii) is an asymptotic law for the nodal variance and (iii) concerns the second-order fluctuations of the normalized version of the nodal volume.

Theorem II.1.1. *Let the above notation prevail. Then the following holds:*

(i) *(Expected nodal volume) For every $n \in S_3$,*

$$\mathbb{E}\left[L_n^{(\ell)}\right] = \begin{cases} \dfrac{2\sqrt{E_n}}{\sqrt{3}\pi}, & \ell = 1 \\ \dfrac{E_n}{3\pi}, & \ell = 2 \\ \dfrac{E_n^{3/2}}{3\sqrt{3}\pi^2}, & \ell = 3 \end{cases}$$

(ii) *(Universal asymptotic nodal variance) As $n \to \infty, n \not\equiv 0, 4, 7 \pmod 8$,*

$$\mathbf{Var}\left[L_n^{(\ell)}\right] \sim \begin{cases} \dfrac{E_n}{\mathcal{N}_n^2} \cdot \dfrac{8}{375\pi^2}, & \ell = 1 \\ \dfrac{E_n^2}{\mathcal{N}_n^2} \cdot \dfrac{316}{3375\pi^2}, & \ell = 2 \\ \dfrac{E_n^3}{\mathcal{N}_n^2} \cdot \dfrac{62}{675\pi^4}, & \ell = 3 \end{cases}$$

(iii) *(Universal asymptotic distribution of the nodal volume) As $n \to \infty, n \not\equiv 0, 4, 7 \pmod 8$,*

$$\widetilde{L_n^{(\ell)}} \xrightarrow{d} \begin{cases} -\dfrac{1}{\sqrt{10}}\hat{\xi}_1(5), & \ell = 1 \\ 5\sqrt{\dfrac{15}{79}} \cdot \left(-\dfrac{1}{50}\hat{\xi}_1(10) - \dfrac{1}{25}\hat{\xi}_2(5) + \dfrac{1}{25}\hat{\xi}_3(5) + \dfrac{1}{50}\hat{\xi}_4(5) - \dfrac{1}{6}\hat{\xi}_5(3) \right), & \ell = 2 \\ 5\sqrt{\dfrac{2}{31}} \cdot \left(-\dfrac{1}{50}\hat{\xi}_1(15) - \dfrac{1}{25}\hat{\xi}_2(15) + \dfrac{1}{25}\hat{\xi}_3(15) + \dfrac{1}{50}\hat{\xi}_4(15) - \dfrac{1}{6}\hat{\xi}_5(9) \right), & \ell = 3 \end{cases}$$

where, in each line, the symbols $\hat{\xi}_i(k_i)$ denote independent centred chi-squared random variables with k_i degrees of freedom.

Remark II.1.2. **(a)** We point out that the results stated separately in Theorem II.1.1 can be written in a compact form. For integers $1 \le \ell \le k$, we set

$$\alpha(\ell, k) := \frac{(k)_\ell \kappa_k}{(2\pi)^{\ell/2} \kappa_{k-\ell}}, \tag{II.1.8}$$

where $(k)_\ell := k!/(k-\ell)!$ and $\kappa_k := \frac{\pi^{k/2}}{\Gamma(1+k/2)}$ stands for the volume of the unit ball in $\mathbb{R}^k$. Note that one can re-write

$$\alpha(\ell, k) = \frac{\ell! \kappa_\ell}{(2\pi)^{\ell/2}} \begin{bmatrix} k \\ \ell \end{bmatrix},$$

where $\begin{bmatrix} k \\ \ell \end{bmatrix} := \binom{k}{\ell} \frac{\kappa_k}{\kappa_{k-\ell}\kappa_\ell}$ are the so-called *flag coefficients* also appearing in the Gaussian Kinematic Formaula (see for instance [AT07, Chapter 13]). Using this definition, the content of Theorem II.1.1 can be restated as follows: for every $\ell \in [3]$, one has that

(i) For every $n \in S_3$,

$$\mathbb{E}\left[L_n^{(\ell)}\right] = \left(\frac{E_n}{3}\right)^{\ell/2} \frac{\alpha(\ell, 3)}{(2\pi)^{\ell/2}}. \tag{II.1.9}$$

(ii) As $n \to \infty, n \not\equiv 0, 4, 7 \pmod 8$,

$$\mathbf{Var}\Big[L_n^{(\ell)}\Big] \sim (c_n^{(\ell)})^2\Big(\ell \cdot \frac{1}{250} + \frac{\ell(\ell-1)}{2} \cdot \frac{76}{375}\Big), \tag{II.1.10}$$

where

$$c_n^{(\ell)} = \left(\frac{E_n}{3}\right)^{\ell/2} \frac{2}{(2\pi)^{\ell/2}} \frac{\alpha(\ell, 3)}{\mathcal{N}_n} .$$

(iii) As $n \to \infty, n \not\equiv 0, 4, 7 \pmod 8$,

$$\widetilde{L_n^{(\ell)}} \xrightarrow{d} \Big(\ell \cdot \frac{1}{250} + \frac{\ell(\ell-1)}{2} \cdot \frac{76}{375}\Big)^{-1/2} Y^{(\ell)} M^{(\ell)} (Y^{(\ell)})^T , \tag{II.1.11}$$

where $Y^{(\ell)} \sim \mathcal{N}_{\ell(9\ell-4)}(0, \mathbf{I}_{\ell(9\ell-4)})$ is a $\ell(9\ell-4)$-dimensional standard Gaussian vector and $M^{(\ell)} \in \mathcal{M}_{\ell(9\ell-4)\times\ell(9\ell-4)}(\mathbb{R})$ is the deterministic matrix given by

$$M^{(\ell)} = \frac{-1}{50}\mathbf{I}_{5\ell} \oplus \frac{-1}{25}\mathbf{I}_{\frac{5\ell(\ell-1)}{2}} \oplus \frac{1}{25}\mathbf{I}_{\frac{5\ell(\ell-1)}{2}} \oplus \frac{1}{50}\mathbf{I}_{\frac{5\ell(\ell-1)}{2}} \oplus \frac{-1}{6}\mathbf{I}_{\frac{3\ell(\ell-1)}{2}} .$$

For the point (iii) above, we observe that $Y^{(\ell)} M^{(\ell)} (Y^{(\ell)})^T$ in (II.1.11) is a diagonal quadratic form that has the same probability distribution as

$$-\frac{1}{50}\hat{\xi}_1(5\ell) - \frac{1}{25}\hat{\xi}_2\Big(\frac{5\ell(\ell-1)}{2}\Big) + \frac{1}{25}\hat{\xi}_3\Big(\frac{5\ell(\ell-1)}{2}\Big) + \frac{1}{50}\hat{\xi}_4\Big(\frac{5\ell(\ell-1)}{2}\Big) - \frac{1}{6}\hat{\xi}_5\Big(\frac{3\ell(\ell-1)}{2}\Big)$$

where $\Big\{\hat{\xi}_i(k_i) : i = 1, \ldots, 5\Big\}$ denote independent centred chi-squared random variables with $k_i \geq 0$ degrees of freedom with the convention $\hat{\xi}_i(0) \equiv 0$. In particular, this shows that for every $\ell \in [3]$, in the high-energy regime, the normalized nodal volume exhibits *universal* and *non-Gaussian* second-order fluctuations.

We also point out that the statements (i) and (ii) of Theorem II.1.1 are sufficient to derive a *universal weak law of large numbers*; it tells that the distribution of the normalized random variable $L_n^{(\ell)}/E_n^{\ell/2}$ is asymptotically concentrated around its mean:

Corollary II.1.3. *For every $\delta > 0$, as $n \to \infty, n \not\equiv 0, 4, 7 \pmod 8$, we have*

$$\mathbb{P}\Big[\Big|\frac{L_n^{(\ell)}}{E_n^{\ell/2}} - \frac{\alpha(\ell, 3)}{3^{\ell/2}(2\pi)^{\ell/2}}\Big| > \delta\Big] = o(1) .$$

This immediately follows from Chebyshev's Inequality: as $n \to \infty, n \not\equiv 0, 4, 7 \pmod 8$,

$$\mathbb{P}\Big[\Big|\frac{L_n^{(\ell)}}{E_n^{\ell/2}} - \frac{\alpha(\ell, 3)}{3^{\ell/2}(2\pi)^{\ell/2}}\Big| > \delta\Big] \leq \frac{1}{\delta^2} \cdot \mathbf{Var}\Bigg[\frac{L_n^{(\ell)}}{E_n^{\ell/2}}\Bigg] = \frac{c_\ell}{\delta^2 \mathcal{N}_n^2}(1 + o(1)) = o(1) ,$$

where c_ℓ is a constant only depending on ℓ.

Remark II.1.4. (*On geometric measures associated with non-zero level sets*) Exploiting similar arguments used in order to prove Theorem II.1.1, it is no more difficult to study geometric quantities associated with non-zero levels. A careful analysis of the second Wiener chaotic projection in this case yields the following statements (compare with Remark II.1.2). Let $u^{(\ell)} \neq 0 \in \mathbb{R}^\ell$ be fixed and γ_ℓ denote the multivariate standard Gaussian density of dimension ℓ. Then,

(i) For every $n \in S_3$

$$\mathbb{E}\left[L_n^{(\ell)}(u^{(\ell)})\right] = \left(\frac{E_n}{3}\right)^{\ell/2} \alpha(\ell, 3)\gamma_\ell(u^{(\ell)}) .$$

(ii) As $n \to \infty, n \not\equiv 0, 4, 7 \pmod 8$,

$$\mathbf{Var}\Big[L_n^{(\ell)}(u^{(\ell)})\Big] \sim \frac{E_n^\ell}{\mathcal{N}_n}\frac{\alpha(\ell,3)^2}{2\cdot 3^\ell}\gamma_\ell(u^{(\ell)})[u_1^4+\ldots+u_\ell^4]. \tag{II.1.12}$$

(iii) As $n \to \infty, n \not\equiv 0 \pmod{4, 7, 8}$, we have

$$\widetilde{L_n^{(\ell)}}(u^{(\ell)}) \xrightarrow{d} N, \tag{II.1.13}$$

where $N \sim \mathcal{N}(0,1)$.

Contrarily to the nodal case, the lower order of magnitude $E_n^\ell/\mathcal{N}_n$ in (II.1.12) as well as the Central Limit Theorem in (II.1.13) both originates from the dominance of the second Wiener chaos projection of $L_n^{(\ell)}(u^{(\ell)})$. We refer the reader to Section II.2.2 for further details on the second-chaotic projections in the non-nodal case.

Remark II.1.5. In the two points listed below, we highlight further technical novelties appearing in the proof of Theorem II.1.1.

(a) The chaos expansions of $L_n^{(\ell)}$ is obtained from the Area/Co-Area formula by an approximation argument similar to those used in [KL01], where the authors discuss Gaussian limit theorems for general level functionals associated with stationary Gaussian fields with integrable covariance function. Our arguments for proving existence in $L^2(\mathbb{P})$ rely on the use of an adequate partition of the torus into singular and non-singular regions, see for instance [ORW08, KKW13]. To the best of our expertise, although such a route has already been effectively exploited for obtaining variance estimates for higher-order chaotic projections of nodal quantities (see [PR18, DNPR19, NPR19]), this approach for proving existence results in $L^2(\mathbb{P})$ for geometric functionals associated with multi-dimensional Gaussian fields appears for the first time in the literature. We also stress that the argument based on almost surely bounding the nodal length $L_n^{(1)}$ associated with a single ARW (see [RW08] and [Cam19]) does not apply in the case of more than one ARW, and therefore requires a different approach.

(b) In order to derive the explicit expression of the fourth-order chaotic projection of $L_n^{(\ell)}$, we compute the Hermite projection coefficients associated with the mapping $\mathbf{X} \mapsto \det(\mathbf{X}\mathbf{X}^T)^{1/2}$, where $\mathbf{X}$ is a $\ell \times 3$ matrix. In order to do this, we tackle the more general task of computing these projection coefficients in the case where $\mathbf{X}$ is a generic $\ell \times k$ matrix. Our techniques build on standard properties of the Gaussian distribution as well as Gramian determinants, and in particular recover the known results obtained in [DNPR19, Lemma 3.3].

II.1.4 Further connection with literature

<u>*Berry's Random Wave Model.*</u> In [Ber77], Berry introduced the so-called *Berry Random Wave model* (BRW), that is, the unique translation-invariant centred Gaussian field $B_j = \left\{B_j(x) : x \in \mathbb{R}^2\right\}$ on the plane with covariance function

$$r_j(x,y) = \mathbb{E}\left[B_j(x)\cdot B_j(y)\right] = J_0\left(\sqrt{\lambda_j}\cdot \|x-y\|\right) =: r_j(x-y)\,, \quad (x,y) \in \mathbb{R}^2\times\mathbb{R}^2 \tag{II.1.14}$$

with J_0 denoting the Bessel function of order 0 of the first kind and $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^2$. Berry conjectured that local aspects of the geometry of zero sets of generic high-energy Laplace eigenfunctions on a two-dimensional manifold can be modelled by the BRW. More precisely, his observation proposes that eigenfunctions of chaotic systems locally 'behave' like a random superposition of plane waves with fixed energy. Since Berry's publication [Ber02], the study of local and non-local features associated with the geometry of nodal and (non-zero) level sets of high-energy Gaussian Laplace eigenfunctions has gained substantial consideration and different models have been studied in recent years, the case of *random spherical harmonics* on the 2-sphere (see e.g. [MRW20, Ros16, Wig10, MP11]) and *arithmetic random waves* on the torus (see e.g. [ORW08, RW08, KKW13, Cam19, DNPR19, MPRW16]) being of particular importance. The study of BRW on $\mathbb{R}^3$ has been initiated in [DEL21]. Therein, the authors consider the nodal length restricted to growing cubes of the complex BRW and distinguish between isotropic and anisotropic covariance functions. In the isotropic case, they show that the limiting distribution of the nodal length is Gaussian whenever the underlying covariance function of the model is square-integrable. The proof of such a Central Limit Theorem, based on the Wiener chaos expansion of the nodal length, reveals in particular that, in this framework, *all* the even chaoses except the second contribute to the limit. As already mentioned, this is in stark contrast with the results presented in this chapter, based on the dominance of fourth chaos projections. Such a discrepancy can be partially explained by comparing the underlying covariance functions of the models, which is nearly monotonically decaying in the Euclidean setting and periodically oscillating on the torus. In [CH20, Zel09], the authors study *monochromatic random waves* on a general smooth compact manifold, that is, Gaussian linear combinations of eigenfunctions associated with eigenvalues ranging in a short interval.

Berry's Cancellation Phenomenon. Berry's cancellation phenomenon was first observed in [Ber02] for nodal sets of BRW. Using the notation introduced in (II.1.14), Berry considered the length $L_j(\mathcal{D})$ of the nodal lines of B_j (Berry random wave for eigenvalue λ_j) and the number of nodal points $N_j(\mathcal{D})$ of the complex version of the BRW, i.e. the random field $\left\{B_j(x) + iB'_j(x) : x \in \mathbb{R}^2\right\}$, with B'_j denoting an independent copy of B_j, when both statistics are restricted to a compact domain $\mathcal{D}$. For these observables, denoting area$(\mathcal{D})$ the area of $\mathcal{D}$, Berry obtained

$$\mathbb{E}\left[L_j(\mathcal{D})\right] = \frac{\text{area}(\mathcal{D})}{2\sqrt{2}}\sqrt{\lambda_j}\,, \quad \mathbb{E}\left[N_j(\mathcal{D})\right] = \frac{\text{area}(\mathcal{D})}{4\pi}\lambda_j\,;$$

as well as variance asymptotics, as $j \to \infty$

$$\begin{aligned}\mathbf{Var}\left[L_j(\mathcal{D})\right] &\sim \frac{\text{area}(\mathcal{D})}{256\pi}\log\left(\sqrt{\lambda_j}\sqrt{\text{area}(\mathcal{D})}\right)\\ \mathbf{Var}\left[N_j(\mathcal{D})\right] &= \frac{11\text{area}(\mathcal{D})}{64\pi^3}\lambda_j\log\left(\sqrt{\lambda_j}\sqrt{\text{area}(\mathcal{D})}\right)\,.\end{aligned} \tag{II.1.15}$$

In [NPR19], the authors recover these results and show that the properly scaled versions of $L_j(\mathcal{D})$ and $N_j(\mathcal{D})$ satisfy a central limit theorem in the high-energy regime. Berry's cancellation phenomenon essentially concerns the order of magnitude of the asymptotic variance in (II.1.15): indeed, its *logarithmic* order is unexpectedly smaller than a natural prediction. Loosely speaking, such a lower order of magnitude originates from the exact cancellation of the leading term in the *Kac-Rice formula* for the variance. A general explanation of such a cancellation, based on the use of Wiener-chaos expansions of the nodal volumes, distilling the main ideas introduced in [MPRW16, DNPR19, NPR19] into a general principle, will be developed in the forthcoming sections.

II.1.5 Overview

In Section II.2, we provide a general result (see Theorem II.2.5) leading to cancellation phenomena in the setting of geometric functionals associated with nodal sets of multiple independent Gaussian fields. Its proof is deferred to Appendix II.A. The proof of Theorem II.1.1 on nodal sets of arithmetic random waves on the three-torus is the content of Section II.3. Appendices II.B-II.E contain proofs of technical results needed for the proof of Theorem II.1.1.

II.2 Wiener Chaos and abstract cancellation phenomena

In this section, we present some general results about non-linear functionals of Gaussian fields that admit an integral representation in terms of Dirac masses and Jacobians. As discussed in Section II.2.2, this contains as special cases exact and partial cancellations discovered in [DNPR19, NPR19, MPRW16, MRW20].

II.2.1 An abstract cancellation phenomenon

We consider a finite measurable space $(Z, \mathscr{Z}, \mu)$ such that $\mu(Z) = 1$. Let $G = \{G(z) : z \in Z\}$ be a real-valued centred Gaussian field indexed by Z. For an integer $\ell \geq 1$, let $G^{(1)}, \dots, G^{(\ell)}$ be i.i.d. copies of G and write $\mathbf{G} = \left\{\mathbf{G}(z) = (G^{(1)}(z), \dots, G^{(\ell)}(z)) : z \in Z\right\}$ to indicate the associated ℓ-dimensional Gaussian field. Additionally, let $W = \{W(z) : z \in Z\}$ be a (not necessarily Gaussian) random field indexed by Z. We denote by δ_u the Dirac mass at $u \in \mathbb{R}$. We introduce the following definition.

Definition II.2.1. For every $u^{(\ell)} := (u_1, \dots, u_\ell) \in \mathbb{R}^\ell$, we define the random variable

$$J(\mathbf{G}, W; u^{(\ell)}) := \int_Z \prod_{i=1}^{\ell} \delta_{u_i}(G^{(i)}(z)) \cdot W(z)\, \mu(dz)$$

$$:= \lim_{\varepsilon \to 0} \int_Z (2\varepsilon)^{-\ell} \prod_{i=1}^{\ell} \mathbb{1}_{[-\varepsilon,\varepsilon]}(G^{(i)}(z) - u_i) \cdot W(z)\, \mu(dz) \tag{II.2.1}$$

whenever the limit exists $\mathbb{P}$-almost surely. In the case where the limit exists in $L^p(\mathbb{P})$ for $p \geq 1$, we say that $J(\mathbf{G}, W; u^{(\ell)})$ is *well-defined* in $L^p(\mathbb{P})$.

Our aim is to study the Wiener-Itô chaos expansion of $J(\mathbf{G}, W; u^{(\ell)})$. As we will prove later (see Lemma II.3.1), the nodal volumes $L_n^{(\ell)}, \ell \in [3]$ defined in (II.1.7) are obtained $\mathbb{P}$-a.s. and in $L^2(\mathbb{P})$ as $L_n^{(\ell)} = J(\mathbf{G}, W, (0, \dots, 0))$, where $\mathbf{G} = \mathbf{T}_n^{(\ell)}$ is as in (II.1.6) and $W(z)$ is the square root of the Gramian determinant of the Jacobian matrix of $\mathbf{T}_n^{(\ell)}$ computed at z.

For integers $1 \leq \ell \leq k$, we use the notation $\mathbf{X} = \left\{X_j^{(i)} : (i, j) \in [\ell] \times [k]\right\}$ to indicate a generic element of the class $\mathcal{M}_{\ell\times k}(\mathbb{R})$ of $\ell \times k$ matrices. The following definition generalizes the notion of Gramian determinants.

Definition II.2.2. We say that a map $\Phi_{\ell,k} : \mathcal{M}_{\ell\times k}(\mathbb{R}) \to \mathbb{R}_+$ satisfies *Assumption A* if it satisfies the following four requirements for every $\mathbf{X} \in \mathcal{M}_{\ell\times k}(\mathbb{R})$:

(A1) $\Phi_{\ell,k}$ is invariant under permutations of columns and rows of $\mathbf{X}$, that is,

$$\Phi_{\ell,k}(\mathbf{X}) = \Phi_{\ell,k}\left(\left\{X_{\sigma(j)}^{(i)} : (i, j) \in [\ell] \times [k]\right\}\right) = \Phi_{\ell,k}\left(\left\{X_j^{(\pi(i))} : (i, j) \in [\ell] \times [k]\right\}\right)$$

for every permutation σ of $[k]$ and π of $[\ell]$.

(A2) $\Phi_{\ell,k}$ is positively homogeneous as a function of the rows of $\mathbf{X}$, that is, for every $c \in \mathbb{R}$ and every $i \in [\ell]$, $|c|\Phi_{\ell,k}(\mathbf{X}) = \Phi_{\ell,k}(\mathbf{X}^*)$, where $\mathbf{X}^*$ denotes the matrix obtained from $\mathbf{X}$ by multiplying the i-th row by c.

(A3) $\Phi_{\ell,k}$ is invariant under sign changes in the columns of $\mathbf{X}$, that is, for every $j \in [k]$, $\Phi_{\ell,k}(\mathbf{X}) = \Phi_{\ell,k}(\mathbf{X}^*)$, where $\mathbf{X}^*$ denotes the matrix obtained from $\mathbf{X}$ by multiplying the j-th column by -1.

(A4) If $\ell \geq 2, \Phi_{\ell,k}$ is invariant under row addition, that is, $\Phi_{\ell,k}(\mathbf{X}) = \Phi_{\ell,k}(\mathbf{X}^*)$, where $\mathbf{X}^*$ denotes the matrix obtained from $\mathbf{X}$ by replacing its i_1-th row by the sum of its i_1-th and i_2-th row for $i_1 \neq i_2 \in [\ell]$.

A prototype example of a function satisfying *Assumption A* above is given by the Gramian determinant $\Phi^*_{\ell,k}(\mathbf{X}) := \det(\mathbf{X}\,\mathbf{X}^T)^{1/2}$ as proved in Lemma II.B.1 of Appendix II.B.

Remark II.2.3 (Role of *Assumption A*)**.** Although in the proof of Theorem II.1.1 we apply the results of the present section only to the particular function $\Phi_{\ell,k}(\mathbf{X}) = \det(\mathbf{X}\mathbf{X}^T)^{1/2}$, we prefer to state our findings in the more general framework of functions verifying *Assumption A*, thus highlighting those features of the mapping $\mathbf{X} \mapsto \det(\mathbf{X}\mathbf{X}^T)^{1/2}$ that determine cancellation phenomena.

To state our result, we introduce the following objects:

- For every $i \in [\ell]$, let

$$\mathbf{X}^{(i)} = \left\{\mathbf{X}^{(i)}(z) := (X_0^{(i)}(z), X_1^{(i)}(z), \dots, X_k^{(i)}(z)) : z \in Z\right\}$$

  be a $(k+1)$-dimensional standard Gaussian field, i.e. $\mathbf{X}^{(i)}$ is a Gaussian family and for every fixed $z \in Z$, the vector $\mathbf{X}^{(i)}(z)$ is a standard $(k+1)$-dimensional Gaussian vector, that is, its coordinates $X_j^{(i)}(z), j = 0, \dots, k$ are independent standard Gaussian random random variables. For $z \in Z$, we let $\mathbf{X}_\star^{(i)}(z) := (X_1^{(i)}(z), \dots, X_k^{(i)}(z))$ and write

$$\mathbf{X}_\star(z) := \left\{X_j^{(i)}(z) : (i,j) \in [\ell] \times [k]\right\} \tag{II.2.2}$$

  for the $\ell \times k$ matrix whose i-th row is given by $\mathbf{X}_\star^{(i)}(z)$. If $\ell \geq 2$, for every $i_1 \neq i_2 \in [\ell]$, we assume that the random fields $\mathbf{X}^{(i_1)}$ and $\mathbf{X}^{(i_2)}$ are stochastically independent.

- For every $i \in [\ell]$, we define the quantities

$$D^{(i)} := \frac{1}{k}\sum_{j=1}^{k}\int_Z X_j^{(i)}(z)^2\,\mu(dz) - \int_Z X_0^{(i)}(z)^2\,\mu(dz)\,, \tag{II.2.3}$$

$$m^{(i)} := \int_Z X_0^{(i)}(z)\,\mu(dz)\,. \tag{II.2.4}$$

- Consider a map $\Phi_{\ell,k} : \mathcal{M}_{\ell\times k}(\mathbb{R}) \to \mathbb{R}_+$ that satisfies *Assumption A* of Definition II.2.2 and such that for every $z \in Z$, $\mathbb{E}\left[\Phi_{\ell,k}(\mathbf{X}_\star(z))^2\right] < \infty$, and set

$$\mathbb{E}\left[\Phi_{\ell,k}(\mathbf{X}_\star(z))\right] =: \alpha_{\ell,k}\,. \tag{II.2.5}$$

Our next result provides the chaotic projections onto the q-th Wiener chaos associated with $\left\{\mathbf{X}^{(1)}, \dots, \mathbf{X}^{(\ell)}\right\}$ of the random variable $J(\mathbf{G}, W; u^{(\ell)})$ defined in Definition II.2.1 in the case where

$$\mathbf{G} = \left\{(X_0^{(1)}(z), \dots, X_0^{(\ell)}(z)) : z \in Z\right\}, \quad W = \{\Phi_{\ell,k}(\mathbf{X}_\star(z)) : z \in Z\}\,. \tag{II.2.6}$$

Note that, for every $z \in Z$, $W(z)$ as defined in (II.2.6) is $\sigma(\mathbf{G})$-measurable and stochastically independent of $\mathbf{G}(z)$. Part (ii) contains a general version of the chaos cancellation phenomenon observed e.g. in [Wig10, MR21, KKW13, DNPR19, MPRW16, NPR19, Cam19]. Its proof is deferred to Appendix II.A.

Remark II.2.4. We stress here that the technical assumption requiring that $\mathbf{G}(z)$ is independent of $W(z)$ for every fixed $z \in Z$ is needed in order to deduce the Wiener chaos expansion of the the random variable $J(\mathbf{G}, W; u^{(\ell)})$. Indeed, exploiting this assumption, the latter will be obtained once (formally) expanding $\prod_{i=1}^{\ell}\delta_{u_i}(G^{(i)}(z))$ and $W(z)$ into Hermite polynomials and then integrating the product over Z (see Section II.A.1 for more details). On the other hand, the assumption that $\mathbf{X}^{(i_1)}$ and $\mathbf{X}^{(i_2)}$ are stochastically independent as random fields is formulated for the later use in the context of nodal volumes of ARW (see in particular example (i) in Section II.2.2).

Theorem II.2.5. *Assume the above setting. Then, we have:*

(i) *(General projection formulae) Fix $u^{(\ell)} := (u_1, \dots, u_\ell) \in \mathbb{R}^\ell$ and assume that $J(\mathbf{G}, W; u^{(\ell)})$ with $(\mathbf{G}, W)$ as in* (II.2.6) *is well-defined in $L^2(\mathbb{P})$ in the sense of Definition II.2.1. Writing $J = J(\mathbf{G}, W; u^{(\ell)})$, we have, for every $q \geq 0$,*

$$\mathrm{proj}_q(J) = \sum_{\substack{j_1,\dots,j_\ell,r\geq 0\\ j_1+\dots+j_\ell+r=q}} \frac{\beta_{j_1}^{(u_1)} \cdots \beta_{j_\ell}^{(u_\ell)}}{j_1! \dots j_\ell!} \int_Z \prod_{i=1}^{\ell} H_{j_i}(G^{(i)}(z)) \cdot \mathrm{proj}_r(W(z))\mu(dz), \tag{II.2.7}$$

where $\left\{\beta_j^{(u_i)} : j \geq 0\right\}$ denote the coefficients associated with the formal Hermite expansion of the Dirac mass δ_{u_i}, given by

$$\beta_j^{(u)} = \int_{\mathbb{R}} \delta_u(y) H_j(y)\gamma(y)dy = H_j(u)\gamma(u)\ .$$

In particular,

$$\mathrm{proj}_0(J) = \mathbb{E}[J] = \alpha_{\ell,k} \cdot \prod_{i=1}^{\ell} \gamma(u_i)\,, \tag{II.2.8}$$

$$\mathrm{proj}_1(J) = \alpha_{\ell,k} \cdot \prod_{i=1}^{\ell} \gamma(u_i) \cdot \sum_{i=1}^{\ell} m^{(i)} u_i\,, \tag{II.2.9}$$

$$\mathrm{proj}_2(J) = \frac{\alpha_{\ell,k}}{2} \cdot \prod_{i=1}^{\ell} \gamma(u_i) \cdot \sum_{i=1}^{\ell} \left(u_i^2 \int_Z (X_0^{(i)}(z)^2 - 1)\ \mu(dz) + D^{(i)}\right). \tag{II.2.10}$$

(ii) *(Abstract cancellation) If $u_i = D^{(i)} = 0$ for every $i \in [\ell]$, then (using* (II.2.5)*)*

$$\mathrm{proj}_0(J) = \mathbb{E}[J] = \frac{\alpha_{\ell,k}}{(2\pi)^{\ell/2}}\,, \tag{II.2.11}$$

$$\mathrm{proj}_{2q+1}(J) = \mathrm{proj}_2(J) = 0\,, \quad q \geq 0\,. \tag{II.2.12}$$

As anticipated, we will apply Theorem II.2.5 to the study of nodal sets of Gaussian Laplace eigenfunctions. The following section deals with two such examples.

II.2.2 Applications to nodal sets of Gaussian Laplace eigenfunctions

We provide two examples of applications of Theorem II.2.5 dealing with nodal volumes associated with (possibly multi-dimensional) stationary Gaussian random fields that are Laplace eigenfunctions. Example (i) deals with ARWs on the d-dimensional torus and is effectively used in the proof of Theorem II.1.1, whereas (ii) is Berry's random wave model in $\mathbb{R}^d$.

(i) ARW on $\mathbb{T}^d$. Let $d \geq 2$ and $(Z, \mathscr{Z}, \mu) = (\mathbb{T}^d, \mathcal{B}(\mathbb{T}^d), dx)$ with dx denoting the Lebesgue measure on $\mathbb{R}^d$. For integers $1 \leq \ell \leq d$, consider independent ARWs $T_n^{(1)}, \dots, T_n^{(\ell)}$ on $\mathbb{T}^d$. By a straightforward computation, we have that, for every $i \in [\ell]$ and $j \in [d]$, the partial derivatives $\partial_j T_n^{(i)}(x)$ are centred Gaussian random variables with variance

$$\mathbf{Var}\left[\partial_j T_n^{(i)}(x)\right] = \frac{E_n}{d}\,, \quad n \in S_d, \quad x \in \mathbb{T}^d, \tag{II.2.13}$$

where $\partial_j := \partial/\partial x_j$. Let $\mathbf{G} = \left\{(T_n^{(1)}(x), \ldots, T_n^{(\ell)}(x)) : x \in \mathbb{T}^d\right\}$ and write $\tilde{\partial}_j := (E_n/d)^{-1/2}\partial_j$ for the normalized derivatives. Denote by $\mathbf{G}_\star(x)$ the normalized Jacobian $\ell \times d$ matrix of $\mathbf{G}$ computed at $x \in \mathbb{T}^d$ and consider the random field $W = \left\{\Phi^*_{\ell,d}(\mathbf{G}_\star(x)) : x \in \mathbb{T}^d\right\}$ where $\Phi^*_{\ell,d}(A) = \det(AA^T)^{1/2}$ for $A \in \mathcal{M}_{\ell\times d}(\mathbb{R})$. Then, using the Area/Co-Area formula (see Proposition I.1.11), the random variable

$$L_n^{(\ell)}(d) := \left(\frac{E_n}{d}\right)^{\ell/2} J(\mathbf{G}, W, (0, \ldots, 0))$$

represents the $(d-\ell)$-dimensional volume of the zero set of $\mathbf{G}$, where J is defined according to Definition II.2.1. Note that $L_n^{(\ell)}(3) = L_n^{(\ell)}, \ell = 1, 2, 3$ as defined in (II.1.7). The continuity result in Theorem II.D.3 shows that the nodal volume is defined $\mathbb{P}$-a.s. The fact that the random variable $L_n^{(1)}(d)$ is well-defined in $L^2(\mathbb{P})$ for $d \geq 2$ is proved in [RW08], whereas the case $(\ell, d) = (2, 2)$ is proved in [DNPR19]. The remaining cases on the three-dimensional torus corresponding to $(\ell, d) = (2, 3), (3, 3)$ will be proved in Lemma II.3.1, the existence in $L^2(\mathbb{P})$ of the nodal volume for arbitrary ℓ and d can be proved by similar arguments, for which we omit the details. Now, for every $i \in [\ell]$, the quantity $D^{(i)}$ in (II.2.3) satisfies (see also [Ros16, MPRW16])

$$\begin{aligned} D^{(i)} &= \frac{1}{d}\sum_{j=1}^{d} \int_{\mathbb{T}^d} \tilde{\partial}_j T_n^{(i)}(x)^2\, dx - \int_{\mathbb{T}^d} T_n^{(i)}(x)^2\, dx \\ &= \frac{1}{d}\int_{\mathbb{T}^d} \|\widetilde{\nabla} T_n^{(i)}(x)\|^2\, dx - \int_{\mathbb{T}^d} T_n^{(i)}(x)^2\, dx \\ &= \frac{1}{d}\int_{\mathbb{T}^d} \langle \widetilde{\nabla} T_n^{(i)}(x), \widetilde{\nabla} T_n^{(i)}(x)\rangle\, dx - \int_{\mathbb{T}^d} T_n^{(i)}(x)^2\, dx \\ &= \frac{1}{E_n}\int_{\mathbb{T}^d} \langle \nabla T_n^{(i)}(x), \nabla T_n^{(i)}(x)\rangle\, dx - \int_{\mathbb{T}^d} T_n^{(i)}(x)^2\, dx\ . \end{aligned}$$

Using Green's first identity (see e.g. [Lee97, p.44]) and the fact that $\Delta T_n^{(i)}(x) = -E_n T_n^{(i)}(x)$, gives

$$D^{(i)} = -\frac{1}{E_n}\int_{\mathbb{T}^3} T_n^{(i)}(x)\Delta T_n^{(i)}(x)\, dx - \int_{\mathbb{T}^3} T_n^{(i)}(x)^2\, dx = 0\ .$$

In particular, we conclude from (II.2.12) that the second chaotic projection of the nodal volume $L_n^{(\ell)}$ is identically zero. On the other hand, if $u^{(\ell)} \neq 0 \in \mathbb{R}^\ell$, then, it follows from (II.2.10) that the second-order chaotic projection of $L_n^{(\ell)}(u^{(\ell)})$ on the three-torus is given by (bearing in mind that $D^{(i)} = 0$ for every $i \in [\ell]$)

$$\mathrm{proj}_2(L_n^{(\ell)}(u^{(\ell)})) = \left(\frac{E_n}{3}\right)^{\ell/2} \frac{\alpha_{\ell,3}}{2} \prod_{i=1}^{\ell} \gamma(u_i) \sum_{i=1}^{\ell} \left(u_1^2 \int_{\mathbb{T}^3} H_2(T_n^{(i)}(x))dx\right),$$

where $H_2(u) = u^2 - 1$ is the second Hermite polynomial. Combining this identity with the orthogonality relation for Hermite polynomials, it is easy to obtain its asymptotic variance (compare with Remark II.1.4)

$$\begin{aligned} \mathbf{Var}\left[\mathrm{proj}_2(L_n^{(\ell)}(u^{(\ell)}))\right] &= \left(\frac{E_n}{3}\right)^{\ell} \frac{\alpha_{\ell,3}^2}{4} \prod_{i=1}^{\ell} \gamma(u_i)^2 2 \sum_{i=1}^{\ell} u_i^4 \int_{\mathbb{T}^3\times\mathbb{T}^3} r_n(x-y)^2 dxdy \\ &= \left(\frac{E_n}{3}\right)^{\ell} \frac{\alpha_{\ell,3}^2}{4} \prod_{i=1}^{\ell} \gamma(u_i)^2 2 \sum_{i=1}^{\ell} u_i^4 \int_{\mathbb{T}^3} r_n(z)^2 dz \\ &= \frac{E_n^\ell}{\mathcal{N}_n} \frac{\alpha_{\ell,3}^2}{3^\ell \cdot 2} \gamma_\ell(u^{(\ell)})^2 (u_1^4 + \ldots + u_\ell^4), \end{aligned}$$

where γ_ℓ stands for the multivariate standard Gaussian density of dimension ℓ and where we used stationarity together with the fact that

$$\int_{\mathbb{T}^3} r_n(z)^2 dz = \int_{\mathbb{T}^3} \frac{1}{\mathcal{N}_n^2} \sum_{\lambda,\lambda' \in \Lambda_n} e_{\lambda+\lambda'}(z) dz = \frac{1}{\mathcal{N}_n^2} \sum_{\lambda,\lambda' \in \Lambda_n} \mathbb{1}_{\lambda=-\lambda'} = \frac{1}{\mathcal{N}_n},$$

in view of the orthogonality relation

$$\int_{\mathbb{T}^3} e_{\lambda+\lambda'}(x) dx = \mathbb{1}_{\lambda=-\lambda'}$$

on the torus. A subsequent Central Limit Theorem for the properly normalized random variable $\mathrm{proj}_2(L_n^{(\ell)}(u^{(\ell)}))$ can be proven by carefully investigating the term $H_2(T_n^{(i)}(x))$. The Central Limit Theorem for the random variable $L_n^{(\ell)}(u^{(\ell)})$ then follows once it is established that the second chaotic projection dominates the Wiener chaos expansion of $L_n^{(\ell)}(u^{(\ell)})$. Such a proof mainly follows from the arguments that will be developped later in order to show that the fourth Wiener chaotic projection is dominant in the nodal case[2].

(ii) BRW on $\mathbb{R}^d$. Let $1 \le \ell \le d$ be as above. Consider a compact convex set $\mathcal{D} \subset \mathbb{R}^d$ with C^1 boundary $\partial\mathcal{D}$ and such that the origin is contained in the interior of $\mathcal{D}$. Let $(Z, \mathscr{Z}, \mu) = (\mathcal{D}, \mathcal{B}(\mathcal{D}), dx)$. Write $\{B_E(x) : x \in \mathcal{D}\}$ to indicate Berry's random wave with parameter $E > 0$ restricted to $\mathcal{D}$, that is, the stationary centred Gaussian Laplace eigenfunction on $\mathbb{R}^d$ with covariance function (see Theorem I.1.8[3])

$$\mathbb{E}\left[B_E(x) \cdot B_E(y)\right] = \frac{J_{(d-2)/2}(2\pi\sqrt{E}\|x-y\|)}{(2\pi\sqrt{E}\|x-y\|)^{(d-2)/2}}, \quad x, y \in \mathcal{D},$$

with J_m denoting the Bessel function of order m of the first kind, and energy $4\pi^2 E$. Consider $B_E^{(1)}, \ldots, B_E^{(\ell)}$ i.i.d. copies of B_E and $\mathbf{G} = \left\{(B_E^{(1)}(x), \ldots, B_E^{(\ell)}(x)) : x \in \mathcal{D}\right\}$. We will show below that for every $i \in [\ell]$ and $j \in [d]$,

$$\mathbf{Var}\left[\partial_j B_E^{(i)}(x)\right] = \frac{4\pi^2 E}{d}, \quad x \in \mathcal{D}. \tag{II.2.14}$$

As in Example (i), we write $\tilde{\partial}_j := (4\pi^2 E/d)^{-1/2}\partial_j$ for the normalized derivatives and consider the random field $W = \{\Phi_{\ell,d}(\mathbf{G}_\star(x)) : x \in \mathcal{D}\}$ with $\Phi_{\ell,d}(A) = \det(AA^T)^{1/2}$ for $A \in \mathcal{M}_{\ell\times d}(\mathbb{R})$. Then, the random variable

$$L_E^{(\ell)}(d) := \left(\frac{4\pi^2 E}{d}\right)^{\ell/2} J(\mathbf{G}, W, (0, \ldots, 0))$$

is the nodal volume of $\mathbf{G}$. Again, by Theorem II.D.3, it is well-defined $\mathbb{P}$-a.s. The existence in $L^2(\mathbb{P})$ is proved in the cases $(\ell, d) = (1, 2), (2, 2)$ in [NPR19]. We omit the proofs for arbitrary integers ℓ and d. Arguing as in the previous example, using Green's identity, the quantity $D^{(i)}$ in (II.2.3) is equal to

$$D^{(i)} = \frac{1}{4\pi^2 E} \int_{\partial\mathcal{D}} B_E^{(i)}(x) \langle \nabla B_E^{(i)}(x), \mathrm{n}(x) \rangle dx, \tag{II.2.15}$$

[2]We omit the details: the main discrepancies with the nodal case lies in the fact that for non-zero levels, the *odd chaotic projections* of $L_n^{(\ell)}(u^{(\ell)})$ do not disappear. This shall however not create issues when adapting our strategies developed for the nodal case. We refer the reader to Section II.3.1 for further details on the proof of Theorem II.1.1.

[3]Such a covariance structure is obtained by setting the Π in Theorem I.1.8 equal to the Dirac mass at $2\pi\sqrt{E}$.

where $\mathrm{n}(x)$ denotes the outward unit normal vector to $\partial\mathcal{D}$ at x. In particular, $D^{(i)}$ and hence the second chaotic component of $L_E^{(\ell)}(d)$ reduce to an integration over the boundary of $\mathcal{D}$, thus recovering the exact expression of the second Wiener chaos of $L_E^{(1)}(2)$ obtained in [NPR19, Lemma 4.1] for $d = 2$. As already pointed out, in [DEL21], the authors study among others the nodal length restricted to growing cubes of the complex BRW on $\mathbb{R}^3$ corresponding to the case $(\ell, d) = (2, 3)$. In particular, applying Green's formula to the expression of the second chaotic component (see [DEL21, Lemma 8]), one can proceed similarly as above to show that it reduces to a boundary integration. In Chapter IV of this thesis, we will present a careful analysis of the boundary integration appearing in the projection on the second Wiener chaos in dimension two.

Let us now prove (II.2.14). For every $i \in [\ell], j \in [d]$ and $x \in \mathcal{D}$, by isotropy, the variance is independent of the chosen point x, so that

$$\mathbf{Var}\Big[\partial_j B_E^{(i)}(x)\Big] = \mathbf{Var}\Big[\partial_1 B_E^{(i)}(x_0)\Big],$$

where x_0 is a fixed point in $\mathcal{D}$. Therefore,

$$\mathbb{E}\left[\int_{\mathcal{D}} \|\nabla B_E^{(i)}(x)\|^2 dx\right] = d\int_{\mathcal{D}} \mathbb{E}\left[\partial_1 B_E^{(i)}(x)^2\right] dx = d \cdot \mathrm{area}(\mathcal{D}) \cdot \mathbf{Var}\Big[\partial_1 B_E^{(i)}(x_0)\Big].$$

On the other hand, by Green's formula and the fact that $\Delta B_E^{(i)}(x) = -4\pi^2 E B_E^{(i)}(x)$, we have

$$\begin{aligned}\mathbb{E}\left[\int_{\mathcal{D}} \|\nabla B_E^{(i)}(x)\|^2 dx\right] &= -\int_{\mathcal{D}} \mathbb{E}\left[B_E^{(i)}(x)\Delta B_E^{(i)}(x)\right] dx + \int_{\partial\mathcal{D}} \mathbb{E}\left[B_E^{(i)}(x)\langle \nabla B_E^{(i)}(x), \mathrm{n}(x)\rangle\right] dx \\ &= 4\pi^2 E\int_{\mathcal{D}} \mathbb{E}\left[B_E^{(i)}(x)^2\right] dx + \int_{\partial\mathcal{D}} \mathbb{E}\left[B_E^{(i)}(x)\langle \nabla B_E^{(i)}(x), \mathrm{n}(x)\rangle\right] dx.\end{aligned}$$

For the first term, since for every $x \in \mathcal{D}$, $B_E^{(i)}(x)$ has unit variance, we have

$$\int_{\mathcal{D}} \mathbb{E}\left[B_E^{(i)}(x)^2\right] dx = \mathrm{area}(\mathcal{D}).$$

For the second term, independence of $B_E^{(i)}(x)$ and $\nabla B_E^{(i)}(x)$ for every fixed $x \in D$, together with the fact that $B_E^{(i)}(x)$ is centred imply that this term is zero. Combining these observations, we deduce that

$$d \cdot \mathrm{area}(\mathcal{D}) \cdot \mathbf{Var}\Big[\partial_1 B_E^{(i)}(x_0)\Big] = 4\pi^2 E \cdot \mathrm{area}(\mathcal{D})$$

which proves (II.2.14).

Remark II.2.6. **(a)** An analogous analysis as in example (i) for ARWs on $\mathbb{T}^d$ can be carried out for the related model of spherical harmonics on the d-sphere, see [MRW20] for the case of the 2-sphere.

(b) From (II.2.10) it follows that, in the case where $D^{(i)} = 0$ for every $i \in [\ell]$ (as in example (i) above), the projection on the second Wiener chaos of the random variable J can be rewritten as (bearing in mind that $\mu(Z) = 1$)

$$\mathrm{proj}_2(J) = \frac{\alpha_{\ell,k}}{2} \cdot \prod_{i=1}^{\ell} \gamma(u_i) \cdot \sum_{i=1}^{\ell} u_i^2 \left\{\|X_0^{(i)}\|_{L^2(Z)}^2 - 1\right\},$$

where $\|g\|_{L^2(Z)}^2 := \int_Z g(z)^2 \mu(dz)$, (see also [MR21] and [CMR20]). In particular, the second order chaotic projection of J is a linear combination of the centred square (random) norms of the fields $X_0^{(i)}, i \in [\ell]$. Therefore, at least heuristically, one expects that in the setting where J is a geometric functional associated with the zero level (such as the random variables $L_n^{(\ell)}$), it should vanish as nodal lines do not depend on scaling factors.

II.3 Proof of Theorem II.1.1

Section II.3.1 contains the proof of Theorem II.1.1: such a proof is based on a number of technical results, whose proofs and discussion are provided in Appendices II.A-II.E. The only exception to this strategy of presentation is given by Proposition II.3.3 and II.3.4: indeed, since these results follow from direct probabilistic arguments, their full proofs will be immediately provided in the forthcoming Section II.3.2.

II.3.1 The proof

II.3.1.1 An integral representation of $L_n^{(\ell)}$

The proof of Theorem II.1.1 is based on the Wiener chaos expansion of the quantities $L_n^{(\ell)}$ defined in (II.1.7). In order to derive this expansion, we will rigorously prove that the nodal volume $L_n^{(\ell)}$ is formally obtained $\mathbb{P}$-almost surely and in $L^2(\mathbb{P})$ as

$$L_n^{(\ell)} = \int_{\mathbb{T}^3} \prod_{i=1}^{\ell} \delta_0(T_n^{(i)}(x)) \cdot \Phi^*_{\ell,3}(\mathrm{jac}_{\mathbf{T}_n^{(\ell)}}(x))\, dx,$$

where $\Phi^*_{\ell,3}(A) = \det(AA^T)^{1/2}$ for $A \in \mathcal{M}_{\ell\times 3}(\mathbb{R})$, and $\mathrm{jac}_{\mathbf{T}_n^{(\ell)}}(x)$ stands for the Jacobian matrix of $\mathbf{T}_n^{(\ell)}$ evaluated at x. More precisely, for $\varepsilon > 0$, we consider the ε-approximations $L_{n,\varepsilon}^{(\ell)}$ of $L_n^{(\ell)}$ given by (compare with Definition II.2.1)

$$L_{n,\varepsilon}^{(\ell)} := \int_{\mathbb{T}^3} (2\varepsilon)^{-\ell} \prod_{i=1}^{\ell} \mathbb{1}_{[-\varepsilon,\varepsilon]}(T_n^{(i)}(x)) \cdot \Phi^*_{\ell,3}(\mathrm{jac}_{\mathbf{T}_n^{(\ell)}}(x))\, dx\,, \quad \varepsilon > 0$$

and prove the following statement.

Lemma II.3.1. *For $\ell \in [3]$ and $n \in S_3$, the random variable $L_{n,\varepsilon}^{(\ell)}$ converges to $L_n^{(\ell)}$ $\mathbb{P}$-a.s and in $L^2(\mathbb{P})$ as $\varepsilon \to 0$.*

The proof of Lemma II.3.1 is presented in Section II.E.2 of Appendix II.E. Note that the case $\ell = 1$ has been investigated in [RW08] for arbitrary dimensions. To deal with the case $\ell = 3$, one can directly adapt the proof of points (i)-(v) of Lemma 3.1 in [NPR19] for the two-dimensional torus, based on universal bounds for the number of solutions of a system of trigonometric polynomials (see e.g. [Kho91]).

The proof of the almost sure convergence relies on a deterministic continuity result for nodal volumes restricted to compact sets on the torus associated with sequences of functions converging to a non-degenerate limit in the C^1-topology (see Appendix II.D). Our proof of the $L^2(\mathbb{P})$ convergence takes advantage of similar techniques as those that will be exposed in the forthcoming Section II.3.1.4, based on partitioning the torus into singular and non-singular subregions. We refer the reader to this part for an overview of our strategy.

II.3.1.2 Wiener-Itô chaos decomposition of $L_n^{(\ell)}$.

The statement of Lemma II.3.1 together with the fact that, for every fixed $x \in \mathbb{T}^3$, the random variables $\mathbf{T}_n^{(\ell)}(x)$ and $\mathrm{jac}_{\mathbf{T}_n^{(\ell)}}(x)$ are stochastically independent, justify the use of the general framework of Theorem II.2.5 to this precise setting, yielding in particular an explicit expression for the chaotic decomposition of $L_n^{(\ell)}$. In view of Example (i) of Section II.2.2 in the case $d = 3$, the quantity $D^{(i)}$ in (II.2.3) is zero for every $i \in [\ell]$. This together with the fact that we study nodal sets, implies that (in view of Theorem II.2.5 (ii)) the second-order as well as the odd-order chaoses identically vanish, yielding

$$L_n^{(\ell)} = \mathbb{E}\left[L_n^{(\ell)}\right] + \sum_{q\geq 2} \mathrm{proj}_{2q}(L_n^{(\ell)})\,, \quad \ell \in [3]\,. \tag{II.3.1}$$

Normalized gradients. Writing $T_n^{(i_1)}(x) = \mathcal{N}_n^{-1/2} \sum_{\lambda\in\Lambda_n} a_{i_1,\lambda} e_\lambda(x)$ for $i_1 \in [\ell]$, in view of (II.2.13), we introduce the scaled partial derivatives having variance 1,

$$T_{n,j}^{(i_1)}(x) := \tilde{\partial}_j T_n^{(i_1)}(x) := \sqrt{\frac{3}{E_n}} \partial_j T_n^{(i_1)}(x) = i\sqrt{\frac{3}{n\mathcal{N}_n}} \sum_{\lambda\in\Lambda_n} \lambda_j a_{i_1,\lambda} e_\lambda(x)\,, \quad j \in [3] \qquad \text{(II.3.2)}$$

and adopt the same notation as in (II.2.2), that is

$$\mathbf{T}_{n\star}^{(\ell)}(x) := \left\{T_{n,j}^{(i)}(x) : (i,j) \in [\ell]\times[3]\right\} \in \mathcal{M}_{\ell\times 3}(\mathbb{R})\,.$$

Using the homogeneity property (A2) in Definition II.2.2 of the map $\Phi^*_{\ell,3}$, it follows that

$$L_{n,\varepsilon}^{(\ell)} = \left(\frac{E_n}{3}\right)^{\ell/2} \int_{\mathbb{T}^3} (2\varepsilon)^{-\ell} \prod_{i=1}^{\ell} \mathbb{1}_{[-\varepsilon,\varepsilon]}(T_n^{(i)}(x)) \cdot \Phi^*_{\ell,3}(\mathbf{T}_{n\star}^{(\ell)}(x))\, dx\,, \quad \varepsilon > 0. \qquad \text{(II.3.3)}$$

Therefore, by virtue of the almost sure convergence stated in Lemma II.3.1, we can write the nodal volume as (recall Definition II.2.1)

$$L_n^{(\ell)} = \left(\frac{E_n}{3}\right)^{\ell/2} J(\mathbf{G}, W; u^{(\ell)}),$$

where

$$\mathbf{G} = \mathbf{T}_n^{(\ell)}\,, \quad W = \left\{\Phi^*_{\ell,3}(\mathbf{T}_{n\star}^{(\ell)}(x)) : x\in\mathbb{T}^3\right\}\,, \quad u^{(\ell)} = (0,\dots,0)\in\mathbb{R}^\ell\,.$$

The following proposition gives the Wiener-Itô chaos expansion of $L_n^{(\ell)}$ and is a direct consequence of Theorem II.2.5.

Proposition II.3.2 (Wiener Chaos expansion of $L_n^{(\ell)}$)**.** *Fix $\ell\in[3]$. For $n\in S_3$, the chaotic projections of $L_n^{(\ell)}$ are given by*

$$\mathrm{proj}_2(L_n^{(\ell)}) = \mathrm{proj}_{2q+1}(L_n^{(\ell)}) = 0\,, \quad q \ge 0\,, \qquad \text{(II.3.4)}$$

while for $q = 0$ and $q \ge 2$,

$$\mathrm{proj}_{2q}(L_n^{(\ell)}) = \left(\frac{E_n}{3}\right)^{\ell/2} \sum_{\substack{p_0^{(1)},\dots,p_3^{(1)}\ge 0 \\ p_0^{(1)}+\dots+p_3^{(1)}+\dots+p_0^{(\ell)}+\dots+p_3^{(\ell)}=2q}} \cdots \sum_{p_0^{(\ell)},\dots,p_3^{(\ell)}\ge 0} \frac{\beta_{p_0^{(1)}}\cdots\beta_{p_0^{(\ell)}}}{p_0^{(1)}!\cdots p_0^{(\ell)}!} \alpha_3^{(\ell)}\left\{p_j^{(i)} : (i,j)\in[\ell]\times[3]\right\}$$

$$\times \int_{\mathbb{T}^3} \prod_{i=1}^{\ell} H_{p_0^{(i)}}(T_n^{(i)}(x)) \prod_{j=0}^{3} H_{p_j^{(i)}}(T_{n,j}^{(i)}(x))\, dx\,,$$

where $\left\{\beta_j : j\ge 0\right\}$ and $\alpha_3^{(\ell)}\{\cdot\}$ are the Wiener chaos projection coefficients of δ_0 and $\Phi^*_{\ell,3}$, that is

$$\beta_{2j+1} = 0\,, \quad \beta_{2j} = \frac{H_{2j}(0)}{\sqrt{2\pi}}\,, \quad j\ge 0\,,$$

and

$$\alpha_k^{(\ell)}\left\{p_j^{(i)} : (i,j)\in[\ell]\times[k]\right\} := \frac{1}{\prod_{i=1}^{\ell}\prod_{j=1}^{k}(p_j^{(i)})!} \cdot \mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})\cdot\prod_{i=1}^{\ell}\prod_{j=1}^{k} H_{p_j^{(i)}}(X_j^{(i)})\right]\,, \quad k\ge\ell$$

respectively. In particular,

$$\mathrm{proj}_0(L_n^{(\ell)}) = \mathbb{E}\left[L_n^{(\ell)}\right] = \left(\frac{E_n}{3}\right)^{\ell/2} \frac{\alpha(\ell,3)}{(2\pi)^{\ell/2}}\,, \qquad \text{(II.3.5)}$$

where

$$\alpha(\ell,k) = \frac{(k)_\ell\,\kappa_k}{(2\pi)^{\ell/2}\kappa_{k-\ell}}\,,$$

is as in (II.1.8).

II.3.1.3 Analysis of the fourth chaotic projection

Our main findings on the high-energy behaviour of the fourth-order chaotic projections $\mathrm{proj}_4(L_n^{(\ell)})$, $\ell \in [3]$ are contained in the next two propositions, whose proofs are presented in Section II.3.2.3:

Proposition II.3.3. *For $\ell \in [3]$, as $n \to \infty, n \not\equiv 0, 4, 7 \pmod 8$,*

$$\mathbf{Var}\left[\mathrm{proj}_4(L_n^{(\ell)})\right] \sim \left(c_n^{(\ell)}\right)^2 \left(\ell \cdot \frac{1}{250} + \frac{\ell(\ell-1)}{2} \cdot \frac{76}{375}\right),$$

where the constant $c_n^{(\ell)}$ is given by

$$c_n^{(\ell)} := \left(\frac{E_n}{3}\right)^{\ell/2} \frac{2}{(2\pi)^{\ell/2}} \frac{\alpha(\ell,3)}{\mathcal{N}_n}.$$

Proposition II.3.4. *For $\ell \in [3]$, we define the normalized fourth-order chaotic component*

$$\left\{\widetilde{\mathrm{proj}_4}(L_n^{(\ell)}) : n \in S_3\right\} := \left\{(v_{n;4}^{(\ell)})^{-1/2}\, \mathrm{proj}_4(L_n^{(\ell)}) : n \in S_3\right\},$$

where $v_{n;4}^{(\ell)} := \mathbf{Var}\left[\mathrm{proj}_4(L_n^{(\ell)})\right]$. As $n \to \infty, n \not\equiv 0, 4, 7 \pmod 8$,

$$\widetilde{\mathrm{proj}_4}(L_n^{(\ell)}) \xrightarrow{d} \left(\ell \cdot \frac{1}{250} + \frac{\ell(\ell-1)}{2} \cdot \frac{76}{375}\right)^{-1/2} Y^{(\ell)} M^{(\ell)} (Y^{(\ell)})^T,$$

where $Y^{(\ell)} \sim \mathcal{N}_{\ell(9\ell-4)}(0, \mathbf{I}_{\ell(9\ell-4)})$ and $M^{(\ell)} \in \mathcal{M}_{\ell(9\ell-4)\times\ell(9\ell-4)}(\mathbb{R})$ is the deterministic matrix given by

$$M^{(\ell)} = \frac{-1}{50}\mathbf{I}_{5\ell} \oplus \frac{-1}{25}\mathbf{I}_{\frac{5\ell(\ell-1)}{2}} \oplus \frac{1}{25}\mathbf{I}_{\frac{5\ell(\ell-1)}{2}} \oplus \frac{1}{50}\mathbf{I}_{\frac{5\ell(\ell-1)}{2}} \oplus \frac{-1}{6}\mathbf{I}_{\frac{3\ell(\ell-1)}{2}}.$$

Such results are proved as follows: In Section II.3.2.1, we provide an exact expression of the fourth-order chaotic projection of $L_n^{(\ell)}$. In order to achieve this, we compute the Fourier-Hermite coefficients of the function $\Phi_{\ell,3}^*$ on the fourth Wiener chaos (see Proposition II.B.5). We then use the orthogonality relation for complex exponentials on the torus

$$\int_{\mathbb{T}^3} e_\lambda(x)\, dx = \mathbb{1}_{\lambda=0}, \tag{II.3.6}$$

and rewrite each integral of multivariate Hermite polynomials evaluated at the arithmetic random waves and its gradient components by means of a useful summation rule over 4-correlations $\mathcal{C}_n(4)$ and non-degenerate 4-correlations $\mathcal{X}_n(4)$ (see (II.3.13) and (II.3.14) for precise definitions).

A subsequent asymptotic analysis of $\mathrm{proj}_4(L_n^{(\ell)})$ is presented in Section II.3.2.2. This analysis is based on a multivariate Central Limit Theorem (see Proposition II.3.19) for the summands composing the explicit expression of $\mathrm{proj}_4(L_n^{(\ell)})$. Such a Central Limit Theorem, already appearing in [MPRW16, DNPR19] for the two-dimensional torus and [Cam19] for the nodal surface on the three-dimensional torus, is obtained by verifying a suitable condition characterising normal convergence of the so-called Fourth Moment Theorem (see Theorem 5.2.7 [NP12a]). Among others, we use the following asymptotic estimate bounding non-degenerate 4-correlations on $\mathbb{T}^3$ (see Theorem 1.6 [BM19]):

$$\mathrm{card}(\mathcal{X}_n(4)) = O(\mathcal{N}_n^{7/4+o(1)}), \quad n \to \infty. \tag{II.3.7}$$

II.3.1.4 Contribution of higher-order chaotic projections

We show that the projection on the fourth Wiener chaos of $L_n^{(\ell)}$ dominates the series in (II.3.1), in the sense that

$$\widetilde{L_n^{(\ell)}} = \widetilde{\mathrm{proj}_4}(L_n^{(\ell)}) + o_{\mathbb{P}}(1)\,,$$

where $o_{\mathbb{P}}(1)$ denotes a sequence of random variables converging to zero in probability as $n \to \infty, n \not\equiv 0, 4, 7 \pmod 8$. This is done by proving the following statement (see Appendix II.E):

Proposition II.3.5. *For $\ell \in [3]$, as $n \to \infty, n \not\equiv 0, 4, 7 \pmod 8$,*

$$\mathbf{Var}\left[\sum_{q\geq 3} \mathrm{proj}_{2q}(L_n^{(\ell)})\right] = o\Big(\mathbf{Var}\big[\mathrm{proj}_4(L_n^{(\ell)})\big]\Big)\,. \tag{II.3.8}$$

The arguments for the proof of Proposition II.3.5 are based on the use of a suitable partition $\mathcal{P}(M)$ (where $M = M(n)$ is proportional to $\sqrt{E_n}$) of the torus into singular and non-singular pairs of subregions (see Definition II.E.1), following the route of [ORW08] and, later, [PR18, DNPR19]. We denote by $L_n^{(\ell)}(Q)$ the nodal volume restricted to a cube Q and by $\mathrm{proj}_{6+}(L_n^{(\ell)}) := \sum_{q\geq 3} \mathrm{proj}_{2q}(L_n^{(\ell)})$ the chaotic projection of $L_n^{(\ell)}$ on Wiener chaoses of order at least 6. This allows us to write the variance of higher-order chaoses as

$$\mathbf{Var}\big[\mathrm{proj}_{6+}(L_n^{(\ell)})\big] = \sum_{(Q,Q')\in\mathcal{P}(M)^2} \mathbf{Cov}\big[\mathrm{proj}_{6+}(L_n^{(\ell)}(Q)), \mathrm{proj}_{6+}(L_n^{(\ell)}(Q'))\big]\,, \tag{II.3.9}$$

where the summation is over all pairs of cubes (Q, Q') of side length $1/M$. Splitting this sum into the singular part $\mathcal{S}$ and the non-singular part $\mathcal{S}^c$, we bound each of the contributions separately. For the singular part, we prove the following bound (see Section II.E.3 of AppendixII.E):

Lemma II.3.6. *For $\ell \in [3]$, as $n \to \infty, n \not\equiv 0, 4, 7 \pmod 8$, we have*

$$\left|S_{n,1}^{(\ell)}\right| := \left|\sum_{(Q,Q')\in\mathcal{S}} \mathbf{Cov}\big[\mathrm{proj}_{6+}(L_n^{(\ell)}(Q)), \mathrm{proj}_{6+}(L_n^{(\ell)}(Q'))\big]\right| = O(E_n^{\ell}\mathcal{R}_n(6))\,.$$

Here, $\mathcal{R}_n(6)$ denotes the integral 6-th moment of the covariance function r_n, see formula (II.3.12) below. We give a brief overview of the proof of Lemma II.3.6. We use the Cauchy-Schwarz inequality and translation-invariance of the model to write

$$\left|S_{n,1}^{(\ell)}\right| \leq E_n^3\mathcal{R}_n(6)\cdot\mathbf{Var}\big[\mathrm{proj}_{6+}(L_n^{(\ell)}(Q_0))\big], \tag{II.3.10}$$

where we used that the number of singular pairs of cubes in the summation index is bounded by $E_n^3\mathcal{R}_n(6)$ and where Q_0 denotes a small cube of side length $1/M$ around the origin. In Lemma II.C.6, we justify the use of Kac-Rice formula in Q_0, so that, writing

$$\mathbf{Var}\big[\mathrm{proj}_{6+}(L_n^{(\ell)}(Q_0))\big] \leq \mathbb{E}\left[L_n^{(\ell)}(Q_0)^2\right],$$

one can use Kac-Rice formulae for moments (see Theorem I.1.12). Doing so, we exploit stationarity to obtain

$$\mathbb{E}\left[L_n^{(\ell)}(Q_0)^2\right] = \int_{Q_0\times Q_0} K^{(\ell)}(x,y;(0,\dots,0))\,dxdy + \mathbb{E}\left[L_n^{(3)}(Q_0)\right]\mathbb{1}_{\ell=3}$$

$$\leq \text{Leb}(Q_0) \int_{2Q_0} K^{(\ell)}(z, 0; (0, \dots, 0))\, dz + \frac{E_n^{3/2}}{M^3} \mathbb{1}_{\ell=3}\,, \qquad \text{(II.3.11)}$$

where $K^{(\ell)}$ is the two-point correlation function defined in (II.C.3) of Appendix II.C . Appendix II.C contains a self-contained study of the two-point correlation function; in particular, in (II.C.4), we derive an upper bound of $K^{(\ell)}$ in terms of the covariance function r_n and its gradient, and subsequently perform a precise Taylor-type expansion near the origin of this expression (see Lemma II.C.5). Using these results then yields the estimate

$$\mathbb{E}\left[L_n^{(\ell)}(Q_0)^2\right] \ll E_n^{-2}\mathbb{1}_{\ell=1} + E_n^{-1}\mathbb{1}_{\ell=2} + \mathbb{1}_{\ell=3},$$

which combined with (II.3.10) establishes Lemma II.3.6.

Concerning the contribution to the variance of the non-singular pairs of cubes, we prove the following proposition (see Section II.E.3 of Appendix II.E):

Lemma II.3.7. *For $\ell \in [3]$, as $n \to \infty, n \not\equiv 0, 4, 7 \pmod 8$, we have*

$$\left|S_{n,2}^{(\ell)}\right| := \left|\sum_{(Q,Q')\in\mathcal{S}^c} \mathbf{Cov}\left[\text{proj}_{6+}(L_n^{(\ell)}(Q)), \text{proj}_{6+}(L_n^{(\ell)}(Q'))\right]\right| = O(E_n^{\ell}\mathcal{R}_n(6))\,.$$

In order to prove Lemma II.3.7, we take advantage of (i) the Wiener-Itô chaos expansion of $L_n^{(\ell)}$ and (ii) a particular version of diagram formula for Hermite polynomials (see Proposition II.E.3) allowing us to handle covariances of products of Hermite polynomials. The desired bound is then obtained by exploiting the fact that the summation is over non-singular pairs of cubes.

Combining the decomposition of the variance in (II.3.9) with Lemma II.3.6 and Lemma II.3.7, the proof of Proposition II.3.5 is then concluded once we derive a bound for the integral 6-th moment of r_n. In order to achieve this, we can again use the orthogonality relation for complex exponentials on the torus (II.3.6) in order to link moments of the covariance function r_n to m-correlations, for $m \geq 1$,

$$\mathcal{R}_n(m) := \int_{\mathbb{T}^3} r_n(z)^m dz = \frac{1}{\mathcal{N}_n^m} \sum_{(\lambda^{(1)},\dots,\lambda^{(m)})\in\Lambda_n^m} \int_{\mathbb{T}^3} e_{\lambda^{(1)}+\dots+\lambda^{(m)}}(z) dz = \frac{\text{card}(C_n(m))}{\mathcal{N}_n^m}\,.$$

Using this formula for $m = 6$ together with the estimate bounding the number of 6-correlations on $\mathbb{T}^3$ (Theorem 1.7 [BM19])

$$\text{card}(C_n(6)) = O(\mathcal{N}_n^{11/3+o(1)})\,, \qquad n \to \infty\,,$$

yields

$$\mathcal{R}_n(6) = \int_{\mathbb{T}^3} r_n(z)^6\, dz = \frac{\text{card}(C_n(6))}{\mathcal{N}_n^6} = O(\mathcal{N}_n^{-7/3+o(1)})\,, \qquad n \to \infty\,. \qquad \text{(II.3.12)}$$

Combining this with the content of Proposition II.3.3, we conclude that $E_n^{\ell}\mathcal{R}_n(6) = o\left(\mathbf{Var}\left[\text{proj}_4(L_n^{(\ell)})\right]\right)$.

Remark II.3.8. As described above, we point out that the combination of the findings in Lemma II.3.6, Lemma II.3.7 and the estimate in (II.3.12) is essential in order to prove that the fourth chaotic component is dominant in the Wiener-Itô chaos decomposition of $L_n^{(\ell)}$. These results should be compared with the analog statements in Section 2.4 of [DNPR19] for the study of ARW on the two-torus, and with the findings in Section 7 of [NPR19] for the somewhat similar approach in the setting of Berry random plane waves.

II.3.1.5 Finishing the proof of Theorem II.1.1

The proof of Theorem II.1.1 is concluded as follows: Relation (II.1.9) follows from (II.3.5) and the distributional identity stated in formula (III.3.49). The asymptotic variance in Proposition II.3.3 together with Proposition II.3.5 prove (IV.1.20). Finally, (II.1.11) follows from the limiting distribution established in Proposition II.3.4 combined with Proposition II.3.5.

II.3.2 Complete study of the fourth chaotic component of $L_n^{(\ell)}$

In this section, we provide the exact expression of the fourth-order chaotic component of $L_n^{(\ell)}$. A subsequent asymptotic analysis of this expression serves as preparation to deriving the limiting distribution of the normalized version of $L_n^{(\ell)}$.

II.3.2.1 Explicit form of $\mathrm{proj}_4(L_n^{(\ell)})$

In order to write the explicit expression of the fourth-order chaotic component of $L_n^{(\ell)}$, we introduce some auxiliary random variables. Fix $\ell \in [3]$.

Definition II.3.9. For $i_1, i_2 \in [\ell]$, $j, k \in [3]$ and $n \in S_3$, we define:

$$
\begin{aligned}
W^{(i_1)}(n) &:= \frac{1}{\sqrt{\mathcal{N}_n}} \sum_{\lambda\in\Lambda_n} (|a_{i_1,\lambda}|^2 - 1)\,, \qquad W^{(i_1)}_{jk}(n) := \frac{1}{n\sqrt{\mathcal{N}_n}} \sum_{\lambda\in\Lambda_n} \lambda_j\lambda_k(|a_{i_1,\lambda}|^2 - 1)\,,\\
M^{(i_1,i_2)}(n) &:= \frac{1}{\sqrt{\mathcal{N}_n}} \sum_{\lambda\in\Lambda_n} a_{i_1,\lambda}\overline{a_{i_2,\lambda}}\,, \qquad i_1 < i_2,\ \ell \in \{2,3\}\,,\\
M^{(i_1,i_2)}_{j}(n) &:= \frac{i}{\sqrt{n\mathcal{N}_n}} \sum_{\lambda\in\Lambda_n} \lambda_j a_{i_1,\lambda}\overline{a_{i_2,\lambda}}\,, \qquad i_1 < i_2,\ \ell \in \{2,3\}\,,\\
M^{(i_1,i_2)}_{jk}(n) &:= \frac{1}{n\sqrt{\mathcal{N}_n}} \sum_{\lambda\in\Lambda_n} \lambda_j\lambda_k a_{i_1,\lambda}\overline{a_{i_2,\lambda}}\,, \qquad i_1 < i_2,\ \ell \in \{2,3\}\,,\\
R^{(i_1,i_2)}(n) &:= \frac{1}{\mathcal{N}_n} \sum_{\lambda\in\Lambda_n} |a_{i_1,\lambda}|^2|a_{i_2,\lambda}|^2\,, \qquad R^{(i_1,i_2)}_{jk}(n) := \frac{1}{n^2\mathcal{N}_n} \sum_{\lambda\in\Lambda_n} \lambda_j^2\lambda_k^2|a_{i_1,\lambda}|^2|a_{i_2,\lambda}|^2\,,\\
S^{(i_1,i_2)}(n) &:= \frac{1}{\mathcal{N}_n} \sum_{\lambda\in\Lambda_n} a_{i_1,\lambda}^2\overline{a_{i_2,\lambda}}^2\,, \qquad S^{(i_1,i_2)}_{jk}(n) := \frac{1}{n^2\mathcal{N}_n} \sum_{\lambda\in\Lambda_n} \lambda_j^2\lambda_k^2 a_{i_1,\lambda}^2\overline{a_{i_2,\lambda}}^2\,,\\
X^{(i_1,i_2)}(n) &:= \frac{1}{\mathcal{N}_n} \sum_{(\lambda,\lambda',\lambda'',\lambda''')\in\mathcal{X}_n(4)} a_{i_1,\lambda}a_{i_1,\lambda'}a_{i_2,\lambda''}a_{i_2,\lambda'''}\,,\\
X^{(i_1,i_2)}_{kk}(n) &:= \frac{1}{n\mathcal{N}_n} \sum_{(\lambda,\lambda',\lambda'',\lambda''')\in\mathcal{X}_n(4)} \lambda_k\lambda'_k a_{i_1,\lambda}a_{i_1,\lambda'}a_{i_2,\lambda''}a_{i_2,\lambda'''}\,,\\
X^{(i_1,i_2)}_{kkjj}(n) &:= \frac{1}{n^2\mathcal{N}_n} \sum_{(\lambda,\lambda',\lambda'',\lambda''')\in\mathcal{X}_n(4)} \lambda_k\lambda'_k\lambda''_j\lambda'''_j a_{i_1,\lambda}a_{i_1,\lambda'}a_{i_2,\lambda''}a_{i_2,\lambda'''}\,.
\end{aligned}
$$

Note that $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = n$ implies the relations

$$
R^{(i_1,i_2)}(n) = \sum_{k,j=1}^{3} R^{(i_1,i_2)}_{jk}(n)\,, \quad S^{(i_1,i_2)}(n) = \sum_{k,j=1}^{3} S^{(i_1,i_2)}_{jk}(n)\,.
$$

Definition II.3.10. For $i_1 \in [\ell]$, and $n \in S_3$, we set

$$a_1^{(i_1)}(n) := \int_{\mathbb{T}^3} H_4(T_n^{(i_1)}(x))\,dx\,, \qquad a_2^{(i_1)}(n) := \sum_{k=1}^3 \int_{\mathbb{T}^3} H_2(T_n^{(i_1)}(x))H_2(T_{n,k}^{(i_1)}(x))\,dx\,,$$

$$a_3^{(i_1)}(n) := \sum_{k=1}^3 \int_{\mathbb{T}^3} H_4(T_{n,k}^{(i_1)}(x))\,dx\,, \qquad a_4^{(i_1)}(n) := \sum_{k<j} \int_{\mathbb{T}^3} H_2(T_{n,k}^{(i_1)}(x))H_2(T_{n,j}^{(i_1)}(x))\,dx\,,$$

and for $\ell \in \{2,3\}$ and $i_1 < i_2 \in [\ell], n \in S_3$,

$$\begin{aligned}
b_1^{(i_1,i_2)}(n) &:= \int_{\mathbb{T}^3} H_2(T_n^{(i_1)}(x))H_2(T_n^{(i_2)}(x))\,dx\,,\\
b_2^{(i_1,i_2)}(n) &:= \sum_{k=1}^3 \int_{\mathbb{T}^3} H_2(T_n^{(i_1)}(x))H_2(T_{n,k}^{(i_2)}(x))\,dx\,,\\
b_2'^{(i_1,i_2)}(n) &:= \sum_{k=1}^3 \int_{\mathbb{T}^3} H_2(T_{n,k}^{(i_1)}(x))H_2(T_n^{(i_2)}(x))\,dx\,,\\
b_3^{(i_1,i_2)}(n) &:= \sum_{k\neq j=1}^3 \int_{\mathbb{T}^3} H_2(T_{n,k}^{(i_1)}(x))H_2(T_{n,j}^{(i_2)}(x))\,dx\,,\\
b_4^{(i_1,i_2)}(n) &:= \sum_{k=1}^3 \int_{\mathbb{T}^3} H_2(T_{n,k}^{(i_1)}(x))H_2(T_{n,k}^{(i_2)}(x))\,dx\,,\\
b_5^{(i_1,i_2)}(n) &:= \sum_{k<j} \int_{\mathbb{T}^3} T_{n,k}^{(i_1)}(x)T_{n,j}^{(i_1)}(x)T_{n,k}^{(i_2)}(x)T_{n,j}^{(i_2)}(x)\,dx\,.
\end{aligned}$$

Spectral correlations on $\mathbb{T}^3$. For $n \in S_3$ and an integer $m \geq 1$, we introduce the set of m-correlations on the torus,

$$C_n(m) := \left\{(\lambda^{(1)},\dots,\lambda^{(m)}) \in \Lambda_n^m : \lambda^{(1)} + \ldots + \lambda^{(m)} = 0\right\} \tag{II.3.13}$$

and the set of non-degenerate m-correlations

$$\mathcal{X}_n(m) := \left\{(\lambda^{(1)},\dots,\lambda^{(m)}) \in C_n(m) : \forall I \subsetneq [m], \sum_{i\in I} \lambda^{(i)} \neq 0\right\} \subsetneq C_n(m)\,. \tag{II.3.14}$$

Recall that $\mathrm{card}(C_n(4)) = 3\mathcal{N}_n^2 - 3\mathcal{N}_n + \mathrm{card}(\mathcal{X}_n(4))$, which is in accordance with the following summation rule (see (3.6) in [Cam19])

$$\begin{aligned}
\sum_{(\lambda,\lambda',\lambda'',\lambda''')\in C_n(4)} &= \sum_{\substack{\lambda=-\lambda'\\ \lambda''=-\lambda'''}} + \sum_{\substack{\lambda=-\lambda''\\ \lambda'=-\lambda'''}} + \sum_{\substack{\lambda=-\lambda'''\\ \lambda'=-\lambda''}}\\
&- \sum_{\lambda=-\lambda'=\lambda''=-\lambda'''} - \sum_{\lambda=\lambda'=-\lambda''=-\lambda'''} - \sum_{\lambda=-\lambda'=-\lambda''=\lambda'''} + \sum_{(\lambda,\lambda',\lambda'',\lambda''')\in\mathcal{X}_n(4)}\,.
\end{aligned} \tag{II.3.15}$$

In the sequel, we will write $(\lambda,\lambda',\lambda'',\lambda''') = (\lambda^{(1)},\lambda^{(2)},\lambda^{(3)},\lambda^{(4)})$ for elements in $C_n(4)$ and $\mathcal{X}_n(4)$ and use the following abbreviations

$$\sum_{\lambda} := \sum_{\lambda\in\Lambda_n}, \quad \sum_{C_n(4)} := \sum_{(\lambda,\lambda',\lambda'',\lambda''')\in C_n(4)}, \quad \sum_{\mathcal{X}_n(4)} := \sum_{(\lambda,\lambda',\lambda'',\lambda''')\in\mathcal{X}_n(4)}\,.$$

The following lemma is a generalization of Lemma 4.5 in [Cam19] (obtained for $\ell = 1$) applying to the setting of multiple independent arithmetic random waves. These formulae follow by carefully applying the summation rule (II.3.15).

Lemma II.3.11. *Fix $\ell \in [3]$. For every $i_1, i_2 \in [\ell]$ and every $j, k \in [3]$, the following formulae hold:*

$$\sum_{\mathcal{C}_n(4)} a_{i_1,\lambda} a_{i_1,\lambda'} a_{i_2,\lambda''} a_{i_2,\lambda'''} = \sum_{\lambda} |a_{i_1,\lambda}|^2 \sum_{\lambda} |a_{i_2,\lambda}|^2 + 2\Big(\sum_{\lambda} a_{i_1,\lambda}\overline{a_{i_2,\lambda}}\Big)^2$$
$$-2\sum_{\lambda} |a_{i_1,\lambda}|^2 |a_{i_2,\lambda}|^2 - \sum_{\lambda} a_{i_1,\lambda}^2 \overline{a_{i_2,\lambda}}^2 + \sum_{\mathcal{X}_n(4)} a_{i_1,\lambda} a_{i_1,\lambda'} a_{i_2,\lambda''} a_{i_2,\lambda'''}\,, \qquad \text{(II.3.16)}$$

$$\sum_{\mathcal{C}_n(4)} \lambda''_k \lambda'''_k a_{i_1,\lambda} a_{i_1,\lambda'} a_{i_2,\lambda''} a_{i_2,\lambda'''} = -\sum_{\lambda} |a_{i_1,\lambda}|^2 \sum_{\lambda} \lambda_k^2 |a_{i_2,\lambda}|^2 + 2\Big(\sum_{\lambda} \lambda_k a_{i_1,\lambda}\overline{a_{i_2,\lambda}}\Big)^2$$
$$+2\sum_{\lambda} \lambda_k^2 |a_{i_1,\lambda}|^2 |a_{i_2,\lambda}|^2 - \sum_{\lambda} \lambda_k^2 a_{i_1,\lambda}^2 \overline{a_{i_2,\lambda}}^2 + \sum_{\mathcal{X}_n(4)} \lambda''_k \lambda'''_k a_{i_1,\lambda} a_{i_1,\lambda'} a_{i_2,\lambda''} a_{i_2,\lambda'''}\,, \qquad \text{(II.3.17)}$$

$$\sum_{\mathcal{C}_n(4)} \lambda_k \lambda'_k \lambda''_j \lambda'''_j a_{i_1,\lambda} a_{i_1,\lambda'} a_{i_2,\lambda''} a_{i_2,\lambda'''} = \sum_{\lambda} \lambda_k^2 |a_{i_1,\lambda}|^2 \sum_{\lambda} \lambda_j^2 |a_{i_2,\lambda}|^2$$
$$+2\Big(\sum_{\lambda} \lambda_k \lambda_j a_{i_1,\lambda}\overline{a_{i_2,\lambda}}\Big)^2 - 2\sum_{\lambda} \lambda_k^2 \lambda_j^2 |a_{i_1,\lambda}|^2 |a_{i_2,\lambda}|^2 - \sum_{\lambda} \lambda_k^2 \lambda_j^2 a_{i_1,\lambda}^2 \overline{a_{i_2,\lambda}}^2$$
$$+ \sum_{\mathcal{X}_n(4)} \lambda_k \lambda'_k \lambda''_j \lambda'''_j a_{i_1,\lambda} a_{i_1,\lambda'} a_{i_2,\lambda''} a_{i_2,\lambda'''}\,, \qquad \text{(II.3.18)}$$

$$\sum_{\mathcal{C}_n(4)} \lambda_k \lambda'_j \lambda''_k \lambda'''_j a_{i_1,\lambda} a_{i_1,\lambda'} a_{i_2,\lambda''} a_{i_2,\lambda'''} = \sum_{\lambda} \lambda_k \lambda_j |a_{i_1,\lambda}|^2 \sum_{\lambda} \lambda_k \lambda_j |a_{i_2,\lambda}|^2$$
$$+\sum_{\lambda} \lambda_k^2 a_{i_1,\lambda}\overline{a_{i_2,\lambda}} \sum_{\lambda} \lambda_j^2 a_{i_1,\lambda}\overline{a_{i_2,\lambda}} + \Big(\sum_{\lambda} \lambda_k \lambda_j a_{i_1,\lambda}\overline{a_{i_2,\lambda}}\Big)^2 - 2\sum_{\lambda} \lambda_k^2 \lambda_j^2 |a_{i_1,\lambda}|^2 |a_{i_2,\lambda}|^2$$
$$-\sum_{\lambda} \lambda_k^2 \lambda_j^2 a_{i_1,\lambda}^2 \overline{a_{i_2,\lambda}}^2 + \sum_{\mathcal{X}_n(4)} \lambda_k \lambda'_j \lambda''_k \lambda'''_j a_{i_1,\lambda} a_{i_1,\lambda'} a_{i_2,\lambda''} a_{i_2,\lambda'''}\,. \qquad \text{(II.3.19)}$$

The next two lemmas express the random variables introduced in Definition II.3.10 in terms of the quantities defined in Definition II.3.9. The following expansions have been proved in Lemma 4.4 of [Cam19].

Lemma II.3.12. *Fix $\ell \in [3]$. For every $i_1 \in [\ell]$, we have*

(i) $a_1^{(i_1)}(n) = \frac{3}{N_n}(W^{(i_1)}(n)^2 - R^{(i_1,i_1)}(n) + \frac{1}{3}X^{(i_1,i_1)}(n))$

(ii) $a_2^{(i_1)}(n) = \frac{3}{N_n}(W^{(i_1)}(n)^2 - R^{(i_1,i_1)}(n) - \sum_{k=1}^3 X_{kk}^{(i_1,i_1)}(n))$

(iii) $a_3^{(i_1)}(n) = \frac{27}{N_n}\sum_{k=1}^3 (W_{kk}^{(i_1)}(n)^2 - R_{kk}^{(i_1,i_1)}(n) + \frac{1}{3}X_{kkkk}^{(i_1,i_1)}(n))$

(iv) $a_4^{(i_1)}(n) = \frac{9}{N_n}\sum_{k<j} (W_{kk}^{(i_1)}(n) W_{jj}^{(i_1)}(n) + 2W_{kj}^{(i_1)}(n)^2 - 3R_{kj}^{(i_1,i_1)}(n) + X_{kkjj}^{(i_1,i_1)}(n))$

The next lemma deals with mixed expressions containing indices $i_1 < i_2$.

Lemma II.3.13. *Fix $\ell \in \{2, 3\}$. For every $i_1 < i_2 \in [\ell]$, we have*

(i) $b_1^{(i_1,i_2)}(n) = \frac{1}{N_n}(W^{(i_1)}(n)W^{(i_2)}(n) + 2M^{(i_1,i_2)}(n)^2 - 2R^{(i_1,i_2)}(n) - S^{(i_1,i_2)}(n) + X^{(i_1,i_2)}(n))$

(ii) $b_2^{(i_1,i_2)}(n) = b_2'^{(i_2,i_1)}(n) = \frac{3}{N_n}(W^{(i_1)}(n)W^{(i_2)}(n) + 2\sum_{k=1}^3 M_k^{(i_1,i_2)}(n)^2 - 2R^{(i_1,i_2)}(n) + S^{(i_1,i_2)}(n) - \sum_{k=1}^3 X_{kk}^{(i_1,i_2)}(n))$

(iii) $b_3^{(i_1,i_2)}(n) = \frac{9}{\mathcal{N}_n}\sum_{k\neq j=1}^3 \left(W_{kk}^{(i_1)}(n)W_{jj}^{(i_2)}(n) + 2M_{kj}^{(i_1,i_2)}(n)^2 - 2R_{kj}^{(i_1,i_2)}(n) - S_{kj}^{(i_1,i_2)}(n) + X_{kkjj}^{(i_1,i_2)}(n)\right)$

(iv) $b_4^{(i_1,i_2)}(n) = \frac{9}{\mathcal{N}_n}\sum_{k=1}^3 \left(W_{kk}^{(i_1)}(n)W_{kk}^{(i_2)}(n) + 2M_{kk}^{(i_1,i_2)}(n)^2 - 2R_{kk}^{(i_1,i_2)}(n) - S_{kk}^{(i_1,i_2)}(n) + X_{kkkk}^{(i_1,i_2)}(n)\right)$

(v) $b_5^{(i_1,i_2)}(n) = \frac{9}{\mathcal{N}_n}\sum_{k<j} \left(W_{kj}^{(i_1)}(n)W_{kj}^{(i_2)}(n) + M_{kk}^{(i_1,i_2)}(n)M_{jj}^{(i_1,i_2)}(n) + M_{kj}^{(i_1,i_2)}(n)^2 - 2R_{kj}^{(i_1,i_2)}(n) - S_{kj}^{(i_1,i_2)}(n) + X_{kkjj}^{(i_1,i_2)}(n)\right)$

Proof. Let $\ell \in \{2,3\}$ be fixed. For (i), by (II.3.16), we have

$$\begin{aligned}
b_1^{(i_1,i_2)}(n) &= \int_{\mathbb{T}^3} H_2(T_n^{(i_1)}(x))H_2(T_n^{(i_2)}(x))\,dx \\
&= \int_{\mathbb{T}^3} (T_n^{(i_1)}(x)^2 T_n^{(i_2)}(x)^2 - T_n^{(i_1)}(x)^2 - T_n^{(i_2)}(x)^2 + 1)\,dx \\
&= \frac{1}{\mathcal{N}_n^2}\sum_{\mathcal{C}_n(4)} a_{i_1,\lambda}a_{i_1,\lambda'}a_{i_2,\lambda''}a_{i_2,\lambda'''} - \frac{1}{\mathcal{N}_n}\sum_\lambda |a_{i_1,\lambda}|^2 - \frac{1}{\mathcal{N}_n}\sum_\lambda |a_{i_2,\lambda}|^2 + 1 \\
&= \frac{1}{\mathcal{N}_n^2}\sum_\lambda |a_{i_1,\lambda}|^2\sum_\lambda |a_{i_2,\lambda}|^2 + \frac{2}{\mathcal{N}_n^2}\Big(\sum_\lambda a_{i_1,\lambda}\overline{a_{i_2,\lambda}}\Big)^2 - \frac{2}{\mathcal{N}_n^2}\sum_\lambda |a_{i_1,\lambda}|^2|a_{i_2,\lambda}|^2 \\
&\quad -\frac{1}{\mathcal{N}_n^2}\sum_\lambda a_{i_1,\lambda}^2\overline{a_{i_2,\lambda}}^2 + \frac{1}{\mathcal{N}_n^2}\sum_{\mathcal{X}_n(4)} a_{i_1,\lambda}a_{i_1,\lambda'}a_{i_2,\lambda''}a_{i_2,\lambda'''} \\
&\quad -\frac{1}{\mathcal{N}_n}\sum_\lambda |a_{i_1,\lambda}|^2 - \frac{1}{\mathcal{N}_n}\sum_\lambda |a_{i_2,\lambda}|^2 + 1\ .
\end{aligned}$$

Now using the relation

$$\begin{aligned}
&\frac{1}{\mathcal{N}_n^2}\sum_\lambda(|a_{i_1,\lambda}|^2 - 1)\sum_\lambda(|a_{i_2,\lambda}|^2 - 1) \\
= &\frac{1}{\mathcal{N}_n^2}\sum_\lambda |a_{i_1,\lambda}|^2\sum_\lambda |a_{i_2,\lambda'}|^2 - \frac{1}{\mathcal{N}_n}\sum_\lambda |a_{i_1,\lambda}|^2 - \frac{1}{\mathcal{N}_n}\sum_\lambda |a_{i_2,\lambda}|^2 + 1\,, \qquad \text{(II.3.20)}
\end{aligned}$$

we can rewrite $b_1^{(i_1,i_2)}(n)$ as

$$\begin{aligned}
&\frac{1}{\mathcal{N}_n^2}\sum_\lambda(|a_{i_1,\lambda}|^2 - 1)\sum_\lambda(|a_{i_2,\lambda}|^2 - 1) + \frac{2}{\mathcal{N}_n^2}\Big(\sum_\lambda a_{i_1,\lambda}\overline{a_{i_2,\lambda}}\Big)^2 \\
&-\frac{2}{\mathcal{N}_n^2}\sum_\lambda |a_{i_1,\lambda}|^2|a_{i_2,\lambda}|^2 - \frac{1}{\mathcal{N}_n^2}\sum_\lambda a_{i_1,\lambda}^2\overline{a_{i_2,\lambda}}^2 + \frac{1}{\mathcal{N}_n^2}\sum_{\mathcal{X}_n(4)} a_{i_1,\lambda}a_{i_1,\lambda'}a_{i_2,\lambda''}a_{i_2,\lambda'''} \\
&= \frac{1}{\mathcal{N}_n}(W^{(i_1)}(n)W^{(i_2)}(n) + 2M^{(i_1,i_2)}(n)^2 - 2R^{(i_1,i_2)}(n) - S^{(i_1,i_2)}(n) + X^{(i_1,i_2)}(n))\ .
\end{aligned}$$

Let us now prove (ii). We start by computing $\int_{\mathbb{T}^3} H_2(T_n^{(i_1)}(x))H_2(T_{n,k}^{(i_2)}(x))dx$ for fixed $k \in [3]$. Bearing in mind that

$$T_{n,k}^{(i_2)}(x) = i\sqrt{\frac{3}{n\mathcal{N}_n}}\sum_\lambda \lambda_k a_{i_2,\lambda}e_\lambda(x)$$

and using (II.3.17), we have

$$\int_{\mathbb{T}^3} H_2(T_n^{(i_1)}(x))H_2(T_{n,k}^{(i_2)}(x))\,dx = \int_{\mathbb{T}^3} (T_n^{(i_1)}(x)^2 T_{n,k}^{(i_2)}(x)^2 - T_n^{(i_1)}(x)^2 - T_{n,k}^{(i_2)}(x)^2 + 1)\,dx$$

$$= \frac{3}{n\mathcal{N}_n^2}\sum_{\lambda}|a_{i_1,\lambda}|^2\sum_{\lambda}\lambda_k^2|a_{i_2,\lambda'}|^2 - \frac{6}{n\mathcal{N}_n^2}\Big(\sum_{\lambda}\lambda_k a_{i_1,\lambda}\overline{a_{i_2,\lambda}}\Big)^2$$
$$-\frac{6}{n\mathcal{N}_n^2}\sum_{\lambda}\lambda_k^2|a_{i_1,\lambda}|^2|a_{i_2,\lambda}|^2 + \frac{3}{n\mathcal{N}_n^2}\sum_{\lambda}\lambda_k^2 a_{i_1,\lambda}^2\overline{a_{i_2,\lambda}}^2$$
$$-\frac{3}{n\mathcal{N}_n^2}\sum_{\mathcal{X}_n(4)}\lambda_k''\lambda_k''' a_{i_1,\lambda}a_{i_1,\lambda'}a_{i_2,\lambda''}a_{i_2,\lambda'''} - \frac{1}{\mathcal{N}_n}\sum_{\lambda}|a_{i_1,\lambda}|^2 - \frac{3}{n\mathcal{N}_n}\sum_{\lambda}|a_{i_2,\lambda}|^2\lambda_k^2 + 1\,.$$

Hence, summing over k and using the fact that $\lambda_1^2+\lambda_2^2+\lambda_3^2 = n$ for $\lambda = (\lambda_1,\lambda_2,\lambda_3)\in\Lambda_n$ yields

$$b_2^{(i_1,i_2)}(n) = \frac{3}{\mathcal{N}_n^2}\sum_{\lambda}|a_{i_1,\lambda}|^2\sum_{\lambda}|a_{i_2,\lambda}|^2 - \frac{6}{n\mathcal{N}_n^2}\sum_{k=1}^{3}\Big(\sum_{\lambda}\lambda_k a_{i_1,\lambda}\overline{a_{i_2,\lambda}}\Big)^2 - \frac{6}{\mathcal{N}_n^2}\sum_{\lambda}|a_{i_1,\lambda}|^2|a_{i_2,\lambda}|^2$$
$$+\frac{3}{\mathcal{N}_n^2}\sum_{\lambda}a_{i_1,\lambda}^2\overline{a_{i_2,\lambda}}^2 - \frac{3}{n\mathcal{N}_n^2}\sum_{k=1}^{3}\sum_{\mathcal{X}_n(4)}\lambda_k''\lambda_k''' a_{i_1,\lambda}a_{i_1,\lambda'}a_{i_2,\lambda''}a_{i_2,\lambda'''}$$
$$-\frac{3}{\mathcal{N}_n}\sum_{\lambda}|a_{i_1,\lambda}|^2 - \frac{3}{\mathcal{N}_n}\sum_{\lambda}|a_{i_2,\lambda}|^2 + 3\,.$$

Note that we can rewrite the second term as

$$-\frac{6}{n\mathcal{N}_n^2}\sum_{k=1}^{3}\Big(\sum_{\lambda}\lambda_k a_{i_1,\lambda}\overline{a_{i_2,\lambda}}\Big)^2 = \frac{6}{\mathcal{N}_n}\sum_{k=1}^{3}\Big(\frac{i}{\sqrt{n\mathcal{N}_n}}\sum_{\lambda}\lambda_k a_{i_1,\lambda}\overline{a_{i_2,\lambda}}\Big)^2 = \frac{6}{\mathcal{N}_n}\sum_{k=1}^{3}M_k^{(i_1,i_2)}(n)^2\,.$$

Substituting (II.3.20) in the computation above shows that $b_2^{(i_1,i_2)}(n)$ is equal to

$$\frac{3}{\mathcal{N}_n}\left(W^{(i_1)}(n)W^{(i_2)}(n) + 2\sum_{k=1}^{3}M_k^{(i_1,i_2)}(n)^2 - 2R^{(i_1,i_2)}(n) + S^{(i_1,i_2)}(n) - \sum_{k=1}^{3}X_{kk}^{(i_1,i_2)}(n)\right),$$

which is the desired equality. Relations (iii)-(v) can be proved by similar arguments. □

Explicit expression of $\mathrm{proj}_4(L_n^{(\ell)})$. We are now in position to provide the precise expression of the fourth-order chaotic component of $L_n^{(\ell)}$. We introduce the following notation: We write $\mathbf{0}_\ell \in \mathcal{M}_{\ell\times 3}(\mathbb{R})$ for the zero-matrix; for an integer $m \geq 1$, we consider the mapping $s_m^{(\ell)} : ([\ell]\times[3])^m \to \mathcal{M}_{\ell\times 3}(\mathbb{R})$ defined by

$$s_m^{(\ell)}((i_1,j_1),\dots,(i_m,j_m)) := \{\mathbb{1}[(i,j)\in\{(i_1,j_1),\dots,(i_m,j_m)\}] : (i,j)\in[\ell]\times[3]\}\ ,$$

that is, $s_m^{(\ell)}((i_1,j_1),\dots,(i_m,j_m))$ is the $\ell\times 3$ matrix whose entry is 1 at positions $(i_1,j_1),\dots,(i_m,j_m)$ and 0 elsewhere. The following Proposition II.3.14 contains the values of all the projection coefficients $\alpha\left\{p_j^{(i)} : (i,j)\in[\ell]\times[3]\right\}$ appearing in the Wiener chaos expansion of $L_n^{(\ell)}$ in (II.3.5) and is a direct consequence of Proposition II.B.5 applied with $\mathbf{X} = \mathbf{T}_{n\star}^{(\ell)}(z), z\in\mathbb{T}^3$ in the three cases $(\ell,k)\in\{(1,3),(2,3),(3,3)\}$; The exact values are entirely determined once we compute (see (II.1.8))

$$\alpha(1,3) = \frac{4}{\sqrt{2\pi}}\,,\quad \alpha(2,3) = 2\,,\quad \alpha(3,3) = \frac{4}{\sqrt{2\pi}}\,.$$

Proposition II.3.14. *For every* $\ell \in [3]$ *and every collection of indices* $I = \{(i_1,j_1)\neq(i_2,j_2)\neq(i_3,j_3)\neq(i_4,j_4)\in[\ell]\times[3]\}$*, we have*

$$\alpha_3^{(\ell)}\{\mathbf{0}_\ell\} = \alpha(\ell,3)\,,$$

$$\alpha_3^{(\ell)}\left\{2s_1^{(\ell)}((i_1,j_1))\right\} = \frac{1}{2!}\frac{1}{3}\alpha(\ell,3) = \frac{1}{6}\alpha(\ell,3)\,,$$
$$\alpha_3^{(\ell)}\left\{4s_1^{(\ell)}((i_1,j_1))\right\} = -\frac{1}{4!}\frac{1}{5}\alpha(\ell,3) = -\frac{1}{120}\alpha(\ell,3)\,,$$
$$\begin{aligned}\alpha_3^{(\ell)}\left\{2s_2^{(\ell)}((i_1,j_1),(i_2,j_2))\right\} = &-\frac{1}{60}\alpha(\ell,3)\mathbb{1}_{i_1=i_2}\\ &-\frac{1}{60}\alpha(\ell,3)\mathbb{1}_{i_1\neq i_2, j_1=j_2}\mathbb{1}_{\ell\in\{2,3\}}\\ &+\frac{1}{20}\alpha(\ell,3)\mathbb{1}_{i_1\neq i_2, j_1\neq j_2}\mathbb{1}_{\ell\in\{2,3\}}\,,\end{aligned}$$
$$\alpha_3^{(\ell)}\left\{s_4^{(\ell)}((i_1,j_1),(i_2,j_2),(i_3,j_3),(i_4,j_4))\right\} = -\frac{2}{15}\alpha(\ell,k)\mathbb{1}_{I\in S}\mathbb{1}_{\ell\in\{2,3\}}\,,$$

where $S = \{\{(i_1,j_1),(i_1,j_2),(i_2,j_1),(i_2,j_2)\} : i_1\neq i_2, j_1\neq j_2\}$.

In particular, from Proposition II.B.5, it becomes clear that the fourth-order chaotic component of $L_n^{(\ell)}$ does not involve (i) any non-linear interaction of the three ARWs simultaneously (for $\ell = 3$), and (ii) any product of odd Hermite polynomials except expressions of the form $H_1(\cdot)H_1(\cdot)H_1(\cdot)H_1(\cdot)$.

Recalling the random variables introduced in Definition II.3.10, we define the following two quantities: for $\ell\in[3]$ and $i_1\in[\ell]$,

$$\begin{aligned}A_{n,\ell}^{(i_1)} \;:=\; &\frac{\beta_4\beta_0^{\ell-1}}{4!}\alpha_3^{(\ell)}\{\mathbf{0}_\ell\}\cdot a_1^{(i_1)}(n) + \frac{\beta_0^{\ell-1}\beta_2}{2!}\alpha_3^{(\ell)}\left\{2s_1^{(\ell)}((1,1))\right\}\cdot a_2^{(i_1)}(n)\\ +\;&\beta_0^{\ell}\alpha_3^{(\ell)}\left\{4s_1^{(\ell)}((1,1))\right\}\cdot a_3^{(i_1)}(n)\\ +\;&\beta_0^{\ell}\alpha_3^{(\ell)}\left\{2s_2^{(\ell)}((1,1),(1,2))\right\}\cdot a_4^{(i_1)}(n)\,;\end{aligned}\tag{II.3.21}$$

and for $\ell\in\{2,3\}$ and $i_1<i_2\in[\ell]$,

$$\begin{aligned}B_{n,\ell}^{(i_1,i_2)} \;:=\; &\left(\frac{\beta_2}{2!}\right)^2\beta_0^{\ell-2}\alpha_3^{(\ell)}\{\mathbf{0}_\ell\}\cdot b_1^{(i_1,i_2)}(n) + \frac{\beta_2\beta_0^{\ell-1}}{2!}\alpha_3^{(\ell)}\left\{2s_1^{(\ell)}((1,1))\right\}\cdot b_2^{(i_1,i_2)}(n)\\ +\;&\frac{\beta_0^{\ell-1}\beta_2}{2!}\alpha_3^{(\ell)}\left\{2s_1^{(\ell)}((1,1))\right\}\cdot b_2'^{(i_1,i_2)}(n)\\ +\;&\beta_0^{\ell}\alpha_3^{(\ell)}\left\{2s_2^{(\ell)}((1,1),(2,2))\right\}\cdot b_3^{(i_1,i_2)}(n)\\ +\;&\beta_0^{\ell}\alpha_3^{(\ell)}\left\{2s_2^{(\ell)}((1,1),(2,1))\right\}\cdot b_4^{(i_1,i_2)}(n)\\ +\;&\beta_0^{\ell}\alpha_3^{(\ell)}\left\{s_4^{(\ell)}((1,1),(1,2),(2,1),(2,2))\right\}\cdot b_5^{(i_1,i_2)}(n)\,.\end{aligned}\tag{II.3.22}$$

Then, the fourth-order chaotic component of $L_n^{(\ell)}$ is given by (recall (II.3.3))

$$\mathrm{proj}_4(L_n^{(\ell)}) = \left(\frac{E_n}{3}\right)^{\ell/2}\left(\sum_{i_1\in[\ell]}A_{n,\ell}^{(i_1)} + \sum_{i_1<i_2\in[\ell]}B_{n,\ell}^{(i_1,i_2)}\right) =: \left(\frac{E_n}{3}\right)^{\ell/2}\cdot S_n^{(\ell)}\,,\tag{II.3.23}$$

with the convention that $\sum_{i_1<i_2\in[\ell]} = 0$ if $\ell = 1$. Using (II.A.4) and Proposition II.3.14, the expressions in (IV.2.9) and (IV.2.10) simplify to

$$A_{n,\ell}^{(i_1)} = \frac{2}{(2\pi)^{\ell/2}}\alpha(\ell,3)\left(\frac{1}{16}a_1^{(i_1)}(n) - \frac{1}{24}a_2^{(i_1)}(n) - \frac{1}{240}a_3^{(i_1)}(n) - \frac{1}{120}a_4^{(i_1)}(n)\right)$$

and

$$B_{n,\ell}^{(i_1,i_2)} = \frac{2}{(2\pi)^{\ell/2}}\alpha(\ell,3)\Big(\frac{1}{8}b_1^{(i_1,i_2)}(n) - \frac{1}{24}b_2^{(i_1,i_2)}(n) - \frac{1}{24}b_2'^{(i_1,i_2)}(n) + \frac{1}{40}b_3^{(i_1,i_2)}(n)$$

$$-\frac{1}{120}b_4^{(i_1,i_2)}(n)-\frac{1}{15}b_5^{(i_1,i_2)}(n)\Big)\,.$$

Using the expansions in Lemma II.3.12 and the fact that $W^{(i_1)}(n)=\sum_{k=1}^3 W_{kk}^{(i_1)}(n)$, we compute

$$A_{n,\ell}^{(i_1)}=\frac{2}{(2\pi)^{\ell/2}}\frac{\alpha(\ell,3)}{\mathcal{N}_n}\Big(-\frac{1}{40}\sum_{k<j}(W_{kk}^{(i_1)}(n)-W_{jj}^{(i_1)})^2-\frac{3}{20}\sum_{k<j}W_{kj}^{(i_1)}(n)^2+\mu^{(i_1)}(n)\Big) \tag{II.3.24}$$

where $\mu^{(i_1)}(n)$ is given by

$$\mu^{(i_1)}(n)=\frac{1}{20}R^{(i_1,i_1)}(n)+\frac{1}{16}X^{(i_1,i_1)}(n)+\frac{1}{8}\sum_{k=1}^{3}X_{kk}^{(i_1,i_1)}(n)-\frac{3}{80}\sum_{k,j=1}^{3}X_{kkjj}^{(i_1,i_1)}(n)\,. \tag{II.3.25}$$

Similarly, if $\ell\in\{2,3\}$, using Lemma II.3.13 together with the fact that $M^{(i_1,i_2)}(n)=\sum_{k=1}^3 M_{kk}^{(i_1,i_2)}(n)$, yields

$$\begin{aligned}B_{n,\ell}^{(i_1,i_2)} \;=\;& \frac{2}{(2\pi)^{\ell/2}}\frac{\alpha(\ell,3)}{\mathcal{N}_n}\Big(-\frac{1}{10}\sum_{k<j}(W_{kk}^{(i_1)}(n)-W_{jj}^{(i_1)}(n))(W_{kk}^{(i_2)}(n)-W_{jj}^{(i_2)}(n))\\ &-\frac{3}{5}\sum_{k<j}W_{kj}^{(i_1)}(n)W_{kj}^{(i_2)}(n)+\frac{1}{10}\sum_{k=1}^{3}M_{kk}^{(i_1,i_2)}(n)^2-\frac{1}{20}\sum_{k\neq j}M_{kk}^{(i_1,i_2)}(n)M_{jj}^{(i_1,i_2)}(n)\\ &-\frac{1}{2}\sum_{k=1}^{3}M_{k}^{(i_1,i_2)}(n)^2+\frac{3}{10}\sum_{k<j}M_{kj}^{(i_1,i_2)}(n)^2+\eta^{(i_1,i_2)}(n)\Big)\end{aligned} \tag{II.3.26}$$

where $\eta^{(i_1,i_2)}(n)$ is given by

$$\begin{aligned}\eta^{(i_1,i_2)}(n) \;=\;& \frac{2}{5}R^{(i_1,i_2)}(n)-\frac{3}{10}S^{(i_1,i_2)}(n)+\frac{1}{8}X^{(i_1,i_2)}(n)\\ &+\frac{1}{4}\sum_{k=1}^{3}X_{kk}^{(i_1,i_2)}(n)-\frac{3}{40}\sum_{k,j=1}^{3}X_{kkjj}^{(i_1,i_2)}(n).\end{aligned} \tag{II.3.27}$$

II.3.2.2 Asymptotic simplification of $\mathrm{proj}_4(L_n^{(\ell)})$

We will now lead an asymptotic study of the fourth chaotic component of $\mathrm{proj}_4(L_n^{(\ell)})$ obtained in (II.3.23). This analysis is based on a multivariate Central Limit Theorem for the summands composing the expressions of $A_{n,\ell}^{(i_1)}$ and $B_{n,\ell}^{(i_1,i_2)}$.

We start by recalling the following formulae (see Lemma 3.3 and Appendix C in [Cam19]), which are a consequence of the asymptotic equidistribution of lattice points projected to the unit two-sphere.

Lemma II.3.15. *For every $j,k,l,m\in[3]$, we have*

$$\frac{1}{n\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}\lambda_k\lambda_j=\frac{1}{3}\mathbb{1}_{k=j}\,, \tag{II.3.28}$$

$$\frac{1}{n^2\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}\lambda_k\lambda_l\lambda_j\lambda_m=\frac{1}{5}\mathbb{1}_{k=l=j=m}+\frac{1}{15}(\mathbb{1}_{k=l,j=m,k\neq j}+\mathbb{1}_{k=j,l=m,k\neq l}+\mathbb{1}_{k=m,l=j,k\neq l})+\varepsilon_n\,, \tag{II.3.29}$$

where $\varepsilon_n=O(n^{-1/28+o(1)})$, as $n\to\infty, n\not\equiv 0,4,7 \pmod 8$.

For the random variables in Definition II.3.9, we prove the following asymptotic relations.

Lemma II.3.16. *Fix $\ell \in [3]$. For every $i_1, i_2 \in [\ell]$, the following holds as $n \to \infty, n \not\equiv 0,4,7 \pmod 8$:*

$$R^{(i_1,i_2)}(n) \xrightarrow{\mathbb{P}} 2\mathbb{1}_{i_1=i_2} + \mathbb{1}_{i_1\neq i_2}\,, \tag{II.3.30}$$

$$S^{(i_1,i_2)}(n) \xrightarrow{\mathbb{P}} 2\mathbb{1}_{i_1=i_2}\,, \tag{II.3.31}$$

$$X^{(i_1,i_2)}(n), X^{(i_1,i_2)}_{kk}(n), X^{(i_1,i_2)}_{kkjj}(n) \xrightarrow{L^2(\mathbb{P})} 0\,. \tag{II.3.32}$$

Proof. We introduce the equivalence relation $\sim$ on Λ_n defined by $\lambda \sim \lambda'$ if and only if $\lambda = -\lambda'$ and write $\Lambda_n/_\sim$ for the set of representatives of the equivalence classes under $\sim$. Then, it follows that $\mathrm{card}(\Lambda_n/_\sim) = \mathcal{N}_n/2$ and the collections $\left\{|a_{i_1,\lambda}|^2|a_{i_2,\lambda}|^2 : \lambda \in \Lambda_n/_\sim\right\}$ resp. $\left\{a^2_{i_1,\lambda}\overline{a_{i_2,\lambda}}^2 : \lambda \in \Lambda_n/_\sim\right\}$ are families of i.i.d. random variables with respective means

$$\mathbb{E}\left[|a_{i_1,\lambda}|^2|a_{i_2,\lambda}|^2\right] = 2\mathbb{1}_{i_1=i_2} + \mathbb{1}_{i_1\neq i_2}\,, \quad \mathbb{E}\left[a^2_{i_1,\lambda}\overline{a_{i_2,\lambda}}^2\right] = 2\mathbb{1}_{i_1=i_2}\,.$$

Thus, relations (II.3.30) and (II.3.31) follow from the Law of Large Numbers: as $n \to \infty, n \not\equiv 0,4,7 \pmod 8$, we have

$$R^{(i_1,i_2)}(n) = \frac{1}{\mathcal{N}_n/2}\sum_{\lambda\in\Lambda_n/_\sim}|a_{i_1,\lambda}|^2|a_{i_2,\lambda}|^2 \xrightarrow{\mathbb{P}} 2\mathbb{1}_{i_1=i_2} + \mathbb{1}_{i_1\neq i_2}\,,$$

and

$$S^{(i_1,i_2)}(n) = \frac{1}{\mathcal{N}_n/2}\sum_{\lambda\in\Lambda_n/_\sim}a^2_{i_1,\lambda}\overline{a_{i_2,\lambda}}^2 \xrightarrow{\mathbb{P}} 2\mathbb{1}_{i_1=i_2}\,.$$

The convergences in (II.3.32) have been proved in [Cam19] in the case $i_1 = i_2$. Using independence and the fact that $a_{i_1,\lambda} = \overline{a_{i_1,-\lambda}}$ for every $i_1 \in [\ell]$ and $\lambda \in \Lambda_n$ yields

$$\begin{aligned}
&\mathbb{E}\left[|X^{(i_1,i_2)}(n)|^2\right] = \mathbb{E}\left[X^{(i_1,i_2)}(n)\overline{X^{(i_1,i_2)}(n)}\right]\\
=\;& \frac{1}{\mathcal{N}_n^2}\sum_{\mathcal{X}_n(4)}\sum_{\mathcal{X}_n(4)}\mathbb{E}\left[a_{i_1,\lambda}a_{i_1,\lambda'}a_{i_1,-\mu}a_{i_1,-\mu'}\right]\mathbb{E}\left[a_{i_2,\lambda''}a_{i_2,\lambda'''}a_{i_2,-\mu''}a_{i_2,-\mu'''}\right]\\
=:\;& \frac{1}{\mathcal{N}_n^2}\sum_{\mathcal{X}_n(4)}\sum_{\mathcal{X}_n(4)}\mathbb{E}\left[z^{(i_1)}_{\lambda,\lambda',\mu,\mu'}\right]\mathbb{E}\left[z^{(i_2)}_{\lambda'',\lambda''',\mu'',\mu'''}\right]\,.
\end{aligned}$$

Let us consider the random variable $z^{(i_1)}_{\lambda,\lambda',-\mu,-\mu'}$. Denote by N the number of pairs of vectors that are equal in absolute value among $\{\lambda, \lambda', \mu, \mu'\}$. Since we consider vectors of $\mathcal{X}_n(4)$, we have that $\lambda + \lambda' \neq 0$ and $\mu + \mu' \neq 0$. Conditional to this observation, we claim that the only non-zero contributions of $\mathbb{E}\left[z^{(i_1)}_{\lambda,\lambda',-\mu,-\mu'}\right]$ arise when $N = 2$ or $N = 4$. Indeed, if $N = 0$, all the vectors are distinct, so that by independence, $\mathbb{E}\left[z^{(i_1)}_{\lambda,\lambda',-\mu,-\mu'}\right] = 0$. If $N = 1$, then $\mathbb{E}\left[z^{(i_1)}_{\lambda,\lambda',-\mu,-\mu'}\right]$ takes one of the forms

$$\mathbb{E}\left[|a_{i_1,s}|^2\right]\mathbb{E}\left[a_{i_1,t}\right]\mathbb{E}\left[a_{i_1,t'}\right] = 0\,, \quad \mathbb{E}\left[a^2_{i_1,s}\right]\mathbb{E}\left[a_{i_1,t}\right]\mathbb{E}\left[a_{i_1,t'}\right] = 0\,, \quad s \neq \pm t \neq \pm t'\,.$$

If $N = 2$, $\mathbb{E}\left[z^{(i_1)}_{\lambda,\lambda',-\mu,-\mu'}\right]$ is of the form

$$\mathbb{E}\left[|a_{i_1,s}|^2\right]\mathbb{E}\left[a^2_{i_1,t}\right] = 0\,, \quad \mathbb{E}\left[|a_{i_1,s}|^2\right]\mathbb{E}\left[|a_{i_1,t}|^2\right] = 1\,, \quad \mathbb{E}\left[a^2_{i_1,s}\right]\mathbb{E}\left[a^2_{i_1,t}\right] = 0\,, \quad s \neq \pm t\,.$$

If $N = 3$, then $\mathbb{E}\left[z^{(i_1)}_{\lambda,\lambda',-\mu,-\mu'}\right]$ is of the form

$$\mathbb{E}\left[a^3_{i_1,s}\right]\mathbb{E}\left[a_{i_1,t}\right] = 0\,, \quad \mathbb{E}\left[|a_{i_1,s}|^2\overline{a_{i_1,s}}\right]\mathbb{E}\left[a_{i_1,t}\right] = 0\,, \quad s \neq \pm t\,.$$

Finally, if $N = 4$, the elements $\lambda, \lambda', \mu, \mu'$ are all the same in absolute value, so that $\mathbb{E}\left[z^{(i_1)}_{\lambda,\lambda',-\mu,-\mu'}\right]$ is of the form $\mathbb{E}\left[|a_{i_1,s}|^4\right] = 2$ or $\mathbb{E}\left[a^4_{i_1,s}\right] = 0$. The same arguments hold for $\mathbb{E}\left[z^{(i_2)}_{\lambda'',\lambda''',\mu'',\mu'''}\right]$. Therefore, in every non-zero contributions, the vector $(\lambda, \lambda', \lambda'', \lambda''')$ determines the choices of $(\mu, \mu', \mu'', \mu''')$, so that

$$\mathbb{E}\left[|X^{(i_1,i_2)}(n)|^2\right] \ll \frac{\operatorname{card}(\mathcal{X}_n(4))}{\mathcal{N}_n^2} \ll \frac{\mathcal{N}_n^{7/4+o(1)}}{\mathcal{N}_n^2} = o(1)\,,$$

as $n \to \infty$, $n \not\equiv 0, 4, 7 \pmod 8$ in view of (II.3.7). □

<u>*A multivariate Central Limit Theorem.*</u> Recalling the random variables defined in Definition II.3.9, we define the following two random vectors for $n \in S_3$: for every $\ell \in [3]$ and $i_1 \in [\ell]$,

$$\mathbb{W}^{(i_1)}(n) := \left(W^{(i_1)}_{11}(n), W^{(i_1)}_{12}(n), W^{(i_1)}_{13}(n), W^{(i_1)}_{22}(n), W^{(i_1)}_{23}(n), W^{(i_1)}_{33}(n)\right) \in \mathbb{R}^6\,,$$

and, for every $\ell \in \{2, 3\}$ and $i_1 < i_2 \in [\ell]$,

$$\begin{aligned}\mathbb{M}^{(i_1,i_2)}(n) := (&M^{(i_1,i_2)}_1(n), M^{(i_1,i_2)}_2(n), M^{(i_1,i_2)}_3(n), M^{(i_1,i_2)}_{11}(n), M^{(i_1,i_2)}_{12}(n), M^{(i_1,i_2)}_{13}(n),\\ &M^{(i_1,i_2)}_{22}(n), M^{(i_1,i_2)}_{23}(n), M^{(i_1,i_2)}_{33}(n)) \in \mathbb{R}^9\,.\end{aligned}$$

The covariance matrix of the vectors $\mathbb{W}^{(i_1)}(n)$ and $\mathbb{M}^{(i_1,i_2)}(n)$ above is computed in the following lemmas.

Lemma II.3.17. *For every $n \in S_3$, $\ell \in [3]$ and every $i_1 \in [\ell]$, the covariance matrix of $\mathbb{W}^{(i_1)}(n)$ is*

$$\Sigma_{\mathbb{W}(n)} = \begin{pmatrix} \frac{2}{5}+\varepsilon_n & 0 & 0 & \frac{2}{15}+\varepsilon_n & 0 & \frac{2}{15}+\varepsilon_n \\ 0 & \frac{2}{15}+\varepsilon_n & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{15}+\varepsilon_n & 0 & 0 & 0 \\ \frac{2}{15}+\varepsilon_n & 0 & 0 & \frac{2}{5}+\varepsilon_n & 0 & \frac{2}{15}+\varepsilon_n \\ 0 & 0 & 0 & 0 & \frac{2}{15}+\varepsilon_n & 0 \\ \frac{2}{15}+\varepsilon_n & 0 & 0 & \frac{2}{15}+\varepsilon_n & 0 & \frac{2}{5}+\varepsilon_n \end{pmatrix}, \qquad \text{(II.3.33)}$$

where $\varepsilon_n = O(n^{-1/28+o(1)})$, as $n \to \infty$, $n \not\equiv 0, 4, 7 \pmod 8$.

Proof. The proof mainly follows from the relations in Lemma II.3.15, together with the fact

$$\mathbb{E}\left[(|a_{i_1,\lambda}|^2 - 1)(|a_{i_1,\lambda'}|^2 - 1)\right] = \mathbb{1}_{\lambda=\pm\lambda'}\,.$$

The covariances of $W^{(i_1)}_{jk}$ for $j, k \in [3]$ have been computed in [Cam19], Appendix II.C. □

Lemma II.3.18. *For every $n \in S_3$, $\ell \in \{2, 3\}$ and every $i_1 < i_2 \in [\ell]$, the covariance matrix of $\mathbb{M}^{(i_1,i_2)}(n)$ is*

$$\Sigma_{\mathbb{M}(n)} = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{5}+\varepsilon_n & 0 & 0 & \frac{1}{15}+\varepsilon_n & 0 & \frac{1}{15}+\varepsilon_n \\ 0 & 0 & 0 & 0 & \frac{1}{15}+\varepsilon_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{15}+\varepsilon_n & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{15}+\varepsilon_n & 0 & 0 & \frac{1}{5}+\varepsilon_n & 0 & \frac{1}{15}+\varepsilon_n \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{15}+\varepsilon_n & 0 \\ 0 & 0 & 0 & \frac{1}{15}+\varepsilon_n & 0 & 0 & \frac{1}{15}+\varepsilon_n & 0 & \frac{1}{5}+\varepsilon_n \end{pmatrix}, \qquad \text{(II.3.34)}$$

where $\varepsilon_n = O(n^{-1/28+o(1)})$, as $n \to \infty$, $n \not\equiv 0, 4, 7 \pmod 8$.

Proof. Similarly as in the proof of Lemma II.3.17, we use Lemma II.3.15 and the fact that, by independence

$$\mathbb{E}\left[a_{i_1,\lambda}\overline{a_{i_2,\lambda}}a_{i_1,\lambda'}\overline{a_{i_2,\lambda'}}\right] = \mathbb{E}\left[a_{i_1,\lambda}a_{i_1,\lambda'}\right]\mathbb{E}\left[\overline{a_{i_2,\lambda}}\;\overline{a_{i_2,\lambda'}}\right] = \mathbb{1}_{\lambda=-\lambda'}\ .$$

Using this identity, it follows that

$$\mathbf{Cov}\Big[M_j^{(i_1,i_2)}(n), M_k^{(i_1,i_2)}(n)\Big] = \mathbb{E}\left[M_j^{(i_1,i_2)}(n)M_k^{(i_1,i_2)}(n)\right] = \frac{1}{n\mathcal{N}_n}\sum_{\lambda}\lambda_j\lambda_k = \frac{1}{3}\mathbb{1}_{j=k}\ ,$$

and

$$\mathbf{Cov}\Big[M_j^{(i_1,i_2)}(n), M_{lm}^{(i_1,i_2)}(n)\Big] = \mathbb{E}\left[M_j^{(i_1,i_2)}(n)M_{lm}^{(i_1,i_2)}(n)\right] = \frac{i}{n\sqrt{n}\mathcal{N}_n}\sum_{\lambda}\lambda_j\lambda_l\lambda_m = 0\ .$$

Moreover,

$$\mathbf{Cov}\Big[M_{jk}^{(i_1,i_2)}(n), M_{lm}^{(i_1,i_2)}(n)\Big] = \mathbb{E}\left[M_{jk}^{(i_1,i_2)}(n)M_{lm}^{(i_1,i_2)}(n)\right] = \frac{1}{n^2\mathcal{N}_n}\sum_{\lambda}\lambda_j\lambda_k\lambda_l\lambda_m$$
$$= \frac{1}{5}\mathbb{1}_{k=l=j=m} + \frac{1}{15}\left(\mathbb{1}_{k=l,j=m,k\neq j} + \mathbb{1}_{k=j,l=m,k\neq l} + \mathbb{1}_{k=m,l=j,k\neq l}\right) + \varepsilon_n\ ,$$

which finishes the proof. □

The following proposition plays a central role in the study of the fourth chaotic component of the nodal volume $L_n^{(\ell)}$ in the high-frequency regime. We define the limiting matrices obtained from (II.3.33) and (II.3.34):

$$\Sigma_{\mathbb{W}} := \lim_{n\to\infty}\Sigma_{\mathbb{W}(n)}\ ,\quad \Sigma_{\mathbb{M}} := \lim_{n\to\infty}\Sigma_{\mathbb{M}(n)}\ ,$$

where for a square matrix $M_n = (m_{ij}(n))$, we set $\lim_n M_n := (\lim_n m_{ij}(n))$.

Proposition II.3.19. *As $n\to\infty$, $n \not\equiv 0,4,7 \pmod 8$, the random vector*

$$\mathbf{V}_{1,2,3}(n) := \left(\mathbb{W}^{(1)}(n), \mathbb{W}^{(2)}(n), \mathbb{W}^{(3)}(n), \mathbb{M}^{(1,2)}(n), \mathbb{M}^{(1,3)}(n), \mathbb{M}^{(2,3)}(n)\right) \in \mathbb{R}^{45}$$

converges in distribution to

$$\mathbf{G}_{1,2,3} := \left(\mathbb{G}^{(1)}, \mathbb{G}^{(2)}, \mathbb{G}^{(3)}, \mathbb{G}^{(1,2)}, \mathbb{G}^{(1,3)}, \mathbb{G}^{(2,3)}\right) \sim \mathcal{N}_{45}(0, \Sigma_{\mathbf{G}_{1,2,3}})\ ,$$

where

$$\Sigma_{\mathbf{G}_{1,2,3}} = \Sigma_{\mathbb{W}}\oplus\Sigma_{\mathbb{W}}\oplus\Sigma_{\mathbb{W}}\oplus\Sigma_{\mathbb{M}}\oplus\Sigma_{\mathbb{M}}\oplus\Sigma_{\mathbb{M}} \in \mathcal{M}_{45\times45}(\mathbb{R})\ .$$

Proof. We start by showing that the covariance matrix of the vector $\mathbf{V}_{1,2,3}(n)$ has the block diagonal form

$$\Sigma_{\mathbf{V}_{1,2,3}(n)} = \Sigma_{\mathbb{W}(n)}\oplus\Sigma_{\mathbb{W}(n)}\oplus\Sigma_{\mathbb{W}(n)}\oplus\Sigma_{\mathbb{M}(n)}\oplus\Sigma_{\mathbb{M}(n)}\oplus\Sigma_{\mathbb{M}(n)}\ .$$

From Lemmas II.3.17 and II.3.18 and by independence, we have

$$\mathbb{E}\left[(|a_{i_1,\lambda}|^2-1)a_{i_1,\lambda'}\overline{a_{i_2,\lambda'}}\right] = \mathbb{E}\left[(|a_{i_1,\lambda}|^2-1)a_{i_1,\lambda'}\right]\mathbb{E}\left[\overline{a_{i_2,\lambda'}}\right] = 0\,,$$

and therefore $\mathbf{Cov}\Big[((\mathbb{W}^{(i_1)}(n))_l, (\mathbb{M}^{(i_1,i_2)}(n))_m\Big] = 0$ for every $l = 1,\dots,6$ and $m = 1,\dots,9$. Similarly, since for every $i_2 \neq i_3$,

$$\mathbb{E}\left[a_{i_1,\lambda}\overline{a_{i_2,\lambda}}a_{i_1,\lambda'}\overline{a_{i_3,\lambda'}}\right] = \mathbb{E}\left[a_{i_1,\lambda}a_{i_1,\lambda'}\right]\mathbb{E}\left[\overline{a_{i_2,\lambda}}\right]\mathbb{E}\left[\overline{a_{i_3,\lambda'}}\right] = 0\,,$$

we have that $\mathbf{Cov}\Big[((\mathbb{M}^{(i_1,i_2)}(n))_l, (\mathbb{M}^{(i_1,i_3)}(n))_m\Big] = 0$ for every $l,m = 1,\dots,9$. Thus, $\mathbf{V}_{1,2,3}(n)$ is of the desired form. Furthermore, we notice that all the components $\left\{(\mathbf{V}_{1,2,3}(n))_l : l = 1,\dots,45\right\}$ of $\mathbf{V}_{1,2,3}(n)$

belong to the second Wiener chaos and that $\Sigma_{\mathbf{V}_{1,2,3}(n)} \to \Sigma_{\mathbf{G}_{1,2,3}}$ entry-wise as $n \to \infty, n \not\equiv 0,4,7 \pmod 8$. Thus, Theorem 6.2.3 of [NP12a] implies that, in order to prove the joint convergence to the Gaussian vector $\mathbf{G}_{1,2,3}$, it suffices to prove that the convergence holds component-wise, that is

$$(\mathbf{V}_{1,2,3}(n))_l \xrightarrow{d} \mathcal{N}(0, (\Sigma_{\mathbf{G}_{1,2,3}})_{ll}), \qquad n \to \infty, n \not\equiv 0,4,7 \pmod 8,$$

for every $l = 1, \dots, 45$. Using the Fourth Moment Theorem (Theorem 5.2.7, [NP12a]), this can be shown by proving that the fourth cumulant of $(\mathbf{V}_{1,2,3}(n))_l$ converges to zero for every $l = 1, \dots, 45$. For the sake of completeness, we include the computations for $W_{jk}^{(i_1)}(n)$ with $j \neq k$ and $M^{(i_1,i_2)}(n)$: writing $\Lambda_n/_\sim$ for the set of all the representatives of the equivalence class of Λ_n under the symmetry $\lambda \mapsto -\lambda$ and using the fact that $j \neq k$, we have

$$W_{jk}^{(i_1)}(n) = \frac{1}{n\sqrt{\mathcal{N}_n}} \sum_{\lambda \in \Lambda_n} \lambda_j \lambda_k (|a_{i_1,\lambda}|^2 - 1) = \frac{2}{n\sqrt{\mathcal{N}_n}} \sum_{\lambda \in \Lambda_n/\sim} \lambda_j \lambda_k |a_{i_1,\lambda}|^2,$$

that is, $W_{jk}^{(i_1)}(n)$ is a sum of i.i.d. random variables. Moreover, for $\lambda \in \Lambda_n/_\sim$,

$$|a_{i_1,\lambda}|^2 \stackrel{d}{=} \frac{u_\lambda^2}{2} + \frac{v_\lambda^2}{2},$$

where $u_\lambda \stackrel{d}{=} v_\lambda$ are independent real $\mathcal{N}(0,1)$ random variables. Thus, using homogeneity and independence properties of cumulants (see e.g. [PT11]), we have, as $n \to \infty, n \not\equiv 0,4,7 \pmod 8$

$$\begin{aligned} \kappa_4\left(W_{jk}^{(i_1)}(n)\right) &= \kappa_4\left(\frac{2}{n\sqrt{\mathcal{N}_n}} \sum_{\lambda \in \Lambda_n/\sim} \lambda_j \lambda_k \left(\frac{u_\lambda^2}{2} + \frac{v_\lambda^2}{2}\right)\right) \\ &= \frac{2^4}{n^4 \mathcal{N}_n^2} \sum_{\lambda \in \Lambda_n/\sim} \lambda_j^4 \lambda_k^4 \left(2^{-4}\kappa_4(u_\lambda^2) + 2^{-4}\kappa_4(v_\lambda^2)\right) \\ &\leq \frac{1}{\mathcal{N}_n^2} \sum_{\lambda \in \Lambda_n/\sim} \left(\kappa_4(u_\lambda^2) + \kappa_4(v_\lambda^2)\right) \ll \frac{1}{\mathcal{N}_n} = o(1), \end{aligned}$$

where we used that $\lambda_k^2 \leq n$ for every $k = 1,2,3$, which implies that $\lambda_j^4 \lambda_k^4 \leq n^4$. Concerning $M^{(i_1,i_2)}(n)$, we write

$$M^{(i_1,i_2)}(n) = \frac{1}{\sqrt{\mathcal{N}_n}} \sum_{\lambda \in \Lambda_n} a_{i_1,\lambda}\overline{a_{i_2,\lambda}} = \frac{2}{\sqrt{\mathcal{N}_n}} \sum_{\lambda \in \Lambda_n/\sim} a_{i_1,\lambda}\overline{a_{i_2,\lambda}}.$$

Noting that for every $\lambda \in \Lambda_n/_\sim$,

$$a_{i_1,\lambda}\overline{a_{i_2,\lambda}} \stackrel{d}{=} \frac{(a_{i_1,\lambda} + \overline{a_{i_2,\lambda}})(a_{i_1,\lambda} - \overline{a_{i_2,\lambda}})}{2} = \frac{a_{i_1,\lambda}^2 - \overline{a_{i_2,\lambda}}^2}{2}$$

and using independence, we infer

$$\begin{aligned} \kappa_4\left(M^{(i_1,i_2)}(n)\right) &= \frac{2^4}{\mathcal{N}_n^2} \sum_{\lambda \in \Lambda_n/\sim} \kappa_4(a_{i_1,\lambda}\overline{a_{i_2,\lambda}}) = \frac{1}{\mathcal{N}_n^2} \sum_{\lambda \in \Lambda_n/\sim} \kappa_4(a_{i_1,\lambda}^2 - \overline{a_{i_2,\lambda}}^2) \\ &= \frac{1}{\mathcal{N}_n^2} \sum_{\lambda \in \Lambda_n/\sim} (\kappa_4(a_{i_1,\lambda}^2) + \kappa_4(\overline{a_{i_2,\lambda}}^2)) \ll \frac{1}{\mathcal{N}_n} = o(1), \end{aligned}$$

as $n \to \infty, n \not\equiv 0,4,7 \pmod 8$. The other computations are done similarly. □

The following corollary follows immediately:

Corollary II.3.20. *For $\ell \in \{2,3\}$ and $i_1 < i_2 \in [\ell]$, as $n \to \infty, n \not\equiv 0,4,7 \pmod 8$, the random vector*

$$\mathbf{V}_{i_1,i_2}(n) := \left(\mathbb{W}^{(i_1)}(n), \mathbb{W}^{(i_2)}(n), \mathbb{M}^{(i_1,i_2)}(n)\right) \in \mathbb{R}^{21}$$

converges in distribution to

$$\mathbf{G}_{i_1,i_2} := \left(\mathbb{G}^{(i_1)}, \mathbb{G}^{(i_2)}, \mathbb{G}^{(i_1,i_2)}\right) \sim \mathcal{N}_{21}(0, \Sigma_{\mathbf{G}_{i_1,i_2}}) ,$$

where

$$\Sigma_{\mathbf{G}_{i_1,i_2}} = \Sigma_{\mathbb{W}} \oplus \Sigma_{\mathbb{W}} \oplus \Sigma_{\mathbb{M}} \in \mathcal{M}_{21\times 21}(\mathbb{R}) .$$

We use the above established CLT in order to derive the limiting distribution of the fourth-order chaotic component of $L_n^{(\ell)}$. From Lemma II.3.16, it follows that, as $n \to \infty, n \not\equiv 0,4,7 \pmod 8$, the sequences in (II.3.25) and (II.3.27) satisfy

$$\mu^{(i_1)}(n) = \frac{1}{10} + o_{\mathbb{P}}(1) , \quad \eta^{(i_1,i_2)}(n) = \frac{2}{5} + o_{\mathbb{P}}(1) , \tag{II.3.35}$$

where $o_{\mathbb{P}}(1)$ denotes a sequence converging to zero in probability. Now, bearing in mind the expressions (II.3.24) and (II.3.26), we define

$$F(\mathbb{W}^{(i_1)}) := -\frac{1}{40}\sum_{k<j}(W_{kk}^{(i_1)}(n) - W_{jj}^{(i_1)}(n))^2 - \frac{3}{20}\sum_{k<j} W_{kj}^{(i_1)}(n)^2 , \quad i_1 \in [\ell]$$

and

$$\begin{aligned} G(\mathbf{V}_{i_1,i_2}) \quad := \quad & -\frac{1}{10}\sum_{k<j}(W_{kk}^{(i_1)}(n) - W_{jj}^{(i_1)}(n))(W_{kk}^{(i_2)}(n) - W_{jj}^{(i_2)}(n)) \\ & -\frac{3}{5}\sum_{k<j} W_{kj}^{(i_1)}(n)W_{kj}^{(i_2)}(n) + \frac{1}{10}\sum_{k=1}^{3} M_{kk}^{(i_1,i_2)}(n)^2 - \frac{1}{20}\sum_{k\neq j} M_{kk}^{(i_1,i_2)}(n)M_{jj}^{(i_1,i_2)}(n) \\ & -\frac{1}{2}\sum_{k=1}^{3} M_{k}^{(i_1,i_2)}(n)^2 + \frac{3}{10}\sum_{k<j} M_{kj}^{(i_1,i_2)}(n)^2 , \quad i_1 < i_2 \in [\ell]. \end{aligned}$$

Combining these definitions with (II.3.35), leads to the asymptotic relations

$$A_{n,\ell}^{(i_1)} = \frac{2}{(2\pi)^{\ell/2}}\frac{\alpha(\ell,3)}{\mathcal{N}_n} \cdot [f(\mathbb{W}^{(i_1)}(n)) + o_{\mathbb{P}}(1)], \quad i_1 \in [\ell] \tag{II.3.36}$$

$$B_{n,\ell}^{(i_1,i_2)} = \frac{2}{(2\pi)^{\ell/2}}\frac{\alpha(\ell,3)}{\mathcal{N}_n} \cdot [g(\mathbf{V}_{i_1,i_2}(n)) + o_{\mathbb{P}}(1)], \quad i_1 < i_2 \in [\ell] \tag{II.3.37}$$

where

$$f(\mathbb{W}^{(i_1)}(n)) := F(\mathbb{W}^{(i_1)}(n)) + \frac{1}{10}, \quad g(\mathbf{V}_{i_1,i_2}(n)) := G(\mathbf{V}_{i_1,i_2}(n)) + \frac{2}{5}. \tag{II.3.38}$$

Plugging (II.3.36) and (II.3.37) into (II.3.23) and using the CLT in Corollary II.3.20, we obtain that, as $n \to \infty, n \not\equiv 0,4,7 \pmod 8$,

$$(c_n^{(\ell)})^{-1} \cdot \mathrm{proj}_4(L_n^{(\ell)}) \xrightarrow{d} \sum_{i_1\in[\ell]} f(\mathbb{G}^{(i_1)}) + \sum_{i_1<i_2\in[\ell]} g(\mathbf{G}_{i_1,i_2}) =: L^{(\ell)} , \tag{II.3.39}$$

where

$$c_n^{(\ell)} := \left(\frac{E_n}{3}\right)^{\ell/2}\frac{2}{(2\pi)^{\ell/2}}\frac{\alpha(\ell,3)}{\mathcal{N}_n} . \tag{II.3.40}$$

II.3.2.3 Proofs of Proposition II.3.3 and II.3.4

From the convergence in distribution stated in (II.3.39), we conclude that the sequence $\left\{Y_n^{(\ell)} := (c_n^{(\ell)})^{-1}\,\mathrm{proj}_4(L_n^{(\ell)}) : n \in S_3\right\}$ living in the fourth Wiener chaos, is tight and thus bounded in $L^p(\mathbb{P})$ for any $p > 0$ by virtue of the hypercontractivity property of Wiener chaoses (see e.g. [NR14, Lemma 2.1]). This implies that the sequence $\left\{(Y_n^{(\ell)})^2 : n \in S_3\right\}$ is uniformly integrable. By Skorohod's Representation Theorem (see e.g. [Bil99, Theorem 25.6]), there exist random variables $\left\{Y_n^{(\ell)*} : n \in S_3\right\}$ and $L^{(\ell)*}$ defined on some auxiliary probability space $(\Omega^*, \mathscr{F}^*, \mathbb{P}^*)$, such that (i) $Y_n^{(\ell)*} \overset{d}{=} Y_n^{(\ell)}$ for every $n \in S_3$ and $L^{(\ell)*} \overset{d}{=} L^{(\ell)}$ and (ii) $Y_n^{(\ell)*} \to L^{(\ell)*}, \mathbb{P}^*$-a.s. as $n \to \infty, n \not\equiv 0,4,7 \pmod 8$. Therefore we conclude that the sequence $\left\{(Y_n^{(\ell)*})^2 : n \in S_3\right\}$ is uniformly integrable. In particular, we infer that $\|Y_n^{(\ell)}\|_{L^2(\mathbb{P})} = \|Y_n^{(\ell)*}\|_{L^2(\mathbb{P}^*)} \to \|L^{(\ell)*}\|_{L^2(\mathbb{P}^*)} = \|L^{(\ell)}\|_{L^2(\mathbb{P})}$, i.e.

$$(c_n^{(\ell)})^{-2}\mathbf{Var}\left[\mathrm{proj}_4(L_n^{(\ell)})\right] \to \mathbf{Var}\left[L^{(\ell)}\right],$$

as $n \to \infty, n \not\equiv 0,4,7 \pmod 8$, or equivalently

$$\mathbf{Var}\left[\mathrm{proj}_4(L_n^{(\ell)})\right] \sim (c_n^{(\ell)})^2 \cdot \mathbf{Var}\left[L^{(\ell)}\right], \tag{II.3.41}$$

as $n \to \infty, n \not\equiv 0,4,7 \pmod 8$. Therefore, the asymptotic variance of $\mathrm{proj}_4(L_n^{(\ell)})$ in Proposition II.3.3 and its asymptotic distribution in Proposition II.3.4 follow respectively from the variance and distribution of $L^{(\ell)}$, given in the following statement.

Proposition II.3.21. *For the random variable $L^{(\ell)}$ appearing in* (II.3.39)*, we have*

$$L^{(\ell)} \overset{d}{=} -\frac{1}{50}\hat{\xi}_1(5\ell) - \frac{1}{25}\hat{\xi}_2\left(\frac{5\ell(\ell-1)}{2}\right) + \frac{1}{25}\hat{\xi}_3\left(\frac{5\ell(\ell-1)}{2}\right) + \frac{1}{50}\hat{\xi}_4\left(\frac{5\ell(\ell-1)}{2}\right) - \frac{1}{6}\hat{\xi}_5\left(\frac{3\ell(\ell-1)}{2}\right),$$

where $\left\{\hat{\xi}(k_i) : i = 1, \ldots, 5\right\}$ is a family of independent centered chi-squared random variables, and therefore

$$\mathbf{Var}\left[L^{(\ell)}\right] = \ell \cdot \frac{1}{250} + \frac{\ell(\ell-1)}{2} \cdot \frac{76}{375}.$$

Proof. The proof is based on lengthy but standard computations involving covariances of Gaussian random variables. We provide a sketch of the proof for the sake of readability. From relation (II.3.39) and the structure of the covariance matrix of $\mathbf{G}_{i_1,i_2}$ in Corollary II.3.20, it follows that

$$\mathbf{Var}\left[L^{(\ell)}\right] = \ell \cdot \mathbf{Var}\left[f(\mathbb{G}^{(1)})\right] + \frac{\ell(\ell-1)}{2} \cdot \mathbf{Var}[g(\mathbf{G}_{1,2})].$$

The variances of $f(\mathbb{G}^{(1)})$ and $g(\mathbf{G}_{1,2})$ are then computed using the explicit expressions of f and g as well as the covariance matrix $\Sigma_{\mathbf{G}_{123}}$ in (II.3.19). The probability distribution of $L^{(\ell)}$ is obtained by a standard diagonalization argument in order to express the latter in terms of independent standard Gaussian random variables, implying in particular the formula for its variance. □

The proof of Propositon II.3.4 is concluded, once we note that the distribution of $L^{(\ell)}$ in Proposition II.3.21 can be written in the form $Y^{(\ell)}M^{(\ell)}(Y^{(\ell)})^T$, where $Y^{(\ell)} \sim \mathcal{N}_{\ell(9\ell-4)}(0, \mathbf{I}_{\ell(9\ell-4)})$ and $M^{(\ell)} \in \mathcal{M}_{\ell(9\ell-4)\times\ell(9\ell-4)}(\mathbb{R})$ is the deterministic matrix given by

$$M^{(\ell)} = \frac{-1}{50}\mathbf{I}_{5\ell} \oplus \frac{-1}{25}\mathbf{I}_{\frac{5\ell(\ell-1)}{2}} \oplus \frac{1}{25}\mathbf{I}_{\frac{5\ell(\ell-1)}{2}} \oplus \frac{1}{50}\mathbf{I}_{\frac{5\ell(\ell-1)}{2}} \oplus \frac{-1}{6}\mathbf{I}_{\frac{3\ell(\ell-1)}{2}},$$

with the convention that, $A \oplus 0 = A$ for any matrix A.

Appendix II.A Proof of Theorem II.2.5 and chaos expansion of level functionals

II.A.1 Wiener-Itô chaos expansion of $J(\mathbf{G}, W; u^{(\ell)})$

We now provide the chaotic decomposition of the random variable $J(\mathbf{G}, W; u^{(\ell)})$ introduced in Definition II.2.1. Informally, the latter is obtained by multiplying the respective chaotic expansions of $\prod_{i=1}^{\ell} \delta_{u_i}$ and W and then integrating the obtained expression over Z.

Formal chaotic expansion of the Dirac mass. For $u \in \mathbb{R}$, denote by $\left\{\beta_j^{(u)} : j \geq 0\right\}$ the Hermite coefficients of the formal expansion in Hermite polynomials of δ_u, that is

$$\delta_u(x) = \sum_{j\geq 0} \frac{\beta_j^{(u)}}{j!} H_j(x) , \quad x \in \mathbb{R}$$

where

$$\beta_j^{(u)} = \int_{\mathbb{R}} \delta_u(y) H_j(y)\gamma(y)dy = H_j(u)\gamma(u) . \tag{II.A.1}$$

Approximating the Dirac mass by indicators $(2\varepsilon)^{-1}\mathbb{1}_{[-\varepsilon,\varepsilon]}(x-u)$ for $\varepsilon > 0$ and denoting by $\left\{\beta_j^{(u)}(\varepsilon) : j \geq 0\right\}$ their associated Fourier-Hermite coefficients, the following lemma (roughly corresponding to [MPRW16, Lemma 3.4]) shows that the coefficients $\left\{\beta_j^{(u)} : j \geq 0\right\}$ in (II.A.1) are obtained from $\left\{\beta_j^{(u)}(\varepsilon) : j \geq 0\right\}$ by letting $\varepsilon \to 0$.

Lemma II.A.1. *For every $u \in \mathbb{R}$ and $\varepsilon > 0$, the following expansion holds in $L^2(\gamma)$:*

$$\frac{1}{2\varepsilon}\mathbb{1}_{[-\varepsilon,\varepsilon]}(x-u) = \sum_{j\geq 0} \frac{\beta_j^{(u)}(\varepsilon)}{j!} H_j(x) , \quad x \in \mathbb{R}$$

where

$$\beta_0^{(u)}(\varepsilon) = \frac{1}{2\varepsilon}\int_{u-\varepsilon}^{u+\varepsilon} \gamma(y)dy ,$$

and for $j \geq 1$,

$$\beta_j^{(u)}(\varepsilon) = -\frac{1}{2\varepsilon}\left(H_{j-1}(u+\varepsilon)\gamma(u+\varepsilon) - H_{j-1}(u-\varepsilon)\gamma(u-\varepsilon)\right) . \tag{II.A.2}$$

In particular, for every $j \geq 0$, as $\varepsilon \to 0$,

$$\beta_j^{(u)}(\varepsilon) \to \beta_j^{(u)} . \tag{II.A.3}$$

For the nodal case corresponding to $u = 0$, we write $\beta_j^{(0)} =: \beta_j$, and compute

$$\beta_{2j+1} = 0 , \quad \beta_{2j} = \frac{H_{2j}(0)}{\sqrt{2\pi}} , \quad j \geq 0 ,$$

where the first equality is a consequence of the symmetry relation $H_k(-x) = (-1)^k H_k(x)$. In particular, we have

$$\beta_0 = \frac{1}{\sqrt{2\pi}} , \quad \beta_2 = -\frac{1}{\sqrt{2\pi}} , \quad \beta_4 = \frac{3}{\sqrt{2\pi}} . \tag{II.A.4}$$

The following standard proposition gives the Wiener-Itô chaos expansion of $J(\mathbf{G}, W; u^{(\ell)})$ defined in Definition II.2.1. Its proof is based on the expansion of $(2\varepsilon)^{-\ell}\prod_{i=1}^{\ell} \mathbb{1}_{[-\varepsilon,\varepsilon]}(\bullet - u_i)$ into Hermite polynomials by means of Lemma II.A.1 and then letting $\varepsilon \to 0$. We omit the proof.

Proposition II.A.2. *Let the above setting prevail. Assume that the random field $W = \{W(z) : z \in Z\}$ is such that (i) $\sup_{z\in Z} \mathbb{E}\left[W(z)^2\right] < \infty$, (ii) $W(z)$ is $\sigma(\mathbf{G})$-measurable for every $z \in Z$, and (iii) $W(z)$ is stochastically independent of $(G^{(1)}(z), \dots, G^{(\ell)}(z))$ for every $z \in Z$. Then, the random variable*

$$J_\varepsilon(\mathbf{G}, W; u^{(\ell)}) := \int_Z (2\varepsilon)^{-\ell} \prod_{i=1}^{\ell} \mathbb{1}_{[-\varepsilon,\varepsilon]}(G^{(i)}(z) - u_i) \cdot W(z)\ \mu(dz)$$

is an element of $L^2(\mathbb{P})$ for every $\varepsilon > 0$. Moreover, if $J(\mathbf{G}, W; u^{(\ell)})$ as in (II.2.1) *is well-defined in $L^2(\mathbb{P})$, then for every $q \geq 0$,*

$$\mathrm{proj}_q(J(\mathbf{G}, W; u^{(\ell)})) = \sum_{\substack{j_1,\dots,j_\ell, r\geq 0\\ j_1+\dots+j_\ell+r=q}} \frac{\beta_{j_1}^{(u_1)} \cdots \beta_{j_\ell}^{(u_\ell)}}{j_1! \cdots j_\ell!} \int_Z \prod_{i=1}^{\ell} H_{j_i}(G^{(i)}(z)) \cdot \mathrm{proj}_r(W(z))\ \mu(dz)\,, \tag{II.A.5}$$

where $\left\{\beta_j^{(u)} : j \geq 0\right\}$ denote the coefficients of the formal Hermite expansion of δ_u given in (II.A.1).

II.A.1.1 Some elementary facts

Let $k \geq 1$ be an integer and $X = (X_1, \dots, X_k)$ a standard k-dimensional Gaussian vector. We write $\|\cdot\|_k$ to indicate the Euclidean norm in $\mathbb{R}^k$. We will need the following standard fact.

Lemma II.A.3. *The random variable $\|X\|_k$ is stochastically independent of $X/\|X\|_k$.*

Proof. Let $g : \mathbb{R}^{k+1} \to \mathbb{R}$ be a bounded continuous function. Then, by passing to polar coordinates, we have

$$\int_{\mathbb{R}^k} g\left(\|x\|, \frac{x}{\|x\|}\right) \gamma_k(x)dx = \int_{\mathbb{R}_+\times\mathbb{S}^{k-1}} g\left(r, \frac{u}{r}\right) r^{k-1} \frac{e^{-r^2/2}}{(2\pi)^{k/2}} \sigma_k(du).$$

This shows that the density function of the vector $(\|X\|_k, X/\|X\|_k)$ is of the form $f_0(r)f_1(u)$, yielding independence. □

For integers $1 \leq \ell \leq k$, we recall the notation introduced in (II.1.8)

$$\alpha(\ell, k) := \frac{(k)_\ell \kappa_k}{(2\pi)^{\ell/2}\kappa_{k-\ell}}\,,$$

where $(k)_\ell := k!/(k-\ell)!$ and $\kappa_k := \frac{\pi^{k/2}}{\Gamma(1+k/2)}$ stands for the volume of the unit ball in $\mathbb{R}^k$. The following lemma contains an expression of the moments of the Euclidean norm of a standard k-dimensional Gaussian vector.

Lemma II.A.4. *For all integers $k \geq 1$ and $n \geq 1$, we have*

$$\mathbb{E}\left[\|X\|_k^n\right] = 2^{n/2}\frac{\Gamma((k+n)/2)}{\Gamma(k/2)}\,. \tag{II.A.6}$$

In particular,

$$\mathbb{E}\left[\|X\|_k\right] = \alpha(1, k)\,, \tag{II.A.7}$$

$$\mathbb{E}\left[\|X\|_k^2\right] = k\,, \tag{II.A.8}$$

$$\mathbb{E}\left[\|X\|_k^3\right] = \alpha(1, k)(k+1)\,, \tag{II.A.9}$$

$$\mathbb{E}\left[\|X\|_k^4\right] = k(k+2)\,, \tag{II.A.10}$$

$$\mathbb{E}\left[\|X\|_k^5\right] = \alpha(1,k)(k+1)(k+3)\,, \tag{II.A.11}$$

so that

$$\frac{\mathbb{E}\left[\|X\|_k^3\right]}{\mathbb{E}\left[\|X\|_k\right]} = k+1\,. \tag{II.A.12}$$

Proof. The law of the random variable $\|X\|_k$ is the chi-distribution with k degrees of freedom, whose density is given by

$$f(x) = \frac{1}{2^{k/2-1}\Gamma(k/2)} x^{k-1} e^{-x^2/2}\,, \qquad x > 0\,.$$

Thus, it follows that, for $n \geq 1$,

$$\mathbb{E}\left[\|X\|_k^n\right] = \int_0^\infty x^n f(x)dx = \frac{1}{2^{k/2-1}\Gamma(k/2)} \int_0^\infty x^{k+n-1} e^{-x^2/2} dx\,.$$

Performing the change of variables $y = x^2/2$ yields

$$\mathbb{E}\left[\|X\|_k^n\right] = \frac{1}{2^{k/2-1}\Gamma(k/2)} \cdot 2^{(k+n)/2-1}\Gamma((k+n)/2) = 2^{n/2}\frac{\Gamma((k+n)/2)}{\Gamma(k/2)}\,,$$

which proves (II.A.6). The identities (II.A.7)-(II.A.11) are obtained from (II.A.6) for $n = 1,\ldots,6$ respectively, together with the relations $\Gamma(z+1) = z\Gamma(z)$ and the definition in (II.1.8). □

II.A.1.2 Wiener-Itô chaos expansion of $\Phi_{\ell,k}$

For integers $1 \leq \ell \leq k$, we consider a generic map $\Phi_{\ell,k}$ as in Definition II.2.2 and a matrix $\mathbf{X} = \left\{X_j^{(i)} : (i,j) \in [\ell]\times[k]\right\} \in \mathcal{M}_{\ell\times k}(\mathbb{R})$ with independent standard normal entries.

The next lemma provides a characterization of the second chaotic projection associated with $\mathbf{X}$ and $\Phi_{\ell,k}(\mathbf{X})$, where we assume that $\mathbb{E}\left[\Phi_{\ell,k}(\mathbf{X})^2\right] < \infty$. As before, we set $\mathbb{E}\left[\Phi_{\ell,k}(\mathbf{X})\right] =: \alpha_{\ell,k}$.

Lemma II.A.5. *Let the above assumptions and notation prevail. Then, the following properties hold:*

(i) *for every $m \geq 1$, $(i_1,j_1),\ldots,(i_m,j_m) \in [\ell]\times[k]$ and $p_1,\ldots,p_m \in \mathbb{N}$ such that $p_1+\ldots+p_m$ is odd, we have*

$$\mathbb{E}\left[\Phi_{\ell,k}(\mathbf{X})\prod_{a=1}^m H_{p_a}(X_{j_a}^{(i_a)})\right] = 0\,;$$

(ii) *for every $(i_1,j_1) \neq (i_2,j_2) \in [\ell]\times[k]$, we have*

$$\mathbb{E}\left[\Phi_{\ell,k}(\mathbf{X})X_{j_1}^{(i_1)}X_{j_2}^{(i_2)}\right] = 0\,;$$

(iii) *for every $(i,j) \in [\ell]\times[k]$, we have*

$$\mathbb{E}\left[\Phi_{\ell,k}(\mathbf{X})H_2(X_j^{(i)})\right] = \frac{1}{k}\alpha_{\ell,k}\,.$$

Proof. Let us prove (i). Writing $p_1+\ldots+p_m = r$ and using the fact that $\mathbf{X} \overset{d}{=} -\mathbf{X}$ together with property (A3) and the symmetry relation $H_k(-x) = (-1)^k H_k(x)$ for odd k, we have

$$\mathbb{E}\left[\Phi_{\ell,k}(\mathbf{X})\prod_{a=1}^m H_{p_a}(X_{j_a}^{(i_a)})\right] = \mathbb{E}\left[\Phi_{\ell,k}(-\mathbf{X})\prod_{a=1}^m H_{p_a}(-X_{j_a}^{(i_a)})\right](-1)^r\mathbb{E}\left[\Phi_{\ell,k}(\mathbf{X})\prod_{a=1}^m H_{p_a}(X_{j_a}^{(i_a)})\right],$$

which implies the claim. Let us now prove (ii). Assume first that $\ell \geq 2$ and $i_1 \neq i_2$. Let $\mathbf{X}^*$ be the matrix obtained from $\mathbf{X}$ by multiplying the i_1-th row by -1. Then, $\mathbf{X} \stackrel{d}{=} \mathbf{X}^*$ together with (A2) applied with $c = -1$ imply

$$J := \mathbb{E}\left[\Phi_{\ell,k}(\mathbf{X}) X^{(i_1)}_{j_1} X^{(i_2)}_{j_2}\right] = \mathbb{E}\left[\Phi_{\ell,k}(\mathbf{X}^*) X^{*(i_1)}_{j_1} X^{*(i_2)}_{j_2}\right] = \mathbb{E}\left[\Phi_{\ell,k}(\mathbf{X})(-X^{(i_1)}_{j_1}) X^{(i_2)}_{j_2}\right] = -J,$$

and therefore $J = 0$. Assume now that $i_1 = i_2$ (and therefore that $j_1 \neq j_2$). Let $\mathbf{X}^{**}$ be the matrix obtained from $\mathbf{X}$ by multiplying the j_1-th column of $\mathbf{X}$ by -1. Then, $\mathbf{X} \stackrel{d}{=} \mathbf{X}^{**}$ together with (A3) imply

$$J := \mathbb{E}\left[\Phi_{\ell,k}(\mathbf{X}) X^{(i_1)}_{j_1} X^{(i_2)}_{j_2}\right] = \mathbb{E}\left[\Phi_{\ell,k}(\mathbf{X}^{**}) X^{**(i_1)}_{j_1} X^{**(i_2)}_{j_2}\right] = \mathbb{E}\left[\Phi_{\ell,k}(\mathbf{X})(-X^{(i_1)}_{j_1}) X^{(i_2)}_{j_2}\right] = -J,$$

which yields the desired conclusion. In order to prove (iii), let $\mathbf{X}^*$ be the matrix obtained from $\mathbf{X}$ by multiplying the i-th row by $c = 1/\|X^{(i)}\|_k$. Then, according to Lemma II.A.3, the i-th row of $\mathbf{X}^*$ is stochastically independent of $\|X^{(i)}\|_k$. We have

$$\mathbb{E}\left[\Phi_{\ell,k}(\mathbf{X}) H_2(X^{(i)}_j)\right] = \frac{1}{k}\mathbb{E}\left[\Phi_{\ell,k}(\mathbf{X})\|X^{(i)}\|_k^2\right] - \mathbb{E}\left[\Phi_{\ell,k}(\mathbf{X})\right] ,$$

so that, using (A2) and the independence mentioned above, yields

$$\begin{aligned}
\mathbb{E}\left[\Phi_{\ell,k}(\mathbf{X}) H_2(X^{(i)}_j)\right] &= \frac{1}{k}\mathbb{E}\left[\Phi_{\ell,k}(\mathbf{X}^*)\|X^{(i)}\|_k^3\right] - \mathbb{E}\left[\Phi_{\ell,k}(\mathbf{X})\right] \\
&= \frac{1}{k}\frac{\mathbb{E}\left[\Phi_{\ell,k}(\mathbf{X})\right]}{\mathbb{E}\left[\|X^{(i)}\|_k\right]}\mathbb{E}\left[\|X^{(i)}\|_k^3\right] - \mathbb{E}\left[\Phi_{\ell,k}(\mathbf{X})\right] = \frac{1}{k}\mathbb{E}\left[\Phi_{\ell,k}(\mathbf{X})\right] = \frac{1}{k}\alpha_{\ell,k} ,
\end{aligned}$$

where the last equality follows from (II.A.12). □

The following proposition combines Lemma II.A.5 with the classical general formula for the chaotic projections of all order of $\Phi_{\ell,k}(\mathbf{X})$.

Proposition II.A.6. *Let $\Phi_{\ell,k} : \mathcal{M}_{\ell\times k}(\mathbb{R}) \to \mathbb{R}_+$ be as in the previous lemma. Then, for $q \geq 0$, the projection of $\Phi_{\ell,k}(\mathbf{X})$ onto the q-th Wiener chaos associated with $\mathbf{X}$ is given by*

$$\mathrm{proj}_q(\Phi_{\ell,k}(\mathbf{X})) = \sum_{\substack{p^{(1)}_1,\dots,p^{(1)}_k \geq 0 \\ p^{(1)}_1+\dots+p^{(1)}_k+\dots+p^{(\ell)}_1+\dots+p^{(\ell)}_k = q}} \cdots \sum_{p^{(\ell)}_1,\dots,p^{(\ell)}_k \geq 0} \alpha^{(\ell)}_k \left\{p^{(i)}_j : (i,j) \in [\ell]\times[k]\right\} \cdot \prod_{i=1}^{\ell}\prod_{j=1}^{k} H_{p^{(i)}_j}(X^{(i)}_j) ,$$

where the coefficients $\alpha^{(\ell)}_k \left\{p^{(i)}_j : (i,j) \in [\ell]\times[k]\right\}$ are given by

$$\alpha^{(\ell)}_k \left\{p^{(i)}_j : (i,j) \in [\ell]\times[k]\right\} := \frac{1}{\prod_{i=1}^{\ell}\prod_{j=1}^{k}(p^{(i)}_j)!} \cdot \mathbb{E}\left[\Phi_{\ell,k}(\mathbf{X}) \cdot \prod_{i=1}^{\ell}\prod_{j=1}^{k} H_{p^{(i)}_j}(X^{(i)}_j)\right] . \quad \text{(II.A.13)}$$

In particular, we have

$$\mathrm{proj}_0(\Phi_{\ell,k}(\mathbf{X})) = \mathbb{E}\left[\Phi_{\ell,k}(\mathbf{X})\right] = \alpha_{\ell,k} , \quad \text{(II.A.14)}$$

$$\mathrm{proj}_2(\Phi_{\ell,k}(\mathbf{X})) = \frac{\alpha_{\ell,k}}{2} \cdot \frac{1}{k}\sum_{i=1}^{\ell}\sum_{j=1}^{k}((X^{(i)}_j)^2 - 1) , \quad \text{(II.A.15)}$$

$$\mathrm{proj}_{2q+1}(\Phi_{\ell,k}(\mathbf{X})) = 0 , \quad q \geq 0 . \quad \text{(II.A.16)}$$

Proof. The formula for $\mathrm{proj}_q(\Phi_{\ell,k}(\mathbf{X}))$ follows from the orthogonal decomposition of $L^2(\mathbb{P})$. For $q = 0$, we have $p^{(i)}_j = 0$ for every $(i,j) \in [\ell]\times[k]$, so that $\mathrm{proj}_0(\Phi_{\ell,k}(\mathbf{X})) = \mathbb{E}\left[\Phi_{\ell,k}(\mathbf{X})\right]$. For $q = 2$, in view of Lemma II.A.5, only the tuples $(p^{(i)}_j : (i,j) \in [\ell]\times[k])$ involving exactly one 2 contribute to the projection on the second chaos and the conclusion then follows from Lemma II.A.5 (iii). Finally, the projections onto Wiener chaoses of odd order vanish in view of Lemma II.A.5 (i). □

II.A.2 Proof of Theorem II.2.5

Part (i) follows from the form of the q-th chaotic projection of J provided in (II.A.5) and Proposition II.A.6 where the random matrix $\mathbf{X}$ is replaced with $\mathbf{X}_\star(z)$. Indeed, by (II.A.14) and the fact that $\mu(Z) = 1$, we have

$$\mathrm{proj}_0(J) = \beta_0^{(u_1)} \cdots \beta_0^{(u_\ell)} \int_Z \prod_{i=1}^{\ell} H_0(X_0^{(i)}(z)) \cdot \mathrm{proj}_0(\Phi_{\ell,k}(\mathbf{X}_\star(z)))\ \mu(dz) = \prod_{i=1}^{\ell} \gamma(u_i) \cdot \alpha_{\ell,k}\ .$$

This proves (II.2.8). For (II.2.9), since $\mathrm{proj}_1(\Phi_{\ell,k}(\mathbf{X}_\star(z))) = 0$ by (II.A.16), we have (recalling the definition of $m^{(i)}$ in (II.2.4))

$$\begin{aligned}
\mathrm{proj}_1(J) &= \sum_{i=1}^{\ell} \beta_1^{(u_i)} \prod_{\substack{j=1\\ j\neq i}}^{\ell} \beta_0^{(u_j)} \int_Z H_0(X_0^{(j)}(z)) H_1(X_0^{(i)}(z)) \cdot \mathrm{proj}_0(\Phi_{\ell,k}(\mathbf{X}_\star(z)))\ \mu(dz) \\
&= \sum_{i=1}^{\ell} \prod_{\substack{j=1\\ j\neq i}}^{\ell} \gamma(u_j)\gamma(u_i) u_i \int_Z X_0^{(i)}(z) \cdot \alpha_{\ell,k}\ \mu(dz) = \prod_{j=1}^{\ell} \gamma(u_j) \cdot \alpha_{\ell,k} \cdot \sum_{i=1}^{\ell} m^{(i)} u_i\ .
\end{aligned}$$

Let us now turn to (II.2.10). We have

$$\begin{aligned}
\mathrm{proj}_2(J) &= \sum_{i=1}^{\ell} \frac{\beta_2^{(u_i)}}{2!} \prod_{\substack{j=1\\ j\neq i}}^{\ell} \beta_0^{(u_j)} \int_Z H_0(X_0^{(j)}(z)) H_2(X_0^{(i)}(z)) \cdot \mathrm{proj}_0(\Phi_{\ell,k}(\mathbf{X}_\star(z)))\ \mu(dz) \\
&\quad + \prod_{i=1}^{\ell} \beta_0^{(u_i)} \int_Z H_0(X_0^{(i)}(z)) \cdot \mathrm{proj}_2(\Phi_{\ell,k}(\mathbf{X}_\star(z)))\ \mu(dz)\ .
\end{aligned}$$

Now, using $\beta_2^{(u_i)} = \gamma(u_i)(u_i^2 - 1)$ and (II.A.15) yields

$$\begin{aligned}
\mathrm{proj}_2(J) &= \frac{\alpha_{\ell,k}}{2} \cdot \prod_{j=1}^{\ell} \gamma(u_j) \cdot \sum_{i=1}^{\ell} (u_i^2 - 1) \int_Z (X_0^{(i)}(z)^2 - 1)\ \mu(dz) \\
&\quad + \frac{\alpha_{\ell,k}}{2} \cdot \prod_{i=1}^{\ell} \gamma(u_i) \int_Z \frac{1}{k} \sum_{i=1}^{\ell} \sum_{j=1}^{k} (X_j^{(i)}(z)^2 - 1)\ \mu(dz) \\
&= \frac{\alpha_{\ell,k}}{2} \cdot \prod_{i=1}^{\ell} \gamma(u_i) \cdot \sum_{i=1}^{\ell} \left\{ (u_i^2 - 1) \int_Z (X_0^{(i)}(z)^2 - 1) + \frac{1}{k} \sum_{j=1}^{k} (X_j^{(i)}(z)^2 - 1)\ \mu(dz) \right\} \\
&= \frac{\alpha_{\ell,k}}{2} \cdot \prod_{i=1}^{\ell} \gamma(u_i) \cdot \sum_{i=1}^{\ell} \left\{ u_i^2 \int_Z (X_0^{(i)}(z)^2 - 1)\ \mu(dz) + D^{(i)} \right\},
\end{aligned}$$

where we used the definition of $D^{(i)}$ in (II.2.3).
For part (ii), set $u_i = D^{(i)} = 0$ for every $i \in [\ell]$. Then, (II.2.11) follows since $\gamma(0) = 1/\sqrt{2\pi}$. By (II.2.10), we have that $\mathrm{proj}_2(J) = 0$. It remains to show that $\mathrm{proj}_{2q+1}(J) = 0$ for $q \geq 0$. The fact that $\beta_{2k+1}^{(0)} = 0$ for every $k \geq 0$ implies that the expansion in (II.A.5) runs over indices $j_1, \ldots, j_\ell$ that are all even. The projection of J onto Wiener chaoses of odd order is therefore of the form

$$\mathrm{proj}_{2q+1}(J) = \sum_{\substack{j_1,\ldots,j_\ell, r \geq 0\\ j_1+\ldots+j_\ell+r=2q+1}} \frac{\beta_{j_1}^{(0)} \cdots \beta_{j_\ell}^{(0)}}{j_1! \cdots j_\ell!} \int_Z \prod_{i=1}^{\ell} H_{j_i}(X_0^{(i)}(z)) \cdot \mathrm{proj}_r(\Phi_{\ell,k}(\mathbf{X}_\star(z)))\ \mu(dz)\ ,$$

where $j_1, \ldots, j_\ell$ are all even and r is odd. The conclusion then follows from (II.A.16).

Appendix II.B Fourier-Hermite coefficients of Gramian determinants on the fourth Wiener chaos

For integers $1 \leq \ell \leq k$ and a $\ell \times k$ matrix $\mathbf{X}$ with i.i.d. standard normal entries, we consider the function

$$\Phi^*_{\ell,k} : \mathcal{M}_{\ell\times k}(\mathbb{R}) \to \mathbb{R}_+ , \quad \mathbf{X} \mapsto \det(\mathbf{X}\,\mathbf{X}^T)^{1/2} . \tag{II.B.1}$$

The following lemma shows that $\Phi^*_{\ell,k}$ defined in (II.B.1) satisfies *Assumption A* of Definition II.2.2. In order to prove this, we recall Cauchy-Binet's identity:

$$\Phi^*_{\ell,k}(\mathbf{X}) = \left[\sum_{j_1<\ldots<j_\ell \in[k]} \det(\mathbf{X}_{j_1,\ldots,j_\ell})^2 \right]^{1/2} , \tag{II.B.2}$$

where, for $j_1 < \ldots < j_\ell \in [k]$, we denote by $\mathbf{X}_{j_1,\ldots,j_\ell} \in \mathcal{M}_{\ell\times\ell}(\mathbb{R})$ the matrix obtained from $\mathbf{X}$ by only keeping columns labeled $j_1, \ldots, j_\ell$. We refer to $\det(\mathbf{X}_{j_1,\ldots,j_\ell})$ as the *minors* of $\mathbf{X}$.

Lemma II.B.1. *The function* $\Phi^*_{\ell,k}$ *in* (II.B.1) *satisfies Assumption A of Definition II.2.2.*

Proof. (A1) Permuting two columns multiplies some of the minors by -1, which is absorbed by taking its square. Permuting two rows multiplies each minor by -1, which is again absorbed by taking its square.

(A2) Let $\mathbf{X}^*$ denote the matrix obtained from $\mathbf{X}$ by multiplying the i-th row by $c \in \mathbb{R}$. Then, for every $j_1 < \ldots < j_\ell \in [k]$, we have $\det(\mathbf{X}^*_{j_1,\ldots,j_\ell})^2 = c^2 \det(\mathbf{X}_{j_1,\ldots,j_\ell})^2$, so that (II.B.2) implies $\Phi^*_{\ell,k}(\mathbf{X}^*) = |c|\Phi^*_{\ell,k}(\mathbf{X})$.

(A3) Let $\mathbf{X}^*$ denote the matrix obtained from $\mathbf{X}$ by multiplying its j-th column by -1. Then, $\mathbf{X}^*(\mathbf{X}^*)^T = \mathbf{X}\,\mathbf{X}^T$, so that trivially $\Phi^*_{\ell,k}(\mathbf{X}) = \Phi^*_{\ell,k}(\mathbf{X}^*)$.

(A4) Let $\mathbf{X}^*$ denote the matrix obtained from $\mathbf{X}$ by replacing its i_1-th row with the sum of its i_1-th and i_2-th row for $i_1 \neq i_2$. Then, the invariance of the determinant under this operation implies that for every $j_1 < \ldots < j_\ell \in [k]$, $\det(\mathbf{X}^*_{j_1,\ldots,j_\ell}) = \det(\mathbf{X}_{j_1,\ldots,j_\ell})$, so that $\Phi^*_{\ell,k}(\mathbf{X}^*) = \Phi^*_{\ell,k}(\mathbf{X})$.

□

II.B.1 A representation of the Gramian determinant

In the forthcoming discussion, our goal is to compute the Fourier-Hermite coefficients within the fourth Wiener chaos associated with the function $\Phi^*_{\ell,k}$ in (II.B.1). Our strategy goes as follows: in Lemma II.B.2, we prove a deterministic identity for Gramian determinants in terms of products of distances between subspaces generated by the matrix based on geometric observations. In Lemma II.B.3, we subsequently characterize the probability distribution of each of the factors and obtain in particular a formula for the expected value of the Gramian matrix associated with a standard Gaussian matrix. We point out that similar techniques based on factorization of Gramian determinants are used in Chapter 13 of [AT07] in order to establish the Gaussian Kinematic Formula. We refer the interested reader to this book for further details.

We start with a deterministic result. Let $v^{(1)}, \ldots, v^{(\ell)} \in \mathbb{R}^k$ be linearly independent vectors and $\mathbf{X}$ the $\ell \times k$ matrix whose i-th row is $v^{(i)}$. For $s = 0, \ldots, \ell - 1$, we write $\mathscr{V}_s := \operatorname{span}\left\{v^{(1)}, \ldots, v^{(s)}\right\}$ to indicate the s-dimensional linear subspace generated by the first s rows of $\mathbf{X}$ with the convention $\mathscr{V}_0 := \{0\}$ and denote by p_s the projection operator onto $\mathscr{V}_s$. Furthermore, we set

$$d(k - s) := \|v^{(s+1)} - p_s(v^{(s+1)})\|_k , \qquad s = 0, \ldots, \ell - 1 ,$$

that is, $d(k-s)$ is the Euclidean distance in $\mathbb{R}^k$ between $v^{(s+1)}$ and $\mathscr{V}_s$. The next lemma yields a useful representation of Gramian determinants.

Lemma II.B.2. *Let the above notation prevail. Then, the map* $\Phi^*_{\ell,k}$ *in* (II.B.1) *admits the representation*

$$\Phi^*_{\ell,k}(\mathbf{X}) = \prod_{s=0}^{\ell-1} d(k-s)\ . \tag{II.B.3}$$

Proof. Applying the Gram-Schmidt orthogonalization process to the vectors $\left\{v^{(1)},\dots,v^{(\ell)}\right\}$ gives rise to a family of orthogonal vectors $\left\{w^{(1)},\dots,w^{(\ell)}\right\}$ such that $\mathrm{span}\left\{w^{(1)},\dots,w^{(\ell)}\right\} = \mathrm{span}\left\{v^{(1)},\dots,v^{(\ell)}\right\}$. These are given recursively by $w^{(1)} = v^{(1)}$ and for $s = 1,\dots,\ell-1$,

$$w^{(s+1)} = v^{(s+1)} - \sum_{i=1}^{s} \frac{\langle v^{(s+1)}, w^{(i)}\rangle}{\|w^{(i)}\|_k^2} w^{(i)} = v^{(s+1)} - p_s(v^{(s+1)})\,,$$

where $\langle\cdot,\cdot\rangle$ denotes the canonical inner product in $\mathbb{R}^d$. Denote by $\mathbf{W}$ the $\ell\times k$ matrix with rows $w^{(1)},\dots,w^{(\ell)}$. There exists an orthogonal $\ell\times\ell$ matrix P such that $\mathbf{W} = P\mathbf{X}$, which implies that $\mathbf{W}\mathbf{W}^T = P\,\mathbf{X}\,\mathbf{X}^T\,P^T$, so that $\Phi^*_{\ell,k}(\mathbf{W}) = \Phi^*_{\ell,k}(\mathbf{X})$. As the rows of $\mathbf{W}$ are mutually orthogonal, we have that

$$\mathbf{W}\mathbf{W}^T = \mathrm{diag}\left(\|w^{(1)}\|_k^2,\dots,\|w^{(\ell)}\|_k^2\right) = \mathrm{diag}\left(d(k)^2,\dots,d(k-(\ell-1))^2\right),$$

and therefore,

$$\Phi^*_{\ell,k}(\mathbf{W}) = \prod_{s=0}^{\ell-1} d(k-s),$$

which is formula (II.B.3). □

We will now pass to the probabilistic setting and replace each of the deterministic vectors $v^{(1)},\dots,v^{(\ell)}$ by independent standard Gaussian vectors $X^{(1)},\dots,X^{(\ell)}$. The following lemma characterizes the probability distribution of the random variables $d(k-s)$.

Lemma II.B.3. *Let the above setting prevail. For every* $s = 0,\dots,\ell-1$, *the random variable* $d(k-s)$ *is chi-distributed with* $k-s$ *degrees of freedom and stochastically independent of* $(X^{(1)},\dots,X^{(s)})$. *In particular,*

$$\alpha_{\ell,k} := \mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})\right] = \prod_{s=0}^{\ell-1} \mathbb{E}\left[d(k-s)\right] = \alpha(\ell,k)\,, \tag{II.B.4}$$

where $\alpha(\ell,k)$ *is defined in* (II.1.8).

Proof. Let $\{e_1,\dots,e_k\}$ denote the canonical basis of $\mathbb{R}^k$. Since $d(k) = \|X^{(1)}\|_k$, the random variable $d(k)$ is clearly chi-distributed with k degrees of freedom. Now fix $s\in\{1,\dots,\ell-1\}$. By the rotational invariance of the Gaussian distribution, the conditional distribution of $d(k-s)$ given $\left\{X^{(1)},\dots,X^{(s)}\right\}$ is precisely the same as the distribution of the distance from $X^{(s+1)}$ to $\mathbb{R}^s$, that is

$$d(k-s)|\left\{X^{(1)},\dots,X^{(s)}\right\} \overset{d}{=} \left(\sum_{j=s+1}^{k} \langle X^{(s+1)}, e_j\rangle^2\right)^{1/2}.$$

Since the coefficients $\langle X^{(s+1)}, e_j \rangle = X_j^{(s+1)}$ are i.i.d. standard Gaussian, we infer that the conditional random variable $d(k-s)|\left\{X^{(1)}, \dots, X^{(s)}\right\}$ is chi-distributed with $k-s$ degrees of freedom. Thus the characteristic function of $d(k-s)^2|\left\{X^{(1)}, \dots, X^{(s)}\right\}$ is

$$\phi_{d(k-s)^2|\{X^{(1)},\dots,X^{(s)}\}}(t) = \mathbb{E}\left[e^{itd(k-s)^2}|X^{(1)}, \dots, X^{(s)}\right] = (1-2it)^{-(k-s)/2}, \quad t \in \mathbb{R}.$$

Therefore, taking expectation

$$\phi_{d(k-s)^2}(t) = \mathbb{E}\left[e^{itd(k-s)^2}\right] = \mathbb{E}\left[\mathbb{E}\left[e^{itd(k-s)^2}|X^{(1)}, \dots, X^{(s)}\right]\right] = (1-2it)^{-(k-s)/2},$$

from which we conclude that $d(k-s)$ is also chi-distributed with $k-s$ degrees of freedom. Moreover, since $d(k-s)|\left\{X^{(1)}, \dots, X^{(s)}\right\} \stackrel{d}{=} d(k-s)$, we deduce that $d(k-s)$ is independent of $\left\{X^{(1)}, \dots, X^{(s)}\right\}$. The identity in (III.3.49) follows from independence, and the fact that by (II.A.7), $\mathbb{E}[d(k-s)] = \alpha(1, k-s)$:

$$\alpha_{\ell,k} = \mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})\right] = \prod_{s=0}^{\ell-1} \mathbb{E}[d(k-s)] = \prod_{s=0}^{\ell-1} \alpha(1, k-s) = \prod_{s=0}^{\ell-1} \frac{(k-s)\kappa_{k-s}}{\sqrt{2\pi}\kappa_{k-s-1}} = \alpha(\ell, k),$$

which finishes the proof. □

II.B.2 Technical computations

The following result entirely characterizes the fourth chaotic component of the function $\Phi^*_{\ell,k}(\mathbf{X})$ defined in (II.B.1) where $\mathbf{X}$ is a $\ell \times k$ matrix with i.i.d. standard Gaussian entries.

Lemma II.B.4. *Let the above notations prevail. The following properties hold:*

(i) *for every* $(i, j) \in [\ell] \times [k]$*, we have*

$$\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})(X_j^{(i)})^4\right] = 3\alpha(\ell, k)\frac{(k+1)(k+3)}{k(k+2)};$$

(ii) *for every* $(i_1, j_1) \neq (i_2, j_2) \in [\ell] \times [k]$*, we have*

$$\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})(X_{j_1}^{(i_1)})^3 X_{j_2}^{(i_2)}\right] = 0,$$

(iii) *for every* $(i_1, j_1) \neq (i_2, j_2) \neq (i_3, j_3) \in [\ell] \times [k]$*, we have*

$$\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})(X_{j_1}^{(i_1)})^2 X_{j_2}^{(i_2)} X_{j_3}^{(i_3)}\right] = 0,$$

(iv) *for every* $(i_1, j_1) \neq (i_2, j_2) \in [\ell] \times [k]$*, we have*

$$\begin{aligned}\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})(X_{j_1}^{(i_1)})^2 (X_{j_2}^{(i_2)})^2\right] &= \alpha(\ell, k)\frac{(k+1)(k+3)}{k(k+2)}\mathbb{1}_{i_1=i_2} \\ &+ \alpha(\ell, k)\frac{(k+1)(k+3)}{k(k+2)}\mathbb{1}_{i_1\neq i_2, j_1=j_2}\mathbb{1}_{\ell\geq 2} \\ + \alpha(\ell, k)(k+1)&\frac{(k+1)(k+2)-(k+3)}{k(k-1)(k+2)}\mathbb{1}_{i_1\neq i_2, j_1\neq j_2}\mathbb{1}_{\ell\geq 2};\end{aligned}$$

(v) *for every collection* $I = \{(i_1, j_1) \neq (i_2, j_2) \neq (i_3, j_3) \neq (i_4, j_4) \in [\ell] \times [k]\}$*, we have*

$$\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X}) X_{j_1}^{(i_1)} X_{j_2}^{(i_2)} X_{j_3}^{(i_3)} X_{j_4}^{(i_4)}\right] = -\alpha(\ell, k)\frac{k+1}{k(k-1)(k+2)}\mathbb{1}_{I\in S}\mathbb{1}_{\ell\geq 2},$$

where $S = \{\{(i_1, j_1), (i_1, j_2), (i_2, j_1), (i_2, j_2)\} : i_1 \neq i_2, j_1 \neq j_2\}$.

Proof. We prove (i). By (A1), without loss of generality, we can assume that $i = 1$. Using the representation in (II.B.3), the fact that $\|X^{(1)}\|_k = d(k)$, as well Lemma II.A.3 and (III.3.49), we have for every $j \in [k]$,

$$\begin{aligned}\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})(X^{(1)}_j)^4\right] &= \mathbb{E}\left[d(k)\prod_{s=1}^{\ell-1} d(k-s)\frac{(X^{(1)}_j)^4}{\|X^{(1)}\|_k^4}\|X^{(1)}\|_k^4\right] = \mathbb{E}\left[d(k)^5\prod_{s=1}^{\ell-1} d(k-s)\frac{(X^{(1)}_j)^4}{d(k)^4}\right]\\ &= \frac{\mathbb{E}\left[d(k)^5\right]}{\mathbb{E}\left[d(k)^4\right]}\prod_{s=1}^{\ell-1}\mathbb{E}\left[d(k-s)\right]\mathbb{E}\left[(X^{(1)}_j)^4\right] = 3\alpha(\ell,k)\frac{(k+1)(k+3)}{k(k+2)},\end{aligned}$$

where the last equality follows from Lemma II.A.4.

We now prove (ii). Assume $i_1 = i_2$ (so that $j_1 \neq j_2$). Multiplying column j_2 by -1 and using (A3) then yields the desired conclusion. If $i_1 \neq i_2$ and $j_1 = j_2$, the result follows from (A2). The case $i_1 \neq i_2, j_1 \neq j_2$ follows either from (A3) or (A2).

The result in (iii) is obtained by arguments similar those in (ii).

For (iv), let us assume that $i_1 = i_2$ (so that $j_1 \neq j_2$). Denote by $\mathbf{X}^*$ the matrix obtained from $\mathbf{X}$ by multiplying the i_1-th row by $1/\|X^{(i_1)}\|_k$. Then, we first observe that by (A2) and Lemma II.A.3,

$$\begin{aligned}\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})\|X^{(i_1)}\|_k^4\right] &= \mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X}^*)\|X^{(i_1)}\|_k^5\right] = \mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X}^*)\right]\mathbb{E}\left[\|X^{(i_1)}\|_k^5\right]\\ &= \frac{\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})\right]}{\mathbb{E}\left[\|X^{(i_1)}\|_k\right]}\mathbb{E}\left[\|X^{(i_1)}\|_k^5\right] = \alpha(\ell,k)(k+1)(k+3),\end{aligned}$$

where we used Lemma II.A.4. On the other hand, we can write

$$\begin{aligned}\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})\|X^{(i_1)}\|_k^4\right] &= \mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})\sum_{j,j'=1}^{k}(X^{(i_1)}_j)^2(X^{(i_1)}_{j'})^2\right]\\ &= \sum_{j=1}^{k}\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})(X^{(i_1)}_j)^4\right] + \sum_{j\neq j'\in[k]}\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})(X^{(i_1)}_j)^2(X^{(i_1)}_{j'})^2\right]\\ &= k\,\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})(X^{(i_1)}_{j_1})^4\right] + k(k-1)\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})(X^{(i_1)}_{j_1})^2(X^{(i_1)}_{j_2})^2\right]\\ &= 3\alpha(\ell,k)\frac{(k+1)(k+3)}{k+2} + k(k-1)\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})(X^{(i_1)}_{j_1})^2(X^{(i_1)}_{j_2})^2\right],\end{aligned}$$

for every $j_1 \neq j_2$, where for the last equality we used the formula proved in (i). Therefore, it follows that for every $j_1 \neq j_2$,

$$\begin{aligned}\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})(X^{(i_1)}_{j_1})^2(X^{(i_1)}_{j_2})^2\right] &= \frac{1}{k(k-1)}\left(\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})\|X^{(i_1)}\|_k^4\right] - 3\alpha(\ell,k)\frac{(k+1)(k+3)}{k+2}\right)\\ &= \frac{1}{k(k-1)}\left(\alpha(\ell,k)(k+1)(k+3) - 3\alpha(\ell,k)\frac{(k+1)(k+3)}{k+2}\right) = \alpha(\ell,k)\frac{(k+1)(k+3)}{k(k+2)}.\end{aligned}$$

Let us now deal with the case $i_1 \neq i_2$ and $j_1 = j_2$, for $\ell \geq 2$. Denote by $\mathbf{X}_\pm$ the matrix obtained from $\mathbf{X}$ as follows:

$$\begin{aligned}(X_\pm)^{(i_1)} &= \frac{1}{\sqrt{2}}(X^{(i_1)} + X^{(i_2)}),\\ (X_\pm)^{(i_2)} &= \frac{1}{\sqrt{2}}\left(-2X^{(i_2)} + (X^{(i_1)} + X^{(i_2)})\right) = \frac{1}{\sqrt{2}}(X^{(i_1)} - X^{(i_2)}),\\ (X_\pm)^{(i)} &= X^{(i)}, \quad i \in [\ell] \setminus \{i_1, i_2\}.\end{aligned}$$

By construction, the rows $(X_\pm)^{(i_1)}$ and $(X_\pm)^{(i_2)}$ are stochastically independent standard Gaussian vectors, so that $\mathbf{X} \overset{d}{=} \mathbf{X}_\pm$. Hence, we have on the one hand

$$\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X}_\pm)((X_\pm)^{(i_1)}_{j_1})^4\right] = \mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})(X^{(i_1)}_{j_1})^4\right] = 3\alpha(\ell,k)\frac{(k+1)(k+3)}{k(k+2)} ,$$

in view of (i), and on the other hand, since $\Phi^*_{\ell,k}(\mathbf{X}_\pm) = \frac{(\sqrt{2})^2}{2}\Phi^*_{\ell,k}(\mathbf{X}) = \Phi^*_{\ell,k}(\mathbf{X})$, we conclude

$$\begin{aligned}\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X}_\pm)((X_\pm)^{(i_1)}_{j_1})^4\right] &= \mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})\Big(\frac{X^{(i_1)}_{j_1}+X^{(i_2)}_{j_1}}{\sqrt{2}}\Big)^4\right]\\ &= \frac{1}{4}\left(2\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})(X^{(i_1)}_{j_1})^4\right] + 6\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})(X^{(i_1)}_{j_1})^2(X^{(i_2)}_{j_1})^2\right]\right) ,\end{aligned}$$

where we used that $\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})X^{(i_1)}_{j_1}(X^{(i_2)}_{j_1})^3\right] = \mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})(X^{(i_1)}_{j_1})^3X^{(i_2)}_{j_1}\right] = 0$ in view of (ii). Therefore,

$$\begin{aligned}\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})(X^{(i_1)}_{j_1})^2(X^{(i_2)}_{j_1})^2\right] &= \frac{1}{6}\left(4\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X}_\pm)((X_\pm)^{(i_1)}_{j_1})^4\right] - 2\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})(X^{(i_1)}_{j_1})^4\right]\right)\\ &= \alpha(\ell,k)\frac{(k+1)(k+3)}{k(k+2)} .\end{aligned}$$

Let us now treat the case $i_1 \neq i_2$ and $j_1 \neq j_2$. Let $\mathbf{X}^*$ be the matrix obtained from $\mathbf{X}$ by multiplying rows $X^{(i_1)}$ resp. $X^{(i_2)}$ by $1/\|X^{(i_1)}\|_k$ resp. $1/\|X^{(i_2)}\|_k$. Then, by independence, we infer

$$\begin{aligned}&\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})\|X^{(i_1)}\|_k^2\|X^{(i_2)}\|_k^2\right] = \mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X}^*)\|X^{(i_1)}\|_k^3\|X^{(i_2)}\|_k^3\right]\\ =\ & \mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X}^*)\right]\mathbb{E}\left[\|X^{(i_1)}\|_k^3\right]\mathbb{E}\left[\|X^{(i_2)}\|_k^3\right] = \frac{\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})\right]}{\mathbb{E}\left[\|X^{(i_1)}\|_k\right]^2}\mathbb{E}\left[\|X^{(i_1)}\|_k^3\right]^2 = \alpha(\ell,k)(k+1)^2 .\end{aligned}$$

Expanding the product of the norms, we can write

$$\begin{aligned}\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})\|X^{(i_1)}\|_k^2\|X^{(i_2)}\|_k^2\right] &= k\,\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})(X^{(i_1)}_{j_1})^2(X^{(i_2)}_{j_1})^2\right] + k(k-1)\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})(X^{(i_1)}_{j_1})^2(X^{(i_2)}_{j_2})^2\right]\\ &= \alpha(\ell,k)\frac{(k+1)(k+3)}{k+2} + k(k-1)\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})(X^{(i_1)}_{j_1})^2(X^{(i_2)}_{j_2})^2\right] ,\end{aligned}$$

where we used the formula proved just before. Hence, we have that for every $j_1 \neq j_2$,

$$\begin{aligned}\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})(X^{(i_1)}_{j_1})^2(X^{(i_2)}_{j_2})^2\right] &= \frac{1}{k(k-1)}\left(\alpha(\ell,k)(k+1)^2 - \alpha(\ell,k)\frac{(k+1)(k+3)}{k+2}\right)\\ &= \alpha(\ell,k)(k+1)\frac{(k+1)(k+2)-(k+3)}{k(k-1)(k+2)} ,\end{aligned}$$

which is the desired formula. The other cases do not contribute as one can mutliply a row or column by -1.

We finally prove (v). First, note that if $I \notin S$, then the expectation is zero. Indeed, we notice that if $I \notin S$, there is at least one row or column of $\mathbf{X}$ that contains only one element corresponding to one of the four pairs of indices of I. Multiplying this row resp. column by -1 and using (A2) gives the desired conclusion. Let us now assume $I \in S$ and denote $E(I) := \mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})X^{(i_1)}_{j_1}X^{(i_2)}_{j_2}X^{(i_3)}_{j_3}X^{(i_4)}_{j_4}\right]\mathbb{1}[I \in S]$. Since $I \in S$, we can write

$$E(I) = \mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})X^{(i_1)}_{j_1}X^{(i_1)}_{j_2}X^{(i_2)}_{j_1}X^{(i_2)}_{j_2}\right] , \quad i_1 \neq i_2, j_1 \neq j_2.$$

Let us again consider the matrix $\mathbf{X}_\pm$ used in part (iv). From formula (iv) in the case $i_1 = i_2$, it follows that

$$\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X}_\pm)((X_\pm)^{(i_1)}_{j_1})^2((X_\pm)^{(i_1)}_{j_2})^2\right] = \alpha(\ell,k)\frac{(k+1)(k+3)}{k(k+2)} \ . \tag{II.B.5}$$

On the other hand, we can write

$$\begin{aligned}\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X}_\pm)((X_\pm)^{(i_1)}_{j_1})^2((X_\pm)^{(i_1)}_{j_2})^2\right] &= \frac{1}{4}\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})(X^{(i_1)}_{j_1}+X^{(i_2)}_{j_1})^2(X^{(i_1)}_{j_2}+X^{(i_2)}_{j_2})^2\right]\\ &= \frac{1}{4}\left(2\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})(X^{(i_1)}_{j_1})^2(X^{(i_1)}_{j_2})^2\right]+2\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})(X^{(i_1)}_{j_1})^2(X^{(i_2)}_{j_2})^2\right]+4E(I)\right) \ .\end{aligned}$$

Notice that the terms of the form $\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})(X^{(i_1)}_{j_1})^2X^{(i_1)}_{j_2}X^{(i_2)}_{j_2}\right]$ are zero, by (iii). Hence, combining (II.B.5) and (II.B.2) together with the results obtained in (iv), we obtain

$$\begin{aligned}E(I) &= \alpha(\ell,k)\left(\frac{(k+1)(k+3)}{k(k+2)}-\frac{1}{2}\frac{(k+1)(k+3)}{k(k+2)}-\frac{1}{2}(k+1)\frac{(k+1)(k+2)-(k+3)}{k(k-1)(k+2)}\right)\\ &= -\alpha(\ell,k)\frac{k+1}{k(k-1)(k+2)} \ ,\end{aligned}$$

which proves the formula. □

The following proposition follows immediately from Lemma II.B.4 and extends the results derived in [DNPR19, Lemma 3.3] (corresponding to $(\ell,k)=(2,2)$ in our notation) to arbitrary integers $1\le\ell\le k$.

Proposition II.B.5. *The following properties hold:*

(i) *for every $(i,j)\in[\ell]\times[k]$, we have*

$$\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})H_4(X^{(i)}_j)\right] = -\frac{3}{k(k+2)}\alpha(\ell,k) \ ;$$

(ii) *for every $(i_1,j_1)\neq(i_2,j_2)\in[\ell]\times[k]$, we have*

$$\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})H_3(X^{(i_1)}_{j_1})H_1(X^{(i_2)}_{j_2})\right] = 0 \, ,$$

(iii) *for every $(i_1,j_1)\neq(i_2,j_2)\neq(i_3,j_3)\in[\ell]\times[k]$, we have*

$$\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})H_2(X^{(i_1)}_{j_1})H_1(X^{(i_2)}_{j_2})H_1(X^{(i_3)}_{j_3})\right] = 0 \, ,$$

(iv) *for every $(i_1,j_1)\neq(i_2,j_2)\in[\ell]\times[k]$, we have*

$$\begin{aligned}\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})H_2(X^{(i_1)}_{j_1})H_2(X^{(i_2)}_{j_2})\right] &= -\frac{1}{k(k+2)}\alpha(\ell,k)\mathbb{1}_{i_1=i_2}\\ &\quad -\frac{1}{k(k+2)}\alpha(\ell,k)\mathbb{1}_{i_1\neq i_2}\mathbb{1}_{j_1=j_2}\mathbb{1}_{\ell\ge 2}\\ &\quad +\frac{k+3}{k(k-1)(k+2)}\alpha(\ell,k)\mathbb{1}_{i_1\neq i_2}\mathbb{1}_{j_1\neq j_2}\mathbb{1}_{\ell\ge 2} \ ;\end{aligned}$$

(v) *for every collection $I=\{(i_1,j_1)\neq(i_2,j_2)\neq(i_3,j_3)\neq(i_4,j_4)\in[\ell]\times[k]\}$, we have*

$$\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})\prod_{a=1}^{4}H_1(X^{(i_a)}_{j_a})\right] = -\frac{k+1}{k(k-1)(k+2)}\alpha(\ell,k)\mathbb{1}_{I\in S}\mathbb{1}_{\ell\ge 2} \, ,$$

where $S=\{\{(i_1,j_1),(i_1,j_2),(i_2,j_1),(i_2,j_2)\}: i_1\neq i_2, j_1\neq j_2\}$.

Proof. These formulae follow when writing $H_4(x) = x^4 - 6x^2 + 3$, $H_3(x) = x^3 - 3x$, $H_2(x) = x^2 - 1$ and $H_1(x) = x$ and then combining the formulae for monomials proved in Lemma II.B.4 with Lemma II.A.5. We include the proof of (i):

$$\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})H_4(X_j^{(i)})\right] = \mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})(X_j^{(i)})^4\right] - 6\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})(X_j^{(i)})^2\right] + 3\mathbb{E}\left[\Phi^*_{\ell,k}(\mathbf{X})\right]$$
$$= 3\alpha(\ell,k)\frac{(k+1)(k+3)}{k(k+2)} - 6\left(\frac{1}{k}+1\right)\alpha(\ell,k) + 3\alpha(\ell,k) = -\frac{3}{k(k+2)}\alpha(\ell,k)\,,$$

where we used (III.3.49). The remaining formulae are proved in the same spirit. □

Appendix II.C On the two-point correlation function

II.C.1 Covariances

Fix $\ell \in [3]$ and $i \in [\ell]$. The following lemma gives the joint distribution of the vector $(\nabla T_n^{(i)}(z), \nabla T_n^{(i)}(0)) \in \mathbb{R}^6$ conditioned on $\left\{T_n^{(i)}(z) = T_n^{(i)}(0) = u_i\right\}$ for $u_i \in \mathbb{R}$ and $0 \neq z \in \mathbb{T}^3$.

Lemma II.C.1. *For every $z \in \mathbb{T}^3$ such that $r_n(z) \neq \pm 1$, the distribution of the vector $(\nabla T_n^{(i)}(z), \nabla T_n^{(i)}(0)) \in \mathbb{R}^6$ conditioned on $\left\{T_n^{(i)}(z) = T_n^{(i)}(0) = u_i\right\}$ is $\mathcal{N}_6(\mu_n^{(i)}, \Omega_n)$, where*

$$\mu_n^{(i)} = \mu_n^{(i)}(z) = \frac{u_i}{1 + r_n(z)}\begin{pmatrix}\nabla r_n(z)^T \\ -\nabla r_n(z)^T\end{pmatrix} \tag{II.C.1}$$

and

$$\Omega_n = \Omega_n(z) = \begin{pmatrix}\Omega_{1,n}(z) & \Omega_{2,n}(z) \\ \Omega_{2,n}(z)^T & \Omega_{1,n}(z)\end{pmatrix}\,, \tag{II.C.2}$$

where

$$\Omega_{1,n} = \Omega_{1,n}(z) = \frac{E_n}{3}\mathbf{I}_3 - \frac{\nabla r_n(z)\nabla r_n(z)^T}{1 - r_n(z)^2}\,;$$
$$\Omega_{2,n} = \Omega_{2,n}(z) = -\mathrm{Hess}(r_n(z)) + \frac{r_n(z)}{1 - r_n(z)^2}\nabla r_n(z)\nabla r_n(z)^T\,,$$

with $\mathrm{Hess}(r_n(z))$ *denoting the Hessian matrix of* $r_n(z)$.

Proof. We write $\partial_a := \partial/\partial_{z_a}$ and $\partial_{ab} := \partial^2/\partial_{z_a}\partial_{z_b}$ for $a, b = 0, 1, 2, 3$ with the convention $\partial_0 = \mathbf{I}$. Computing the covariance $\mathbb{E}\left[\partial_a T_n^{(i)}(z) \cdot \partial_b T_n^{(i)}(0)\right]$ and relating it to the covariance function r_n given in (II.1.2), we obtain that the covariance matrix of the vector $(\nabla T_n^{(i)}(z), \nabla T_n^{(i)}(0), T_n^{(i)}(z), T_n^{(i)}(0)) \in \mathbb{R}^8$ is given by

$$\begin{pmatrix}A_n & B_n \\ B_n^T & C_n\end{pmatrix}\,,$$

where

$$A_n = A_n(z) = \begin{pmatrix}E_n/3\,\mathbf{I}_3 & -\mathrm{Hess}(r_n(z)) \\ -\mathrm{Hess}(r_n(z)) & E_n/3\,\mathbf{I}_3\end{pmatrix}\,, \quad B_n = B_n(z) = \begin{pmatrix}\mathbf{0}^T & \nabla r_n(z)^T \\ -\nabla r_n(z)^T & \mathbf{0}^T\end{pmatrix}\,,$$
$$C_n = C_n(z) = \begin{pmatrix}1 & r_n(z) \\ r_n(z) & 1\end{pmatrix}\,,$$

and $\mathbf{0} := (0,0,0)$. Thus, the covariance matrix of $(\nabla T_n^{(i)}(z), \nabla T_n^{(i)}(0))$ conditioned on $\left\{T_n^{(i)}(z) = T_n^{(i)}(0) = u_i\right\}$ is given by $\Omega_n = \Omega_n(z) = A_n - B_n C_n^{-1} B_n^T$, which yields the matrix in (II.C.2) after a standard computation. Its mean is given by

$$\mu_n^{(i)} = \mu_n^{(i)}(z) = B_n C_n^{-1} \begin{pmatrix} u_i \\ u_i \end{pmatrix} = \frac{u_i}{1 + r_n(z)} \begin{pmatrix} \nabla r_n(z)^T \\ -\nabla r_n(z)^T \end{pmatrix} .$$

□

II.C.2 Two-point correlation function

For $\ell \in [3]$, we fix $u^{(\ell)} := (u_1, \dots, u_\ell) \in \mathbb{R}^\ell$. The two-point correlation function associated with the random field $\mathbf{T}_n^{(\ell)}$ is given by

$$K^{(\ell)}(x, y; u^{(\ell)}) := \mathbb{E}\left[\Phi_{\ell,3}^*(\mathrm{jac}_{\mathbf{T}_n^{(\ell)}}(x))\Phi_{\ell,3}^*(\mathrm{jac}_{\mathbf{T}_n^{(\ell)}}(y)) | \, \mathbf{T}_n^{(\ell)}(x) = \mathbf{T}_n^{(\ell)}(y) = u^{(\ell)}\right] \times p_{(\mathbf{T}_n^{(\ell)}(x), \mathbf{T}_n^{(\ell)}(y))}(u^{(\ell)}, u^{(\ell)}), \qquad \text{(II.C.3)}$$

where $p_{(\mathbf{T}_n^{(\ell)}(x), \mathbf{T}_n^{(\ell)}(y))}(\cdot,\cdot)$ denotes the density function of the vector $(\mathbf{T}_n^{(\ell)}(x), \mathbf{T}_n^{(\ell)}(y)) \in \mathbb{R}^{2\ell}$ and $\Phi_{\ell,3}^*(A) = \sqrt{\det(AA^T)}$ for $A \in \mathcal{M}_{\ell \times 3}(\mathbb{R})$. The function $K^{(\ell)}$ is defined whenever the distribution of $(\mathbf{T}_n^{(\ell)}(x), \mathbf{T}_n^{(\ell)}(y))$ is non-degenerate, that is, whenever $r_n(x-y) \neq \pm 1$.

The following lemma gives an upper bound for $K^{(\ell)}(z, 0; u^{(\ell)})$ for $z \in \mathbb{T}^3$ in terms of the covariance function r_n and the norm of its gradient.

Lemma II.C.2. *For every $z \in \mathbb{T}^3$ such that $r_n(z) \neq \pm 1$, we have*

$$\begin{aligned} K^{(\ell)}(z, 0; u^{(\ell)}) &\leq (1 - r_n(z)^2)^{-\ell/2} \cdot (3)_\ell \left(\frac{E_n}{3}\right)^{\ell-1} \left(\frac{E_n}{3} - \frac{\ell}{3} \frac{\|\nabla r_n(z)\|^2}{1 - r_n(z)^2} + \frac{\|u^{(\ell)}\|^2}{3} \frac{\|\nabla r_n(z)\|^2}{(1 + r_n(z))^2}\right) \\ &=: q^{(\ell)}(z, 0; \|u^{(\ell)}\|). \end{aligned} \qquad \text{(II.C.4)}$$

Proof. By independence, the density factorizes as follows

$$p_{(\mathbf{T}_n^{(\ell)}(z), \mathbf{T}_n^{(\ell)}(0))}(u^{(\ell)}, u^{(\ell)}) = \prod_{i=1}^{\ell} p_{(T_n^{(i)}(z), T_n^{(i)}(0))}(u_i, u_i) ,$$

and moreover satisfies

$$p_{(\mathbf{T}_n^{(\ell)}(z), \mathbf{T}_n^{(\ell)}(0))}(u^{(\ell)}, u^{(\ell)}) \leq \prod_{i=1}^{\ell} p_{(T_n^{(i)}(z), T_n^{(i)}(0))}(0,0) \leq (1 - r_n(z)^2)^{-\ell/2} . \qquad \text{(II.C.5)}$$

We now deal with the conditional expectation in (II.C.3). First, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\mathbb{E}\left[\Phi_{\ell,3}^*(\mathrm{jac}_{\mathbf{T}_n^{(\ell)}}(z))\Phi_{\ell,3}^*(\mathrm{jac}_{\mathbf{T}_n^{(\ell)}}(0)) | \, \mathbf{T}_n^{(\ell)}(z) = \mathbf{T}_n^{(\ell)}(0) = u^{(\ell)}\right] \\ &\leq \mathbb{E}\left[\Phi_{\ell,3}^*(\mathrm{jac}_{\mathbf{T}_n^{(\ell)}}(z))^2 | \, \mathbf{T}_n^{(\ell)}(z) = \mathbf{T}_n^{(\ell)}(0) = u^{(\ell)}\right]^{1/2} \cdot \mathbb{E}\left[\Phi_{\ell,3}^*(\mathrm{jac}_{\mathbf{T}_n^{(\ell)}}(0))^2 | \, \mathbf{T}_n^{(\ell)}(z) = \mathbf{T}_n^{(\ell)}(0) = u^{(\ell)}\right]^{1/2} . \end{aligned}$$

By symmetry, we conclude that the two expectations above coincide, yielding

$$\begin{aligned} &\mathbb{E}\left[\Phi_{\ell,3}^*(\mathrm{jac}_{\mathbf{T}_n^{(\ell)}}(z))\Phi_{\ell,3}^*(\mathrm{jac}_{\mathbf{T}_n^{(\ell)}}(0)) | \, \mathbf{T}_n^{(\ell)}(z) = \mathbf{T}_n^{(\ell)}(0) = u^{(\ell)}\right] \\ &\leq \mathbb{E}\left[\Phi_{\ell,3}^*(\mathrm{jac}_{\mathbf{T}_n^{(\ell)}}(z))^2 | \, \mathbf{T}_n^{(\ell)}(z) = \mathbf{T}_n^{(\ell)}(0) = u^{(\ell)}\right] =: \mathbb{E}\left[\Phi_{\ell,3}^*(X(z, u^{(\ell)}))^2\right], \end{aligned} \qquad \text{(II.C.6)}$$

where $X(z,u^{(\ell)}) = \left\{X_j^{(i)}(z,u^{(\ell)}) : (i,j) \in [\ell]\times[3]\right\} \in \mathcal{M}_{\ell\times 3}(\mathbb{R})$ is a random matrix having the same distribution as $\mathrm{jac}_{\mathbf{T}_n^{(\ell)}}(z)$ conditionally on $\left\{\mathbf{T}_n^{(\ell)}(z) = \mathbf{T}_n^{(\ell)}(0) = u^{(\ell)}\right\}$. Now, the Cauchy Binet formula (II.B.2) yields

$$\Phi^*_{\ell,3}(X(z,u^{(\ell)}))^2 = \sum_{j_1<\ldots<j_\ell\in[3]} \det\left(X(z,u^{(\ell)})_{j_1,\ldots,j_\ell}\right)^2 ,$$

where, as previously, $X(z,u^{(\ell)})_{j_1,\ldots,j_\ell}$ is the matrix obtained from $X(z,u^{(\ell)})$ by only keeping the columns labeled $j_1,\ldots,j_\ell$. By definition of the determinant, we have

$$\det\left(X(z,u^{(\ell)})_{j_1,\ldots,j_\ell}\right) = \sum_{\sigma\in\mathfrak{S}_\ell} \varepsilon(\sigma)\prod_{i=1}^{\ell} X^{(i)}_{j_{\sigma(i)}}(z,u^{(\ell)}) ,$$

where $\varepsilon(\sigma)$ denotes the signature of the permutation $\sigma\in\mathfrak{S}_\ell$. Then, developing the square, taking expectations and using independence,

$$\begin{aligned}
\mathbb{E}\left[\Phi^*_{\ell,3}(X(z,u^{(\ell)}))^2\right] &= \sum_{j_1<\ldots<j_\ell\in[3]} \mathbb{E}\left[\det\left(X(z,u^{(\ell)})_{j_1,\ldots,j_\ell}\right)^2\right]\\
&= \sum_{j_1<\ldots<j_\ell\in[3]}\ \sum_{\sigma,\sigma'\in\mathfrak{S}_\ell} \varepsilon(\sigma)\varepsilon(\sigma')\mathbb{E}\left[\prod_{i=1}^{\ell} X^{(i)}_{j_{\sigma(i)}}(z,u^{(\ell)})\cdot\prod_{l=1}^{\ell} X^{(l)}_{j_{\sigma'(l)}}(z,u^{(\ell)})\right]\\
&= \sum_{j_1<\ldots<j_\ell\in[3]}\ \sum_{\sigma,\sigma'\in\mathfrak{S}_\ell} \varepsilon(\sigma)\varepsilon(\sigma')\prod_{i=1}^{\ell}\mathbb{E}\left[X^{(i)}_{j_{\sigma(i)}}(z,u^{(\ell)})\cdot X^{(i)}_{j_{\sigma'(i)}}(z,u^{(\ell)})\right]. \qquad \text{(II.C.7)}
\end{aligned}$$

For notational ease, we write

$$\mathbf{E}^{(i)}_{\ell,ab} = \mathbf{E}^{(i)}_{\ell,ab}(z,u^{(\ell)}) := \mathbb{E}\left[X^{(i)}_a(z,u^{(\ell)})X^{(i)}_b(z,u^{(\ell)})\right] , \quad i\in[\ell], a,b\in[3] .$$

Exploiting once more the independence of the fields $T_n^{(1)},\ldots,T_n^{(\ell)}$, we have that

$$\mathbf{E}^{(i)}_{\ell,ab} = \mathbb{E}\left[\partial_a T_n^{(i)}(z)\partial_b T_n^{(i)}(z)|\,\mathbf{T}_n^{(\ell)}(z) = \mathbf{T}_n^{(\ell)}(0) = u^{(\ell)}\right] = \mathbb{E}\left[\partial_a T_n^{(i)}(z)\partial_b T_n^{(i)}(z)|T_n^{(i)}(z) = T_n^{(i)}(0) = u_i\right].$$

Writing formula (II.C.7) for $\ell = 1,2,3$ gives the respective relations

$$\mathbb{E}\left[\Phi^*_{1,3}(X(z,u^{(1)}))^2\right] = \sum_{a\in[3]}\mathbf{E}^{(1)}_{1,aa} , \qquad \text{(II.C.8)}$$

$$\mathbb{E}\left[\Phi^*_{2,3}(X(z,u^{(2)}))^2\right] = \sum_{a\neq b\in[3]}\left\{\mathbf{E}^{(1)}_{2,aa}\mathbf{E}^{(2)}_{2,bb} - \mathbf{E}^{(1)}_{2,ab}\mathbf{E}^{(2)}_{2,ab}\right\} \qquad \text{(II.C.9)}$$

and

$$\begin{aligned}
\mathbb{E}\left[\Phi^*_{3,3}(X(z,u^{(3)}))^2\right] = \sum_{a\neq b\neq c\neq a\in[3]}\Big(&\mathbf{E}^{(1)}_{3,aa}\mathbf{E}^{(2)}_{3,bb}\mathbf{E}^{(3)}_{3,cc} \qquad \text{(II.C.10)}\\
&-\left(\mathbf{E}^{(1)}_{3,cc}\mathbf{E}^{(2)}_{3,ab}\mathbf{E}^{(3)}_{3,ab} + \mathbf{E}^{(1)}_{3,ab}\mathbf{E}^{(2)}_{3,cc}\mathbf{E}^{(3)}_{3,ab} + \mathbf{E}^{(1)}_{3,ab}\mathbf{E}^{(2)}_{3,ab}\mathbf{E}^{(3)}_{3,cc}\right) + 2\mathbf{E}^{(1)}_{3,ab}\mathbf{E}^{(2)}_{3,bc}\mathbf{E}^{(3)}_{3,ac}\Big) .
\end{aligned}$$

We will now provide an explicit expression for the formulae on the right hand side of (II.C.8), (II.C.9) and (II.C.10). For $z = (z_1,z_2,z_3)\in\mathbb{T}^3$ and $(a,b)\in[3]\times[3]$, we use the shorthand notations

$$\partial_a r_n(z) := \frac{\partial}{\partial z_a} r_n(z) ; \quad \partial_{ab} r_n(z) := \frac{\partial^2}{\partial z_a\partial z_b} r_n(z)$$

and

$$\rho_{ab} = \rho_{n,ab}(z) := \frac{\partial_a r_n(z) \cdot \partial_b r_n(z)}{1 - r_n(z)^2} \; ; \quad \mu_{ab} = \mu_{n,ab}(z) := \frac{\partial_a r_n(z) \cdot \partial_b r_n(z)}{(1 + r_n(z))^2} \; .$$

Note that

$$\rho_{ab}^2 = \rho_{aa}\rho_{bb} \; , \quad \mu_{ab}^2 = \mu_{aa}\mu_{bb} \; , \quad \rho_{aa}\mu_{bb} = \rho_{ab}\mu_{ab} \; . \tag{II.C.11}$$

From Lemma II.C.1, it follows that for every $i \in [\ell]$ and $(a,b) \in [3] \times [3]$,

$$\mathbf{E}^{(i)}_{\ell,aa} = \mathbf{Var}\left[X^{(i)}_a(z, u^{(\ell)})\right] + \mathbb{E}\left[X^{(i)}_a(z, u^{(\ell)})\right]^2 = \frac{E_n}{3} - \rho_{aa} + u_i^2 \mu_{aa} \tag{II.C.12}$$

and for $a \neq b$,

$$\begin{aligned} \mathbf{E}^{(i)}_{\ell,ab} &= \mathbf{Cov}\left[X^{(i)}_a(z, u^{(\ell)}), X^{(i)}_b(z, u^{(\ell)})\right] + \mathbb{E}\left[X^{(i)}_a(z, u^{(\ell)})\right] \mathbb{E}\left[X^{(i)}_b(z, u^{(\ell)})\right] \\ &= -\rho_{ab} + u_i^2 \mu_{ab} \; . \end{aligned} \tag{II.C.13}$$

Then, it is immediate that

$$\mathbb{E}\left[\Phi^*_{1,3}(X(z, u^{(1)}))^2\right] = \sum_{a \in [3]} \mathbf{E}^{(1)}_{1,aa} = \sum_{a \in [3]} \left\{ \frac{E_n}{3} - \rho_{aa} + u_1^2 \mu_{aa} \right\} \; .$$

Similarly, using (II.C.12) and (II.C.13) in (II.C.9) and (II.C.10) and exploiting the identities in (II.C.11) yields after simplifications

$$\begin{aligned} &\mathbb{E}\left[\Phi^*_{2,3}(X(z, u^{(2)}))^2\right] \\ = &\sum_{a \neq b \in [3]} \left\{ \left(\frac{E_n}{3} - \rho_{aa} + u_1^2 \mu_{aa}\right)\left(\frac{E_n}{3} - \rho_{bb} + u_2^2 \mu_{bb}\right) - \left(-\rho_{ab} + u_1^2 \mu_{ab}\right)\left(-\rho_{ab} + u_2^2 \mu_{ab}\right) \right\} \\ = &\sum_{a \neq b \in [3]} \left\{ \left(\frac{E_n}{3}\right)^2 - \frac{E_n}{3}\left(\rho_{aa} + \rho_{bb}\right) + \frac{E_n}{3} u_1^2 \mu_{aa} + \frac{E_n}{3} u_2^2 \mu_{bb} \right\} \; . \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}\left[\Phi^*_{3,3}(X(z, u^{(3)}))^2\right] = &\sum_{a \neq b \neq c \neq a \in [3]} \left(\left(\frac{E_n}{3}\right)^3 - \left(\frac{E_n}{3}\right)^2 (\rho_{aa} + \rho_{bb} + \rho_{cc}) \right. \\ &\left. + \left(\frac{E_n}{3}\right)^2 u_1^2 \mu_{aa} + \left(\frac{E_n}{3}\right)^2 u_2^2 \mu_{bb} + \left(\frac{E_n}{3}\right)^2 u_3^2 \mu_{cc} \right) \end{aligned}$$

respectively. Then, we note that for every $\ell \in [3]$, writing

$$\Delta_\ell := \left\{ i^{(\ell)} = (i_1, \ldots, i_\ell) \in [3]^\ell : i_a \neq i_b, \forall a \neq b \in [\ell] \right\},$$

the following identities hold

$$\begin{aligned} \sum_{i^{(\ell)} \in \Delta_\ell} 1 &= \mathrm{card}(\Delta_\ell) = (3)_\ell \; ; \\ \sum_{i^{(\ell)} \in \Delta_\ell} (\rho_{i_1 i_1} + \ldots + \rho_{i_\ell i_\ell}) &= \ell \sum_{i^{(\ell)} \in \Delta_\ell} \rho_{i_1 i_1} = \ell \frac{(3)_\ell}{3} \frac{\|\nabla r_n(z)^2\|}{1 - r_n(z)^2} \; ; \\ \sum_{i^{(\ell)} \in \Delta_\ell} (u_1^2 \mu_{i_1 i_1} + \ldots + u_\ell^2 \mu_{i_\ell i_\ell}) &= u_1^2 \left[\sum_{i^{(\ell)} \in \Delta_\ell} \mu_{i_1 i_1} \right] + \ldots + u_\ell^2 \left[\sum_{i^{(\ell)} \in \Delta_\ell} \mu_{i_\ell i_\ell} \right] \end{aligned}$$

$$= (u_1^2 + \ldots + u_\ell^2) \sum_{i^{(\ell)} \in \Delta_\ell} \mu_{i_1 i_1} = \|u^{(\ell)}\|^2 \frac{(3)_\ell}{3} \frac{\|\nabla r_n(z)\|^2}{(1 + r_n(z))^2} .$$

Using these identities, (II.C.8), (II.C.9) and (II.C.10) finally reduce to

$$\begin{aligned} \mathbb{E}\left[\Phi_{\ell,3}(X(z, u^{(\ell)}))^2\right] &= (3)_\ell \left(\frac{E_n}{3}\right)^\ell - \left(\frac{E_n}{3}\right)^{\ell-1} \ell \frac{(3)_\ell}{3} \frac{\|\nabla r_n(z)\|^2}{1 - r_n(z)^2} + \left(\frac{E_n}{3}\right)^{\ell-1} \|u^{(\ell)}\|^2 \frac{(3)_\ell}{3} \frac{\|\nabla r_n(z)\|^2}{(1 + r_n(z))^2} \\ &= (3)_\ell \left(\frac{E_n}{3}\right)^{\ell-1} \left(\frac{E_n}{3} - \frac{\ell}{3} \frac{\|\nabla r_n(z)\|^2}{1 - r_n(z)^2} + \frac{\|u^{(\ell)}\|^2}{3} \frac{\|\nabla r_n(z)\|^2}{(1 + r_n(z))^2}\right) . \end{aligned} \tag{II.C.14}$$

Plugging the bounds obtained in (II.C.5) and (II.C.2) into (II.C.3) yields the desired upper bound for the two-point correlation function in (II.C.4). □

Lemma II.C.3. *For every fixed* $(x, y) \in \mathbb{T}^3 \times \mathbb{T}^3$ *such that* $r_n(x - y) \neq \pm 1$*, the function* $u^{(\ell)} := (u_1, \ldots, u_\ell) \mapsto K^{(\ell)}(x, y; u^{(\ell)})$ *is continuous.*

Proof. Denoting by $\Sigma = \Sigma(x - y)$ the covariance matrix of the vector $(T_n^{(i)}(x), T_n^{(i)}(y))$ for $i \in [\ell]$, the Gaussian density is given by

$$\begin{aligned} p_{(\mathbf{T}_n^{(\ell)}(x), \mathbf{T}_n^{(\ell)}(y))}(u^{(\ell)}, u^{(\ell)}) &= \left(\frac{1}{2\pi \sqrt{1 - r_n(x - y)^2}}\right)^\ell \prod_{i=1}^{\ell} \exp\left\{-\frac{1}{2}(u_i, u_i)^T \Sigma^{-1} (u_i, u_i)\right\} \\ &= \left(\frac{1}{2\pi \sqrt{1 - r_n(x - y)^2}}\right)^\ell \prod_{i=1}^{\ell} \exp\left\{-\frac{u_i^2}{2(1 + r_n(x - y))}\right\} , \end{aligned}$$

which is a continuous function of $u^{(\ell)}$. We will now argue that the conditional expectation appearing in (II.C.3) is a continuous function of $u^{(\ell)}$. It can be rewritten as

$$\mathbb{E}\left[\Phi_{\ell,3}^*(\mathrm{jac}_{\mathbf{T}_n^{(\ell)}}(x)) \Phi_{\ell,3}^*(\mathrm{jac}_{\mathbf{T}_n^{(\ell)}}(y)) | \, \mathbf{T}_n^{(\ell)}(x) = \mathbf{T}_n^{(\ell)}(y) = u^{(\ell)}\right] = \mathbb{E}\left[\Phi_{\ell,3}^*(X(x, u^{(\ell)})) \Phi_{\ell,3}^*(X(y, u^{(\ell)}))\right],$$

where, for every $x \in \mathbb{T}^3$, the random $\ell \times 3$ matrix $X(x, u^{(\ell)}) = \left\{X_j^{(i)}(x, u^{(\ell)}) : (i, j) \in [\ell] \times [3]\right\}$ has the same distribution as $\mathrm{jac}_{\mathbf{T}_n^{(\ell)}}(x)$ conditionally on $\left\{\mathbf{T}_n^{(\ell)}(x) = \mathbf{T}_n^{(\ell)}(y) = u^{(\ell)}\right\}$. From Lemma II.C.1, it follows that the mean in (II.C.1) depends linearly on $u^{(\ell)}$. In view of the definition of $\Phi_{\ell,3}^*$, and the structure of the covariance function in (II.C.2), we conclude that the above expected value is also a continuous function of $u^{(\ell)}$, showing that $K^{(\ell)}(x, y; u^{(\ell)})$ is a continuous function with variable $u^{(\ell)}$. □

II.C.3 Taylor expansions

We compute an expansion of $q^{(\ell)}(z, 0; \|u^{(\ell)}\|)$ in (II.C.4) around $z = 0$. In order to do so, we start by deriving the Taylor expansions of r_n and its first-order partial derivatives near $z = 0$. For $n \in S_3$, let

$$\Psi_n := \frac{1}{n^2 \mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \lambda_k^4 , \qquad k = 1, 2, 3 . \tag{II.C.15}$$

and set $e_n := E_n/3$. Note that $\Psi_n \leq 1$ since $\lambda_k^4 \leq n^2$.

Lemma II.C.4. *For* $z = (z_1, z_2, z_3) \in \mathbb{T}^3$ *and every* $k \in [3]$*, the following Taylor expansions hold near* $z = 0$*:*

$$r_n(z) = 1 - \frac{E_n}{6}\|z\|^2 + \frac{E_n^2}{24} \Psi_n \sum_{j=1}^{3} z_j^4 + \frac{E_n^2}{4}\left(\frac{1}{6} - \frac{1}{2}\Psi_n\right) \sum_{i<j \in [3]} z_i^2 z_j^2 + R_n^{(0)}$$

$$=: \quad 1 - \frac{e_n}{2}\|z\|^2 + t_n(z) + R_n^{(0)} \tag{II.C.16}$$

$$\partial_k r_n(z) \quad = \quad -\frac{E_n}{3} z_k + \frac{E_n^2}{6}\Psi_n \sum_{j=1}^{3} z_j^3 + \frac{E_n^2}{2}\left(\frac{1}{6} - \frac{1}{2}\Psi_n\right) \sum_{i\neq j\in[3]} z_j z_i^2 + R_n^{(k)}$$

$$=: \quad -e_n z_k + u_{n,k}(z) + R_n^{(k)}\,, \tag{II.C.17}$$

where $R_n^{(0)} = E_n^3 O(\|z\|^6)$ and $R_n^{(k)} = E_n^3 O(\|z\|^5)$, and the constants involved in the 'big-O' notation are independent of n.

Proof. These expansions follow from direct computations of partial derivatives. Note that all derivatives of odd (resp. even) order of r_n (resp. $\partial_k r_n$) vanish in view of the fact that, by symmetry, $\sum_{\lambda\in\Lambda_n} \lambda_j^\alpha$ is zero whenever α is odd. Also, we note that

$$\frac{1}{n^2 \mathcal{N}_n} \sum_{\lambda\in\Lambda_n} \lambda_a^2 \lambda_b^2 = \frac{1}{6} - \frac{1}{2}\Psi_n$$

for $a \neq b \in [3]$, where Ψ_n is as in (II.C.15). The remainders are of the form $R_n^{(0)} = O(\|\partial^6 r_n\|_\infty \|z\|^6)$ and $R_n^{(k)} = O(\|\partial^6 r_n\|_\infty \|z\|^5)$, where

$$\partial^6 r_n := \sup_{i_1,\dots,i_6\in[3]} \partial_{i_1,\dots,i_6} r_n$$

and $\partial_{i_1,\dots,i_6} r_n(z)$ denotes partial derivatives of r_n of cumulative order equal to 6. Observe that for every $z \in \mathbb{T}^3$,

$$\left|\partial^6 r_n(z)\right| \leq \frac{(2\pi)^6}{\mathcal{N}_n} \sum_{\lambda\in\Lambda_n} \lambda_1^\alpha \lambda_2^\beta \lambda_3^\gamma\,,$$

where α, β, γ are non-negative even integers such that $\alpha+\beta+\gamma = 6$. Therefore, we can write $\lambda_1^\alpha \lambda_2^\beta \lambda_3^\gamma = \lambda_a^2 \lambda_b^2 \lambda_c^2$ for $a, b, c \in \{1, 2, 3\}$ not necessarily distinct. Then it follows that $\lambda_a^2\lambda_b^2\lambda_c^2 \leq \lambda_a^6/3 + \lambda_b^6/3 + \lambda_c^6/3$, so that

$$\left|\partial^6 r_n(z)\right| \leq \frac{(2\pi)^6}{\mathcal{N}_n} \sum_{\lambda\in\Lambda_n} \lambda_1^6 \leq \frac{(2\pi)^6}{\mathcal{N}_n} \sum_{\lambda\in\Lambda_n} (\lambda_1^2+\lambda_2^2+\lambda_3^2)^3 = (2\pi)^6 n^3 \leq E_n^3\,,$$

which concludes the proof. □

The following result contains the expansion around zero of $q^{(\ell)}(z, 0; \|u^{(\ell)}\|)$. In particular, we remark a singularity in the coefficient of $\|z\|^{-\ell}$ in the case $\ell = 3$, which is consistent with the fact that the mapping $z \mapsto \|z\|^{-3}$ is not integrable on $\mathbb{T}^3$.

Lemma II.C.5. *For $\ell \in [3]$, as $\|z\| \to 0$, we have*

$$q^{(\ell)}(z, 0; \|u^{(\ell)}\|) = (3)_\ell \left(1 - \frac{\ell}{3}\right) e_n^{\ell/2} \|z\|^{-\ell} + (3)_\ell \left(1 + \|u^{(\ell)}\|^2\right) E_n^{\ell/2+1} O(\|z\|^{2-\ell}), \tag{II.C.18}$$

where the constants involved in the 'big-O' notation are independent of n.

Proof. From the expansion in (II.C.16) we obtain that

$$\begin{aligned} 1 - r_n(z)^2 &= (1 - r_n(z))(1 + r_n(z)) \\ &= \left(\frac{e_n}{2}\|z\|^2 - t_n(z) + E_n^3 O(\|z\|^6)\right)\left(2 - \frac{e_n}{2}\|z\|^2 + t_n(z) + E_n^3 O(\|z\|^6)\right) \\ &= e_n\|z\|^2 - \left[(\frac{e_n}{2})^2\|z\|^4 + 2t_n(z)\right] + E_n^3 O(\|z\|^6) \end{aligned}$$

$$=: \quad e_n\|z\|^2 - f_n(z) + E_n^3 O(\|z\|^6)\,, \tag{II.C.19}$$

and

$$\begin{aligned}(1+r_n(z))^2 &= \left(2 - \frac{e_n}{2}\|z\|^2 + t_n(z) + E_n^3O(\|z\|^6)\right)^2\\ &= 4 - 2e_n\|z\|^2 + \left[(\frac{e_n}{2})^2\|z\|^4 + 4t_n(z)\right] + E_n^3O(\|z\|^6)\\ &=: 4 - 2e_n\|z\|^2 + h_n(z) + E_n^3O(\|z\|^6)\,,\end{aligned} \tag{II.C.20}$$

where $t_n(z)$ is as in (II.C.16). Note that since $\Psi_n \le 1$, we have $t_n(z) = E_n^2O(\|z\|^4)$ where the constant in the big-O notation is independent of n. Therefore, we have $f_n(z) := (e_n/2)^2\|z\|^4 + 2t_n(z) = E_n^2O(\|z\|^4)$ and $h_n(z) := (e_n/2)^2\|z\|^4 + 4t_n(z) = E_n^2O(\|z\|^4)$. From (II.C.17), we have

$$\partial_k r_n(z)^2 = \left(-e_n z_k + u_{n,k}(z) + E_n^3O(\|z\|^5)\right)^2 = e_n^2 z_k^2 - 2e_n z_k u_{n,k}(z) + E_n^4O(\|z\|^6)\,,$$

so that summing over $k = 1, 2, 3$ leads to

$$\|\nabla r_n(z)\|^2 = e_n^2\|z\|^2 - 2e_n\sum_{k=1}^{3} z_k u_{n,k}(z) + E_n^4O(\|z\|^6) =: e_n^2\|z\|^2 + g_n(z) + E_n^4O(\|z\|^6)\,, \tag{II.C.21}$$

where $g_n(z) = E_n^3O(\|z\|^4)$ and again the constant in the big-O notation does not depend on n. Hence we obtain the expansions of the quotient

$$\begin{aligned}\frac{\|\nabla r_n(z)\|^2}{1-r_n(z)^2} &= \frac{e_n^2\|z\|^2 + g_n(z) + E_n^4O(\|z\|^6)}{e_n\|z\|^2 - f_n(z) + E_n^3O(\|z\|^6)}\\ &= e_n\frac{1 + e_n^{-2}\|z\|^{-2}g_n(z) + E_n^2O(\|z\|^4)}{1 - e_n^{-1}\|z\|^{-2}f_n(z) + E_n^2O(\|z\|^4)}\\ &= e_n\left(1 + \frac{g_n(z)}{e_n^2\|z\|^2} + E_n^2O(\|z\|^4)\right)\left(1 + \frac{f_n(z)}{e_n\|z\|^2} + E_n^2O(\|z\|^4)\right)\\ &= e_n\left(1 + \frac{g_n(z)}{e_n^2\|z\|^2} + \frac{f_n(z)}{e_n\|z\|^2} + E_n^2O(\|z\|^4)\right)\\ &= e_n + \frac{g_n(z)}{e_n\|z\|^2} + \frac{f_n(z)}{\|z\|^2} + E_n^3O(\|z\|^4) = e_n + E_n^2O(\|z\|^2),\end{aligned} \tag{II.C.22}$$

since $e_n^{-1}\|z\|^{-2}g_n(z) + \|z\|^{-2}f_n(z) = E_n^2O(\|z\|^2)$ and similar computations yield

$$\frac{\|\nabla r_n(z)\|^2}{(1+r_n(z))^2} = \left(\frac{e_n}{2}\right)^2\|z\|^2 + E_n^4O(\|z\|^4)\,. \tag{II.C.23}$$

Combining (II.C.22) and (II.C.23), the expansion around zero of $\mathbb{E}\left[\Phi^*_{\ell,3}(X(z,u^{(\ell)}))^2\right]$ is then obtained from (II.C.2):

$$\begin{aligned}\mathbb{E}\left[\Phi_{\ell,3}(X(z,u^{(\ell)}))^2\right] &= (3)_\ell e_n^{\ell-1}\left(e_n - \frac{\ell}{3}\frac{\|\nabla r_n(z)\|^2}{1-r_n(z)^2} + \frac{\|u^{(\ell)}\|^2}{3}\frac{\|\nabla r_n(z)\|^2}{(1+r_n(z))^2}\right)\\ &= (3)_\ell e_n^{\ell-1}(e_n - \frac{\ell}{3}\left(e_n + E_n^2O(\|z\|^2)\right) + \frac{\|u^{(\ell)}\|^2}{3}\left\{\left(\frac{e_n}{2}\right)^2\|z\|^2 + E_n^4O(\|z\|^4)\right\})\\ &= (3)_\ell e_n^{\ell-1}\left(e_n\left(1-\frac{\ell}{3}\right) + \left(1 + \|u^{(\ell)}\|^2\right)E_n^2O(\|z\|^2)\right)\end{aligned}$$

$$= \quad (3)_\ell \left(1 - \frac{\ell}{3}\right) e_n^\ell + (3)_\ell \left(1 + \|u^{(\ell)}\|^2\right) E_n^{\ell+1} O(\|z\|^2). \tag{II.C.24}$$

Then, using $1 - r_n(z)^2 = e_n \|z\|^2 (1 + E_n O(\|z\|^2))$ gives

$$\begin{aligned} & q^{(\ell)}(z, 0; \|u^{(\ell)}\|) = (1 - r_n(z)^2)^{-\ell/2} \cdot \mathbb{E}\left[\Phi_{\ell,3}(X(z, u^{(\ell)}))^2\right] \\ = \quad & e_n^{-\ell/2} \|z\|^{-\ell} \mathbb{E}\left[\Phi_{\ell,3}(X(z, u^{(\ell)}))^2\right] (1 + E_n O(\|z\|^2)) \\ = \quad & (3)_\ell \left(1 - \frac{\ell}{3}\right) e_n^{\ell/2} \|z\|^{-\ell} + (3)_\ell \left(1 + \|u^{(\ell)}\|^2\right) E_n^{\ell/2+1} O(\|z\|^{2-\ell}), \end{aligned}$$

which has the desired form. □

The following lemma justifies the use of Kac-Rice formulae in a sufficiently small cube around the origin, Q_0.

Lemma II.C.6. *For every $n \in S_3$, there exists a sufficiently small constant $c_0 > 0$ such that for every $(x, y) \in \mathbb{T}^3 \times \mathbb{T}^3$ satisfying $0 < \|x - y\| < c_0/\sqrt{E_n}$, we have $r_n(x - y) \neq \pm 1$.*

Proof. We set $z = x - y$ and perform a Taylor expansion of $1 - r_n(z)^2$ around $z = 0$. From (II.C.19), we have

$$1 - r_n(z)^2 = \frac{E_n}{3} \|z\|^2 + E_n^2 O(\|z\|^4) = \frac{E_n}{3} \|z\|^2 (1 + E_n O(\|z\|^2)) \ .$$

Thus, for every $0 < \|z\| \ll 1/\sqrt{E_n}$, we obtain

$$1 - r_n(z)^2 = \frac{E_n}{3} \frac{C^2}{E_n} (1 + O(1)) = \frac{C^2}{3} (1 + O(1)),$$

for some absolute constant $C > 0$, so that there exists a sufficiently small constant $c_0 > 0$ such that $1 - r_n(z)^2 > 0$ for every $0 < \|z\| < c_0/\sqrt{E_n}$. □

Appendix II.D Continuity of nodal volumes

In this section, we prove a more general version of the continuity theorem proved in Theorem 3 of [APP18]. Our version applies to vector-valued functions on the torus. For completeness, we give the arguments for the d-dimensional torus $\mathbb{T}^d$, $d \geq 2$. Recall that $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d \simeq [0,1]^d/_\sim$, where $\sim$ denotes the equivalence relation given by $(x_1, \ldots, x_d) \sim (x_1', \ldots, x_d')$ if and only if $x_i - x_i' \in \mathbb{Z}$ for every $i = 1, \ldots, d$. Let us introduce some notation.

Topology on $\mathbb{T}^d$. (see e.g. [Sha14]) Denote by $\pi_d : [0,1]^d \to \mathbb{T}^d$ the quotient map associated with $\sim$. We endow the torus with the quotient topology, that is, the open (closed) subsets of $\mathbb{T}^d$ are precisely the subsets $U \subset \mathbb{T}^d$ such that $\pi_d^{-1}(U) \subset [0,1]^d$ are open (closed) in $[0,1]^d$ for the Euclidean topology. Moreover, we equip the torus with the quotient metric given by

$$\mathrm{dist}_d(\pi_d(x), \pi_d(x')) = \inf_{a \in \mathbb{Z}^d} \|x - x' + a\|_d \ , \quad x, x' \in [0,1]^d \ ,$$

where $\|\cdot\|_d$ denotes the standard Euclidean norm in $\mathbb{R}^d$. From now on, we will write x instead of $\pi_d(x)$ for a point on the torus. Since the equivalence relation $\sim$ is defined coordinate-wise, we will implicitly use the fact that the $\mathbb{T}^d$ is a realisation of the cartesian product of d copies of $\mathbb{T}^1$.

Banach space of continuous functions on $\mathbb{T}^d$. For integers $1 \le k < d$, let $E = C^1(\mathbb{T}^d, \mathbb{R}^k)$ be the set of C^1 real vector-valued functions on $\mathbb{T}^d$. Then, for a compact space $K \subset \mathbb{T}^d$ (note that a compact subset on the torus has the form $\pi_d(\tilde{K})$ for some compact $\tilde{K} \subset [0,1]^d$), and $F = (F^{(1)}, \dots, F^{(k)}) \in E$, we define the norm

$$\|F\|_K := \max_{i=1,\dots,k} \sup_{x\in K} \left(|F^{(i)}(x)| + \sum_{j=1}^d |\partial_j F^{(i)}(x)| \right).$$

We will use the following version of the Implicit Function Theorem for Banach spaces (see e.g. [Edw94, p.417]).

Lemma II.D.1 (Implicit Function Theorem for Banach spaces)**.** *Let X, Y, Z be Banach spaces and $f : X \times Y \to Z$ be a function of class C^1. Let $(x_0, y_0) \in X \times Y$ such that $f(x_0, y_0) = 0$ and $(d_y f)_{(x_0,y_0)} : Y \to Z$ is an isomorphism. Then there exist neighborhoods $U(x_0) \subset X$ of x_0 and $U(x_0, y_0) \subset X \times Y$ of (x_0, y_0) and a function $g : U(x_0) \to Z$ of class C^1 such that*

$$((x,y) \in U(x_0,y_0), x \in U(x_0)) \Rightarrow (f(x,y) = 0 \iff y = g(x)).$$

Here $(d_y f)_{(x_0,y_0)}$ denotes the partial differential of f with respect to $y \in Y$ computed at (x_0, y_0).

Some notation. For $F \in E$, let $Z_K(F)$ be the set of zeros of F lying in the compact $K \subset \mathbb{T}^d$, i.e. $Z_K(F) = \{x \in K : F(x) = 0\}$. We denote by $\mathrm{vol}(Z_K(F)) := \mathcal{H}_{d-k}(Z_K(F))$ the $(d-k)$-dimensional Hausdorff measure of $Z_K(F)$. As usual, we write $\mathrm{jac}_F(x) \in \mathcal{M}_{k\times d}(\mathbb{R})$ to indicate the Jacobian matrix of F computed at x. We introduce the set $D_k := \{J \subset [d] : \mathrm{card}(J) = k\}$, that is, the set of all subsets of $[d]$ that have cardinality k. For $J \in D_k$ and $x \in \mathbb{T}^d$, we denote $x_J := (x_l : l \in J)$ and $p_J(x) := \hat{x}_J := (x_l : l \notin J)$. For x_J as just defined, we write jac_{F,x_J} for the $k \times k$ Jacobian matrix obtained when differentiating with respect to the variable x_J. We say that F is non-degenerate on K if $\mathrm{jac}_F(x_0)$ has full rank k whenever $x_0 \in Z_F(K)$, that is, whenever there exists $J = J(x_0) \in D_k$ such that $\mathrm{jac}_{F,x_J}(x_0)$ is invertible.

We first prove the following lemma, adapted from [APP18].

Lemma II.D.2. *Let $(F_n)_{n\ge1} \subset E$ and $F \in E$ be such that $F_n \to F$ in the C^1 topology on $K \subset \mathbb{T}^d$ as $n \to \infty$. Then, for n sufficiently large and for every $\varepsilon > 0$, we have that $Z_K(F_n) \subset Z_K^{+\varepsilon}(F)$, where*

$$Z_K^{+\varepsilon}(F) := \{x \in K : \mathrm{dist}_d(x, Z_K(F)) \le \varepsilon\}.$$

Proof. We proceed by contradiction. Assume that there exists $\varepsilon > 0$ such that $Z_K(F_n)$ is not a subset of $Z_K^{+\varepsilon}(F)$ for n big enough, i.e. such that for every $N \ge 1$, there exists $n \ge N$ and $x_n \in Z_K(F_n)$ with $\mathrm{dist}_d(x_n, Z_K(F)) > \varepsilon$. As $(x_n)_{n\ge N} \subset K$ and K is compact, we can extract a converging subsequence $(x_{n_j})_{j\ge1}$; denote $x_\infty := \lim_j x_{n_j} \in K$ and note that $\mathrm{dist}_d(x_\infty, Z_K(F)) > \varepsilon$ by assumption. Then, using the triangular inequality, we have for every $j \ge 1$,

$$\begin{aligned} \|F(x_\infty)\|_k &= \|F(x_\infty) - F_{n_j}(x_{n_j})\|_k \le \sum_{i=1}^k |F^{(i)}(x_\infty) - F_{n_j}^{(i)}(x_{n_j})| \\ &\le \sum_{i=1}^k |F^{(i)}(x_\infty) - F_{n_j}^{(i)}(x_\infty)| + \sum_{i=1}^k |F_{n_j}^{(i)}(x_\infty) - F_{n_j}^{(i)}(x_{n_j})| \\ &\le k \cdot \|F - F_{n_j}\|_K + \lambda \cdot \mathrm{dist}_d(x_{n_j}, x_\infty), \end{aligned} \tag{II.D.1}$$

where

$$\lambda := \sum_{i=1}^k \sum_{l=1}^d \sup_{x\in K} |\partial_l F_{n_j}^{(i)}(x)| \le k \cdot \max_{i=1,\dots,k} \sum_{l=1}^d \sup_{x\in K} |\partial_l F_{n_j}^{(i)}(x)| \le k \cdot \|F_{n_j}\|_K < \infty,$$

because $(F_n)_{n\ge1} \subset E$. Letting $j \to \infty$ in (II.D.1) leads to $F(x_\infty) = 0$, since $F_{n_j} \to F$ in the C^1 topology on K and $x_{n_j} \to x_\infty$. Hence $x_\infty \in Z_K(F)$, but this contradicts the fact that $\mathrm{dist}_d(x_\infty, Z_K(F)) \ge \varepsilon > 0$. □

We now prove the continuity result about nodal volumes. The strategy of our proof is inspired by the proof in [APP18].

Theorem II.D.3 (Continuity of the nodal volume)**.** *Let $(F_n)_{n\geq 1} \subset E$ and $F \in E$ be such that F is non-degenerate on a compact $K \subset \mathbb{T}^d$ and $F_n \to F$ in the C^1 topology on K as $n \to \infty$. Then, as $n \to \infty$,*

$$\mathrm{vol}(Z_K(F_n)) \to \mathrm{vol}(Z_K(F)) .$$

Proof. Denote by $\phi : E \times \mathbb{T}^d \to \mathbb{R}^k$ the evaluation map $\phi(f, x) := f(x)$. Since F is non-degenerate, for all $x_0 \in K$ such that $\phi(F, x_0) = 0$, there exists $J_0 = J_0(x_0) \in D_k$ such that $\mathrm{jac}_{F,x_{J_0}}(x_0)$ is invertible, that is, the linear map $(d_{x_{J_0}}\phi)_{(F,x_0)} : \mathbb{T}^k \to \mathbb{R}^k$ is an isomorphism. Therefore, by the Implicit Function Theorem stated in Lemma II.D.1, there exist open neighborhoods $U(F) \subset E$ of F, $U((x_0)_{J_0}) \subset \mathbb{T}^k$ of $(x_0)_{J_0}$ and $U((\hat{x_0})_{J_0}) \subset \mathbb{T}^{d-k}$ of $(\hat{x_0})_{J_0}$ as well as a function $X_0 : E \times \mathbb{T}^{d-k} \to \mathbb{R}^k$ of class C^1 such that

$$(f \in U(F), x_{J_0} \in U((x_0)_{J_0}), \hat{x}_{J_0} \in U((\hat{x_0})_{J_0})) \Rightarrow (\phi(f, x) = 0 \iff x_{J_0} = X_0(f, \hat{x}_{J_0})). \tag{II.D.2}$$

Now denote $W_0 = W_0(J_0) \subset \mathbb{T}^d$ the set of points of $x \in \mathbb{T}^d$ such that $x_{J_0} \in U((\hat{x_0})_{J_0})$ and $\hat{x}_{J_0} \in U((\hat{x_0})_{J_0})$. Then, choosing $f = F$ in (II.D.2), we obtain that $Z_K(F)$ restricted to W_0 is the $(d-k)$-dimensional submanifold of $\mathbb{T}^d$

$$Z_K(F) \cap W_0 = \left\{x \in W_0 : x_{J_0} = X_0(F, \hat{x}_{J_0}) = (X_0^{(1)}(F, \hat{x}_{J_0}), \ldots, X_0^{(k)}(F, \hat{x}_{J_0}))\right\}$$

parametrized by

$$g_0 = g_0(J_0) : \mathbb{T}^{d-k} \to \mathbb{T}^{d-k} \times \mathbb{R}^k , \quad \hat{x}_{J_0} \mapsto (\hat{x}_{J_0}, X_0(F, \hat{x}_{J_0})) . \tag{II.D.3}$$

Exploiting the compactness of $Z_K(F)$ together with the Implicit Function Theorem, there is $m \geq 1$ such that for every $j \in [m]$, there are $x_j \in Z_K(F)$, $J_j = J_j(x_j) \in D_k$ and $W_j = W_j(J_j) \subset \mathbb{T}^d$, such that

$$Z_K(F) \subset \bigcup_{j=1}^{m} W_j,$$

and moreover, for every $j \in [m]$, the Implicit Function Theorem ensures the existence of an implicit function X_j of class C^1 that yields a local parametrization

$$g_j = g_j(J_j) : \mathbb{T}^{d-k} \to \mathbb{T}^{d-k} \times \mathbb{R}^k , \quad \hat{x}_{J_j} \mapsto (\hat{x}_{J_j}, X_j(F, \hat{x}_{J_j}))$$

of $Z_K(F) \cap W_j$. Hence, if $T = \{j_1, \ldots, j_r\} \subset [m]$ for $r \leq m$ and $\bigcap_{j\in T} W_j \neq \emptyset$, then

$$\Gamma_T(F) := Z_K(F) \cap \Big(\bigcap_{j\in T} \overline{W_j}\Big) \tag{II.D.4}$$

describes a $(d-k)$-dimensional surface whose volume is computed when integrating the corresponding volume element $y \mapsto \sqrt{\det(\mathrm{jac}^T_{g_{j_1}}(y)\,\mathrm{jac}_{g_{j_1}}(y))}$ (see e.g. [HJ20, Section 10.4]). An application of the chain rule gives

$$\mathrm{vol}(\Gamma_T(F)) = \int_{Y_T} \sqrt{\det(\mathrm{jac}^T_{g_{j_1}}(y)\,\mathrm{jac}_{g_{j_1}}(y))}\, dy = \int_{Y_T} \sqrt{1 + \sum_{i\in[k]} \|\nabla X_{j_1}^{(i)}(F, y)\|_k^2}\, dy ,$$

where the region of integration is $Y_T = p_{J_1}\big(\bigcap_{j\in T} \overline{W_j}\big)$. The total volume of $Z_K(F)$ is then computed by

$$\mathrm{vol}(Z_K(F)) = \sum_{\emptyset\neq T\subset[m]} (-1)^{\mathrm{card}(T)} \mathrm{vol}(\Gamma_T(F)) . \tag{II.D.5}$$

Now we can find $\varepsilon > 0$ small enough such that $Z_K^{+\varepsilon}(F) \subset \bigcup_{j=1}^m W_j$ and in view of Lemma II.D.2, it follows that $Z_K(F_n) \subset \bigcup_{j=1}^m W_j$ for n sufficiently large, so that

$$Z_K(F_n) = \bigcup_{j=1}^m \left(Z_K(F_n) \cap \overline{W_j}\right) .$$

Since for $T = \{j_1, \dots, j_r\} \subset [m]$, $\Gamma_T(F_n)$ as defined in (II.D.4) identifies with a $(d-k)$-dimensional surface of volume $\mathrm{vol}(\Gamma_T(F_n))$, the total nodal volume of F_n in K is given by

$$\mathrm{vol}(Z_K(F_n)) = \sum_{\emptyset \neq T \subset [m]} (-1)^{\mathrm{card}(T)} \mathrm{vol}(\Gamma_T(F_n)) .$$

Using Lipschitz continuity of $x \mapsto \sqrt{1+x}$ for $x > 0$, it follows that

$$\begin{aligned}
&\left|\mathrm{vol}(Z_K(F_n)) - \mathrm{vol}(Z_K(F))\right| \\
\leq \;& \sum_{\emptyset \neq T \subset [m]} \int_{Y_T} \left| \sqrt{1 + \sum_{i \in [k]} \|\nabla X_{j_1}^{(i)}(F_n, y)\|_k^2} - \sqrt{1 + \sum_{i \in [k]} \|\nabla X_{j_1}^{(i)}(F, y)\|_k^2} \right| dy \\
\leq \;& \sum_{\emptyset \neq T \subset [m]} \int_{Y_T} \sum_{i \in [k]} \left| \|\nabla X_{j_1}^{(i)}(F_n, y)\|_k^2 - \|\nabla X_{j_1}^{(i)}(F, y)\|_k^2 \right| dy .
\end{aligned}$$

Now, using the reversed triangular inequality $|\|u\| - \|v\|| \leq \|u - v\|$ yields

$$\begin{aligned}
&\left| \|\nabla X_{j_1}^{(i)}(F_n, y)\|_k^2 - \|\nabla X_{j_1}^{(i)}(F, y)\|_k^2 \right| \\
= \;& \left| \|\nabla X_{j_1}^{(i)}(F_n, y)\|_k - \|\nabla X_{j_1}^{(i)}(F, y)\|_k \right| \cdot \left(\|\nabla X_{j_1}^{(i)}(F_n, y)\|_k + \|\nabla X_{j_1}^{(i)}(F, y)\|_k \right) \\
\leq \;& \|\nabla X_{j_1}^{(i)}(F_n, y) - \nabla X_{j_1}^{(i)}(F, y)\|_k \cdot \left(\|\nabla X_{j_1}^{(i)}(F_n, y)\|_k + \|\nabla X_{j_1}^{(i)}(F, y)\|_k \right) .
\end{aligned}$$

In order to conclude, it suffices to show that the first factor converges to 0 uniformly on Y_T as $n \to \infty$. Consider the equation

$$F(\hat{y}_{J_1}, y_{J_1}) = F(\hat{y}_{J_1}, X_{j_1}(F, \hat{y}_{J_1})) = 0, \tag{II.D.6}$$

where, for the vector $(\hat{y}_{J_1}, y_{J_1})$ it is implicitly understood that coordinates with indices in J_1 are located in the corresponding position. Differentiating (II.D.6) with respect to the coordinates $\hat{y}_{J_1}$, we obtain

$$\mathrm{jac}_{F,\hat{y}_{J_1}}(\hat{y}_{J_1}, y_{J_1}) \cdot \mathbf{I}_{d-k} + \mathrm{jac}_{F,j_{J_1}}(\hat{y}_{J_1}, y_{J_1}) \cdot \mathrm{jac}_{X_{j_1},\hat{y}_{J_1}}(F, \hat{y}_{J_1}) = 0,$$

where the zero in the right-hand side denotes the zero $k \times (d-k)$ matrix. Therefore, since $\mathrm{jac}_{F,y_{J_1}}(\hat{y}_{J_1}, y_{J_1})$ is invertible,

$$\mathrm{jac}_{X_{j_1},\hat{y}_{J_1}}(F, \hat{y}_{J_1}) = -\left[\mathrm{jac}_{F,y_{J_1}}(\hat{y}_{J_1}, y_{J_1})\right]^{-1} \cdot \mathrm{jac}_{F,\hat{y}_{J_1}}(\hat{y}_{J_1}, y_{J_1}) . \tag{II.D.7}$$

Since F_n converges to F in the C^1 topology, we have that, for n sufficiently large, (II.D.7) holds true for F_n. Writing out the i-th row for $i \in [k]$ of this relation, and using the fact that all the partial derivatives of F_n converge uniformly to the corresponding partial derivatives of F (as $F_n \to F$), we conclude that $\|\nabla X_{j_1}^{(i)}(F_n, \hat{y}_{J_1}) - \nabla X_{j_1}^{(i)}(F, \hat{y}_{J_1})\|_k$ converges to zero uniformly on Y_T as $n \to \infty$, proving the statement. □

Appendix II.E Singular and non-singular cubes

II.E.1 Definitions and ancillary results

II.E.1.1 Singular and non-singular pairs of points and cubes

For every $n \in S_3$, we partition the torus into a disjoint union of cubes of length $1/M$, where $M = M_n \geq 1$ is an integer proportional to $\sqrt{E_n}$ as follows: Let $Q_0 = [0, 1/M)^3$; then we consider the partition of $\mathbb{T}^3$ obtained by translating Q_0 in the directions $k/M, k \in \mathbb{Z}^3$. Denote by $\mathcal{P}(M)$ the partition of $\mathbb{T}^3$ that is obtained in this way. By construction, $\text{card}(\mathcal{P}(M)) = M^3$. By linearity, we can decompose the random variable $L_n^{(\ell)}$ as

$$L_n^{(\ell)} = \sum_{Q \in \mathcal{P}(M)} L_n^{(\ell)}(Q), \quad \ell \in [3] \tag{II.E.1}$$

where $L_n^{(\ell)}(Q)$ denotes the nodal volume restricted to Q. From now on, we fix a small number $0 < \eta < 10^{-10}$. In the forthcoming definition, we define singular pairs of points and cubes.

Definition II.E.1 (Singular pairs of points and cubes)**.** A pair of points $(x, y) \in \mathbb{T}^3 \times \mathbb{T}^3$ is called a *singular pair of points* if one of the following inequalities is satisfied:

$$|r_n(x - y)| > \eta, \quad |\partial_i r_n(x - y)| > \eta \sqrt{E_n/3}, \quad |\partial_{ij} r_n(x - y)| > \eta E_n/3$$

for $(i, j) \in [3] \times [3]$. A pair of cubes $(Q, Q') \in \mathcal{P}(M)^2$ is called a *singular pair of cubes* if the product $Q \times Q'$ contains a singular pair of points. We denote by $\mathcal{S} = \mathcal{S}(M) \subset \mathcal{P}(M)^2$ the set of singular pairs of cubes. A pair of cubes $(Q, Q') \in \mathcal{S}^c$ is called *non-singular*. By construction, $\mathcal{P}(M)^2 = \mathcal{S} \cup \mathcal{S}^c$.

For fixed $Q \in \mathcal{P}(M)$, let us furthermore denote by $\mathcal{B}_Q$ the union over all cubes $Q' \in \mathcal{P}(M)$ such that $(Q, Q') \in \mathcal{S}$. In particular, analogously as in Lemma 6.3 of [DNPR19], we have

$$\text{Leb}(\mathcal{B}_Q) = O(\mathcal{R}_n(6)), \tag{II.E.2}$$

where $\mathcal{R}_n(6) = \int_{\mathbb{T}^3} r_n(z)^6 dz$. We write

$$\tilde{r}_{a,b}(x - y) := \mathbb{E}\left[\tilde{\partial}_a T_n^{(i)}(x) \cdot \tilde{\partial}_b T_n^{(i)}(y)\right], \quad a, b = 0, 1, 2, 3, \quad i \in [\ell],$$

where, we recall that $\tilde{\partial}_a = (E_n/3)^{-1/2}\partial_a$ with the convention $\tilde{\partial}_0 := \mathbf{I}$. Note that $\tilde{r}_{0,0} = r_n$ and that we dropped the dependence on n in order to simplify notations. We need the following lemma: its proof is based on differentiating the expression of r_n and the orthogonality relations for complex exponentials on the full torus. We omit the details.

Lemma II.E.2. *For every $a, b \in \{0, 1, 2, 3\}$ and every integer $m \geq 1$,*

$$\int_{\mathbb{T}^3} \tilde{r}_{a,b}(z)^{2m}\, dz \ll \int_{\mathbb{T}^3} r_n(z)^{2m}\, dz = \mathcal{R}_n(2m), \tag{II.E.3}$$

where the constant involved in the '$\ll$' notation depends only on m.

II.E.1.2 A diagram formula

The proofs to be presented in the forthcoming sections are based on the following diagram formula. Such a formula is counterpart to Proposition 8.1 in [DNPR19], and is based on the Leonov-Shiryaev formulae (see e.g. [PT11, Proposition 3.2.1]). We introduce some notation: For $i \in [\ell]$, write

$$\left(X_0^{(i)}(x), X_1^{(i)}(x), X_2^{(i)}(x), X_3^{(i)}(x)\right) := \left(T_n^{(i)}(x), \widetilde{\nabla} T_n^{(i)}(x)\right), \quad x \in \mathbb{T}^3 \tag{II.E.4}$$

and consider families of non-negative integers

$$p^{(i)} = \left\{p_j^{(i)} : j = 0,1,2,3\right\} , \quad q^{(i)} = \left\{q_j^{(i)} : j = 0,1,2,3\right\}$$

for which we write

$$S(p^{(i)}) := \sum_{j=0}^{3} p_j^{(i)} , \quad S(q^{(i)}) := \sum_{j=0}^{3} q_j^{(i)} . \tag{II.E.5}$$

For $m \in \left\{p^{(i)}, q^{(i)}\right\}$, we also define the vector of $\mathbb{R}^{m_0} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{m_3}$ given by

$$X_m^{(i)}(x) := \left([X_0^{(i)}(x)]_{m_0}, [X_1^{(i)}(x)]_{m_1}, [X_2^{(i)}(x)]_{m_2}, [X_3^{(i)}(x)]_{m_3}\right) ,$$

where for an integer $n \geq 1$ and a real number N, we write $[N]_n := (N, \dots, N) \in \mathbb{R}^n$.

Proposition II.E.3. *For $i \in [\ell]$, consider families of non-negative integers $p^{(i)} = \left\{p_j^{(i)} : j = 0,1,2,3\right\}$ and $q^{(i)} = \left\{q_j^{(i)} : j = 0,1,2,3\right\}$ as above, as well as $x, y \in \mathbb{T}^3$. Then,*

$$\begin{aligned} \mathbb{E}\left[\prod_{i=1}^{\ell}\prod_{j=0}^{3} H_{p_j^{(i)}}\left(X_j^{(i)}(x)\right) \cdot H_{q_j^{(i)}}\left(X_j^{(i)}(y)\right)\right] &= \prod_{i=1}^{\ell}\mathbb{E}\left[\prod_{j=0}^{3} H_{p_j^{(i)}}\left(X_j^{(i)}(x)\right) \cdot H_{q_j^{(i)}}\left(X_j^{(i)}(y)\right)\right] \\ &= \prod_{i=1}^{\ell}\mathbb{1}[S(p^{(i)}) = S(q^{(i)})]\sum_{\sigma_i}\prod_{j=1}^{S(p^{(i)})}\mathbb{E}\left[\left(X_{p^{(i)}}^{(i)}\right)_j(x) \cdot \left(X_{q^{(i)}}^{(i)}\right)_{\sigma_i(j)}(y)\right] , \end{aligned}$$

where the sum runs over all permutations σ_i of $\left\{1, \dots, S(p^{(i)})\right\}$.

II.E.2 Proof of Lemma II.3.1

Proof of the almost sure convergence: In the case $\ell = 3$, one can argue similarly as in the proof of Lemma 3.1 in [DNPR19]. We present the arguments for $\ell = 2$. Since, $\mathbf{T}_n^{(2)}$ is of class C^∞, Sard's Theorem (see e.g. [Sar42]) implies that its set of critical values has almost surely zero Lebesgue measure. Therefore, applying the Co-Area formula (see Proposition I.1.11) to the functions $f = \mathbf{T}_n^{(2)} : \mathbb{T}^3 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}$, $g(x_1, x_2) = (2\varepsilon)^{-2}\prod_{i=1}^{2}\mathbb{1}_{[-\varepsilon,\varepsilon]}(x_i)$ yields

$$L_{n,\varepsilon}^{(2)} = (2\varepsilon)^{-2}\int_{[-\varepsilon,\varepsilon]^2} L_n^{(2)}(\mathbb{T}^3; (u_1, u_2))\, du_1 du_2 , \tag{II.E.6}$$

where for $B \subset \mathbb{T}^3$, we set $L_n^{(2)}(B; (u_1, u_2)) = \mathcal{H}_1\left\{(\mathbf{T}_n^{(2)})^{-1}(\{(u_1, u_2)\}) \cap B\right\}$. Now, as $(u_1, u_2) \to (0,0)$, the shifted random field $\mathbf{T}_n^{(2)} - (u_1, u_2)$ converges in the C^1 topology on $\mathbb{T}^3$ to the random field $\mathbf{T}_n^{(2)}$, which is non-degenerate - as can be seen e.g. by checking the assumptions of Proposition 6.12 in [AW09] - so that by the continuity of the nodal volume proved in Theorem II.D.3,

$$\lim_{(u_1,u_2)\to(0,0)} \mathcal{H}_1\left\{(\mathbf{T}_n^{(2)} - (u_1, u_2))^{-1}(\{(0,0)\})\right\} = \mathcal{H}_1\left\{(\mathbf{T}_n^{(2)})^{-1}(\{(0,0)\})\right\} = L_n^{(2)}(\mathbb{T}^3; (u_1, u_2)) .$$

This proves the continuity of $L_n^{(2)}(\mathbb{T}^3; (u_1, u_2))$ at $(u_1, u_2) = (0,0)$. The almost sure convergence then follows by letting $\varepsilon \to 0$ in (II.E.6).

Proof of the $L^2(\mathbb{P})$-convergence: We now prove that the convergence also takes place in $L^2(\mathbb{P})$. For completeness, we include the three cases corresponding to $\ell = 1, 2, 3$ in our proof. We start by proving an auxiliary result. Recall that Q_0 is the small cube around the origin of side length $1/M$.

Lemma II.E.4. *The map* $(u_1,\dots,u_\ell)\mapsto \mathbb{E}\left[L_n^{(\ell)}(Q_0;(u_1,\dots,u_\ell))^2\right]$ *is continuous at* $(0,\dots,0)$.

Proof. Writing $u^{(\ell)} := (u_1,\dots,u_\ell)$, we will prove that

$$\lim_{u^{(\ell)}\to(0,\dots,0)} \mathbb{E}\left[L_n^{(\ell)}(Q_0;u^{(\ell)})^2\right] = \mathbb{E}\left[L_n^{(\ell)}(Q_0;(0,\dots,0))^2\right] . \tag{II.E.7}$$

By virtue of Lemma II.C.6 the random field $(\mathbf{T}_n^{(\ell)}(x),\mathbf{T}_n^{(\ell)}(y))$ is non-degenerate in Q_0 so that we may use Kac-Rice formulae in the cube Q_0. For $\ell=1,2$, we write,

$$\mathbb{E}\left[L_n^{(\ell)}(Q_0;u^{(\ell)})^2\right] = \int_{Q_0\times Q_0} K^{(\ell)}(x,y;u^{(\ell)})\,dxdy\,,$$

where $K^{(\ell)}$ is as in (II.C.3), whereas for $\ell=3$, we write

$$\mathbb{E}\left[L_n^{(3)}(Q_0;u^{(3)})^2\right] = \mathbb{E}\left[L_n^{(3)}(Q_0;u^{(3)})(L_n^{(3)}(Q_0;u^{(3)})-1)\right] + \mathbb{E}\left[L_n^{(3)}(Q_0;u^{(3)})\right] ,$$

and apply Theorem I.1.12 to the respective summands, so that

$$\begin{aligned}
&\mathbb{E}\left[L_n^{(3)}(Q_0;u^{(3)})^2\right]\\
&= \int_{Q_0\times Q_0} K^{(3)}(x,y;u^{(3)})dxdy + \int_{Q_0}\mathbb{E}\left[\Phi^*_{\ell,3}(\mathrm{jac}_{\mathbf{T}_n^{(3)}}(x))\,|\,\mathbf{T}_n^{(3)}(x)=u^{(3)}\right]\cdot p_{\mathbf{T}_n^{(3)}(x)}(u^{(3)})dx\\
&= \int_{Q_0\times Q_0} K^{(3)}(x,y;u^{(3)})dxdy + \int_{Q_0}\mathbb{E}\left[\Phi^*_{\ell,3}(\mathrm{jac}_{\mathbf{T}_n^{(3)}}(x))\right]\cdot p_{\mathbf{T}_n^{(3)}(x)}(u^{(3)})\,dx\,,
\end{aligned}$$

where the last line follows from the independence of $\mathbf{T}_n^{(3)}(x)$ and $\mathrm{jac}_{\mathbf{T}_n^{(3)}}(x)$. Thus, the L.H.S of (II.E.7) reduces to

$$\begin{aligned}
\lim_{u^{(\ell)}\to(0,\dots,0)} \mathbb{E}\left[L_n^{(\ell)}(Q_0;u^{(\ell)})^2\right] &= \lim_{u^{(\ell)}\to(0,\dots,0)}\Bigg(\int_{Q_0\times Q_0} K^{(\ell)}(x,y;u^{(\ell)})dxdy\\
&\quad +\mathbb{1}_{\ell=3}\times\int_{Q_0}\mathbb{E}\left[\Phi^*_{\ell,3}(\mathrm{jac}_{\mathbf{T}_n^{(3)}}(x))\right]\cdot p_{\mathbf{T}_n^{(3)}(x)}(u^{(3)})dx\Bigg).
\end{aligned}\tag{II.E.8}$$

Let us deal with the additional term appearing in the case $\ell=3$: The Hadamard inequality (see e.g. [RWH17]) and independence yield

$$\mathbb{E}\left[\Phi^*_{\ell,3}(\mathrm{jac}_{\mathbf{T}_n^{(3)}}(x))\right] \le \prod_{i=1}^{3}\mathbb{E}\left[\|\nabla T_n^{(i)}(x)\|\right] \le \mathbb{E}\left[\|\nabla T_n^{(1)}(x)\|^2\right]^{3/2} = E_n^{3/2}.$$

Moreover, the Gaussian probability density $u^{(3)}\mapsto p_{\mathbf{T}_n^{(3)}(x)}(u^{(3)})$ satisfies

$$p_{\mathbf{T}_n^{(3)}(x)}(u^{(3)}) = \prod_{i=1}^{3} p_{T_n^{(i)}(x)}(u_i) \le (p_{T_n^{(1)}(x)}(0))^3 = (2\pi)^{-3/2}\,.$$

Therefore, applying dominated convergence yields,

$$\begin{aligned}
&\lim_{u^{(\ell)}\to(0,\dots,0)}\int_{Q_0}\mathbb{E}\left[\Phi^*_{\ell,3}(\mathrm{jac}_{\mathbf{T}_n^{(3)}}(x))\right]\cdot p_{\mathbf{T}_n^{(3)}(x)}(u^{(3)})dx\\
&= \int_{Q_0}\mathbb{E}\left[\Phi^*_{\ell,3}(\mathrm{jac}_{\mathbf{T}_n^{(3)}}(x))\right]p_{\mathbf{T}_n^{(3)}(x)}(0,0,0)\,dx = \mathbb{E}\left[L_n^{(3)}(Q_0;(0,0,0))\right] .
\end{aligned}$$

We now deal with the first summand in the R.H.S of (II.E.8). By stationarity,

$$\int_{Q_0\times Q_0} K^{(\ell)}(x,y;u^{(\ell)})dxdy = \int_{Q_0-Q_0} \mathrm{Leb}(Q_0\cap Q_0 - z)K^{(\ell)}(z,0;u^{(\ell)})dz.$$

Now, for every $u^{(\ell)}$ in a neighbourhood of $(0,\dots,0)$, say $\|u^{(\ell)}\| < \delta$, for some $\delta > 0$, in view of (II.C.4), we have $K^{(\ell)}(z,0;u^{(\ell)}) \le q^{(\ell)}(z,0;\|u^{(\ell)}\|) < q^{(\ell)}(z,0;\delta)$ for every z. Therefore, again by dominated convergence, we infer

$$\begin{aligned}\lim_{u^{(\ell)}\to(0,\dots,0)} \int_{Q_0\times Q_0} K^{(\ell)}(x,y;u^{(\ell)})dxdy &= \int_{Q_0\times Q_0} \lim_{u^{(\ell)}\to(0,\dots,0)} K^{(\ell)}(x,y;u^{(\ell)})dxdy \\ &= \mathbb{E}\left[L_n^{(\ell)}(\mathbb{T}^3;(0,\dots,0))^2\right],\end{aligned}$$

where, in the last line we used the continuity result proved in Lemma II.C.3. □

Now, for a domain $B\subset\mathbb{T}^3$, we set $L_n^{(\ell)}(B) := L_n^{(\ell)}(B;(0,\dots,0))$ and for $\varepsilon > 0$, we write $L_{n,\varepsilon}^{(\ell)}(B) := L_{n,\varepsilon}^{(\ell)}(B;(0,\dots,0))$ for the ε-approximation of $L_n^{(\ell)}(B)$ (recall Definition (II.3.3)). We define the random variable

$$A_n^{(\ell)}(B;\varepsilon,\varepsilon') := L_{n,\varepsilon}^{(\ell)}(B) - L_{n,\varepsilon'}^{(\ell)}(B)\,,\quad n\in S_3,\ \varepsilon>0,\ \varepsilon'>0. \tag{II.E.9}$$

Proving that $L_{n,\varepsilon}^{(\ell)}$ converges to $L_n^{(\ell)}$ in $L^2(\mathbb{P})$ as $\varepsilon\to 0$ is equivalent to showing that for every $n\in S_3$, the random variable $A_n^{(\ell)}(\mathbb{T}^3;\varepsilon,\varepsilon')$ converges to zero in $L^2(\mathbb{P})$ as $\varepsilon,\varepsilon'\to 0$. We first show that the latter convergence holds in the small cube Q_0 around the origin.

Lemma II.E.5. *For every $n\in S_3$, one has that $A_n^{(\ell)}(Q_0;\varepsilon,\varepsilon')\to 0$ in $L^2(\mathbb{P})$ as $\varepsilon,\varepsilon'\to 0$.*

Proof. We will show that, for every $n\in S_3$, the sequence $\left\{L_{n,\varepsilon}^{(\ell)}(Q_0):\varepsilon>0\right\}$ converges in $L^2(\mathbb{P})$ to $L_n^{(\ell)}(Q_0)$ as $\varepsilon\to 0$. This implies that $\left\{L_{n,\varepsilon}^{(\ell)}(Q_0):\varepsilon>0\right\}$ is a Cauchy sequence in $L^2(\mathbb{P})$, and therefore $A_n^{(\ell)}(Q_0;\varepsilon,\varepsilon')\to 0$ in $L^2(\mathbb{P})$ as $\varepsilon,\varepsilon'\to 0$. Since almost sure convergence together with convergence of norms implies convergence in $L^2(\mathbb{P})$ (see e.g. [Rud87, p.73]), it suffices to show that $\mathbb{E}\left[L_{n,\varepsilon}^{(\ell)}(Q_0)^2\right]\to\mathbb{E}\left[L_n^{(\ell)}(Q_0)^2\right]$ as $\varepsilon\to 0$. We start by proving that $L_{n,\varepsilon}^{(\ell)}(Q_0)\in L^2(\mathbb{P})$ for every $\varepsilon>0$: Using the definition of $L_{n,\varepsilon}^{(\ell)}(Q_0)$ and the Hadamard inequality, we have

$$\begin{aligned}L_{n,\varepsilon}^{(\ell)}(Q_0) &\le (2\varepsilon)^{-\ell}\int_{Q_0}\Phi_{\ell,3}^{*}(\mathrm{Jac}_{\mathbf{T}_n^{(2)}}(x))\,dx \le (2\varepsilon)^{-\ell}\int_{Q_0}\prod_{i=1}^{\ell}\|\nabla T_n^{(i)}(x)\|dx \\ &\le (2\varepsilon)^{-\ell}\int_{\mathbb{T}^3}\prod_{i=1}^{\ell}\|\nabla T_n^{(i)}(x)\|dx\,,\end{aligned}$$

and hence, using Jensen's inequality,

$$\mathbb{E}\left[L_{n,\varepsilon}^{(\ell)}(Q_0)^2\right] \le (2\varepsilon)^{-2\ell}\mathbb{E}\left[\left(\int_{\mathbb{T}^3}\prod_{i=1}^{\ell}\|\nabla T_n^{(i)}(x)\|dx\right)^2\right]$$

$$\le (2\varepsilon)^{-2\ell}\mathbb{E}\left[\int_{\mathbb{T}^3}\prod_{i=1}^{\ell}\|\nabla T_n^{(i)}(x)\|^2\,dx\right] = (2\varepsilon)^{-2\ell}\int_{\mathbb{T}^3}\mathbb{E}\left[\|\nabla T_n^{(1)}(x)\|^2\right]^{\ell}\,dx = (2\varepsilon)^{-2\ell}E_n^{\ell} < +\infty.$$

In order to prove that $L_n^{(\ell)}(Q_0)$ is in $L^2(\mathbb{P})$, we use Kac-Rice formulae for second moments and proceed as in the proof of Lemma II.E.4: For $\ell=3$, we write

$$\mathbb{E}\left[L_n^{(3)}(Q_0)^2\right] = \mathbb{E}\left[L_n^{(3)}(Q_0)(L_n^{(3)}(Q_0)-1)\right] + \mathbb{E}\left[L_n^{(3)}(Q_0)\right],$$

and apply Kac-Rice formula for moments and use stationarity, to write

$$\mathbb{E}\left[L_n^{(\ell)}(Q_0)^2\right] = \int_{Q_0\times Q_0} K^{(\ell)}(x,y;(0,\dots,0))\,dxdy + \mathbb{E}\left[L_n^{(3)}(Q_0)\right]\mathbb{1}_{\ell=3}$$
$$\le \text{Leb}(Q_0)\int_{2Q_0} K^{(\ell)}(z,0;(0,\dots,0))\,dz + \frac{E_n^{3/2}}{M^3}\mathbb{1}_{\ell=3}\,, \tag{II.E.10}$$

where the last line follows from the fact that $\mathbb{E}\left[L_n^{(3)}(Q_0)\right] = \text{Leb}(Q_0)\mathbb{E}\left[L_n^{(3)}\right] \ll M^{-3}E_n^{3/2}$. From (II.C.4) and the Taylor expansion in Lemma II.C.5, we can upper bound (II.E.10) by

$$\begin{aligned}
&\le \frac{1}{M^3}\int_{2Q_0} q^{(\ell)}(z,0;0)\,dz + \frac{E_n^{3/2}}{M^3}\mathbb{1}_{\ell=3}\\
&\ll \frac{1}{M^3}\int_0^{1/M}\left[(3)_\ell\left(1-\frac{\ell}{3}\right)e_n^{\ell/2}r^{2-\ell} + (3)_\ell\left(1+\|u^{(\ell)}\|^2\right)E_n^{\ell/2+1}r^{4-\ell}\right]dr + \frac{E_n^{3/2}}{M^3}\mathbb{1}_{\ell=3}\\
&\ll E_n^{-2}\mathbb{1}_{\ell=1} + E_n^{-1}\mathbb{1}_{\ell=2} + \mathbb{1}_{\ell=3}.
\end{aligned} \tag{II.E.11}$$

This proves that $L_n^{(\ell)}(Q_0)$ is an element of $L^2(\mathbb{P})$. In order to show that the convergence holds in $L^2(\mathbb{P})$, we will prove the inequalities

$$\mathbb{E}\left[L_n^{(\ell)}(Q_0)^2\right] \le \lim_{\varepsilon\to0}\mathbb{E}\left[L_{n,\varepsilon}^{(\ell)}(Q_0)^2\right] \le \mathbb{E}\left[L_n^{(\ell)}(Q_0)^2\right].$$

For the first inequality, we use the almost sure convergence proved above and Fatou's Lemma to write

$$\mathbb{E}\left[L_n^{(\ell)}(Q_0)^2\right] = \mathbb{E}\left[\liminf_{\varepsilon\to0} L_{n,\varepsilon}^{(\ell)}(Q_0)^2\right] \le \liminf_{\varepsilon\to0}\mathbb{E}\left[L_{n,\varepsilon}^{(\ell)}(Q_0)^2\right] = \lim_{\varepsilon\to0}\mathbb{E}\left[L_{n,\varepsilon}^{(\ell)}(Q_0)^2\right].$$

The second inequality is proved as follows: Applying the Co-Area formula (see Proposition I.1.11) and then the Cauchy-Schwarz inequality

$$\begin{aligned}
\mathbb{E}\left[L_{n,\varepsilon}^{(\ell)}(Q_0)^2\right] &= (2\varepsilon)^{-2\ell}\int_{[-\varepsilon,\varepsilon]^\ell\times[-\varepsilon,\varepsilon]^\ell}\mathbb{E}\left[L_n^{(\ell)}(Q_0;u^{(\ell)})\cdot L_n^{(\ell)}(Q_0;v^{(\ell)})\right]du^{(\ell)}dv^{(\ell)}\\
&\le \left((2\varepsilon)^{-\ell}\int_{[-\varepsilon,\varepsilon]^\ell}\mathbb{E}\left[L_n^{(\ell)}(Q_0;u^{(\ell)})^2\right]^{1/2}\,du^{(\ell)}\right)^2,
\end{aligned}$$

where $u^{(\ell)} = (u_1,\dots,u_\ell)$ and $v^{(\ell)} = (v_1,\dots,v_\ell)$. By Lemma II.E.4, the map $u^{(\ell)} \mapsto \mathbb{E}\left[L_n^{(\ell)}(\mathbb{T}^3;u^{(\ell)})^2\right]$ is continuous at $(0,\dots,0)$, so that letting $\varepsilon\to0$ yields the desired inequality. □

Taking advantage of the partition of the torus introduced in Section II.E.1.1, we decompose

$$\mathbb{E}\left[A_n^{(\ell)}(\mathbb{T}^3;\varepsilon,\varepsilon')^2\right] = \sum_{(Q,Q')\in\mathcal{P}(M)^2}\mathbb{E}\left[A_n^{(\ell)}(Q;\varepsilon,\varepsilon')A_n^{(\ell)}(Q';\varepsilon,\varepsilon')\right]$$
$$= \left\{\sum_{(Q,Q')\in\mathcal{S}} + \sum_{(Q,Q')\in\mathcal{S}^c}\right\}\mathbb{E}\left[A_n^{(\ell)}(Q;\varepsilon,\varepsilon')A_n^{(\ell)}(Q';\varepsilon,\varepsilon')\right] =: S_{n,1}^{(\ell)}(\varepsilon,\varepsilon') + S_{n,2}^{(\ell)}(\varepsilon,\varepsilon')$$

and control each term separately. This is the content of the next two lemmas.

Lemma II.E.6. *For every $n\in S_3$, one has that $|S_{n,1}^{(\ell)}(\varepsilon,\varepsilon')|\to0$ as $\varepsilon,\varepsilon'\to0$.*

Proof. Using the triangular inequality and then the Cauchy-Schwarz inequality, we can write

$$\left|S^{(\ell)}_{n,1}(\varepsilon,\varepsilon')\right| \le \sum_{(Q,Q')\in\mathcal{S}} \sqrt{\mathbb{E}\left[A^{(\ell)}_n(Q;\varepsilon,\varepsilon')^2\right]\mathbb{E}\left[A^{(\ell)}_n(Q';\varepsilon,\varepsilon')^2\right]} = \operatorname{card}(\mathcal{S})\cdot\mathbb{E}\left[A^{(\ell)}_n(Q_0;\varepsilon,\varepsilon')^2\right], \tag{II.E.12}$$

where we used translation-invariance of $\mathbf{T}^{(\ell)}_n$ in order to reduce the arguments over the cube Q_0. Now, thanks to (II.E.2) and the fact that we are summing over pairs of cubes yields $\operatorname{card}(\mathcal{S}) = M^6\cdot \operatorname{Leb}(\mathcal{B}_Q) = O(E_n^3\mathcal{R}_n(6))$. By Lemma II.E.5, $\mathbb{E}\left[A^{(\ell)}_n(Q_0;\varepsilon,\varepsilon')^2\right]$ converges to 0 as $\varepsilon,\varepsilon'\to 0$, which yields the desired conclusion. □

Lemma II.E.7. *For every $n\in S_3$, one has that $|S^{(\ell)}_{n,2}(\varepsilon,\varepsilon')|\to 0$ as $\varepsilon,\varepsilon'\to 0$.*

Proof. Adopting the same notation as in Section II.E.3, we write $\mathbf{p}$ for multi-indices of the form $\left\{p^{(i)}_j : (i,j)\in[\ell]\times\{0,1,2,3\}\right\}$ and set $S(\mathbf{p}) = \sum_{i=1}^{\ell}\sum_{j=0}^{3} p^{(i)}_j$. The Wiener-chaos decomposition of $A^{(\ell)}_n(Q;\varepsilon,\varepsilon')$ in (II.E.9) is obtained from that of $L^{(\ell)}_n$ in (II.3.5) by replacing $\mathbb{T}^3$ with Q and the coefficients $\beta_{p_0^{(1)}}\cdots\beta_{p_0^{(\ell)}}$ with

$$\delta_{p_0^{(1)},\dots,p_0^{(\ell)}}(\varepsilon,\varepsilon') := \prod_{i=1}^{\ell}\beta_{p_0^{(i)}}(\varepsilon) - \prod_{i=1}^{\ell}\beta_{p_0^{(i)}}(\varepsilon'),$$

where the coefficients $\beta_j(\varepsilon)$ are as in (II.A.2). Moreover, using the notation in (II.E.4) and writing

$$\gamma^{(\ell)}_3\left\{p^{(i)}_j\right\} = \gamma^{(\ell)}_3\left\{p^{(i)}_j : (i,j)\in[\ell]\times[3]\right\} := \alpha^{(\ell)}_3\left\{p^{(i)}_j : (i,j)\in[\ell]\times[3]\right\}\cdot\prod_{i=1}^{\ell}\prod_{j=1}^{3} p^{(i)}_j! \tag{II.E.13}$$

for the Fourier-Hermite coefficients of the function $\Phi^*_{\ell,3}$, we infer that

$$\begin{aligned}\left|S^{(\ell)}_{n,2}(\varepsilon,\varepsilon')\right| &\le \left(\frac{E_n}{3}\right)^{\ell}\sum_{q\ge 0}\sum_{\mathbf{p},\mathbf{q}}\left|\frac{\delta_{p_0^{(1)},\dots,p_0^{(\ell)}}(\varepsilon,\varepsilon')}{p_0^{(1)}!\dots p_0^{(\ell)}!}\frac{\gamma^{(\ell)}_3\left\{p^{(i)}_j\right\}}{\prod_{i=1}^{\ell}\prod_{j=1}^{3}p^{(i)}_j!}\frac{\delta_{q_0^{(1)},\dots,q_0^{(\ell)}}(\varepsilon,\varepsilon')}{q_0^{(1)}!\dots q_0^{(\ell)}!}\frac{\gamma^{(\ell)}_3\left\{q^{(i)}_j\right\}}{\prod_{i=1}^{\ell}\prod_{j=1}^{3}q^{(i)}_j!}\right| \\ &\qquad\times \mathbb{1}_{S(\mathbf{p})=2q}\mathbb{1}_{S(\mathbf{q})=2q}|W(\mathbf{p},\mathbf{q})| \\ &=: \left(\frac{E_n}{3}\right)^{\ell}\times B^{(\ell)}_n(\varepsilon,\varepsilon'),\end{aligned}$$

where

$$W(\mathbf{p},\mathbf{q}) = \sum_{(Q,Q')\in\mathcal{S}^c}\int_Q\int_{Q'}\mathbb{E}\left[\prod_{i=1}^{\ell}\prod_{j=0}^{3}H_{p^{(i)}_j}\left(X^{(i)}_j(x)\right)H_{q^{(i)}_j}\left(X^{(i)}_j(y)\right)\right]dxdy. \tag{II.E.14}$$

Applying Proposition II.E.3, using that $\mathbb{1}_{\bullet}\le 1$ and the fact that $S(p^{(1)})!\cdots S(p^{(\ell)})! \le (S(p^{(1)})+\ldots+S(p^{(\ell)}))! = S(\mathbf{p})! = (2q)!$, we see that $W(\mathbf{p},\mathbf{q})$ is a sum of at most $(2q)!$ terms of the type

$$w = \sum_{(Q,Q')\in\mathcal{S}^c}\int_Q\int_{Q'}\prod_{j=1}^{2q}\tilde{r}_{a_j,b_j}(x-y)\,dxdy \tag{II.E.15}$$

for some $a_1,b_1,\dots,a_{2q},b_{2q}\in\{0,1,2,3\}$. Now, using the fact that for every $(x,y)\in Q\times Q'\subset\mathcal{S}^c$ and every $a,b\in\{0,1,2,3\}$, we have $|\tilde{r}_{a,b}(x,y)|\le\eta$, we infer that $|W(\mathbf{p},\mathbf{q})|\le(2q)!\times\eta^{2q}$. Using

$$\sum_{q\ge 0}\sum_{\mathbf{p},\mathbf{q}}(2q)!\cdot\mathbb{1}[S(\mathbf{p})=2q]\mathbb{1}[S(\mathbf{q})=2q] \le \sum_{\mathbf{p},\mathbf{q}}\sqrt{S(\mathbf{p})!}\sqrt{S(\mathbf{q})!},$$

we obtain

$$\begin{aligned}\left|B_n^{(\ell)}(\varepsilon,\varepsilon')\right| &\leq \sum_{\mathbf{p},\mathbf{q}} \left| \frac{\delta_{p_0^{(1)},\dots,p_0^{(\ell)}}(\varepsilon,\varepsilon')}{p_0^{(1)}!\dots p_0^{(\ell)}!} \frac{\gamma_3^{(\ell)}\left\{p_j^{(i)}\right\}}{\prod_{i=1}^{\ell}\prod_{j=1}^{3} p_j^{(i)}!}\right| \cdot \sqrt{S(\mathbf{p})!}\,\sqrt{\eta}^{\frac{S(\mathbf{p})+S(\mathbf{q})}{2}} \\ &\quad \cdot \left| \frac{\delta_{q_0^{(1)},\dots,q_0^{(\ell)}}(\varepsilon,\varepsilon')}{q_0^{(1)}!\dots q_0^{(\ell)}!} \frac{\gamma_3^{(\ell)}\left\{q_j^{(i)}\right\}}{\prod_{i=1}^{\ell}\prod_{j=1}^{3} q_j^{(i)}!}\right| \cdot \sqrt{S(\mathbf{q})!}\,\sqrt{\eta}^{\frac{S(\mathbf{p})+S(\mathbf{q})}{2}} \\ &\leq \sum_{\mathbf{p},\mathbf{q}} \frac{\left[\delta_{p_0^{(1)},\dots,p_0^{(\ell)}}(\varepsilon,\varepsilon')\right]^2}{p_0^{(1)}!\dots p_0^{(\ell)}!} \frac{\gamma_3^{(\ell)}\left\{p_j^{(i)}\right\}^2}{\prod_{i=1}^{\ell}\prod_{j=1}^{3} p_j^{(i)}!} \cdot \frac{S(\mathbf{p})!}{\prod_{i=1}^{\ell}\prod_{j=0}^{3} p_j^{(i)}!} \sqrt{\eta}^{S(\mathbf{p})+S(\mathbf{q})}\,,\end{aligned} \tag{II.E.16}$$

where the last inequality follows from an application of the Cauchy-Schwarz inequality to the symmetric measure $(\mathbf{p},\mathbf{q}) \mapsto \sqrt{\eta}^{S(\mathbf{p})+S(\mathbf{q})}$. We now argue that $|B_n^{(\ell)}(\varepsilon,\varepsilon')| \to 0$ as $\varepsilon,\varepsilon' \to 0$. First, the estimate (see e.g. [AS92, formula 22.14.16]),

$$|\beta_j(\varepsilon)| \leq \gamma\left(\frac{\varepsilon}{\sqrt{2}}\right) \frac{j!}{2^{j/2}(j/2)!} < |\beta_j|, \quad \varepsilon > 0,\ j \geq 1,$$

implies that

$$|\delta_{p_0^{(1)},\dots,p_0^{(\ell)}}(\varepsilon,\varepsilon')| < 2 \cdot |\beta_{p_0^{(1)}} \dots \beta_{p_0^{(\ell)}}|$$

so that we can apply dominated convergence and use the fact that $\delta_{p_0^{(1)},\dots,p_0^{(\ell)}}(\varepsilon,\varepsilon') \to 0$ as $\varepsilon,\varepsilon' \to 0$ in view of (II.A.3). We will now prove that the remaining series over $\mathbf{p},\mathbf{q}$ is finite. We note that (i) for every $\mathbf{p}$, the quantity

$$\frac{\beta^2_{p_0^{(1)}} \cdots \beta^2_{p_0^{(\ell)}}}{p_0^{(1)}!\dots p_0^{(\ell)}!} \frac{\gamma_3^{(\ell)}\left\{p_j^{(i)}\right\}^2}{\prod_{i=1}^{\ell}\prod_{j=1}^{3} p_j^{(i)}!}$$

is bounded, and (ii) using the multinomial theorem

$$\frac{S(\mathbf{p})!}{\prod_{i=1}^{\ell}\prod_{j=0}^{3} p_j^{(i)}!} \leq \sum_{\substack{\mathbf{m}=(m_j^{(i)}):\\ S(\mathbf{m})=S(\mathbf{p})}} \frac{S(\mathbf{p})!}{\prod_{i=1}^{\ell}\prod_{j=0}^{3} m_j^{(i)}!} \cdot \prod_{i=1}^{\ell}\prod_{j=0}^{3} 1^{m_j^{(i)}} = (4\ell)^{S(\mathbf{p})}\,.$$

Plugging (i) and (ii) into (II.E.16) and using the fact that $4\ell\sqrt{\eta} < 1$, gives

$$\left|B_n^{(\ell)}(\varepsilon,\varepsilon')\right| \ll \sum_{\mathbf{p},\mathbf{q}} (4\ell)^{S(\mathbf{p})} \sqrt{\eta}^{S(\mathbf{p})+S(\mathbf{q})} < +\infty.$$

This finishes the proof. □

II.E.3 Proofs of Lemma II.3.6 and Lemma II.3.7

Proof of Lemma II.3.6. Arguing as in (II.E.12), we have

$$\left|S_{n,1}^{(\ell)}\right| \leq \mathrm{card}(\mathcal{S}) \cdot \mathbf{Var}\left[\mathrm{proj}_{6+}(L_n^{(\ell)}(Q_0))\right] \ll E_n^3 \mathcal{R}_n(6) \cdot \mathbf{Var}\left[\mathrm{proj}_{6+}(L_n^{(\ell)}(Q_0))\right], \tag{II.E.17}$$

where Q_0 is the cube around the origin. Now we notice that

$$\mathbf{Var}\Big[\mathrm{proj}_{6+}(L_n^{(\ell)}(Q_0))\Big] \le \mathbf{Var}\Big[L_n^{(\ell)}(Q_0)\Big] \le \mathbb{E}\left[L_n^{(\ell)}(Q_0)^2\right] .$$

Using Kac-Rice formulae and reasoning as in (II.E.10) and (II.E.11), we obtain that

$$\mathbb{E}\left[L_n^{(\ell)}(Q_0)^2\right] \le E_n^{-2}\mathbb{1}_{\ell=1} + E_n^{-1}\mathbb{1}_{\ell=2} + \mathbb{1}_{\ell=3}.$$

Combining this with the estimate in (II.E.17), yields the desired conclusion. □

Proof of Lemma II.3.7. Using the fact that $\mathrm{proj}_{6+}(L_n^{(\ell)}(Q))$ is centred and the triangular inequality, we first write

$$\left|S_{n,2}^{(\ell)}\right| \le \sum_{(Q,Q')\in\mathcal{S}^c} \mathbb{E}\left[\mathrm{proj}_{6+}(L_n^{(\ell)}(Q)) \cdot \mathrm{proj}_{6+}(L_n^{(\ell)}(Q'))\right].$$

For a family of non-negative integers $\mathbf{p} := \left\{p_j^{(i)} : (i,j) \in [\ell]\times\{0,1,2,3\}\right\}$, we write $S(\mathbf{p}) := \sum_{i=1}^{\ell}\sum_{j=0}^{3} p_j^{(i)}$. Adopting the notation introduced in (II.E.4), it follows from the chaotic expansion in Proposition II.3.2 that,

$$\left|S_{n,2}^{(\ell)}\right| \le \left(\frac{E_n}{3}\right)^{\ell} \sum_{q\ge 3}\sum_{\mathbf{p},\mathbf{q}} \left| \frac{\beta_{p_0^{(1)}}\cdots\beta_{p_0^{(\ell)}}}{p_0^{(1)}!\cdots p_0^{(\ell)}!} \frac{\gamma_3^{(\ell)}\left\{p_j^{(i)}\right\}}{\prod_{i=1}^{\ell}\prod_{j=1}^{3} p_j^{(i)}!} \frac{\beta_{q_0^{(1)}}\cdots\beta_{q_0^{(\ell)}}}{q_0^{(1)}!\cdots q_0^{(\ell)}!} \frac{\gamma_3^{(\ell)}\left\{q_j^{(i)}\right\}}{\prod_{i=1}^{\ell}\prod_{j=1}^{3} q_j^{(i)}!}\right| \times \mathbb{1}_{S(\mathbf{p})=2q}\mathbb{1}_{S(\mathbf{q})=2q}|W(\mathbf{p},\mathbf{q})|, \tag{II.E.18}$$

where $\gamma_3^{(\ell)}\{\cdot\}$ is as in (II.E.13) and $W(\mathbf{p},\mathbf{q})$ as in (II.E.14). Arguing as in the proof of Lemma II.E.7, we see that $W(\mathbf{p},\mathbf{q})$ is a sum of at most $(2q)!$ terms of the type

$$w = \sum_{(Q,Q')\in\mathcal{S}^c} \int_Q\int_{Q'} \prod_{j=1}^{2q} \tilde{r}_{a_j,b_j}(x-y)\, dxdy$$

for some $a_1, b_1, \dots, a_{2q}, b_{2q} \in \{0,1,2,3\}$. Now, using the fact that for every $(x,y)\in Q\times Q' \subset \mathcal{S}^c$ and every $a,b\in\{0,1,2,3\}$, we have $|\tilde{r}_{a,b}(x,y)| \le \eta$, we deduce that

$$|w| \le \eta^{2q-6} \sum_{(Q,Q')\in\mathcal{S}^c} \int_Q\int_{Q'} \prod_{j=1}^{6} \left|\tilde{r}_{a_j,b_j}(x-y)\right| dxdy \le \eta^{2q-6}\int_{\mathbb{T}^3}\prod_{j=1}^{6}\left|\tilde{r}_{a_j,b_j}(z)\right| dz .$$

Then, by the Cauchy-Schwarz inequality, we have that $|\tilde{r}_{a_j,b_j}(z)| \le 1$ for every $z\in\mathbb{T}^3$. Since $\tilde{r}_{a_j,b_j} \in L^6(dz)$ for every $j\in[6]$, an application of the generalized Hölder inequality yields

$$\begin{aligned} |w| &\le \eta^{2q-6}\prod_{j=1}^{6}\left(\int_{\mathbb{T}^3}\tilde{r}_{a_j,b_j}(z)^6 dz\right)^{1/6} \\ &\ll \eta^{2q-6}\cdot\mathcal{R}_n(6) = \frac{\mathcal{R}_n(6)}{\eta^6}\cdot\sqrt{\eta}^{\frac{S(\mathbf{p})+S(\mathbf{q})}{2}}\sqrt{\eta}^{\frac{S(\mathbf{p})+S(\mathbf{q})}{2}}, \end{aligned} \tag{II.E.19}$$

where we used Lemma II.E.2 and the fact that $S(\mathbf{p}) = S(\mathbf{q}) = 2q$. Then, arguing exactly as in (II.E.16), we write

$$|W(\mathbf{p},\mathbf{q})| \le (2q)!\cdot\frac{\mathcal{R}_n(6)}{\eta^6}\cdot\sqrt{\eta}^{\frac{S(\mathbf{p})+S(\mathbf{q})}{2}}\sqrt{\eta}^{\frac{S(\mathbf{p})+S(\mathbf{q})}{2}} = \frac{\mathcal{R}_n(6)}{\eta^6}\cdot\sqrt{S(\mathbf{p})!}\sqrt{S(\mathbf{q})!}\sqrt{\eta}^{\frac{S(\mathbf{p})+S(\mathbf{q})}{2}}\sqrt{\eta}^{\frac{S(\mathbf{p})+S(\mathbf{q})}{2}},$$

and obtain that

$$\left|S_{n,2}^{(\ell)}\right| \ll \left(\frac{E_n}{3}\right)^{\ell} \frac{\mathcal{R}_n(6)}{\eta^6} \sum_{\mathbf{p},\mathbf{q}} \frac{\beta^2_{p_0^{(1)}} \cdots \beta^2_{p_0^{(\ell)}}}{p_0^{(1)}! \dots p_0^{(\ell)}!} \frac{\gamma_3^{(\ell)} \left\{p_j^{(i)}\right\}^2}{\prod_{i=1}^{\ell} \prod_{j=1}^{3} p_j^{(i)}!} \cdot \frac{S(\mathbf{p})!}{\prod_{i=1}^{\ell} \prod_{j=0}^{3} p_j^{(i)}!} \sqrt{\eta}^{S(\mathbf{p})+S(\mathbf{q})}.$$

Proceeding exactly as in the end of the proof of Lemma II.E.7, shows that the series over **p**, **q** converges, which finishes the proof. □

Chapter III

Matrix-Hermite polynomials, random determinants and the geometry of Gaussian fields

We study generalized Hermite polynomials with rectangular matrix arguments arising in multivariate statistical analysis and the theory of zonal polynomials. We show that these are well-suited for expressing the Wiener-Itô chaos expansion of functionals of the spectral measure associated with Gaussian matrices. In particular, we obtain the Wiener chaos expansion of Gaussian determinants of the form $\det(XX^T)^{1/2}$ and prove that, in the setting where the rows of X are i.i.d. centred Gaussian vectors with a given covariance matrix, its projection coefficients admit a geometric interpretation in terms of intrinsic volumes of ellipsoids, thus extending the findings by Kabluchko and Zaporozhets ([ZK12]) to arbitrary chaotic projection coefficients. Our proofs are based on a crucial relation between generalized Hermite polynomials and generalized Laguerre polynomials. In a second part, we introduce the matrix analog of the classical Mehler's formula for the Ornstein-Uhlenbeck semigroup and prove that matrix-variate Hermite polynomials are eigenfunctions of these operators. As a byproduct, we derive an orthogonality relation for Hermite polynomials evaluated at correlated Gaussian matrices. We apply our results to vectors of independent arithmetic random waves on the three-torus, proving in particular a CLT in the high-energy regime for a generalized notion of total variation on the full torus.

Notation. For integers $\ell, n \geq 1$, we write $[n] := \{1, \dots, n\}$ and $\mathbb{R}^{\ell \times n}$ to indicate the ℓn-dimensional vector space of $\ell \times n$ matrices with entries in $\mathbb{R}$ with $\mathbf{I}_n$ denoting the identity matrix of dimension n. We write $\mathcal{P}_n(\mathbb{R})$ for the space of positive-definite matrices of dimension n. For $X \in \mathbb{R}^{\ell \times n}$, we denote by $\mathrm{Vec}(X)$ its vectorisation, that is the vector in $\mathbb{R}^{\ell n}$ obtained from X by juxtaposing its columns and $\mathrm{etr}\,(X) := e^{\mathrm{tr}(X)}$, where $\mathrm{tr}(X)$ is the trace of X. We write $\gamma^{(\ell,n)}$ for the probability density function of $X \in \mathbb{R}^{\ell \times n}$ with i.i.d. real standard Gaussian entries, given by

$$\gamma^{(\ell,n)}(X) = (2\pi)^{-n\ell/2}\mathrm{etr}\left(-2^{-1}XX^T\right).$$

In this case, we write $X \sim \mathcal{N}_{\ell\times n}(0, \mathbf{I}_\ell \otimes \mathbf{I}_n)$ and refer to it as the *standard normal matrix distribution*. Here, $\otimes$ denotes the usual Kronecker product of matrices. When $\ell = n = 1$, we write $\gamma^{(1,1)} =: \gamma$ for the standard Gaussian density on $\mathbb{R}$.

As usual, for numerical sequences $\{a_n\}, \{b_n\}$, we write $a_n = O(b_n)$ or $a_n \ll b_n$ to indicate that there exists an absolute constant $C > 0$ such that $|a_n| \leq C|b_n|$ and $a_n = o(b_n)$ to indicate that $a_n/b_n \to 0$ as $n \to \infty$.

III.1 Introduction

In applications to stochastic geometry dealing with the asymptotic analysis of local geometric quantities associated with Gaussian random fields on manifolds, one is often confronted with expressions involving quantities of the type $F(X)$, where X is a rectangular Gaussian matrix and F is a certain *spectral* function, depending on the spectral measure associated with the matrix XX^T. For instance, as already discussed in Chapter II, if $Z = \{Z(x) : x \in \mathcal{M}\}$ is a ℓ-dimensional stationary Gaussian field on a manifold $\mathcal{M}$ of dimension n (with $1 \leq \ell \leq n$), the *nodal volume* of Z over a region $R \subset \mathcal{M}$ has typically the form (see in particular the Area/Co-Area formula in Proposition I.1.11)

$$\int_R \delta_0(Z(x))F(J_Z(x))\mathrm{vol}_{\mathcal{M}}(dx),$$

where δ_0 indicates the Dirac mass at zero, $J_Z(x)$ stands for the Jacobian matrix of Z computed at x and $F(X) = \sqrt{\det(XX^T)}$. We refer the reader for instance to [Wig10, MPRW16, MRW20, NPR19] for more works in this direction. While objects of this type are amenable to analysis by Wiener-Itô chaos expansions (which involves in particular the decomposition of $F(J_Z(x))$ into Hermite polynomials having the entries of $J_Z(x)$ as arguments, as discussed in Chapter II), it is to be expected that such a technique will generate combinatorially untractable expressions for large values of the dimensions ℓ and n. The aim of this chapter is to tackle directly such a difficulty by initiating a systematic study of chaotic expansions for spectral random variables $F(X)$ as above by using *matrix-variate Hermite polynomials*, that is, a collection of orthogonal polynomials with matrix entries which are indexed by partitions of integers, obtained by orthogonalizing matrix monomials of the type $\mathrm{tr}([XX^T]^s)$ with respect to the law of a Gaussian matrix. We will see that matrix-variate Hermite polynomials inherit the rich combinatorial structure and actually can be defined in terms of *zonal polynomials* introduced in [Jam61], thus allowing one to deduce explicit formulae in any dimension. We now describe the principal achievements of the present chapter.

(a) In [Tha93] (see also the related work [Koc96]), the author studies Hermite expansions of functions of the form $F(x) = f_0(\|x\|)P(x)$ on $\mathbb{R}^n$, where f_0 is a function depending only on the norm $\|x\|$ and P is a harmonic polynomial. In particular, in such a work, the author provides explicit formulae for the projection coefficients associated with the Wiener-Itô chaos expansion of functionals F as above in terms of *Laguerre polynomials* on the real line. In Theorem III.3.2, we extend this framework by studying matrix-Hermite expansions of radial functionals of the type $F(X) = f_0(XX^T)$ on matrix spaces. Our results involve *generalized Laguerre polynomials* with matrix argument, thus yielding a natural counterpart to the work by Thangavelu [Tha93] in higher dimensions.

(b) In [ZK12, Theorem 1.1], Kabluchko and Zaporozhets establish a formula for the expected value of Gaussian determinants of the form $F(X) = \sqrt{\det(XX^T)}$ in terms of mixed volumes and intrinsic volumes of ellipsoids associated with the covariance matrices of the underlying Gaussian vectors, yielding in particular an expression for the projection of F onto the Gaussian Wiener chaos of order zero associated with X. In Theorem III.3.5 and Theorem III.3.6 of the present chapter, we substantially extend their framework by considering *arbitrary* projection coefficients of the form $\mathbb{E}\left[F(X)H_\kappa^{(\ell,n)}(X)\right]$ (where X is a Gaussian matrix of dimension $\ell \times n$) associated with such random determinants. Our results can be formulated using integrations on the so-called *Stiefel manifold* (see Theorem III.3.5), which can subsequently be interpreted in terms of mixed and intrinsic volumes (see Theorem III.3.6).

(c) In Section III.3.3, we introduce a collection of operators on matrix spaces via a *Mehler-type* formula, whose definition is amenable to that of the classical Ornstein-Uhlenbeck semigroup on the Euclidean space $\mathbb{R}^n$. In Theorem III.3.10, we characterize the action of the generalized

Ornstein-Uhlenbeck operators on matrix-Hermite polynomials: it turns out that matrix-Hermite polynomials are eigenfunctions of this semigroup, yielding a direct analog of the action of the Ornstein-Uhlenbeck semigroup on classical Hermite polynomials on the real line (see Proposition I.1.22). We subsequently use Theorem III.3.10 in order to deduce an intrinsic orthogonality relation between two matrix-Hermite polynomials evaluated in correlated Gaussian matrices. Such a result extends the classical orthogonality relation for matrix-Hermite polynomials as well as the case of Hermite polynomials on the real line. Conjecturally, the objects and techniques introduced in Section III.3.3 generate a basis for a special Malliavin Calculus on matrix spaces via the introduction of further operators, such as Malliavin derivatives, adjoints and generators of the Ornstein-Uhlenbeck semigroup (see e.g. [Nua95, NP12a]).

(d) In Section III.3.4, we apply our results to the study of the *generalized total variation* of multi-dimensional Gaussian random fields, defined as the integral of the square root of the Gramian determinant of its normalized Jacobian matrix. More specifically, we study the high-energy behaviour of the generalized total variation of multiple independent Arithmetic Random Waves on the three-torus. In particular, in Theorem III.3.17 we establish its expected mean, an asymptotic law for its variance and a Central Limit Theorem for the suitably normalized total variation. Our arguments rely on the expansion in matrix-Hermite polynomials of the total variation, allowing us to prove that its probabilistic fluctuations are entirely characterized by its projection on the second Wiener chaos. Throughout this application, we also make use of variance expansions of radial functionals by means of its projection coefficients (see Proposition III.3.3). Our findings are to be compared with Theorem 1 of Peccati and Rossi [PR18], where the authors prove a CLT for the *Leray measure* of Arithmetic Random Waves on the two-torus.

The organization of chapter is as follows: In Section III.2, we present preliminary notions that will be used in our proofs, notably on zonal polynomials and generalized Laguerre polynomials (Section III.2.1), polar matrix factorizations (Section III.2.2) and tools from integral geometry such as mixed volumes, intrinsic volumes of convex bodies and general facts about ellipsoids (Section III.2.3). Our main contributions are presented in Section III.3. Finally, the entire Section III.4 is devoted to the proofs of our results. In Appendix III.A and III.B, we present two proofs of technical results for completeness.

III.2 Preliminaries

III.2.1 Zonal polynomials and generalized Laguerre polynomials

Zonal polynomials. Zonal polynomials with matrix argument were introduced in [Jam61], using group representation theory, as certain homogeneous symmetric functions of the eigenvalues (also called the *latent roots*) of the matrix. We give a brief overview of zonal polynomials and their properties; the reader is referred for instance to the books by Mathai, Provost and Hayakawa [MPH95] and Chikuse [Chi03] for a thorough introduction to zonal polynomials. Let us now fix integers $\ell \geq 1$ and $k \geq 0$. We write $\kappa \vdash k$ to denote a *partition* κ of k into no more than ℓ integer parts (note that such a notation does not involve the integer ℓ, whose role should be understood from the context), that is

$$\kappa = (k_1, \ldots, k_\ell), \quad k_1 \geq k_2 \geq \ldots \geq k_\ell \geq 0, \quad k_1 + \ldots + k_\ell = k.$$

For instance, if $\ell = 1$, then $\kappa = (k)$ is the only partition of an integer k; if $\ell \geq 2$, then $\kappa = (2)$ and $\kappa = (1, 1)$ are the only partitions of $k = 2$. Sometimes it is useful to represent the partition $\kappa \vdash k$ as $\kappa = (1^{\nu_1} 2^{\nu_2} \ldots k^{\nu_k})$, to indicate that the integer j occurs with multiplicity ν_j; in particular $\nu_1 + 2\nu_2 + \ldots + k\nu_k = k$. With this notation, we have for instance $(1, 1) = (1^2) \vdash 2$ and $(1, 2, 3, 3) = (1^1 2^1 3^2) \vdash 9$.

Let $S \in \mathbb{R}^{\ell\times\ell}$ be a symmetric matrix with eigenvalues $s_1, \dots, s_\ell$. For an integer $k \geq 1$, we denote by $\mathrm{Pol}_k(S)$ the space of homogeneous polynomials of degree k in the $\ell(\ell+1)/2$ variables of S. For an invertible matrix $L \in \mathbb{R}^{\ell\times\ell}$, the transformation $S \to LSL^T$ induces a representation π of $\mathrm{GL}_\ell(\mathbb{R})$ into the vector space $\mathrm{GL}(\mathrm{Pol}_k(S))$ of isomorphisms from $\mathrm{Pol}_k(S)$ to itself ([Chi03, Eq.(A.2.1)]):

$$\pi : \mathrm{GL}_\ell(\mathbb{R}) \to \mathrm{GL}(\mathrm{Pol}_k(S)) \ ; \quad L \to \pi(L) \,,$$

given by $\pi(L)(P) := P(L^{-1}S(L^{-1})^T)$. It can be shown that $\mathrm{Pol}_k(S)$ can be decomposed as direct sum ([Chi03, p.297])

$$\mathrm{Pol}_k(S) = \bigoplus_{\kappa\vdash k} V_\kappa(S), \tag{III.2.1}$$

where $\{V_\kappa(S) : \kappa \vdash k\}$ are irreducible and π-invariant subspaces. Since $\mathrm{tr}(S)^k$ is a homogeneous symmetric polynomial of degree k in the eigenvalues of S, it can accordingly be decomposed in the spaces $V_\kappa(S)$ as follows ([MPH95, Eq.(4.3.38)]),

$$\mathrm{tr}(S)^k = (s_1 + \ldots + s_\ell)^k = \sum_{\kappa\vdash k} C_\kappa(S), \tag{III.2.2}$$

where $C_\kappa(S)$ denotes the *zonal polynomial* associated with the partition κ of k, that is, $C_\kappa(S)$ is the projection of $\mathrm{tr}(S)^k$ onto the space $V_\kappa(S)$. Applying (III.2.2) with $\ell = 1$ gives $C_{(k)}(s) = s^k$, so that zonal polynomials can be interpreted as a generalization of classical monomials. In particular, evaluating at $s = 1$ yields $C_{(k)}(1) = 1$. Zonal polynomials satisfy a generalized binomial formula ([MPH95, Eq.(4.5.1)]),

$$\frac{C_\kappa(S + \mathbf{I}_\ell)}{C_\kappa(\mathbf{I}_\ell)} = \sum_{s=0}^{k}\sum_{\sigma\vdash s} \binom{\kappa}{\sigma} \frac{C_\sigma(S)}{C_\sigma(\mathbf{I}_\ell)}, \quad \kappa \vdash k. \tag{III.2.3}$$

This relation in particular defines the generalized binomial coefficients $\binom{\kappa}{\sigma}$. Taking $S = a\,\mathbf{I}_\ell$ for $a \in \mathbb{R}$ in (III.2.3) yields

$$\frac{C_\kappa((a+1)\,\mathbf{I}_\ell)}{C_\kappa(\mathbf{I}_\ell)} = \sum_{s=0}^{k}\sum_{\sigma\vdash s} \binom{\kappa}{\sigma} \frac{C_\sigma(a\,\mathbf{I}_\ell)}{C_\sigma(\mathbf{I}_\ell)},$$

so that, using the homogeneity property of zonal polynomials gives

$$(a+1)^k = \sum_{s=0}^{k}\sum_{\sigma\vdash s} a^s \binom{\kappa}{\sigma}.$$

In particular, using the usual binomial formula for real numbers on the left-hand side, one deduces a relation linking classical and generalized binomial coefficients ([MPH95, Eq.(4.5.2)]):

$$\binom{k}{s} = \sum_{\sigma\vdash s} \binom{\kappa}{\sigma}.$$

A table with generalized binomial coefficients up to order 5 can be found in [MPH95, Table 4.4.1]. For $X \in \mathbb{R}^{\ell\times n}$, zonal polynomials associated with partition $\kappa \vdash k$ and matrix argument XX^T can be decomposed as ([MPH95, Theorem 4.3.6])

$$C_\kappa(XX^T) = \sum_{(1^{\nu_1}2^{\nu_2}\ldots k^{\nu_k})\vdash k} z_{\kappa\nu}^{(k)} t_1(X)^{\nu_1} \ldots t_k(X)^{\nu_k}, \tag{III.2.4}$$

where

$$t_s(X) := \operatorname{tr}([XX^T]^s), \quad s \geq 1 \tag{III.2.5}$$

and $z^{(k)}_{\kappa\nu}$ are numerical constants. Writing $t_j := t_j(X)$, the zonal polynomials associated with partitions up to order 3 are given by (see e.g. [MPH95, Table 4.3.1])

$$C_{(1)}(XX^T) = t_1$$
$$C_{(2)}(XX^T) = \frac{1}{3}(t_1^2 + 2t_2), \quad C_{(1,1)}(XX^T) = \frac{2}{3}(t_1^2 - t_2)$$
$$C_{(3)}(XX^T) = \frac{1}{15}(t_1^3 + 6t_1t_2 + 8t_3), \quad C_{(2,1)}(XX^T) = \frac{3}{5}(t_1^3 + t_1t_2 - 2t_3)$$
$$C_{(1,1,1)}(XX^T) = \frac{1}{3}(t_1^3 - 3t_1t_2 + 2t_3).$$

In particular, since for every $j \in [k]$, $t_j(X)^{\nu_j}$ is a homogeneous polynomial of degree $2j\nu_j$ in the entries of X, it follows from (III.2.4) that $C_\kappa(XX^T)$ is a homogeneous polynomial of degree $2k$ in the entries of X, that is,

$$C_\kappa(XX^T) = \sum_{|\alpha|=2k} z^\kappa_\alpha \prod_{i=1}^{\ell} \prod_{j=1}^{n} X_{ij}^{\alpha_{ij}}, \tag{III.2.6}$$

where $\alpha = (\alpha_{ij}) \in \mathbb{N}^{\ell \times n}$ is a multi-index such that $|\alpha| = \sum_{i=1}^{\ell} \sum_{j=1}^{n} \alpha_{ij} = 2k$ and z^κ_α is a numerical constant depending on α and κ. Zonal polynomials evaluated at the identity matrix $\mathbf{I}_\ell$ can be computed to be ([Chi03, Eq.(A.2.7)])

$$C_\kappa(\mathbf{I}_\ell) = 2^{2k} k! \left(\frac{\ell}{2}\right)_\kappa \frac{\prod_{i<j}^{p}(2k_i - 2k_j - i + j)}{\prod_{j=1}^{p}(2k_j + p - j)!},$$

where $p = p(\kappa)$ is the number of non-zero parts in κ, and $(a)_\kappa$ stands for the generalized Pochammer symbol ([Chi03, Eq.(A.2.4)])

$$(a)_\kappa := \prod_{j=1}^{\ell} \left(a - \frac{j-1}{2}\right)_{k_j}, \quad (a)_n = a(a+1)\cdots(a+n-1) \tag{III.2.7}$$

defined in terms of classical Pochammer symbols $(a)_n$. The product of two zonal polynomials associated with partitions $\tau \vdash t$ and $\sigma \vdash s$ respectively, is given by ([MPH95, Eq.(4.3.65)])

$$C_\tau(S)C_\sigma(S) = \sum_{\kappa \vdash t+s} a^\kappa_{\tau,\sigma} C_\kappa(S), \tag{III.2.8}$$

for some uniquely determined coefficients $a^\kappa_{\tau,\sigma}$. A table for these coefficients is found in Table 4.3.2(a) of [MPH95]. Moreover, for positive-definite matrices S and T, zonal polynomials enjoy the property ([MPH95, Eq.(4.3.18)])

$$C_\kappa(S^{1/2}TS^{1/2}) = C_\kappa(ST) = C_\kappa(TS) = C_\kappa(T^{1/2}ST^{1/2}). \tag{III.2.9}$$

Generalized Laguerre polynomials. For a symmetric matrix $S \in \mathbb{R}^{\ell \times \ell}$, the generalized Laguerre polynomial of order $\gamma > -1$ associated with a partition κ of k and matrix variable S is defined as ([MPH95, Eq.(4.6.5)])

$$L^{(\gamma)}_\kappa(S) = \left(\gamma + \frac{\ell+1}{2}\right)_\kappa C_\kappa(\mathbf{I}_\ell) \sum_{s=0}^{k} \sum_{\sigma \vdash s} \binom{\kappa}{\sigma} \frac{(-1)^s}{(\gamma + \frac{\ell+1}{2})_\sigma} \frac{C_\sigma(S)}{C_\sigma(\mathbf{I}_\ell)}. \tag{III.2.10}$$

The first Laguerre polynomials associated with partitions up to order three are listed in [MPH95, Eq.(4.6.8)]. The generalized Laguerre polynomials define a class of orthogonal polynomials on $\mathcal{P}_\ell(\mathbb{R})$ with respect to the weight function $\operatorname{etr}(-R)\det(R)^\gamma$, that is, for every integers $k, l \geq 0$ and every partitions $\kappa \vdash k, \sigma \vdash l$, one has ([MPH95, Theorem 4.6.4])

$$\begin{aligned}&\int_{\mathcal{P}_\ell(\mathbb{R})} L_\kappa^{(\gamma)}(R) L_\sigma^{(\gamma)}(R) \operatorname{etr}(-R)\det(R)^\gamma \nu(dR)\\ &= \mathbb{1}_{\kappa=\sigma}\cdot k! C_\kappa(\mathbf{I}_\ell)\Gamma_\ell\left(\gamma+\frac{\ell+1}{2}\right)\left(\gamma+\frac{\ell+1}{2}\right)_\kappa,\end{aligned} \tag{III.2.11}$$

where $\nu(dR)$ denotes the Lebesgue measure on $\mathcal{P}_\ell(\mathbb{R})$, for $a \in \mathbb{R}$, $\Gamma_\ell(a)$ denotes the multivariate Gamma function defined by

$$\Gamma_\ell(a) := \pi^{\ell(\ell-1)/4}\prod_{i=1}^{\ell}\Gamma(a-2^{-1}(i-1)), \quad \ell \geq 1,$$

where $\Gamma(\cdot)$ is the usual Gamma function. A useful formula that we will use at several occasions is the following (see e.g. [MPH95, Theorem 4.4.1])

$$\int_{\mathcal{P}_\ell(\mathbb{R})} \operatorname{etr}(-AR)\det(R)^{t-\frac{\ell+1}{2}} C_\kappa(RB)\nu(dR) = (t)_\kappa \Gamma_\ell(t)\det(A)^{-t} C_\kappa(BA^{-1}), \tag{III.2.12}$$

where $A \in \mathbb{C}^{\ell\times\ell}$ is a complex symmetric matrix with positive real part, $B \in \mathbb{C}^{\ell\times\ell}$ is a complex symmetric matrix and t is such that $\Re(t) > (\ell-1)/2$.

III.2.2 Polar decomposition for matrices

Let $1 \leq \ell \leq n$ be integers. For $X = (X_{ij}) \in \mathbb{R}^{\ell\times n}$, we denote by $dX := (dX_{ij})$ its associated differential matrix. We endow the spaces $\mathbb{R}^{\ell\times n}$ and $\mathcal{P}_\ell(\mathbb{R})$ with the measures

$$(dX) := \prod_{i=1}^{\ell}\prod_{j=1}^{n} dX_{ij}, \quad \nu(dX) := \prod_{1\leq i\leq j\leq \ell} dX_{ij}$$

respectively. Assuming that the rows of X are linearly independent, the *polar decomposition* of X is uniquely given by (see for instance [Dow72])

$$X = R^{1/2}\cdot U, \quad R = XX^T \in \mathcal{P}_\ell(\mathbb{R}), \qquad U = (XX^T)^{-1/2}X \in O(n,\ell), \tag{III.2.13}$$

where $R^{1/2}$ denotes the positive square root of R, that is the unique matrix B such that $B^2 = R$. We also define $R^{-1/2} := (R^{1/2})^{-1}$. The space $O(n,\ell)$ in (III.2.13) denotes the so-called *Stiefel manifold* of matrices $Y \in \mathbb{R}^{\ell\times n}$ such that $YY^T = \mathbf{I}_\ell$, that is, Y has orthonormal rows. An element of $O(n,\ell)$ is called an *ℓ-frame in* $\mathbb{R}^n$, see for instance [Chi03, p.8]. The matrices R and U in (III.2.13) are seen to be the radial part and orientation of X, respectively and hence the decomposition $X = R^{1/2}U$ is a generalization of the standard polar factorization for vectors (obtained for $\ell = 1$).

Haar measure on the Stiefel manifold. The family of Stiefel manifolds $O(n,\ell)$ contains as special cases the n-shpere $O(n,1) = \mathbb{S}^{n-1}$ and the orthogonal group $O(n,n) = O(n)$. The space $O(n,\ell)$ is the compact manifold of dimension $n\ell - \ell - \ell(\ell-1)/2$ realized as the homogeneous space $O(n)/O(n-\ell)$. The Stiefel manifold is endowed with a left and right-invariant Haar measure μ, that is, for every $P \in O(n)$ and every $Q \in O(\ell)$,

$$\mu(UP) = \mu(U) = \mu(QU),$$

for every $U \in O(n, \ell)$. Remark that our notation of μ is independent of ℓ and n, and should be understood from the context. We refer the reader for instance to [Chi03] or [Mui82] for details on the construction of such a measure. The total volume of $O(n, \ell)$ is given by ([Chi03, Eq.(1.4.8)])

$$v(n, \ell) := \mu(O(n, \ell)) = \int_{O(n,\ell)} \mu(dU) = \frac{2^\ell \pi^{n\ell/2}}{\Gamma_\ell(n/2)}.$$

The normalized measure

$$\tilde{\mu}(dU) := \frac{1}{v(n, \ell)} \mu(dU) \tag{III.2.14}$$

hence defines a left and right invariant probability measure on $O(n, \ell)$. We call it the *Haar probability measure* on $O(n, \ell)$.

III.2.3 Intrinsic volumes, mixed volumes and ellipsoids

<u>*Intrinsic volumes and mixed volumes.*</u> We present two important notions from integral geometry: intrinsic and mixed volumes. We mainly follow the book by Schneider and Weil [SW08] for this part (see in particular Section 14.2 therein). For an integer $n \geq 1$, we denote by $\mathbb{K}^n$ the set of convex bodies in $\mathbb{R}^n$. We write $\mathbb{B}_n$ for the unit ball in $\mathbb{R}^n$ and vol_n for the n-dimensional volume measure in $\mathbb{R}^n$. For $K \in \mathbb{K}^n$ and $\varepsilon > 0$, we write

$$K^\varepsilon := K + \varepsilon \mathbb{B}_n = \{x \in \mathbb{R}^n : \mathrm{dist}(x, K) \leq \varepsilon\}$$

for the parallel body of K at distance ε. *Steiner's formula* ([SW08, Eq.(14.5)]) asserts that its volume is a polynomial of degree n in ε,

$$\mathrm{vol}_n(K^\varepsilon) = \sum_{j=0}^{n} \varepsilon^{n-j} \kappa_{n-j} V_j(K), \tag{III.2.15}$$

where the coefficients $\left\{V_j(K), j = 0, \ldots, n\right\}$ denote the *intrinsic volumes* of K. We set $V_j(\emptyset) := 0$. For instance, when $n = 2$, $V_2(K)$ is the area, $V_1(K)$ is half the boundary length and $V_0(K)$ is the Euler characteristic of K. Moreover, for every $n \geq 1$, we have $V_n(K) = \mathrm{vol}_n(K)$, that is, the n-th intrinsic volume coincides with the n-dimensional volume measure. The intrinsic volumes of the unit ball $\mathbb{B}_n$ can be computed to be ([SW08, Eq.(14.8)])

$$V_j(\mathbb{B}_n) = \binom{n}{j} \frac{\kappa_n}{\kappa_{n-j}}, \quad \kappa_n = \frac{\pi^{n/2}}{\Gamma(1 + n/2)}.$$

For an integer $1 \leq j \leq n$, we denote by $G(n, j)$ the Grassmannian of j-dimensional linear subspaces of $\mathbb{R}^n$. It carries a unique invariant Haar probability measure $\nu_{n,j}$. One possible way to realize Grassmannians is as the quotient space $G(n, j) = O(n, j)/O(j)$, where two elements $U_1, U_2 \in O(n, j)$ are equivalent if and only if there exists an orthogonal matrix $Q \in O(j)$ such that $U_1 = QU_2$, see for instance [Chi03, p.8-9]. Intrinsic volumes admit a useful integral representation, known as *Kubota's formula* ([SW08, Eq.(6.11)]),

$$V_j(K) = \binom{n}{j} \frac{\kappa_n}{\kappa_j \kappa_{n-j}} \int_{G(n,j)} \mathrm{vol}_j(K|\mathscr{L}) \nu_{n,j}(d\mathscr{L}), \tag{III.2.16}$$

where $K|\mathscr{L}$ stands for the image of the orthogonal projection of K onto $\mathscr{L} \in G(n, j)$, and integration is with respect to the Haar probability measure on $G(n, j)$.

Let $m \geq 1$ and consider m convex bodies $K_1, \ldots, K_m \in \mathbb{K}^n$. Then, for real numbers $\lambda_1, \ldots, \lambda_m \geq 0$, the n-dimensional volume of the Minkowski sum $\lambda_1 K_1 + \ldots \lambda_m K_m$ is a homogeneous polynomial of degree n in the variables $\lambda_1, \ldots, \lambda_m$ ([SW08, Eq.(14.7)]),

$$\mathrm{vol}_n(\lambda_1 K_1 + \ldots \lambda_m K_m) = \sum_{i_1, \ldots, i_n = 1}^{m} V(K_{i_1}, \ldots, K_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n},$$

for uniquely determined symmetric coefficients $V(K_{i_1}, \ldots, K_{i_n})$. These coefficients are called the *mixed volumes* of the convex bodies $K_{i_1}, \ldots, K_{i_n}$. This formula is a generalization of Steiner's formula in (III.2.15). Whenever we have mixed volumes involving only two distinct convex bodies K_1 and K_2, we use the short-hand notation

$$V(\underbrace{K_1, \ldots, K_1}_{\ell \text{ times}}, \underbrace{K_2, \ldots, K_2}_{n-\ell \text{ times}}) =: V(K_1[\ell], K_2[n-\ell]), \quad \ell \geq 1. \tag{III.2.17}$$

Intrinsic volumes of a convex body $K \in \mathbb{K}^n$ are related to mixed volume by the relation ([SW08, Eq.(14.18)])

$$V_j(K) = \frac{\binom{n}{j}}{\kappa_{n-j}} V(K[j], \mathbb{B}_n[n-j]), \quad j = 1, \ldots, n, \tag{III.2.18}$$

where we used notation (III.2.17).

General facts about ellipsoids. We will need some preliminaries about a particular type of convex bodies, namely ellipsoids, see e.g. [ZK12]. Let $\Sigma \in \mathbb{R}^{n \times n}$ be a non-singular symmetric matrix. We define the ellipsoid $\mathcal{E}_\Sigma = \left\{x \in \mathbb{R}^n : x^T \Sigma^{-1} x \leq 1\right\}$ of $\mathbb{R}^n$ represented by the matrix Σ, obtained as an affinity of the unit n-dimensional ball $\mathbb{B}_n$, that is $\mathcal{E}_\Sigma = \left\{\Sigma^{1/2} y : y \in \mathbb{B}_n\right\}$. In particular, its n-dimensional volume is given by

$$\mathrm{vol}_n(\mathcal{E}_\Sigma) = \kappa_n \det(\Sigma)^{1/2}. \tag{III.2.19}$$

For any non-degenerate linear transformation represented by a matrix $A \in \mathbb{R}^{n \times n}$, the ellipsoid $A\mathcal{E}_\Sigma = \{Ax : x \in \mathcal{E}_\Sigma\}$ is represented by the matrix $A \Sigma A^T$, that is

$$A\mathcal{E}_\Sigma = \left\{x \in \mathbb{R}^n : x^T (A \Sigma A^T)^{-1} x \leq 1\right\} = \mathcal{E}_{A \Sigma A^T}.$$

Let $\mathscr{L} \in G(n, \ell)$ be a ℓ-dimensional linear subspace in $\mathbb{R}^n$ and denote by $L \in O(n, \ell)$ any matrix whose rows form an orthonormal basis of $\mathscr{L}$. Then, the image of the orthogonal projection of $\mathcal{E}_\Sigma$ onto $\mathscr{L}$, written $\mathcal{E}_\Sigma | \mathscr{L}$, is an ellipsoid in $\mathbb{R}^\ell$ that is represented by the matrix $L \Sigma L^T \in \mathbb{R}^{\ell \times \ell}$. In particlar, it follows from (III.2.19) that its ℓ-dimensional volume is $\mathrm{vol}_\ell(\mathcal{E}_\Sigma | \mathscr{L}) = \kappa_\ell \det(L \Sigma L^T)^{1/2}$.

III.3 Main results

III.3.1 Wiener-chaos expansion of matrix-variate functions

Hermite polynomials on the real line. Let $m \geq 1$ be an integer and $\mathbf{X} = (X_1, \ldots, X_m)$ be a standard m-dimensional Gaussian vector. For $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$, we write $\alpha! := \alpha_1! \cdots \alpha_m!$ and $|\alpha| := \alpha_1 + \ldots + \alpha_m$ and define the *multivariate Hermite polynomials* associated with the vector $(X_1, \ldots, X_m)$ as the tensor product of univariate Hermite polynomials, that is

$$H_\alpha^{\otimes m}(X_1, \ldots, X_m) := \prod_{l=1}^{m} H_{\alpha_l}(X_l),$$

where H_{α_l} denotes the Hermite polynomial of order α_l on the real line. It is well-known that the normalized Hermite polynomials $\left\{(k!)^{-1/2}H_k : k \geq 0\right\}$ form a complete orthonormal system of $L^2(\gamma) := L^2(\mathbb{R}, \gamma(z)dz)$. This implies that the collection of normalized multivariate Hermite polynomials

$$\mathbb{H}_{[m]} = \left\{(\alpha!)^{-1/2}H_\alpha^{\otimes m} : \alpha \in \mathbb{N}^m\right\} \tag{III.3.1}$$

form a complete orthonormal system of $L^2(\gamma^{\otimes m})$, where $\gamma^{\otimes m}$ stands for the standard m-dimensional Gaussian measure. In particular, every random variable $F \in L^2(\phi^{\otimes m})$ admits a unique decomposition

$$F = \sum_{k\geq 0} \sum_{|\alpha|=k} \widehat{F}(\alpha) H_\alpha^{\otimes m}, \tag{III.3.2}$$

where

$$\widehat{F}(\alpha) := (\alpha!)^{-1} \int_{\mathbb{R}^m} F(x_1, \ldots, x_m) H_\alpha^{\otimes m}(x_1, \ldots, x_m) \phi^{\otimes m}(dx_1, \ldots, dx_m) \tag{III.3.3}$$

denotes the Fourier-Hermite coefficients of F associated with the multi-index α. For $k \geq 0$, we write

$$C_k^{\mathbf{X}} = \overline{\operatorname{span}_{\mathbb{R}}} \left\{H_\alpha^{\otimes m}(X_1, \ldots, X_m) : |\alpha| = k\right\} \tag{III.3.4}$$

for the closed linear subspace of $L^2(\mathbb{P})$ generated by multivariate Hermite polynomials of cumulative degree k. The space $C_k^{\mathbf{X}}$ is the so-called *k-th Wiener chaos* associated with the vector $\mathbf{X} = (X_1, \ldots, X_m)$. We have that $C_0^{\mathbf{X}} = \mathbb{R}$. For $F \in L^2(\gamma^{\otimes m})$, we denote by $\operatorname{proj}(F|C_k^{\mathbf{X}})$ the projection of F onto C_k^X, that is, (III.3.2) can be rewritten as the $L^2(\mathbb{P})$-converging series

$$F = \sum_{k\geq 0} \operatorname{proj}(F|C_k^{\mathbf{X}}).$$

This decomposition is known as the *Wiener-Itô chaos expansion of F*. We refer the reader to Section I.1.2 of Chapter I for a concise introduction on this.

Matrix-variate Hermite polynomials. Matrix-variate Hermite polynomials on the matrix space $\mathbb{R}^{\ell\times n}$ are introduced in [Chi92] and admit an expansion in *zonal polynomials*. More specifically, the matrix-variate Hermite polynomials associated with the partition $\kappa \vdash k$ of an integer $k \geq 0$, written $H_\kappa^{(\ell,n)}$, is given by ([Chi92, Eq. (4.11)]):

$$H_\kappa^{(\ell,n)}(X) = k! C_\kappa(\mathbf{I}_\ell) \sum_{s=0}^{k} \sum_{\sigma\vdash s} \sum_{\tau\vdash k-s} \frac{a_{\tau,\sigma}^\kappa}{(-2)^{k+s}(k-s)!} \frac{C_\sigma(XX^T)}{s!(\frac{n}{2})_\sigma C_\sigma(\mathbf{I}_\ell)}, \tag{III.3.5}$$

where the coefficients $a_{\tau,\sigma}^\kappa$ are defined by the relation (III.2.8) and $(\ell/2)_\sigma$ denotes the generalized Pochammer symbol, formally defined in (III.2.7). Zonal polynomials being generalizations of monomials, the expansion in (III.3.5) is to be compared to the classical expansion of univariate Hermite polynomials in the basis of monomials (see e.g. [NP12a, p.19])

$$H_k(x) = \sum_{n=0}^{\lfloor k/2 \rfloor} \frac{k!(-1)^n}{n!(k-2n)!2^n} x^{k-2n}.$$

Alternatively, $H_\kappa^{(\ell,n)}$ are defined by Rodrigues formula ([Chi92, Eq.(4.9)])

$$H_\kappa^{(\ell,n)}(X)\gamma^{(\ell,n)}(X) = 4^{-k} \left(\frac{n}{2}\right)_\kappa^{-1} C_\kappa(\partial X \partial X^T) \gamma^{(\ell,n)}(X), \tag{III.3.6}$$

where, for $X = (X_{ij}) \in \mathbb{R}^{\ell\times n}$, the differential matrix ∂X is given by $\partial X = \left(\frac{\partial}{\partial X_{ij}}\right)$. We note that (III.3.6) is a generalization of the classical well-known Rodrigues formula for univariate Hermite polynomials (see (I.1.17))

$$H_k(x)\gamma(x) = (-1)^k \frac{d^k}{dx^k}\gamma(x), \quad k \geq 0. \tag{III.3.7}$$

Matrix-variate Hermite polynomials are linked to the generalized matrix-variate Laguerre polynomials by the relation ([Chi92, Eq. (5.16)] and [Hay69, Eq. (10)])

$$\gamma_\kappa \cdot L_\kappa^{(\frac{n-\ell-1}{2})}(XX^T) = H_\kappa^{(\ell,n)}(\sqrt{2}X), \quad \gamma_\kappa := (-2)^{-k}\left(\frac{n}{2}\right)_\kappa^{-1}, \quad \kappa \vdash k. \tag{III.3.8}$$

A proof of this fact is presented in Appendix III.B. Moreover, matrix-variate Hermite polynomials are orthonormal on $\mathbb{R}^{\ell\times n}$ with respect to the matrix-normal density function $\gamma^{(\ell,n)}$, that is for every integers $k, l \geq 0$ and every partitions $\kappa \vdash k, \sigma \vdash l$ (see e.g. [Hay69, Corollary 3]),

$$\int_{\mathbb{R}^{\ell\times n}} H_\kappa^{(\ell,n)}(X) H_\sigma^{(\ell,n)}(X)\gamma^{(\ell,n)}(X)(dX) = \mathbb{1}_{\kappa=\sigma} \cdot 4^{-k}\left(\frac{n}{2}\right)_\kappa^{-1} k! C_\kappa(\mathbf{I}_\ell). \tag{III.3.9}$$

Let now $X \sim \mathcal{N}_{\ell\times n}(0, \mathbf{I}_\ell \otimes \mathbf{I}_n)$ and write $s_1, \ldots, s_\ell$ for the eigenvalues of XX^T. The *spectral measure* of XX^T associated with the matrix X is the measure

$$\mu_X(ds) := \sum_{i=1}^{\ell} \delta_{s_i}(ds),$$

supported on the spectrum of XX^T, where δ_y is the Dirac mass at y. We write $L^2(\mu_X) := L^2(\Omega, \sigma(\mu_X), \mathbb{P})$ to indicate the subspace of $L^2(\gamma^{(\ell,n)}) := L^2(\mathbb{R}^{\ell\times n}, \gamma^{(\ell,n)}(X)(dX))$ consisting of those random variables that are measurable with respect to the sigma algebra generated by μ_X. By this, we mean the subspace of $L^2(\gamma^{(\ell,n)})$ of random variables that are generated by elements of the type

$$\int_{\mathbb{R}} f(t)\mu_X(dt) =: \mu_X(f). \tag{III.3.10}$$

Since matrix-variate Hermite polynomials in (III.3.5) admit an expansion into zonal polynomials, they are themselves symmetric functionals of the eigenvalues $s_1, \ldots, s_\ell$. This fact together with the orthogonality relation (III.3.9), implies that the family of normalized matrix-variate Hermite polynomials

$$\mathbb{H}_{[\ell\times n]} := \left\{c(\kappa)^{-1/2} H_\kappa^{(\ell,n)} : \kappa \vdash k, k \geq 0\right\}, \quad c(\kappa) := 4^{-k}\left(\frac{n}{2}\right)_\kappa^{-1} k! C_\kappa(\mathbf{I}_\ell) \tag{III.3.11}$$

forms an orthonormal system in $L^2(\mu_X)$. In Appendix III.A, we prove the following Proposition, stating that this system is also complete in $L^2(\mu_X)$.

Proposition III.3.1. *The system* $\mathbb{H}_{[\ell\times n]}$ *in* (III.3.11) *is complete in* $L^2(\mu_X)$.

Therefore, every $F \in L^2(\mu_X)$ admits a unique decomposition in the basis (III.3.11),

$$F = \sum_{k\geq 0}\sum_{\kappa\vdash k} \widehat{F}(\kappa) H_\kappa^{(\ell,n)}, \tag{III.3.12}$$

where

$$\widehat{F}(\kappa) := c(\kappa)^{-1}\int_{\mathbb{R}^{\ell\times n}} F(X) H_\kappa^{(\ell,n)}(X)\gamma^{(\ell,n)}(X)(dX) \tag{III.3.13}$$

is the Fourier-Hermite coefficient of F associated with the partition κ and $c(\kappa)$ is as in (III.3.11). To state our result, we introduce some further notation. For an integer $s \geq 0$ and $X \sim \mathcal{N}_{\ell\times n}(0, \mathbf{I}_\ell \otimes \mathbf{I}_n)$, we recall the notation $t_s(X) := \operatorname{tr}([XX^T]^s)$ introduced in (III.2.5) and define the spaces

$$\mathcal{U}_0^X := \mathbb{R}, \quad \mathcal{U}_k^X := \overline{\operatorname{span}_{\mathbb{R}}} \left\{ \prod_{j=1}^m t_{s_j}(X) : s_1 + \ldots + s_m \leq k, m \geq 1 \right\}, \ k \geq 1,$$

where the closure is with respect to $L^2(\mu_X)$. By construction, we have that $\mathcal{U}_k^X \subset \mathcal{U}_{k+1}^X$. We let

$$\mathbf{U}_k^X := \mathcal{U}_k^X \ominus \mathcal{U}_{k-1}^X := \mathcal{U}_k^X \cap (\mathcal{U}_{k-1}^X)^\perp,$$

that is, $\mathbf{U}_k^X$ is the space of those random variables in $\mathcal{U}_k^X$ that are orthogonal in $L^2(\mathbb{P})$ to elements of $\mathcal{U}_{k-1}^X$. Expanding matrix-Hermite polynomials into zonal polynomials by (III.3.5) and subsequently zonal polynomials into monomials of the type $t_s(X)$ by (III.2.4) shows that Hermite polynomials admit an expansion into monomials $t_s(X)$. In particular, since Hermite polynomials are orthogonal in view of (III.3.9), it follows that

$$\mathbf{U}_k^X = \overline{\operatorname{span}_{\mathbb{R}}} \left\{ H_\kappa^{(\ell,n)}(X) : \kappa \vdash k \right\}.$$

The following result links matrix-variate Hermite polynomials with the classical Wiener-Itô decomposition in (III.3.2). In particular, we establish an explicit formula for projection coefficients associated with radial functionals of the form $F(X) = f_0(XX^T) \in L^2(\mu_X)$ (where $X \sim \mathcal{N}_{\ell\times n}(0, \mathbf{I}_\ell \otimes \mathbf{I}_n)$) in terms of generalized Laguerre polynomials (see Section III.2.1 for definitions). Such a formula is to be compared to [Tha93, Koc96], where the authors study Hermite expansions of functions of the form $F(x) = f_0(\|X\|)P(x)$ on $\mathbb{R}^n$, where P is a harmonic polynomial.

Theorem III.3.2. *For integers $1 \leq \ell \leq n$, let $X \sim \mathcal{N}_{\ell\times n}(0, \mathbf{I}_\ell \otimes \mathbf{I}_n)$ and write $\mathbf{X} = \operatorname{Vec}(X)$. Then, for every integer $k \geq 0$ and every partition $\kappa \vdash k$, we have that $H_\kappa^{(\ell,n)}(X)$ is an element of $C_{2k}^{\mathbf{X}}$ and for every $F \in L^2(\mu_X)$,*

$$\operatorname{proj}(F|C_{2k}^{\mathbf{X}}) = \operatorname{proj}(F|\mathbf{U}_k^X) = \sum_{\kappa\vdash k} \widehat{F}(\kappa) H_\kappa^{(\ell,n)}(X), \tag{III.3.14}$$

where $\widehat{F}(\kappa)$ is as in (III.3.13). *In particular, we have that* $\operatorname{proj}(F|C_{2k+1}^{\mathbf{X}}) = 0$. *Moreover, if $F(X) = f_0(XX^T)$, then*

$$\widehat{F}(\kappa) = \frac{1}{2^{n\ell/2}\Gamma_\ell(\frac{n}{2})} \frac{(-2)^k}{k! C_\kappa(\mathbf{I}_\ell)} \int_{\mathcal{P}_\ell(\mathbb{R})} f_0(R) L_\kappa^{(\frac{n-\ell-1}{2})}(2^{-1}R) \operatorname{etr}\left(-2^{-1}R\right) \det(R)^{\frac{n-\ell-1}{2}} \nu(dR), \tag{III.3.15}$$

where $L_\kappa^{(\gamma)}$ denotes the generalized Laguerre polynomial of order $\gamma > -1$ associated with the partition κ, defined in (III.2.10) *and $\nu(dR)$ is the Lebesgue measure on $\mathcal{P}_\ell(\mathbb{R})$.*

Our proof of Theorem III.3.2 suggests that, combining the generalized Rodrigues formula (III.3.6) with the univariate Rodrigues formula (III.3.7), matrix-variate Hermite polynomials can be expressed in terms of multivariate Hermite polynomials. For instance, combining (III.3.6) with (III.3.7) in the case $\ell = n = 1$ (so that for every integer $k \geq 0$, $\kappa = (k)$ is the only partition of k) and writing $\gamma = \gamma^{(1,1)}$ for the standard Gaussian density function yields for every $k \geq 0$

$$H_{(k)}^{(1,1)}(X)\phi(X) = 4^{-k} \left(\frac{n}{2}\right)_{(k)}^{-1} C_{(k)}([\partial X]^2)\phi(X) = 4^{-k} \left(\frac{n}{2}\right)_k^{-1} \left(\frac{\partial}{\partial X}\right)^{2k} \phi(X) = 4^{-k} \left(\frac{n}{2}\right)_k^{-1} H_{2k}(X)\phi(X),$$

where we used that $\left(\frac{n}{2}\right)_{(k)} = \left(\frac{n}{2}\right)_k$, $C_{(k)}(a) = a^k$ for $a \in \mathbb{R}$ and the Rodrigues formula for classical Hermite polynomials in (III.3.7). This shows in particular that

$$H^{(1,1)}_{(k)}(X) = 4^{-k}\left(\frac{n}{2}\right)_k^{-1} H_{2k}(X).$$

Proceeding similarly for arbitrary dimensions ℓ and n, we compute the first matrix-variate Hermite polynomials associated with partitions of order up to 2 to be

$$\begin{aligned}
H^{(\ell,n)}_{(1)}(X) &= \frac{1}{2n}\sum_{i\in[\ell]}\sum_{j\in[n]} H_2(X_{ij}),\\
H^{(\ell,n)}_{(2)}(X) &= \frac{1}{12n(n+2)}\Big(3\sum_{i\in[\ell]}\sum_{j\in[n]} H_4(X_{ij}) + 3\sum_{i_1\neq i_2\in[\ell]}\sum_{j\in[n]} H_2(X_{i_1j})H_2(X_{i_2j})\\
&\quad +3\sum_{i\in[\ell]}\sum_{j_1\neq j_2\in[n]} H_2(X_{ij_1})H_2(X_{ij_2}) + \sum_{i_1\neq i_2\in[\ell]}\sum_{j_1\neq j_2\in[n]} H_2(X_{i_1j_1})H_2(X_{i_2j_2})\\
&\quad +2\sum_{i_1\neq i_2\in[\ell]}\sum_{j_1\neq j_2\in[n]} H_1(X_{i_1j_1})H_1(X_{i_2j_1})H_1(X_{i_1j_2})H_1(X_{i_2j_2})\Big),\\
H^{(\ell,n)}_{(1,1)}(X) &= \frac{1}{6n(n-1)}\Big(\sum_{i_1\neq i_2\in[\ell]}\sum_{j_1\neq j_2\in[n]} H_2(X_{i_1j_1})H_2(X_{i_2j_2})\\
&\quad -\sum_{i_1\neq i_2\in[\ell]}\sum_{j_1\neq j_2\in[n]} H_1(X_{i_1j_1})H_1(X_{i_2j_1})H_1(X_{i_1j_2})H_1(X_{i_2j_2})\Big).
\end{aligned} \tag{III.3.16}$$

Combining the content of Theorem III.3.2 with the orthogonality relation (III.3.9), allows one to derive variance expansions of spectral variables $F(X) \in L^2(\mu_X)$ where $X \sim \mathcal{N}_{\ell\times n}(0, \mathbf{I}_\ell \otimes \mathbf{I}_n)$ as a converging series in terms of its Fourier-Hermite coefficients.

Proposition III.3.3. *For integers $1 \le \ell \le n$, let $X \sim \mathcal{N}_{\ell\times n}(0, \mathbf{I}_\ell \otimes \mathbf{I}_n)$ and $F(X) \in L^2(\mu_X)$. Then,*

$$\mathbf{Var}[F(X)] = \sum_{k\ge 1}\sum_{\kappa\vdash k} \frac{4^k(\frac{n}{2})_\kappa}{k!C_\kappa(\mathbf{I}_\ell)}\mathbb{E}\left[F(X)H^{(\ell,n)}_\kappa(X)\right]^2,$$

where the convergence of the series is part of the conclusion.

III.3.2 Fourier-Hermite coefficients of Gaussian determinants as intrinsic volumes of ellipsoids

In this section, we consider rectangular Gaussian matrices X and provide the Wiener chaos expansion of determinants of the form $\det(XX^T)^{1/2}$. In [ZK12], the authors consider the case where $X \in \mathbb{R}^{\ell\times n}$ has independent rows with respective covariance matrices $\Sigma_1, \ldots, \Sigma_\ell$, and prove that (see in particular [ZK12, Theorem 1.1])

$$\mathbb{E}\left[\det(XX^T)^{1/2}\right] = \frac{(n)_\ell}{(2\pi)^{\ell/2}\kappa_{n-\ell}} V(\mathcal{E}_{\Sigma_1}, \ldots, \mathcal{E}_{\Sigma_\ell}, \mathbb{B}_n, \ldots, \mathbb{B}_n), \tag{III.3.17}$$

where $V(\mathcal{E}_{\Sigma_1}, \ldots, \mathcal{E}_{\Sigma_\ell}, \mathbb{B}_n, \ldots, \mathbb{B}_n)$ denotes the mixed volume of the ellipsoids $\mathcal{E}_{\Sigma_i}$, $i = 1, \ldots, \ell$ associated with matrices Σ_i and $\mathbb{B}_n$ denotes the unit ball in $\mathbb{R}^n$ with volume $\kappa_n = \pi^{n/2}/\Gamma(1 + n/2)$. We also refer the reader to [Vit91, Theorem 3.2], where the author proves a similar formula linking the expected absolute determinant of a matrix with i.i.d. copies of a random vector to the volume of the zonoid associated with the random distribution.

In Theorem III.3.6 below, we substantially extend this framework to arbitrary projection coefficients associated with the Wiener chaos expansion of such Gaussian determinants in the case where the rows of X are i.i.d Gaussian vectors with the same covariance matrix Σ.

Let $\Sigma \in \mathbb{R}^{n\times n}$ be a symmetric positive-definite matrix and $\left\{X^{(i)} = (X_1^{(i)}, \dots, X_n^{(i)}) : i \in [\ell]\right\}$ a collection of independent Gaussian vectors with covariance matrix Σ. We write $X \in \mathbb{R}^{\ell\times n}$ for the matrix whose i-th row is $X^{(i)}$. It follows that X has distribution $\mathcal{N}_{\ell\times n}(0, \mathbf{I}_\ell \otimes \Sigma)$ with density function

$$\gamma_\Sigma^{(\ell,n)}(X) = (2\pi)^{-n\ell/2} \det(\Sigma)^{-\ell/2} \mathrm{etr}\left(-2^{-1} X\Sigma^{-1}X^T\right). \tag{III.3.18}$$

As a consequence, the matrix $X\Sigma^{-1/2}$ has the $\mathcal{N}_{\ell\times n}(0, \mathbf{I}_\ell \otimes \mathbf{I}_n)$ distribution (see e.g. [GN00, Theorem 2.3.10]). Based on the matrix-variate Hermite polynomials $H_\kappa^{(\ell,n)}$ and their orthogonality relation with respect to $\gamma^{(\ell,n)}$ in (III.3.9), we define

$$H_\kappa^{(\ell,n)}(X;\Sigma) := \det(\Sigma)^{\ell k} H_\kappa^{(\ell,n)}(X\Sigma^{-1/2}). \tag{III.3.19}$$

In particular, we note that $H_\kappa^{(\ell,n)}(\bullet, \mathbf{I}_n) = H_\kappa^{(\ell,n)}(\bullet)$. The following proposition states that $H_\kappa^{(\ell,n)}(\bullet, \Sigma)$ are orthogonal with respect to the density $\gamma_\Sigma^{(\ell,n)}$ in (III.3.18).

Proposition III.3.4. *For every integers $k, l \geq 0$ and every partitions $\kappa \vdash k, \sigma \vdash l$, we have*

$$\int_{\mathbb{R}^{\ell\times n}} H_\kappa^{(\ell,n)}(X;\Sigma) H_\sigma^{(\ell,n)}(X;\Sigma) \gamma_\Sigma^{(\ell,n)}(X)(dX) = \mathbb{1}_{\kappa=\sigma} \cdot \det(\Sigma)^{2\ell k} 4^{-k} \left(\frac{n}{2}\right)_\kappa^{-1} k! C_\kappa(\mathbf{I}_\ell).$$

Therefore, the family of normalized polynomials

$$\mathbb{H}_\Sigma := \left\{c(\kappa;\Sigma)^{-1/2} H_\kappa^{(\ell,n)}(\bullet;\Sigma) : \kappa \vdash k \geq 0\right\}, \quad c(\kappa;\Sigma) := \det(\Sigma)^{2\ell k} 4^{-k} \left(\frac{n}{2}\right)_\kappa^{-1} k! C_\kappa(\mathbf{I}_\ell) \tag{III.3.20}$$

forms an orthonormal system of $L^2(\mu_X)$, where μ_X denotes the spectral measure of XX^T associated with X. Hence, for every $F \in L^2(\mu_X)$, one has the expansion

$$F(X) = \sum_{k\geq 0} \sum_{\kappa \vdash k} \widehat{F}(\kappa;\Sigma) H_\kappa^{(\ell,n)}(X;\Sigma),$$

where the projection coefficients are given by

$$\begin{aligned}
\widehat{F}(\kappa;\Sigma) &= c(\kappa;\Sigma)^{-1} \int_{\mathbb{R}^{\ell\times n}} F(X) H_{\kappa,\Sigma}^{(\ell,n)}(X) \phi_\Sigma^{(\ell,n)}(X)(dX) \\
&= c(\kappa;\Sigma)^{-1} \mathbb{E}_X \left[F(X) H_\kappa^{(\ell,n)}(X;\Sigma)\right], \quad X \sim \mathcal{N}_{\ell\times n}(0, \mathbf{I}_\ell \otimes \Sigma).
\end{aligned} \tag{III.3.21}$$

The next result provides an explicit formula for the projection coefficients $\widehat{F}(\kappa;\Sigma)$ in the special case where $F(X) = \det(XX^T)^{1/2}$. Here, $\binom{\kappa}{\sigma}$ denote the generalized binomial coefficients defined by (III.2.3), and $\tilde{\mu}$ stands for the Haar probability measure on the Stiefel manifold $O(n, \ell)$ of ℓ-frames in $\mathbb{R}^n$.

Theorem III.3.5. *For integers $1 \leq \ell \leq n$ and $\Sigma \in \mathbb{R}^{n\times n}$ positive-definite symmetric, let $X \sim \mathcal{N}_{\ell\times n}(0, \mathbf{I}_\ell \otimes \Sigma)$. Then, $F(X) = \det(XX^T)^{1/2}$ is an element of $L^2(\mu_X)$, and one has the decomposition*

$$F(X) = \sum_{k\geq 0} \sum_{\kappa \vdash k} \widehat{F}(\kappa;\Sigma) H_\kappa^{(\ell,n)}(X;\Sigma),$$

where the Fourier-Hermite coefficients of F are given by the formula

$$\begin{aligned}
\widehat{F}(\kappa;\Sigma) = \frac{(-2)^k}{\det(\Sigma)^{\ell k} k!} \left(\frac{n}{2}\right)_\kappa \sum_{s=0}^{k} \sum_{\sigma \vdash s} \binom{\kappa}{\sigma} (-1)^s \frac{(\frac{n+1}{2})_\sigma}{(\frac{n}{2})_\sigma} \\
\times \det(\Sigma)^{-\ell/2} 2^{\ell/2} \frac{\Gamma_\ell(\frac{n+1}{2})}{\Gamma_\ell(\frac{n}{2})} \int_{O(n,\ell)} \det(U\Sigma^{-1}U^T)^{-(n+1)/2} \tilde{\mu}(dU).
\end{aligned} \tag{III.3.22}$$

As anticipated, our next result yields a geometric interpretation of the projection coefficients $\widehat{F}(\kappa;\Sigma)$ appearing in (III.3.22) in terms of mixed volumes and intrinsic volumes of ellipsoids (see Section III.2.3 for preliminaries on these notions).

Theorem III.3.6. *For integers $1 \leq \ell \leq n$ and $\Sigma \in \mathbb{R}^{n\times n}$ positive-definite symmetric, let $X \sim \mathcal{N}_{\ell\times n}(0, \mathbf{I}_\ell \otimes \Sigma)$. Then, for $F(X) = \det(XX^T)^{1/2}$, we have*

$$\widehat{F}(\kappa;\Sigma) = \mathcal{M}(\kappa;\Sigma,\ell,n)\cdot V(\mathcal{E}_\Sigma[\ell],\mathbb{B}_n[n-\ell]) \tag{III.3.23}$$

$$= \mathcal{M}(\kappa;\Sigma,\ell,n)\cdot \kappa_{n-\ell}\binom{n}{\ell}^{-1} V_\ell(\mathcal{E}_\Sigma), \tag{III.3.24}$$

where

$$\mathcal{M}(\kappa;\Sigma,\ell,n) := \frac{(-2)^k}{\det(\Sigma)^{\ell k}k!}\left(\frac{n}{2}\right)_\kappa \sum_{s=0}^{k}\sum_{\sigma\vdash s}\binom{\kappa}{\sigma}(-1)^s\frac{(\frac{n+1}{2})_\sigma}{(\frac{n}{2})_\sigma}\frac{(n)_\ell}{(2\pi)^{\ell/2}\kappa_{n-\ell}}$$

and where $V(\bullet,\dots,\bullet)$ and $V_\ell(\bullet)$ stand for the mixed and ℓ-th intrinsic volumes, respectively (see also notation (III.2.17)*), $\mathbb{B}_n$ denotes the unit ball in $\mathbb{R}^n$ and $\kappa_n = \pi^{n/2}/\Gamma(1+n/2)$ denotes its volume. In particular, for $\kappa = (0)$,*

$$\mathbb{E}\left[\det(XX^T)^{1/2}\right] = \frac{(n)_\ell}{(2\pi)^{n\ell/2}\kappa_{n-\ell}}V(\mathcal{E}_\Sigma[\ell],\mathbb{B}_n[n-\ell]) = \frac{(n)_\ell}{(2\pi)^{\ell/2}}\binom{n}{\ell}^{-1} V_\ell(\mathcal{E}_\Sigma). \tag{III.3.25}$$

Remark III.3.7. **(a)** We point out that (III.3.25) coincides with (III.3.17) in the case where $\Sigma_i = \Sigma$ for $i = 1,\dots,\ell$. In this sense, relations (III.3.23) and (III.3.24) therefore generalize the content of [ZK12, Theorem 1.1] to arbitrary chaotic projection coefficients $\widehat{F}(\kappa;\Sigma)$ associated with partitions κ of order $k \geq 1$.

(b) Our proof of Theorem III.3.6 suggests the following new identity for intrinsic volumes of ellipsoids

$$V_\ell(\mathcal{E}_\Sigma) = \binom{n}{\ell}\frac{\kappa_n}{\kappa_{n-\ell}}\det(\Sigma)^{-\ell/2}\int_{O(n,\ell)}\det(U\Sigma^{-1}U^T)^{-(n+1)/2}\tilde{\mu}(dU), \quad 1 \leq \ell \leq n,$$

where $\tilde{\mu}$ indicates the Haar probability measure on the Stiefel manifold $O(n,\ell)$.

(c) In Section IV.2.6, we sketch an attempt to further generalize the findings of Kabluchko and Zaporozhets to the more general setting where the rows of X are independent with respective covariance matrices $\Sigma_1,\dots,\Sigma_\ell$. As we will explain, we are not successful to adapt our techniques employed in the proof of Theorems III.3.5 and III.3.6 to this more general framework. Such a difficulty may be explained by the fact that the polynomials defined in (III.4.17) are not easily tractable for matrix calculus, as we have to deal with each row separately.

The following corollary is obtained from Theorem III.3.5 applied with $\Sigma = \mathbf{I}_n$, that is, when X has independent rows with independent coordinates. In this case, we have $H_\kappa^{(\ell,n)}(X;\mathbf{I}_n) = H_\kappa^{(\ell,n)}(X)$ and $\widehat{F}(\kappa;\mathbf{I}_n) = \widehat{F}(\kappa)$ as in (III.3.13).

Corollary III.3.8. *For integers $1 \leq \ell \leq n$, let $X \sim \mathcal{N}_{\ell\times n}(0, \mathbf{I}_\ell \otimes \mathbf{I}_n)$. Then, $F(X) = \det(XX^T)^{1/2}$ is an element of $L^2(\mu_X)$, and one has the decomposition*

$$F(X) = \sum_{k\geq 0}\sum_{\kappa\vdash k}\widehat{F}(\kappa)H_\kappa^{(\ell,n)}(X),$$

where the Fourier-Hermite coefficients of F are given by the formula

$$\widehat{F}(\kappa) = 2^{\ell/2}\frac{\Gamma_\ell(\frac{n+1}{2})}{\Gamma_\ell(\frac{n}{2})}\frac{(-2)^k}{k!}\left(\frac{n}{2}\right)_\kappa \sum_{s=0}^{k}\sum_{\sigma\vdash s}\binom{\kappa}{\sigma}(-1)^s\frac{(\frac{n+1}{2})_\sigma}{(\frac{n}{2})_\sigma}. \tag{III.3.26}$$

In particular,

$$\mathbb{E}\left[\det(XX^T)^{1/2}\right] = 2^{\ell/2}\frac{\Gamma_\ell(\frac{n+1}{2})}{\Gamma_\ell(\frac{n}{2})}. \tag{III.3.27}$$

Remark III.3.9. **(a)** Combining the contents of Corollary III.3.8 and Theorem III.3.2, we see that (III.3.26) provides the chaotic projection coefficients associated with the Wiener-chaos decomposition of $\det(XX^T)^{1/2}$. In Section III.3.4, we consider functionals of multi-dimensional Gaussian fields arising in stochastic geometry, that admit a certain integral representation in terms of Jacobian determinants, and effectively use formula (III.3.26) to obtain a compact expression of their Wiener-Itô chaos expansions.

(b) Formula (III.3.27) is to be compared with Remark II.1.2 **(a)** of Chapter II for a link to *flag coefficients* $\begin{bmatrix} n \\ \ell \end{bmatrix} := \binom{n}{\ell}\frac{\kappa_n}{\kappa_{n-\ell}\kappa_\ell}$, also appearing in the Gaussian Kinematic formula (see for instance Chapter 13 in [AT07]). In particular, one has that

$$\mathbb{E}\left[\det(XX^T)^{1/2}\right] = \frac{\ell!\kappa_\ell}{(2\pi)^{\ell/2}}\begin{bmatrix} n \\ \ell \end{bmatrix}.$$

III.3.3 Generalized Ornstein-Uhlenbeck semigroup

III.3.3.1 A Mehler-type representation

In this section, we provide the equivalent counterpart on matrix spaces of the classical Ornstein-Uhlenbeck semigroup $\{P_t : t \geq 0\}$ on $\mathbb{R}$ defined via *Mehler's formula* (see Proposition I.1.20)

$$P_t f(x) = \mathbb{E}\left[f(e^{-t}x + \sqrt{1-e^{-2t}}X_0)\right], \quad X_0 \sim \mathcal{N}(0,1), \quad x \in \mathbb{R}, \quad t \geq 0. \tag{III.3.28}$$

For an integer $d \geq 1$ and $f : \mathbb{R}^d \to \mathbb{R}$, we write

$$P_t^{(d)} f(x) = \mathbb{E}\left[f(e^{-t}x + \sqrt{1-e^{-2t}}X_0)\right], \quad X_0 \sim \mathscr{N}_d(0, \mathbf{I}_d), \quad x \in \mathbb{R}^d, \quad t \geq 0 \tag{III.3.29}$$

for the Ornstein-Uhlenbeck operator in dimension d, in such a way that $P_t = P_t^{(1)}$. We fix integers $1 \leq \ell \leq n$, and define the space

$$\Pi(\ell, n) = \left\{f : \mathbb{R}^{\ell\times n} \to \mathbb{R} : f(XH) = f(X) \text{ for every } H \in O(n)\right\}, \tag{III.3.30}$$

that is, an element of $\Pi(\ell, n)$ is a matrix-variate function that is right-invariant under orthogonal transformations. For a diagonal matrix $A = \operatorname{diag}(a_1, \ldots, a_n) \in \mathbb{R}^{n\times n}$ with $a_1, \ldots, a_n \geq 0$ and $f \in \Pi(\ell, n)$, we introduce the operator

$$O_{t;A}^{(\ell,n)} f(X) = \mathbb{E}\left[\int_{O(n)} f(XHe^{-tA} + X_0(\mathbf{I}_n - e^{-2tA})^{1/2})\tilde{\mu}(dH)\middle| X\right], \quad t \geq 0 \tag{III.3.31}$$

where the expectation is taken with respect to $X_0 \sim \mathcal{N}_{\ell\times n}(0, \mathbf{I}_\ell \otimes \mathbf{I}_n)$, for a matrix $M \in \mathbb{R}^{n\times n}$,

$$e^{tM} = \sum_{p\geq 0}\frac{1}{p!}(tM)^p$$

denotes the matrix exponential of M, and $\tilde{\mu}$ indicates the probability Haar measure on the orthogonal group $O(n)$.

Our next result specifies the action of $O_{t;A}^{(\ell,n)}$ on matrix-variate Hermite polynomials and naturally complements the action of P_t on Hermite polynomials on $\mathbb{R}$ given derived in (I.1.21),

$$P_t H_k(x) = e^{-kt} H_k(x). \tag{III.3.32}$$

Theorem III.3.10. *For every diagonal matrix $A = \operatorname{diag}(a_1, \dots, a_n) \in \mathbb{R}^{n\times n}$ such that $a_1, \dots, a_n \geq 0$, every integer $k \geq 0$ and every partition $\kappa \vdash k$, we have that*

$$O_{t;A}^{(\ell,n)} H_\kappa^{(\ell,n)}(X) = \frac{C_\kappa(e^{-2tA})}{C_\kappa(\mathbf{I}_n)} H_\kappa^{(\ell,n)}(X). \tag{III.3.33}$$

In particular, the family $\left\{O_{t;A}^{(\ell,n)} : t \geq 0\right\}$ is a semigroup on the class $\Pi(\ell, n)$ if and only if $a_1 = \dots = a_n = a$. More precisely, in this case, $O_{t;A}^{(\ell,n)}$ coincides with $P_{at}^{(\ell n)}$ on the class $\Pi(\ell, n)$.

In particular, it becomes clear that the polynomials $H_\kappa^{(\ell,n)}$ are eigenfunctions of $O_{t;A}^{(\ell,n)}$ with respective eigenvalue $C_\kappa(e^{-2tA})C_\kappa(\mathbf{I}_n)^{-1}$. Moreover, if $F \in L^2(\mu_X)$ admits the expansion (III.3.12), then $O_{t;A}^{(\ell,n)} F \in L^2(\mu_X)$ and

$$O_{t;A}^{(\ell,n)} F = \sum_{k\geq 0} \sum_{\kappa\vdash k} \widehat{F}(\kappa) \frac{C_\kappa(e^{-2tA})}{C_\kappa(\mathbf{I}_n)} H_\kappa^{(\ell,n)},$$

that is, the projection coefficients of $O_{t;A}^{(\ell,n)} F$ are obtained from those of F by multiplying by $C_\kappa(e^{-2tA})C_\kappa(\mathbf{I}_n)^{-1}$.

Remark III.3.11. **(a)** Using the fact that $H_\kappa^{(\ell,n)}$ is an element of the class $\Pi(\ell, n)$ (as can be seen for instance from (III.3.8)), we deduce from (III.3.33) applied with $A = \mathbf{I}_n$ that

$$P_t^{(\ell n)} H_\kappa^{(\ell,n)}(X) = O_{t;\mathbf{I}_n}^{(\ell,n)} H_\kappa^{(\ell,n)}(X) = \frac{C_\kappa(e^{-2t\mathbf{I}_n})}{C_\kappa(\mathbf{I}_n)} H_\kappa^{(\ell,n)}(X) = e^{-2kt} H_\kappa^{(\ell,n)}(X), \tag{III.3.34}$$

where we used that $C_\kappa(e^{-2t\mathbf{I}_n}) = e^{-2kt} C_\kappa(\mathbf{I}_n)$ by homogeneity. Recalling that $H_\kappa^{(\ell,n)}(X)$ is an element of the $2k$-th Wiener chaos associated with $\operatorname{Vec}(X)$, it is clear that the classical Ornstein-Uhlenbeck semigroup $\left\{P_t^{(\ell n)} : t \geq 0\right\}$ acts on the entries X_{ij} of X via the relation $P_t^{(\ell n)} H_\kappa^{(\ell,n)}(X) = e^{-2kt} H_\kappa^{(\ell,n)}(X)$, which is consistent with (III.3.34).

(b) Let us assume that $A = \operatorname{diag}(a, \dots, a)$, $a \geq 0$. Then the relation in (III.3.33) reduces to

$$O_{t;A}^{(\ell,n)} H_\kappa^{(\ell,n)}(X) = e^{-2ta} H_\kappa^{(\ell,n)}(X),$$

in view of the identity $C_\kappa(e^{-2tA}) = e^{-2ta} C_\kappa(\mathbf{I}_n)$. In particular, from this identity, one can directly verify the semigroup property verified by $O_{t;A}^{(\ell,n)}$ on matrix-Hermite polynomials, as for every $s, t \geq 0$,

$$O_{t+s;A}^{(\ell,n)} H_\kappa^{(\ell,n)}(X) = e^{-2(t+s)a} H_\kappa^{(\ell,n)}(X) = e^{-2ta} e^{-2sa} H_\kappa^{(\ell,n)}(X) = O_{t;A}^{(\ell,n)} O_{s;A}^{(\ell,n)} H_\kappa^{(\ell,n)}(X).$$

Combining this relation with (III.3.33) in particular suggests the identity

$$\frac{C_\kappa(e^{-2(t+s)A})}{C_\kappa(\mathbf{I}_n)} = \frac{C_\kappa(e^{-2tA}) C_\kappa(e^{-2sA})}{C_\kappa(\mathbf{I}_n)^2},$$

which fails to hold in the case where the diagonal entries of A are not all equal. Indeed, for simplicity a direct computation in the case $\ell = n = 2, \kappa = (1), a_1 = 1, a_2 = 2$ shows that the left and right-hand sides of the above relation are respectively given by

$$\frac{1}{2}[e^{-2(t+s)} + e^{-4(t+s)}], \quad \frac{1}{4}e^{-2t}e^{-4s},$$

which are different.

III.3.3.2 An extension of the orthogonality relation for matrix-variate Hermite polynomials

We recall that for jointly standardized Gaussian random variables X, Y such that $\mathbb{E}[XY] = \rho$, the univariate Hermite polynomials on the real line satisfy the orthogonality relation (see Proposition I.1.22)

$$\mathbb{E}[H_k(X)H_l(Y)] = \mathbb{1}_{k=l} \cdot k!\rho^k. \tag{III.3.35}$$

Exploiting the action of the semigroup $O_{t;A}^{(\ell,n)}$ on matrix-variate Hermite polynomials derived in Theorem III.3.10 allows us to establish the matrix-counterpart of the orthogonality relation (III.3.35) in the setting where the correlation of the Gaussian matrix entries X and Y is reflected in a matrix R. This is the content of the following theorem.

Theorem III.3.12. *Let $X, X_0 \sim \mathcal{N}_{\ell\times n}(0, \mathbf{I}_\ell \otimes \mathbf{I}_n)$ be independent and R be a deterministic matrix of dimension $n\times n$. Let $Y \overset{d}{=} XR + X_0(\mathbf{I}_n - R^2)^{1/2}$ in distribution. Then, for every integers $k, l \geq 0$ and every partitions $\kappa \vdash k, \sigma \vdash l$, we have*

$$\mathbb{E}\left[H_\kappa^{(\ell,n)}(X)H_\sigma^{(\ell,n)}(Y)\right] = \mathbb{1}_{\kappa=\sigma} \cdot 4^{-k}\left(\frac{n}{2}\right)_\kappa^{-1} k!C_\kappa(R^2)\frac{C_\kappa(\mathbf{I}_\ell)}{C_\kappa(\mathbf{I}_n)}. \tag{III.3.36}$$

Some remarks concerning Theorem III.3.12 are in order.

Remark III.3.13. **(a)** By independence of X and X_0 and the distributional identity $Y \overset{d}{=} XR + X_0(\mathbf{I}_n - R^2)^{1/2}$, we have that for every $i, i' \in [\ell], j, j' \in [n]$,

$$\mathbb{E}\left[X_{ij}Y_{i'j'}\right] = \sum_{k=1}^n \mathbb{E}\left[X_{ij}X_{i'k}\right]R_{kj'} = \sum_{k=1}^n \mathbb{1}[i = i', j = k]R_{kj'} = \mathbb{1}[i = i']R_{jj'},$$

where we used that $X \sim \mathcal{N}_{\ell\times n}(0, \mathbf{I}_\ell \otimes \mathbf{I}_n)$, yielding that, for every $j, j' = 1, \ldots, n, |R_{jj'}| \leq 1$ by virtue of the Cauchy-Schwarz inequality. The above observation implies that R is necessarily symmetric and positive-semidefinite as a covariance matrix, and therefore has non-negative eigenvalues $r_1, \ldots, r_n$. Note that, if $R = \Delta = \operatorname{diag}(r_1, \ldots, r_n)$ is diagonal, we therefore necessarily have $|r_j| \leq 1$ for every $j = 1, \ldots, n$, so that $(\mathbf{I}_n - R^2)^{1/2}$ is well-defined. Our arguments to prove Theorem III.3.12 are based on the following general reduction argument: for $f, g \in \Pi(\ell, n)$, writing $R = O\Delta O^T$ with $O \in O(n)$ and $\Delta = \operatorname{diag}(r_1, \ldots, r_n)$,

$$\begin{aligned}\mathbb{E}\left[f(X)g(XO\Delta O^T + X_0O(\mathbf{I}_n - R^2)^{1/2}O^T)\right] &= \mathbb{E}\left[f(XO)g(XO\Delta + X_0O(\mathbf{I}_n - \Delta^2)^{1/2})\right] \\ &= \mathbb{E}\left[f(X)g(X\Delta + X_0(\mathbf{I}_n - \Delta^2)^{1/2})\right],\end{aligned}$$

where we used that $f(X) = f(XO)$ and $g(XO^T) = g(X)$ since $f, g \in \Pi(\ell, n)$ as well as the fact that $(XO, X_0O) \overset{d}{=} (X, X_0)$, showing in particular that $|r_j| \leq 1$ for every $j = 1, \ldots, n$.

(b) We point out two particular cases of Theorem III.3.12: (1) if $R = \mathbf{I}_n$, relation (III.3.36) reduces to the orthogonality of Hermite polynomials stated in (III.3.9) and (2) when $R = \mathrm{diag}(\rho, \dots, \rho)$, (III.3.36) gives

$$\mathbb{E}\left[H_\kappa^{(\ell,n)}(X)H_\sigma^{(\ell,n)}(\rho X + \sqrt{1-\rho^2}X_0)\right] = \mathbb{1}_{\kappa=\sigma} \cdot 4^{-k}\left(\frac{n}{2}\right)_\kappa^{-1} k! C_\kappa(\rho^2\,\mathbf{I}_n)\frac{C_\kappa(\mathbf{I}_\ell)}{C_\kappa(\mathbf{I}_n)}$$
$$= \mathbb{1}_{\kappa=\sigma} \cdot 4^{-k}\left(\frac{n}{2}\right)_\kappa^{-1} k!\rho^{2k} C_\kappa(\mathbf{I}_\ell), \qquad \text{(III.3.37)}$$

where we used homogeneity of zonal polynomials. For completeness, in Example III.3.14, we present three explicit examples of (III.3.37) for the Hermite polynomials in (III.3.16) by relying on moment formulae for products of univariate Hermite polynomials.

(c) Writing $\mathbf{X} = \mathrm{Vec}(X)$ and $\mathbf{Y} = \mathrm{Vec}(Y)$, we know by Theorem III.3.2 that $H_\kappa^{(\ell,n)}(X) \in C_{2k}^{\mathbf{X}}$ and $H_\kappa^{(\ell,n)}(Y) \in C_{2k}^{\mathbf{Y}}$. In particular by orthogonality of Wiener chaoses, it is clear that $H_\kappa^{(\ell,n)}(X)$ and $H_\sigma^{(\ell,n)}(Y)$ are orthogonal in $L^2(\mathbb{P})$ when $k \neq l$. Remarkably relation (III.3.36) yields a stronger orthogonality in the sense that, even if $k = l$, the elements $H_\kappa^{(\ell,n)}(X)$ and $H_\sigma^{(\ell,n)}(Y)$, belonging both to the Wiener chaos of order $2k$, are orthogonal as soon as $\kappa \neq \sigma$.

(d) Combining relation (III.3.8) with (III.3.36), we deduce an extended orthogonality relation for generalized matrix-variate Laguerre polynomials

$$\mathbb{E}\left[L_\kappa^{(\frac{n-\ell-1}{2})}(2^{-1}XX^T)L_\sigma^{(\frac{n-\ell-1}{2})}(2^{-1}YY^T)\right] = \gamma_\kappa^{-1}\gamma_\sigma^{-1}\mathbb{E}\left[H_\kappa^{(\ell,n)}(X)H_\sigma^{(\ell,n)}(Y)\right]$$
$$= \mathbb{1}_{\kappa=\sigma} \cdot \left(\frac{n}{2}\right)_\kappa^{-1} k! C_\kappa(R^2)\frac{C_\kappa(\mathbf{I}_\ell)}{C_\kappa(\mathbf{I}_n)},$$

where we used that $\gamma_\kappa := (-2)^{-k}(\frac{n}{2})_\kappa^{-1}$, thus extending the orthogonality relation in (III.2.11) obtained for $R = \mathbf{I}_n$.

Example III.3.14. In this example, we explicitly compute the covariance

$$\mathbb{E}\left[H_\kappa^{(\ell,n)}(X)H_\sigma^{(\ell,n)}(Y)\right], \quad Y := \rho X + \sqrt{1-\rho^2}X_0$$

in the three examples (i) $\kappa = \sigma = (1)$, (ii) $\kappa = (2), \sigma = (1,1)$ and (iii) $\kappa = \sigma = (1,1)$ by relying on the explicit expansions of the corresponding matrix-Hermite polynomials in terms of univariate Hermite polynomials in (III.3.16) and product formulae for the latter. Our computations developed below are consistent with (III.3.37). In view of the covariance structure between X and Y, we have that $\mathbb{E}\left[X_{ij}Y_{i'j'}\right] = \mathbb{1}[i = i', j = j']\rho$. We start with (i). Using the covariance structure together with the expression for $H_{(1)}^{(\ell,n)}$ in (III.3.16) yields

$$\mathbb{E}\left[H_{(1)}^{(\ell,n)}(X)H_{(1)}^{(\ell,n)}(Y)\right] = \frac{1}{4n^2}\sum_{i_1,i_2\in[\ell]}\sum_{j_1,j_2\in[n]}\mathbb{E}\left[H_2(X_{i_1j_1})H_2(Y_{i_2j_2})\right]$$
$$= \frac{\rho^2}{2n^2}\sum_{i_1,i_2\in[\ell]}\sum_{j_1,j_2\in[n]}\mathbb{1}[i_1 = i_2, j_1 = j_2] = \frac{\rho^2}{2n^2}\ell n = \rho^2\frac{\ell}{2n},$$

where we used (III.3.35). On the other hand, using that $(n/2)_{(1)} = n/2$ and $C_{(1)}(\mathbf{I}_\ell) = \mathrm{tr}(\mathbf{I}_\ell) = \ell$ yields from (III.3.37)

$$\mathbb{E}\left[H_{(1)}^{(\ell,n)}(X)H_{(1)}^{(\ell,n)}(Y)\right] = 4^{-1}\frac{2}{n}\rho^2\ell = \rho^2\frac{\ell}{2n},$$

which coincides with the above. Let us now treat (ii). In view of (III.3.16), we can write

$$H^{(\ell,n)}_{(2)}(X) := \frac{1}{12n(n+2)}\sum_{i=1}^{5} A_i(X), \quad H^{(\ell,n)}_{(1,1)}(Y) := \frac{1}{6n(n-1)}(B_1(Y)+B_2(Y)),$$

where $A_1(X),\dots,A_5(X)$ and $B_1(Y),B_2(Y)$ are the double summations appearing in the respective definitions of $H^{(\ell,n)}_{(2)}(X)$ and $H^{(\ell,n)}_{(1,1)}(Y)$ (including their multiplicative coefficient). We can thus compute

$$\mathbb{E}\left[H^{(\ell,n)}_{(2)}(X)H^{(\ell,n)}_{(1,1)}(Y)\right] = \frac{1}{12n(n+2)}\frac{1}{6n(n-1)}\sum_{i=1}^{5}\sum_{j=1}^{2}\mathbb{E}\left[A_i(X)B_j(Y)\right], \tag{III.3.38}$$

which is a sum of ten terms. First we recall the following relations for jointly standard Gaussian random variables N_1, N_2, Z_1, Z_2 such that $\mathbb{E}[N_1N_2] = \mathbb{E}[Z_1Z_2] = 0$,

$$\begin{aligned}
\mathbb{E}[H_4(N_1)H_2(Z_1)H_2(Z_2)] &= 24\mathbb{E}[N_1Z_1]^2\,\mathbb{E}[N_1Z_2]^2,\\
\mathbb{E}[H_2(N_1)Z_1Z_2] &= 2\mathbb{E}[N_1Z_1]\,\mathbb{E}[N_1Z_2],\\
\mathbb{E}[N_1N_2Z_1Z_2] &= \mathbb{E}[N_1Z_1]\,\mathbb{E}[N_2Z_2] + \mathbb{E}[N_1Z_2]\,\mathbb{E}[N_2Z_1].
\end{aligned}$$

Combining these relations with the covariance structure between X and Y, one verifies that

$$\mathbb{E}\left[A_i(X)B_j(Y)\right] = 0, \quad \forall (i,j)\notin\{(5,2),(4,1)\}$$

and

$$\mathbb{E}[A_4(X)B_1(Y)] = -\mathbb{E}[A_5(X)B_2(Y)] = 8\ell(\ell-1)n(n-1)\rho^4,$$

implying in particular that $\mathbb{E}\left[H^{(\ell,n)}_{(2)}(X)H^{(\ell,n)}_{(1,1)}(Y)\right] = 0$ in view of (III.3.38). Proceeding similarly for example (iii), we write

$$\mathbb{E}\left[H^{(\ell,n)}_{(1,1)}(X)H^{(\ell,n)}_{(1,1)}(Y)\right] = \frac{1}{36n^2(n-1)^2}\sum_{i,j=1}^{2}\mathbb{E}\left[B_i(X)B_j(Y)\right]$$

where B_1 and B_2 are as above, for which we compute

$$\begin{aligned}
&\mathbb{E}[B_1(X)B_1(Y)] = \mathbb{E}[A_4(X)B_1(X)] = 8\ell(\ell-1)n(n-1)\rho^4,\\
&\mathbb{E}[B_1(X)B_2(Y)] = \mathbb{E}[B_2(X)B_1(Y)] = \mathbb{E}[A_4(X)B_2(Y)] = 0,\\
&\mathbb{E}[B_2(X)B_2(Y)] = -\frac{1}{2}\mathbb{E}[A_5(X)B_2(Y)] = 4\ell(\ell-1)n(n-1)\rho^4,
\end{aligned}$$

where A_4 and A_5 are the terms appearing in $H^{(\ell,n)}_{(2)}$. Summing these terms yields

$$\mathbb{E}\left[H^{(\ell,n)}_{(1,1)}(X)H^{(\ell,n)}_{(1,1)}(Y)\right] = \frac{1}{36n^2(n-1)^2}\sum_{i,j=1}^{2}\mathbb{E}\left[B_i(X)B_j(Y)\right] = \frac{1}{3n(n-1)}\ell(\ell-1)\rho^4. \tag{III.3.39}$$

On the other hand, computing $(n/2)_{(1,1)} = n(n-1)/4$ and $C_{(1,1)}(\mathbf{I}_\ell) = \frac{2}{3}\ell(\ell-1)$ yields from (III.3.37)

$$\mathbb{E}\left[H^{(\ell,n)}_{(1,1)}(X)H^{(\ell,n)}_{(1,1)}(Y)\right] = 4^{-2}\left(\frac{n}{2}\right)^{-1}_{(1,1)} 2!\rho^4 C_{(1,1)}(\mathbf{I}_\ell) = \frac{1}{3n(n-1)}\ell(\ell-1)\rho^4,$$

which is consistent with (III.3.39).

III.3.4 Applications to geometric functionals of Gaussian random fields

In this section, we apply our main results of Sections III.3.1, III.3.2 and III.3.3 to the study of geometric functionals of multidimensional Gaussian fields.

In Section III.3.4.1, we consider random variables admitting an integral representation in terms of Jacobian determinants associated with multi-dimensional Gaussian fields. We argue that such a definition can be interpreted as the *total variation* of vector-valued functions, generalizing the classical definition of total variation of multi-variate functions. More specifically, in the setting of a certain matrix correlation structure between two Jacobian matrices, appearing notably in the study of Gaussian Laplace eigenfunctions, we exploit the findings of Theorem III.3.12 to obtain a precise expression for the variance of the total variation in terms of integrals of zonal polynomials.

In Section III.3.4.2, we apply the general framework of Section III.3.4.1 to vectors of independent arithmetic random waves with the same eigenvalue on the three-dimensional torus, and prove a CLT in the high-energy regime for their generalized total variation on the full torus.

In Section III.3.4.3, we consider the nodal volumes associated with vectors of independent arithmetic random waves on the three torus. In particular, we provide its Wiener-Itô chaos expansions in terms of both, multivariate and matrix-variate Hermite polynomials, and provide some insight for variance estimates of its chaotic components.

III.3.4.1 Generalized total variation of vector-valued functions

Let $n \geq 1$ be an integer and consider a centred smooth Gaussian field $\mathfrak{f} = \{\mathfrak{f}(z) : z \in \mathbb{R}^n\}$ on $\mathbb{R}^n$. For $1 \leq \ell \leq n$, we consider ℓ i.i.d copies $\mathfrak{f}^{(1)}, \dots, \mathfrak{f}^{(\ell)}$ of $\mathfrak{f}$ and are interested in the ℓ-dimensional Gaussian field

$$\mathfrak{f}_\ell = \left\{\mathfrak{f}_\ell(z) = (\mathfrak{f}^{(1)}(z), \dots, \mathfrak{f}^{(\ell)}(z)) : z \in \mathbb{R}^n\right\}.$$

We denote by $\mathfrak{f}_\ell'(z) \in \mathbb{R}^{\ell\times n}$ the Jacobian matrix of $\mathfrak{f}_\ell$ evaluated at $z \in \mathbb{R}^n$. Moreover, we assume that (i) for every $z \in \mathbb{R}^n$, the distribution of $\mathfrak{f}_\ell(z)$ is non-degenerate and (ii) for every $z \in \mathbb{R}^n$, $\mathfrak{f}_\ell'(z) \sim \mathcal{N}_{\ell\times n}(0, \mathbf{I}_\ell \otimes \mathbf{I}_n)$. We define the following random variable.

Definition III.3.15. For a compact domain $U \subset \mathbb{R}^n$, we define

$$\mathbf{V}_{\ell,n}(\mathfrak{f}_\ell; U) := \int_U \Phi(\mathfrak{f}_\ell'(z))dz, \tag{III.3.40}$$

where $\Phi(M) := \det(MM^T)^{1/2}$ for $M \in \mathbb{R}^{\ell\times n}$.

We note that the above integral is well-defined since U is compact and $\det(\mathfrak{f}_\ell'(z))$ is a multivariate polynomial in the entries of $\mathfrak{f}_\ell'(z)$. We remark that the random variable $\mathbf{V}_{\ell,n}(\mathfrak{f}_\ell; U)$ can be seen as a generalization of the total variation of vector-valued functions. Indeed, for $\ell = 1$, (III.3.40) coincides with the definition of the total variation for functions $\mathbb{R}^n \to \mathbb{R}$. For $\ell = n$, [FFM04, DP12] consider a *relaxed total variation* of the Jacobian given by the Area formula (see Proposition I.1.11)

$$\mathbf{V}_{n,n}(\mathfrak{f}_n; U) = \int_U |\det(\mathfrak{f}_n'(z))|\, dz = \int_{\mathbb{R}^n} N_y(\mathfrak{f}_n; U)dy,$$

where $N_y(\mathfrak{f}_n; U) = \mathrm{card}(\{z \in U : \mathfrak{f}_n(z) = y\})$. Using the Co-area formula in (III.3.40) shows that

$$\mathbf{V}_{\ell,n}(\mathfrak{f}_\ell; U) = \int_{\mathbb{R}^\ell} \sigma_y(\mathfrak{f}_\ell; U)dy,$$

where $\sigma_y(\mathfrak{f}_\ell; U)$ denotes the $(n-\ell)$-dimensional Hausdorff measure of the level set $\{z \in U : \mathfrak{f}_\ell(z) = y\}$. Thus, the definition (III.3.40) generalizes the above setting to functions $\mathbb{R}^n \to \mathbb{R}^\ell$ with $\ell < n$.

From now on, $1 \le \ell \le n$ are fixed and we write $\mathbf{V}(\mathfrak{f}_\ell; U) = \mathbf{V}_{\ell,n}(\mathfrak{f}_\ell; U)$. The fact that, for every $z \in \mathbb{R}^n$, $\Phi(\mathfrak{f}'_\ell(z))$ is an element of $L^2(\mu_{\mathfrak{f}'_\ell(z)})$ implies that $\mathbf{V}(\mathfrak{f}_\ell; U)$ can be expanded in matrix-variate Hermite polynomials by means of Corollary III.3.8, yielding its Wiener chaos expansion

$$\mathbf{V}(\mathfrak{f}_\ell; U) = \sum_{k\ge 0} \mathbf{V}(\mathfrak{f}_\ell; U)[2k], \quad \mathbf{V}(\mathfrak{f}_\ell; U)[2k] = \sum_{\kappa\vdash k} \widehat{\Phi}(\kappa) \int_U H_\kappa^{(\ell,n)}(\mathfrak{f}'_\ell(z))dz, \tag{III.3.41}$$

where $\widehat{\Phi}(\kappa)$ is as in (III.3.26) and $\mathbf{V}(\mathfrak{f}_\ell; U)[2k]$ denotes the projection of $\mathbf{V}(\mathfrak{f}_\ell; U)$ onto the Wiener chaos of order $2k$ associated with $\mathfrak{f}_\ell$. In the following proposition, we compute the variance of the total variation of $\mathfrak{f}_\ell$ on U in the specific framework, where the matrices $\mathfrak{f}'_\ell(z)$ and $\mathfrak{f}'_\ell(z')$ satisfy a certain matrix correlation structure for every $z, z' \in \mathbb{R}^n$ (see (III.3.42) below).

Proposition III.3.16. *Let the above notation prevail. Assume furthermore that for every $z, z' \in \mathbb{R}^n$,*

$$\mathfrak{f}'_\ell(z') \stackrel{d}{=} \mathfrak{f}'_\ell(z)R(z,z') + X_0(\mathbf{I}_n - R(z,z')^2)^{1/2}, \tag{III.3.42}$$

in distribution, where $X_0 = X_0(z,z')$ is an independent copy of $\mathfrak{f}'_\ell(z)$ and $R(z,z')$ is a deterministic matrix. Then,

$$\mathbf{Var}[\mathbf{V}(\mathfrak{f}_\ell; U)] = \sum_{k\ge 1}\sum_{\kappa\vdash k} \widehat{\Phi}(\kappa)^2 4^{-k} \left(\frac{n}{2}\right)_\kappa^{-1} k! \frac{C_\kappa(\mathbf{I}_\ell)}{C_\kappa(\mathbf{I}_n)} \int_{U\times U} C_\kappa(R(z,z')^2)dzdz', \tag{III.3.43}$$

where $\widehat{\Phi}(\kappa)$ is as in (III.3.26).

III.3.4.2 Applications to Arithmetic Random Waves on the three-torus

In this section, we apply the general framework presented in Section III.3.4.1 to the setting of vectors of independent arithmetic random waves on the three-torus, $\mathbb{T}^3$. Recall that Arithmetic Random Waves T_n are defined for integers $n \in S_3$ (that is n is an integer expressible as the sum of three integer squares) as the stationary Gaussian process $\left\{T_n(z) : z \in \mathbb{T}^3\right\}$ on the three-torus with covariance function

$$r^{(n)}(z,z') := \mathbb{E}\left[T_n(z)T_n(z')\right] = \frac{1}{\mathcal{N}_n}\sum_{\lambda\in\Lambda_n} e_\lambda(z-z') =: r^{(n)}(z-z'), \quad z,z' \in \mathbb{T}^3, \tag{III.3.44}$$

where $e_\lambda(z) := \exp(2\pi i\langle\lambda, z\rangle)$, Λ_n denotes the *set of frequencies* and $\mathcal{N}_n$ is its cardinality. For thorough introduction on Arithmetic Random Waves, we refer the reader to Section II.1.2 of Chapter II.

Total variation of vectors of ARW on $\mathbb{T}^3$. For an integer $1 \le \ell \le 3$ and $n \in S_3$, we consider i.i.d copies $T_n^{(1)}, \dots, T_n^{(\ell)}$ of T_n and consider the associated ℓ-dimensional Gaussian field

$$\mathbf{T}_n^{(\ell)} := \left\{\mathbf{T}_n^{(\ell)}(z) = \left(T_n^{(1)}(z), \dots, T_n^{(\ell)}(z)\right) : z \in \mathbb{T}^3\right\}. \tag{III.3.45}$$

Our specific goal is to study the high-energy behaviour (that is, when $\mathcal{N}_n \to \infty$) of the total variation $\mathbf{V}(\mathbf{T}_n^{(\ell)}; \mathbb{T}^3)$ (as defined in (III.3.40)) of $\mathbf{T}_n^{(\ell)}$ on the full torus. Rescaling the Jacobian matrix of $\mathbf{T}_n^{(\ell)}$ to make its entries have unit variance and according to (III.3.40), we use the homogeneity of the determinant in order to rewrite the total variation as

$$\mathbf{V}(\mathbf{T}_n^{(\ell)}; \mathbb{T}^3) = \left(\frac{E_n}{3}\right)^{\ell/2} \int_{\mathbb{T}^3} \Phi(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z))dz, \tag{III.3.46}$$

where $\Phi(M) := \sqrt{\det(MM^T)}$ and $\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z)$ denotes its normalized Jacobian matrix evaluated at z. We furthermore adopt the notation

$$r_{j,j'}^{(n)}(z) := \frac{\partial^2}{\partial z_j \partial z_j'} r^{(n)}(z), \quad \tilde{r}_{j,j'}^{(n)}(z) := \left(\frac{E_n}{3}\right)^{-1} r_{j,j'}^{(n)}(z). \tag{III.3.47}$$

The statement of our result is divided into three parts: (i) gives the expected total variation of vector-valued ARWs on the full torus, (ii) is an exact variance asymptotic and (iii) is a Central Limit Theorem in the high-energy regime for the normalized total variation

$$\widehat{\mathbf{V}}(\mathbf{T}_n^{(\ell)}; \mathbb{T}^3) := \frac{\mathbf{V}(\mathbf{T}_n^{(\ell)}; \mathbb{T}^3) - \mathbb{E}\left[\mathbf{V}(\mathbf{T}_n^{(\ell)}; \mathbb{T}^3)\right]}{\sqrt{\mathbf{Var}\left[\mathbf{V}(\mathbf{T}_n^{(\ell)}; \mathbb{T}^3)\right]}}. \tag{III.3.48}$$

Theorem III.3.17. *Let the above notation prevail.*

(i) *(Expected total variation) For every $n \in S_3$, we have*

$$\mathbb{E}\left[\mathbf{V}(\mathbf{T}_n^{(\ell)}; \mathbb{T}^3)\right] = \left(\frac{E_n}{3}\right)^{\ell/2} 2^{\ell/2} \frac{\Gamma_\ell(2)}{\Gamma_\ell(\frac{3}{2})} \tag{III.3.49}$$

(ii) *(Asymptotic variance) As $n \to \infty, n \not\equiv 0, 4, 7 \pmod 8$,*

$$\mathbf{Var}\left[\mathbf{V}(\mathbf{T}_n^{(\ell)}; \mathbb{T}^3)\right] = \left(\frac{E_n}{3}\right)^{\ell} 2^{\ell} \frac{\Gamma_\ell(2)^2}{\Gamma_\ell(\frac{3}{2})^2} \frac{\ell}{2\mathcal{N}_n} \left(1 + O(n^{-1/28+o(1)})\right) \tag{III.3.50}$$

(iii) *(CLT) As $n \to \infty, n \not\equiv 0, 4, 7 \pmod 8$,*

$$\widehat{\mathbf{V}}(\mathbf{T}_n^{(\ell)}; \mathbb{T}^3) \xrightarrow{d} \mathcal{N}(0, 1), \tag{III.3.51}$$

where $\xrightarrow{d}$ denotes convergence in distribution.

We remark that (III.3.49) and (III.3.50) imply that the normalized total variation $\mathbf{V}(\mathbf{T}_n^{(\ell)}; \mathbb{T}^3)/E_n^\ell$ converges in probability to $\left(\frac{2}{3}\right)^{\ell/2} \frac{\Gamma_\ell(2)}{\Gamma_\ell(3/2)}$ as $n \to \infty, n \not\equiv 0, 4, 7 \pmod 8$. Our proof of Theorem III.3.17 is based on the expansion of the total variation in (III.3.46) into matrix-variate Hermite polynomials by means of Corollary III.3.8 (see also (III.3.41)). As we will prove, the high-energy distributional behaviour of the normalized total variation is entirely characterized by its projection on the second Wiener chaos, which explains the underlying Gaussian fluctuations. In order to prove the negligibility of higer-order Wiener chaoses with respect to the second one, we rely on fine estimates for the second and sixth integral moments of $r^{(n)}$ derived in [BM19].

III.3.4.3 Wiener chaos expansion of nodal volumes associated with ARW

In this section, we consider the same framework of Section III.3.4.2 and provide the Wiener chaos expansion of the $(3-\ell)$-dimensional volume $L_n^{(\ell)} := \mathcal{H}_{3-\ell}((\mathbf{T}_n^{(\ell)})^{-1}(0))$ of the nodal sets associated with $\mathbf{T}_n^{(\ell)}$ in (III.3.45), that we extensively studied in Chapter II.

Recall that, as discussed in Section II.3.1 of Chapter II, using the Area/Co-Area formulae (see Proposition I.1.11), the random variable $L_n^{(\ell)}$ is defined $\mathbb{P}$-almost surely and in $L^2(\mathbb{P})$ by the integral representation

$$L_n^{(\ell)} = \left(\frac{E_n}{3}\right)^{\ell/2} \int_{\mathbb{T}^3} \delta_{(0,\dots,0)}(\mathbf{T}_n^{(\ell)}(z)) \Phi(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z)) dz, \tag{III.3.52}$$

where Φ is as in (III.3.46) and for $x = (x_1, \dots, x_\ell) \in \mathbb{R}^\ell$, $\delta_{(0,\dots,0)}(x) := \delta_0(x_1)\cdots\delta_0(x_\ell)$ denotes the multiple Dirac mass at the origin. Here below we show how matrix-variate Hermite polynomials allow one to obtain compact forms of the Wiener-Itô chaos expansion of $L_n^{(\ell)}$.

In view of definition (III.3.52), we use the stochastic independence of $\mathbf{T}_n^{(\ell)}(z)$ and $\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z)$ for every fixed $z \in \mathbb{T}^3$, to derive the chaos expansion of the nodal volume. Indeed, the latter is obtained by multiplying the respective Hermite expansions of the multivariate Dirac function and of Φ and then integrate over the torus.

Formal Wiener chaos expansion of multiple Dirac mass. The multivariate Dirac mass admits the formal expansion into multivariate Hermite polynomials (see Lemma II.A.1 of Chapter II)

$$\delta_{(0,\dots,0)}(\mathbf{T}_n^{(\ell)}(z)) = \sum_{q\geq 0}\sum_{|\alpha|=q} \frac{\tilde{\beta}_\alpha}{\alpha!} H_\alpha^{\otimes \ell}(\mathbf{T}_n^{(\ell)}(z)), \quad z \in \mathbb{T}^3$$

where $\alpha := (\alpha_1, \dots, \alpha_\ell) \in 2\mathbb{N}^\ell$, $|\alpha| := \sum_{i=1}^\ell \alpha_i$, $\alpha! := \prod_{i=1}^\ell \alpha_i!$ and

$$\tilde{\beta}_\alpha := \prod_{i=1}^{\ell} \beta_{\alpha_i}; \quad \beta_{\alpha_i} := \int_{\mathbb{R}} \delta_0(u) H_{\alpha_i}(u)\gamma(du).$$

Wiener chaos expansion of Gramian determinant. By Corollary III.3.8, we have the expansion of Φ in matrix-variate Hermite polynomials

$$\Phi(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z)) = \sum_{k\geq 0}\sum_{\kappa\vdash k} \widehat{\Phi}(\kappa) H_\kappa^{(\ell,3)}(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z)), \quad z \in \mathbb{T}^3$$

where the projection coefficients are obtained from (III.3.26) applied with $n = 3$,

$$\widehat{\Phi}(\kappa) = (-2)^k \left(\frac{3}{2}\right)_\kappa 2^{\ell/2} \cdot \frac{1}{k!} \frac{\Gamma_\ell(2)}{\Gamma_\ell(\frac{3}{2})} \sum_{s=0}^{k} \sum_{\sigma\vdash s} \binom{\kappa}{\sigma} (-1)^s \frac{(2)_\sigma}{(\frac{3}{2})_\sigma}. \tag{III.3.53}$$

Wiener chaos expansion of the nodal volume. Combining the two previous expansions and using independence, the Wiener chaos expansion of $L_n^{(\ell)}$ is given by the $L^2(\mathbb{P})$-converging series $L_n^{(\ell)} = \sum_{q\geq 0} L_n^{(\ell)}[2q]$ where for $q \geq 0$, $L_n^{(\ell)}[2q]$ is the chaotic component of order $2q$ given by

$$\begin{aligned} &L_n^{(\ell)}[2q] \\ &= \left(\frac{E_n}{3}\right)^{\ell/2} \sum_{q_1+2q_2=2q} \int_{\mathbb{T}^3} \sum_{|\alpha|=q_1} \frac{\tilde{\beta}_\alpha}{\alpha!} H_\alpha^{\otimes\ell}(\mathbf{T}_n^{(\ell)}(z)) \sum_{\kappa\vdash q_2} \widehat{\Phi}(\kappa) H_\kappa^{(\ell,3)}(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z))dz \\ &= \left(\frac{E_n}{3}\right)^{\ell/2} \sum_{q_1+2q_2=2q} \sum_{|\alpha|=q_1} \sum_{\kappa\vdash q_2} \frac{\tilde{\beta}_\alpha}{\alpha!} \widehat{\Phi}(\kappa) \int_{\mathbb{T}^3} H_\alpha^{\otimes\ell}(\mathbf{T}_n^{(\ell)}(z)) H_\kappa^{(\ell,3)}(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z))dz. \end{aligned} \tag{III.3.54}$$

We remark that, unlike the Wiener chaos expansion of the generalized total variation in (III.3.41), it becomes clear that the presence of the multiple Dirac mass leads to an expression containing both multivariate and matrix-variate Hermite polynomials. We write out the expressions obtained from (III.3.54) for $q \in \{0, 1, 2\}$, corresponding to the projections of $L_n^{(\ell)}$ on chaoses of order $0, 2$ and 4, respectively. For $q = 0$, we obtain the expected nodal volume

$$L_n^{(\ell)}[0] = \mathbb{E}\left[L_n^{(\ell)}\right] = \left(\frac{E_n}{3}\right)^{\ell/2} \beta_0^\ell \widehat{\Phi}((0)) = \left(\frac{E_n}{3}\right)^{\ell/2} \frac{1}{(2\pi)^{\ell/2}} 2^{\ell/2} \frac{\Gamma_\ell(2)}{\Gamma_\ell(\frac{3}{2})},$$

thus recovering the value of the expected nodal volume established in part (i) of our Theorem II.1.1. The projection on the second Wiener chaos is obtained for $q = 1$,

$$L_n^{(\ell)}[2] = \left(\frac{E_n}{3}\right)^{\ell/2} \int_{\mathbb{T}^3} \left[\sum_{i=1}^{\ell} \frac{\beta_2}{2!} H_2(T_n^{(i)}(z)) + \widehat{\Phi}((1)) H_{(1)}^{(\ell,3)}(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z)) \right] dz.$$

In Section II.2 of Chapter II, we presented a general principle, based on an application of Green's formula on manifolds, leading to cancellation phenomena in the study of nodal sets associated with Gaussian fields. In particular, we derived that that $L_n[2] = 0$ (see Section II.2.2 for details). An equivalent reformulation of this fact in terms of the matrix-variate Hermite polynomials $H_{(1)}^{(\ell,3)}$ then leads to the identity

$$\int_{\mathbb{T}^3} \left[\sum_{i=1}^{\ell} \frac{\beta_2}{2!} H_2(T_n^{(i)}(z)) + \widehat{\Phi}((1)) H_{(1)}^{(\ell,3)}(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z)) \right] dz = 0.$$

Using the definition of $H_{(1)}^{(\ell,3)}$ in terms of univariate Hermite polynomials H_2 (see (III.3.16)), and subsequently applying Green's integration by part formula on manifolds (see e.g. [Lee97, p.44]) eventually recovers this cancellation phenomenon. The fourth-order chaotic component of the nodal volume is obtained from (III.3.54) with $q = 2$, namely writing $\tilde{\beta}_0 := \prod_{i=1}^{\ell} \beta_0 = \beta_0^{\ell}$

$$\begin{aligned}
& L_n^{(\ell)}[4] \\
= & \left(\frac{E_n}{3}\right)^{\ell/2} \int_{\mathbb{T}^3} \Big[\widehat{\Phi}((0)) \sum_{|\alpha|=4} \frac{\tilde{\beta}_\alpha}{\alpha!} H_\alpha^{\otimes \ell}(\mathbf{T}_n^{(\ell)}(z)) + \sum_{|\alpha|=2} \frac{\tilde{\beta}_\alpha}{\alpha!} \widehat{\Phi}((1)) H_\alpha^{\otimes \ell}(\mathbf{T}_n^{(\ell)}(z)) H_{(1)}^{(\ell,3)}(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z)) \\
& \quad + \tilde{\beta}_0 \widehat{\Phi}((2)) H_{(2)}^{(\ell,3)}(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z)) + \tilde{\beta}_0 \widehat{\Phi}((1,1)) \int_{\mathbb{T}^3} H_{(1,1)}^{(\ell,3)}(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z)) \Big] dz \\
= & \left(\frac{E_n}{3}\right)^{\ell/2} \Big[\widehat{\Phi}((0)) \frac{\beta_4}{4!} \sum_{i \in [\ell]} \int_{\mathbb{T}^3} H_4(T_n^{(i)}(z)) dz + \left(\frac{\beta_2}{2!}\right)^2 \widehat{\Phi}((0)) \sum_{i<j \in [\ell]} \int_{\mathbb{T}^3} H_2(T_n^{(i)}(z)) H_2(T_n^{(j)}(z)) dz \\
& \quad + \frac{\beta_2}{2!} \widehat{\Phi}((1)) \sum_{i \in [\ell]} \int_{\mathbb{T}^3} H_2(T_n^{(i)}(z)) H_{(1)}^{(\ell,3)}(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z)) dz \\
& \quad + \tilde{\beta}_0 \widehat{\Phi}((2)) \int_{\mathbb{T}^3} H_{(2)}^{(\ell,3)}(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z)) dz + \tilde{\beta}_0 \widehat{\Phi}((1,1)) \int_{\mathbb{T}^3} H_{(1,1)}^{(\ell,3)}(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z)) dz \Big] \\
=: & \left(\frac{E_n}{3}\right)^{\ell/2} \left[T_1^{(\ell)}(n) + \ldots + T_5^{(\ell)}(n) \right], \qquad \text{(III.3.55)}
\end{aligned}$$

where

$$\begin{aligned}
T_1^{(\ell)}(n) &:= \frac{\beta_4}{4!} \widehat{\Phi}((0)) \sum_{i \in [\ell]} \int_{\mathbb{T}^3} H_4(T_n^{(i)}(z)) dz \\
T_2^{(\ell)}(n) &:= \left(\frac{\beta_2}{2!}\right)^2 \widehat{\Phi}((0)) \sum_{i<j \in [\ell]} \int_{\mathbb{T}^3} H_2(T_n^{(i)}(z)) H_2(T_n^{(j)}(z)) dz \\
T_3^{(\ell)}(n) &:= \frac{\beta_2}{2!} \widehat{\Phi}((1)) \sum_{i \in [\ell]} \int_{\mathbb{T}^3} H_2(T_n^{(i)}(z)) H_{(1)}^{(\ell,3)}(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z)) dz \\
T_4^{(\ell)}(n) &:= \tilde{\beta}_0 \widehat{\Phi}((2)) \int_{\mathbb{T}^3} H_{(2)}^{(\ell,3)}(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z)) dz \\
T_5^{(\ell)}(n) &:= \tilde{\beta}_0 \widehat{\Phi}((1,1)) \int_{\mathbb{T}^3} H_{(1,1)}^{(\ell,3)}(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z)) dz,
\end{aligned}$$

that is, the fourth-order chaotic component can be written compactly as a sum of 5 terms. This expression is to be compared with Equation (II.3.23). Furthermore, when $\ell = 1$, we have that $T_5^{(1)}(n) = 0$, since in this case, only projection coefficients $\widehat{\Phi}(\kappa)$ associated with partitions κ of length one contribute to the chaotic expansion of $L_n^{(\ell)}$. Writing out the explicit values of the projection coefficients in (III.3.53) for partitions $\kappa \in \{(2),(1,1)\}$ and using the expressions for matrix-variate Hermite polynomials in (III.3.16), we recover the projection coefficients on the fourth Wiener chaos associated with Φ appearing in Proposition II.B.5 of Chapter II (see also Proposition II.3.14).

In Chapter II, we established variance asymptotics of the fourth Wiener chaos by computing variances and covariances of each terms appearing in its expression. Here below, we give some insight to deal with the variance of the fourth Wiener chaos in a more compact form. From (III.3.55), the variance of $L_n^{(\ell)}[4]$ is given by

$$\mathbf{Var}\left[L_n^{(\ell)}[4]\right] = \left(\frac{E_n}{3}\right)^{\ell}\left\{\sum_{p=1}^{5}\mathbf{Var}\left[T_p^{(\ell)}(n)\right] + 2\sum_{1\le p<q\le 5}\mathbb{E}\left[T_p^{(\ell)}(n)T_q^{(\ell)}(n)\right]\right\}.$$

In order to deal with the variances of the random variables $T_p^{(\ell)}(n), p = 1,2,3$, one can rely on the classical diagram formulae for Hermite polynomials (see e.g. [PT11]), whereas for the variance of $T_p^{(\ell)}(n), p = 4,5$, one has

$$\begin{aligned}\mathbf{Var}\left[T_4^{(\ell)}(n)\right] &= \tilde{\beta}_0^2\widehat{\Phi}((2))^2\int_{\mathbb{T}^3\times\mathbb{T}^3}\mathbb{E}\left[H_{(2)}^{(\ell,3)}(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z))H_{(2)}^{(\ell,3)}(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z'))\right]dzdz'\\ \mathbf{Var}\left[T_5^{(\ell)}(n)\right] &= \tilde{\beta}_0^2\widehat{\Phi}((1,1))^2\int_{\mathbb{T}^3\times\mathbb{T}^3}\mathbb{E}\left[H_{(1,1)}^{(\ell,3)}(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z))H_{(1,1)}^{(\ell,3)}(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z'))\right]dzdz'.\end{aligned}$$

Using the fact that the Jacobian matrices $\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z)$ and $\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z')$ are correlated according to (III.3.42) where $R(z,z') = R_n(z-z')$ with (see also the proof of Lemma III.4.6 for more details on this)

$$R_n(z-z') := \left(\tilde{r}_{j,j'}^{(n)}(z-z')\right)_{j,j'\in[3]},\quad \tilde{r}_{j,j'}^{(n)}(z-z') := \left(\frac{E_n}{3}\right)^{-1}\frac{\partial^2}{\partial z_j\partial z_{j'}}r^{(n)}(z-z'),$$

we can use Theorem III.3.12 to infer (for $T_4^{(\ell)}(n)$ and similarly for $T_5^{(\ell)}(n)$)

$$\begin{aligned}\mathbf{Var}\left[T_4^{(\ell)}(n)\right] &= \tilde{\beta}_0^2\widehat{\Phi}((2))^2 4^{-2}\left(\frac{3}{2}\right)_{(2)}^{-1}2!\frac{C_{(2)}(\mathbf{I}_\ell)}{C_{(2)}(\mathbf{I}_3)}\int_{\mathbb{T}^3\times\mathbb{T}^3}C_{(2)}(R_n(z-z')^2)dzdz'\\ &= \tilde{\beta}_0^2\widehat{\Phi}((2))^2 4^{-2}\left(\frac{3}{2}\right)_{(2)}^{-1}2!\frac{C_{(2)}(\mathbf{I}_\ell)}{C_{(2)}(\mathbf{I}_3)}\int_{\mathbb{T}^3}C_{(2)}(R_n(z)^2)dz,\end{aligned}$$

where the last identity follows from stationarity (and a similar expression involving the zonal polynomial $C_{(1,1)}$ holds for the variance of $T_5^{(\ell)}(n)$). Moreover since $T_4^{(\ell)}(n)$ and $T_5^{(\ell)}(n)$ involve different partitions of the integer 2, Theorem III.3.12 implies that $\mathbb{E}\left[T_4^{(\ell)}(n)T_5^{(\ell)}(n)\right] = 0$, that is, the random variables $T_4(n)$ and $T_5(n)$ are orthogonal in $L^2(\mathbb{P})$. It should be remarked that $T_4^{(\ell)}(n)$ and $T_5^{(\ell)}(n)$ are however not orthogonal in $L^2(\mathbb{P})$ to the remaining terms $T_p^{(\ell)}(n), p = 1,2,3$, as can be seen for instance from

$$\mathbb{E}\left[T_4^{(\ell)}(n)T_1^{(\ell)}(n)\right] = \left(\frac{\beta_4}{4!}\right)^2\widehat{\Phi}((2))^2\sum_{i\in[\ell]}\int_{\mathbb{T}^3\times\mathbb{T}^3}\mathbb{E}\left[H_4(T_n^{(i)}(z))H_{(2)}^{(\ell,3)}(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z'))\right]dzdz'$$

involving covariances of both univariate and matrix-variate Hermite polynomials. Writing out $H_{(2)}^{(\ell,3)}$ in univariate Hermite polynomials in (III.3.16), and using independence, we obtain that

$$\sum_{i=1}^{\ell}\mathbb{E}\left[H_4(T_n^{(i)}(z))H_{(2)}^{(\ell,3)}(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z'))\right] = \ell\cdot\mathbb{E}\left[H_4(T_n^{(1)}(z))H_{(2)}^{(\ell,3)}(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z'))\right]$$

$$= \frac{\ell}{36\cdot 5}\Big(3\sum_{j\in[3]}\mathbb{E}\left[H_4(T_n^{(1)}(z))H_4(\tilde{\partial}_j T_n^{(1)}(z'))\right]$$
$$+3\sum_{j_1\neq j_2\in[3]}\mathbb{E}\left[H_4(T_n^{(1)}(z))H_2(\tilde{\partial}_{j_1}T_n^{(1)}(z'))H_2(\tilde{\partial}_{j_2}T_n^{(1)}(z'))\right]\Big).$$

In order to deal with this expression, one can again use the classical diagram formulae for univariate Hermite polynomials (see for instance the relations in Example III.3.14). We thus believe that matrix-variate Hermite polynomials can be adequately used in order to handle covariances of terms involving only the gradient components of $\mathbf{T}_n^{(\ell)}$.

III.4 Proofs of main results

III.4.1 Proofs of Section III.3.1

Proof of Theorem III.3.2

Since $F\in L^2(\mu_X)\subset L^2(\gamma^{(\ell,n)})$, we can expand it in the two orthonormal systems $\mathbb{H}_{[\ell n]}$ and $\mathbb{H}_{[\ell\times n]}$ defined in (III.3.1) and (III.3.11) respectively, yielding

$$F=\sum_{k\geq 0}\mathrm{proj}(F|C_k^{\mathbf{X}})=\sum_{k\geq 0}\sum_{\kappa\vdash k}\widehat{F}(\kappa)H_\kappa^{(\ell,n)}. \tag{III.4.1}$$

Using the representation of zonal polynomials in (III.2.6), we write $C_\kappa(XX^T)$ as a homogeneous polynomial of degree $2k$ in the entries of $X=(X_{ij})$, that is

$$C_\kappa(XX^T)=\sum_{|\alpha|=2k}z_\alpha^\kappa\prod_{i=1}^{\ell}\prod_{j=1}^{n}X_{ij}^{\alpha_{ij}},$$

where $\alpha\in\mathbb{N}^{\ell\times n}$ is a multi-index such that $|\alpha|=2k$ and z_α^κ is an explicit constant depending on α and κ. Using the above representation of zonal polynomials in the generalized Rodrigues formula (III.3.6), it follows that

$$\begin{aligned}H_\kappa^{(\ell,n)}(X) &= 4^{-k}\left(\frac{n}{2}\right)_\kappa^{-1}[\gamma^{(\ell,n)}(X)]^{-1}C_\kappa(\partial X\partial X^T)\gamma^{(\ell,n)}(X)\\ &= 4^{-k}\left(\frac{n}{2}\right)_\kappa^{-1}[\gamma^{(\ell,n)}(X)]^{-1}\sum_{|\alpha|=2k}z_\alpha^\kappa\prod_{i=1}^{\ell}\prod_{j=1}^{n}\frac{\partial^{\alpha_{ij}}}{\partial X_{ij}^{\alpha_{ij}}}\gamma^{(\ell,n)}(X)\\ &= 4^{-k}\left(\frac{n}{2}\right)_\kappa^{-1}\sum_{|\alpha|=2k}z_\alpha^\kappa\prod_{i=1}^{\ell}\prod_{j=1}^{n}[\gamma(X_{ij})]^{-1}\frac{\partial^{\alpha_{ij}}}{\partial X_{ij}^{\alpha_{ij}}}\gamma(X_{ij}).\end{aligned}$$

Then, using the classical Rodrigues formula for Hermite polynomials on the real line (III.3.7) for every $i\in[\ell]$, $j\in[n]$, we infer that

$$[\gamma(X_{ij})]^{-1}\frac{\partial^{\alpha_{ij}}}{\partial X_{ij}^{\alpha_{ij}}}\gamma(X_{ij})=(-1)^{\alpha_{ij}}H_{\alpha_{ij}}(X_{ij}),$$

so that, using the fact that $|\alpha|=2k$,

$$H_\kappa^{(\ell,n)}(X)=4^{-k}\left(\frac{n}{2}\right)_\kappa^{-1}\sum_{|\alpha|=2k}z_\alpha^\kappa\prod_{i=1}^{\ell}\prod_{j=1}^{n}(-1)^{\alpha_{ij}}H_{\alpha_{ij}}(X_{ij})=4^{-k}\left(\frac{n}{2}\right)_\kappa^{-1}\sum_{|\alpha|=2k}z_\alpha^\kappa H_\alpha^{\otimes\ell n}(X_{11},\dots,X_{\ell n}).$$

The above expression yields the expansion of $H_\kappa^{(\ell,n)}(X)$ into multivariate Hermite polynomials and implies in particular that $H_\kappa^{(\ell,n)}(X)$ is an element of the Wiener chaos of order $2k$ associated with the vector $\mathbf{X} = \mathrm{Vec}(X)$. The formula for the projection of F onto $C_{2k}^{\mathbf{X}}$ in (III.3.14) then follows summing over all partitions of k. The fact that the projection of F onto Wiener chaos of odd order is zero follows from the fact that the R.H.S of (III.4.1) does not involve any multivariate Hermite polynomials of cumulative odd order since $H_\kappa^{(\ell,n)}(X) \in C_{2k}^{\mathbf{X}}$.

In order to prove formula (III.3.15), we use the identity (III.3.8) and subsequently apply the polar decomposition $X = R^{1/2}U$ according to (III.2.13), yielding $(dX) = \frac{\pi^{n\ell/2}}{\Gamma_\ell(\frac{n}{2})}\det(R)^{\frac{n-\ell-1}{2}}\nu(dR)\tilde{\mu}(dU)$, (see e.g. [Chi03, Theorem 1.5.2]). Therefore, we have from (III.3.13)

$$\begin{aligned}
&\hat{F}(\kappa) = c(\kappa)^{-1}\int_{\mathbb{R}^{\ell\times n}} F(X)H_\kappa^{(\ell,n)}(X)\phi^{(\ell,n)}(X)(dX)\\
=\;& c(\kappa)^{-1}\gamma_\kappa(2\pi)^{-n\ell/2}\int_{\mathbb{R}^{\ell\times n}} f_0(XX^T)L_\kappa^{(\frac{n-\ell-1}{2})}(2^{-1}XX^T)\mathrm{etr}\left(-2^{-1}XX^T\right)(dX)\\
=\;& c(\kappa)^{-1}\gamma_\kappa(2\pi)^{-n\ell/2}\int_{O(n,\ell)}\int_{\mathcal{P}_\ell(\mathbb{R})} f_0(R)L_\kappa^{(\frac{n-\ell-1}{2})}(2^{-1}R)\mathrm{etr}\left(-2^{-1}R\right)\frac{\pi^{n\ell/2}}{\Gamma_\ell(\frac{n}{2})}\det(R)^{\frac{n-\ell-1}{2}}\nu(dR)\tilde{\mu}(dU)\\
=\;& c(\kappa)^{-1}\gamma_\kappa(2\pi)^{-n\ell/2}\int_{\mathcal{P}_\ell(\mathbb{R})} f_0(R)L_\kappa^{(\frac{n-\ell-1}{2})}(2^{-1}R)\mathrm{etr}\left(-2^{-1}R\right)\frac{\pi^{n\ell/2}}{\Gamma_\ell(\frac{n}{2})}\det(R)^{\frac{n-\ell-1}{2}}\nu(dR)\\
=\;& \frac{1}{2^{n\ell/2}\Gamma_\ell(\frac{n}{2})}\frac{(-2)^k}{k!C_\kappa(\mathbf{I}_\ell)}\int_{\mathcal{P}_\ell(\mathbb{R})} f_0(R)L_\kappa^{(\frac{n-\ell-1}{2})}(2^{-1}R)\mathrm{etr}\left(-2^{-1}R\right)\det(R)^{\frac{n-\ell-1}{2}}\nu(dR),
\end{aligned}$$

where we used that $\tilde{\mu}$ is a probability measure on $O(n,\ell)$ and the definitions of $c(\kappa)$ and γ_κ in (III.3.11) and (III.3.8), respectively. This finishes the proof.

Proof of Proposition III.3.3

By Theorem III.3.2, the Wiener-Itô chaos expansion of $F(X)$ is given by

$$F(X) = \sum_{k\geq 0}\sum_{\kappa\vdash k}\widehat{F}(\kappa)H_\kappa^{(\ell,n)}(X), \tag{III.4.2}$$

where $\widehat{F}(\kappa)$ is as in (III.3.13). Computing the $L^2(\mathbb{P})$-norm on both sides of (III.4.2) and using the orthogonality relation (III.3.9) then yields

$$\begin{aligned}
\mathbb{E}\left[F(X)^2\right] &= \sum_{k\geq 0}\sum_{l\geq 0}\sum_{\kappa\vdash k}\sum_{\sigma\vdash l}\widehat{F}(\kappa)\widehat{F}(\sigma)\mathbb{E}\left[H_\kappa^{(\ell,n)}(X)H_\sigma^{(\ell,n)}(X)\right]\\
&= \sum_{k\geq 0}\sum_{\kappa\vdash k}\widehat{F}(\kappa)^2 4^{-k}\left(\frac{n}{2}\right)_\kappa^{-1}k!C_\kappa(\mathbf{I}_\ell) = \widehat{F}((0))^2 + \sum_{k\geq 1}\sum_{\kappa\vdash k}\widehat{F}(\kappa)^2 4^{-k}\left(\frac{n}{2}\right)_\kappa^{-1}k!C_\kappa(\mathbf{I}_\ell).
\end{aligned}$$

Since $\widehat{F}((0)) = \mathbb{E}[F(X)]$, we obtain the expansion for the variance of $F(X)$,

$$\begin{aligned}
\mathbf{Var}[F(X)] &= \sum_{k\geq 1}\sum_{\kappa\vdash k}\widehat{F}(\kappa)^2 4^{-k}\left(\frac{n}{2}\right)_\kappa^{-1}k!C_\kappa(\mathbf{I}_\ell) = \sum_{k\geq 1}\sum_{\kappa\vdash k}\left[4^{-k}\left(\frac{n}{2}\right)_\kappa^{-1}k!C_\kappa(\mathbf{I}_\ell)\right]^{-1}\mathbb{E}\left[F(X)H_\kappa^{(\ell,n)}(X)\right]^2\\
&= \sum_{k\geq 1}\sum_{\kappa\vdash k}\frac{4^k(\frac{n}{2})_\kappa}{k!C_\kappa(\mathbf{I}_\ell)}\mathbb{E}\left[F(X)H_\kappa^{(\ell,n)}(X)\right]^2,
\end{aligned}$$

where we used (III.3.11).

III.4.2 Proofs of Section III.3.2

Polar decomposition of Gaussian rectangular matrices

Let us assume that X has the $\mathcal{N}_{\ell\times n}(0, \mathbf{I}_\ell \otimes \Sigma)$ distribution with density function $\gamma_\Sigma^{(\ell,n)}(X)$ defined in (III.3.18), and write $X = R^{1/2}U$ for its polar decomposition according to (III.2.13). In the following lemma, we compute the joint probability density function of the pair (R, U).

Lemma III.4.1. *If $X \sim \mathcal{N}_{\ell\times n}(0, \mathbf{I}_\ell \otimes \Sigma)$, the joint probability density of the pair (R, U) is given by*

$$f_{(R,U)}(R, U) = \frac{1}{\Gamma_\ell(\frac{n}{2})} \frac{1}{2^{n\ell/2}} \det(\Sigma)^{-\ell/2} \det(R)^{\frac{n-\ell-1}{2}} \operatorname{etr}\left(-2^{-1}U\Sigma^{-1}U^T R\right). \tag{III.4.3}$$

Proof. Applying the polar change of variable $X = R^{1/2}U$ gives $(dX) = \frac{\pi^{n\ell/2}}{\Gamma_\ell(\frac{n}{2})} \det(R)^{\frac{n-\ell-1}{2}} \nu(dR)\tilde{\mu}(dU)$ (see e.g. [Chi03, Theorem 1.5.2]), so that

$$\begin{aligned}
\gamma_\Sigma(X)(dX) &= \gamma_\Sigma(R^{1/2}U) \frac{\pi^{n\ell/2}}{\Gamma_\ell(\frac{n}{2})} \det(R)^{\frac{n-\ell-1}{2}} \nu(dR)\tilde{\mu}(dU) \\
&= \frac{1}{\Gamma_\ell(\frac{n}{2})} \frac{1}{2^{n\ell/2}} \det(\Sigma)^{-\ell/2} \det(R)^{\frac{n-\ell-1}{2}} \operatorname{etr}\left(-2^{-1}U\Sigma^{-1}U^T R\right) \nu(dR)\tilde{\mu}(dU),
\end{aligned}$$

where we used that $\operatorname{etr}\left(-2^{-1}R^{1/2}U\Sigma^{-1}U^T R^{1/2}\right) = \operatorname{etr}\left(-2^{-1}U\Sigma^{-1}U^T R\right)$. □

The following lemma (see [Chi03, Theorem 2.4.2]) gives the marginal density functions of R and U, respectively. These are obtained when integrating the joint density $f_{(R,U)}(R, U)$ with respect to U and R, respectively.

Lemma III.4.2 (Theorem 2.4.2, [Chi03])**.** *Assume that $X \sim \mathcal{N}_{\ell\times n}(0, \mathbf{I}_\ell \otimes \Sigma)$ and write $X = R^{1/2}U$. Then, the marginal density functions of R and U are respectively given by*

$$f_R(R) = \frac{1}{2^{n\ell/2}\Gamma_\ell(\frac{n}{2}) \det(\Sigma)^{\ell/2}} {}_0F_0\left(;; -2^{-1}\Sigma^{-1}, R\right) \det(R)^{\frac{n-\ell-1}{2}}, \tag{III.4.4}$$

where

$${}_pF_q\left(a_1, \dots, a_p; b_1, \dots, b_q; S, T\right) = \sum_{k\geq 0} \sum_{\kappa\vdash k} \frac{(a_1)_\kappa \cdots (a_p)_\kappa}{(b_1)_\kappa \cdots (b_q)_\kappa} \frac{C_\kappa(S)C_\kappa(T)}{k! C_\kappa(\mathbf{I}_\ell)}$$

denotes the hypergeometric function with two matrix arguments (see e.g. [Chi03, Appendix A.6]) and

$$f_U(U) = \det(\Sigma)^{-\ell/2} \det(U\Sigma^{-1}U^T)^{-n/2}. \tag{III.4.5}$$

The density function of U in (III.4.5) is referred to as the *matrix angular central distribution with parameter* Σ on $O(n, \ell)$. We also point out that, when $\Sigma = \mathbf{I}_n$, the matrix R follows the *Wishart distribution* with density function

$$\frac{1}{2^{n\ell/2}\Gamma_\ell(\frac{n}{2})} {}_0F_0\left(;; -2^{-1}\,\mathbf{I}_\ell, R\right) \det(R)^{\frac{n-\ell-1}{2}} = \frac{1}{2^{n\ell/2}\Gamma_\ell(\frac{n}{2})} \operatorname{etr}\left(-2^{-1}R\right) \det(R)^{\frac{n-\ell-1}{2}}$$

and the matrix angular central distribution of U reduces to the uniform distribution on $O(n, \ell)$. Moreover, it follows from (III.4.3), that in this case, R and U are independent.

Combining (III.4.3) with (III.4.5), we obtain the conditional probability density of R given U:

$$\frac{f_{(R,U)}(R, U)}{f_U(U)} = \frac{1}{\Gamma_\ell(\frac{n}{2})} \frac{1}{2^{n\ell/2}} \det(R)^{\frac{n-\ell-1}{2}} \operatorname{etr}\left(-2^{-1}U\Sigma^{-1}U^T R\right) \det(U\Sigma^{-1}U^T)^{n/2}. \tag{III.4.6}$$

In the forthcoming sections, whenever Z is a random variable, we often write $\mathbb{E}_Z[\cdot]$ to indicate mathematical expectation with respect to the law of Z.

Proof of Proposition III.3.4

We observe that the following relation holds

$$\gamma_{\Sigma}^{(\ell,n)}(X) = \det(\Sigma)^{-\ell/2}\gamma^{(\ell,n)}(X\Sigma^{-1/2}), \tag{III.4.7}$$

where $\gamma^{(\ell,n)}$ denotes the standard Gaussian density on $\mathbb{R}^{\ell\times n}$. From the definition (III.3.19) and the relation (III.4.7), it hence follows that

$$\begin{aligned}&\int_{\mathbb{R}^{\ell\times n}} H_\kappa^{(\ell,n)}(X;\Sigma)H_\sigma^{(\ell,n)}(X;\Sigma)\gamma_\Sigma^{(\ell,n)}(X)(dX)\\ &= \det(\Sigma)^{\ell k+\ell l-\ell/2}\int_{\mathbb{R}^{\ell\times n}} H_\kappa^{(\ell,n)}(X\Sigma^{-1/2})H_\sigma^{(\ell,n)}(X\Sigma^{-1/2})\gamma^{(\ell,n)}(X\Sigma^{-1/2})(dX).\end{aligned}$$

Applying the change of variables $Y = X\Sigma^{-1/2}$, we have $(dY) = \det(\Sigma^{-1/2})^{\ell}(dX) = \det(\Sigma)^{-\ell/2}(dX)$ (see e.g. [Mui82, Theorem 2.1.5]), i.e. $(dX) = \det(\Sigma)^{\ell/2}(dY)$, so that the integral above becomes

$$\begin{aligned}&\det(\Sigma)^{\ell k+\ell l-\ell/2}\int_{\mathbb{R}^{\ell\times n}} H_\kappa^{(\ell,n)}(X\Sigma^{-1/2})H_\sigma^{(\ell,n)}(X\Sigma^{-1/2})\gamma^{(\ell,n)}(X\Sigma^{-1/2})(dX)\\ =\ &\det(\Sigma)^{\ell k+\ell l}\int_{\mathbb{R}^{\ell\times n}} H_\kappa^{(\ell,n)}(Y)H_\sigma^{(\ell,n)}(Y)\gamma^{(\ell,n)}(Y)(dY) = \mathbb{1}_{\kappa=\sigma}\det(\Sigma)^{2\ell k}4^{-k}\left(\frac{n}{2}\right)_\kappa^{-1} k!C_\kappa(\mathbf{I}_\ell),\end{aligned}$$

where we used (III.3.9). This proves the statement.

Proof of Theorem III.3.5

The proof of Theorem III.3.5 is based on the following key identity.

Lemma III.4.3. *Let $A\in\mathbb{C}^{\ell\times\ell}$ be a complex symmetric matrix with positive real part, $B\in\mathbb{C}^{\ell\times\ell}$ a complex symmetric matrix and $t\in\mathbb{C}$ such that $\Re(t) > (\ell-1)/2$. Then, we have*

$$\begin{aligned}&\int_{\mathcal{P}_\ell(\mathbb{R})} \operatorname{etr}(-AR)\det(R)^{t-\frac{\ell+1}{2}}L_\kappa^{(\gamma)}(RB)\nu(dR) \\ &= \left(\gamma+\frac{\ell+1}{2}\right)_\kappa C_\kappa(\mathbf{I}_\ell)\Gamma_\ell(t)\det(A)^{-t}\sum_{s=0}^{k}\sum_{\sigma\vdash s}\binom{\kappa}{\sigma}\frac{(-1)^s}{(\gamma+\frac{\ell+1}{2})_\sigma}\frac{1}{C_\sigma(\mathbf{I}_\ell)}(t)_\sigma C_\sigma(BA^{-1}).\end{aligned} \tag{III.4.8}$$

Proof. This identity follows directly from the definition of Laguerre polynomials in (III.2.10): indeed by linearity, it suffices to apply relation (III.2.12) on each zonal polynomial C_σ appearing in the expansion of $L_\kappa^{(\gamma)}$. □

We are now in position to prove Theorem III.3.5.

Proof of Theorem III.3.5. The fact that the random variable $F(X) = \det(XX^T)^{1/2}$ is an element of $L^2(\mu_X)$ follows from the following observation: Denoting by $s_1,\ldots,s_\ell$ the eigenvalues of XX^T, we have that

$$\det(XX^T) = \prod_{i=1}^{\ell} s_i = \frac{1}{\ell!}\sum_{i_1\neq\ldots\neq i_\ell\in[\ell]} s_{i_1}\cdots s_{i_\ell} = \int_{\mathbb{R}}\cdots\int_{\mathbb{R}}\frac{1}{\ell!}\mathbb{1}[t_i\neq t_j,\ \forall i\neq j\in[\ell]]\mu_X(dt_1)\ldots\mu_X(dt_\ell).$$

This justifies the decomposition into matrix-variate Hermite polynomials of F. We now prove formula (III.3.22). Using the definition of the polynomials $H_\kappa^{(\ell,n)}(X;\Sigma)$ in (III.3.19) and the relation (III.3.8), we obtain from (III.3.21)

$$\widehat{F}(\kappa;\Sigma) = c(\kappa;\Sigma)^{-1}\mathbb{E}_X\left[F(X)H_\kappa^{(\ell,n)}(X;\Sigma)\right]$$

$$= \quad c(\kappa;\Sigma)^{-1}\det(\Sigma)^{\ell k}\gamma_\kappa \mathbb{E}_X\left[F(X)L_\kappa^{(\frac{n-\ell-1}{2})}(2^{-1}X\Sigma^{-1}X^T)\right], \quad X \sim \mathcal{N}_{\ell\times n}(0,\mathbf{I}_\ell\otimes\Sigma).$$

Applying the polar decomposition $X = R^{1/2}U$ and noting that $F(R^{1/2}U) = \det(R)^{1/2}$, we have

$$\begin{aligned}\widehat{F}(\kappa;\Sigma) &= c(\kappa;\Sigma)^{-1}\det(\Sigma)^{\ell k}\gamma_\kappa\mathbb{E}_{(R,U)}\left[\det(R)^{1/2}L_\kappa^{(\frac{n-\ell-1}{2})}(2^{-1}R^{1/2}U\Sigma^{-1}U^TR^{1/2})\right]\\ &= c(\kappa;\Sigma)^{-1}\det(\Sigma)^{\ell k}\gamma_\kappa\mathbb{E}_{(R,U)}\left[\det(R)^{1/2}L_\kappa^{(\frac{n-\ell-1}{2})}(2^{-1}U\Sigma^{-1}U^TR)\right]\end{aligned}$$

where in the last line we used the fact that $L_\kappa^{(\frac{n-\ell-1}{2})}(2^{-1}R^{1/2}U\Sigma^{-1}U^TR^{1/2}) = L_\kappa^{(\frac{n-\ell-1}{2})}(2^{-1}U\Sigma^{-1}U^TR)$, which is a consequence of the permutation invariance property (III.2.9) of zonal polynomials appearing in the definition of matrix-variate Laguerre polynomials (III.2.10). By conditioning on U, we can rewrite the above expectation as

$$\begin{aligned}&\mathbb{E}_{(R,U)}\left[\det(R)^{1/2}L_\kappa^{(\frac{n-\ell-1}{2})}(2^{-1}U\Sigma^{-1}U^TR)\right]\\ =\ &\mathbb{E}_U\left[\mathbb{E}_{R|U}\left[\det(R)^{1/2}L_\kappa^{(\frac{n-\ell-1}{2})}(2^{-1}U\Sigma^{-1}U^TR)\right]\right] =: \mathbb{E}_U\left[Z_\kappa(U;\Sigma)\right],\end{aligned}$$

where

$$Z_\kappa(U;\Sigma) := \mathbb{E}_{R|U}\left[\det(R)^{1/2}L_\kappa^{(\frac{n-\ell-1}{2})}(2^{-1}U\Sigma^{-1}U^TR)\right],$$

so that

$$\widehat{F}(\kappa;\Sigma) = c(\kappa;\Sigma)^{-1}\det(\Sigma)^{\ell k}\gamma_\kappa\mathbb{E}_U\left[Z_\kappa(U;\Sigma)\right]. \tag{III.4.9}$$

We start by computing $Z_\kappa(U;\Sigma)$. Using the conditional probability density of R given U in (III.4.6), we have

$$\begin{aligned}Z_\kappa(U;\Sigma) &= \int_{\mathcal{P}_\ell(\mathbb{R})}\det(R)^{1/2}L_\kappa^{(\frac{n-\ell-1}{2})}(2^{-1}U\Sigma^{-1}U^TR)\frac{f_{(R,U)}(R,U)}{f_U(U)}\nu(dR)\\ &= \int_{\mathcal{P}_\ell(\mathbb{R})}\det(R)^{1/2}L_\kappa^{(\frac{n-\ell-1}{2})}(2^{-1}U\Sigma^{-1}U^TR)\frac{1}{\Gamma_\ell(\frac{n}{2})}\frac{1}{2^{n\ell/2}}\det(R)^{\frac{n-\ell-1}{2}}\operatorname{etr}\left(-2^{-1}U\Sigma^{-1}U^TR\right)\\ &\qquad\times\det(U\Sigma^{-1}U^T)^{n/2}\nu(dR)\\ &= \int_{\mathcal{P}_\ell(\mathbb{R})}\det(R)^{\frac{n-\ell}{2}}L_\kappa^{(\frac{n-\ell-1}{2})}(2^{-1}U\Sigma^{-1}U^TR)\operatorname{etr}\left(-2^{-1}U\Sigma^{-1}U^TR\right)\nu(dR)\\ &\qquad\times\frac{1}{\Gamma_\ell(\frac{n}{2})}\frac{1}{2^{n\ell/2}}\det(U\Sigma^{-1}U^T)^{n/2}\\ &=: \frac{1}{\Gamma_\ell(\frac{n}{2})}\frac{1}{2^{n\ell/2}}\det(U\Sigma^{-1}U^T)^{n/2}\cdot I_\kappa(U;\Sigma),\end{aligned} \tag{III.4.10}$$

where

$$I_\kappa(U;\Sigma) = \int_{\mathcal{P}_\ell(\mathbb{R})}\det(R)^{\frac{n-\ell}{2}}L_\kappa^{(\frac{n-\ell-1}{2})}(2^{-1}U\Sigma^{-1}U^TR)\operatorname{etr}\left(-2^{-1}U\Sigma^{-1}U^TR\right)\nu(dR). \tag{III.4.11}$$

Exploiting the identity (III.4.8) with $\gamma = (n-\ell-1)/2, t = (n+1)/2$ and $A = B = 2^{-1}U\Sigma^{-1}U^T$ yields

$$I_\kappa(U;\Sigma) = \left(\frac{n}{2}\right)_\kappa C_\kappa(\mathbf{I}_\ell)\sum_{s=0}^{k}\sum_{\sigma\vdash s}\binom{\kappa}{\sigma}\frac{(-1)^s}{(\frac{n}{2})_\sigma}\frac{1}{C_\sigma(\mathbf{I}_\ell)}\left(\frac{n+1}{2}\right)_\sigma\Gamma_\ell\left(\frac{n+1}{2}\right)$$

$$
\begin{aligned}
& \times \det(2^{-1}U\Sigma^{-1}U^T)^{-(n+1)/2} C_\sigma(\mathbf{I}_\ell) \\
= \;& \det(2^{-1}U\Sigma^{-1}U^T)^{-(n+1)/2} \left(\frac{n}{2}\right)_\kappa C_\kappa(\mathbf{I}_\ell)\Gamma_\ell\left(\frac{n+1}{2}\right) \sum_{s=0}^{k}\sum_{\sigma\vdash s}\binom{\kappa}{\sigma}(-1)^s \frac{(\frac{n+1}{2})_\sigma}{(\frac{n}{2})_\sigma} \\
= \;& \det(U\Sigma^{-1}U^T)^{-(n+1)/2} 2^{\ell(n+1)/2} \left(\frac{n}{2}\right)_\kappa C_\kappa(\mathbf{I}_\ell)\Gamma_\ell\left(\frac{n+1}{2}\right) \sum_{s=0}^{k}\sum_{\sigma\vdash s}\binom{\kappa}{\sigma}(-1)^s \frac{(\frac{n+1}{2})_\sigma}{(\frac{n}{2})_\sigma} \\
= \;& \det(U\Sigma^{-1}U^T)^{-(n+1)/2} \cdot d_\kappa,
\end{aligned}
$$

where

$$
d_\kappa := 2^{\ell(n+1)/2} \left(\frac{n}{2}\right)_\kappa C_\kappa(\mathbf{I}_\ell)\Gamma_\ell\left(\frac{n+1}{2}\right) \sum_{s=0}^{k}\sum_{\sigma\vdash s}\binom{\kappa}{\sigma}(-1)^s \frac{(\frac{n+1}{2})_\sigma}{(\frac{n}{2})_\sigma}. \tag{III.4.12}
$$

Replacing this expression into the R.H.S of (III.4.10) eventually gives

$$
\begin{aligned}
Z_\kappa(U;\Sigma) &= \frac{1}{\Gamma_\ell(\frac{n}{2})}\frac{1}{2^{n\ell/2}} \det(U\Sigma^{-1}U^T)^{n/2} \det(U\Sigma^{-1}U^T)^{-(n+1)/2} \cdot d_\kappa \\
&= d_\kappa \frac{1}{\Gamma_\ell(\frac{n}{2})}\frac{1}{2^{n\ell/2}} \det(U\Sigma^{-1}U^T)^{-1/2}.
\end{aligned}
$$

Taking expectations with respect to U gives from (III.4.9)

$$
\begin{aligned}
\widehat{F}(\kappa;\Sigma) &= c(\kappa;\Sigma)^{-1}\det(\Sigma)^{\ell k}\gamma_\kappa \mathbb{E}_U\left[Z_\kappa(U;\Sigma)\right] \\
&= c(\kappa;\Sigma)^{-1}\det(\Sigma)^{\ell k}\gamma_\kappa d_\kappa \frac{1}{\Gamma_\ell(\frac{n}{2})}\frac{1}{2^{n\ell/2}} \mathbb{E}_U\left[\det(U\Sigma^{-1}U^T)^{-1/2}\right].
\end{aligned}
$$

The expectation with respect to U is computed using (III.4.5),

$$
\begin{aligned}
\mathbb{E}_U\left[\det(U\Sigma^{-1}U^T)^{-1/2}\right] &= \int_{O(\ell,n)} \det(U\Sigma^{-1}U^T)^{-1/2} f_U(U)\tilde{\mu}(dU) \\
&= \det(\Sigma)^{-\ell/2}\int_{O(\ell,n)} \det(U\Sigma^{-1}U^T)^{-(n+1)/2}\tilde{\mu}(dU).
\end{aligned}
$$

Replacing this expression into the previous relation, we conclude that

$$
\begin{aligned}
&\widehat{F}(\kappa;\Sigma) = c(\kappa;\Sigma)^{-1}\det(\Sigma)^{\ell k-\ell/2}\gamma_\kappa d_\kappa \frac{1}{\Gamma_\ell(\frac{n}{2})}\frac{1}{2^{n\ell/2}} \int_{O(\ell,n)} \det(U\Sigma^{-1}U^T)^{-(n+1)/2}\tilde{\mu}(dU) \\
&= \det(\Sigma)^{-\ell k}4^k\left(\frac{n}{2}\right)_\kappa \frac{1}{k!C_\kappa(\mathbf{I}_\ell)} \det(\Sigma)^{-\ell/2}\gamma_\kappa d_\kappa \frac{1}{\Gamma_\ell(\frac{n}{2})}\frac{1}{2^{n\ell/2}} \int_{O(n,\ell)} \det(U\Sigma^{-1}U^T)^{-(n+1)/2}\tilde{\mu}(dU),
\end{aligned}
$$

where we used the definition of $c(\kappa;\Sigma)$ in (III.3.20). Combining this expression with the definitions of γ_κ in (III.3.8) and d_κ in (III.4.12), yields after simplications

$$
\begin{aligned}
\widehat{F}(\kappa;\Sigma) &= \frac{(-2)^k}{\det(\Sigma)^{\ell k}k!}\left(\frac{n}{2}\right)_\kappa \sum_{s=0}^{k}\sum_{\sigma\vdash s}\binom{\kappa}{\sigma}(-1)^s \frac{(\frac{n+1}{2})_\sigma}{(\frac{n}{2})_\sigma} \\
&\quad \times \det(\Sigma)^{-\ell/2}2^{\ell/2}\frac{\Gamma_\ell(\frac{n+1}{2})}{\Gamma_\ell(\frac{n}{2})} \int_{O(n,\ell)} \det(U\Sigma^{-1}U^T)^{-(n+1)/2}\tilde{\mu}(dU),
\end{aligned}
$$

which finishes the proof. □

Proof of Theorem III.3.6

In order to prove (III.3.23), it is sufficient to prove the relation

$$\det(\Sigma)^{-\ell/2}2^{\ell/2}\frac{\Gamma_\ell(\frac{n+1}{2})}{\Gamma_\ell(\frac{n}{2})}\int_{O(n,\ell)}\det(U\Sigma^{-1}U^T)^{-(n+1)/2}\tilde{\mu}(dU)$$
$$= \frac{(n)_\ell}{(2\pi)^{\ell/2}\kappa_{n-\ell}}V(\mathcal{E}_\Sigma[\ell],\mathbb{B}_n[n-\ell]), \tag{III.4.13}$$

since then (III.3.23) directly follows after combining (III.4.13) with (III.3.22). Let us now prove (III.4.13). A direct computation shows that

$$2^{\ell/2}\frac{\Gamma_\ell(\frac{n+1}{2})}{\Gamma_\ell(\frac{n}{2})} = \frac{(n)_\ell}{(2\pi)^{\ell/2}}\frac{\kappa_n}{\kappa_{n-\ell}}, \quad \kappa_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)}. \tag{III.4.14}$$

Since the mixed volume on the R.H.S of (III.4.13) only involves the convex bodies $\mathcal{E}_\Sigma$ and $\mathbb{B}_n$, we can use (III.2.18) to represent it as an intrinsic volume,

$$V(\mathcal{E}_\Sigma[\ell],\mathbb{B}_n[n-\ell]) = \frac{\kappa_{n-\ell}}{\binom{n}{\ell}}V_\ell(\mathcal{E}_\Sigma). \tag{III.4.15}$$

Using the integral representation (III.2.16) for the ℓ-th intrinsic volume yields

$$V_\ell(\mathcal{E}_\Sigma) = \binom{n}{\ell}\frac{\kappa_n}{\kappa_\ell\kappa_{n-\ell}}\int_{G(n,\ell)}\mathrm{vol}_\ell(\mathcal{E}_\Sigma|\mathscr{U})\nu_{n,\ell}(d\mathscr{U}),$$

where $\nu_{n,\ell}$ is the Haar probability measure on the Grassmannian $G(n,\ell)$. Combining this with (III.4.14) shows that the identity in (III.4.13) is equivalent to

$$\det(\Sigma)^{-\ell/2}\int_{O(n,\ell)}\det(U\Sigma^{-1}U^T)^{-(n+1)/2}\tilde{\mu}(dU) = \frac{1}{\kappa_\ell}\int_{G(n,\ell)}\mathrm{vol}_\ell(\mathcal{E}_\Sigma|\mathscr{U})\nu_{n,\ell}(d\mathscr{U}). \tag{III.4.16}$$

Therefore it remains to prove (III.4.16). We rewrite the L.H.S of (III.4.16) as follows

$$\int_{O(n,\ell)}\det(U\Sigma^{-1}U^T)^{-1/2}\Pi_{n,\ell}(dU) = \int_{O(n,\ell)}\det([U\Sigma^{-1}U^T]^{-1})^{1/2}\Pi_{n,\ell}(dU),$$

where $\Pi_{n,\ell}(dU) = \det(\Sigma)^{-\ell/2}\det(U\Sigma^{-1}U^T)^{-n/2}\tilde{\mu}(dU)$ is a probability measure on $O(n,\ell)$ by virtue of (III.4.5). We now argue that

$$\int_{O(n,\ell)}\det([U\Sigma^{-1}U^T]^{-1})^{1/2}\Pi_{n,\ell}(dU) = \int_{O(n,\ell)}\det(U\Sigma U^T)^{1/2}\Pi_{n,\ell}(dU).$$

In order to see this, let us write $\Sigma = O\Lambda O^T$ for $O \in O(n), \Lambda = \mathrm{diag}(\lambda_1,\dots,\lambda_n)$. Then, we have $\det(U\Sigma^{-1}U^T) = \det(W\Lambda^{-1}W^T)$ with $W = UO \in O(n,\ell)$ since $WW^T = UO(UO)^T = \mathbf{I}_\ell$. Therefore, it suffices to consider the case where $\Sigma = \Lambda$ is diagonal. Moreover, since $W \in O(n,\ell)$ we have that for every $Q \in O(\ell)$, $(QW)(QW)^T = QWW^TQ^T = \mathbf{I}_\ell$, that is, $QW \in O(n,\ell)$. This implies that, up to rotating the matrix $W = UO$, we can assume that the rows of W coincide with the ℓ first canonical basis vectors $e_1,\dots,e_\ell$ in $\mathbb{R}^n$. Then, we compute

$$\det([W\Lambda^{-1}W^T]^{-1}) = \det(W\Lambda^{-1}W^T)^{-1} = \left(\prod_{i=1}^{\ell}\lambda_i^{-1}\right)^{-1} = \prod_{i=1}^{\ell}\lambda_i = \det(W\Lambda W^T).$$

Therefore, integrating on $O(n,\ell)$ and noting that $\Pi_{n,\ell}(d(QU)) = \Pi_{n,\ell}(dU)$ for every $Q \in O(\ell)$, yields the claim. Now, since $\Pi_{n,\ell}$ is left-invariant by orthogonal transformations, it can be viewed as a probability measure on $O(n,\ell)/O(\ell) \simeq G(n,\ell)$, where two elements U_1, U_2 in $O(n,\ell)$ are equivalent if and only if there exists $Q \in O(\ell)$ such that $U_1 = QU_2$. Thus, since $\nu_{n,\ell}$ is the unique left and right-invariant Haar probability measure on $G(n,\ell)$, it must coincide with $\Pi_{n,\ell}$. Writing $\mathscr{U}$ for the ℓ-dimensional linear subspace generated by the rows of U, we have that the matrix $U\Sigma U^T$ represents the ellipsoid $\mathcal{E}_\Sigma|\mathscr{U}$ of volume $\mathrm{vol}_\ell(\mathcal{E}_\Sigma|\mathscr{U}) = \kappa_\ell \det(U\Sigma U^T)^{1/2}$, implying in turn

$$\int_{O(n,\ell)} \det(U\Sigma U^T)^{1/2}\Pi_{n,\ell}(dU) = \int_{G(n,\ell)} \frac{1}{\kappa_\ell}\mathrm{vol}_\ell(\mathcal{E}_\Sigma|\mathscr{U})\nu_{n,\ell}(d\mathscr{U}).$$

This proves (III.4.16) and thus (III.4.13). Formula (III.3.24) follows from (III.3.23) and relation (III.4.15). Formula (III.3.25) is obtained when setting $\kappa = (0)$ in (III.3.23) and (III.3.24), respectively, and using the fact that $\widehat{F}((0);\Sigma) = \mathbb{E}_X[F(X)]$.

III.4.2.1 An attempt at generalizing to distinct covariance matrices

In this section, we try to generalize the results of Theorem III.3.5 and Theorem III.3.6 to the more general setting where the rows of X are independent Gaussian vectors with distinct covariance matrices. Let $\{\Sigma_i \in \mathbb{R}^{n\times n} : i \in [\ell]\}$ be positive-definite symmetric matrices and $\left\{X^{(i)} = (X_1^{(i)}, \dots, X_n^{(i)}) : i \in [\ell]\right\}$ a collection of ℓ independent Gaussian vectors with respective covariance matrices $\Sigma_1, \dots, \Sigma_\ell$. We write X for the $\ell \times n$ matrix whose i-th row is $X^{(i)}$. Then, the vector $\mathrm{Vec}(X^T)$ has the multivariate normal distribution $\mathcal{N}_{\ell n}(0,\Omega)$, where

$$\Omega = \sum_{i=1}^{\ell} \left(e_i e_i^T \otimes \Sigma_i\right) = \mathrm{diag}(\Sigma_1, \dots, \Sigma_\ell) = \Sigma_1 \oplus \dots \oplus \Sigma_\ell,$$

with $e_i \in \mathbb{R}^\ell$ denoting the i-th canonical basis vector. The density function of X is given by

$$\gamma_\Omega(\mathrm{Vec}(X^T)) = (2\pi)^{-n\ell/2} \det(\Omega)^{-n\ell/2} \mathrm{etr}\left(-\frac{1}{2}\Omega^{-1}\mathrm{Vec}(X^T)\mathrm{Vec}(X^T)^T\right).$$

If X is distributed as above, a computation shows that the $\ell \times n$ matrix

$$Y_X := (\mathbf{I}_\ell \otimes \mathrm{Vec}(X^T)^T\Omega^{-1/2})(\mathrm{Vec}(\mathbf{I}_\ell) \otimes \mathbf{I}_n)$$

has the standard matrix normal distribution. Therefore, we consider the matrix-variate polynomials

$$H_\kappa^{(\ell,n)}(X;\Omega) = \det(\Omega)^k H_\kappa^{(\ell,n)}(Y_X), \quad \kappa \vdash k \tag{III.4.17}$$

satisfying the orthogonality relation (similar as in the proof of Proposition III.3.4)

$$\int_{\mathbb{R}^{\ell\times n}} H_\kappa^{(\ell,n)}(X;\Omega) H_\sigma^{(\ell,n)}(X;\Omega)\gamma_\Omega(\mathrm{Vec}(X^T))(dX) = \mathbb{1}_{\kappa=\sigma} \cdot \det(\Omega)^{2k} 4^{-k} \left(\frac{n}{2}\right)_\kappa^{-1} k! C_\kappa(\mathbf{I}_\ell),$$

and thus the family

$$\mathbb{H}_\Omega := \left\{c(\kappa;\Omega)^{-1/2} H_\kappa^{(\ell,n)}(\cdot;\Omega) : \kappa \vdash k \geq 0\right\}, \quad c(\kappa;\Omega) := \det(\Omega)^{2k} 4^{-k} \left(\frac{n}{2}\right)_\kappa^{-1} k! C_\kappa(\mathbf{I}_\ell)$$

forms an orthonormal system of $L^2(\mu_X)$, where, as usual, μ_X indicates the spectral measure associated with XX^T. Expanding the function $F(X) = \det(XX^T)^{1/2} \in L^2(\mu_X)$ in the basis $\mathbb{H}_\Omega$, using the relation

$$\gamma_\Omega(\mathrm{Vec}(X^T)) = \det(\Omega)^{-1/2}\gamma^{(\ell,n)}(Y_X)$$

and the definition of $H_\kappa^{(\ell,n)}(\cdot;\Omega)$, we have that the associated projection coefficients are

$$\begin{aligned}
\widehat{F}(\kappa;\Omega) &= c(\kappa;\Omega)^{-1}\int_{\mathbb{R}^{\ell\times n}} F(X)H_\kappa^{(\ell,n)}(X;\Omega)\gamma_\Omega(\mathrm{Vec}(X^T))(dX)\\
&= \det(\Omega)^{k-1/2}\int_{\mathbb{R}^{\ell\times n}} F(X)H_\kappa^{(\ell,n)}(Y_X)\gamma^{(\ell,n)}(Y_X)(dX)\\
&= \det(\Omega)^{k-1/2}\gamma_\kappa\int_{\mathbb{R}^{\ell\times n}} F(X)L_\kappa^{(\frac{n-\ell-1}{2})}(2^{-1}Y_XY_X^T)(2\pi)^{-n\ell/2}\mathrm{etr}\left(-2^{-1}Y_XY_X^T\right)(dX).
\end{aligned}$$

The idea is now to perform the polar change of variables $X = R^{1/2}U$. In order to do so, we compute $\mathrm{tr}(Y_XY_X^T)$:

$$\mathrm{tr}(Y_XY_X^T) = \mathrm{tr}(\Omega^{-1}\mathrm{Vec}(X^T)\mathrm{Vec}(X^T)^T) = \mathrm{Vec}(X^T)^T\Omega^{-1}\mathrm{Vec}(X^T)$$

$$= \mathrm{Vec}(X^T)^T\sum_{i=1}^{\ell}\left(e_ie_i^T\otimes\Sigma_i^{-1}\right)\mathrm{Vec}(X^T) = \sum_{i=1}^{\ell}\mathrm{Vec}(X^T)^T\left(e_ie_i^T\otimes\Sigma_i^{-1}\right)\mathrm{Vec}(X^T).$$

Then, using the relation $\mathrm{Vec}(S)^T(BD\otimes E)\mathrm{Vec}(S) = \mathrm{tr}(DS^TESB)$ (see e.g. [GN00, Theorem 1.2.22]), we obtain

$$\begin{aligned}
\mathrm{tr}(Y_XY_X^T) &= \sum_{i=1}^{\ell}\mathrm{Vec}(X^T)^T\left(e_ie_i^T\otimes\Sigma_i^{-1}\right)\mathrm{Vec}(X^T) = \sum_{i=1}^{\ell}\mathrm{tr}\left(e_i^TR^{1/2}U\Sigma_i^{-1}U^TR^{1/2}e_i\right)\\
&= \sum_{i=1}^{\ell}\mathrm{tr}\left(e_ie_i^TR^{1/2}U\Sigma_i^{-1}U^TR^{1/2}\right) = \mathrm{tr}\left(\sum_{i=1}^{\ell}e_ie_i^TR^{1/2}U\Sigma_i^{-1}U^TR^{1/2}\right).
\end{aligned}$$

The difficulty to proceed now is the following: the above computation suggests that we cannot write $\mathrm{tr}(Y_XY_X^T)$ as $\mathrm{tr}(AR)$ for some matrix A, due to the fact that one cannot exploit the permutation invariance of the trace in view of presence of the matrix $e_ie_i^T$. We remark that, when $\Sigma_i = \Sigma$ for every $i = 1,\dots,\ell$, the above formula gives $\mathrm{tr}(Y_XY_X^T) = \mathrm{tr}(\mathbf{I}_\ell\, U\Sigma^{-1}U^T) = \mathrm{tr}(U\Sigma^{-1}U^T)$, which coincides with our computations in the proof of Theorem III.3.5. This observation makes it in particular difficult to directly apply the integration formula (III.2.12), and thus hints to the fact that the polynomials $H_\kappa^{(\ell,n)}(\cdot;\Omega)$ are not easily amenable to matrix calculus.

III.4.3 Proofs of Section III.3.3

Proofs of Theorem III.3.10 and Theorem III.3.12

Our proofs of Theorem III.3.10 and Theorem III.3.12 involve auxiliary polynomials introduced by Hayakawa in [Hay69]. For $X\in\mathbb{R}^{\ell\times n}$ and $A\in\mathbb{R}^{n\times n}$ symmetric, we consider the polynomials $P_\kappa(X,A)$, $\kappa\vdash k$ defined by (see [Hay69, Eq.(34)])

$$\mathrm{etr}\left(-XX^T\right)P_\kappa(X,A) = \frac{(-1)^k}{\pi^{n\ell/2}}\int_{\mathbb{R}^{\ell\times n}}\mathrm{etr}\left(-2iXU^T\right)\mathrm{etr}\left(-UU^T\right)C_\kappa(UAU^T)(dU). \tag{III.4.18}$$

These polynomials have the following properties (see e.g. [MPH95, p.229] and [Hay69, Section 6]).

Lemma III.4.4. *For every integer $k\geq 0$, every $\kappa\vdash k$ and every symmetric $A\in\mathbb{R}^{n\times n}$, we have*

$$P_\kappa(X,\mathbf{I}_n) = 2^k\left(\frac{n}{2}\right)_\kappa H_\kappa^{(\ell,n)}(\sqrt{2}X) \tag{III.4.19}$$

$$\int_{O(n)}P_\kappa(XH,A)\tilde{\mu}(dH) = \int_{O(n)}P_\kappa(X,HAH^T)\tilde{\mu}(dH) = \frac{C_\kappa(A)}{C_\kappa(\mathbf{I}_n)}P_\kappa(X,\mathbf{I}_n) \tag{III.4.20}$$

$$P_\kappa(X,A) = \mathbb{E}_V\left[C_\kappa((X+iV)A(X+iV)^T)\right],\quad V\sim\mathcal{N}_{\ell\times n}(0,\mathbf{I}_\ell/2\otimes\mathbf{I}_n). \tag{III.4.21}$$

In order to prove Theorem III.3.10, we shall first show the following Lemma, linking the conditional expectation of $H_\kappa^{(\ell,n)}$ with the polynomial P_κ.

Lemma III.4.5. *Let $k \geq 0$ be an integer, $\kappa \vdash k$ a partition of k and $\Delta = \operatorname{diag}(d_1, \ldots, d_n)$ a diagonal matrix with $|d_i| \leq 1$ for $i = 1, \ldots, n$. Then, for $X_0 \sim \mathcal{N}_{\ell\times n}(0, \mathbf{I}_\ell \otimes \mathbf{I}_n)$, we have for every $X \in \mathbb{R}^{\ell\times n}$,*

$$\mathbb{E}_{X_0}\left[H_\kappa^{(\ell,n)}(X\Delta + X_0(\mathbf{I}_n - \Delta^2)^{1/2})\right] = 2^{-k}\left(\frac{n}{2}\right)_\kappa^{-1} P_\kappa\left(\frac{X}{\sqrt{2}}, \Delta^2\right). \tag{III.4.22}$$

Proof. For $W = (W_{lj}) \in \mathbb{R}^{\ell\times n}$ we use the implicit representation (III.2.6) of $C_\kappa(WW^T)$ as homogeneous polynomials of degree $2k$ in the entries of W,

$$C_\kappa(WW^T) = \sum_{|\alpha|=2k} z_\alpha^\kappa \prod_{l=1}^{\ell}\prod_{j=1}^{n} W_{lj}^{\alpha_{lj}}.$$

Then, using (III.4.21) with $B = \operatorname{diag}(b_1, \ldots, b_n)$ such that $b_1, \ldots, b_n \geq 0$, we can write

$$\begin{aligned}
P_\kappa(W, B) &= \mathbb{E}_V\left[C_\kappa((W + iV)B(W + iV)^T)\right] \\
&= \mathbb{E}_V\left[C_\kappa((WB^{1/2} + iVB^{1/2})(WB^{1/2} + iVB^{1/2})^T)\right] \\
&= \sum_{|\alpha|=2k} z_\alpha^\kappa \prod_{l=1}^{\ell}\prod_{j=1}^{n} \mathbb{E}_{V_{lj}}\left[\left(W_{lj}\sqrt{b_j} + iV_{lj}\sqrt{b_j}\right)^{\alpha_{lj}}\right] \\
&= \sum_{|\alpha|=2k} z_\alpha^\kappa \prod_{l=1}^{\ell}\prod_{j=1}^{n} b_j^{\alpha_{lj}/2}\mathbb{E}_{V_{lj}}\left[(W_{lj} + iV_{lj})^{\alpha_{lj}}\right].
\end{aligned}$$

Using the one-dimensional representation of Hermite polynomials as Gaussian expectation,

$$\mathbb{E}_{V_{lj}}\left[(W_{lj} + iV_{lj})^{\alpha_{lj}}\right] = 2^{-\alpha_{lj}/2}H_{\alpha_{lj}}(\sqrt{2}W_{lj})$$

leads to

$$\begin{aligned}
P_\kappa(W, B) &= \sum_{|\alpha|=2k} z_\alpha^\kappa \prod_{l=1}^{\ell}\prod_{j=1}^{n} b_j^{\alpha_{lj}/2} 2^{-\alpha_{lj}/2} H_{\alpha_{lj}}(\sqrt{2}W_{lj}) \\
&= 2^{-k}\sum_{|\alpha|=2k} z_\alpha^\kappa \prod_{l=1}^{\ell}\prod_{j=1}^{n} b_j^{\alpha_{lj}/2} H_{2\alpha_{lj}}(\sqrt{2}W_{lj}).
\end{aligned} \tag{III.4.23}$$

Applying (III.4.23) with $W = X/\sqrt{2}$ and $B = \Delta^2$, yields

$$P_\kappa\left(\frac{X}{\sqrt{2}}, \Delta^2\right) = 2^{-k}\sum_{|\alpha|=2k} z_\alpha^\kappa \prod_{l=1}^{\ell}\prod_{j=1}^{n} d_j^{\alpha_{lj}} H_{\alpha_{lj}}(X_{lj}). \tag{III.4.24}$$

On the other hand, applying (III.4.23) with $W = (X/\sqrt{2})\Delta + (X_0/\sqrt{2})(\mathbf{I}_n - \Delta^2)^{1/2}$ and $B = \mathbf{I}_n$, we have

$$P_\kappa\left(\frac{X}{\sqrt{2}}\Delta + \frac{X_0}{\sqrt{2}}(\mathbf{I}_n - \Delta^2)^{1/2}, \mathbf{I}_n\right) = 2^{-k}\sum_{|\alpha|=2k} z_\alpha^\kappa \prod_{l=1}^{\ell}\prod_{j=1}^{n} H_{\alpha_{lj}}\left(d_jX_{lj} + \sqrt{1 - d_j^2}X_{0,lj}\right). \tag{III.4.25}$$

Taking expectation with respect to X_0 in (III.4.25), we infer

$$\mathbb{E}_{X_0}\left[H_\kappa^{(\ell,n)}(X\Delta + X_0(\mathbf{I}_n - \Delta^2)^{1/2})\right]$$

$$
\begin{aligned}
&= 2^{-k}\left(\frac{n}{2}\right)_\kappa^{-1} \mathbb{E}_{X_0}\left[P_\kappa\left(\frac{X}{\sqrt{2}}\Delta + \frac{X_0}{\sqrt{2}}(\mathbf{I}_n - \Delta^2)^{1/2}, \mathbf{I}_n\right)\right] && \text{(by (III.4.19))}\\
&= 2^{-k}\left(\frac{n}{2}\right)_\kappa^{-1} \mathbb{E}_{X_0}\left[2^{-k}\sum_{|\alpha|=2k} z_\alpha^\kappa \prod_{l=1}^{\ell}\prod_{j=1}^{n} H_{\alpha_{lj}}\left(d_j X_{lj} + \sqrt{1-d_j^2}X_{0,lj}\right)\right] && \text{(by (III.4.25))}\\
&= 2^{-k}\left(\frac{n}{2}\right)_\kappa^{-1} 2^{-k}\sum_{|\alpha|=2k} z_\alpha^\kappa \prod_{l=1}^{\ell}\prod_{j=1}^{n} \mathbb{E}_{X_{0,lj}}\left[H_{\alpha_{lj}}\left(d_j X_{lj} + \sqrt{1-d_j^2}X_{0,lj}\right)\right] && \text{(by independence)}\\
&= 2^{-k}\left(\frac{n}{2}\right)_\kappa^{-1} 2^{-k}\sum_{|\alpha|=2k} z_\alpha^\kappa \prod_{l=1}^{\ell}\prod_{j=1}^{n} d_j^{\alpha_{lj}} H_{\alpha_{lj}}(X_{lj}) && \text{(by (III.3.32))}\\
&= 2^{-k}\left(\frac{n}{2}\right)_\kappa^{-1} P_\kappa\left(\frac{X}{\sqrt{2}}, \Delta^2\right), && \text{(by (III.4.24))}
\end{aligned}
$$

which proves relation (III.4.22). □

We are now in position to prove Theorem III.3.10.

Proof of Theorem III.3.10. In order to prove (III.3.33), we use Fubini and apply (III.4.22) with $\Delta = e^{-tA}$ and integrate both sides with respect to the Haar measure on $O(n)$ to obtain:

$$
\begin{aligned}
O_{t;A}^{(\ell,n)} H_\kappa^{(\ell,n)}(X) &= \mathbb{E}\left[\int_{O(n)} H_\kappa^{(\ell,n)}(XHe^{-tA} + X_0(\mathbf{I}_n - e^{-2tA})^{1/2})\tilde{\mu}(dH)\middle| X\right]\\
&= \int_{O(n)} \mathbb{E}\left[H_\kappa^{(\ell,n)}(XHe^{-tA} + X_0(\mathbf{I}_n - e^{-2tA})^{1/2})\middle| X\right]\tilde{\mu}(dH)\\
&= 2^{-k}\left(\frac{n}{2}\right)_\kappa^{-1}\int_{O(n)} P_\kappa\left(\frac{XH}{\sqrt{2}}, e^{-2tA}\right)\tilde{\mu}(dH)\\
&= 2^{-k}\left(\frac{n}{2}\right)_\kappa^{-1}\frac{C_\kappa(e^{-2tA})}{C_\kappa(\mathbf{I}_n)} P_\kappa\left(\frac{X}{\sqrt{2}}, \mathbf{I}_n\right) = \frac{C_\kappa(e^{-2tA})}{C_\kappa(\mathbf{I}_n)} H_\kappa^{(\ell,n)}(X),
\end{aligned}
$$

where we used (III.4.19) and (III.4.20). This finishes the proof of the first part of the statement. Let us now prove the second part: Assume first that $A = \operatorname{diag}(a, \dots, a)$ and let $f \in \Pi(\ell, n)$ (see (III.3.30)). Then, one has that

$$
\begin{aligned}
O_{t;A}^{(\ell,n)} f(X) &= \int_{O(n)} \mathbb{E}\left[f(e^{-at}XH + \sqrt{1-e^{-2at}}X_0H)\middle| X\right]\tilde{\mu}(dH)\\
&= \int_{O(n)} \mathbb{E}\left[f(e^{-at}X + \sqrt{1-e^{-2at}}X_0)\middle| X\right]\tilde{\mu}(dH) = P_{at}^{(\ell n)} f(X),
\end{aligned}
$$

where we used the facts that $X_0 \overset{d}{=} X_0H$ for $H \in O(n)$, f is an element of $\Pi(\ell, n)$ and $\tilde{\mu}$ is a probability measure on $O(n)$. Finally, if the a_i's are not all equal, then arguing as in Remark III.3.11 **(b)**, one can derive a relation contradicting the semigroup property of $O_{t;A}^{(\ell,n)}$. □

Proof of Theorem III.3.12. We proceed in two steps. In view of Remark III.3.13, the matrix R is necessarily symmetric and has non-negative eigenvalues. We start by showing that (III.3.36) holds for diagonal matrices $R = \operatorname{diag}(r_1, \dots, r_n)$. The statement for arbitrary symmetric matrices will then follow from the diagonal case by a reduction argument.

Step 1: R is diagonal. Let us first assume that $r_1, \dots, r_n > 0$. Since $X \overset{d}{=} XH, H \in O(n)$ and using the fact that $H_\kappa^{(\ell,n)}(XH) = H_\kappa^{(\ell,n)}(X)$ for every $H \in O(n)$ (as can be seen e.g. from (III.3.5) or (III.3.8)), we have

$$\begin{aligned} & \mathbb{E}\left[H_\kappa^{(\ell,n)}(X)H_\sigma^{(\ell,n)}(XR + X_0(\mathbf{I}_n - R^2)^{1/2})\right] \\ = \; & \mathbb{E}\left[H_\kappa^{(\ell,n)}(X)\mathbb{E}\left[H_\sigma^{(\ell,n)}(XR + X_0(\mathbf{I}_n - R^2)^{1/2})|X\right]\right] \\ = \; & \mathbb{E}\left[\int_{O(n)} H_\kappa^{(\ell,n)}(XH)\mathbb{E}\left[H_\sigma^{(\ell,n)}(XHR + X_0(\mathbf{I}_n - R^2)^{1/2})|X\right]\tilde{\mu}(dH)\right] \\ = \; & \mathbb{E}\left[H_\kappa^{(\ell,n)}(X)\mathbb{E}\left[\int_{O(n)} H_\sigma^{(\ell,n)}(XHR + X_0(\mathbf{I}_n - R^2)^{1/2})\tilde{\mu}(dH)\Big|X\right]\right] \\ = \; & \mathbb{E}\left[H_\kappa^{(\ell,n)}(X)O_{1;R_*}^{(\ell,n)}H_\sigma^{(\ell,n)}(X)\right], \end{aligned} \tag{III.4.26}$$

where $R_* := \operatorname{diag}(\ln(1/r_1), \ldots, \ln(1/r_n))$. Then, exploiting the action of $O_{t;R_*}^{(\ell,n)}$ on matrix-variate Hermite polynomials given in (III.3.33) we infer

$$\begin{aligned} \mathbb{E}\left[H_\kappa^{(\ell,n)}(X)H_\sigma^{(\ell,n)}(XR + X_0(\mathbf{I}_n - R^2)^{1/2})\right] &= \frac{C_\kappa(e^{-2R_*})}{C_\kappa(\mathbf{I}_n)}\mathbb{E}\left[H_\kappa^{(\ell,n)}(X)H_\sigma^{(\ell,n)}(X)\right] \\ = \frac{C_\kappa(R^2)}{C_\kappa(\mathbf{I}_n)}\mathbb{E}\left[H_\kappa^{(\ell,n)}(X)H_\sigma^{(\ell,n)}(X)\right] &= \mathbb{1}_{\kappa=\sigma} \cdot 4^{-k}\left(\frac{n}{2}\right)_\kappa^{-1} k! C_\kappa(R^2)\frac{C_\kappa(\mathbf{I}_\ell)}{C_\kappa(\mathbf{I}_n)}, \end{aligned}$$

where we used that $e^{-2R_*} = R^2$ and the orthogonality relation for Hermite polynomials (III.3.9). If some of $r_1, \ldots, r_n$ are equal to zero, the conclusion remains valid, as in this case, from (III.4.26), we can use (III.4.22) and (III.4.20) yielding the same conclusion.

Step 2: R is symmetric. Since R is symmetric, there exists $O \in O(n)$ such that $R = O\Delta_R O^T$, where Δ_R is diagonal. Moreover, since $R^2 = O\Delta_R^2 O^T$, we have

$$\mathbf{I}_n - R^2 = \mathbf{I}_n - O\Delta_R^2 O^T = OO^T - O\Delta_R^2 O^T = O(\mathbf{I}_n - \Delta_R^2)O^T$$

yielding $(\mathbf{I}_n - R^2)^{1/2} = O(\mathbf{I}_n - \Delta_R^2)^{1/2}O^T$, as can be seen from the computation

$$[O(\mathbf{I}_n - \Delta_R^2)^{1/2}O^T]^2 = [O(\mathbf{I}_n - \Delta_R^2)^{1/2}O^T][O(\mathbf{I}_n - \Delta_R^2)^{1/2}O^T] = O(\mathbf{I}_n - \Delta_R^2)O^T = \mathbf{I}_n - R^2.$$

Exploiting once more the fact that $H_\kappa^{(\ell,n)}(XO) = H_\kappa^{(\ell,n)}(X)$ for every $O \in O(n)$, we have

$$\begin{aligned} & \mathbb{E}\left[H_\kappa^{(\ell,n)}(X)H_\sigma^{(\ell,n)}(XR + X_0(\mathbf{I}_n - R^2)^{1/2})\right] \\ = \; & \mathbb{E}\left[H_\kappa^{(\ell,n)}(X)H_\sigma^{(\ell,n)}(XO\Delta_R O^T + X_0 O(\mathbf{I}_n - \Delta_R^2)^{1/2}O^T)\right] \\ = \; & \mathbb{E}\left[H_\kappa^{(\ell,n)}(X)H_\sigma^{(\ell,n)}((XO\Delta_R + X_0 O(\mathbf{I}_n - \Delta_R^2)^{1/2})O^T)\right] \\ = \; & \mathbb{E}\left[H_\kappa^{(\ell,n)}(XO)H_\sigma^{(\ell,n)}(XO\Delta_R + X_0 O(\mathbf{I}_n - \Delta_R^2)^{1/2})\right] \\ = \; & \mathbb{E}\left[H_\kappa^{(\ell,n)}(X)H_\sigma^{(\ell,n)}(X\Delta_R + X_0(\mathbf{I}_n - \Delta_R^2)^{1/2})\right], \end{aligned}$$

where the last equality follows from the fact that the pair (X, X_0) has the same distribution as the pair (XO, X_0O). Since Δ_R is diagonal, we can apply the conclusion of Step 1 to infer

$$\begin{aligned} \mathbb{E}\left[H_\kappa^{(\ell,n)}(X)H_\sigma^{(\ell,n)}(X\Delta_R + X_0(\mathbf{I}_n - \Delta_R^2)^{1/2})\right] &= \mathbb{1}_{\kappa=\sigma} \cdot 4^{-k}\left(\frac{n}{2}\right)_\kappa^{-1} k! C_\kappa(\Delta_R^2)\frac{C_\kappa(\mathbf{I}_\ell)}{C_\kappa(\mathbf{I}_n)} \\ &= \mathbb{1}_{\kappa=\sigma} \cdot 4^{-k}\left(\frac{n}{2}\right)_\kappa^{-1} k! C_\kappa(R^2)\frac{C_\kappa(\mathbf{I}_\ell)}{C_\kappa(\mathbf{I}_n)}, \end{aligned}$$

where in the last line, we used the fact that $C_\kappa(\Delta_R^2) = C_\kappa(O\Delta_R^2 O^T) = C_\kappa(R^2)$. This finishes the proof. □

III.4.4 Proofs of Section III.3.4

Proof of Proposition III.3.16

The variance of the total variation is obtained from (III.3.41). Using the orthogonality of Wiener chaoses, the variance of $\mathbf{V}(\mathfrak{f}_\ell; U)$ is computed to be

$$\mathbf{Var}[\mathbf{V}(\mathfrak{f}_\ell; U)] = \mathbf{Var}\left[\sum_{k\geq 1} \mathbf{V}(\mathfrak{f}_\ell; U)[2k]\right] = \sum_{k\geq 1} \mathbf{Var}[\mathbf{V}(\mathfrak{f}_\ell; U)[2k]] \tag{III.4.27}$$

where

$$\mathbf{Var}[\mathbf{V}(\mathfrak{f}_\ell; U)[2k]] = \sum_{\kappa\vdash k}\sum_{\sigma\vdash k} \widehat{\Phi}(\kappa)\widehat{\Phi}(\sigma)\int_{U^2} \mathbb{E}\left[H_\kappa^{(\ell,n)}(\mathfrak{f}_\ell'(z))H_\sigma^{(\ell,n)}(\mathfrak{f}_\ell'(z'))\right] dzdz',$$

with $\widehat{\Phi}(\kappa)$ as in (III.3.26). Now, in view of (III.3.42), we can apply Theorem III.3.12 with $R = R(z, z')$ to infer

$$\mathbb{E}\left[H_\kappa^{(\ell,n)}(\mathfrak{f}_\ell'(z))H_\sigma^{(\ell,n)}(\mathfrak{f}_\ell'(z'))\right] = \mathbb{1}_{\kappa=\sigma} \cdot 4^{-k}\left(\frac{n}{2}\right)_\kappa^{-1} k!C_\kappa(R(z,z')^2)\frac{C_\kappa(\mathbf{I}_\ell)}{C_\kappa(\mathbf{I}_n)},$$

yielding

$$\mathbf{Var}[\mathbf{V}(\mathfrak{f}_\ell; U)[2k]] = \sum_{\kappa\vdash k} \widehat{\Phi}(\kappa)^2 4^{-k}\left(\frac{n}{2}\right)_\kappa^{-1} k!\frac{C_\kappa(\mathbf{I}_\ell)}{C_\kappa(\mathbf{I}_n)}\int_{U^2} C_\kappa(R(z,z')^2)dzdz'.$$

The relation in (III.3.43) then follows from (III.4.27).

Proof of Theorem III.3.17

The Wiener chaos expansion of $\mathbf{V}(\mathbf{T}_n^{(\ell)}; \mathbb{T}^3)$ is given by (III.3.41):

$$\mathbf{V}(\mathbf{T}_n^{(\ell)}; \mathbb{T}^3) = \left(\frac{E_n}{3}\right)^{\ell/2} \sum_{k\geq 0} \mathbf{V}(\mathbf{T}_n^{(\ell)}; \mathbb{T}^3)[2k], \tag{III.4.28}$$

where for $k \geq 0$,

$$\mathbf{V}(\mathbf{T}_n^{(\ell)}; \mathbb{T}^3)[2k] = \sum_{\kappa\vdash k} \widehat{\Phi}(\kappa)\int_{\mathbb{T}^3} H_\kappa^{(\ell,3)}(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z))dz$$

and $\widehat{\Phi}(\kappa)$ is as in (III.3.26). In particular, for $k = 0$, we have by (III.3.27),

$$\mathbb{E}\left[\mathbf{V}(\mathbf{T}_n^{(\ell)}; \mathbb{T}^3)\right] = \left(\frac{E_n}{3}\right)^{\ell/2} \widehat{\Phi}((0)) = \left(\frac{E_n}{3}\right)^{\ell/2} 2^{\ell/2}\frac{\Gamma_\ell(2)}{\Gamma_\ell(\frac{3}{2})},$$

which proves (III.3.49).

<u>*Second Wiener chaos component.*</u> The second Wiener chaos of $\mathbf{V}(\mathbf{T}_n^{(\ell)}; \mathbb{T}^3)$ is given by

$$\mathbf{V}(\mathbf{T}_n^{(\ell)}; \mathbb{T}^3)[2] = \left(\frac{E_n}{3}\right)^{\ell/2} \widehat{\Phi}((1))\int_{\mathbb{T}^3} H_{(1)}^{(\ell,3)}(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z))dz. \tag{III.4.29}$$

In the following lemma, we establish the asymptotic variance of the second Wiener chaos in the high-energy regime:

Lemma III.4.6. *As $n \to \infty, n \not\equiv 0, 4, 7 \pmod 8$, we have*

$$\mathbf{Var}\Big[\mathbf{V}(\mathbf{T}_n^{(\ell)}; \mathbb{T}^3)[2]\Big] = \left(\frac{E_n}{3}\right)^{\ell} 2^{\ell} \frac{\Gamma_\ell(2)^2}{\Gamma_\ell(\frac{3}{2})^2} \frac{\ell}{2\mathcal{N}_n} \left(1 + O(n^{-1/28+o(1)})\right).$$

Proof. Since, for every $z, z' \in \mathbb{T}^3$, we have

$$\mathbb{E}\left[\tilde{\partial}_j T_n^{(i)}(z) \cdot \tilde{\partial}_{j'} T_n^{(i')}(z')\right] = \mathbb{1}_{i=i'} \cdot \left(\frac{E_n}{3}\right)^{-1} r_{j,j'}^{(n)}(z - z') \tag{III.4.30}$$

we note that the matrices $\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z)$ and $\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z')$ are such that

$$\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z') \overset{d}{=} \widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z) R_n(z - z') + X_0 (\mathbf{I}_3 - R_n(z - z')^2)^{1/2}, \tag{III.4.31}$$

where $X_0 = X_0(z, z')$ is an independent copy of $\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z)$ and the matrix $R_n(z - z')$ is given by

$$R_n(z - z') := \left(\tilde{r}_{j,j'}^{(n)}(z - z')\right)_{j,j' \in [3]}, \quad \tilde{r}_{j,j'}^{(n)}(z - z') := \left(\frac{E_n}{3}\right)^{-1} \frac{\partial^2}{\partial z_j \partial z_{j'}} r^{(n)}(z - z').$$

Indeed, by (III.4.31) it follows by Remark III.3.13 part **(a)**, that

$$\mathbb{E}\left[\tilde{\partial}_j T_n^{(i)}(z) \cdot \tilde{\partial}_{j'} T_n^{(i')}(z')\right] = \mathbb{1}_{i=i'} \cdot \tilde{r}_{j,j'}^{(n)}(z - z'),$$

which is (III.4.30). In particular, the variance of the second Wiener chaos component is computed by Proposition III.3.16,

$$\begin{aligned}
\mathbf{Var}\Big[\mathbf{V}(\mathbf{T}_n^{(\ell)}; \mathbb{T}^3)[2]\Big] &= \left(\frac{E_n}{3}\right)^{\ell} \widehat{\Phi}((1))^2 4^{-1} \left(\frac{3}{2}\right)_{(1)}^{-1} \frac{C_{(1)}(\mathbf{I}_\ell)}{C_{(1)}(\mathbf{I}_3)} \int_{\mathbb{T}^3 \times \mathbb{T}^3} C_{(1)}(R_n(z - z')^2) dz dz' \\
&= \left(\frac{E_n}{3}\right)^{\ell} \widehat{\Phi}((1))^2 4^{-1} \frac{2}{3} \frac{\ell}{3} \int_{\mathbb{T}^3} \mathrm{tr}(R_n(z)^2) dz \\
&= \left(\frac{E_n}{3}\right)^{\ell} \widehat{\Phi}((1))^2 \frac{\ell}{18} \int_{\mathbb{T}^3} \mathrm{tr}(R_n(z)^2) dz,
\end{aligned} \tag{III.4.32}$$

where we used that $C_{(1)}(A) = \mathrm{tr}(A)$ and stationarity of $\mathbf{T}_n^{(\ell)}$ to reduce integrations on $\mathbb{T}^3 \times \mathbb{T}^3$ to $\mathbb{T}^3$. A direct computation gives

$$\mathrm{tr}(R_n(z)^2) = \sum_{j,j' \in [3]} \left(\tilde{r}_{j,j'}^{(n)}(z)\right)^2.$$

Now, in view of (III.3.44) and (III.3.47), we have

$$\tilde{r}_{j,j'}^{(n)}(z) = \left(\frac{E_n}{3}\right)^{-1} (-4\pi^2) \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \lambda_j \lambda_{j'} e_\lambda(z).$$

Integrating over $\mathbb{T}^3$ and using the orthogonality relation for complex exponentials on the torus

$$\int_{\mathbb{T}^3} e_\lambda(z) dz = \mathbb{1}_{\lambda=0} \tag{III.4.33}$$

then yields

$$\begin{aligned}
&\int_{\mathbb{T}^3} \mathrm{tr}(R_n(z)^2)dz \\
&= \int_{\mathbb{T}^3} \sum_{j,j'\in[3]} \left(\tilde{r}^{(n)}_{j,j'}(z)\right)^2 dz = \left(\frac{E_n}{3}\right)^{-2} \sum_{j,j'\in[3]} 16\pi^4 \frac{1}{\mathcal{N}_n^2} \sum_{\lambda,\lambda'\in\Lambda_n} \lambda_j\lambda_{j'}\lambda'_j\lambda'_{j'} \int_{\mathbb{T}^3} e_{\lambda+\lambda'}(z)dz \\
&= \left(\frac{E_n}{3}\right)^{-2} 16\pi^4 \frac{1}{\mathcal{N}_n^2} \sum_{j,j'\in[3]} \sum_{\lambda\in\Lambda_n} \lambda_j^2\lambda_{j'}^2 = \frac{9}{n^2\mathcal{N}_n^2} \sum_{j,j'\in[3]} \sum_{\lambda\in\Lambda_n} \lambda_j^2\lambda_{j'}^2 \\
&= \frac{9}{\mathcal{N}_n} \frac{1}{n^2\mathcal{N}_n} \sum_{j,j'\in[3]} \sum_{\lambda\in\Lambda_n} \lambda_j^2\lambda_{j'}^2.
\end{aligned}$$

Now, using the relation (see e.g. [Cam19, Appendix C])

$$\frac{1}{n^2\mathcal{N}_n} \sum_{\lambda\in\Lambda_n} \lambda_j^2\lambda_{j'}^2 = \frac{1}{5}\mathbb{1}_{j=j'} + \frac{1}{15}\mathbb{1}_{j\neq j'} + O(n^{-1/28+o(1)})$$

gives

$$\int_{\mathbb{T}^3} \mathrm{tr}(R_n(z)^2)dz = \frac{9}{\mathcal{N}_n}\left(\frac{3}{5} + \frac{6}{15} + O(n^{-1/28+o(1)})\right) = \frac{9}{\mathcal{N}_n}\left(1 + O(n^{-1/28+o(1)})\right),$$

so that, computing $\widehat{\Phi}((1)) = 2^{\ell/2}\frac{\Gamma_\ell(2)}{\Gamma_\ell(\frac{3}{2})}$ from (III.3.26) gives by (III.4.32)

$$\begin{aligned}
\mathbf{Var}\left[\mathbf{V}(\mathbf{T}_n^{(\ell)};\mathbb{T}^3)[2]\right] &= \left(\frac{E_n}{3}\right)^{\ell} \widehat{\Phi}((1))^2 \frac{\ell}{18}\frac{9}{\mathcal{N}_n}\left(1 + O(n^{-1/28+o(1)})\right) \\
&= \left(\frac{E_n}{3}\right)^{\ell} 2^{\ell}\frac{\Gamma_\ell(2)^2}{\Gamma_\ell(\frac{3}{2})^2}\frac{\ell}{2\mathcal{N}_n}\left(1 + O(n^{-1/28+o(1)})\right),
\end{aligned}$$

which finishes the proof. □

<u>*Higher-order chaotic components.*</u> We prove the following statement, dealing with the variance of the tail of the Wiener chaos expansion of $\mathbf{V}(\mathbf{T}_n^{(\ell)};\mathbb{T}^3)$.

Proposition III.4.7. *As $n\to\infty, n \not\equiv 0,4,7 \pmod 8$, we have*

$$\mathbf{Var}\left[\sum_{k\geq 2}\mathbf{V}(\mathbf{T}_n^{(\ell)};\mathbb{T}^3)[2k]\right] = O(E_n^{\ell}\mathcal{N}_n^{-5/3+o(1)}) = o\left(\mathbf{Var}\left[\mathbf{V}(\mathbf{T}_n^{(\ell)};\mathbb{T}^3)[2]\right]\right). \qquad \text{(III.4.34)}$$

In particular, as $n\to\infty, n\not\equiv 0,4,7 \pmod 8$,

$$\mathbf{V}(\mathbf{T}_n^{(\ell)};\mathbb{T}^3) = \mathbf{V}(\mathbf{T}_n^{(\ell)};\mathbb{T}^3)[2] + o_{\mathbb{P}}(1), \qquad \text{(III.4.35)}$$

where $o_{\mathbb{P}}(1)$ denotes a sequence of random variables converging to zero in probability, that is, in the high-energy regime, the random variable $\mathbf{V}(\mathbf{T}_n^{(\ell)};\mathbb{T}^3)$ is dominated in the $L^2(\mathbb{P})$-sense by its projection on the second Wiener chaos.

The proof of Proposition III.4.7 is based on a suitable partition of the torus into *singular* and *non-singular* pairs of cubes, as already exploited in Appendix II.E of Chapter II (see also [ORW08, DNPR19, PR18] for further references using this approach).

We now describe this partition for the convenience of the reader. For every $n \in S_3$, we partition the torus into a disjoint union of cubes of length $1/M$, where $M = M_n \geq 1$ is an integer proportional to $\sqrt{E_n}$, as follows: Let $Q_0 = [0, 1/M)^3$; then we consider the partition of $\mathbb{T}^3$ obtained by translating Q_0 in the directions k/M, $k \in \mathbb{Z}^3$. Denote by $\mathcal{P}(M)$ the partition of $\mathbb{T}^3$ that is obtained in this way. By construction, we have that $\text{card}(\mathcal{P}(M)) = M^3$. Let us now denote by

$$\mathbf{V}(\mathbf{T}_n^{(\ell)}; \mathbb{T}^3)[4^+] := \sum_{k \geq 2} \mathbf{V}(\mathbf{T}_n^{(\ell)}; \mathbb{T}^3)[2k]$$

the projection of $\mathbf{V}(\mathbf{T}_n^{(\ell)}; \mathbb{T}^3)$ onto Wiener chaoses of order at least 4. By linearity, we can write

$$\mathbf{V}(\mathbf{T}_n^{(\ell)}; \mathbb{T}^3)[4^+] = \sum_{Q \in \mathcal{P}(M)} \mathbf{V}(\mathbf{T}_n^{(\ell)}; Q)[4^+], \quad \ell \in [3] \tag{III.4.36}$$

where $\mathbf{V}(\mathbf{T}_n^{(\ell)}; Q)$ denotes the total variation of $\mathbf{T}_n^{(\ell)}$ in the cube Q. From now on, we fix a small number $0 < \eta < 10^{-10}$. In the forthcoming definition, we define singular pairs of points and cubes (see also Definition II.E.1). Recall the notations

$$r_i^{(n)}(z) := \frac{\partial}{\partial z_i} r^{(n)}(z), \quad r_{i,j}^{(n)}(z) := \frac{\partial^2}{\partial z_i \partial z_j} r^{(n)}(z), \quad (i, j) \in [3] \times [3].$$

Definition III.4.8 (Singular pairs of points and cubes)**.** A pair of points $(z, z') \in \mathbb{T}^3 \times \mathbb{T}^3$ is called a *singular pair of points* if one of the following inequalities is satisfied:

$$|r^{(n)}(z - z')| > \eta, \quad |r_i^{(n)}(z - z')| > \eta \sqrt{E_n/3}, \quad |r_{i,j}^{(n)}(z - z')| > \eta E_n/3$$

for $(i, j) \in [3] \times [3]$. A pair of cubes $(Q, Q') \in \mathcal{P}(M)^2$ is called a *singular pair of cubes* if the product $Q \times Q'$ contains a singular pair of points. We denote by $\mathcal{S} = \mathcal{S}(M) \subset \mathcal{P}(M)^2$ the set of singular pairs of cubes. A pair of cubes $(Q, Q') \in \mathcal{S}^c$ is called *non-singular*. By construction, $\mathcal{P}(M)^2 = \mathcal{S} \cup \mathcal{S}^c$.

For fixed $Q \in \mathcal{P}(M)$, let us furthermore denote by $\mathcal{B}_Q$ the union over all cubes $Q' \in \mathcal{P}(M)$ such that $(Q, Q') \in \mathcal{S}$. Arguing as in (II.E.2), we have that

$$\text{Leb}(\mathcal{B}_Q) = O(\mathcal{R}_n(6)), \tag{III.4.37}$$

where $\mathcal{R}_n(6) = \int_{\mathbb{T}^3} [r^{(n)}(z)]^6 dz$. In view of (III.4.36), we can thus split the variance into its singular and non-singular contribution as follows

$$\mathbf{Var}\Big[\mathbf{V}(\mathbf{T}_n^{(\ell)}; \mathbb{T}^3)[4^+]\Big] = \left\{ \sum_{(Q,Q') \in \mathcal{S}} + \sum_{(Q,Q') \in \mathcal{S}^c} \right\} \mathbb{E}\left[\mathbf{V}(\mathbf{T}_n^{(\ell)}; Q)[4^+] \cdot \mathbf{V}(\mathbf{T}_n^{(\ell)}; Q')[4^+]\right] := \Delta_{n,1}^{(\ell)} + \Delta_{n,2}^{(\ell)}.$$

The contributions to the variance of the terms $\Delta_{n,j}^{(\ell)}$, $j = 1, 2$ are given in Lemma III.4.9 and III.4.10 below. The combination of both results proves Proposition III.4.7.

Lemma III.4.9 (Singular part)**.** *As $n \to \infty$, $n \not\equiv 0, 4, 7 \pmod 8$, we have that*

$$\left|\Delta_{n,1}^{(\ell)}\right| = o\left(\mathbf{Var}\Big[\mathbf{V}(\mathbf{T}_n^{(\ell)}; \mathbb{T}^3)[2]\Big]\right).$$

Proof. Using the triangle inequality, the Cauchy-Schwarz inequality, and (III.4.37), we can write

$$\left|\Delta_{n,1}^{(\ell)}\right| \leq \sum_{(Q,Q') \in \mathcal{S}} \sqrt{\mathbf{Var}\Big[\mathbf{V}(\mathbf{T}_n^{(\ell)}; Q)[4^+]\Big]} \sqrt{\mathbf{Var}\Big[\mathbf{V}(\mathbf{T}_n^{(\ell)}; Q')[4^+]\Big]}$$

$$\ll \quad E_n^3 \mathcal{R}_n(6) \cdot \mathbf{Var}\Big[\mathbf{V}(\mathbf{T}_n^{(\ell)}; Q_0)[4^+]\Big], \tag{III.4.38}$$

where we exploited stationarity of $\mathbf{T}_n^{(\ell)}$ and where Q_0 denotes the cube around the origin. Now we notice that

$$\mathbf{Var}\Big[\mathbf{V}(\mathbf{T}_n^{(\ell)}; Q_0)[4^+]\Big] \leq \mathbf{Var}\Big[\mathbf{V}(\mathbf{T}_n^{(\ell)}; Q_0)\Big] \leq \mathbb{E}\left[\mathbf{V}(\mathbf{T}_n^{(\ell)}; Q_0)^2\right] .$$

By definition of the total variation (III.3.46), we can write

$$\mathbb{E}\left[\mathbf{V}(\mathbf{T}_n^{(\ell)}; Q_0)^2\right] = \left(\frac{E_n}{3}\right)^{\ell} \int_{Q_0 \times Q_0} \mathbb{E}\left[\Phi(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z))\Phi(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z'))\right] dz dz'.$$

Now for every fixed $z, z' \in Q_0$, we have by the Cauchy-Schwarz inequality

$$\mathbb{E}\left[\Phi(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z))\Phi(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z'))\right] \leq \sqrt{\mathbb{E}\left[\Phi(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z))^2\right]\mathbb{E}\left[\Phi(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z'))^2\right]} = \mathbb{E}\left[\Phi(\mathbf{N})^2\right] = O(1),$$

where $\mathbf{N} \overset{d}{=} \mathcal{N}_{\ell\times 3}(0, \mathbf{I}_\ell \otimes \mathbf{I}_3)$. Therefore, bearing in mind that $\mathrm{Leb}(Q_0) = M^{-3} = O(E_n^{-3/2})$, it follows that

$$\mathbb{E}\left[\mathbf{V}(\mathbf{T}_n^{(\ell)}; Q_0)^2\right] = O\left(E_n^{\ell} M^{-6}\right) = O(E_n^{\ell-3}).$$

Combining this with the estimate in (III.4.38) yields $\left|\Delta_{n,1}^{(\ell)}\right| \ll E_n^3 \mathcal{R}_n(6) E_n^{\ell-3} \ll E_n^{\ell}\mathcal{R}_n(6)$. By [BM19, Eq.(1.18)], we have that $\mathcal{R}_n(6) \ll \mathcal{N}_n^{-7/3+o(1)}$, as $n \to \infty, n \not\equiv 0, 4, 7 \pmod 8$. Combining this with the estimate in Lemma III.4.6 yields the desired conclusion. □

Lemma III.4.10 (Non-singular part)**.** *As $n \to \infty, n \not\equiv 0, 4, 7 \pmod 8$, we have that*

$$\left|\Delta_{n,2}^{(\ell)}\right| = o\left(\mathbf{Var}\Big[\mathbf{V}(\mathbf{T}_n^{(\ell)}; \mathbb{T}^3)[2]\Big]\right).$$

Proof. Using the expansion in matrix-Hermite polynomials and arguing as in the proof of Proposition III.3.16, we have that

$$\left|\Delta_{n,2}^{(\ell)}\right| \leq E_n^{\ell} \sum_{k\geq 2} \sum_{\kappa \vdash k} \widehat{\Phi}(\kappa)^2 4^{-k} \left(\frac{n}{2}\right)_{\kappa}^{-1} k! \frac{C_\kappa(\mathbf{I}_\ell)}{C_\kappa(\mathbf{I}_3)} \sum_{(Q,Q')\in \mathcal{S}^c} \int_{Q\times Q'} \left|C_\kappa(R_n(x-y)^2)\right| dx dy.$$

Now for a matrix $S \in \mathbb{C}^{m\times m}$, we denote by $\rho(S) := \max(|\lambda_1|, \dots, |\lambda_m|)$, where λ_i denote the eigenvalues of S. We now use the following two facts: (i) For every partition $\kappa \vdash k$, every matrix $S \in \mathbb{C}^{m\times m}$ and every x such that $\rho(S) < x$, one has that $|C_\kappa(S)| \leq x^k C_\kappa(\mathbf{I}_m)$ (see for instance [MPH95], p.197) and (ii) by Gerschgorin's Theorem (see for instance [GR14], p.1084), writing $S = (s_{ij})$,

$$\rho(S) \leq \min\left(\max_{i=1,\dots,m} \sum_{j=1}^{m} |s_{ij}|, \max_{j=1,\dots,m} \sum_{i=1}^{m} |s_{ij}|\right) =: \tilde{\rho}(S).$$

Applying the facts above with the symmetric matrix $S = R_n(x-y)^2$ and $x = 2\tilde{\rho}(S)$ yields

$$\left|\Delta_{n,2}^{(\ell)}\right| \leq E_n^{\ell} \sum_{k\geq 2} \sum_{\kappa \vdash k} \widehat{\Phi}(\kappa)^2 4^{-k} \left(\frac{n}{2}\right)_{\kappa}^{-1} k! C_\kappa(\mathbf{I}_\ell) \sum_{(Q,Q')\in \mathcal{S}^c} \int_{Q\times Q'} \left(2\tilde{\rho}(R_n(x-y)^2)\right)^k dx dy. \tag{III.4.39}$$

By definition of $\tilde{\rho}$ and the triangular inequality, we have that

$$2\tilde{\rho}(R_n(x-y)^2) \leq 2 \max_{i=1,2,3} \sum_{j,l=1}^{3} \left|\tilde{r}_{il}^{(n)}(x-y)\right|\left|\tilde{r}_{lj}^{(n)}(x-y)\right|, \tag{III.4.40}$$

which is bounded by $18\eta^2 < 1$ on the non-singular regions. Combining this with the fact that we are summing over integers $k \geq 2$ yields

$$\begin{aligned}
& \sum_{(Q,Q')\in\mathcal{S}^c} \int_{Q\times Q'} \left(2\tilde{\rho}(R_n(x-y)^2)\right)^k dxdy \\
= & \sum_{(Q,Q')\in\mathcal{S}^c} \int_{Q\times Q'} \left(2\tilde{\rho}(R_n(x-y)^2)\right)^{k-2} \left(2\tilde{\rho}(R_n(x-y)^2)\right)^2 dxdy \\
\ll & \int_{\mathbb{T}^3} \left(\tilde{\rho}(R_n(z)^2)\right)^2 dz.
\end{aligned}$$

Combining (III.4.40) with the Cauchy-Schwarz inequality and the estimate

$$\int_{\mathbb{T}^3} [\tilde{r}^{(n)}_{j,j'}(z)]^{2p} dz = O(\mathcal{R}_n(2p)), \quad p \geq 1,$$

where the constant involved in the 'big-O' notation depends only on p (see Lemma II.E.2 in Chapter II), we deduce that

$$\int_{\mathbb{T}^3} \left(\tilde{\rho}(R_n(z)^2)\right)^2 dz \ll \int_{\mathbb{T}^3} \max_{i,l=1,2,3} \left|\tilde{r}^{(n)}_{il}(z)\right|^2 \max_{j,l=1,2,3} \left|\tilde{r}^{(n)}_{jl}(z)\right|^2 dz = \int_{\mathbb{T}^3} \max_{j,l=1,2,3} \left|\tilde{r}^{(n)}_{jl}(z)\right|^4 dz \ll \mathcal{R}_n(4).$$

Therefore, in view of the estimate (III.4.39), and the fact that $\Phi \in L^2(\mu_X)$ for $X \sim \mathscr{N}_{\ell\times n}(0, \mathbf{I}_\ell \otimes \mathbf{I}_n)$, we conclude that

$$\left|\Delta^{(\ell)}_{n,2}\right| \ll E^\ell_n \mathcal{R}_n(4) \sum_{k\geq 2} \sum_{\kappa\vdash k} \widehat{\Phi}(\kappa)^2 4^{-k} \left(\frac{n}{2}\right)^{-1}_\kappa k! C_\kappa(\mathbf{I}_\ell) \leq E^\ell_n \mathcal{R}_n(4) \mathbb{E}\left[\Phi(X)^2\right] \ll E^\ell_n \mathcal{R}_n(4).$$

Now, the Cauchy-Schwarz inequality implies that

$$\mathcal{R}_n(4) = \int_{\mathbb{T}^3} [r^{(n)}(z)]^4 dz \leq \left(\int_{\mathbb{T}^3} [r^{(n)}(z)]^2 dz \int_{\mathbb{T}^3} [r^{(n)}(z)]^6 dz\right)^{1/2} = \sqrt{\mathcal{R}_n(2)\mathcal{R}_n(6)}.$$

Using the estimates (see [BM19, Eq.(1.16) and (1.18)])

$$\mathcal{R}_n(2) = \frac{1}{\mathcal{N}_n}, \quad \mathcal{R}_n(6) \ll \mathcal{N}_n^{-7/3+o(1)}, \quad n \to \infty, n \not\equiv 0,4,7 \pmod 8$$

implies that $\sqrt{\mathcal{R}_n(2)\mathcal{R}_n(6)} \ll \mathcal{N}_n^{-5/3+o(1)}$. The fact that $E^\ell_n \mathcal{N}_n^{-5/3+o(1)} = o\left(\mathbf{Var}\left[\mathbf{V}(\mathbf{T}^{(\ell)}_n, \mathbb{T}^3)\right]\right)$ follows from the order of the variance of the second Wiener chaos in Lemma III.4.6. □

Limiting distribution of the normalized total variation. The next proposition establishes a CLT in the high-frequency regime for normalized version of the second chaotic component of the total variation

$$\widehat{\mathbf{V}}(\mathbf{T}^{(\ell)}_n; \mathbb{T}^3)[2] := \frac{\mathbf{V}(\mathbf{T}^{(\ell)}_n; \mathbb{T}^3)[2]}{\sqrt{\mathbf{Var}\left[\mathbf{V}(\mathbf{T}^{(\ell)}_n; \mathbb{T}^3)[2]\right]}}. \tag{III.4.41}$$

Proposition III.4.11. *As $n \to \infty, n \not\equiv 0,4,7 \pmod 8$, we have*

$$\widehat{\mathbf{V}}(\mathbf{T}^{(\ell)}_n; \mathbb{T}^3)[2] \xrightarrow{d} \mathcal{N}(0,1).$$

Proof. The expression of $\mathbf{V}(\mathbf{T}_n^{(\ell)};\mathbb{T}^3)[2]$ is given in (III.4.29). Normalising that expression by the square root of the order of the variance in (III.3.50) yields

$$\widehat{\mathbf{V}}(\mathbf{T}_n^{(\ell)};\mathbb{T}^3)[2] = \frac{\sqrt{2}\sqrt{\mathcal{N}_n}}{\sqrt{\ell}} \int_{\mathbb{T}^3} H_{(1)}^{(\ell,3)}(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z))dz.$$

Using (III.3.16), we can rewrite

$$H_{(1)}^{(\ell,3)}(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z)) = \frac{1}{6}\sum_{k=1}^{\ell}\sum_{j=1}^{3} H_2(\tilde{\partial}_j T_n^{(k)}(z)).$$

Now, for every $k \in [\ell]$, we write $H_2(u) = u^2 - 1$ and exploit once more the orthogonality relations for complex exponentials on the torus (III.4.33) in order to write

$$\begin{aligned}\sum_{j=1}^{3}\int_{\mathbb{T}^3} H_2(\tilde{\partial}_j T_n^{(k)})dz &= \sum_{j=1}^{3}\int_{\mathbb{T}^3}\left[\left(\tilde{\partial}_j T_n^{(k)}(z)\right)^2 - 1\right]dz\\ &= \sum_{j=1}^{3}\frac{3}{n\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}\lambda_j^2(|a_{k,\lambda}|^2 - 1) = \frac{3}{\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}(|a_{k,\lambda}|^2 - 1),\end{aligned}$$

where we used that $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = n$, so that

$$\begin{aligned}\widehat{\mathbf{V}}(\mathbf{T}_n^{(\ell)};\mathbb{T}^3)[2] &= \frac{\sqrt{2}\sqrt{\mathcal{N}_n}}{\sqrt{\ell}}\int_{\mathbb{T}^3} H_{(1)}^{(\ell,3)}(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(z))dz = \frac{1}{\sqrt{\ell}}\frac{1}{\sqrt{2}}\sum_{k=1}^{\ell}\frac{1}{\sqrt{\mathcal{N}_n}}\sum_{\lambda\in\Lambda_n}(|a_{k,\lambda}|^2 - 1)\\ &= \frac{1}{\sqrt{\ell}}\sum_{k=1}^{\ell}\frac{1}{\sqrt{\mathcal{N}_n/2}}\sum_{\lambda\in\Lambda_n/\sim}(|a_{k,\lambda}|^2 - 1), \qquad \text{(III.4.42)}\end{aligned}$$

where $\Lambda_n/_\sim$ stands the equivalence classes in Λ_n obtained by identifying λ with $-\lambda$, so that $|\Lambda_n/_\sim| = \mathcal{N}_n/2$. Note that the random variables $\left\{|a_{k,\lambda}|^2 - 1 : \lambda \in \Lambda_n/_\sim\right\}$ are i.i.d, centered and have unit variance. The classical CLT thus implies that, as $n \to \infty, n \not\equiv 0, 4, 7 \pmod 8$,

$$\sum_{k=1}^{\ell}\frac{1}{\sqrt{\mathcal{N}_n/2}}\sum_{\lambda\in\Lambda_n/\sim}(|a_{k,\lambda}|^2 - 1) \xrightarrow{d} \sum_{k=1}^{\ell} Z_k,$$

where $(Z_1, \ldots, Z_\ell)$ is a standard Gaussian vector. The statement then follows from (III.4.42). □

End of the proof of Theorem III.3.17. Relation (III.4.35) implies that the second chaotic component of the total variation dominates the Wiener chaos expansion in (III.4.28). In particular, (III.4.35) implies that, as $n \to \infty, n \not\equiv 0, 4, 7 \pmod 8$

$$\mathbf{Var}\left[\mathbf{V}(\mathbf{T}_n^{(\ell)};\mathbb{T}^3)\right] = \mathbf{Var}\left[\mathbf{V}(\mathbf{T}_n^{(\ell)};\mathbb{T}^3)[2]\right](1 + o(1)),$$

and that the normalized sequences of random variables

$$\left\{\widehat{\mathbf{V}}(\mathbf{T}_n^{(\ell)};\mathbb{T}^3) : n \in S_3\right\}, \quad \left\{\widehat{\mathbf{V}}(\mathbf{T}_n^{(\ell)};\mathbb{T}^3)[2] : n \in S_3\right\}$$

defined in (III.3.48) and (III.4.41) respectively, have the same limiting distribution. Combining this with the asymptotic variance for the second chaos in Lemma III.4.6 proves the variance estimate in (III.3.50). Finally, the CLT in (III.3.51) for the total variation follows when combining (III.4.35) with the content of Proposition III.4.11. This concludes the proof of Theorem III.3.17.

Appendix III.A Proof of Proposition III.3.1

We recall that $L^2(\mu_X) = L^2(\Omega, \sigma(\mu_X), \mathbb{P})$, where $\sigma(\mu_X)$ is the σ-field generated by random variables of the form

$$\int_{\mathbb{R}} f(x)\mu_X(dx)$$

where f is a finite linear combination of trigonometric functions of the form $\cos(ax), \sin(bx)$ with $a, b \in \mathbb{R}$. Since $f(x)$ is equal to the limit of its Taylor expansion for every $x \in \mathbb{R}$, and since the support of μ_X consists of at most ℓ points, we deduce that $\sigma(\mu_X)$ is generated by random variables as above where $f = p$ is a polynomial. Our goal is now to prove that, if $F \in L^2(\mu_X)$ is such that $\mathbb{E}\left[FH_\kappa^{(\ell,n)}(X)\right] = 0$ for every $\kappa \vdash k$ and every $k \geq 0$, then $F = 0$, $\mathbb{P}$-almost everywhere. In order to obtain the desired conclusion, we will use the following three facts: (i) zonal polynomials can be expanded into a finite linear combination of matrix-Hermite polynomials (see e.g [Chi92, Eq.(4.12)]), (ii) the product of finitely many zonal polynomials is a finite linear combination of zonal polynomials (see (III.2.8)) and (iii) every monomial of the form $t_k(X) := \operatorname{tr}([XX^T]^k) = s_1^k + \ldots + s_\ell^k$ (where $s_1, \ldots, s_\ell$ denote the eigenvalues of XX^T) can be represented as a linear combination of zonal polynomials (see (III.2.1)). Using these three facts shows that, whenever F satisfies the assumption above, then $\mathbb{E}\left[Ft_{j_0}(X)^{a_0} \ldots t_{j_M}(X)^{a_M}\right] = 0$ for every finite $M \geq 1$ and every collection of integers $j_0, \ldots, j_M \geq 0$ and $a_0, \ldots, a_M \geq 0$. In particular, writing $p(x) = \sum_{j=0}^M c_j x^j$ for a polynomial of degree M, one has that

$$\mathbb{E}\left[F\exp\left(i\int_{\mathbb{R}} p(x)\mu_X(dx)\right)\right] = \mathbb{E}\left[F\exp\left(i\sum_{j=0}^M c_j(s_1^j + \ldots + s_\ell^j)\right)\right] = \mathbb{E}\left[F\exp\left(i\sum_{j=0}^M c_j t_j(X)\right)\right]$$
$$= \mathbb{E}\left[F\prod_{j=0}^M \exp\left(ic_j t_j(X)\right)\right] = \sum_{a_0,\ldots,a_M\geq 0} \frac{(ic_0)^{a_0}\cdots(ic_M)^{a_M}}{a_0!\cdots a_M!}\mathbb{E}\left[Ft_0(X)^{a_0}\ldots t_M(X)^{a_M}\right] = 0$$

by assumption. By a standard approximation argument, we therefore deduce that $\mathbb{E}\left[F|\sigma(\mu_X)\right] = 0$, yielding the desired conclusion.

Appendix III.B A relation between Hermite and Laguerre polynomials

In this section, we present a proof of relation (III.3.8) between matrix-variate Hermite polynomials and generalized Laguerre polynomials, inspired by [Hay69]. We state the result in the following theorem for convenience.

Theorem III.B.1. *For every integer $k \geq 0$ and $\kappa \vdash k$, we have*

$$\gamma_\kappa \cdot L_\kappa^{(\frac{n-\ell-1}{2})}(XX^T) = H_\kappa^{(\ell,n)}(\sqrt{2}X), \quad \gamma_\kappa := (-2)^{-k}\left(\frac{n}{2}\right)_\kappa^{-1}.$$

The following notion of matrix-variate Bessel function is defined in [Hay69, Eq. (6)].

Definition III.B.2. Let $R \in \mathbb{R}^{\ell\times\ell}$ be a symmetric matrix. For $\gamma > -1$ a real number, the Bessel function of matrix argument R is defined as

$$A_\gamma(R) = \frac{1}{\Gamma_\ell(\gamma + \frac{\ell+1}{2})}\sum_{k\geq 0}\sum_{\kappa\vdash k}\frac{C_\kappa(-R)}{(\gamma + \frac{\ell+1}{2})_\kappa k!}. \tag{III.B.1}$$

In [Hay69, Eq.(7)], it is shown that Laguerre polynomials can be recovered by means of Bessel functions as follows

$$\mathrm{etr}\,(-S)\, L_\kappa^{(\gamma)}(S) = \int_{\mathcal{P}_\ell(\mathbb{R})} A_\gamma(RS)\mathrm{etr}\,(-R)\det(R)^\gamma C_\kappa(R)\nu(dR). \tag{III.B.2}$$

The following auxiliary Lemma is needed for the proof of Theorem III.B.1.

Lemma III.B.3. *Let $f : \mathbb{R}^{\ell\times\ell} \to \mathbb{R}$ be a function and $X \in \mathbb{R}^{\ell\times n}$. Then, we have*

$$\int_{\mathbb{R}^{\ell\times n}} \mathrm{etr}\left(-2iXU^T\right) f(UU^T)(dU) = \pi^{n\ell/2}\cdot \int_{\mathcal{P}_\ell(\mathbb{R})} f(R)A_{\frac{n-\ell-1}{2}}(XX^TR)\det(R)^{\frac{n-\ell-1}{2}}\nu(dR).$$

Proof. By integrating on $O(n)$, we can write

$$\begin{aligned}\int_{\mathbb{R}^{\ell\times n}} \mathrm{etr}\left(-2iXU^T\right) f(UU^T)(dU) &= \int_{\mathbb{R}^{\ell\times n}}\int_{O(n)} \mathrm{etr}\left(-2iXH^TU^T\right) f(UHH^TU^T)\tilde{\mu}(dH)(dU)\\ &= \int_{\mathbb{R}^{\ell\times n}} f(UU^T)\int_{O(n)} \mathrm{etr}\left(-2iXH^TU^T\right)\tilde{\mu}(dH)(dU).\end{aligned} \tag{III.B.3}$$

We now compute the inner integral on $O(n)$. Using the relations (see for instance [MPH95, Theorem 4.3.3])

$$\int_{O(n)} \mathrm{tr}(AH)^{2k+1}\tilde{\mu}(dH) = 0, \quad \int_{O(n)} \mathrm{tr}(AH)^{2k}\tilde{\mu}(dH) = \sum_{\kappa\vdash k}\frac{(2k)!}{4^k k!(\frac{n}{2})_\kappa}C_\kappa(AA^T), \quad k\geq 0,$$

we can write

$$\int_{O(n)} \mathrm{etr}\left(-2iXH^TU^T\right)\tilde{\mu}(dH) = \sum_{k\geq 0}\frac{1}{k!}\int_{O(n)}(-2i)^k\,\mathrm{tr}(X^TUH)^k\tilde{\mu}(dH)$$

$$= \sum_{k\geq 0}\frac{1}{(2k)!}(-4)^k\int_{O(n)}\mathrm{tr}(X^TUH)^{2k}\tilde{\mu}(dH) = \sum_{k\geq 0}\frac{1}{(2k)!}(-4)^k\sum_{\kappa\vdash k}\frac{(2k)!}{4^k k!(\frac{n}{2})_\kappa}C_\kappa(X^TUU^TX)$$

$$= \sum_{k\geq 0}\sum_{\kappa\vdash k}\frac{1}{k!(\frac{n}{2})_\kappa}C_\kappa(-X^TUU^TX).$$

Now using (III.B.1), the last line can be re-written as

$$\int_{O(n)} \mathrm{etr}\left(-2iXH^TU^T\right)\tilde{\mu}(dH) = \Gamma_\ell\left(\frac{n}{2}\right)A_{\frac{n-\ell-1}{2}}(X^TUU^TX).$$

The proof is then completed from (III.B.3) and the change of variable $UU^T = R$. □

We are now in position to prove Theorme III.B.1.

Proof of Lemma III.B.1. Setting $A = \mathbf{I}_n$ in (III.4.18) and combining it with (III.4.19), we deduce the relation

$$\mathrm{etr}\left(-XX^T\right)2^k\left(\frac{n}{2}\right)_\kappa H_\kappa^{(\ell,n)}(\sqrt{2}X) = \frac{(-1)^k}{\pi^{n\ell/2}}\int_{\mathbb{R}^{\ell\times n}}\mathrm{etr}\left(-2iXU^T\right)\mathrm{etr}\left(-UU^T\right)C_\kappa(UU^T)(dU).$$

Applying Lemma III.B.3 to the R.H.S with $f(UU^T) = \mathrm{etr}\left(-UU^T\right)C_\kappa(UU^T)$ then leads to

$$\begin{aligned}&\mathrm{etr}\left(-XX^T\right)2^k\left(\frac{n}{2}\right)_\kappa H_\kappa^{(\ell,n)}(\sqrt{2}X)\\ = \;& (-1)^k\int_{\mathcal{P}_\ell(\mathbb{R})}\mathrm{etr}\,(-R)\,C_\kappa(R)A_{\frac{n-\ell-1}{2}}(XX^TR)\det(R)^{\frac{n-\ell-1}{2}}\nu(dR)\\ = \;& (-1)^k\mathrm{etr}\left(-XX^T\right)L_\kappa^{(\frac{n-\ell-1}{2})}(XX^T),\end{aligned}$$

where we used (III.B.2). This finishes the proof. □

Chapter IV

Some functional convergence results related to Berry's nodal lengths on the plane

In this chapter, we study the high-energy behaviour of the nodal length process indexed by rectangles in the unit square associated with the two-dimensional Berry random field. Such a model of Gaussian eigenfunctions has been introduced by Berry [Ber77, Ber02] and studied later in a number of works (see for instance [NPR19, KW18, BCW17, CH20, PV20, DNPR20] and references therein). In [NPR19] and [PV20], the authors prove that, in the high-energy limit, the nodal length restricted to a fixed planar domain exhibits Gaussian fluctuations (see [NPR19]), and present subsequent multi-dimensional extensions for random vectors associated with a collection of domains (see [PV20]), yielding in particular finite-dimensional convergence results, suggesting a functional limit theorem. In [PV20], a partial weak convergence result towards a standard two-parameter Wiener sheet is obtained for the dominant projection of the nodal length on the fourth Wiener chaos. An extension of such a limit theorem to the entire nodal length process was not fully solved in [PV20], due to technical problems arising when studying certain boundary terms appearing in the projections of the nodal length on the second Wiener chaos. In this chapter, we present some progress towards such a global functional limit theorem, allowing one to deduce novel probabilistic limit theorems for semi-local functionals associated with nodal length processes. In order to achieve such a task, we study the second chaos components independently, highlighting in particular an intrinsic relation with a Gaussian total disorder process (see Corollary IV.1.5). Based on a tightness criterion by Davydov and Zitikis [DZ08] for proving weak convergence of multivariate processes, our findings allow us to show that the laws of the second chaos projections are tight and converge weakly to zero (see Corollary IV.1.6). Combining this result with a chaining technique (inspired by Dehling and Taqqu [DT89] and Marinucci and Wigman [MW11]) for dealing with higher-order chaotic projections, allows us in particular to formulate a weak convergence result, towards a Wiener sheet, of a *discretized version* of the nodal length, obtained by introducing refining partitions of the unit square (see Theorem IV.1.8 and Corollary IV.1.9). We also derive weak convergence results for the *truncated* nodal length process in the high-energy limit (see Corollary IV.1.15). As a by-product of our results, we deduce a number of new limit theorems involving suprema of the discretized and truncated nodal length process. We believe that our findings are important steps towards a fully general functional limit theorem for the nodal length process.

IV.1 Introduction and main results

For a parameter $E > 0$, we consider the real-valued Berry Random Wave Model with energy $4\pi^2 E$, that is the centred stationary and isotropic Gaussian field on $\mathbb{R}^2$, $B_E = \left\{B_E(x) : x \in \mathbb{R}^2\right\}$, whose covariance function is given by

$$r^E(x, y) = r^E(x - y) := J_0(2\pi\sqrt{E}\|x - y\|), \quad x, y \in \mathbb{R}^2, \tag{IV.1.1}$$

where J_0 is the Bessel function of the first kind of order 0, see for instance [Ber77, Ber02, NPR19, PV20]. Let us denote by $\mathscr{A}$ the collection of all piecewise C^1 simply connected compact subsets of $\mathbb{R}^2$ having non-empty interior. For $\mathcal{D} \in \mathscr{A}$, we write

$$\mathscr{L}_E(\mathcal{D}) := \mathcal{H}^1(B_E^{-1}(0) \cap \mathcal{D}),$$

for the length of the zero set of B_E inside $\mathcal{D}$. In [NPR19, Theorem 1.1], it is shown that for a fixed domain $\mathcal{D}$, the nodal length verifies

$$\mathbb{E}\left[\mathscr{L}_E(\mathcal{D})\right] = \text{area}(\mathcal{D})\frac{\pi}{\sqrt{2}}\sqrt{E}, \quad \mathbf{Var}[\mathscr{L}_E(\mathcal{D})] \sim \text{area}(\mathcal{D})\frac{\log E}{512\pi}$$

as $E \to \infty$, and the subsequent one-dimensional Central Limit Theorem is derived

$$\widetilde{\mathscr{L}_E}(\mathcal{D}) := \frac{\mathscr{L}_E(\mathcal{D}) - \mathbb{E}\left[\mathscr{L}_E(\mathcal{D})\right]}{\sqrt{\mathbf{Var}[\mathscr{L}_E(\mathcal{D})]}} \xrightarrow{d} N \sim \mathcal{N}(0, 1). \tag{IV.1.2}$$

Such a limit theorem originates from the fact that the Wiener-Itô chaos expansion of $\mathscr{L}_E(\mathcal{D})$ is dominated in the $L^2(\mathbb{P})$-sense by its projection on the fourth Wiener chaos. In [PV20, Theorem 3.2], the authors prove the following multivariate extension of (IV.1.2).

Theorem IV.1.1. *For every integer $d \geq 1$ and every fixed $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_d \in \mathscr{A}$, we define the $d \times d$ matrix $\Sigma = \{\Sigma(i, j) : i, j = 1, \ldots, d\}$ by the relation*

$$\Sigma(i, j) := \frac{\text{area}(\mathcal{D}_i \cap \mathcal{D}_j)}{\sqrt{\text{area}(\mathcal{D}_i)\,\text{area}(\mathcal{D}_j)}}. \tag{IV.1.3}$$

Then, as $E \to \infty$, one has that

$$\left(\widetilde{\mathscr{L}_E}(\mathcal{D}_1), \widetilde{\mathscr{L}_E}(\mathcal{D}_2), \ldots, \widetilde{\mathscr{L}_E}(\mathcal{D}_d)\right) \xrightarrow{d} Z \sim \mathcal{N}_d(0, \Sigma), \tag{IV.1.4}$$

where Z is a centred d-dimensional Gaussian vector with covariance matrix Σ.

Such a result shows that, in the high-energy limit, the finite-dimensional distributions of the process $\left\{\widetilde{\mathscr{L}_E}(\mathcal{D}) : \mathcal{D} \in \mathscr{A}\right\}$ converge to those of a Gaussian process with a limiting covariance structure of a homogeneous independently scattered random measure with unit intensity. Given the limiting covariance structure appearing in (IV.1.3), Theorem IV.1.1 immediately implies that, when restricting the nodal length to rectangles of the type $[0, t_1] \times [0, t_2]$, the process $X_E = \left\{X_E(t_1, t_2) : (t_1, t_2) \in [0, 1]^2\right\}$ defined by

$$X_E(t_1, t_2) := \sqrt{\frac{512\pi}{\log E}}\Big(\mathscr{L}_E([0, t_1] \times [0, t_2]) - \mathbb{E}\left[\mathscr{L}_E([0, t_1] \times [0, t_2])\right]\Big), \tag{IV.1.5}$$

converges in the sense of finite-dimensional distributions to a standard *Wiener sheet*, that is, to a centred Gaussian process $\mathbf{W} = \left\{\mathbf{W}(t_1, t_2) : (t_1, t_2) \in [0, 1]^2\right\}$ with covariance function $\mathbb{E}\left[\mathbf{W}(t_1, t_2)\mathbf{W}(s_1, s_2)\right] = (t_1 \wedge s_1)(t_2 \wedge s_2)$. We refer the reader for instance to [RY99, p.39] for an introduction to such an object. The following partial weak convergence result for the projection of X_E on the fourth Wiener chaos is obtained in [PV20, Theorem 3.4]. We denote by $X_E[q](t_1, t_2)$ the projection of $X_E(t_1, t_2)$ on the q-th Wiener chaos associated with B_E.

Theorem IV.1.2. *For every fixed $(t_1, t_2) \in [0,1]^2$, one has that, as $E \to \infty$,*

$$\mathbb{E}\left[(X_E(t_1,t_2) - X_E[4](t_1,t_2))^2\right] \to 0.$$

Moreover, the random mappings $X_E[4] : (t_1,t_2) \mapsto X_E[4](t_1,t_2)$ belong almost surely to the class $C([0,1]^2,\mathbb{R})$ of continuous processes on $[0,1]^2$, and as $E \to \infty$, $X_E[4]$ converges weakly to a standard Wiener sheet $\mathbf{W}$ on $[0,1]^2$ in the Skorohod space $\mathbf{D}_2 = D([0,1]^2,\mathbb{R})$.

Results regarding a global functional convergence in the Skorohod space were not obtained in [PV20]. The proof of such weak convergence results would typically allow one to derive new probabilistic limit theorems involving semi-local functionals associated to nodal length processes, such as for instance the supremum of X_E. The difficulties of extending such a weak convergence from the fourth projection to the entire process X_E was partially explained by the presence of certain boundary terms appearing in the expression of the second chaotic projection $X_E[2]$, see in particular [PV20, Remark 3.2]. The goal of this chapter is to substantially (albeit not completely!) fill such gaps by presenting a careful analysis of these residual terms and dealing with the remainder term formed by higher-order chaotic projections associated with X_E. In the forthcoming sections, we describe our findings.

IV.1.1 Some weak convergence results

We now describe the main contributions of this chapter. We write $\mathbf{t} = (t_1,t_2) \in [0,1]^2$ and consider the normalized nodal length process

$$X_E = \left\{X_E(\mathbf{t}) : \mathbf{t} \in [0,1]^2\right\}$$

introduced in (IV.1.5), whose Wiener-Itô chaos expansion is given by

$$X_E = X_E[2] + X_E[4] + R_E, \quad R_E := \sum_{q \geq 3} X_E[2q] \tag{IV.1.6}$$

where $X_E[q]$ indicates the projection of X_E on the q-th Wiener chaos. In what follows, we specify to which functions spaces our random objects of interest belong. Consider the unit square $[0,1]^2$ and define the following four regions for every fixed $\mathbf{t} = (t_1,t_2) \in [0,1]^2$,

$$\begin{aligned}
Q(\mathbf{t}, NE) &:= \left\{\mathbf{s} = (s_1,s_2) \in [0,1]^2 : s_1 > t_1, s_2 > t_2\right\}\\
Q(\mathbf{t}, NW) &:= \left\{\mathbf{s} = (s_1,s_2) \in [0,1]^2 : s_1 < t_1, s_2 > t_2\right\}\\
Q(\mathbf{t}, SW) &:= \left\{\mathbf{s} = (s_1,s_2) \in [0,1]^2 : s_1 < t_1, s_2 < t_2\right\}\\
Q(\mathbf{t}, SE) &:= \left\{\mathbf{s} = (s_1,s_2) \in [0,1]^2 : s_1 > t_1, s_2 < t_2\right\}.
\end{aligned}$$

We remark that some of these regions may be empty, in the case where $\mathbf{t}$ belongs to the boundary of $[0,1]^2$. The Skorohod space $\mathbf{D}_2 = D([0,1]^2,\mathbb{R})$ is the class of functions $f : [0,1]^2 \to \mathbb{R}$ verifying the following continuity property for every $\mathbf{t} \in [0,1]^2$: for every $R \in \{NE, NW, SW, SE\}$ and every sequence $\{\mathbf{t}_n : n \geq 1\} \subset Q(\mathbf{t}, R)$ such that $\mathbf{t}_n \to \mathbf{t}$ as $n \to \infty$, the limit $\lim_{n\to\infty} f(\mathbf{t}_n)$ exists and is finite, and, moreover for every sequence $\{\mathbf{t}_n : n \geq 1\} \subset Q(\mathbf{t}, NE)$ such that $\mathbf{t}_n \to \mathbf{t}$ as $n \to \infty$, we have that $\lim_{n\to\infty} f(\mathbf{t}_n) = f(\mathbf{t})$.

We endow the space $\mathbf{D}_2$ both with the σ-field generated by coordinate projections, and with the Skorohod topology described in Neuhaus [Neu71, p.1289]. We also define $\mathbf{C}_2 = C([0,1]^2,\mathbb{R})$ to be the subset of $\mathbf{D}_2$ composed of continuous mappings.

We note that, in the case where $f : [0,1]^2 \to \mathbb{R}$ takes the form $f(\mathbf{t}) := \mu([0,t_1] \times [0,t_2])$ for some finite measure μ on $[0,1]^2$, it follows from an application of the dominated convergence theorem that

$f \in \mathbf{D}_2$. In view of the above discussion, the nodal length processes $\{X_E : E > 0\}$ are $\mathbf{D}_2$-valued random mappings. Our strategy for proving a weak convergence result for the process X_E is based on the following lemma. Its proof is postponed to Appendix IV.A.

Lemma IV.1.3. *Let $\{X, X_n : n \geq 1\}$ be a sequence of stochastic processes with values in $\mathbf{D}_2$ such that $P(X \in \mathbf{C}_2) = 1$. We assume that for every $n \geq 1$, the process X_n can be written as $X_n = U_n + V_n + W_n$, where the processes U_n, V_n, W_n are such that*

(i) as $n \to \infty$, U_n converges weakly to X in $\mathbf{D}_2$,

(ii) as $n \to \infty$, V_n converges weakly to zero in $\mathbf{D}_2$,

(iii) for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left\{ \sup_{\mathbf{t}\in[0,1]^2} |W_n(\mathbf{t})| > \varepsilon \right\} = 0,$$

Then, X_n converges weakly to X in $\mathbf{D}_2$.

In order order to prove a global functional convergence result for the normalized nodal length process X_E, we apply Lemma IV.1.3 with $(X_n, U_n, V_n, W_n) = (X_E, X_E[4], X_E[2], R_E)$ in (IV.1.6). We note that $\mathbb{P}\{X_E[2] \in \mathbf{C}_2\} = \mathbb{P}\{X_E[4] \in \mathbf{C}_2\} = 1$, for every $E > 0$. Then, one has to deal with the following three steps:

(i) proving that the projection $X_E[4]$ converges weakly to a standard Wiener sheet as $E \to \infty$, which was fully solved in [PV20] (see also Theorem IV.1.2 above),

(ii) proving that the second chaotic projection $X_E[2]$ associated with X_E converges weakly to zero, as $E \to \infty$,

(iii) proving that the residue term formed by the series of higher-order chaotic projections R_E associated with X_E converges uniformly to zero in probability, as $E \to \infty$.

In this chapter, our principal aim is to deal with points **(ii)** and **(iii)** above, that were left open in [PV20]. For part **(ii)**, we present an auxiliary study of certain integral expressions amenable to the exact expression of $X_E[2]$ (see Section IV.1.2), and are in particular able to prove the following result. Its proof is a consequence of Theorem IV.1.21 and Remark IV.1.18 **(a)**.

For $\mathbf{t} = (t_1, t_2)$, we set $\mathcal{D}_\mathbf{t} := [0, t_1] \times [0, t_2]$ and write $\mathscr{L}_E[2](\mathcal{D}_\mathbf{t})$ for the projection of $\mathscr{L}_E(\mathcal{D}_\mathbf{t})$ on the second Wiener chaos.

Theorem IV.1.4. *For every $\mathbf{t} \in [0, 1]^2$, we set*

$$Y_E(\mathbf{t}) := \frac{\mathscr{L}_E[2](\mathcal{D}_\mathbf{t})}{\sqrt{\mathbf{Var}[\mathscr{L}_E[2](\mathcal{D}_\mathbf{t})]}}.$$

For every integer $d \geq 1$ and every collection of $\mathbf{t}_1, \dots, \mathbf{t}_d \in [0, 1]^2$, we have that, as $E \to \infty$

$$(Y_E(\mathbf{t}_1), \dots, Y_E(\mathbf{t}_d)) \xrightarrow{d} Z \sim \mathcal{N}_d(0, \Sigma)$$

where Z is a centred d-dimensional Gaussian vector with covariance matrix $\Sigma = \{\Sigma(i, j) : i, j = 1, \dots, d\}$ given by the relation

$$\Sigma(i, j) = \frac{\lambda(\partial\mathcal{D}_{\mathbf{t}_i}, \partial\mathcal{D}_{\mathbf{t}_j})}{\sqrt{\lambda(\partial\mathcal{D}_{\mathbf{t}_i}, \partial\mathcal{D}_{\mathbf{t}_i})\lambda(\partial\mathcal{D}_{\mathbf{t}_j}, \partial\mathcal{D}_{\mathbf{t}_j})}},$$

where $\lambda(\partial\mathcal{D}_{\mathbf{t}_i}, \partial\mathcal{D}_{\mathbf{t}_j})$ denotes the signed length of $\partial\mathcal{D}_{\mathbf{t}_i} \cap \partial\mathcal{D}_{\mathbf{t}_j}$ (see Section IV.1.2 for more details).

In particular, specializing the findings of Theorem IV.1.4 to the setting of concentric squares $R_t := [1/2 - t, 1/2 - t]^2, 0 < t < 1/2$ centred at the point $(1/2, 1/2)$, verifying $\partial R_t \cap \partial R_s = \emptyset$ for $t \neq s$, yields the following characterization of Y_E as a total disorder process. Here, we call *total disorder process* any Gaussian process whose linear span contains an uncountable collection of i.i.d standard Gaussian random variables. We refer the reader for instance to [RY99, Section 3] for more details on such processes.

Corollary IV.1.5. *The limiting process of* $\left\{Y_E(\mathbf{t}) : \mathbf{t} \in [0,1]^2\right\}$ *is a total disorder process.*

Total disorders appear as the limiting distribution in a number of works. We refer the reader to Section IV.1.2.1 for an overview. Our arguments for proving Theorem IV.1.4, are based on a preliminary study of the second chaotic component (see Section IV.1.2), in which we prove asymptotic variance estimates (see Theorem IV.1.20) and a multivariate Central Limit Theorem in the high-energy regime (see Theorem IV.1.21).

Combining the variance estimates for second chaotic projections in Theorem IV.1.20 with a suitable tightness criterion by Davydov and Zitikis [DZ08] (see Proposition IV.2.4) for proving weak convergence of stochastic processes on $[0,1]^d$, and some moment estimates for suprema of stationary Gaussian fields (see Proposition IV.2.6), implies the following weak convergence result for the normalized second order projection $X_E[2]$, which solves **(ii)**.

Corollary IV.1.6. *As* $E \to \infty$*, the process* $\left\{X_E[2](\mathbf{t}) : \mathbf{t} \in [0,1]^2\right\}$ *converges weakly to zero in* $\mathbf{D}_2$.

Concerning part **(iii)** above, we are able to present partial solutions dealing with *discretized* and *truncated* nodal length processes.

Discretized nodal length process. Let us first introduce some notation. For $K \geq 1$, we indicate by Π_K the partition of the $[0,1]^2$ formed by the collection of squares of side length 2^{-K}. For every vector $i = (i_1, i_2) \in \left\{0, \ldots, 2^K\right\}^2$, we define the *partition points* $\mathbf{p}_i(K,K) := (p_{i_1}(K), p_{i_2}(K)) \in [0,1]^2$ by

$$p_{i_1}(K) := \frac{i_1}{2^K}, \quad p_{i_2}(K) := \frac{i_2}{2^K}, \quad i_1, i_2 = 0, 1, \ldots, 2^K.$$

For $\mathbf{t} = (t_1, t_2) \in [0,1]^2$, we write $i_{K,K}(\mathbf{t}) = (i_{1,K}(t_1), i_{2,K}(t_2))$ for the vector verifying

$$p_{i_{1,K}(t_1)} \leq t_1 \leq p_{i_{1,K}(t_1)+1}, \quad p_{i_{2,K}(t_2)} \leq t_2 \leq p_{i_{2,K}(t_2)+1},$$

that is, the vector $i_{K,K}(\mathbf{t})$ is such that $\mathbf{p}_{i_{K,K}(\mathbf{t})}(K,K)$ is the closest partition point to $\mathbf{t}$ on the left. We introduce the following notion of *discretized nodal length*.

Definition IV.1.7. (*Discretized nodal length process*) Let $K \geq 1$ be an integer and Π_K a partition of $[0,1]^2$ as described above. For $\mathbf{t} \in [0,1]^2$, we define the *discretized nodal length* by

$$\mathscr{L}_E^K([0,t_1] \times [0,t_2]) := \mathscr{L}_E\left([0, p_{i_{1,K}(t_1)}(K)] \times [0, p_{i_{2,K}(t_2)}(K)]\right)$$

and write X_E^K for its normalized version

$$X_E^K(\mathbf{t}) = \sqrt{\frac{512\pi}{\log E}}\left(\mathscr{L}_E^K([0,t_1] \times [0,t_2]) - \mathbb{E}\left[\mathscr{L}_E^K([0,t_1] \times [0,t_2])\right]\right).$$

As usual, we write $X_E^K[q]$ for the projection of X_E^K on the q-th Wiener chaos and set $R_E^K = \sum_{q \geq 3} X_E^K[2q]$.

In the light of the above definition, $X_E^K(\mathbf{t})$ represents the normalized nodal length contained in the rectangle formed by the partition coordinates that are closest to $\mathbf{t}$, thus yielding a discrete approximation of $X_E(\mathbf{t})$. Moreover, $\mathscr{L}_E^K$ is $\mathbb{P}$-almost surely an element of $\mathbf{D}_2$. The following result shows that there exists a suitable partition Π_K of $[0,1]^2$ associated with a sequence $K = K(E)$ such that the discretized residue process R_E^K converges to zero uniformly on the unit square, thus showing a *discretized* version of **(iii)**. The proof of IV.1.8 relies on a *planar* chaining argument inspired by Dehling and Taqqu [DT89] and Marinucci and Wigman [MW11].

Theorem IV.1.8. *Let $\{K(E) : E > 0\}$ be a numerical sequence such that $K(E) \to \infty$ and $K(E) = o((\log E)^{1/10})$ as $E \to \infty$. Then, for every $\varepsilon > 0$,*

$$\mathbb{P}\left\{\sup_{\mathbf{t}\in[0,1]^2}\left|R_E^{K(E)}(\mathbf{t})\right| > \varepsilon\right\} \to 0.$$

Combining the findings of Theorem IV.1.8, Corollary IV.1.6 and the weak convergence of $X_E[4]$ proved in [PV20] (see Theorem IV.1.2), allows us to deduce the following weak convergence result for the discretized nodal length process.

Corollary IV.1.9. *Let $\{K(E) : E > 0\}$ be a numerical sequence such that $K(E) \to \infty$ and $K(E) = o((\log E)^{1/10})$ as $E \to \infty$. Then, as $E \to \infty$, the normalized process $X_E^{K(E)}$ converges weakly to a standard Wiener sheet $\mathbf{W}$ on $[0,1]^2$ in $\mathbf{D}_2$.*

Corollary IV.1.9 gives access to a number of new limit theorems dealing with specific functionals of the discretized nodal length process. Of particular interest is, for instance, the asymptotic behaviour of the maximal discrepancy between the discretized nodal length and its expectation, given by the supremum of X_E^K. Such statistics provide global indications on how the nodal length process deviates from its mean and are intimately related to overcrowding estimates and concentration inequalities. We refer the reader for instance to [Pri20] for the study of such events in the framework of zero counts and nodal length associated with stationary Gaussian processes.

The following result provides an answer in this direction.

Corollary IV.1.10. *Let $\{K(E) : E > 0\}$ be a numerical sequence such that $K(E) \to \infty$ and $K(E) = o((\log E)^{1/10})$ as $E \to \infty$. Then, as $E \to \infty$, we have that*

$$\sup_{\mathbf{t}\in[0,1]^2}\left|X_E^{K(E)}(\mathbf{t})\right| \xrightarrow{d} \sup_{\mathbf{t}\in[0,1]^2}|\mathbf{W}(\mathbf{t})|.$$

To the best of our expertise, the probability distribution of the supremum of the Wiener sheet is not known. In [PP73], the authors provide a number of explicit expressions for the probability distribution function of the supremum of Wiener sheets restricted to the boundary of planar domains. For instance, the following statement is a direct consequence of Corollary IV.1.10 and [PP73, Theorem 3], yielding a closed formula for the asymptotic distribution function of the supremum of X_E^K on the boundary of the unit square. We refer the reader to [PP73] for more examples in this direction.

Corollary IV.1.11. *Let $\{K(E) : E > 0\}$ be a numerical sequence such that $K(E) \to \infty$ and $K(E) = o((\log E)^{1/10})$ as $E \to \infty$. Then, for every $z \in \mathbb{R}$, we have that, as $E \to \infty$,*

$$\mathbb{P}\left\{\sup_{\mathbf{t}\in\partial[0,1]^2}\left|X_E^{K(E)}(\mathbf{t})\right| \le z\right\} \to \mathbb{P}\left\{\sup_{\mathbf{t}\in\partial[0,1]^2}|\mathbf{W}(\mathbf{t})| \le z\right\} = 1 - 3\Phi(-z) + e^{4z^2}\Phi(-3z),$$

where $\Phi(z) := \mathbb{P}\{N \le z\}, N \sim \mathcal{N}(0,1)$.

Remark IV.1.12. The findings described above are not sufficient to obtain a weak convergence result for the process X_E and thus to fully solve part **(iii)**. Our main difficulty for directly dealing with the residue term R_E (instead of its discretized version R_E^K) appears in the chaining argument used in the proof of Theorem IV.1.8 and is essentially explained by the fact that the expectation of X_E (which is of order $\sqrt{E/\log E}$) grows considerably faster than the normalizing factor $\log E$. Carrying out the planar chaining argument with R_E typically requires the quantity

$$\left|\mathbb{E}\left[X_E(\mathbf{t})\right] - \mathbb{E}\left[X_E(\mathbf{p}_{i_{K,K}(\mathbf{t})}(K,K))\right]\right| \approx \frac{\sqrt{E}}{\sqrt{\log E}}\frac{1}{2^K}$$

to be bounded, thus imposing $K = K(E)$ to be of logarithmic order. Such a requirement is however incompatible with the choice $o((\log E)^{1/10})$, as is needed in the above statements. Such a difficulty is eschewed when dealing with the discretized versions, since in this case $\mathbb{E}\left[X_E^K(\mathbf{t})\right] = \mathbb{E}\left[X_E^K(\mathbf{p}_{i_{K,K}(\mathbf{t})}(K,K))\right]$ by construction of X_E^K, implying that the above difference is zero. One possible strategy for providing a complete answer to **(iii)** would be to prove that for every $\varepsilon > 0$

$$\mathbb{P}\left\{\sup_{\mathbf{t}\in[0,1]^2}\left|R_E^{K(E)}(\mathbf{t}) - R_E(\mathbf{t})\right| > \varepsilon\right\} \to 0,$$

as $E \to \infty$, where $K(E)$ is as in Theorem IV.1.8. However, our arguments allow to prove that such an asymptotic relation only holds pointwise in the $L^2(\mathbb{P})$-sense

$$\mathbb{E}\left[\left(R_E^{K(E)}(\mathbf{t}) - R_E(\mathbf{t})\right)^2\right] \leq c_1\frac{1}{\log E}\frac{1}{2^{K(E)}}$$

where $c_1 > 0$ is some absolute constant, thus converging to zero in view of our choice of $K(E)$ (see in particular Lemma IV.2.9).

Truncated nodal length process. We also point out that our results on the second Wiener chaos are sufficient to formulate a weak convergence result for *truncated nodal lengths* of increasing degree, defined as follows.

Definition IV.1.13. (*Truncated nodal length*) For an integer $N \geq 1$, we define the *truncated nodal length* of order N by

$$\mathscr{L}_E(\mathcal{D};N) := \sum_{q=0}^{N}\mathscr{L}_E[2q](\mathcal{D}).$$

We write $X_E(\mathbf{t};N)$ for the normalized version of $\mathscr{L}_E([0,t_1]\times[0,t_2];N)$ and $R_E(\mathbf{t};N) := \sum_{q=3}^{N} X_E[2q](\mathbf{t})$ for its chaotic projections of order 6 to N.

The following result shows that the process $R_E(\cdot;N)$ converges to zero for a well-chosen $N = N(E)$, and is a consequence of the hypercontractivity property on Wiener chaoses (see (I.1.29)).

Proposition IV.1.14. *Let $N(E) = \log_5(\log E)$. Then, as $E \to \infty$, the process $\left\{R_E(\mathbf{t};N(E)) : \mathbf{t} \in [0,1]^2\right\}$ converges weakly to zero in $\mathbf{D}_2$.*

Combining this result with the weak convergence to zero of the second chaotic projections $X_E[2]$ (see Corollary IV.1.6) and the weak convergence of $X_E[4]$, is sufficient to derive the following functional limit theorem for the *truncated nodal length process* of order $N = N(E)$.

Corollary IV.1.15. *Let* $N(E) = \log_5(\log E)$. *Then, as* $E \to \infty$, *the process* $\left\{X_E(\mathbf{t}; N(E)) : \mathbf{t} \in [0,1]^2\right\}$ *converges weakly to a standard Wiener sheet* $\mathbf{W}$ *on* $[0,1]^2$ *in* $\mathbf{D}_2$.

Remark IV.1.16. In order to prove **(iii)**, it remains to show that the *tail series* $X_E - R_E(\bullet, N(E))$ formed by chaotic projections exceeding $N(E)$, converges weakly to zero, as $E \to \infty$. A possible route for tackling such a problem is to divide the remainder term in *singular* and *non-singular* pair of cells as introduced in [ORW08] and, later exploited in [PR18, DNPR19, NPR19] (see in particular Appendix II.E of Chapter II) and investigate each of their contributions separately.

IV.1.2 Study of the second Wiener chaos

In this section, we present our preliminary results on the second Wiener chaos, allowing one to prove Theorem IV.1.4.

Let us consider a convex planar domain $\mathcal{D}$ with piecewise C^1 boundary $\partial\mathcal{D}$, which we assume to be oriented clock-wise (see also Remark IV.1.18). In [NPR19, Lemma 4.1], the authors prove that the projection on the second Wiener chaos of the nodal length associated with Berry's random field can be written as

$$\mathscr{L}_E[2](\mathcal{D}) = \frac{1}{8\pi\sqrt{2E}} \int_{\partial\mathcal{D}} B_E(x)\langle \nabla B_E(x), \mathrm{n}_{\mathcal{D}}(x)\rangle dx, \tag{IV.1.7}$$

where $\mathrm{n}_{\mathcal{D}}(x) = (\mathrm{n}^1_{\mathcal{D}}(x), \mathrm{n}^2_{\mathcal{D}}(x))$ is the outward unit normal vector to $\partial\mathcal{D}$ at x and dx indicates the one-dimensional Hausdorff measure. We recall that relation (IV.1.7) is immaterial on manifolds without boundary as for instance in the related models of random spherical harmonics and toral arithmetic random waves, where the second chaotic projection of the nodal length is almost surely zero, see for instance [Wig10, KKW13] and in particular the characterization of the second Wiener chaos presented in Theorem II.2.5 of Chapter II.

In order to study the second chaotic projections in (IV.1.7), we introduce *abstract* random variables whose expression is amenable to (IV.1.7). Let $\mathscr{C}$ be the collection of all polygonal planar curves of finite length, that is, *formal sums* of the form

$$C = \sum_{j=1}^{r} \alpha_j S_j, \quad r \geq 1$$

where $\alpha_1, \ldots, \alpha_r \in \mathbb{R}$ and $S_1, \ldots, S_r$ are oriented line segments. In particular, the above formal sum has a clear geometric meaning when the coefficients α_j are equal to one and the line segments S_j are adjacent. Also, the sign of the coefficients α_j determines the orientation of the corresponding line segment S_j: if $\alpha_j < 0$, then the orientation of S_j is reversed, whereas if $\alpha_j > 0$, then $\alpha_j S_j$ is the line segment having same orientation as S_j and of length α_j times the length of S_j.

Definition IV.1.17. For $C \in \mathscr{C}$ and $E > 0$, we define the random variable

$$\phi_E(C) = \frac{1}{8\pi\sqrt{2E}} \int_C B_E(x)\langle \nabla B_E(x), \mathrm{n}_C(x)\rangle dx, \tag{IV.1.8}$$

where dx indicates one-dimensional Hausdorff measure on C and n_C denotes the unit normal vector to C computed at x.

We make some remarks about Definition IV.1.17.

Remark IV.1.18. **(a)** Comparing (IV.1.7) with (IV.1.8), it becomes clear that when the curve C is the boundary of a compact planar domain $\mathcal{D}$, then $\phi_E(\partial\mathcal{D}) = \mathscr{L}_E[2](\mathcal{D})$, that is, $\phi_E(\partial\mathcal{D})$ coincides with the projection on the second Wiener chaos of the nodal length restricted to $\mathcal{D}$, associated with B_E. This observation motivates the study of random variables $\phi_E(C)$.

(b) If S is a line segment directed by its tangent vector $\tau_S = (\tau_1, \tau_2) \in \mathbb{R}^2$, then $\mathrm{n}_S = \|\tau\|^{-1}(-\tau_2, \tau_1)$. Since we deal with line segments, we remark that $\mathrm{n}_S(x)$ is independent of the chosen point $x \in S$. The line integration over a segment S in (IV.1.8) is defined as follows: Let $\gamma_S : [0, L] \to S$ be a unit speed parametrization of S, that is γ_S is such that $\|\dot{\gamma_S}(t)\| = 1$ for every $t \in [0, L]$, (where $\dot{\gamma_S}$ denotes the derivative of γ_S), showing in particular that S has length L. We define

$$\phi_E(S) = \frac{1}{8\pi\sqrt{2E}} \int_0^L B_E(\gamma(t))\langle\nabla B_E(\gamma(t)), \mathrm{n}_S(\gamma(t))\rangle dt, \tag{IV.1.9}$$

where n_S is as described above. We then extend the definition of ϕ_E to the class C by linearity

$$\phi_E(C) := \sum_{j=1}^r \alpha_j \phi_E(S_j), \quad C = \sum_{j=1}^r \alpha_j S_j. \tag{IV.1.10}$$

(c) It is easy to see that for every $C \in \mathscr{C}$, the random variable $\phi_E(C)$ is an element of the second Wiener chaos associated with B_E. Indeed, denoting by I_p the Wiener isometry of order p, and writing $B_E(x) = I_1(f_0^E(x, \cdot)), \partial_j B_E(x) = \sqrt{2\pi^2 E} I_1(f_j^E(x, \cdot)), j = 1, 2$ for suitable kernels $f_0^E(x, \cdot), j = 0, 1, 2$ defined on the Hilbert space $L^2([0, 1], \lambda)$ (with λ denoting Lebesgue measure), an application of the product formula for Wiener integrals (see (I.1.28)) allows to write $\phi_E(C) = I_2(u^E(C))$, where

$$u^E(C) = \frac{1}{8} \sum_{j=1}^2 \int_C f_0^E(x, \cdot) \widetilde{\otimes} f_j^E(x, \cdot) \mathrm{n}_C^j(x) dx, \tag{IV.1.11}$$

and where $\widetilde{\otimes}$ denotes the canonical symmetrization of the tensor product. We refer the reader to the proof of Proposition IV.2.3 for more details.

In order to state our main results, we endow the product space $\mathscr{C} \times \mathscr{C}$ with the an inner product product. For line segments S_1 and S_2, we define $\lambda(S_1, S_2)$ to be the *signed length* of $S_1 \cap S_2$, that is, the length of their intersection multiplied if S_1 and S_2 have the same orientation and its opposite if they have opposite orientations. For arbitrary $C_1 = \sum_{j=1}^r \alpha_j S_j, C_2 = \sum_{k=1}^s \beta_k T_k \in \mathscr{C}$ (for collections of line segments $\left\{S_j : j = 1, \ldots, r\right\}$ and $\{T_k : k = 1, \ldots, s\}$, we extend this definition by bilinearity,

$$\lambda(C_1, C_2) := \sum_{j=1}^r \sum_{k=1}^s \alpha_j \beta_k \lambda(S_j, T_k) \tag{IV.1.12}$$

and refer to it as the *signed length* of $C_1 \cap C_2$. In particular, choosing $C_1 = C_2 = C$ shows that $\lambda(C, C)$ coincides with the one-dimensional Hausdorff measure, that is, the length of C, for which we will write $\lambda(C) := \lambda(C, C)$. We point out that such a scalar product already appears in [BS17, Definition 3], where the authors study fluctuations of increments of the Gaussian entire function along smooth curves. We refer the reader to section IV.1.2.1 for more details on this and related works.

Remark IV.1.19. We remark that in [BS17], the authors consider the class of all simple regular curves, as opposed to our definition of $\mathscr{C}$, only restricted to polygonal chains. Our reason for only dealing with this restricted family of curves originates from a number of difficulties encountered when trying to extend our findings to the setting of arbitrary planar curves. Our main idea for dealing with this more general framework, is to approximate a planar curve by infinitesimal line segments $S_1, \dots, S_N$ of length converging to zero as $N \to \infty$. However, our results stated below, typically require uniform estimates in E or N, which for the moment have eluded our attempts.

We now state our main results concerning the random variables $\phi_E(C)$. The first statement gives the asymptotic covariance structure in the high-energy regime.

Theorem IV.1.20 (Asymptotic covariance structure)**.** *For every $C_1, C_2 \in \mathscr{C}$, as $E \to \infty$,*

$$\mathbf{Cov}[\phi_E(C_1), \phi_E(C_2)] = \frac{\lambda(C_1, C_2)}{16\pi^2\sqrt{E}} + o\left(\frac{1}{\sqrt{E}}\right), \tag{IV.1.13}$$

where $\lambda(C_1, C_2)$ indicates the signed length of $C_1 \cap C_2$. In particular, as $E \to \infty$, we have that

$$\mathbf{Var}[\phi_E(C)] = \frac{\lambda(C)}{16\pi^2\sqrt{E}} + o\left(\frac{1}{\sqrt{E}}\right), \tag{IV.1.14}$$

where $\lambda(C)$ denotes the length of C.

Specifying the content of Theorem IV.1.20 to the case where $C = \partial\mathcal{D}$ is the boundary of a polygonal compact domain $\mathcal{D} \subset \mathbb{R}^2$ (as for instance rectangles of the type $[0, t_1] \times [0, t_2]$) and bearing in mind Remark IV.1.18 **(a)**, we deduce the following variance estimate for the second chaotic projection of the nodal length associated with the Berry random field

$$\mathbf{Var}[\mathscr{L}_E[2](\mathcal{D})] = \frac{\lambda(\partial\mathcal{D})}{16\pi^2\sqrt{E}} + o\left(\frac{1}{\sqrt{E}}\right),$$

as $E \to \infty$. This estimate refines the upper bound $O(1)$ for the variance of $\mathscr{L}_E[2](\mathcal{D})$ derived in [NPR19, Lemma 4.1] and in particular shows that $\mathscr{L}_E[2](\mathcal{D})$ is a degenerate random variable in the high-energy limit.

In view of (IV.1.14), we introduce the normalized version of $\phi_E(C)$,

$$\widetilde{\phi}_E(C) := 4\pi E^{1/4}\phi_E(C). \tag{IV.1.15}$$

The following result states a weak convergence result in the sense of finite-dimensional distributions for the random field $\left\{\widetilde{\phi}_E(C) : C \in \mathscr{C}\right\}$.

Theorem IV.1.21. *For every integer $d \geq 1$ and every $C_1, \dots, C_d \in \mathscr{C}$, we have that, as $E \to \infty$,*

$$\left(\widetilde{\phi}_E(C_1), \dots, \widetilde{\phi}_E(C_d)\right) \xrightarrow{d} \mathcal{N}_d(0, \Sigma),$$

where $\Sigma = \{\Sigma(i, j) : i, j = 1, \dots, d\}$ is the $d \times d$ matrix defined by

$$\Sigma(i, j) := \lambda(C_i, C_j), \quad i, j = 1, \dots, d,$$

where $\lambda(C_i, C_j)$ denotes the signed length of $C_i \cap C_j$.

The statement of Theorem IV.1.21 shows that, in the high-frequency regime, the covariance of $\widetilde{\phi}_E(C_1)$ and $\widetilde{\phi}_E(C_2)$ depends on the geometry of C_1 and C_2 through the signed length of their intersection. In particular, it becomes apparent that, whenever $\lambda(C_1, C_2)$ is zero (which is the case when C_1 and C_2 intersect in finitely many isolated points or are disjoint), the random variables $\widetilde{\phi}_E(C_1)$ and $\widetilde{\phi}_E(C_2)$ are asymptotically independent Gaussian with variance $\lambda(C_1)$ and $\lambda(C_2)$, respectively.

IV.1.2.1 The total disorder process in the literature

In Corollary IV.1.5, we prove that the limiting structure of the second chaotic projection restricted to concentric squares is of total disorder, which is a consequence of the asymptotic dependence structure derived in Theorem IV.1.20. A similar structure appears as the limiting covariance in several works in the literature, that we will briefly describe here below.

As already mentioned above, in [BS17], the authors study the fluctuations of the increment of the Gaussian entire function along planar curves. More precisely, denoting by $\Delta_R(\Gamma)$ the increment of the Gaussian entire function along a curve Γ, their results show that, properly normalized, $\Delta_R(\Gamma)$ exhibits Gaussian fluctuations, and moreover, the random variables $\Delta_R(\Gamma_1)$ and $\Delta_R(\Gamma_2)$ are jointly Gaussian as $R \to \infty$ with limiting covariance proportional to the *signed length* of the intersection of Γ_1 and Γ_2. Specifying their setting to the case where Γ_i is a positively oriented boundary of a bounded domain G_i and writing $n_R(G_i)$ for the number of zeros of the Gaussian entire function in the domain RG_i, their results allow to conclude that the limiting covariance between $n_R(G_i)$ and $n_R(G_j)$ is proportional to the signed length of $\partial G_i \cap \partial G_j$. In this respect, our findings should be naturally confronted with the main contributions of [BS17].

A similar limiting covariance structure arises in the physics paper [Leb83] by Lebowitz on charge fluctuations for Coulomb systems. Therein, the author considers the net electric charge Q_Λ contained in a subregion Λ of an infinite equilibrium system and studies the asymptotic covariance between Q_{Λ_1} and Q_{Λ_2} where Λ_1, Λ_2 are growing regions. For instance, for cubes Λ_1, Λ_2 of side length $L \to \infty$, it turns out that the limiting covariance is only non-zero when the cubes share a pair of adjacent faces.

Total disorder processes also appear in several works in random matrix theory. In [Wie02] (see also [DE01, Theorem 6.3]) the authors consider the number $N_n(\alpha, \beta)$ of eigenvalues lying in a circular interval $(e^{i\alpha}, e^{i\beta})$ of random $n \times n$ unitary matrices sampled according to the Haar measure. It is shown that the finite-dimensional distributions of the normalized process

$$\left\{ \frac{N_n(\alpha,\beta) - \mathbb{E}\left[N_n(\alpha,\beta)\right]}{\pi^{-1}\sqrt{\log n}} : 0 < \alpha < \beta < 2\pi \right\}$$

converge to those of a centred Gaussian process $\{Z(\alpha, \beta) : 0 < \alpha < \beta < 2\pi\}$ with covariance function

$$\mathbb{E}\left[Z(\alpha,\beta)Z(\alpha',\beta')\right] = \begin{cases} 1 & \alpha = \alpha', \beta = \beta' \\ -1 & \alpha = \beta', \alpha' = \beta \\ 1/2 & \alpha = \alpha' \text{ or } \beta = \beta' \text{ but not both} \\ -1/2 & \alpha = \beta' \text{ or } \beta = \alpha' \text{ but not both} \\ 0 & \alpha, \beta, \alpha', \beta' \text{ distinct} \end{cases}.$$

From such a covariance structure, it becomes clear that, unless two intervals $(e^{i\alpha}, e^{i\beta})$ and $(e^{i\alpha'}, e^{i\beta'})$ have at least one endpoint in common, the limiting random variables $Z(\alpha, \beta)$ and $Z(\alpha', \beta')$ are independent.

Finally, in [HNY08] the authors prove that the finite-dimensional distributions of a complex Gaussian total disorder process appear as the limiting distribution of the multi-dimensional extension of Selberg's Central Limit Theorem for the logarithm of the Riemann Zeta function (see [Sel46, Sel92]).

IV.2 Proof of the main results

In the forthcoming part, we briefly expose the main ideas for proving our findings in this chapter.

Theorem IV.1.20 and Theorem IV.1.21 are proved in Section IV.2.1. Corollary IV.1.6 is proved in Section IV.2.2 and finally the proof of Theorem IV.1.8 is deferred to Section IV.2.3.

IV.2.1 Proofs of Theorem IV.1.20 and Theorem IV.1.21

In order to prove Theorem IV.1.20 and Theorem IV.1.21, we first prove their analog statements when the polygonal curves are replaced with straight line segments. More specifically, in Section IV.2.1.1, we investigate the limiting covariance structure of ϕ_E when restricted to line segments, by carefully taking into account all possible spatial configurations of two line segments. In Proposition IV.2.1, we show that, in the high energy limit, this covariance is non-zero only when the line segments have a non-trivial intersection, that is, when the line segments are adjacent to each other. In Proposition IV.2.3, we establish a multi-dimensional Gaussian limit theorem for random vectors of the form $(\widetilde{\phi}_E(S_1), \dots, \widetilde{\phi}_E(S_d))$, where $S_1, \dots, S_d$ is a collection of line segments. Our methods rely on both the Fourth Moment Theorem (see Theorem I.1.30) for proving normal approximations of chaotic sequences and the Peccati-Tudor Theorem (see Theorem I.1.31) for deducing multi-dimensional extensions. The proofs of Theorems IV.1.20 and IV.1.21 are then obtained by the linearity property of ϕ_E in (IV.1.10).

IV.2.1.1 Study of line segments

Let S_1 and S_2 be two line segments in $\mathbb{R}^2$, and consider the random variables $\phi_E(S_i)$, $i = 1, 2$. Our principal aim of this section is to prove the following result.

Proposition IV.2.1. *Let S_1 and S_2 be two line segments. Then, we have that, as $E \to \infty$,*

$$\mathbf{Cov}[\phi_E(S_1), \phi_E(S_2)] = \frac{\lambda(S_1, S_2)}{16\pi^2\sqrt{E}} + o\left(\frac{1}{\sqrt{E}}\right), \tag{IV.2.1}$$

where $\lambda(S_1, S_2)$ is the signed length of $S_1 \cap S_2$.

In order to reduce the length of the proof of Proposition IV.2.1, we start with some ancillary computations.

Introducing normalized derivatives $\tilde{\partial}_i := \sqrt{2\pi^2 E}\partial_i$, $i = 1, 2$ (where $\partial_i := \partial_{x_i} = \partial/\partial x_i$) and exploiting the definition of ϕ_E in (IV.1.7), we have for every $C_1, C_2 \in \mathscr{C}$

$$\begin{aligned}
&\mathbf{Cov}[\phi_E(C_1), \phi_E(C_2)] = \mathbb{E}\left[\phi_E(C_1)\phi_E(C_2)\right] \\
= \;& \frac{1}{128\pi^2 E}\int_{C_1\times C_2} \mathbb{E}\left[B_E(x)\langle\nabla B_E(x), \mathrm{n}_{C_1}(x)\rangle B_E(y)\langle\nabla B_E(y), \mathrm{n}_{C_2}(y)\rangle\right] dxdy \\
= \;& \frac{1}{128\pi^2 E}\sum_{i,j=1}^{2}\int_{C_1\times C_2} \mathbb{E}\left[B_E(x)B_E(y)\partial_i B_E(x)\partial_j B_E(y)\right] \mathrm{n}^i_{C_1}(x)\mathrm{n}^j_{C_2}(y)dxdy \\
= \;& \frac{2\pi^2 E}{128\pi^2 E}\sum_{i,j=1}^{2}\int_{C_1\times C_2} \mathbb{E}\left[B_E(x)B_E(y)\tilde{\partial}_i B_E(x)\tilde{\partial}_j B_E(y)\right] \mathrm{n}^i_{C_1}(x)\mathrm{n}^j_{C_2}(y)dxdy \\
=: \;& \frac{1}{64}\sum_{i,j=1}^{2}\int_{C_1\times C_2} \psi^E_{i,j}(x,y)\mathrm{n}^i_{C_1}(x)\mathrm{n}^j_{C_2}(y)dxdy,
\end{aligned} \tag{IV.2.2}$$

where $\psi^E_{i,j} : \mathbb{R}^2 \to \mathbb{R}$ is the function

$$\psi^E_{i,j}(x,y) := \mathbb{E}\left[B_E(x)B_E(y)\tilde{\partial}_i B_E(x)\tilde{\partial}_j B_E(y)\right], \quad i,j = 1,2.$$

Since for every $x, y \in \mathbb{R}^2$ and every $i = 1, 2$, $(B_E(x), B_E(y), \tilde{\partial}_i B_E(x), \tilde{\partial}_j B_E(y))$ is a centred Gaussian vector, we recall Feynmann's formula (see for instance [MP11, Proposition 4.15]) in order to simplify the above expression: for jointly centred Gaussian random variables $Z_1, \dots, Z_4$, we have

$$\mathbb{E}\left[Z_1Z_2Z_3Z_4\right] = \gamma_{12}\gamma_{34} + \gamma_{13}\gamma_{24} + \gamma_{14}\gamma_{23}, \quad \gamma_{ij} := \mathbb{E}\left[Z_iZ_j\right].$$

Therefore, exploiting the covariance structure of the vector $(B_E(x), B_E(y), \tilde{\partial}_i B_E(x), \tilde{\partial}_j B_E(y))$ (see [NPR19, Lemma 3.1]), we obtain

$$\begin{aligned}\psi^E_{i,j}(x,y) &= \mathbb{E}\left[B_E(x)B_E(y)\right]\mathbb{E}\left[\tilde{\partial}_i B_E(x)\tilde{\partial}_j B_E(y)\right] + \mathbb{E}\left[B_E(x)\tilde{\partial}_i B_E(x)\right]\mathbb{E}\left[B_E(y)\tilde{\partial}_j B_E(y)\right] \\ &\quad + \mathbb{E}\left[B_E(x)\tilde{\partial}_j B_E(y)\right]\mathbb{E}\left[B_E(y)\tilde{\partial}_i B_E(x)\right] \\ &= r^E(x-y)\tilde{r}^E_{i,j}(x-y) - \tilde{r}^E_{0,j}(x-y)\tilde{r}^E_{0,i}(x-y), \end{aligned} \tag{IV.2.3}$$

where the second term is equal to zero by independence of $B_E(x)$ and $\nabla B_E(x)$ for every fixed $x \in \mathbb{R}^2$ and we set

$$\tilde{r}^E_{i,j}(x-y) := \tilde{\partial}_{x_i}\tilde{\partial}_{y_i} r^E(x-y), \quad i,j = 0,1,2$$

where r^E is as in (IV.1.1) and we adopt the convention that $\tilde{\partial}_0$ is the identity operator. We now restrict (IV.2.2) to the case where $C_i = S_i, i = 1,2$ are straight line segments. Denoting by

$$B^{(\theta)}_E(x) := B_E(R_\theta x), \qquad \hat{B}^{(L)}_E(x) := B_E(x+L), \quad x \in \mathbb{R}^2, \theta \in [0,2\pi], L \in \mathbb{R}^2$$

where $R_\theta \in \mathcal{M}_{2\times 2}(\mathbb{R})$ stands for the rotation matrix associated with angle θ, it follows by isotropy and stationarity of Berry's random field that $B_E \overset{d}{=} B^{(\theta)}_E$ and $B_E \overset{d}{=} \hat{B}^{(L)}_E$, where $\overset{d}{=}$ denotes equality in distribution of random fields. These observations imply that for every choice of $x, y \in \mathbb{R}^2$, the pairs $(B^{(\theta)}_E(x), B^{(\theta)}_E(y))$ and $(\hat{B}^{(L)}_E(x), \hat{B}^{(L)}_E(y))$ have the same distribution as $(B_E(x), B_E(y))$. As a consequence, we reduce our investigations to line segments given by the unit speed parametrizations

$$\gamma_1 : [0,\lambda_1] \to S_1, \quad t \mapsto \gamma_1(t) := te_1 \tag{IV.2.4}$$

$$\gamma_2 : [0,\lambda_2] \to S_2, \quad t \mapsto \gamma_2(t) := p + t\rho(\theta) \tag{IV.2.5}$$

where $\lambda_i > 0, i = 1,2$, e_i is the i-th canonical basis vector of $\mathbb{R}^2$, $p = (p_1, p_2) \in \mathbb{R}^2$ and $\rho(\theta) := (\cos\theta, \sin\theta)$ for $\theta \in [0,2\pi]$. In particular, for $i = 1,2$, S_i is a line segment of length λ_i.

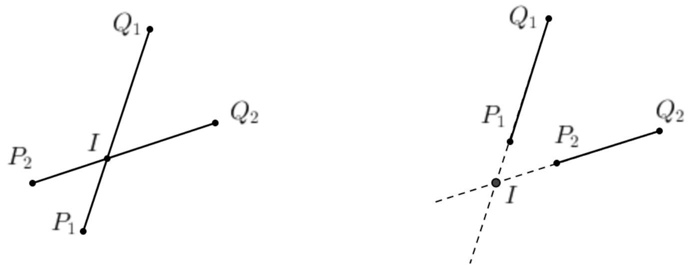
Remark IV.2.2. In view of the linearity property of ϕ_E in (IV.1.10), it follows that whenever S is a line segment given by a union of line segments $S = S' \cup S''$ sharing only one point, then we have $\phi_E(S) = \phi_E(S') + \phi_E(S'')$. It follows that, one can always express the covariance associated with arbitrary line segments as a linear combination involving only covariances associated with line segments that have the same origin. This implies that, in (IV.2.5), we can consistently reduce to the case $p = (0,0)$, that is when S_1 and S_2 have the same origin, except when S_1 and S_2 are parallel but disjoint. Indeed, in order to see this, let us assume that $S_1 = [P_1, Q_1]$ and $S_2 = [P_2, Q_2]$ for points $P_1, P_2, Q_1, Q_2 \in \mathbb{R}^2$ such that $p = P_2 \neq (0,0)$. (Here for $A, B \in \mathbb{R}^2$, we use the notation $[A,B]$ to indicate the line segment joining A and B.) Denote by ℓ_1 and ℓ_2 the lines directed by S_1 and S_2 respectively and let $I = \ell_1 \cap \ell_2$. If $S_1 \cap S_2 = \{I\}$, we consider the four line segments $[P_2, I], [I, Q_2], [P_1, I]$ and $[I, Q_1]$. By linearity, we thus have $\phi_E(S_1) = \phi_E([P_1, I]) + \phi_E([I, Q_1])$ and $\phi_E(S_2) = \phi_E([P_2, I]) + \phi_E([I, Q_2])$, so that

$$\begin{aligned}\mathbf{Cov}[\phi_E(S_1), \phi_E(S_2)] &= \mathbf{Cov}[\phi_E([P_1,I]), \phi_E([P_2,I])] + \mathbf{Cov}[\phi_E([P_1,I]), \phi_E([I,Q_2])] \\ &\quad + \mathbf{Cov}[\phi_E([I,Q_1]), \phi_E([P_2,I])] + \mathbf{Cov}[\phi_E([I,Q_1]), \phi_E([I,Q_2])]\end{aligned}$$

and each of these covariances contains only line segments with common point I, which one can set to be the origin by translation invariance of the Berry random field. Similarly, if $S_1 \cap S_2 = \emptyset$, we consider the line segments $[I, P_1]$ and $[I, P_2]$. Then, again by linearity we can write (up to sign, which is determined by the orientation of S_1) $\phi_E([I,Q_1]) - \phi_E([I,P_1]) = \phi_E(S_1)$ and $\phi_E([I,Q_2]) - \phi_E([I,P_2]) = \phi_E(S_2)$, so that

$$\begin{aligned}\mathbf{Cov}[\phi_E(S_1), \phi_E(S_2)] &= \mathbf{Cov}[\phi_E([I,Q_1]), \phi_E([I,Q_2])] - \mathbf{Cov}[\phi_E([I,Q_1]), \phi_E([I,P_2])] \\ &\quad - \mathbf{Cov}[\phi_E([I,P_1]), \phi_E([I,Q_2])] + \mathbf{Cov}[\phi_E([I,P_1]), \phi_E([I,P_2])],\end{aligned}$$

and the covariances on the right hand side can be dealt with setting $I = (0,0)$ as before.



In view of the above reductions, throughout this section, we will assume that S_1 and S_2 are parametrized as in (IV.2.4) and (IV.2.5), respectively with $p = (0,0)$ and $\theta \in [0, 2\pi]$. The fact that $\mathrm{n}_{S_1}(x) = e_2$ for every $x \in S_1$ and $\mathrm{n}_{S_2}(x) = \rho(\theta)^\perp = (-\sin\theta, \cos\theta)$ for every $x \in S_2$, together with the integral transform (IV.1.9) yields

$$\mathbf{Cov}[\phi_E(S_1), \phi_E(S_2)] = \frac{1}{64}\int_{S_1\times S_2} [\psi^E_{2,2}(x,y)\cos\theta - \psi^E_{2,1}(x,y)\sin\theta]dxdy$$
$$= \frac{1}{64}\int_0^{\lambda_1} dt \int_0^{\lambda_2} ds[\psi^E_{2,2}(\gamma_1(t),\gamma_2(s))\cos\theta - \psi^E_{2,1}(\gamma_1(t),\gamma_2(s))\sin\theta], \qquad \text{(IV.2.6)}$$

where $\psi^E_{i,j}$ is as in (IV.2.3). From the parametrizations in (IV.2.4) and (IV.2.5), it follows that

$$\|\gamma_1(t) - \gamma_2(s)\|^2 = \|te_1 - s\rho(\theta)\|^2 = t^2 + s^2 - 2st\langle e_1, \rho(\theta)\rangle = t^2 + s^2 - 2st\cos\theta. \qquad \text{(IV.2.7)}$$

Now, computations based on the explicit expressions of the functions $r^E, \tilde{r}^E_{i,j}$ for $i, j = 0, 1, 2$ in terms of Bessel functions (see [NPR19, Lemma 3.1]), lead to (for $\gamma_1(t) \neq \gamma_2(s)$)

$$\begin{aligned}\psi^E_{2,2}(\gamma_1(t),\gamma_2(s)) &= J_0(\tau^E(t,s))\left(J_0(\tau^E(t,s)) + J_2(\tau^E(t,s))\right)\\ &\quad -2\frac{s^2\sin^2\theta}{\|\gamma_1(t)-\gamma_2(s)\|^2}\left(J_0(\tau^E(t,s))J_2(\tau^E(t,s)) + J_1(\tau^E(t,s))^2\right)\end{aligned}$$

and

$$\psi^E_{2,1}(\gamma_1(t),\gamma_2(s)) = 2\frac{(t - s\cos\theta)s\sin\theta}{\|\gamma_1(t)-\gamma_2(s)\|^2}\left(J_0(\tau^E(t,s))J_2(\tau^E(t,s)) + J_1(\tau^E(t,s))^2\right)$$

where we set $\tau^E(t,s) := 2\pi\sqrt{E}\|\gamma_1(t) - \gamma_2(s)\|$. As a consequence, by (IV.2.7), we have that

$$\begin{aligned}&\psi^E_{2,2}(\gamma_1(t),\gamma_2(s))\cos\theta - \psi^E_{2,1}(\gamma_1(t),\gamma_2(s))\sin\theta\\ = \;&\cos\theta J_0(\tau^E(t,s))\left(J_0(\tau^E(t,s)) + J_2(\tau^E(t,s))\right)\\ &-\left(\frac{2s^2\sin^2\theta\cos\theta + 2(t - s\cos\theta)s\sin^2\theta}{\|\gamma_1(t)-\gamma_2(s)\|^2}\right)\left(J_0(\tau^E(t,s))J_2(\tau^E(t,s)) + J_1(\tau^E(t,s))^2\right)\\ = \;&\cos\theta J_0(\tau^E(t,s))\left(J_0(\tau^E(t,s)) + J_2(\tau^E(t,s))\right)\\ &-\frac{2ts\sin^2\theta}{t^2 + s^2 - 2st\cos\theta}\left(J_0(\tau^E(t,s))J_2(\tau^E(t,s)) + J_1(\tau^E(t,s))^2\right)\\ = \;&2\cos\theta\frac{J_0(\tau^E(t,s))J_1(\tau^E(t,s))}{\tau^E(t,s))} - \frac{2ts\sin^2\theta}{t^2 + s^2 - 2st\cos\theta}\left(J_0(\tau^E(t,s))J_2(\tau^E(t,s)) + J_1(\tau^E(t,s))^2\right),\end{aligned}$$

where in the last line, we exploited the recurrence relation $J_{n+1}(x) + J_{n-1}(x) = 2nJ_n(x)/x, n > 0, x \in \mathbb{R}$ (see e.g. [Sze39, Equation (1.71.5)]) implying the useful identity

$$J_0(x) + J_2(x) = 2\frac{J_1(x)}{x}. \qquad \text{(IV.2.8)}$$

Inserting this expression into (IV.2.6), we obtain that

$$\begin{aligned}\mathbf{Cov}[\phi_E(S_1), \phi_E(S_2)] &= \frac{2\cos\theta}{64}\int_0^{\lambda_1} dt \int_0^{\lambda_2} ds \frac{J_0(\tau^E(t,s))J_1(\tau^E(t,s))}{\tau^E(t,s)} \\ &\quad -\frac{2\sin^2\theta}{64}\int_0^{\lambda_1} dt \int_0^{\lambda_2} ds \frac{ts\left(J_0(\tau^E(t,s))J_2(\tau^E(t,s)) + J_1(\tau^E(t,s))^2\right)}{t^2+s^2-2st\cos\theta} \\ &=: A_E(\lambda_1,\lambda_2,\theta) + B_E(\lambda_1,\lambda_2,\theta),\end{aligned}$$

where

$$A_E(\lambda_1,\lambda_2,\theta) := \frac{\cos\theta}{32}\int_0^{\lambda_1} dt \int_0^{\lambda_2} ds \frac{J_0(\tau^E(t,s))J_1(\tau^E(t,s))}{\tau^E(t,s)} \tag{IV.2.9}$$

$$B_E(\lambda_1,\lambda_2,\theta) := -\frac{\sin^2\theta}{32}\int_0^{\lambda_1} dt \int_0^{\lambda_2} ds \frac{ts\left(J_0(\tau^E(t,s))J_2(\tau^E(t,s)) + J_1(\tau^E(t,s))^2\right)}{t^2+s^2-2st\cos\theta} \tag{IV.2.10}$$

and where we recall the notation $\tau^E(t,s) = 2\pi\sqrt{E}\sqrt{t^2+s^2-2st\cos\theta}$. We note that if $\theta \in \{0,\pi\}$, then $B_E(\lambda_1,\lambda_2,\theta) = 0$, so that for parallel line segments we immediately deduce that

$$\mathbf{Cov}[\phi_E(S_1), \phi_E(S_2)] = A_E(\lambda_1,\lambda_2,\theta).$$

We are now in position to prove Proposition IV.2.1.

Proof of Proposition IV.2.1. Throughout the proof, we can and will assume without loss of generality, that S_1 and S_2 are both oriented in the same way. Indeed, if S is a line segment, then $-S$ is the same line segment with opposite orientation to S and therefore $\phi_E(-S) = -\phi_E(S)$.

In order to prove the statement, we distinguish two cases: *(A)* S_1 and S_2 are parallel, and *(B)* S_1 and S_2 are not parallel.

Case (A): We treat the case where S_1 and S_2 are parallel line segments. Let $\gamma_1 : t \in [a,b] \mapsto te_1$ and $\gamma_2 : t \in [c,d] \mapsto te_1 + Le_2$ where $0 \le a < b, 0 \le c < d$ and $L \ge 0$ are fixed real numbers be the respective parametrizations of S_1 and S_2. Note that the case $L = 0$ corresponds to the configuration where S_1 and S_2 are supported by the same line, whereas, the case $L > 0$ corresponds to the case where S_1 and S_2 are supported by parallel distinct lines. We have that $\|\gamma_1(t) - \gamma_2(s)\|^2 = \|(t,0) - (s,L)\|^2 = (t-s)^2 + L^2$. Therefore, performing the linear change of variables $(u,v) = (t, t-s)$, we infer

$$\begin{aligned}&\mathbf{Cov}[\phi_E(S_1), \phi_E(S_2)] \\ =\ & \frac{1}{32}\int_a^b dt \int_c^d ds \frac{J_0\left(2\pi\sqrt{E}\sqrt{(t-s)^2+L^2}\right) J_1\left(2\pi\sqrt{E}\sqrt{(t-s)^2+L^2}\right)}{2\pi\sqrt{E}\sqrt{(t-s)^2+L^2}} \\ =\ & \frac{1}{32}\int_a^b du \int_{u-d}^{u-c} dv \frac{J_0\left(2\pi\sqrt{E}\sqrt{v^2+L^2}\right) J_1\left(2\pi\sqrt{E}\sqrt{v^2+L^2}\right)}{2\pi\sqrt{E}\sqrt{v^2+L^2}} \\ =\ & \frac{1}{32}\int_a^b du \frac{1}{2\pi\sqrt{E}}\int_{2\pi\sqrt{E}(u-d)}^{2\pi\sqrt{E}(u-c)} dv \frac{J_0\left(\sqrt{v^2+(2\pi\sqrt{E}L)^2}\right) J_1\left(\sqrt{v^2+(2\pi\sqrt{E}L)^2}\right)}{\sqrt{v^2+(2\pi\sqrt{E}L)^2}} \\ =:\ & \frac{1}{32}\int_a^b du K^E(u; L, c, d),\end{aligned} \tag{IV.2.11}$$

where we set

$$K^E(u;L,c,d) := \frac{1}{2\pi\sqrt{E}}\int_{2\pi\sqrt{E}(u-d)}^{2\pi\sqrt{E}(u-c)} dv \frac{J_0\left(\sqrt{v^2+(2\pi^2\sqrt{E}L)^2}\right)J_1\left(\sqrt{v^2+(2\pi\sqrt{E}L)^2}\right)}{\sqrt{v^2+(2\pi\sqrt{E}L)^2}}. \quad \text{(IV.2.12)}$$

Note that, using (IV.2.8) and the bound $|J_\nu(x)| \le 1$, $\nu = 0,1,2$ implies that

$$\left|\frac{J_0(x)J_1(x)}{x}\right| = \frac{1}{2}\left|J_0(x)\,(J_0(x)+J_2(x))\right| \le 1$$

so that

$$\left|K^E(u;L,c,d)\right| \le \frac{1}{2\pi\sqrt{E}}\int_{2\pi\sqrt{E}(u-d)}^{2\pi\sqrt{E}(u-c)} dv = d-c,$$

for every $u \in (a,b)$ and every $c < d$, so that $K^E(u;L,c,d)$ is integrable on (a,b). We now study the two cases $L > 0$ and $L = 0$ separately.

<u>*Case (A.1):* $L > 0$.</u> Fix $L > 0$. We show that $K^E(u;L,c,d) = o(1/\sqrt{E})$ as $E \to \infty$ uniformly for $u \in (a,b)$. Indeed, using the fact that $|J_\nu(x)| = O(x^{-1/2})$ for $x > 0$ and $\nu = 0,1,2$, we infer from (IV.2.12) that for every $u \in (a,b)$,

$$\sqrt{E}\left|K^E(u;L,c,d)\right| \le \frac{O(1)}{2\pi}\int_{2\pi\sqrt{E}(u-d)}^{2\pi\sqrt{E}(u-c)} dv \frac{1}{v^2+(2\pi\sqrt{E}L)^2} \le \frac{O(1)}{2\pi}\int_{2\pi\sqrt{E}(u-d)}^{2\pi\sqrt{E}(u-c)} dv\frac{1}{E} = O(E^{-1/2}),$$

where the constant involved in the 'big-O' notation does not depend on u. Thus $\sqrt{E}K^E(u;L,c,d) \to 0$ as $E \to \infty$ uniformly on $[a,b]$, and therefore we infer from (IV.2.11)

$$\mathbf{Cov}[\phi_E(S_1),\phi_E(S_2)] = O\left(\frac{1}{E}\right)$$

as $E \to \infty$, which gives the desired conclusion.

<u>*Case (A.2):* $L = 0$.</u> Setting $L = 0$ in (IV.2.12), we obtain

$$K^E(u;0,c,d) = \frac{1}{2\pi\sqrt{E}}\int_{2\pi\sqrt{E}(u-d)}^{2\pi\sqrt{E}(u-c)} dv\frac{J_0(|v|)J_1(|v|)}{|v|}.$$

In order to show (IV.2.1), we treat the two cases (i) $[a,b]\cap[c,d] = \emptyset$ and (ii) $[a,b]\cap[c,d] \neq \emptyset$. We start by case (i). This is the case when $a < b < c < d$ or $c < d < a < b$. We only treat the case $a < b < c < d$ as the other case is dealt with similarly. The assumption $a < b < c < d$ implies that $u - c < 0$ and $u - d < 0$ for every $u \in (a,b)$. Then, using the fact that $|J_\nu(x)| = O(x^{-1/2})$ for $x > 0$, we have

$$\begin{aligned}\sqrt{E}\left|K^E(u;0,c,d)\right| &\le \frac{1}{2\pi}\int_{2\pi\sqrt{E}(u-d)}^{2\pi\sqrt{E}(u-c)} dv\frac{|J_0(|v|)J_1(|v|)|}{|v|}\\ &= \frac{O(1)}{2\pi}\int_{2\pi\sqrt{E}(u-d)}^{2\pi\sqrt{E}(u-c)}\frac{dv}{v^2} = O\left(\frac{1}{\sqrt{E}}\right)\left(\frac{1}{c-u}-\frac{1}{d-u}\right),\end{aligned}$$

which goes to zero as $E \to \infty$, uniformly for $u \in (a,b)$. Thus, from (IV.2.11) it follows that in this case

$$\mathbf{Cov}[\phi_E(S_1),\phi_E(S_2)] = O\left(\frac{1}{E}\right),$$

which implies (IV.2.1).
We now study the case (ii) and start with the special case $(c,d)=(a,b)$, that is when $S_1=S_2$. From (IV.2.11), we write

$$\sqrt{E}\mathbf{Cov}[\phi(S_1),\phi(S_1)]=\frac{1}{32}\int_a^b du\,\sqrt{E}K^E(u;0,a,b)$$

with

$$\sqrt{E}K^E(u;0,a,b)=\frac{1}{2\pi}\int_{2\pi\sqrt{E}(u-b)}^{2\pi\sqrt{E}(u-a)}dv\frac{J_0(|v|)J_1(|v|)}{|v|}=\frac{1}{2\pi}\int_{2\pi\sqrt{E}(u-b)}^{2\pi\sqrt{E}(u-a)}dv\frac{J_0(v)J_1(v)}{v},$$

where we used that J_0 is even and J_1 is odd. Now computations based on differentiation of Bessel functions imply that $\frac{d}{dv}[v(J_0(v)^2+J_1(v)^2)-J_0(v)J_1(v)]=J_0(v)J_1(v)/v$, so that

$$\sqrt{E}K^E(u;0,a,b)=\frac{1}{2\pi}\left[v(J_0(v)^2+J_1(v)^2)-J_0(v)J_1(v)\right]_{2\pi\sqrt{E}(u-b)}^{2\pi\sqrt{E}(u-a)}$$

and therefore

$$\begin{aligned}\sqrt{E}\mathbf{Cov}[\phi_E(S_1),\phi_E(S_1)] &= \frac{1}{64\pi}\int_a^b du\left[v(J_0(v)^2+J_1(v)^2)\right]_{2\pi\sqrt{E}(u-b)}^{2\pi\sqrt{E}(u-a)}\\ &\quad -\frac{1}{64\pi}\int_a^b du\,[J_0(v)J_1(v)]_{2\pi\sqrt{E}(u-b)}^{2\pi\sqrt{E}(u-a)}.\end{aligned} \tag{IV.2.13}$$

For the first term, we use the dominated convergence theorem: since $|J_\nu(x)|\le C_\nu x^{-1/2}$, $x>0$ for some constant $C_\nu>0$, it follows that

$$\left|\left[v(J_0(v)^2+J_1(v)^2)\right]_{2\pi\sqrt{E}(u-b)}^{2\pi\sqrt{E}(u-a)}\right|\le 2[C_0^2+C_1^2]$$

which is integrable on (a,b). Setting $\mathfrak{f}(v):=v(J_0(v)^2+J_1(v)^2)$, we have $\mathfrak{f}(-v)=-\mathfrak{f}(v)$ and

$$\left|\left[v(J_0(v)^2+J_1(v)^2)\right]_{2\pi\sqrt{E}(u-b)}^{2\pi\sqrt{E}(u-a)}\right|=\mathfrak{f}(2\pi\sqrt{E}(u-a))-\mathfrak{f}(2\pi\sqrt{E}(u-b)),$$

so that

$$\lim_{E\to\infty}\left|\left[v(J_0(v)^2+J_1(v)^2)\right]_{2\pi\sqrt{E}(u-b)}^{2\pi\sqrt{E}(u-a)}\right|=\lim_{E\to\infty}2\mathfrak{f}(2\pi\sqrt{E}(u-a))=\lim_{y\to\infty}2\mathfrak{f}(y)$$

since $u-a>0$. Now, the asymptotic expansion of Bessel functions (see for instance [Kra14])

$$J_\nu(y)=\sqrt{\frac{2}{\pi y}}\cos(y-\omega_\nu)+O(y^{-3/2}),\quad \omega_\nu:=(2\nu+1)\frac{\pi}{4},\quad y\to\infty \tag{IV.2.14}$$

yield

$$2\mathfrak{f}(y)\sim 2y\frac{2}{\pi y}\left[\cos\left(x-\frac{\pi}{4}\right)^2+\cos\left(x-\frac{3\pi}{4}\right)^2\right]=2y\frac{2}{\pi y}\left[\cos\left(x-\frac{\pi}{4}\right)^2+\sin\left(x-\frac{\pi}{4}\right)^2\right]=\frac{4}{\pi}$$

as $y\to\infty$. Thus, by dominated convergence, we obtain

$$\frac{1}{64\pi}\int_a^b du\left[v(J_0(v)^2+J_1(v)^2)\right]_{2\pi\sqrt{E}(u-b)}^{2\pi\sqrt{E}(u-a)}\to\frac{1}{64\pi}\int_a^b\frac{4}{\pi}du=\frac{1}{16\pi^2}(b-a)$$

as $E \to \infty$. For the remainder term in (IV.2.13), we use the bound $|J_0(x)| \le C_0 x^{-1/2}, x \ge 0$ and $|J_1(x)| \le 1$ to obtain

$$\begin{aligned}\int_a^b du\,[J_0(v)J_1(v)]_{2\pi\sqrt{E}(u-b)}^{2\pi\sqrt{E}(u-a)} &\le \int_a^b \left|J_0(2\pi\sqrt{E}(u-a))\right|\left|J_1(2\pi\sqrt{E}(u-a))\right| \\ &\quad + \left|J_0(2\pi\sqrt{E}(b-u))\right|\left|J_1(2\pi\sqrt{E}(b-u))\right| du \\ &\le c\int_a^b \frac{1}{E^{1/4}\sqrt{u-a}} + \frac{1}{E^{1/4}\sqrt{b-u}} \le c\frac{1}{E^{1/4}}\sqrt{b-a}\end{aligned}$$

for some constant $c > 0$. This proves that

$$\mathbf{Cov}[\phi_E(S_1), \phi_E(S_1)] = \frac{1}{16\pi^2}\frac{b-a}{\sqrt{E}} + o\left(\frac{1}{\sqrt{E}}\right).$$

Let us now assume that $S_1 \neq S_2$ but $S_1 \cap S_2 \neq \emptyset$. Without loss of generality, assume that $0 < a < c \le b < d$, that is $S_1 \cap S_2 = [c, b] \times \{0\}$. Exploiting the linearity of ϕ_E, we write

$$\phi_E(S_2) = \phi_E([c, b] \times \{0\}) + \phi_E([b, d] \times \{0\})$$

and use the previous observations to obtain

$$\begin{aligned} & \mathbf{Cov}[\phi_E(S_1), \phi_E(S_2)] \\ = \; & \mathbf{Cov}[\phi_E([a, c] \times \{0\}), \phi_E([c, b] \times \{0\})] + \mathbf{Cov}[\phi_E([a, c] \times \{0\}), \phi_E([b, d] \times \{0\})] \\ & + \mathbf{Cov}[\phi_E([c, b] \times \{0\}), \phi_E([c, b] \times \{0\})] + \mathbf{Cov}[\phi_E([c, b] \times \{0\}), \phi_E([b, d] \times \{0\})] \\ = \; & \mathbf{Cov}[\phi_E([c, b] \times \{0\}), \phi_E([c, b] \times \{0\})] + o\left(\frac{1}{\sqrt{E}}\right) \\ = \; & \frac{\lambda(S_1, S_2)}{16\pi^2\sqrt{E}} + o\left(\frac{1}{\sqrt{E}}\right), \end{aligned}$$

which is (IV.2.1).

Case (B): We now treat the case where the line segments are not parallel. We will use the parametrizations (IV.2.4) and (IV.2.5) of S_1 and S_2 respectively with

$$p = (0, 0), \quad \theta \in [0, 2\pi) \setminus \{0, \pi\}.$$

Moreover, in this case

$$\mathbf{Cov}[\phi_E(S_1), \phi_E(S_2)] = A_E(\lambda_1, \lambda_2, \theta) + B_E(\lambda_1, \lambda_2, \theta) \tag{IV.2.15}$$

where $A_E(\lambda_1, \lambda_2, \theta)$ and $B_E(\lambda_1, \lambda_2, \theta)$ are given in (IV.2.9) and (IV.2.10), respectively. We show that both the contributions of $A_E(\lambda_1, \lambda_2, \theta)$ and $B_E(\lambda_1, \lambda_2, \theta)$ to the covariance are of order $o(E^{-1/2})$ in the high-energy regime. By (IV.2.8), we can write

$$A_E(\lambda_1, \lambda_2, \theta) = \frac{\cos\theta}{64}\int_0^{\lambda_1} dt \int_0^{\lambda_2} ds J_0(\tau^E(t, s))\left(J_0(\tau^E(t, s)) + J_2(\tau^E(t, s))\right)$$

where we recall $\tau^E(t, s) = 2\pi\sqrt{E}\sqrt{t^2 + s^2 - 2st\cos\theta}$. Passing to polar coordinates $(t, s) = (\rho\cos\phi, \rho\sin\phi)$, we have

$$\tau^E(\rho\cos\phi, \rho\sin\phi) = 2\pi\sqrt{E}\sqrt{\rho^2 - 2\rho^2\sin\phi\cos\phi\cos\theta} = 2\pi\sqrt{E}\rho\sqrt{1 - \sin(2\phi)\cos\theta}$$

$$=: \quad \tilde{\tau}^E(\rho,\phi). \tag{IV.2.16}$$

We note that $\tilde{\tau}^E(\rho,\phi) > 0$ and $\cos\theta \neq 0$ for $\theta \in [0,2\pi) \setminus \{0,\pi\}$. Using polar coordinates (ρ,ϕ) on rectangle $[0,\lambda_1]\times[0,\lambda_2]$ and the fact that the line joining the origin and the point (λ_1,λ_2) forms an angle of $\arctan(\lambda_2/\lambda_1)$ shows that the range of integration is parametrized according to

$$\int_0^{\lambda_1} dt \int_0^{\lambda_2} ds = \int_0^{\alpha_{1,2}} d\phi \int_0^{\lambda_1/\cos\phi} \rho d\rho + \int_{\alpha_{1,2}}^{\pi/2} d\phi \int_0^{\lambda_2/\sin\phi} \rho d\rho, \tag{IV.2.17}$$

where we set $\alpha_{1,2} := \arctan(\lambda_2/\lambda_1) \in (0,\pi/2)$. We split the integral

$$\begin{aligned} A_E(\lambda_1,\lambda_2,\theta) &= \frac{\cos\theta}{64}\int_0^{\lambda_1} dt \int_0^{\lambda_2} ds J_0(\tau^E(t,s))\left(J_0(\tau^E(t,s)) + J_2(\tau^E(t,s))\right) \\ = &\frac{\cos\theta}{64}\int_0^{\alpha_{1,2}} d\phi \int_0^{\lambda_1/\cos\phi} \rho d\rho J_0(\tilde{\tau}^E(\rho,\phi))\left(J_0(\tilde{\tau}^E(\rho,\phi)) + J_2(\tilde{\tau}^E(\rho,\phi))\right) \\ &+\frac{\cos\theta}{64}\int_{\alpha_{1,2}}^{\pi/2} d\phi \int_0^{\lambda_2/\sin\phi} \rho d\rho J_0(\tilde{\tau}^E(\rho,\phi))\left(J_0(\tilde{\tau}^E(\rho,\phi)) + J_2(\tilde{\tau}^E(\rho,\phi))\right) \\ =: &\ A_{E,1}(\lambda_1,\lambda_2,\theta) + A_{E,2}(\lambda_1,\lambda_2,\theta). \end{aligned}$$

We focus on the term $A_{E,1}(\lambda_1,\lambda_2,\theta)$. For fixed $\phi \in (0,\alpha_{1,2})$, we perform the change of variable $\psi = \tilde{\tau}^E(\rho,\phi)$ with $d\psi = \tilde{\tau}^E(1,\phi)d\rho$, yielding

$$\begin{aligned} A_{E,1}(\lambda_1,\lambda_2,\theta) &= \frac{\cos\theta}{64}\int_0^{\alpha_{1,2}} \frac{d\phi}{(\tilde{\tau}^E(1,\phi))^2}\int_0^{\frac{\tilde{\tau}^E(1,\phi)\lambda_1}{\cos\phi}} d\psi\psi J_0(\psi)\left(J_0(\psi)+J_2(\psi)\right) \\ &=: \frac{\cos\theta}{64}\int_0^{\alpha_{1,2}} d\phi K^E(\phi;\lambda_1,\theta), \end{aligned} \tag{IV.2.18}$$

with

$$\begin{aligned} K^E(\phi;\lambda_1,\theta) &= \frac{1}{(\tilde{\tau}^E(1,\phi))^2}\int_0^{\frac{\tilde{\tau}^E(1,\phi)\lambda_1}{\cos\phi}} d\psi\psi J_0(\psi)\left(J_0(\psi)+J_2(\psi)\right) \\ = \frac{2}{(\tilde{\tau}^E(1,\phi))^2}&\int_0^{\frac{\tilde{\tau}^E(1,\phi)\lambda_1}{\cos\phi}} d\psi J_0(\psi)J_1(\psi) = \frac{2}{(\tilde{\tau}^E(1,\phi))^2}\left[-\frac{J_0(\psi)^2}{2}\right]_0^{\frac{\tilde{\tau}^E(1,\phi)\lambda_1}{\cos\phi}}, \end{aligned} \tag{IV.2.19}$$

where we used (IV.2.8) and the fact that $\frac{d}{d\psi}J_0(\psi) = -J_1(\psi)$. Thus, it follows that (since $|J_0(x)| \leq 1$)

$$\left|K^E(\phi;\lambda_1,\theta)\right| \leq \frac{1}{(\tilde{\tau}^E(1,\phi))^2}\left|J_0\left(\frac{\tilde{\tau}^E(1,\phi)\lambda_1}{\cos\phi}\right)^2 - J_0(0)^2\right| \leq \frac{2}{(\tilde{\tau}^E(1,\phi))^2},$$

so that by (IV.2.18)

$$|A_{E,1}(\lambda_1,\lambda_2,\theta)| \leq \int_0^{\alpha_{1,2}} d\phi \frac{2}{(\tilde{\tau}^E(1,\phi))^2} = \frac{2}{4\pi^2 E}\int_0^{\alpha_{1,2}} \frac{d\phi}{1-\sin(2\phi)\cos(\theta)} = O(E^{-1})$$

where the last upper bound is justified by the reverse triangular inequality $|x-y| \geq ||x|-|y||$, the assumption $|\cos\theta| \neq 1$ and

$$\int_0^{\alpha_{1,2}} \frac{d\phi}{|1-\sin(2\phi)\cos\theta|} \leq \int_0^{\alpha_{1,2}} \frac{d\phi}{1-|\sin(2\phi)||\cos\theta|} \leq \frac{\alpha_{1,2}}{1-|\cos\theta|} < \infty. \tag{IV.2.20}$$

Arguing similarly for the term $A_{E,2}(\lambda_1, \lambda_2, \theta)$, we obtain that $|A_{E,2}(\lambda_1, \lambda_2, \theta)| = O(E^{-1})$, so that $|A_E(\lambda_1, \lambda_2, \theta)| = O(E^{-1})$ as $E \to \infty$. We now treat the term $B_E(\lambda_1, \lambda_2, \theta)$. From (IV.2.10), we have

$$B_E(\lambda_1, \lambda_2, \theta) := -\frac{\sin^2\theta}{32}\int_0^{\lambda_1} dt \int_0^{\lambda_2} ds \frac{ts\left(J_0(\tau^E(t,s))J_2(\tau^E(t,s)) + J_1(\tau^E(t,s))^2\right)}{t^2+s^2-2st\cos\theta}$$

Passing to polar coordinates and using (IV.2.17), we write

$$B_E(\lambda_1, \lambda_2, \theta) = B_{E,1}(\lambda_1, \lambda_2, \theta) + B_{E,2}(\lambda_1, \lambda_2, \theta),$$

where

$$B_{E,1}(\lambda_1, \lambda_2, \theta) := -\frac{\sin^2\theta}{32}\int_0^{\alpha_{1,2}} \frac{d\phi}{(\tilde{\tau}^E(1,\phi))^2}\frac{\sin\phi\cos\phi}{1-\sin(2\phi)\cos\theta}\int_0^{\frac{\tilde{\tau}^E(1,\phi)\lambda_1}{\cos\phi}} d\psi\psi\left(J_0(\psi)J_2(\psi)+J_1(\psi)^2\right),$$

and

$$B_{E,2}(\lambda_1, \lambda_2, \theta) := -\frac{\sin^2\theta}{32}\int_{\alpha_{1,2}}^{\pi/2} \frac{d\phi}{(\tilde{\tau}^E(1,\phi))^2}\frac{\sin\phi\cos\phi}{1-\sin(2\phi)\cos\theta}\int_0^{\frac{\tilde{\tau}^E(1,\phi)\lambda_2}{\sin\phi}} d\psi\psi\left(J_0(\psi)J_2(\psi)+J_1(\psi)^2\right).$$

We treat the first term $B_{E,1}(\lambda_1, \lambda_2, \theta)$. We write

$$B_{E,1}(\lambda_1, \lambda_2, \theta) := -\frac{\sin^2\theta}{32}\int_0^{\alpha_{1,2}} d\phi M^E(\phi; \lambda_1, \theta), \tag{IV.2.21}$$

where

$$M^E(\phi;\lambda_1,\theta) := \frac{1}{(\tilde{\tau}^E(1,\phi))^2}\frac{\sin\phi\cos\phi}{1-\sin(2\phi)\cos\theta}\int_0^{\frac{\tilde{\tau}^E(1,\phi)\lambda_1}{\cos\phi}} d\psi\psi\left(J_0(\psi)J_2(\psi)+J_1(\psi)^2\right).$$

Using the asymptotic expansion of Bessel functions in (IV.2.14) yields

$$\psi\left(J_0(\psi)J_2(\psi)+J_1(\psi)^2\right) = \frac{2}{\pi}\cos\left(2\psi+\frac{\pi}{2}\right) + O(\psi^{-1}) = \frac{2}{\pi}\sin(2\psi) + O(\psi^{-1})$$

as $\psi \to \infty$, so that for large E

$$\begin{aligned}
&\int_0^{\frac{\tilde{\tau}^E(1,\phi)\lambda_2}{\cos\phi}} d\psi\psi\left(J_0(\psi)J_2(\psi)+J_1(\psi)^2\right)\\
&= \int_0^1 d\psi\psi\left(J_0(\psi)J_2(\psi)+J_1(\psi)^2\right) + \int_1^{\frac{\tilde{\tau}^E(1,\phi)\lambda_2}{\cos\phi}} d\psi\psi\left(J_0(\psi)J_2(\psi)+J_1(\psi)^2\right)\\
&= O(1) + \frac{2}{\pi}\int_1^{\frac{\tilde{\tau}^E(1,\phi)\lambda_2}{\cos\phi}} d\psi\left(\sin(2\psi)+O(\psi^{-1})\right)\\
&= \frac{1}{\pi}\cos\left(\frac{2\tilde{\tau}^E(1,\phi)\lambda_2}{\cos\phi}\right) + O(1)\left(1+\log\left(\frac{\tilde{\tau}^E(1,\phi)\lambda_2}{\cos\phi}\right)\right)\\
&= O(1)\left(1+\log\left(\frac{\tilde{\tau}^E(1,\phi)\lambda_2}{\cos\phi}\right)\right).
\end{aligned}$$

Therefore, we conclude by (IV.2.21)

$$B_{E,1}(\lambda_1, \lambda_2, \theta) = -\frac{\sin^2\theta}{32}\int_0^{\alpha_{1,2}} \frac{d\phi}{(\tilde{\tau}^E(1,\phi))^2}\frac{\sin\phi\cos\phi}{1-\sin(2\phi)\cos\theta}\left\{O(1)\left(1+\log\left(\frac{\tilde{\tau}^E(1,\phi)\lambda_2}{\cos\phi}\right)\right)\right\}$$

$$
\begin{aligned}
&= -\frac{\sin^2\theta}{32\pi}\frac{O(1)}{4\pi^2 E}\int_0^{\alpha_{1,2}} d\phi \frac{\sin\phi\cos\phi}{(1-\sin(2\phi)\cos\theta)^2} \\
&\quad -\frac{\sin^2\theta}{32\pi}\frac{O(1)}{4\pi^2 E}\int_0^{\alpha_{1,2}} d\phi \frac{\sin\phi\cos\phi}{(1-\sin(2\phi)\cos\theta)^2}\log\left(\tilde{\tau}^E(1,\phi)\lambda_2\right) \\
&\quad +\frac{\sin^2\theta}{32\pi}\frac{O(1)}{4\pi^2 E}\int_0^{\alpha_{1,2}} d\phi \frac{\sin\phi\cos\phi}{(1-\sin(2\phi)\cos\theta)^2}\log(\cos\phi) \\
&=: b_E^1 + b_E^2 + b_E^3.
\end{aligned}
$$

Clearly we have $|b_E^1| = O(E^{-1})$ since (arguing similarly as in (IV.2.20))

$$
\int_0^{\alpha_{1,2}} d\phi \left|\frac{\sin\phi\cos\phi}{(1-\sin(2\phi)\cos\theta)^2}\right| \le \int_0^{\alpha_{1,2}} \frac{d\phi}{(1-|\cos\theta|)^2} = \frac{\alpha_{1,2}}{(1-|\cos\theta|)^2} < \infty
$$

since $|\cos\theta| \neq 1$. Let us now consider the term b_E^2. Using (IV.2.16), we write

$$
\begin{aligned}
\log\left(\tilde{\tau}^E(1,\phi)\lambda_2\right) &= \log\left(2\pi\lambda_2\sqrt{E}\sqrt{1-\sin(2\phi)\cos\theta}\right) \\
&= 2^{-1}\log E + \log\left(2\pi\lambda_2\sqrt{1-\sin(2\phi)\cos\theta}\right) = O(\log E) + O(1),
\end{aligned}
$$

where we used the fact that the map $\phi \mapsto \log\left(2\pi\lambda_2\sqrt{1-\sin(2\phi)\cos\theta}\right)$ is bounded. Thus, arguing as above shows that $|b_E^2| = O(E^{-1} + \log(E)/E) = O(\log(E)/E)$. For the term b_E^3, we show that $|b_E^3| = O(E^{-1})$. Indeed, since $\alpha_{1,2} = \arctan(\lambda_2/\lambda_1) \le \pi/2$, we have

$$
\int_0^{\alpha_{1,2}} d\phi \left|\frac{\sin\phi\cos\phi}{(1-\sin(2\phi)\cos\theta)^2}\log(\cos\phi)\right| \le \frac{1}{(1-|\cos\theta|)^2}\int_0^{\pi/2} d\phi|\log(\cos\phi)| < \infty
$$

since it is straightforward to check that

$$
\int_0^{\pi/2} d\phi|\log(\cos\phi)| = \frac{\pi\log 2}{2}.
$$

Indeed, the last integral is obtained as follows: Since $\log(\cos\phi)) \le 0$ on $(0,\pi/2)$, we have

$$
\int_0^{\pi/2} d\phi|\log(\cos\phi)| = -\int_0^{\pi/2} d\phi\log(\cos\phi) =: -I
$$

By changing variable $u = \pi/2 - \phi$, we have that $I = \int_0^{\pi/2} d\phi\log(\sin\phi)$ and also by symmetry $I = \int_{\pi/2}^{\pi} d\phi\log(\sin\phi)$. Therefore, we have

$$
\begin{aligned}
2I &= \int_0^{\pi/2} d\phi\log(\cos\phi\sin\phi) = \int_0^{\pi/2} d\phi\,(\log(\sin 2\phi) - \log 2) \\
&= \frac{1}{2}\int_0^{\pi}\log(\sin\phi)d\phi - \frac{\pi\log 2}{2} = \frac{1}{2}\times 2I - \frac{\pi\log 2}{2} = I - \frac{\pi\log 2}{2},
\end{aligned}
$$

so that $I = -\frac{\pi\log 2}{2}$ as desired.

Combining the contributions of each of the terms b_E^j, $j = 1,2,3$, we conclude that $B_{E,1}(\lambda_1,\lambda_2,\theta) = O(E^{-1}\log E)$. The analysis for $B_{E,2}(\lambda_1,\lambda_2,\theta)$ is done analogously, so that $B_{E,2}(\lambda_1,\lambda_2,\theta) = O(E^{-1}\log E)$. We conclude from (IV.2.15) that, as $E \to \infty$,

$$
\mathbf{Cov}[\phi_E(S_1),\phi_E(S_2)] = A_E(\lambda_1,\lambda_2,\theta) + B_E(\lambda_1,\lambda_2,\theta) = O\left(\frac{\log E}{E}\right) = o\left(\frac{1}{\sqrt{E}}\right),
$$

which proves the statement. This concludes the proof. □

We now prove the following multivariate Central Limit Theorem for the normalized random variables $\widetilde{\phi}_E(S) = 4\pi E^{1/4}\phi_E(S)$.

Proposition IV.2.3 (Multi-dimensional CLT for line segments)**.** *For every integer $d \geq 1$ and every line segments $S_1, \dots, S_d$, we have that, as $E \to \infty$,*

$$\left(\widetilde{\phi}_E(S_1), \dots, \widetilde{\phi}_E(S_d)\right) \xrightarrow{d} \mathcal{N}_d(0, \Sigma),$$

where $\Sigma = \{\Sigma(i,j) : i,j = 1,\dots,d\}$ is the $d \times d$ matrix defined by

$$\Sigma(i,j) := \lambda(S_i, S_j)\,, \quad i,j = 1,\dots,d,$$

where $\lambda(S_i, S_j)$ is the signed length of $S_i \cap S_j$.

Proof. Using the fact that for every line segment S, $\widetilde{\phi}_E(S)$ is an element of the second Wiener chaos and we proved that the covariances $\mathbb{E}\left[\widetilde{\phi}_E(S_1)\widetilde{\phi}_E(S_2)\right]$ converge to $\lambda(S_i, S_j)$ as $E \to \infty$ (see Proposition IV.2.1), it is sufficient to prove the statement for $d = 1$, since in view of Peccati-Tudor Theorem I.1.31, joint convergence is equivalent to marginal convergence for chaotic sequences. By invariance under rigid motions of the plane of Berry's random wave model, we can assume without loss of generality, that $S_1 = [0, L] \times \{0\}$ for $L > 0$. Using the fact that $\mathrm{n}_{S_1} = e_1$, we have

$$\widetilde{\phi}_E(S_1) = \frac{\pi E^{1/4}}{2}\int_0^L B_E(x,0)\tilde{\partial}_2 B_E(x,0)dx. \tag{IV.2.22}$$

We now represent $\widetilde{\phi}_E(S_1)$ as a multiple integral of order 2 with respect to an isonormal Gaussian process on the Hilbert space $L^2([0,1],\lambda)$, where λ denotes Lebesgue measure (see Remark IV.1.18 **(c)**). For $(x,0) \in \mathbb{R}^2$, let $f_0^E(x,\cdot), f_2^E(x,\cdot) : [0,1] \to \mathbb{R}$ be such that

$$B_E(x,0) = I_1(f_0^E(x,\cdot)), \quad \tilde{\partial}_2 B_E(x) = I_1(f_2^E(x,\cdot)),$$

where I_α denotes the Wiener-Itô isometry of order α. Using the product formula for multiple integrals (see (I.1.28)) and independence, we can write

$$B_E(x,0)\tilde{\partial}_2 B_E(x,0) = I_2\left(f_0^E(x,\cdot)\widetilde{\otimes} f_2^E(x,\cdot)\right) + I_0\left(f_0^E(x,\cdot)\widetilde{\otimes}_1 f_2^E(x,\cdot)\right) = I_2\left(f_0^E(x,\cdot)\widetilde{\otimes} f_2^E(x,\cdot)\right),$$

where the symbols $\otimes_r$ and $\widetilde{\otimes}_r$ denote the contraction operator of order r and its symmetrization respectively (see (I.1.22)). In particular, for $r = 0$ and $r = 1$, these are given by (writing $\lambda(du) = du$)

$$\begin{aligned} f_0^E(x,\cdot)\otimes_0 f_2^E(x,\cdot) &= f_0^E(x,\cdot)\otimes f_2^E(x,\cdot), \\ f_0^E(x,\cdot)\otimes_1 f_2^E(x,\cdot) &= \int_0^1 f_0^E(x,u) f_2^E(x,u)du = \langle f_0^E(x,\cdot), f_2^E(x,\cdot)\rangle_{L^2([0,1],\lambda)} = 0, \end{aligned}$$

where the last identity follows from the isometry property for Wiener integrals and independence. It follows from (IV.2.22) that,

$$\widetilde{\phi}_E(S_1) = \frac{\pi E^{1/4}}{2}\int_0^L I_2\left(f_0^E(x,\cdot)\widetilde{\otimes} f_2^E(x,\cdot)\right)dx =: I_2\left(k^E\right),$$

where (Sym denotes the symmetrization operator)

$$k^E(u,v) = \frac{\pi E^{1/4}}{2}\mathrm{Sym}\left\{\int_0^L f_0^E(x,u) f_2^E(x,v)dx\right\}$$

$$= \frac{\pi E^{1/4}}{4}\left\{\int_0^L f_0^E(x,u)f_2^E(x,v)dx + \int_a^b f_0^E(x,v)f_2^E(x,u)dx\right\}. \qquad \text{(IV.2.23)}$$

In order to show that $\widetilde{\phi}_E(S_1)$ satisfies a CLT as $E \to \infty$, it suffices to show that $\|k^E \otimes_1 k^E\|_{L^2([0,1]^2,\lambda^{\otimes 2})}$ converges to zero as $E \to \infty$, in view of the Fourth Moment Theorem I.1.30. From (IV.2.23) it follows that

$$\begin{aligned} k^E \otimes_1 k^E(u,v) &= \int_0^1 dz k^E(u,z)k^E(v,z) \\ &= \frac{\pi^2\sqrt{E}}{16}\int_0^1 dz\left\{\int_0^L f_0^E(x,u)f_2^E(x,z)dx + \int_0^L f_0^E(x,z)f_2^E(x,u)dx\right\} \\ &\quad \times\left\{\int_0^L f_0^E(y,v)f_2^E(y,z)dy + \int_0^L f_0^E(y,z)f_2^E(y,v)dy\right\}, \end{aligned}$$

that is, after expanding, a sum of four terms, among which one of them is (ignoring multiplicative constants that are independent of E)

$$\begin{aligned} (u,v) \;\mapsto\; & \sqrt{E}\int_0^L dx\int_0^L dy f_0^E(x,u)f_0^E(y,v)\int_0^1 dz f_2^E(x,z)f_2^E(y,z) \\ =\; & \sqrt{E}\int_0^L dx\int_0^L dy f_0^E(x,u)f_0^E(y,v)\mathbb{E}\left[\tilde{\partial}_2 B_E(x,0)\tilde{\partial}_2 B_E(y,0)\right] \\ =\; & \sqrt{E}\int_0^L dx\int_0^L dy f_0^E(x,u)f_0^E(y,v)\tilde{r}_{2,2}^E(x-y,0) \end{aligned}$$

by isometry. From this, we compute the squared norm

$$\|k^E \otimes_1 k^E\|^2_{L^2([0,1]^2,\lambda^{\otimes 2})} = \int_0^1 du\int_0^1 dv\left[k^E \otimes_1 k^E(u,v)\right]^2,$$

which is given by a sum of 16 terms that have all the same behaviour. We will expose the details for one of them (which is representative of the difficulty), the others can be treated similarly. Exploiting once more the isometry property of Wiener integrals, one among them (corresponding to the computation above) is given by

$$E\int_{[0,L]^4} dx_1\ldots dx_4\, \tilde{r}_{2,2}^E(x_1-x_2,0)\tilde{r}_{2,2}^E(x_2-x_3,0)r^E(x_3-x_4,0)r^E(x_4-x_1,0). \qquad \text{(IV.2.24)}$$

We now show that the integral in (IV.2.24) converges to zero as $E \to \infty$. Performing the change of variables $(u_1,u_2,u_3,u_4) = (x_1-x_2, x_2-x_3, x_3-x_4, x_4)$ yields that the integral in (IV.2.24) is equal to

$$\begin{aligned} & E\int_0^L du_4\int_{-u_4}^{L-u_4} du_3\int_{-(u_3+u_4)}^{L-(u_3+u_4)} du_2\int_{-(u_2+u_3+u_4)}^{L-(u_2+u_3+u_4)} du_1\, \tilde{r}_{2,2}^E(u_1,0)\tilde{r}_{2,2}^E(u_2,0)r^E(u_3,0) \\ & \qquad\qquad \cdot r^E(-u_1-u_2-u_3,0) \\ \leq\; & EL\int_{-L}^L du_3\int_{-2L}^{2L} du_2\int_{-4L}^{4L} du_1\left|\tilde{r}_{2,2}^E(u_3,0)\tilde{r}_{2,2}^E(u_2,0)r^E(u_1,0)\right|, \end{aligned} \qquad \text{(IV.2.25)}$$

where in the second line, we used the fact that $|r^E(\cdot)| \leq 1$ and uniformly bounded the regions of integrations. Now using [NPR19, Lemma 3.1] and the relation (IV.2.8) yields

$$\tilde{r}_{2,2}^E(u_3,0) = J_0(2\pi\sqrt{E}|u_3|) + J_2(2\pi\sqrt{E}|u_3|) = 2\frac{J_1(2\pi\sqrt{E}|u_3|)}{2\pi\sqrt{E}|u_3|},$$

so that changing variable $v = 2\pi\sqrt{E}u_3$,

$$\int_{-L}^{L} du_3 |\tilde{r}_{2,2}^E(u_3, 0)| = \frac{1}{2\pi\sqrt{E}} \int_{-2\pi\sqrt{E}L}^{2\pi\sqrt{E}L} 2\frac{|J_1(|v|)|}{|v|} dv.$$

Splitting the region of integrations and using that $|\tilde{r}_{2,2}^1(\cdot)| \leq 2$ yields

$$\frac{1}{2\pi\sqrt{E}L} \int_{-2\pi\sqrt{E}L}^{2\pi\sqrt{E}} 2\frac{|J_1(|v|)|}{|v|} dv = \frac{1}{2\pi\sqrt{E}} \int_{-1}^{1} 2\frac{|J_1(|v|)|}{|v|} dv + \frac{2}{2\pi\sqrt{E}} \int_{1}^{2\pi\sqrt{E}L} 2\frac{|J_1(v)|}{v} dv.$$

The first term is $O(E^{-1/2})$. For the second term, we use the bound $|J_1(v)| \leq v^{-1/2}$, to obtain

$$\frac{2}{2\pi\sqrt{E}} \int_{1}^{2\pi\sqrt{E}L} 2\frac{|J_1(v)|}{v} dv \leq \frac{2}{2\pi\sqrt{E}} \int_{1}^{2\pi\sqrt{E}L} \frac{1}{v^{3/2}} = O(E^{-1/2}).$$

For the integration with respect to u_1 in (IV.2.25), we have that

$$\begin{aligned} \int_{-4L}^{4L} |r^E(u_1, 0)| du_1 &= \frac{1}{2\pi\sqrt{E}} \left(O(1) + \int_{1}^{8\pi\sqrt{E}L} |J_0(v)| dv \right) \\ &\leq \frac{1}{2\pi\sqrt{E}} \left(O(1) + \int_{1}^{8\pi\sqrt{E}L} \frac{1}{\sqrt{v}} dv \right) = O(E^{-1/4}). \end{aligned}$$

From this, we deduce that the integral in (IV.2.24) is $O(E \cdot E^{-1/2} E^{-1/2} E^{-1/4}) = O(E^{-1/4})$, which suffices. □

We now conclude the proofs of Theorem IV.1.20 and Theorem IV.1.21. These will essentially follow from the findings of Proposition IV.2.1 and the linearity property of ϕ_E in (IV.1.10).

Proof of Theorem IV.1.20. Assume that $C_1, C_2 \in \mathscr{C}$ are given by $C_1 = \sum_{j=1}^{r_1} \alpha_j S_j$ and $C_2 = \sum_{j=1}^{r_2} \beta_j T_j$ for $\alpha_j, \beta_j \in \mathbb{R}$ and line segments $S_1, \ldots, S_{r_1}, T_1, \ldots, T_{r_2}$. Then, by the additivity rule (IV.1.10) and in view of the covariance structure for line segments proved in Proposition IV.2.1, it follows that, as $E \to \infty$,

$$\begin{aligned} \mathbf{Cov}[\phi_E(C_1), \phi_E(C_2)] &= \sum_{j=1}^{r_1} \sum_{k=1}^{r_2} \alpha_j \beta_k \mathbf{Cov}\left[\phi_E(S_j), \phi_E(T_k)\right] \\ = \sum_{j=1}^{r_1} \sum_{k=1}^{r_2} \alpha_j \beta_k \frac{\lambda(S_j, T_k)}{16\pi^2\sqrt{E}} + o\left(\frac{1}{\sqrt{E}}\right) &= \frac{\lambda(C_1, C_2)}{16\pi^2\sqrt{E}} + o\left(\frac{1}{\sqrt{E}}\right), \end{aligned}$$

where we used the bilinearity of $\lambda(\cdot, \cdot)$. The variance estimate follows after setting $C_1 = C_2$ above. This finishes the proof. □

Proof of Theorem IV.1.21. Thanks to the estimate in (IV.1.13) and the Peccati-Tudor Theorem I.1.31, it suffices to prove the statement for $d = 1$. Write $C = \sum_{j=1}^{r} \alpha_j S_j$ for line segments $S_1, \ldots, S_r$ and $\alpha_1, \ldots, \alpha_r \in \mathbb{R}$. The additivity property (IV.1.10) and the fact that the random vector $(\widetilde{\phi}_E(S_1), \ldots, \widetilde{\phi}_E(S_r))$ converges in distribution to a Gaussian vector with covariance matrix $\Sigma(i, j) = \lambda(S_i, S_j), i, j = 1, \ldots, r$ (in view of Proposition IV.2.3) imply that $\widetilde{\phi}_E(C)$ converges in distribution to a Gaussian random variable with variance $\lambda(C, C)$. □

IV.2.2 Proof of Corollary IV.1.6

In view of the variance estimate in Theorem IV.1.20, and taking into account the normalization in the definition of X_E (see (IV.1.5)), we deduce that the finite-dimensional distributions of $X_E[2]$ converge to zero. We are thus left to show that the laws of $\{X_E[2] : E > 0\}$ are tight, which is the content of the following proposition.

Proposition IV.2.4. *The laws of the random functions* $\{X_E[2] : E > 0\}$ *are tight in* $\mathbf{D}_2$.

The proof of Proposition IV.2.4 is based on the following criterion by Davydov and Zitikis [DZ08, Theorem 1] for proving weak convergence of processes on $[0, 1]^d$.

Theorem IV.2.5 (see [DZ08])**.** *Let* $Y_n = \left\{Y_n(t) : t \in [0, 1]^d\right\}, n \geq 1$ *be a collection of real-valued stochastic processes on* $[0, 1]^d$ *such that its paths belong* $\mathbb{P}$*-almost surely to* $C([0, 1]^d, \mathbb{R})$*. Assume furthermore that*

(a) the finite-dimensional distributions of Y_n *converge to those of some stochastic process* Y*,*

(b) there exist $\alpha \geq \beta > d, c > 0$ *and a numerical sequence* $\{\alpha_n : n \geq 1\}$ *such that* $\alpha_n \to 0$ *as* $n \to \infty$*,* $\mathbb{E}\left[|Y_n(0)|^\alpha\right] \leq c$ *for every* $n \geq 1$ *and*

$$\mathbb{E}\left[|Y_n(t) - Y_n(s)|^\alpha\right] \leq c\|t - s\|^\beta, \quad \forall t, s \in [0, 1]^d : \|t - s\| \geq \alpha_n, \tag{IV.2.26}$$

(c) for the sequence $\{\alpha_n : n \geq 1\}$ *at point (b), we have as* $n \to \infty$*,*

$$\omega_{Y_n}(\alpha_n) := \sup_{\|t-s\| \leq \alpha_n} |Y_n(t) - Y_n(s)| \xrightarrow{\mathbb{P}} 0. \tag{IV.2.27}$$

Then, as $n \to \infty$*,* Y_n *converges weakly to* Y *and* Y *has continuous paths* $\mathbb{P}$*-almost surely.*

In order to prove that the process $X_E[2]$ verifies the assumptions (IV.2.26) and (IV.2.27), our arguments make use of the following moment estimates for suprema of stationary Gaussian random fields. Here, for a function $f : \mathbb{R}^d \to \mathbb{R}$, a domain $\mathcal{D} \subset \mathbb{R}^d$ and an integer $j \geq 0$, we denote by

$$\|f\|_{C^j(\mathcal{D})} := \sup_{x \in \mathcal{D}} \sup_{|\alpha| \leq j} |\partial_\alpha f(x)|$$

where $\partial_\alpha f(x) := \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d} f(x)$, for $\alpha := (\alpha_1, \dots, \alpha_d)$ with $|\alpha| := \sum_{k=1}^d \alpha_k$. The proof of Proposition IV.2.6 is postponed to Appendix IV.B.

Proposition IV.2.6. *Let* G *be a stationary Gaussian random field on* $\mathbb{R}^d$*. Assume that for every* $m \geq 0$*, there exists a constant* $\tilde{\sigma}^2(m) < \infty$ *such that*

$$\mathbb{E}\left[(\partial_\alpha G(x))^2\right] \leq \tilde{\sigma}^2(m), \quad \forall \alpha \in \mathbb{N}^d, |\alpha| \leq m. \tag{IV.2.28}$$

Then, for any $p \geq 1$*,*

$$\mathbb{E}\left[\|G\|_{C^j(\mathcal{D})}^p\right] \leq C \left\{\log(\mathrm{vol}(\mathcal{D}))\right\}^{p/2}$$

where $C > 0$ *is an absolute constant depending on* p *and* j*, and* $\mathrm{vol}(\mathcal{D})$ *is the volume of* $\mathcal{D}$*.*

The following auxiliary results are needed to complete the proof Proposition IV.2.4.

Lemma IV.2.7. *For* $\mathbf{t} = (t_1, t_2) \in [0,1]^2$*, we write* $\mathcal{D}_{\mathbf{t}} := [0, t_1] \times [0, t_2]$*. Then, for every continuous function* $f : [0,1]^2 \to \mathbb{R}$ *and every* $\mathbf{t}, \mathbf{s} \in [0,1]^2$*, we have*

$$\left| \int_{\mathcal{D}_{\mathbf{t}}} f(x)dx - \int_{\mathcal{D}_{\mathbf{s}}} f(x)dx \right| \leq C \sup_{x \in [0,1]^2} |f(x)| \|\mathbf{t} - \mathbf{s}\|,$$

for some absolute constant $C > 0$.

Proof. This follows directly from the fact that $\text{area}(\mathcal{D}_{\mathbf{t}} \setminus \mathcal{D}_{\mathbf{s}}) \leq C\|\mathbf{t} - \mathbf{s}\|$, for some constant $C > 0$ which is independent of $\mathbf{t}$ and $\mathbf{s}$. □

Lemma IV.2.8. *For every* $p \geq 1$ *and* $E > 0$*, we have*

$$\mathbb{E}\left[\sup_{x \in [0,1]^2} |B_E(x)|^p\right] + \mathbb{E}\left[\sup_{x \in [0,1]^2} \|\widetilde{\nabla} B_E(x)\|^p\right] \leq C(\log E)^{p/2},$$

where $C > 0$ *is some absolute constant depending only on* p.

Proof. We use the fact that $B_E \stackrel{d}{=} B_1(2\pi\sqrt{E}\cdot)$ as random fields, so that

$$\mathbb{E}\left[\sup_{x \in [0,1]^2} B_E(x)^p\right] = \mathbb{E}\left[\sup_{y \in [0, 2\pi\sqrt{E}]^2} B_1(y)^p\right].$$

It is easy to see that the assumption (IV.2.28) of Proposition IV.2.6 is satisfied by B_1. Applying the estimate in Proposition IV.2.6 with $\mathcal{D} = [0, 2\pi\sqrt{E}]^2 \subset \mathbb{R}^2$ yields the desired conclusion. The second supremum involving the normalized gradient is dealt with in the same way. □

We are now in the position to prove Proposition IV.2.4.

Proof of Proposition IV.2.4. In view of Theorem IV.2.5 and the fact that the finite-dimensional distributions of $X_E[2]$ converge to those of the zero-process, it is sufficient to prove that there exists a numerical sequence $\{a_E : E > 0\}$ such that $a_E \to 0$ as $E \to \infty$ and (i) there exist absolute constants $\alpha \geq \beta > 2, c > 0$ such that

$$\mathbb{E}\left[|X_E[2](\mathbf{t}) - X_E[2](\mathbf{s})|^\alpha\right] \leq c\|\mathbf{t} - \mathbf{s}\|^\beta, \quad \forall \mathbf{t}, \mathbf{s} : \|\mathbf{t} - \mathbf{s}\| \geq a_E \tag{IV.2.29}$$

and (ii)

$$\omega(E) := \sup_{\|\mathbf{t} - \mathbf{s}\| \leq a_E} |X_E[2](\mathbf{t}) - X_E[2](\mathbf{s})| \stackrel{\mathbb{P}}{\to} 0, \tag{IV.2.30}$$

as $E \to \infty$. We claim that choosing $a_E := (\sqrt{E}\log E)^{-1}$ verifies the above conditions (i) and (ii). Let us prove that (i) holds. The variance estimate in Theorem IV.1.20 implies that there exists an absolute constant $K > 0$ such that for every $E > 0$ and every $\mathbf{t} \in [0,1]^2$, $\mathbf{Var}[X_E[2](\mathbf{t})] \leq K(\sqrt{E}\log E)^{-1}$. Therefore, choosing $\alpha = 2$ in (IV.2.29), we infer that for every $\mathbf{t}, \mathbf{s}$ such that $\|\mathbf{t} - \mathbf{s}\| \geq (\sqrt{E}\log E)^{-1}$,

$$\mathbb{E}\left[(X_E[2](\mathbf{t}) - X_E[2](\mathbf{s}))^2\right] \leq 2\mathbf{Var}[X_E[2](\mathbf{t})] + 2\mathbf{Var}[X_E[2](\mathbf{s})] \leq \frac{2K}{\sqrt{E}\log E} \leq c\|\mathbf{t} - \mathbf{s}\|.$$

Since for every $\mathbf{t} \in [0,1]^2$, $X_E[2](\mathbf{t})$ is an element of the second Wiener chaos associated with B_E, we exploit the hypercontractivity property for multiple Wiener integrals (see (I.1.29)) to obtain (for some absolute constant $C > 0$)

$$\mathbb{E}\left[(X_E[2](\mathbf{t}) - X_E[2](\mathbf{s}))^p\right] \leq C\mathbb{E}\left[(X_E[2](\mathbf{t}) - X_E[2](\mathbf{s}))^2\right]^{p/2} \leq C\|\mathbf{t} - \mathbf{s}\|^{p/2}$$

for every $p > 4$, which gives the desired estimate in (IV.2.29) since $p/2 > 2$. Let us now argue that (ii) holds. By [NPR19, Eq. (4.58)], we can write

$$\begin{aligned} X_E[2](\mathbf{t}) &= \sqrt{\frac{512\pi}{\log E}} \mathscr{L}_E[2](\mathcal{D}_{\mathbf{t}}) = \sqrt{\frac{512\pi}{\log E}} \frac{\pi}{8} \sqrt{2E} \left[-2 \int_{\mathcal{D}_{\mathbf{t}}} B_E(x)^2 dx + \int_{\mathcal{D}_{\mathbf{t}}} \|\widetilde{\nabla} B_E(x)\|^2 dx \right] \\ &= 4\pi^{3/2} \sqrt{\frac{E}{\log E}} \left[-2 \int_{\mathcal{D}_{\mathbf{t}}} B_E(x)^2 dx + \int_{\mathcal{D}_{\mathbf{t}}} \|\widetilde{\nabla} B_E(x)\|^2 dx \right]. \end{aligned}$$

Combining this expression with Lemma IV.2.7 applied to $f = B_E(\cdot)^2$ and $f = \|\widetilde{\nabla} B_E(\cdot)\|^2$, yields for every choice of $\mathbf{t}, \mathbf{s}$ such that $\|\mathbf{t} - \mathbf{s}\| \le a_E$ (denoting by C an absolute constant whose value varies from line to line)

$$\begin{aligned} &|X_E[2](\mathbf{t}) - X_E[2](\mathbf{s})| \\ &\le C \sqrt{\frac{E}{\log E}} \left\{ \left| \int_{\mathcal{D}_{\mathbf{t}}} B_E(x)^2 dx - \int_{\mathcal{D}_{\mathbf{s}}} B_E(x)^2 dx \right| + \left| \int_{\mathcal{D}_{\mathbf{t}}} \|\widetilde{\nabla} B_E(x)\|^2 dx - \int_{\mathcal{D}_{\mathbf{s}}} \|\widetilde{\nabla} B_E(x)\|^2 \right| \right\} \\ &\le C \sqrt{\frac{E}{\log E}} \left\{ \sup_{x \in [0,1]^2} |B_E(x)|^2 + \sup_{x \in [0,1]^2} \|\widetilde{\nabla} B_E(x)\|^2 \right\} \|\mathbf{t} - \mathbf{s}\| \\ &\le \frac{C}{(\log E)^{3/2}} \left\{ \sup_{x \in [0,1]^2} |B_E(x)|^2 + \sup_{x \in [0,1]^2} \|\widetilde{\nabla} B_E(x)\|^2 \right\}, \end{aligned}$$

where we used the definition of a_E. This implies that

$$\begin{aligned} \mathbb{E}[\omega(E)] &= \mathbb{E}\left[\sup_{\|\mathbf{t}-\mathbf{s}\| \le a_E} |X_E[2](\mathbf{t}) - X_E[2](\mathbf{s})| \right] \\ &\le \frac{C}{(\log E)^{3/2}} \left\{ \mathbb{E}\left[\sup_{x \in [0,1]^2} |B_E(x)|^2 \right] + \mathbb{E}\left[\sup_{x \in [0,1]^2} \|\widetilde{\nabla} B_E(x)\|^2 dx \right] \right\} \\ &\le \frac{C}{(\log E)^{3/2}} \cdot \log E = \frac{C}{\sqrt{\log E}} \end{aligned}$$

where we used Lemma IV.2.8 with $p = 2$. Therefore, by the Markov inequality we have for every $\eta > 0$,

$$\mathbb{P}\{\omega(E) > \eta\} \le \eta^{-1} \mathbb{E}[\omega(E)] \le \frac{C}{\eta \sqrt{\log E}},$$

which proves the validity of (ii). □

IV.2.3 Proof of Theorem IV.1.8 and Corollary IV.1.9

Our proof of Theorem IV.1.8 is based on a planar chaining argument, similar to the one presented in [DT89] and [MW11] in dimension one for a study related to empirical processes.

We start with a preliminary lemma, yielding a L^2 bound for increments of $R_E = \sum_{q \ge 3} X_E[2q]$ along rectangles of the form $[s_1, t_1] \times [s_2, t_2] \subset [0, 1]^2$.

Lemma IV.2.9. *For every* $0 \le s_i \le t_i \in [0, 1], i = 1, 2,$ *and* $E > 0$, *we have that*

$$\mathbb{E}\left[(R_E(t_1, t_2) - R_E(s_1, t_2) - R_E(t_1, s_2) + R_E(s_1, s_2))^2 \right] \le \frac{C}{\log E} [(t_1 - s_1)(t_2 - s_2)],$$

where $C > 0$ *is some absolute constant (independent of* t_1, t_2, s_1, s_2 *and* E*).*

Proof. Let $\mathcal{D}(\mathbf{t},\mathbf{s}) := [s_1,t_1]\times[s_2,t_2]$. By definition of R_E and the additivity of the nodal length, we have

$$R_E(t_1,t_2) - R_E(s_1,t_2) - R_E(t_1,s_2) + R_E(s_1,s_2) = \sqrt{\frac{512\pi}{\log E}}\sum_{q\geq 3}\mathscr{L}_E[2q](\mathcal{D}(\mathbf{t},\mathbf{s})). \quad \text{(IV.2.31)}$$

By inspection of the arguments used in the proofs of [NPR19, Lemmas 7.6,7.8,7.9], one verifies that there is an absolute constant $C_1 > 0$ (independent of $\mathbf{t},\mathbf{s}$ and E) such that

$$\mathbf{Var}\left[\sum_{q\geq 3}\mathscr{L}_E[2q](\mathcal{D}(\mathbf{t},\mathbf{s}))\right] \leq C_1\text{area}(\mathcal{D}(\mathbf{t},\mathbf{s})).$$

Taking the square of $L^2(\mathbb{P})$-norm in (IV.2.31), and exploiting the above upper bound, we obtain

$$\begin{aligned}\mathbb{E}\left[(R_E(t_1,t_2) - R_E(s_1,t_2) - R_E(t_1,s_2) + R_E(s_1,s_2))^2\right] &= \frac{512\pi}{\log E}\mathbf{Var}\left[\sum_{q\geq 3}\mathscr{L}_E[2q](\mathcal{D}(\mathbf{t},\mathbf{s}))\right]\\ &= C_1\frac{512\pi}{\log E}\text{area}(\mathcal{D}(\mathbf{t},\mathbf{s})) = \frac{C}{\log E}(t_1-s_1)(t_2-s_2),\end{aligned}$$

which gives the desired conclusion. □

We are now in the position to prove Proposition IV.1.8.

Proof of Theorem IV.1.8. We start by introducing refining partitions of the unit square.

Refining partitions of the unit square. Let us fix a large integer K (whose exact value will be chosen later as a function of E). For every integers $k,k' = 0,\dots,K$, and every vector $i = (i_1,i_2) \in \left\{0,\dots,2^k\right\}\times\left\{0,\dots,2^{k'}\right\}$, we define the *partition points*

$$\mathbf{p}_i(k,k') := (p_{i_1}(k), p_{i_2}(k')) = \left(\frac{i_1}{2^k},\frac{i_2}{2^{k'}}\right) \in [0,1]^2.$$

Moreover, for every $\mathbf{t} = (t_1,t_2) \in [0,1]^2$ and $k,k' = 0,\dots,K$, we define the vector $i_{k,k'}(\mathbf{t}) = (i_{1,k}(t_1), i_{2,k'}(t_2))$ to be such that

$$p_{i_{1,k}(t_1)}(k) \leq t_1 \leq p_{i_{1,k}(t_1)+1}(k), \quad p_{i_{2,k'}(t_2)}(k') \leq t_2 \leq p_{i_{2,k'}(t_2)+1}(k'),$$

that is, for every $\mathbf{t}\in[0,1]^2$, the vector $i_{k,k'}(\mathbf{t})$ is such that $\mathbf{p}_{i_{k,k'}(\mathbf{t})}(k,k')$ is the closest partition point to $\mathbf{t}$ on the left.

We introduce the following operators.

Definition IV.2.10. Given a function $f:[0,1]^2\to\mathbb{R}$ a point $\mathbf{t} = (t_1,t_2)\in[0,1]^2$, and $k,k'\in\{0,\dots,K-1\}$, we define the *difference operator*

$$\begin{aligned}\Delta_{k,k'}f(\mathbf{t}) \;:=\;& f\left(p_{i_{1,k+1}(t_1)}(k+1), p_{i_{2,k'+1}(t_2)}(k'+1)\right) - f\left(p_{i_{1,k+1}(t_1)}(k+1), p_{i_{2,k'}(t_2)}(k')\right)\\ &- f\left(p_{i_{1,k}(t_1)}(k), p_{i_{2,k'+1}(t_2)}(k'+1)\right) + f\left(p_{i_{1,k}(t_1)}(k), p_{i_{2,k'}(t_2)}(k')\right).\end{aligned}$$

Also, for $k,k'\in\{0,\dots,K-1\}$, we set

$$\Delta_{K,k'}f(\mathbf{t}) \;:=\; f\left(t_1, p_{i_{2,k'+1}(t_2)}(k'+1)) - f(t_1, p_{i_{2,k'}(t_2)}(k')\right)$$

$$
\begin{aligned}
& -f\left(p_{i_{1,K}(t_1)}(K), p_{i_{2,k'+1}(t_2)}(k'+1)\right) + f\left(p_{i_{1,K}(t_1)}(K), p_{i_{2,k'}(t_2)}(k')\right), \\
\Delta_{k,K} f(\mathbf{t}) \quad := \quad & f\left(p_{i_{1,k+1}(t_1)}(k+1), t_2\right) - f\left(p_{i_{1,k+1}(t_1)}(k+1), p_{i_{2,K}(t_2)}(K)\right) \\
& -f\left(p_{i_{1,k}(t_1)}(k), t_2\right) + f\left(p_{i_{1,k}(t_1)}(k), p_{i_{2,K}(t_2)}(K)\right)
\end{aligned}
$$

and finally

$$
\begin{aligned}
\Delta_{K,K} f(\mathbf{t}) \quad := \quad & f(t_1, t_2) - f\left(t_1, p_{i_{2,K}(t_2)}(K)\right) \\
& -f\left(p_{i_{1,K}(t_1)}(K), t_2\right) + f\left(p_{i_{1,K}(t_1)}(K), p_{i_{2,K}(t_2)}(K)\right).
\end{aligned}
$$

Also, we use the notations $\Delta^{+}_{k,k'}, \Delta^{+}_{K,k'}$ and $\Delta^{+}_{k,K}$ to indicate the operators obtained from the relations above by replacing t_1 and t_2 with $p_{i_{1,k}(t_1)+1}(k)$ and $p_{i_{2,k'}(t_2)+1}(k')$, respectively.

We remark that, by construction of the refining partitions, we have either $p_{i_{1,k}}(k) = p_{i_{1,k+1}(t_1)}(k+1)$ or $p_{i_{1,k+1}(t_1)}(k+1) - p_{i_{1,k}(t_1)}(k) = 2^{-(k+1)}$ (and similarly for partition coordinates involving the index i_2) which yields in particular

$$
\begin{aligned}
& \left|\Delta_{k,k'} f(t_1, t_2)\right| \\
\leq \quad & \left| f\left(p_{i_{1,k}(t_1)}(k) + \frac{1}{2^{k+1}}, p_{i_{2,k'}(t_2)}(k') + \frac{1}{2^{k'+1}}\right) - f\left(p_{i_{1,k}(t_1)}(k) + \frac{1}{2^{k+1}}, p_{i_{2,k'}(t_2)}(k')\right) \right. \\
& \left. -f\left(p_{i_{1,k}(t_1)}(k), p_{i_{2,k'}(t_2)}(k') + \frac{1}{2^{k'+1}}\right) + f\left(p_{i_{1,k}(t_1)}(k), p_{i_{2,k'}(t_2)}(k')\right) \right|. \qquad \text{(IV.2.32)}
\end{aligned}
$$

In view of the above defined difference operators, the following bivariate telescopic formula holds

$$
f(t_1, t_2) = \sum_{k,k'=0}^{K} \Delta_{k,k'} f(t_1, t_2) \qquad \text{(IV.2.33)}
$$

for every $f : [0,1]^2 \to \mathbb{R}$.

Let us now write R_E^K for the discretized version of R_E associated with the above partition. Applying (IV.2.33) to R_E^K, we can write for every $\mathbf{t} \in [0,1]^2$,

$$
\left|R_E^K(\mathbf{t})\right| = \left|\sum_{k,k'=0}^{K} \Delta_{k,k'} R_E^K(\mathbf{t})\right| \leq \sum_{(k,k') \in B(K)^c} \left|\Delta_{k,k'} R_E^K(\mathbf{t})\right| + \left|\sum_{(k,k') \in B(K)} \Delta_{k,k'} R_E^K(\mathbf{t})\right|, \qquad \text{(IV.2.34)}
$$

where we set $B(K) := \left\{(k,k') \in \{0,\dots,K\}^2 : \max(k,k') = K\right\}$. Note that the second term in the R.H.S of (IV.2.34) vanishes by definition of the operators $\Delta_{K,k'}, \Delta_{k,K}$ and $\Delta_{K,K}$, and the fact that we consider the discretized remainder R_E^K: indeed, for every $(k,k') \in B(K)$, we have that $\Delta_{k,k'} R_E^K(\mathbf{t}) = 0$. From this, we conclude that, for every $\varepsilon > 0$,

$$
\mathbb{P}\left\{\sup_{\mathbf{t} \in [0,1]^2} \left|R_E^K(\mathbf{t})\right| > \varepsilon\right\} \leq \mathbb{P}\left\{\sup_{\mathbf{t} \in [0,1]^2} \sum_{(k,k') \in B(K)^c} \left|\Delta_{k,k'} R_E^K(\mathbf{t})\right| > \varepsilon\right\}. \qquad \text{(IV.2.35)}
$$

We remark that the R.H.S involves the increments on closest partition points associated with $\mathbf{t}$. Now, using the fact that

$$
\sum_{k,k' \geq 0} \frac{\varepsilon}{(k+3)^2 (k'+3)^2} \leq \varepsilon
$$

and the Chebychev inequality, we can bound the probability in (IV.2.35) by

$$\sum_{(k,k')\in B(K)^c} \mathbb{P}\left\{\sup_{\mathbf{t}\in[0,1]^2}\left|\Delta_{k,k'}R_E^K(\mathbf{t})\right| > \frac{\varepsilon}{(k+3)^2(k'+3)^2}\right\}$$

$$\leq \sum_{(k,k')\in B(K)^c} \sum_{i_1=0}^{2^k}\sum_{i_2=0}^{2^{k'}} \mathbb{P}\left\{\left|\Delta_{k,k'}R_E^K(\mathbf{p}_{(i_1,i_2)}(k+1,k+1))\right| > \frac{\varepsilon}{(k+3)^2(k'+3)^2}\right\}$$

$$\leq \sum_{(k,k')\in B(K)^c} \frac{(k+3)^4(k'+3)^4}{\varepsilon^2}\sum_{i_1=0}^{2^k}\sum_{i_2=0}^{2^{k'}} \frac{C}{\log E}\frac{1}{2^{k+1}}\frac{1}{2^{k'+1}} \leq \frac{C'}{\log E}K^{10},$$

where we used Lemma IV.2.9. Therefore, this probability converges to zero once we chose $K = K(E)$ in such a way that $K(E)\to\infty$ and $K(E) = o((\log E)^{1/10})$ as $E\to\infty$. This concludes the proof. □

Proof of Corollary IV.1.9. Let us choose $K = K(E)$ as in the statement. By the Wiener chaos expansion of X_E^K, we can write

$$X_E^K = X_E[4] + \left(X_E^K[4] - X_E[4]\right) + X_E[2] + \left(X_E^K[2] - X_E[2]\right) + R_E^K.$$

We use the same strategy used to prove Lemma IV.1.3. The process $X_E[4]$ converges weakly to a standard Wiener sheet in the space $\mathbf{D}_2$, in view of [PV20, Theorem 3.4]. The residue process R_E^K converges to zero uniformly in probability, in view of Theorem IV.1.8. The second chaotic projections converge weakly to zero in view of Corollary IV.1.6. For the term $X_E^K[4] - X_E[4]$ we argue that, for every $\varepsilon > 0$,

$$\mathbb{P}\left\{\sup_{\mathbf{t}\in[0,1]^2}\left|X_E^K[4](\mathbf{t}) - X_E[4](\mathbf{t})\right| > \varepsilon\right\} \to 0$$

as $E\to\infty$. By definition of $X_E^K[4]$, we can rewrite

$$\mathbb{P}\left\{\sup_{\mathbf{t}\in[0,1]^2}\left|X_E^K[4](\mathbf{t}) - X_E[4](\mathbf{t})\right| > \varepsilon\right\} = \mathbb{P}\left\{\sup_{\mathbf{t}\in[0,1]^2}\left|X_E[4](\mathbf{p}_{i_{K,K}(\mathbf{t})}(K,K)) - X_E[4](\mathbf{t})\right| > \varepsilon\right\}.$$

Since both $X_E[4]$ and $\mathbf{W}$ belong to the space $\mathbf{C}_2$, $\mathbb{P}$-almost surely, we can apply the Skorohod Representation Theorem [Dud02, Theorem 11.7.2]. Thus, on some probability space $(\Omega', \mathscr{F}', \mathbb{P}')$, there exist $\left\{Y_E' : E > 0\right\}, Z' \in \mathbf{C}_2$ such that $Y_E' \overset{d}{=} X_E[4], Z' \overset{d}{=} \mathbf{W}$ and $\sup_{\mathbf{t}\in[0,1]^2}|Y_E'(\mathbf{t}) - Z'(\mathbf{t})| \to 0, \mathbb{P}'$-almost surely as $E\to\infty$. Therefore, denoting by $\Delta\in P_E$ a cell of a partition P_E of $[0,1]^2$ with mesh $|P_E|\to 0$ as $E\to\infty$, we can write

$$\mathbb{P}\left\{\sup_{\mathbf{t}\in[0,1]^2}\left|X_E[4](\mathbf{p}_{i_{K,K}(\mathbf{t})}(K,K)) - X_E[4](\mathbf{t})\right| > \varepsilon\right\}$$

$$\leq \mathbb{P}'\left\{\sup_{\Delta\in P_E}\sup_{\mathbf{t},\mathbf{s}\in\Delta}|X_E'(\mathbf{t}) - X_E'(\mathbf{s})| > \varepsilon\right\}$$

$$\leq \mathbb{P}'\left\{\sup_{\Delta\in P_E}\sup_{\mathbf{t},\mathbf{s}\in\Delta}\left(|X_E'(\mathbf{t}) - Z'(\mathbf{t})| + |Z'(\mathbf{s}) - X_E'(\mathbf{s})|\right) > \frac{\varepsilon}{2}\right\}$$

$$+\mathbb{P}'\left\{\sup_{\Delta\in P_E}\sup_{\mathbf{t},\mathbf{s}\in\Delta}|Z'(\mathbf{t}) - Z'(\mathbf{s})| > \frac{\varepsilon}{2}\right\}$$

where we used the triangle inequality. For the first probability, we use the fact that X_E' converges to Z' uniformly on $[0,1]^2$, $\mathbb{P}'$-almost surely, implying that

$$\sup_{\Delta\in P_E}\sup_{\mathbf{t},\mathbf{s}\in\Delta}\left(|X_E'(\mathbf{t}) - Z'(\mathbf{t})| + |Z'(\mathbf{s}) - X_E'(\mathbf{s})|\right) \leq 2\sup_{\mathbf{t}\in[0,1]^2}|X_E'(\mathbf{t}) - Z'(\mathbf{t})| \to 0,$$

$\mathbb{P}'$-almost surely. For the second term, we notice that Z' is uniformly continuous on $[0,1]^2$ (being continuous on $[0,1]^2$), so that $\sup_{\mathbf{t},\mathbf{s}\in\Delta}|Z'(\mathbf{t}) - Z'(\mathbf{s})| \to 0$, $\mathbb{P}'$-almost surely. □

IV.2.4 Proof of Proposition IV.1.14

By Lemma IV.2.9, we deduce that for every $\mathbf{t} \in [0,1]^2$,

$$\mathbf{Var}[R_E(\mathbf{t})] = O\left(\frac{1}{\log E}\right),$$

where the constants involved in the 'big-O' notation are independent of $\mathbf{t}$ and E. This implies that the finite-dimensional distributions of the process $R_E(\bullet, N)$ converge to zero for every fixed $N \geq 4$. Therefore, in order to obtain the desired conclusion, it is sufficient to prove that the laws of the random mappings $\{R_E(\bullet; N) : E > 0\}$ (for $N = N(E)$ as in the statement) verify a Kolmogorov type estimate of the form

$$\mathbb{E}\left[(R_E(\mathbf{t}; N) - R_E(\mathbf{s}; N))^\alpha\right] \leq c\|\mathbf{t} - \mathbf{s}\|^{2+\beta}, \quad \forall \mathbf{t}, \mathbf{s} \in [0,1]^2 \tag{IV.2.36}$$

for some constants $\alpha, \beta > 0$ and $c > 0$ that are independent of E. Denoting by $\mathcal{D}_{\mathbf{t}} := [0, t_1] \times [0, t_2]$ and $\mathcal{D}_{\mathbf{t},\mathbf{s}} := \mathcal{D}_{\mathbf{t}} \setminus \mathcal{D}_{\mathbf{s}}$, we have that for every integer N (to be chosen later as a function of E) and every $p > 2$,

$$\mathbb{E}\left[(R_E(\mathbf{t}; N) - R_E(\mathbf{s}; N))^p\right]^{1/p} = \frac{512\pi}{\log E}\mathbb{E}\left[\left(\sum_{q=3}^{N} \mathscr{L}_E[2q](\mathcal{D}_{\mathbf{t},\mathbf{s}})\right)^p\right]^{1/p}.$$

Since $\sum_{q=3}^{N} \mathscr{L}_E[2q](\mathcal{D}_{\mathbf{t},\mathbf{s}})$ is a random variable living in the orthogonal sum of Wiener chaoses up to order $2N$, we use the hypercontractivity property (I.1.29) together with Lemma IV.2.9, to deduce that

$$\begin{aligned}\mathbb{E}\left[(R_E(\mathbf{t}; N) - R_E(\mathbf{s}; N))^p\right]^{1/p} &\leq \frac{512\pi}{\log E}(p-1)^N \mathbf{Var}\left[\sum_{q=3}^{N} \mathscr{L}_E[2q](\mathcal{D}_{\mathbf{t},\mathbf{s}})\right]^{1/2} \\ &\leq \frac{c_p}{\log E}(p-1)^N \|t - s\|^{1/2},\end{aligned}$$

where c_p is some absolute constant only depending on p. In particular, for $p = 6$ we obtain the estimate

$$\mathbb{E}\left[(R_E(\mathbf{t}; N) - R_E(\mathbf{s}; N))^6\right] \leq \left\{\frac{c_6}{\log E}5^N \|t - s\|^{1/2}\right\}^6 = c\frac{5^{6N}}{(\log E)^6}\|t - s\|^3,$$

for some absolute constant $c > 0$. Thus in order to prove (IV.2.36), it is sufficient to choose $N = N(E)$ such that (the constant 1 is not important here)

$$\frac{5^{6N(E)}}{(\log E)^6} = 1,$$

yielding that $N(E) = 6^{-1}\log_5((\log E)^6) = \log_5(\log E)$. This proves the claim.

Appendix IV.A Proof of Lemma IV.1.3

Since U_n and V_n converge weakly to X and zero in $\mathbf{D}_2$, respectively, we use for instance [Wic69, Theorem 2], to deduce that, for every $\varepsilon > 0$,

$$\lim_{\delta\to 0}\limsup_{n\to\infty} \mathbb{P}\{\omega_\delta(U_n) > \varepsilon\} = 0, \quad \lim_{\delta\to 0}\limsup_{n\to\infty} \mathbb{P}\{\omega_\delta(V_n) > \varepsilon\} = 0 \tag{IV.A.1}$$

where $\omega_\delta(f) := \sup\{|f(\mathbf{t}) - f(\mathbf{s})| : \|\mathbf{t} - \mathbf{s}\| < \delta\}$. To obtain the desired conclusion, by virtue of the discussion contained in [Neu71, p.1291], it is sufficient to show that

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \mathbb{P}\{\omega_\delta(X_n) > \varepsilon\} = 0.$$

By the triangle inequality, we can write for every $\delta > 0$,

$$\omega_\delta(X_n) = \omega_\delta(U_n + V_n + W_n) \le \omega_\delta(U_n) + \omega_\delta(V_n) + \omega_\delta(W_n),$$

in such a way that

$$\mathbb{P}\{\omega_\delta(X_n) > \varepsilon\} \le \mathbb{P}\{\omega_\delta(U_n) > \varepsilon/3\} + \mathbb{P}\{\omega_\delta(V_n) > \varepsilon/3\} + \mathbb{P}\{\omega_\delta(W_n) > \varepsilon/3\}.$$

Using the estimate $\omega_\delta(W_n) \le 2\sup_{\mathbf{t}\in[0,1]^2} |W_n(\mathbf{t})|$ and letting $n \to \infty$ and $\delta \to 0$ then implies the desired conclusion from (IV.A.1) and assumption (iii) in the statement.

Appendix IV.B Moment estimates for suprema of Gaussian fields

In what follows we consider a centred smooth stationary Gaussian field $G = \left\{G(x) : x \in \mathbb{R}^d\right\}$ on $\mathbb{R}^d$ with covariance function $\mathbb{E}\left[G(x)G(y)\right] = \kappa(x - y)$. For an integer $j \ge 0$ and $\mathcal{D} \subset \mathbb{R}^d$, we write

$$\sigma^2(\mathcal{D}; j) := \sup_{x\in\mathcal{D}} \sup_{|\alpha|\le j} \mathbb{E}\left[(\partial_\alpha G(x))^2\right],$$

where $\partial_\alpha G(x) := \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d} G(x)$, for $\alpha := (\alpha_1, \dots, \alpha_d)$ with $|\alpha| := \sum_{k=1}^d \alpha_k$. Moreover, for $\mathcal{D} \subset \mathbb{R}^d$ and $\varepsilon > 0$, we write $\mathcal{D}^{(\varepsilon)}$ for the ε-enlargement of $\mathcal{D}$. Finally, we use the notation

$$\|f\|_{C^j(\mathcal{D})} := \sup_{x\in\mathcal{D}} \sup_{|\alpha|\le j} |\partial_\alpha f(x)|$$

for $f : \mathbb{R}^d \to \mathbb{R}$. The goal of this section is to prove Proposition IV.2.6, whose statement we recall for convenience.

Proposition IV.B.1. *Let the above setting prevail. Assume that for every $m \ge 0$, there exists $\tilde{\sigma}^2(m) < \infty$ such that*

$$\mathbb{E}\left[(\partial_\alpha G(x))^2\right] \le \tilde{\sigma}^2(m), \quad \forall \alpha \in \mathbb{N}^d, |\alpha| \le m. \tag{IV.B.1}$$

Then, for every $p \ge 1$ and $j \ge 0$

$$\mathbb{E}\left[\|G\|_{C^j(\mathcal{D})}^p\right] \le C\left\{\log(\mathrm{vol}(\mathcal{D}))\right\}^{p/2}$$

where $C > 0$ is an absolute constant depending on p and j, and $\mathrm{vol}(\mathcal{D})$ *is the d-dimensional volume of $\mathcal{D}$.*

We remark that assumption (IV.B.1) in particular implies that $\sigma^2(\mathcal{D}; j) \le \tilde{\sigma}^2(j)$ for every $j \ge 0$.

IV.B.1 Proof of Proposition IV.B.1

The proof[1] of Proposition IV.B.1 is based on several classical concentration inequalities for suprema of Gaussian fields, that we state here below. The first statement is an estimate for the first moment of $\|G\|_{C^j(\mathcal{D})}$ (see [NS16, Appendix A.9]).

Proposition IV.B.2. *Let the above setting prevail.*

$$\mathbb{E}\left[\|G\|_{C^j(\mathcal{D})}\right] \le c_1(\mathcal{D})\sigma(\mathcal{D}^{(1)}; j+1), \tag{IV.B.2}$$

where $c_1(\mathcal{D})$ is a constant depending on $\mathcal{D}$.

The following inequality is the so-called *Borell-TIS inequality* applied to the Gaussian field $\partial_\alpha G$, see for instance [AT07, Theorem 2.1.1].

Proposition IV.B.3. *For every $\alpha \in \mathbb{N}^d$ and $u > 0$, we have*

$$\mathbb{P}\left\{\sup_{x\in D} \partial_\alpha G(x) > \mathbb{E}\left[\sup_{x\in\mathcal{D}} \partial_\alpha G(x)\right] + u\right\} \le e^{-\frac{u^2}{2\sigma^2(\mathcal{D};|\alpha|)}}. \tag{IV.B.3}$$

Combining the contents of Propositions IV.B.2 and IV.B.3, we deduce that for every $\alpha \in \mathbb{N}^d$ with $|\alpha| \le j$ and $u > 0$

$$\mathbb{P}\left\{\sup_{x\in\mathcal{D}} \partial_\alpha G(x) > c_1(\mathcal{D})\tilde{\sigma}(j+1) + u\right\} \le e^{-\frac{u^2}{2\tilde{\sigma}^2(j+1)}}$$

which implies (by symmetry)

$$\mathbb{P}\left\{\sup_{x\in\mathcal{D}} |\partial_\alpha G(x)| > c_1(\mathcal{D})\tilde{\sigma}(j+1) + u\right\} \le 2e^{-\frac{u^2}{2\tilde{\sigma}^2(j+1)}}.$$

Therefore summing over all possible α with $|\alpha| \le j$,

$$\mathbb{P}\left\{\|G\|_{C^j(\mathcal{D})} > c_1(\mathcal{D})\tilde{\sigma}(j+1) + u\right\} \le k(j,d)e^{-\frac{u^2}{2\sigma^2(j+1)}} \tag{IV.B.4}$$

where $k(j,d) := 2\mathrm{card}\left\{\alpha \in \mathbb{N}^d : |\alpha| = j\right\}$.

We can now prove Proposition IV.B.1.

Proof of Proposition IV.B.1. By stationarity of G it follows that, if $\mathcal{D}'$ is a translation of $\mathcal{D}$, then necessarily $c_1(\mathcal{D}) = c_1(\mathcal{D}')$, where $c_1(\mathcal{D})$ is the constant appearing in (IV.B.2). In particular, applying (IV.B.2) in the case where $\mathcal{D}$ is a ball $\mathbb{B}$ with unit radius and exploiting the moment assumption (IV.B.1) on G, we deduce that

$$\mathbb{E}\left[\|G\|_{C^j(\mathbb{B})}\right] \le c_1\tilde{\sigma}(j+1),$$

where c_1 is a universal constant. Therefore, applying (IV.B.4) with $\mathcal{D} = \mathbb{B}$ yields

$$\mathbb{P}\left\{\|G\|_{C^j(\mathbb{B})} > c_1\tilde{\sigma}(j+1) + u\right\} \le k(j,d)e^{-\frac{u^2}{2\tilde{\sigma}^2(j+1)}}, \quad u > 0.$$

Now, using the above inequality with $u = t - c_1\tilde{\sigma}(j+1)$, we can write for every $b > 0$ (setting $k := k(j,d), \tilde{\sigma} := \tilde{\sigma}(j+1)$),

$$\mathbb{E}\left[e^{b\|G\|_{C^j(\mathbb{B})}}\right] = 1 + b\int_0^\infty e^{tb}\mathbb{P}\left\{\|G\|_{C^j(\mathbb{B})} > c_1\tilde{\sigma} + (t - c_1\tilde{\sigma})\right\} dt$$

[1] I acknowledge preliminary computations by Giovanni Peccati for this part.

$$
\begin{aligned}
&= e^{bc_1\tilde{\sigma}} + b\int_{c_1\tilde{\sigma}}^{\infty} e^{tb}\mathbb{P}\left\{\|G\|_{C^j(\mathbb{B})} > c_1\tilde{\sigma} + (t - c_1\tilde{\sigma})\right\} dt \\
&\leq e^{bc_1\tilde{\sigma}} + bk\int_{c_1\tilde{\sigma}}^{\infty} e^{tb}e^{-\frac{(t-c_1\tilde{\sigma})^2}{2\tilde{\sigma}^2}} dt \leq e^{bc_1\tilde{\sigma}} + bk\int_{\mathbb{R}} e^{tb}e^{-\frac{(t-c_1\tilde{\sigma})^2}{2\tilde{\sigma}^2}} dt \\
&= e^{bc_1\tilde{\sigma}} + bk\sqrt{2\pi}\tilde{\sigma}\mathbb{E}\left[e^{bZ}\right], \quad Z \sim \mathcal{N}(c_1\tilde{\sigma}, \tilde{\sigma}^2) \\
&= e^{bc_1\tilde{\sigma}} + bk\sqrt{2\pi}\tilde{\sigma}\left(e^{bc_1\tilde{\sigma}+b^2\tilde{\sigma}^2/2}\right) = e^{bc_1\tilde{\sigma}}(1 + bk\sqrt{2\pi}\tilde{\sigma}e^{b^2\tilde{\sigma}^2/2}) \\
&\leq e^{bc_1\tilde{\sigma}+b^2\tilde{\sigma}^2/2}(1 + bk\sqrt{2\pi}\tilde{\sigma}) \leq e^{bc_1\tilde{\sigma}+b^2\tilde{\sigma}^2/2+bk\sqrt{2\pi}\tilde{\sigma}} \\
&= e^{b\tilde{\sigma}(c_1+k\sqrt{2\pi})+b^2\tilde{\sigma}^2/2}, \qquad \text{(IV.B.5)}
\end{aligned}
$$

where we used that $1 + x \leq e^x$. Now for $\mathcal{D} \subset \mathbb{R}^d$ we denote by $N_{\mathcal{D}}$ the minimal number of unit balls needed to cover $\mathcal{D}$ and by $\mathcal{B}_{\mathcal{D}} := \{\mathbb{B}_1, \ldots, \mathbb{B}_{N_{\mathcal{D}}}\}$ the collection of all unit balls covering $\mathcal{D}$ in such a way that $\mathrm{card}(\mathcal{B}_{\mathcal{D}}) = N_{\mathcal{D}}$. Then, we have that, for every $b > 0$

$$
\begin{aligned}
\mathbb{E}\left[\|G\|_{C^j(\mathcal{D})}\right] &= \mathbb{E}\left[\log\exp(b^{-1}b\|G\|_{C^j(\mathcal{D})})\right] = b^{-1}\mathbb{E}\left[\log e^{b\|G\|_{C^j(\mathcal{D})}}\right] \\
&\leq b^{-1}\log\mathbb{E}\left[e^{b\|G\|_{C^j(\mathcal{D})}}\right] \leq b^{-1}\log\sum_{l=1}^{N_D}\mathbb{E}\left[e^{b\|G\|_{C^j(\mathbb{B}_l)}}\right] \\
&\leq b^{-1}\log\left(N_{\mathcal{D}}\mathbb{E}\left[e^{b\|G\|_{C^j(\mathbb{B}_1)}}\right]\right) \\
&\leq b^{-1}\log\left(N_{\mathcal{D}}e^{b\tilde{\sigma}(c_1+k\sqrt{2\pi})+b^2\tilde{\sigma}^2/2}\right) \qquad \text{using (IV.B.5)} \\
&= b^{-1}\log(N_{\mathcal{D}}) + \tilde{\sigma}(c_1 + k\sqrt{2\pi}) + b\frac{\tilde{\sigma}^2}{2} =: h(b).
\end{aligned}
$$

Differentiating h with respect to b, we find that $h(b) \leq h(b_0)$ for $b_0 = \sqrt{2}\sqrt{\log(N_{\mathcal{D}})}/\tilde{\sigma}$ and thus

$$
\mathbb{E}\left[\|G\|_{C^j(\mathcal{D})}\right] \leq h(b_0) = \sqrt{2}\tilde{\sigma}\sqrt{\log(N_{\mathcal{D}})} + \tilde{\sigma}(c_1 + k\sqrt{2\pi}) =: \mu. \qquad \text{(IV.B.6)}
$$

Now let $p \geq 1$. Then, using the inequality

$$
\mathbb{P}\left\{\|G\|_{C^j(\mathcal{D})} > \mu + u\right\} \leq ke^{-\frac{u^2}{2\tilde{\sigma}^2}}, \quad u > 0
$$

together with (IV.B.6), yields

$$
\begin{aligned}
\mathbb{E}\left[\|G\|^p_{C^j(\mathcal{D})}\right] &= p\int_0^{\infty} t^{p-1}\mathbb{P}\left\{\|G\|_{C^j(\mathcal{D})} > \mu + (t - \mu)\right\} dt \\
&\leq \mu^p + pk\int_{\mu}^{\infty} t^{p-1}e^{-\frac{(t-\mu)^2}{2\tilde{\sigma}^2}} dt \leq \mu^p + pk\int_{\mathbb{R}} |t|^{p-1}e^{-\frac{(t-\mu)^2}{2\tilde{\sigma}^2}} dt \\
&= \mu^p + pk\sqrt{2\pi}\tilde{\sigma}\mathbb{E}\left[|Z|^{p-1}\right], \qquad Z \sim \mathcal{N}(\mu, \tilde{\sigma}^2).
\end{aligned}
$$

Now for $Z \sim \mathcal{N}(\mu, \tilde{\sigma}^2)$ and $Z' := (Z - \mu)/\tilde{\sigma} \sim \mathcal{N}(0, 1)$,

$$
\mathbb{E}\left[|Z|^{p-1}\right] = \tilde{\sigma}^{p-1}\mathbb{E}\left[|Z' + \mu/\tilde{\sigma}|^{p-1}\right] \leq 2^{p-2}\tilde{\sigma}^{p-1}\left(\mathbb{E}\left[|Z'|^{p-1}\right] + (\mu/\tilde{\sigma})^{p-1}\right) =: C_p(\tilde{\sigma}^{p-1} + \mu^{p-1}),
$$

where $C_p := 2^{p-2}\mathbb{E}\left[|Z'|^{p-1}\right]$ depends only on p, so that

$$
\mathbb{E}\left[\|G\|^p_{C^j(\mathcal{D})}\right] \leq \mu^p + pk\sqrt{2\pi}\tilde{\sigma}C_p(\tilde{\sigma}^{p-1} + \mu^{p-1}).
$$

The conclusion follows from the definition of μ in (IV.B.6) and the fact that there are constants $C_1, C_2 > 0$ such that $C_1\mathrm{vol}(\mathcal{D}) \leq N_{\mathcal{D}} \leq C_2\mathrm{vol}(\mathcal{D})$. □

Chapter V

On non-linear functionals associated with the d-dimensional Berry Random Wave Model

In this chapter, we study non-linear functionals associated with the d-dimensional Berry Random Wave Model, already studied in Chapter IV in dimension two. More precisely, writing $B_E^{(d)} = \left\{B_E^{(d)}(x) : x \in \mathbb{R}^d\right\}$ for Berry's random field in $\mathbb{R}^d$, we establish variance estimates in the high-energy limit (as $E \to \infty$) for random variables admitting an integral representation of the type

$$Z_E(d, q; \mathcal{D}) := \int_{\mathcal{D}} H_q(B_E^{(d)}(x))dx,$$

where $d \geq 2$ and $q \geq 3$ are integers, $\mathcal{D}$ is a convex d-dimensional domain and H_q is the q-th Hermite polynomial on the real line defined by the relation

$$H_q(x)e^{-x^2/2} = (-1)^q \frac{d^q}{dx^q} e^{-x^2/2}.$$

Our main results, involving the study of the asymptotic behaviour of powers of Bessel functions, show that, when $(d, q) = (2, 4)$, the variance exhibits a logarithmic behaviour, whereas, when $(d, q) \neq (2, 4)$, the variance is of constant lower order. Such an observation is consistent with the related model of spherical harmonics on the d-sphere investigated in [MW14] (for the two-dimensional sphere) and [MR15] (for arbitrary dimensions) and is tightly connected to the Berry cancellation phenomenon discussed in Sections II.1.4 and II.2 of Chapter II. In Theorem V.1.2, we subsequently prove quantitative Central Limit Theorems for the normalized versions of the random variables $Z_E(d, q; \mathcal{D})$.

V.1 Variance estimates and Central Limit Theorems

In view of Theorem I.1.8 (see also Example (ii) in Section II.2.2 of Chapter II), the covariance function of the d-dimensional Berry random wave is given by

$$r_d^E(x - y) = \mathbb{E}\left[B_E^{(d)}(x)B_E^{(d)}(y)\right] = \frac{J_{\frac{d-2}{2}}(2\pi\sqrt{E}\|x - y\|)}{(2\pi\sqrt{E}\|x - y\|)^{\frac{d-2}{2}}}, \quad x, y \in \mathbb{R}^d, \tag{V.1.1}$$

where J_α denotes the Bessel function of the first kind of order α. We now consider the random variable

$$Z_E(d, q; \mathcal{D}) := \int_{\mathcal{D}} H_q(B_E^{(d)}(x))dx, \tag{V.1.2}$$

where H_q indicates the q-th Hermite polynomial. In Theorem V.1.1 below, we provide asymptotic laws for the variance of $Z_E(q,d;\mathcal{D})$ when (d,q) are pairs belonging to the set

$$S := \{(d,q) : d \geq 2, q \geq 3\} \setminus \{(2,3),(3,3)\}\,. \tag{V.1.3}$$

We write vol_d to indicate the d-dimensional volume and $\kappa_d := \frac{\pi^{d/2}}{\Gamma(d/2+1)}$ for the volume of the unit ball in dimension d.

Theorem V.1.1. *Let $(d,q) \in S$. If $(d,q) = (2,4)$, we have that, as $E \to \infty$,*

$$\mathbf{Var}\left[Z_E(d,q;\mathcal{D})\right] \sim \frac{9}{\pi^3}\mathrm{vol}_2(\mathcal{D})\frac{\log E}{E}\,,$$

whereas, if $(d,q) \neq (2,4)$, we have that, as $E \to \infty$

$$\mathbf{Var}\left[Z_E(d,q;\mathcal{D})\right] \sim \alpha(d;q)\frac{1}{E^{d/2}}\,,$$

where

$$\alpha(d;q) := \mathrm{vol}_d(\mathcal{D})q!d\kappa_d(2\pi)^{-\frac{q}{2}(d-2)}\int_0^\infty d\psi\, J_{\frac{d-2}{2}}(2\pi\psi)^q\psi^{(d-1)(1-\frac{q}{2})+\frac{q}{2}}. \tag{V.1.4}$$

We prove quantitative Central Limit Theorems for the normalized random variables $Z_E(d,q;\mathcal{D})$ in the high-energy limit. Since the case of $Z_E(2,4;\mathcal{D})$ is solved in [NPR19] (see the random variable $a_{1,E}$ p.124 therein and Proposition 8.2) and this is the unique case where the variance is not of order $E^{-d/2}$ for pairs of integers $(d,q) \in S$, we consider the normalizations

$$\widetilde{Z}_E(d,q;\mathcal{D}) := E^{d/4}Z_E(d,q;\mathcal{D})$$

for every pair $(d,q) \neq (2,4)$. In the following theorem, we prove quantitative CLTs in Kolmogorov, total variation and Wasserstein distance (see Section I.1.2.3 for their definitions) for the random variables $\widetilde{Z}_E(d,q;\mathcal{D})$. Note that, in the statement below, the exponent $d - q\frac{d-1}{2} < 0$ for pairs $(d,q) \in S$ such that $(d,q) \neq (2,4)$, which allows to ensure asymptotic Gaussianity.

Theorem V.1.2. *Assume that $(d,q) \in S$ is such that $(d,q) \neq (2,4)$ and let $N \sim \mathcal{N}(0,\alpha(d;q)^2)$, where $\alpha(d;q)$ is as in* (V.1.4). *Then, for* $\mathrm{U} \in \{\mathrm{Kol},\mathrm{TV},\mathrm{W}\}$*, we have that*

$$d_{\mathrm{U}}\left(\widetilde{Z}_E(d,q;\mathcal{D}),N\right) \leq c_1(\log E)^{3/2}\sqrt{E}^{d-q\frac{d-1}{2}},$$

where c_1 is some absolute constant that is independent of E. In particular, $\widetilde{Z}_E(d,q;\mathcal{D})$ converges in distribution to N, as $E \to \infty$.

V.1.1 Comparison with [MR15] and [DEL21] and some remarks

In [MR15], the authors present a similar study for spherical harmonics on the d-sphere. More precisely, they consider the random variables

$$h_{\ell;q,d} := \int_{\mathbb{S}^d} H_q(T_\ell^{(d)}(x))dx$$

where $T_\ell^{(d)}$ is a random spherical harmonic of order ℓ, that is a centred stationary Gaussian process with covariance function

$$\mathbb{E}\left[T_\ell^{(d)}(x)T_\ell^{(d)}(y)\right] = G_{\ell;d}(\cos\theta(x,y)), \quad x,y \in \mathbb{S}^d.$$

Here $\theta(x, y)$ denotes the spherical distance between x and y, and $G_{\ell;d}$ indicates the Gegenbauer polynomial of order ℓ, defined by

$$G_{\ell;d}(x) = P_\ell^{(\frac{d}{2}-1,\frac{d}{2}-1)}(x),$$

where $P_\ell^{(a,b)}$ are the Jacobi polynomials which are orthogonal with respect to the weight function $t \mapsto (1-t)^a(1+t)^b$ for $t \in [-1, 1]$, see for [Sze39] for more details. The authors show that for $d, q \geq 3$ (see [MR15, Proposition 1.1])

$$\mathbf{Var}\left[h_{\ell;q,d}\right] = \frac{1}{\ell^d} 2q!\mu_d\mu_{d-1} \int_0^{\pi/2} G_{\ell;d}(\cos\theta)^q (\sin\theta)^{d-1} d\theta \sim \frac{1}{\ell^d} C(d, q), \quad \ell \to \infty$$

where

$$C(d, q) := 2q!\mu_d\mu_{d-1} \left(2^{\frac{d}{2}-1}\left(\frac{d}{2}-1\right)!\right)^q \int_0^\infty d\psi\, J_{\frac{d}{2}-1}(\psi)^q \psi^{(d-1)(1-q/2)+q/2}$$

with $\mu_d := \frac{2\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})}$. This shows that the order of magnitude $E^{-d/2}$ derived in Theorem V.1.1 is consistent with the findings of [MR15], as the energy $\sqrt{E}$ on the Berry random wave model corresponds to ℓ in the case of spherical harmonics. The case $(d, q) = (2, 4)$ was investigated in [MW14], where it is shown that its variance is of order $\ell^{-2}\log\ell$. Furthermore, we remark that our constants $\alpha(d; q)$ in (V.1.4) involve exactly the same integral expressions in terms of Bessel functions, and differ from $C(d, q)$ only by multiplicative constants, appearing notably as an artifact of change of variables on the sphere and the *Hilb's asymptotic* formula for Jacobi polynomials (see for instance [Sze39, Theorem 8.21.12])

$$(\sin\theta)^{\frac{d}{2}-1} G_{\ell;d}(\cos\theta) = \frac{2^{\frac{d}{2}-1}}{\binom{\ell+\frac{d}{2}-1}{\ell}} \left(\frac{\Gamma(\ell+\frac{d}{2})}{(\ell+\frac{d-1}{2})^{\frac{d}{2}-1}\ell!}\left(\frac{\theta}{\sin\theta}\right)^{1/2} J_{\frac{d}{2}-1}(L\theta) + R_{\ell,d}(\theta)\right),$$

where $L := \ell + \frac{d-1}{2}$ and $R_{\ell,d}(\theta)$ is a certain remainder. We also point out that the constants $\alpha(d; q)$ are non-negative for every $(d, q) \in S$ and strictly positive for all even q. Conjecturally, this is the case for all pairs (d, q), see also [MR15]. Also, our findings in Theorem V.1.2 are to be compared to Theorem 1.2 and Corollary 1.3 therein. We remark that the logarithmic factor appearing in the quantitative bound of Theorem V.1.2 does not appear in each of the cases studied in [MR15], and is presumably not optimal, but sufficient to ensure a Central Limit Theorem.

In [DEL21], the authors study the asymptotic behaviour of *dislocation lines* of zero sets associated with two independent copies of Berry's random field with fixed energy on growing cubes. More specifically, denoting by B_1 and $\hat{B}_1$ two independent copies of Berry's random fields with unit energy and by $\mathcal{N}_1(Q_E)$ the length of the dislocation lines restricted to the cube $Q_E = [-E, E]^3$, they show that (see [DEL21, Proposition 3.4], note that therein the energy E is replaced with n) $\mathbf{Var}[\mathcal{N}_1(Q_E)] \sim V\text{vol}(Q_E) \approx VE^{3/2}$, as $E \to \infty$, where $V \geq 0$ is some finite constant. We remark that such a result is consistent with our findings: Indeed, using the equality in distribution $B_E(x) \overset{d}{=} B_1(2\pi\sqrt{E}x)$, the length of dislocation lines associated with B_E and $\hat{B}_E$ in some fixed domain $\mathcal{D} \subset \mathbb{R}^d$ is given by

$$\begin{aligned}
\mathcal{N}_E(\mathcal{D}) &= \int_{\mathcal{D}} \delta_0(B_E(x))\delta_0(\hat{B}_E(x))|\det(\text{jac}_{B_E,\hat{B}_E}(x))|dx \\
&\overset{d}{=} 4\pi^2 E \int_{\mathcal{D}} \delta_0(B_1(2\pi\sqrt{E}x))\delta_0(\hat{B}_1(2\pi\sqrt{E}x)) \left|\det(\text{jac}_{B_1,\hat{B}_1}(2\pi\sqrt{E}x))\right| dx \\
&= \frac{4\pi^2}{(2\pi)^{d/2}} E^{1-d/2} \int_{2\pi\sqrt{E}\mathcal{D}} \delta_0(B_1(y))\delta_0(\hat{B}_1(y)) \left|\det(\text{jac}_{B_1,\hat{B}_1}(y))\right| dy
\end{aligned}$$

$$= \frac{4\pi^2}{(2\pi)^{d/2}} E^{1-d/2} \mathcal{N}_1(2\pi\sqrt{E}\mathcal{D}),$$

where we performed the change of variable $y = 2\pi\sqrt{E}x$, so that

$$\mathbf{Var}\Big[\mathcal{N}_1(2\pi\sqrt{E}\mathcal{D})\Big] = \frac{(2\pi)^d}{16\pi^4} E^{d-2}\mathbf{Var}[\mathcal{N}_E(\mathcal{D})].$$

Since, by Theorem V.1.1, the random variables $Z_E(d,q;\mathcal{D})$ have the same order of variance as $d \geq 3$, and $E \cdot Z_E(d,q;\mathcal{D})$ are typical elements appearing in the projection of $\mathcal{N}_E(\mathcal{D})$ on the q-th Wiener chaos, it is natural to expect that the variance order of the q-th Wiener chaos of $\mathcal{N}_1(2\pi\sqrt{E}\mathcal{D})$ is $E^{d-2}E^2E^{-d/2} = E^{d/2} \approx \mathrm{vol}([-\sqrt{E},\sqrt{E}]^d)$, showing that our findings are consistent with those of [DEL21] (in the case $d = 3$). In view of the reduction principles discussed later in Section V.3, it is to be expected that the fourth Wiener chaos of $\mathcal{N}_1(2\pi\sqrt{E}\mathcal{D})$ is asymptotically equivalent to a scalar multiple of the random variable $EZ_E(3,4;\mathcal{D})$, whose variance is strictly positive since $\alpha(3;4) > 0$. This heuristic observation may justify the fact that the constant V, appearing in [DEL21, Proposition 3.2] and discussed above, is strictly positive, which was not proven in [DEL21]. Thanks to Theorem V.1.2, we furthermore know that the normalized random variable $\widetilde{Z}_E(3,4;\mathcal{D})$ satisfies a Central Limit Theorem, which is consistent with the findings of [DEL21, Proposition 3.2], once it is shown that $V > 0$.

Let us make some further comments on our results.

Remark V.1.3. **(a)** Our techniques for proving Theorem V.1.1 rely on the asymptotic study of moments of Bessel functions and make use of the decay property $|J_\alpha(z)| = O(z^{-1/2})$, for $\alpha \geq 0$ and $z > 0$. Consequently, our computations naturally require the comparison of the quantity $(d-1)(1-q/2)$ with -1. We note that (i) if $(d,q) \in S$, then $(d-1)(1-q/2) \leq -1$ and (ii) the pair $(d,q) = (2,4)$ is the only pair in S verifying the condition $(d-1)(1-q/2) = -1$. Carrying out our computations more generally for the case $(d-1)(1-q/2) = -1$, our arguments show that the first part of Theorem V.1.1 can be written as

$$\mathbf{Var}[Z_E(d,q;\mathcal{D})] \sim q!d\kappa_d \mathrm{vol}_d(\mathcal{D})c(d;q)\frac{1}{2}\mathrm{vol}_2(\mathcal{D})\frac{\log E}{E},$$

where $c(d;q) := \pi^{-q}(2\pi)^{-\frac{q}{2}(d-2)}\frac{1}{2^q}\binom{q}{\frac{q}{2}}$. Specifying that $(d,q) = (2,4)$ yields

$$q!d\kappa_d c(d;q)\frac{1}{2} = 4!2(2\pi)c(2;4)\frac{1}{2} = \frac{9}{\pi^3},$$

as stated in Theorem V.1.1. Such a result is consistent with the findings of [NPR19] (see in particular the random variable $a_{1,E}$ and Lemma 8.5 therein), where the authors study the nodal length of the Berry random wave in $\mathbb{R}^2$.

(b) The reader might wonder why we restrict the choice of (d,q) to the set S. The reason for this is the following: when dealing with the cases $(d,q) \notin S$ (corresponding to the pairs $(d \geq 2, q = 1)$, $(d \geq 2, q = 2)$, $(d,q) = (2,3)$ and $(d,q) = (3,3)$), the same techniques used to prove Theorem V.1.1 allow us to obtain a variance estimate of the form

$$\mathbf{Var}[Z_E(d,q;\mathcal{D})] = A(d;q)a_E(d;q) + O(b_E(d;q)),$$

where $A(d;q)$ is a constant depending only on d,q and $\mathcal{D}$ and $\{a_E(d;q) : E > 0\}, \{b_E(d;q) : E > 0\}$ are typically convergence rates (depending on d

and q) such that $a_E(d;q) = o(b_E(d;q))$ or $a_E(d;q) \approx b_E(d;q)$ (of the same order), thus only resulting in upper bounds and not true asymptotics. More specifically, computations show that

$$\begin{aligned}
\mathbf{Var}[Z_E(d,1;\mathcal{D})] &= \frac{C^E_{d,1}}{E^{(d+1)/4}} + O(E^{-(d-1)/4}), \quad d \geq 2 \\
\mathbf{Var}[Z_E(d,2;\mathcal{D})] &= \frac{C_{d,2}}{E^{(d-1)/2}} + O(E^{-(d-1)/2}), \quad d \geq 2 \\
\mathbf{Var}[Z_E(2,3;\mathcal{D})] &= \frac{\alpha(2;3)}{E} + O(E^{-3/4}), \\
\mathbf{Var}[Z_E(3,3;\mathcal{D})] &= \frac{\alpha(3;3)}{E^{3/2}} + O(E^{-3/2}),
\end{aligned}$$

where $C^E_{d,1}$ is a constant depending on d and E such that $C^E_{d,1} = O(1)$ and $C_{d,2}$ is some explicitly computable constant that is independent of E. We remark that the case $q = 1$ and $q = 2$ behave differently than the cases $q \geq 3$. We remark that the case of $Z_E(d,2;\mathcal{D})$ is consistent with the findings of [MR15], where it is shown that the variance of $h_{\ell;2,d}$ is of magnitude $\ell^{-(d-1)}$. We conjecture that in the above estimates, the true asymptotic behaviour is given by $a_E(d;q)$. By inspection of our proof, we believe that such a difficulty emerges from bounding certain residual terms (see in particular Proposition V.2.2) by means of the *Steiner formula* for convex bodies, which presumably leads to suboptimal bounds in the afore-mentioned cases and expect that their study can be carried out separately. We also point out that the idea of using the Steiner formula originates from [NPR19].

V.2 Proof of our main results

V.2.1 Proof of Theorem V.1.1

The variance of $Z_E(d,q;\mathcal{D})$ is computed using the covariance relation for Hermite polynomials in Proposition I.1.22. Indeed, this relation yields

$$\begin{aligned}
\mathbf{Var}\left[Z_E(d,q;\mathcal{D})\right] &= \int_{\mathcal{D}^2} \mathbb{E}\left[H_q(B_E(x))H_q(B_E(y))\right] dxdy \\
&= q! \int_{\mathcal{D}^2} \mathbb{E}\left[B_E(x)B_E(y)\right]^q dxdy = q! \int_{\mathcal{D}^2} r^E_d(x-y)^q dxdy, \qquad \text{(V.2.1)}
\end{aligned}$$

where r^E_d is as in (V.1.1). We introduce some notation.

Notation V.2.1.

- *For* $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, *we use hyperspherical coordinates* $(\phi, \varphi) := (\phi, \varphi_1, \dots, \varphi_{d-2}, \varphi_{d-1}) \in \mathbb{R}_+ \times [0,\pi]^{d-2} \times [0,2\pi]$ *given by*

$$\begin{aligned}
&x_1 = \phi\cos(\varphi_1), \quad x_2 = \phi\sin(\varphi_1)\cos(\varphi_2), \quad x_3 = \phi\sin(\varphi_1)\sin(\varphi_2)\cos(\varphi_3), \dots, \\
&x_{d-1} = \phi\sin(\varphi_1)\sin(\varphi_2)\cdots\sin(\varphi_{d-2})\cos(\varphi_{d-1}), \quad x_d = \phi\sin(\varphi_1)\sin(\varphi_2)\cdots\sin(\varphi_{d-2})\sin(\varphi_{d-1})
\end{aligned}$$

  with Jacobian transform $dx = \phi^{d-1}\prod_{k=1}^{d-2}\sin(\varphi_k)^{d-k-1} d\phi d\varphi_1 \cdots d\varphi_{d-1}$.

- *For* $x \in \mathbb{R}^d$ *and* $R > 0$, *we denote by* $\mathbb{B}(x,R)$ *the* d*-dimensional ball of radius* R *centred at* x *and* $\partial\mathbb{B}(x,R)$ *its boundary. The diameter and the inner radius of a domain* $\mathcal{D}$ *are defined by*

$$\operatorname{diam}(\mathcal{D}) = \sup_{x,y\in\mathcal{D}} \|x-y\|, \quad \operatorname{inrad}(\mathcal{D}) = \sup\{R > 0 : \exists x \in \mathcal{D} : \mathbb{B}(x,R) \subseteq \mathcal{D}\},$$

  respectively. Furthermore, for $\phi \in [0, \operatorname{inrad}(\mathcal{D}))$, *we let* $\mathcal{D}_\phi := \{x \in \mathcal{D} : B(x,\phi) \subseteq \mathcal{D}\}$.

- *As usual, for numerical sequences* $\{a_E : E > 0\}, \{b_E : E > 0\}$*, we use the notation* $a_E = O(b_E)$ *or* $a_E \ll b_E$ *to indicate that* $|a_E| \leq C|b_E|$ *for some absolute constant* $C > 0$.

The following preliminary proposition gives a reformulation of the double integral appearing in the R.H.S in (V.2.1), needed to complete the proof of Theorem V.1.1.

Proposition V.2.2. *For every pair* $(d, q) \in S$*, we have as* $E \to \infty$,

$$\int_{\mathcal{D}^2} r_d^E(x-y)^q dxdy = \text{vol}_d(\mathcal{D}) \int_0^{\text{diam}(\mathcal{D})} d\phi \int_{\mathcal{R}_d} d\varphi\, \hat{r}_d^E(\phi, \varphi)\phi^{d-1} \prod_{k=1}^{d-2} \sin(\varphi_k)^{d-k-1} + O\left(\beta_E(d; q)\right),$$

where $\mathcal{R}_d := [0, \pi]^{d-2} \times [0, 2\pi]$*,* $d\varphi = d\varphi_1 \cdots d\varphi_{d-1}$*,*

$$\hat{r}_d^E(\phi, \varphi) := r_d^E(\phi \cos(\varphi_1), \dots, \phi \sin(\varphi_1)\sin(\varphi_2)\sin(\varphi_3)\cdots\sin(\varphi_{d-1}))$$

and

$$\beta_E(d; q) = \begin{cases} E^{-\frac{q}{4}(d-1)} & ; (d-1)(1-\frac{q}{2}) = -1 \\ E^{-\frac{d+1}{2}} + E^{-\frac{d+\ell-1}{2}} \log E & ; (d-1)(1-\frac{q}{2}) = -\ell, \ell > 1 \end{cases}.$$

Proof. By the Co-Area formula (see Proposition I.1.11), we have

$$\int_{\mathcal{D}^2} r_d^E(x-y)^q dxdy = \int_0^{\text{diam}(\mathcal{D})} d\phi \int_{\mathcal{D}} dx \int_{\partial\mathbb{B}(x,\phi)\cap\mathcal{D}} dy\, r_d^E(x-y)^q = \int_0^{\text{diam}(\mathcal{D})} d\phi F^E(\phi),$$

where we set

$$F^E(\phi) = \int_{\mathcal{D}} dx \int_{\partial\mathbb{B}(x,\phi)\cap\mathcal{D}} dy\, r_d^E(x-y)^q.$$

For $\phi \in (0, \text{inrad}(\mathcal{D}))$, we decompose

$$F^E(\phi) = \int_{\mathcal{D}_\phi} dx \int_{\partial B(x,\phi)} dy\, r_d^E(x-y)^q + \int_{\mathcal{D}\setminus\mathcal{D}_\phi} dx \int_{\partial B(x,\phi)\cap\mathcal{D}} dy\, r_d^E(x-y)^q =: F_I^E(\phi) + F_{II}^E(\phi).$$

We start by considering $F_I^E(\phi)$. Using hyperspherical coordinates (ϕ, φ) for y on $\partial\mathbb{B}(x, \phi)$, we can rewrite $F_I^E(\phi)$ as

$$\begin{aligned} F_I^E(\phi) &= \int_{\mathcal{D}_\phi} dx \int_{\mathcal{R}_d} d\varphi\, \hat{r}_d^E(\phi, \varphi)^q \phi^{d-1} \prod_{k=1}^{d-2} \sin(\varphi_k)^{d-k-1} \\ &= \text{vol}_d(\mathcal{D}_\phi) \int_{\mathcal{R}_d} d\varphi\, \hat{r}_d^E(\phi, \varphi)^q \phi^{d-1} \prod_{k=1}^{d-2} \sin(\varphi_k)^{d-k-1}, \end{aligned}$$

where $\mathcal{R}_d := [0, \pi]^{d-2} \times [0, 2\pi]$ and $d\varphi = d\varphi_1 \cdots d\varphi_{d-1}$ and we set

$$\hat{r}_d^E(\phi, \varphi) := r_d^E(\phi \cos(\varphi_1), \dots, \phi \sin(\varphi_1)\sin(\varphi_2)\sin(\varphi_3)\cdots\sin(\varphi_{d-1})).$$

Now, since $\mathcal{D}_\phi \subseteq \mathcal{D}$, it follows that $\text{vol}_d(\mathcal{D}_\phi) = \text{vol}_d(\mathcal{D}) - \text{vol}_d(\mathcal{D} \setminus \mathcal{D}_\phi)$. By linearity, we can thus write $F_I^E(\phi) = F_{I_1}^E(\phi) + F_{I_2}^E(\phi)$, where

$$F_{I_1}^E(\phi) = \text{vol}_d(\mathcal{D}) \int_{\mathcal{R}_d} d\varphi\, \hat{r}_d^E(\phi, \varphi)^q \phi^{d-1} \prod_{k=1}^{d-2} \sin(\varphi_k)^{d-k-1},$$

$$F_{I_2}^E(\phi) \quad = \quad -\mathrm{vol}_d(\mathcal{D} \setminus \mathcal{D}_\phi) \int_{\mathcal{R}_d} d\varphi\, \hat{r}_d^E(\phi,\varphi)^q \phi^{d-1} \prod_{k=1}^{d-2} \sin(\varphi_k)^{d-k-1}.$$

For $\phi \in (0, \mathrm{diam}(\mathcal{D}))$, we decompose

$$\begin{aligned} F^E(\phi) &= \left(F_{I_1}^E(\phi) + F_{I_2}^E(\phi) + F_{II}^E(\phi)\right) \mathbb{1}[\phi \in (0, \mathrm{inrad}(\mathcal{D}))] + F^E(\phi)\mathbb{1}[\phi \in (\mathrm{inrad}(\mathcal{D}), \mathrm{diam}(\mathcal{D}))] \\ &= F_{I_1}^E(\phi)\mathbb{1}[\phi \in (0, \mathrm{diam}(\mathcal{D}))] + \left(F^E(\phi) - F_{I_1}^E(\phi)\right) \mathbb{1}[\phi \in (\mathrm{inrad}(\mathcal{D}), \mathrm{diam}(\mathcal{D}))] \\ &\quad + \left(F_{I_2}^E(\phi) + F_{II}^E(\phi)\right) \mathbb{1}[\phi \in (0, \mathrm{inrad}(\mathcal{D}))]. \end{aligned} \tag{V.2.2}$$

The proof is concluded once we prove that

$$\left(F^E(\phi) - F_{I_1}^E(\phi)\right) \mathbb{1}[\phi \in (\mathrm{inrad}(\mathcal{D}), \mathrm{diam}(\mathcal{D}))] + \left(F_{I_2}^E(\phi) + F_{II}^E(\phi)\right) \mathbb{1}[\phi \in (0, \mathrm{inrad}(\mathcal{D}))]$$

is bounded by the quantity $\beta_E(d;q)$ appearing in the statement. In order to prove this, we will rely on the following geometric observation[1]: Since $\mathcal{D} \subseteq \mathcal{D}_\phi + 2\phi\mathbb{B}(0,1)$, we have by the Steiner formula (see (III.2.15) and also [SW08, Eq.(14.6)]) that (denoting by $W_j(K)$, $j = 0, 1, \dots, d$ the j-th quermassintegral of $K \subset \mathbb{R}^d$),

$$\begin{aligned} &\mathrm{vol}_d(\mathcal{D} \setminus \mathcal{D}_\phi) = \mathrm{vol}_d(\mathcal{D}) - \mathrm{vol}_d(\mathcal{D}_\phi) \le \mathrm{vol}_d(\mathcal{D}_\phi + 2\phi\mathbb{B}(0,1)) - \mathrm{vol}_d(\mathcal{D}_\phi) \\ = &\sum_{k=0}^{d} \binom{d}{k} W_k(\mathcal{D}_\phi) 2^k \phi^k - W_0(\mathcal{D}_\phi) = \sum_{k=1}^{d} \binom{d}{k} W_k(\mathcal{D}_\phi) 2^k \phi^k \le \sum_{k=1}^{d} \binom{d}{k} W_k(\mathcal{D}) 2^k \phi^k, \end{aligned} \tag{V.2.3}$$

where we used that $W_k(\mathcal{D}_\phi) \le W_k(\mathcal{D})$ since $\mathcal{D}_\phi \subset \mathcal{D}$ and $W_0(\cdot) = \mathrm{vol}_d(\cdot)$. We now distinguish cases $(d-1)(1-\frac{q}{2}) = -1$ and $(d-1)(1-\frac{q}{2}) < -1$.

<u>*Case* $(d-1)(1-\frac{q}{2}) = -1$:</u> Using the bound $|J_{\frac{d-2}{2}}(z)| \ll z^{-1/2}$, we obtain

$$\left(\frac{J_{\frac{d-2}{2}}(2\pi\sqrt{E}\phi)}{(2\pi\sqrt{E}\phi)^{\frac{d-2}{2}}}\right)^q \ll \frac{(2\pi\sqrt{E}\phi)^{-q/2}}{(2\pi\sqrt{E}\phi)^{q\frac{d-2}{2}}} \ll E^{-\frac{q}{4}(d-1)} \phi^{-\frac{q}{2}(d-1)} \tag{V.2.4}$$

so that, using (V.2.3) yields

$$\begin{aligned} |F_{I_2}^E(\phi)| &\le \mathrm{vol}_d(\mathcal{D} \setminus \mathcal{D}_\phi) \int_{\mathcal{R}_d} d\varphi\, |\hat{r}_d^E(\phi,\varphi)|^q \phi^{d-1} \left| \prod_{k=1}^{d-2} \sin(\varphi_k)^{d-k-1} \right| \\ &\ll \sum_{k=1}^{d} \binom{d}{k} W_k(\mathcal{D}) \phi^k \int_{\mathcal{R}_d} d\varphi\, E^{-\frac{q}{4}(d-1)} \phi^{-\frac{q}{2}(d-1)} \phi^{d-1} \\ &\ll E^{-\frac{q}{4}(d-1)} \sum_{k=1}^{d} \phi^{k-\frac{q}{2}(d-1)+d-1} = E^{-\frac{q}{4}(d-1)} \sum_{k=1}^{d} \phi^{k+(d-1)(1-\frac{q}{2})} = E^{-\frac{q}{4}(d-1)} \sum_{k=1}^{d} \phi^{k-1}, \end{aligned}$$

since $(d-1)(1-q/2) = -1$, which is integrable on $(0, \mathrm{inrad}(\mathcal{D}))$. This argument is also valid to bound the contribution of $F_{II}^E(\phi)$:

$$|F_{II}^E(\phi)| \le \int_{\mathcal{D}\setminus\mathcal{D}_\phi} dx \int_{\partial\mathbb{B}(x,\phi)} dy\, |r_d^E(x-y)|^q \ll E^{-\frac{q}{4}(d-1)} \sum_{k=1}^{d} \phi^{k-1},$$

[1] Such a technique was exploited in [NPR19, Proposition 5.1] for the study in dimension two.

where we have used hyperspherical coordinates on $\partial\mathbb{B}(x,\phi)$ and the bounds (V.2.3) and (V.2.4). Let us now bound the contribution of $|F^E(\phi) - F^E_{I_1}(\phi)|$ on $(\mathrm{inrad}(\mathcal{D}), \mathrm{diam}(\mathcal{D}))$. By the triangle inequality and (V.2.4), we have

$$\begin{aligned}
&|F^E(\phi) - F^E_{I_1}(\phi)|\mathbb{1}[\phi \in (\mathrm{inrad}(\mathcal{D}), \mathrm{diam}(\mathcal{D}))] \\
\ll &\left(\frac{J_{\frac{d-2}{2}}(2\pi\sqrt{E}\phi)}{(2\pi\sqrt{E}\phi)^{\frac{d-2}{2}}}\right)^q \phi^{d-1}\frac{\phi}{\phi} \ll \left(\frac{J_{\frac{d-2}{2}}(2\pi\sqrt{E}\phi)}{(2\pi\sqrt{E}\phi)^{\frac{d-2}{2}}}\right)^q \phi^d \ll E^{-\frac{q}{4}(d-1)}\phi^{d-\frac{q}{2}(d-1)},
\end{aligned}$$

which is integrable on $(\mathrm{inrad}(\mathcal{D}), \mathrm{diam}(\mathcal{D}))$. The desired conclusion then follows from (V.2.2) by integrating the above estimates on $(0, \mathrm{diam}(\mathcal{D}))$.

<u>*Case* $(d-1)(1-\frac{q}{2}) < -1$:</u> Let us write $(d-1)(1-\frac{q}{2}) := -\ell < -1$. For the integral of $F^E_{I_2}$, we use (V.2.3) to estimate

$$\begin{aligned}
\left|\int_0^{\mathrm{inrad}(\mathcal{D})} F^E_{I_2}(\phi)d\phi\right| &\ll \int_0^{\mathrm{inrad}(\mathcal{D})} \mathrm{vol}_d(\mathcal{D}\setminus\mathcal{D}_\phi)\left|r^E_d(\phi)\right|^q \phi^{d-1}d\phi \ll \sum_{k=1}^d \int_0^{\mathrm{inrad}(\mathcal{D})} \left|r^E_d(\phi)\right|^q \phi^{d-1+k}d\phi \\
&= \sum_{k=1}^d \left(\frac{1}{E^{\frac{d+k}{2}}}\int_0^1 \left|r^1_d(\psi)\right|^q \psi^{d-1+k}d\psi + \frac{1}{E^{\frac{d+k}{2}}}\int_1^{\sqrt{E}\mathrm{inrad}(\mathcal{D})} \left|r^1_d(\psi)\right|^q \psi^{d-1+k}d\psi\right) =: X_E + Y_E,
\end{aligned}$$

where we performed the change of variables $\psi = \sqrt{E}\phi$. For X_E, we use that $\left|r^1_d(\psi)\right| = O(1)$, uniformly in ψ, so that

$$X_E \ll \sum_{k=1}^d \frac{1}{E^{\frac{d+k}{2}}}\int_0^1 \psi^{d-1+k}d\psi \ll \frac{1}{E^{\frac{d+1}{2}}},$$

since $d-1+k \geq 2$ for $k = 1,\ldots,d$. For Y_E, we use the fact that $|J_{\frac{d-2}{2}}(z)| \ll z^{-1/2}$ implying that $\left|r^1_d(\psi)\right|^q \psi^{d-1+k} \ll \psi^{(d-1)(1-\frac{q}{2})+k} = \psi^{-\ell+k}$. Therefore, it follows that

$$\begin{aligned}
Y_E &\ll \sum_{k=1}^d \frac{1}{E^{\frac{d+k}{2}}}\int_1^{\sqrt{E}\mathrm{inrad}(\mathcal{D})} \psi^{-\ell+k}d\psi(\mathbb{1}[-\ell+k \neq -1] + \mathbb{1}[-\ell+k = -1]) \\
&\ll \sum_{k=1}^d \frac{1}{E^{\frac{d+k}{2}}}\left(E^{\frac{-\ell+k+1}{2}} + \log E + O(1)\right) \ll E^{\frac{-\ell+1-d}{2}} + \frac{\log E}{E^{\frac{d+\ell-1}{2}}} \ll \frac{\log E}{E^{\frac{d+\ell-1}{2}}}. \qquad \text{(V.2.5)}
\end{aligned}$$

This shows that $X_E + Y_E \ll \beta_E(d;q)$. The same bound is valid for $F^E_{II}(\phi)$. For the remaining part, we treat the term

$$\left|\int_{\mathrm{inrad}(\mathcal{D})}^{\mathrm{diam}(\mathcal{D})} F^E(\phi) - F^E_{I_1}(\phi)d\phi\right| \leq \int_{\mathrm{inrad}(\mathcal{D})}^{\mathrm{diam}(\mathcal{D})} \left|F^E(\phi) - F^E_{I_1}(\phi)\right|d\phi.$$

Using the fact that on $(\mathrm{inrad}(\mathcal{D}), \mathrm{diam}(\mathcal{D}))$, we have for every $k = 1,\ldots,d$,

$$\left|F^E(\phi) - F^E_{I_1}(\phi)\right| \ll \left(\frac{J_{\frac{d-2}{2}}(2\pi\sqrt{E}\phi)}{(2\pi\sqrt{E}\phi)^{\frac{d-2}{2}}}\right)^q \phi^{d-1} \ll \left(\frac{J_{\frac{d-2}{2}}(2\pi\sqrt{E}\phi)}{(2\pi\sqrt{E}\phi)^{\frac{d-2}{2}}}\right)^q \phi^{d-1+k}$$

yields after changing variable $\psi = \sqrt{E}\phi$,

$$\left|\int_{\mathrm{inrad}(\mathcal{D})}^{\mathrm{diam}(\mathcal{D})} F^E(\phi) - F^E_{I_1}(\phi)d\phi\right| \ll \frac{1}{E^{\frac{d+k}{2}}}\int_{\sqrt{E}\mathrm{inrad}(\mathcal{D})}^{\sqrt{E}\mathrm{diam}(\mathcal{D})} \left(\frac{J_{\frac{d-2}{2}}(2\pi\psi)}{(2\pi\psi)^{\frac{d-2}{2}}}\right)^q \psi^{d-1+k}d\psi,$$

which yields the same bound as in (V.2.5). Therefore, the conclusion follows from (V.2.2). □

Proof of Theorem V.1.1. Using Proposition V.2.2 and performing the change of variable $\psi = \sqrt{E}\phi$, we obtain

$$\begin{aligned}
&\int_{\mathcal{D}^2} r_d^E(x-y)^q\,dxdy \\
=\ & \frac{\mathrm{vol}_d(\mathcal{D})}{E^{d/2}} \int_0^{\sqrt{E}\mathrm{diam}(\mathcal{D})} d\psi \int_{\mathcal{R}_d} d\varphi\, \hat{r}_d^1(\psi,\varphi)^q \psi^{d-1} \prod_{k=1}^{d-2} \sin(\varphi_k)^{d-k-1} + O\left(\beta_E(d;q)\right) \\
=:\ & U_E + V_E + O\left(\beta_E(d;q)\right),
\end{aligned} \tag{V.2.6}$$

where in U_E and V_E the variable ψ ranges in $[0,1]$ and $[1, \sqrt{E}\mathrm{diam}(\mathcal{D}))$, respectively. We now distinguish two cases.
<u>*Case* $(d-1)(1-\frac{q}{2}) = -1$:</u> We start with U_E. Since near zero, we have $J_{\frac{d-2}{2}}(z) \sim z^{\frac{d-2}{2}}$, we obtain that

$$\begin{aligned}
U_E &= \frac{\mathrm{vol}_d(\mathcal{D})}{E^{d/2}} \int_0^1 d\psi \int_{\mathcal{R}_d} d\varphi \left(\frac{J_{\frac{d-2}{2}}(2\pi\psi)}{(2\pi\psi)^{\frac{d-2}{2}}}\right)^q \psi^{d-1} \prod_{k=1}^{d-2} \sin(\varphi_k)^{d-k-1} \\
&\ll \frac{1}{E^{d/2}} \int_0^1 d\psi \psi^{q\frac{d-2}{2}} \psi^{-q\frac{d-2}{2}} \psi^{d-1} \ll E^{-d/2}.
\end{aligned}$$

For V_E, we make use of the asymptotic expansion of Bessel functions (see for instance [Kra14])

$$J_\nu(t) = \sqrt{\frac{2}{\pi t}} \cos(t-\omega_\nu) + R_\nu(t), \quad t \to \infty$$

where $\omega_\nu := (2\nu+1)\pi/4$ and $|R_\nu(t)| \le c_\nu t^{-3/2}$. Raising to the power q (letting $R(\psi) := R_{d-2/2}(\psi) \ll \psi^{-3/2}$) yields

$$\begin{aligned}
\left(\frac{J_{\frac{d-2}{2}}(2\pi\psi)}{(2\pi\psi)^{\frac{d-2}{2}}}\right)^q &= (2\pi\psi)^{-\frac{q}{2}(d-2)} \left(\frac{\cos(2\pi\psi-(d-1)\frac{\pi}{4})^q}{\pi^q \psi^{q/2}} + \left[\frac{\cos(2\pi\psi-(d-1)\frac{\pi}{4})}{\pi\sqrt{\psi}}\right]^{q-1} R(\psi) + o(R(\psi))\right) \\
&= \pi^{-q}\,(2\pi)^{-\frac{q}{2}(d-2)} \frac{\cos(2\pi\psi-(d-1)\frac{\pi}{4})^q}{\psi^{\frac{q}{2}(d-1)}} + O\left(\psi^{-\frac{q}{2}(d-2)-\frac{q-1}{2}-\frac{3}{2}}\right).
\end{aligned}$$

Moreover, exploiting the relation (see [GR14, p. 31])

$$\begin{aligned}
\cos(y)^{2m} &= \frac{1}{2^{2m}}\binom{2m}{m} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m}{k} \cos(2(m-k)y) \\
\cos(y)^{2m+1} &= \frac{1}{2^{2m}} \sum_{k=0}^{m} \binom{2m+1}{k} \cos((2m+1-2k)y)
\end{aligned} \tag{V.2.7}$$

leads to (for q even):

$$\begin{aligned}
&\int_1^{\sqrt{E}\mathrm{diam}(\mathcal{D})} d\psi \left(\frac{J_{\frac{d-2}{2}}(2\pi\psi)}{(2\pi\psi)^{\frac{d-2}{2}}}\right)^q \psi^{d-1} \\
=\ & \int_1^{\sqrt{E}\mathrm{diam}(\mathcal{D})} d\psi \left[\pi^{-q}\,(2\pi)^{-\frac{q}{2}(d-2)} \frac{\cos(2\pi\psi-(d-1)\frac{\pi}{4})^q}{\psi^{\frac{q}{2}(d-1)}} + O\left(\psi^{-\frac{q}{2}(d-2)-\frac{q-1}{2}-\frac{3}{2}}\right)\right] \psi^{d-1} \\
=\ & \int_1^{\sqrt{E}\mathrm{diam}(\mathcal{D})} d\psi \pi^{-q}\,(2\pi)^{-\frac{q}{2}(d-2)} \cos\left(2\pi\psi-(d-1)\frac{\pi}{4}\right)^q \psi^{(d-1)(1-\frac{q}{2})}
\end{aligned}$$

$$+O\left(\int_1^{\sqrt{E}\mathrm{diam}(\mathcal{D})} d\psi\psi^{-\frac{q}{2}(d-2)-\frac{q-1}{2}-\frac{3}{2}+d-1}\right) =: A(E)+B(E). \tag{V.2.8}$$

For $A(E)$, we have by (V.2.7),

$$\begin{aligned} A(E) &= \pi^{-q}\,(2\pi)^{-\frac{q}{2}(d-2)}\,\frac{1}{2^q}\binom{q}{\frac{q}{2}}\int_1^{\sqrt{E}\mathrm{diam}(\mathcal{D})} d\psi\psi^{(d-1)(1-\frac{q}{2})} \\ &\quad+\frac{\pi^{-q}\,(2\pi)^{-\frac{q}{2}(d-2)}}{2^{q-1}}\sum_{k=0}^{q/2-1}\binom{q}{k}\int_1^{\sqrt{E}\mathrm{diam}(\mathcal{D})}\cos\left(2(q-k)[2\pi\psi-(d-1)\frac{\pi}{4}]\right)\psi^{(d-1)(1-\frac{q}{2})}d\psi \\ &=: A_1(E)+A_2(E). \end{aligned}$$

Denoting by $c(d;q) := \pi^{-q}\,(2\pi)^{-\frac{q}{2}(d-2)}\,\frac{1}{2^q}\binom{q}{\frac{q}{2}}$, we obtain

$$A_1(E) = c(d;q)\int_1^{\sqrt{E}\mathrm{diam}(\mathcal{D})} d\psi\psi^{(d-1)(1-\frac{q}{2})} = \frac{c(d;q)}{2}\log E + O(1)$$

since $(d-1)(1-q/2) = -1$. Now we consider the term $A_2(E)$. Let $k = 0,\dots,q/2-1$ be fixed. Using the fact that $(d-1)(1-\frac{q}{2}) = -1$ and integrating by parts shows that

$$\int_1^{\sqrt{E}\mathrm{diam}(\mathcal{D})} d\psi\cos\left(2(q-k)[2\pi\psi-(d-1)\frac{\pi}{4}]\right)\psi^{(d-1)(1-\frac{q}{2})} = O(E^{-1/2})+O(1),$$

proving that $A_2(E) \ll 1+E^{-1/2}$. For the error term $B(E)$ in (V.2.8), we have

$$B(E) \ll \int_1^{\sqrt{E}\mathrm{diam}(\mathcal{D})} d\psi\psi^{(d-1)(1-\frac{q}{2})-1} = O(E^{-1/2}).$$

Summing the contributions of the terms $A(E)$ and $B(E)$ and using the relation

$$\int_{\mathcal{R}_d} d\varphi\prod_{k=1}^{d-2}\sin(\varphi_k)^{d-k-1} = \mathrm{vol}_d(\mathbb{B}(0,1))\times\left[\int_0^1 dss^{d-1}\right]^{-1} = d\frac{\pi^{d/2}}{\Gamma(d/2+1)} = d\kappa_d,$$

it follows that,

$$\begin{aligned} V_E &= \frac{\mathrm{vol}_d(\mathcal{D})}{E^{d/2}}\int_1^{\sqrt{E}\mathrm{diam}(\mathcal{D})} d\psi\left(\frac{J_{\frac{d-2}{2}}(2\pi\psi)}{(2\pi\psi)^{\frac{d-2}{2}}}\right)^q\psi^{d-1}\int_{\mathcal{R}_d} d\varphi\prod_{k=1}^{d-2}\sin(\varphi_k)^{d-k-1} \\ &= d\kappa_d\frac{\mathrm{vol}_d(\mathcal{D})}{E^{d/2}}\int_1^{\sqrt{E}\mathrm{diam}(\mathcal{D})} d\psi\left(\frac{J_{\frac{d-2}{2}}(2\pi\psi)}{(2\pi\psi)^{\frac{d-2}{2}}}\right)^q\psi^{d-1} = d\kappa_d\frac{\mathrm{vol}_d(\mathcal{D})}{E^{d/2}}\,[A(E)+B(E)], \end{aligned}$$

where $A(E)$ and $B(E)$ are as in (V.2.8). Therefore, combining the above observations with (V.2.6) yields

$$\begin{aligned} \int_{\mathcal{D}^2} r_d^E(x-y)^q dxdy &= d\kappa_d\mathrm{vol}_d(\mathcal{D})c(d;q)\frac{1}{2}\frac{\log E}{E^{d/2}} + O(E^{-d/2}) + O\left(E^{-\frac{q}{4}(d-1)}\right) \\ &= d\kappa_d\mathrm{vol}_d(\mathcal{D})c(d;q)\frac{1}{2}\frac{\log E}{E^{d/2}} + o\left(\frac{\log E}{E^{d/2}}\right) \end{aligned}$$

where we used that for $(d-1)(1-\frac{q}{2}) = -1$, we have $E^{-\frac{q}{4}(d-1)} = E^{\frac{1}{2}(d-1)(1-\frac{q}{2})-\frac{d-1}{2}} = E^{-\frac{1}{2}-\frac{d-1}{2}} = E^{-d/2}$.

Case $(d-1)(1-\frac{q}{2}) < -1$: Using the same notation as above, we have for the term U_E in (V.2.6),

$$U_E = d\kappa_d \frac{\text{vol}_d(\mathcal{D})}{E^{d/2}} (2\pi)^{-\frac{q}{2}(d-2)} \int_0^1 d\psi \left(J_{\frac{d-2}{2}}(2\pi\psi) \right)^q \psi^{(d-1)(1-\frac{q}{2})+\frac{q}{2}}.$$

Moreover, using the bound $|J_\alpha(z)| \ll z^{-1/2}$ yields for the term V_E

$$\int_1^{\sqrt{E}\text{diam}(\mathcal{D})} d\psi \left(\frac{J_{\frac{d-2}{2}}(2\pi\psi)}{(2\pi\psi)^{\frac{d-2}{2}}} \right)^q \psi^{d-1} \ll \int_1^{\sqrt{E}\text{diam}(\mathcal{D})} d\psi \psi^{(d-1)(1-q/2)},$$

which converges to a constant as $E \to \infty$, since $(d-1)(1-q/2) < -1$. Therefore, in this case it follows from (V.2.6)

$$\int_{\mathcal{D}^2} r_d^E(x-y)^q dxdy = d\kappa_d \frac{\text{vol}_d(\mathcal{D})}{E^{d/2}} (2\pi)^{-\frac{q}{2}(d-2)} \int_0^\infty d\psi \left(J_{\frac{d-2}{2}}(2\pi\psi) \right)^q \psi^{(d-1)(1-\frac{q}{2})+\frac{q}{2}} + o(E^{-\frac{d}{2}}),$$

where we used the fact that $\beta_E(d;q) = o(E^{-\frac{d}{2}})$ since $\ell > 1$. □

V.2.2 Proof of Theorem V.1.2

We represent the random variable $\widetilde{Z}_E(d,q;\mathcal{D})$ as a multiple integral of order q with respect to some isonormal Gaussian process on $H := L^2([0,1],\lambda)$, with λ denoting Lebesgue measure. In order to do this, we write $B_E^{(d)}(x) = I_1(f_{x,E})$, for fixed $E > 0$ and $x \in \mathbb{R}^d$ where $f_{x,E} \in H$ (note that we drop the dependence on d for $f_{x,E}$). Using (I.1.25), yields $H_q(B_E^{(d)}(x)) = I_q(f_{x,E}^{\otimes q})$, where $f_{x,E}^{\otimes q} \in H^{\odot q}$ stands for the q-fold tensor product of $f_{x,E}$ with itself. Therefore, we have that

$$\widetilde{Z}_E(d,q;\mathcal{D}) = E^{d/4} \int_{\mathcal{D}} H_q(B_E^{(d)}(x))dx = I_q\left(E^{d/4} \int_{\mathcal{D}} f_{x,E}^{\otimes q} dx \right) := I_q(z_{d;E}),$$

where

$$z_{d;E}(u_1,\ldots,u_q) := E^{d/4} \int_{\mathcal{D}} f_{x,E}^{\otimes q}(u_1,\ldots,u_q)dx.$$

Since $\widetilde{Z}_E(d,q;\mathcal{D})$ is an element of the q-th Wiener chaos, in order to prove that it verifies a Central Limit Theorem, we have to prove that the contraction norms

$$\|z_{d;E} \otimes_r z_{d;E}\|_{H^{\otimes(2q-2r)}}, \quad r = 1,\ldots,q-1$$

converge to zero as $E \to \infty$ (see Theorem I.1.30). Using the symmetry relation $\|z_{d;E} \otimes_r z_{d;E}\|_{H^{\otimes(2q-2r)}} = \|z_{d;E} \otimes_{q-r} z_{d;E}\|_{H^{\otimes 2r}}$, we can reduce our investigations to the range $r = 1,\ldots,[q/2]$. We have that

$$\begin{aligned}
& (z_{d;E} \otimes_r z_{d;E})(u_1,\ldots,u_{2q-2r}) \\
= & \int_{[0,1]^r} ds_1 \ldots ds_r z_{d;E}(s_1,\ldots,s_r,u_1,\ldots,u_{q-r}) z_{d;E}(s_1,\ldots,s_r,u_{q-r+1},\ldots,u_{2q-2r}) \\
= & E^{d/2} \int_{[0,1]^r} ds_1 \ldots ds_r \int_{\mathcal{D}} dx \int_{\mathcal{D}} dy f_{x,E}(s_1)\cdots f_{x,E}(s_r) f_{x,E}(u_1)\cdots f_{x,E}(u_{q-r}) \\
& \qquad f_{y,E}(s_1)\cdots f_{y,E}(s_r) f_{y,E}(u_{q-r+1})\cdots f_{y,E}(u_{2q-2r}) \\
= & E^{d/2} \int_{\mathcal{D}\times\mathcal{D}} dxdy r_E^d(x-y)^r f_{x,E}(u_1)\cdots f_{x,E}(u_{q-r}) f_{y,E}(u_{q-r+1})\cdots f_{y,E}(u_{2q-2r}),
\end{aligned}$$

where we used the isometry property of multiple integrals. Taking squared norms and using isometry yields

$$\begin{aligned}
&\|z_{d;E} \otimes_r z_{d;E}\|^2_{H^{\otimes(2q-2r)}} \\
=\ & \int_{[0,1]^{2q-2r}} \left(z_{d;E} \otimes_r z_{d;E}(u_1,\dots,u_{2q-2r})\right)^2 du_1 \dots du_{2q-2r} \\
=\ & E^d \int_{\mathcal{D}^4} dxdydx'dy' \int_{[0,1]^{2q-2r}} du_1 \dots du_{2q-2r} r_E^d(x-y)^r r_E^d(x'-y')^r \\
& f_{x,E}(u_1)\cdots f_{x,E}(u_{q-r}) f_{y,E}(u_{q-r+1})\cdots f_{y,E}(u_{2q-2r}) \\
& f_{x',E}(u_1)\cdots f_{x',E}(u_{q-r}) f_{y',E}(u_{q-r+1})\cdots f_{y',E}(u_{2q-2r}) \\
=\ & E^d \int_{\mathcal{D}^4} dxdydx'dy' r_E^d(x-y)^r r_E^d(x'-y')^r r_E^d(x-x')^{q-r} r_E^d(y-y')^{q-r}.
\end{aligned}$$

Now since $|r_E^d(\cdot)| \le 1$, we use the inequality $x^a y^b \le x^{a+b} + y^{a+b}, 0 < x, y \le 1$, so that we can bound the above integral by two similar terms of the form

$$\begin{aligned}
& E^d \int_{\mathcal{D}^4} dxdydx'dy' |r_E^d(x-y)|^{2r} |r_E^d(x-x')|^{q-r} |r_E^d(y-y')|^{q-r} \\
=\ & \frac{1}{E^d} \int_{(\sqrt{E}\mathcal{D})^4} dxdydx'dy' |r_1^d(x-y)|^{2r} |r_1^d(x-x')|^{q-r} |r_1^d(y-y')|^{q-r},
\end{aligned}$$

where we performed a change of variable. Now by the linear change of variables $w_1 = x - y, w_2 = x - x', w_3 = y - y', w_4 = x$, we infer that (up to multiplicative constants)

$$\begin{aligned}
& \frac{1}{E^d} \int_{(\sqrt{E}\mathcal{D})^4} dxdydx'dy' |r_1^d(x-y)|^{2r} |r_1^d(x-x')|^{q-r} |r_1^d(y-y')|^{q-r} \qquad \text{(V.2.9)} \\
=\ & \frac{\text{area}(\mathcal{D})}{E^{d/2}} \int_{\sqrt{E}(\mathcal{D}-\mathcal{D})} dw_1 |r_1^d(w_1)|^{2r} \int_{\sqrt{E}(\mathcal{D}-\mathcal{D})} dw_2 |r_1^d(w_2)|^{q-r} \int_{\sqrt{E}(\mathcal{D}-\mathcal{D})} dw_3 |r_1^d(w_3)|^{q-r}.
\end{aligned}$$

Now using the explicit representation of r_1 in terms of Bessel functions yields the bound

$$|r^1(u)| = \frac{\left|J_{\frac{d-2}{2}}(2\pi\|u\|)\right|}{(2\pi\|u\|)^{\frac{d-2}{2}}} \le c\|u\|^{\frac{1-d}{2}}, \quad u \in \mathbb{R}^d,$$

for some constant $c > 0$. Therefore, we have that for every integer $m \ge 1$,

$$\begin{aligned}
\int_{\sqrt{E}(\mathcal{D}-\mathcal{D})} du |r_1^d(u)|^m &\ll \int_{\sqrt{E}(\mathcal{D}-\mathcal{D})} du \|u\|^{m\frac{1-d}{2}} \ll \int_{c_1}^{\sqrt{E}\text{diam}(\mathcal{D}-\mathcal{D})} d\rho \rho^{(d-1)(1-\frac{m}{2})} \\
&\ll \mathbb{1}_{(d-1)(1-\frac{m}{2})=-1} \log E + \mathbb{1}_{(d-1)(1-\frac{m}{2})\neq -1} \sqrt{E}^{(d-1)(1-\frac{m}{2})+1} \\
&\ll \log E \sqrt{E}^{(d-1)(1-\frac{m}{2})+1} := K_m(E).
\end{aligned}$$

Using this bound, we can bound the expression in (V.2.9) by

$$\begin{aligned}
\frac{1}{E^{d/2}} K_{2r}(E) K_{q-r}(E)^2 &\ll (\log E)^3 E^{-d/2} \sqrt{E}^{(d-1)(1-r)+1} E^{(d-1)(1-\frac{q-r}{2})+1} \\
&= (\log E)^3 E^{\frac{(d-1)}{2}(3-q)+\frac{3-d}{2}} = (\log E)^3 E^{d-q\frac{d-1}{2}}.
\end{aligned}$$

The claim then follows from [NP12a, Theorem 5.1.3] and the fact that for $F = I_q(f)$ with $\mathbf{Var}[F] = \sigma^2$,

$$\mathbb{E}\left[\left|\sigma^2 - q^{-1}\|DF\|_H^2\right|\right] \le \sqrt{\frac{1}{q^2}\sum_{r=1}^{q-1} r^2 r! \binom{q}{r}^4 (2q-2r)! \|f\widetilde{\otimes}_r f\|^2_{H^{\otimes(2q-2r)}}},$$

and $\|f\widetilde{\otimes}_r f\|^2_{H^{\otimes(2q-2r)}} \leq \|f\otimes_r f\|^2_{H^{\otimes(2q-2r)}}$ in view of the triangle inequality. Since the quantity $d-q\frac{d-1}{2} < 0$ for every $(d,q) \in S$ such that $(d,q) \neq (2,4)$ (indeed, this condition is equivalent to $d > \frac{q}{q-2}$, which is always satisfied), we deduce that $\widetilde{Z}_E(d,q;\mathcal{D})$ verifies a Central Limit Theorem. This concludes the proof.

V.3 Digression: reduction principles on Wiener chaoses and variance estimates for the nodal length

In this section, we make some comments on reduction principles on Wiener chaoses in the framework of nodal volumes associated with Berry's random field, and point out heuristic directions based on our results in Theorem V.1.1 and Theorem V.1.2.

We consider the nodal length associated with zero sets of the d-dimensional Berry random wave restricted to a domain $\mathcal{D} \subset \mathbb{R}^d$,

$$\mathscr{L}_E^{(d)}(\mathcal{D}) = \mathcal{H}_1([B_E^{(d)}]^{-1}(0) \cap \mathcal{D}).$$

In view of Proposition I.1.11, $\mathscr{L}_E^{(d)}(\mathcal{D})$ is obtained $\mathbb{P}$-almost surely and in $L^2(\mathbb{P})$ as

$$\mathscr{L}_E^{(d)}(\mathcal{D}) = \sqrt{\frac{4\pi^2 E}{d}} \int_{\mathcal{D}} \delta_0(B_E^{(d)}(x)) \|\widetilde{\nabla} B_E^{(d)}(x)\| dx$$

where $\nabla B_E^{(d)}(x)$ denotes the gradient of $B_E^{(d)}$ computed at x and $\widetilde{\nabla} := (\tilde{\partial}_1, \dots, \tilde{\partial}_d)$ is the normalized gradient with $\tilde{\partial}_j := (4\pi^2 E/d)^{-1/2}\partial_j$, for $j = 1, \dots, d$. The Wiener-Itô chaos expansion of $\mathscr{L}_E^{(d)}(\mathcal{D})$ is the $L^2(\mathbb{P})$-converging series

$$\mathscr{L}_E^{(d)}(\mathcal{D}) = \sum_{q\geq 0} \mathscr{L}_E^{(d)}(\mathcal{D})[2q] \tag{V.3.1}$$

where for $q \geq 0$, $\mathscr{L}_E(\mathcal{D})[2q]$ indicates the projection of $\mathscr{L}_E(\mathcal{D})$ on the $2q$-th Wiener chaos given by

$$\mathscr{L}_E^{(d)}(\mathcal{D})[2q] = \sqrt{\frac{4\pi^2 E}{d}} \sum_{|Q|=2q} \frac{\beta_{2q_0}}{(2q_0)!} \frac{\eta_{2q_1,\dots,2q_d}}{(2q_1)!\dots(2q_d)!} \int_{\mathcal{D}} H_{2q_0}(B_E^{(d)}(x)) \prod_{j=1}^d H_{2q_j}(\widetilde{\partial}_j B_E^{(d)}(x)) dx$$

where the sum runs over multi-indices $Q = (q_0, \dots, q_d) \in \mathbb{N}^{d+1}$ with $|Q| = 2q$ and

$$\beta_{2q_0} := \mathbb{E}\left[\delta_0(N_1) H_{2q_0}(N_1)\right], \quad \eta_{2q_1,\dots,2q_d} := \mathbb{E}\left[\|N\| \prod_{j=1}^d H_{2q_j}(N_j)\right],$$

for $N = (N_1, \dots, N_d) \sim \mathcal{N}_d(0, \mathbf{I}_d)$ denote the projection coefficients of δ_0 and $\|\bullet\|$, respectively. In view of the variance estimates in Theorem V.1.1 for the random variables $Z_E(d,q;\mathcal{D})$ appearing in this expansion, it is to be expected that for every $d \geq 3$ and every $q \geq 2$, the variance of the $2q$-th chaotic component is of order $E^{1-d/2}$. Such a claim might be turned rigorous by evoking certain reduction principles on Wiener chaoses. For instance, in [Vid21] (see also the earlier contribution [MRW20] for a reduction principle for the nodal length of spherical Gaussian eigenfunctions in dimension two, therein the integral of the fourth Hermite polynomials is referred to as the *sample trispectrum*) it is shown that in dimension 2, the projection on the fourth Wiener chaos of the standardized nodal length is asymptotically fully correlated with the integral of the fourth Hermite polynomial. More precisely, defining the normalized random variables

$$\widetilde{\mathscr{L}_E^{(2)}}(\mathcal{D}) := \frac{\mathscr{L}_E^{(2)}(\mathcal{D}) - \mathbb{E}\left[\mathscr{L}_E^{(2)}\right](\mathcal{D})}{\sqrt{\mathbf{Var}\left[\mathscr{L}_E^{(2)}(\mathcal{D})\right]}},$$

and

$$\mathcal{M}_E(2,4;\mathcal{D}) := -\frac{\sqrt{2\pi^2 E}}{96} Z_E(2,4;\mathcal{D}), \quad \widetilde{\mathcal{M}_E}(2,4;\mathcal{D}) := \frac{\mathcal{M}_E(2,4;\mathcal{D})}{\sqrt{\mathbf{Var}[\mathcal{M}_E(2,4;\mathcal{D})]}},$$

it is shown that (see [Vid21, Theorem 1.1]), as $E \to \infty$

$$\widetilde{\mathscr{L}_E^{(2)}}(\mathcal{D}) = \widetilde{\mathcal{M}_E}(2,4;\mathcal{D}) + o_{L^2(\mathbb{P})}(1)$$

where $o_{L^2(\mathbb{P})}(1)$ denotes a sequence of random variables converging to zero in $L^2(\mathbb{P})$. Such a reduction principle in particular gives a more direct proof of the Central Limit Theorem in dimension two (although crucially relying on a number of covariance estimates on the fourth Wiener chaos available in [NPR19]). Although our findings in Theorem V.1.1 seem to exclude the fact that a single chaotic component dominates the series (V.3.1) for $d \geq 3$, we believe that similar reduction principles as above are valid within fixed Wiener chaoses. More precisely, we expect that for fixed $d \geq 3$ and every $q \geq 3$, there exists a constant $k(d,q)$ depending on d and q such that

$$\mathbb{E}\left[\left(\mathscr{L}_E^{(d)}[2q] - \mathcal{M}_E(d,2q;\mathcal{D})\right)^2\right] = O(R_E(d,q)) \tag{V.3.2}$$

where $\{R_E(d,q) : E > 0\}$ is a numerical sequence such that $R_E(d,q) \to 0$ as $E \to \infty$ and

$$\mathcal{M}_E(d,q;\mathcal{D}) := k(d,q)\sqrt{\frac{4\pi^2 E}{d}} Z_E(d,q;\mathcal{D}).$$

Note that the asymptotic variance of $\mathcal{M}_E(d,q;\mathcal{D})$ is obtained by Theorem V.1.1. If an asymptotic estimate as in (V.3.2) is valid for every $q \geq 3$, then one has that for every finite integer $N \geq 2$,

$$\sum_{q=2}^{N} \mathscr{L}_E^{(d)}[2q] = \sum_{q=2}^{N} \mathcal{M}_E(d,2q;\mathcal{D}) + o_{L^2(\mathbb{P})}(1), \tag{V.3.3}$$

thus, heuristically suggesting that the nodal length is asymptotically equivalent to the series of the *sample polyspectra*. Writing $\mathscr{L}_E^{(d)}(\mathcal{D};N) := \sum_{q=0}^{N} \mathscr{L}_E^{(d)}[2q]$ for the *truncated* nodal length of order N obtained by summing the projection of the nodal length on Wiener chaoses up to order $2N$, we deduce the following variance estimate

$$\begin{aligned}\mathbf{Var}\left[\mathscr{L}_E^{(d)}(\mathcal{D};N)\right] &= \mathbf{Var}\left[\mathscr{L}_E^{(d)}(\mathcal{D})[2]\right] + \frac{4\pi^2 E}{d}\sum_{q=2}^{N}(2q)!k(d,2q)^2\mathbf{Var}[Z_E(d,2q;\mathcal{D})] \\ &\sim \mathbf{Var}\left[\mathscr{L}_E^{(d)}(\mathcal{D})[2]\right] + \frac{4\pi^2}{d}E^{1-d/2}\sum_{q=2}^{N}(2q)!k(d,2q)^2\alpha(d;2q), \end{aligned} \tag{V.3.4}$$

where $\alpha(d;2q)$ is as in (V.1.4). Also, we remark that for $q \geq 3$, the random variables $\mathcal{M}_E(d,\mathcal{D})$ are asymptotically Gaussian in view of Theorem V.1.2. By means of Green's formula on manifolds, the second chaotic projection can be re-written as (see also Theorem II.2.5 and Section II.2.2 of Chapter II)

$$\mathscr{L}_E^{(d)}(\mathcal{D})[2] = \frac{1}{\sqrt{2d}\sqrt{2\pi}}\frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})}\frac{1}{\sqrt{4\pi^2 E}}\int_{\partial\mathcal{D}} B_E^{(d)}(x)\langle\nabla B_E^{(d)}(x), \mathrm{n}(x)\rangle_{\mathbb{R}^d}\,dx, \tag{V.3.5}$$

where we combined (II.2.10) with (II.2.15) and (II.A.6) (with $n = 1$), and where $\mathrm{n}(x)$ indicates the outward unit normal vector to $\partial\mathcal{D}$ at x. In particular, using the Cauchy-Schwarz inequality and the fact that $B_E^{(d)}(x)$ is independent of $\nabla B_E^{(d)}(x)$ for fixed x, leads to

$$\mathbf{Var}\left[\mathscr{L}_E^{(d)}(\mathcal{D})[2]\right] \leq \frac{1}{2d}\frac{1}{2\pi}\frac{\Gamma(\frac{d+1}{2})^2}{\Gamma(\frac{d}{2})^2}\frac{1}{4\pi^2 E}4\pi^2 E\mathcal{H}_{d-1}(\partial\mathcal{D})^2 = O(1), \tag{V.3.6}$$

where $\mathcal{H}_{d-1}(\partial\mathcal{D})$ stands for the $(d-1)$-dimensional measure of $\partial\mathcal{D}$. This shows that in every dimension d, the variance of the second chaotic component contributes at most $O(1)$. In Theorem IV.1.20 of Chapter IV, we proved that, in dimension two, the true variance order of the second chaos is of commensurate to $E^{-1/2}$. We note that the bound $O(1)$ in (V.3.6) is sufficient to deduce the negligibility of the second chaos component in dimension two, as in this case the variance of the fourth order chaos is of magnitude $\log E$. We also point out that, in view of the conjectured variance estimates in Remark V.1.3 **(b)** for the random variable $Z_E(d,2;\mathcal{D})$ (which is a typical element of the second chaotic projection, up to a multiplicative factor $\sqrt{E}$), we deduce that $\mathbf{Var}[E\cdot Z_E(2,2;\mathcal{D})] \approx EE^{-1/2} = E^{1/2}$, which is considerably bigger than the true order $E^{-1/2}$ derived in Chapter IV. Such an observation is due to the cancellation phenomenon described in Chapter II. Unfortunately, the (rather rough) estimate in (V.3.6) is not enough to deduce that, in dimension $d \geq 3$, the second chaos is negligible with respect to higher-order chaoses, as comparing to (V.3.4) yields $E^{1-d/2} = o(1)$. In order to confirm such a phenomenon, one can study precise variance asymptotics of the second-order chaos components in higher dimensions by means of formula (V.3.5).

Chapter VI

Optimality of convergence rates for Gamma approximations

VI.1 Stein's Method and preliminary notions

We start this chapter with a review of Stein's method for the Gaussian distribution and the Gamma distribution. Although the content of this chapter deals with optimal rates for Gamma approximations, we shall do several comparisons with the Gaussian setting. In Section VI.1.4, we present some ancillary results from Gaussian analysis on the second Wiener chaos, that will be needed in our approach.

VI.1.1 Stein's Method for normal approximations

To start with, we present the principal ideas behind Stein's method for the Gaussian distribution initiated in the landmark contribution [Ste72]. For a thorough introduction, the reader is referred for instance to the monographs by Stein [Ste86], Chen, Goldstein and Shao [CGS11], Nourdin and Peccati [NP12a] and the survey by Nathan Ross [Ros11].

In (I.1.15), we have seen that whenever $N \sim \mathcal{N}(0,1)$, then for a sufficiently regular function f, we have $\mathbb{E}\left[f'(N)\right] = \mathbb{E}\left[Nf(N)\right]$. *Stein's Lemma* (see for instance [NP12a, Lemma 3.1.2]) asserts that the reverse implication is also true, that is, if Z is a random variable such that $\mathbb{E}\left[f'(Z)\right] = \mathbb{E}\left[Zf(Z)\right]$ holds for a sufficiently large class of functions f, then necessarily $Z \sim \mathcal{N}(0,1)$. This observation is the starting point of Stein's method: Heuristically, one expects that if Z is a random variable such that the difference $\mathbb{E}\left[f'(Z)\right] - \mathbb{E}\left[Zf(Z)\right]$ is close to zero, then one should have that the law of Z is close to the standard Gaussian distribution. This is known as *Stein's heuristic*. To make the above observation rigorous, Stein introduces the so-called *Stein's equation*, an ordinary differential equation of the form

$$f'(x) - xf(x) = h(x) - \mathbb{E}\left[h(N)\right], \quad N \sim \mathcal{N}(0,1), \tag{VI.1.1}$$

where the unknown is f and $h : \mathbb{R} \to \mathbb{R}$ is some test function. Denoting by f_h the solution to (VI.1.1) (keeping h fixed), replacing x with a random variable F and taking expectations on both sides then yields

$$\mathbb{E}\left[f_h'(F) - Ff_h(F)\right] = \mathbb{E}\left[h(F)\right] - \mathbb{E}\left[h(N)\right]. \tag{VI.1.2}$$

The idea is now to control the R.H.S, yielding bounds on certain probability metrics introduced in Section I.1.2.3: in order to do this, Stein's idea is to uniformly control the L.H.S of (VI.1.2) by relying on appropriate regularity conditions available for the Stein solution f_h. The reader might wonder what the advantage is of deriving upper bounds for $\mathbb{E}\left[f'(F) - Ff(F)\right]$ rather than $\mathbb{E}\left[h(F) - h(N)\right]$: a first observation is that the former does not depend explicitly on the target random variable N anymore[1]; but more importantly the variational techniques from Malliavin calculus introduced in Section I.1.2.2 allow to methodically deal with expressions of the type $\mathbb{E}\left[f'(F) - Ff(F)\right]$. For instance, using the integration by part formula (I.1.31), one has that $\mathbb{E}\left[Ff(F)\right] = \mathbb{E}\left[f'(F)\langle DF, -DL^{-1}F\rangle_H\right]$ (where D and L^{-1} denote the Malliavin derivative and the pseudo-inverse of the generator of the Ornstein-Uhlenbeck semigroup), from which the link to Malliavin calculus becomes evident. The next theorem (see e.g. [NP12a, Proposition 3.2.2]) characterizes the solutions of the Stein equation in (VI.1.1).

Theorem VI.1.1. *Every solutions of* (VI.1.1) *takes the form*

$$f(x) = ce^{x^2/2} + e^{x^2/2}\int_{-\infty}^{x} \left(h(y) - \mathbb{E}\left[h(N)\right]\right) e^{-y^2/2} dy, \quad x, c \in \mathbb{R}.$$

In particular, the function

$$f_h(x) = e^{x^2/2}\int_{-\infty}^{x} \left(h(y) - \mathbb{E}\left[h(N)\right]\right) e^{-y^2/2} dy, \quad x \in \mathbb{R} \tag{VI.1.3}$$

is the unique solution to (VI.1.1) *verifying* $e^{-x^2/2} f_h(x) \to 0$ *as* $|x| \to \infty$.

The following statement (see for instance [NP12a, Theorem 3.4.2]) yields regularity properties on the solution f_h in the setting of normal approximations in Kolmogorov distance[2], $d_{\mathrm{Kol}}(F, N) = \sup\left\{\left|\mathbb{P}\left\{X \le z\right\} - \mathbb{P}\left\{N \le z\right\}\right| : z \in \mathbb{R}\right\}$.

Theorem VI.1.2. *Let* $h(x) = \mathbb{1}_{(-\infty, z]}(x)$ *for* $z \in \mathbb{R}$. *Denoting by* f_z *the solution to* (VI.1.1) *associated with h, we have that* $\|f_z\|_\infty \le \sqrt{2\pi}/4$ *and* $\|f_z'\|_\infty \le 1$. *In particular,*

$$d_{\mathrm{Kol}}(F, N) \le \sup_{f \in \mathscr{F}_{\mathrm{Kol}}} \left|\mathbb{E}\left[f'(F) - Ff(F)\right]\right|, \tag{VI.1.4}$$

where $\mathscr{F}_{\mathrm{Kol}} := \left\{f : \|f\|_\infty \le \sqrt{2\pi}/4, \|f'\|_\infty \le 1\right\}$.

The upper bound for the Kolmogorov distance in (VI.1.4) is customarily referred to as *Stein's bound*. The proof of Theorem VI.1.2 is based on rewriting the explicit solution f_z in (VI.1.3) as

$$f_z(x) = \sqrt{2\pi} e^{x^2/2} \left(\Phi(z)(1 - \Phi(x))\mathbb{1}_{x \ge z} + \Phi(x)(1 - \Phi(z))\mathbb{1}_{x \le z}\right).$$

VI.1.2 Stein's Method for Gamma approximations

We present preliminary results of Stein's method for Gamma approximation. For real numbers $\lambda, r > 0$, we say that a random variable $X_{r,\lambda}$ follows the Gamma distribution with parameters r, λ (denoted $X_{r,\lambda} \sim \Gamma(r, \lambda)$) if its density is given by

$$f_{r,\lambda}(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x} \mathbb{1}_{x>0},$$

where $\Gamma(s)$ denotes the Gamma function evaluated at s. Simple computations show that $\mathbb{E}\left[X_{r,\lambda}\right] = r/\lambda$ and $\mathbf{Var}\left[X_{r,\lambda}\right] = r/\lambda^2$. Moreover, it is a known fact that, whenever $Y \sim \Gamma(r, \lambda)$, then for $c > 0$, we have that $cY \sim \Gamma(r, \lambda/c)$. For a real number $\nu > 0$, we let

$$G(\nu) := 2X_{\nu/2,1} - \nu, \quad X_{\nu/2,1} \sim \Gamma(\nu/2, 1).$$

[1]Actually, one can rewrite $\mathbb{E}\left[f'(F) - Ff(F)\right] = \mathbb{E}\left[\tau_N(F)\right]$, where $\tau_N(W) := f'(W) - Wf(W)$ is the *Stein kernel* for the normal distribution, so that the dependence on N is implicit.

[2]Similar results are available for the total variation and Wasserstein distance. We refer the reader to [NP12a, Chapter 3] for further details.

Then, $\mathbb{E}[G(\nu)] = 0$ and $\mathbf{Var}[G(\nu)] = 2\nu$, that is $G(\nu)$ is a *centred Gamma distribution* with parameter ν. The density of $G(\nu)$ is given by

$$p_\nu(x) = \frac{2^{-\nu/2}}{\Gamma(\nu/2)}(x+\nu)^{\nu/2-1}e^{-(x+\nu)/2}\mathbb{1}_{x>-\nu}.$$

We remark that if $\nu \in \mathbb{N}$ is an integer, then $G(\nu) \overset{d}{=} \chi_c^2(\nu)$ has the centred chi-squared distribution with ν degrees of freedom, that is $G(\nu) \overset{d}{=} \sum_{k=1}^{\nu}(N_k^2 - 1)$, where $(N_1, \dots, N_\nu)$ is a standard Gaussian vector in dimension ν. Stein's Lemma for the Gamma distribution was introduced by Luk in [Luk94]: A real-valued random variable X has the $\Gamma(r, \lambda)$ distribution if and only if

$$\mathbb{E}[Xf'(X)] = -\mathbb{E}[(r - \lambda X)f(X)]$$

for a large class of test functions f. This leads to the following Stein equation[3] associated with the Gamma distribution $\Gamma(r,\lambda)$

$$xf'(x) + (r - \lambda x)f(x) = h(x) - \mathbb{E}[h(X_{r,\lambda})], \quad x \in \mathbb{R} \tag{VI.1.5}$$

where h is a test function in a suitable class of functions. The Stein equation for the centred Gamma distribution is

$$2(x+\nu)f'(x) - xf(x) = h(x) - \mathbb{E}[h(G(\nu))]. \tag{VI.1.6}$$

From this it is easy to verify that if g_h is a solution of (VI.1.5) associated with test function h, then the function

$$f_h(x) = \frac{1}{2}g_{h_1}\left(\frac{x+\nu}{2}\right), \quad h_1(x) := h(2x - \nu)$$

is a solution of (VI.1.6). Explicitly it is given by (see e.g [AEK20, Eq. (3.5)])

$$f_h(x) = \int_{-\nu}^{x}\left(\frac{q(t)}{2(x+\nu)q(x)}\mathbb{1}_{x\le-\nu} + \frac{p_\nu(t)}{2(x+\nu)p_\nu(x)}\mathbb{1}_{x>-\nu}\right)(h(t) - \mathbb{E}[h(G(\nu))])\,dt,$$

where $q(x) = (-1)^{\nu/2}2^{-\nu/2}(x+\nu)^{\nu/2}e^{-(x+\nu)/2}$ and p_ν is the density function of $G(\nu)$. The following regularity bounds on the solution of (VI.1.6), due to Döbler and Peccati, [DP18, Theorem 2.3, Theorem 1.7], will be useful for our analysis in Section VI.2.

Theorem VI.1.3. *If $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function, then there exists a unique Lipschitz solution f_h to* (VI.1.6) *with*

$$\|f_h\|_\infty \le \|h'\|_\infty, \quad \|f_h'\|_\infty \le \max(1, 2/\nu)\|h'\|_\infty. \tag{VI.1.7}$$

In particular, for $F \in \mathbb{D}_{1,2}$ such that $\mathbb{E}[F] = 0$, we have that, for fixed $\nu > 0$,

$$d_{\mathrm{W}}(F, G(\nu)) \le \max(1, 2/\nu)\sqrt{\mathbb{E}\left[(2(F+\nu) - \langle DF, -DL^{-1}F\rangle_{\mathcal{H}})^2\right]}. \tag{VI.1.8}$$

[3] Several other Stein's equations for the Gamma distribution are discussed in the literature: for instance, the classical *density approach* (see e.g. [CGS11, Section 13.1]) leads to the equation $f'(x) - \psi(x)f(x) = h(x) - \mathbb{E}\left[h(X_{r,\lambda})\right]$, where $\psi(x) := \frac{d}{dx}\log p_{r,\lambda}(x)$ is a rational function. However, for the conclusions of Theorem VI.1.3 to hold, it is particularly convenient to have a *linear* coefficient of $f(x)$, as is the case in (VI.1.5). We refer the reader to [DP18] for more details.

VI.1.3 Optimal convergence rates in probabilistic approximations

Our principal aim in this chapter is to detect *optimal rates of convergence* for Gamma approximations on a Gaussian space. Let $\{F_n : n \geq 1\}$ and F_∞ be random variables and $d_{\mathscr{H}}$ a probability metric (associated with a class of test functions[4] $\mathscr{H}$) characterizing convergence in distribution. Let us formulate the following definition.

Definition VI.1.4. A strictly positive numerical sequence $\{\phi(n) : n \geq 1\}$ converging to zero as $n \to \infty$ is said to be

(i) an *optimal rate of convergence* for the probability metric $d_{\mathscr{H}}$ if there exist two constants $0 < c_1 < c_2 < \infty$ and an integer $n_0 \geq 1$ such that

$$c_1\phi(n) \leq d_{\mathscr{H}}(F_n, F_\infty) \leq c_2\phi(n), \quad \forall n \geq n_0.$$

(ii) a *strongly optimal rate of convergence* for the probability metric $d_{\mathscr{H}}$ if there exist $h \in \mathscr{H}$, constants $0 < c_1 < c_2 < \infty$ and an integer $n_0 \geq 1$ such that

$$c_1\phi(n) \leq |\mathbb{E}[h(F_n)] - \mathbb{E}[h(F_\infty)]| \leq c_2\phi(n), \quad \forall n \geq n_0.$$

Obviously, we have that a strongly optimal rate is also optimal. In the light of the above definition, the convergence rate $\phi(n)$ is optimal as soon as the ratio $\frac{d(F_n,F_\infty)}{\phi(n)}$ converges to a finite non-zero limit. In [NP09b], Nourdin and Peccati study optimal convergence rates for the Kolmogorov distance in the Gaussian setting, that is, when the target random variable $F_\infty = N$ is standard Gaussian and $d_{\mathscr{H}} = d_{\text{Kol}}$ is the Kolmogorov distance. Combining the integration by part formula (I.1.31) with the Stein Equation (VI.1.1), it follows that the quantity $\mathbb{P}\{F_n \leq z\} - \mathbb{P}\{N \leq z\}$ can be re-written as

$$\mathbb{P}\{F_n \leq z\} - \mathbb{P}\{N \leq z\} = \mathbb{E}\left[f_z'(F_n) - F_n f_z(F_n)\right] = \mathbb{E}\left[f_z'(F_n)\left(1 - \langle DF_n, -DL^{-1}F_n\rangle_H\right)\right].$$

Using the fact that f_z' is bounded by 1 (in view of Theorem VI.1.2), it therefore follows from an application of the Cauchy-Schwarz inequality that

$$d_{\text{Kol}}(F_n, N) \leq \sqrt{\mathbb{E}\left[(1 - \langle DF_n, -DL^{-1}F_n\rangle_H)^2\right]} =: \phi(n). \tag{VI.1.9}$$

In Theorem 3.1 of [NP09b], Nourdin and Peccati provide a set of sufficient conditions under which the rate $\phi(n)$ in (VI.1.9) is optimal (in the sense of Definition VI.1.4) for the Kolmogorov distance. Here, as usual, we consider a real separable Hilbert space H as well as an isonormal Gaussian process $X = \{X(h) : h \in H\}$, and assume that the random variables F_n are certain functionals of X.

Theorem VI.1.5. *Let $\{F_n : n \geq 1\}$ be a sequence of centred random variables such that* $\mathbf{Var}[F_n] \to 1$ *as $n \to \infty$. Assume that*

(i) for every $n \geq 1$ one has that $F_n \in \mathbb{D}_{1,2}$, and the law of F_n is absolutely continuous with respect to the Lebesgue measure,

(ii) the numerical sequence $\phi(n)$ defined in (VI.1.9) *satisfies $\phi(n) < \infty, \forall n \geq 1$, $\phi(n) \to 0, n \to \infty$ and there exists an integer $m \geq 1$ such that $\phi(n) > 0, \forall n \geq m$,*

(iii) the two-dimensional random vector $\left(F_n, \frac{1-\langle DF_n,-DL^{-1}F_n\rangle_H}{\phi(n)}\right)$ converges in distribution to a centred two-dimensional Gaussian vector (N_1, N_2) such that $\mathbf{Var}[N_1] = \mathbf{Var}[N_2] = 1$ *and* $\mathbb{E}[N_1N_2] = \rho$ *for some real number ρ.*

[4]For instance, when dealing with the Wasserstein distance, we have that the class $\mathscr{H}$ coincides with that of 1-Lipschitz functions.

Then $d_{\mathrm{Kol}}(F_n, N) \leq \phi(n)$ and for every $z \in \mathbb{R}$, one has that, as $n \to \infty$,

$$\frac{\mathbb{P}\{F_n \leq z\} - \mathbb{P}\{N \leq z\}}{\phi(n)} \to \frac{\rho}{3}(z^2 - 1)\frac{e^{-z^2/2}}{\sqrt{2\pi}}.$$

In particular, if $\rho \neq 0$, the sequence $\{\phi(n) : n \geq 1\}$ is a strongly optimal rate of convergence for d_{Kol} in the sense of Definition VI.1.4.

When specializing the above findings to the case where each F_n is an element of the second Wiener chaos associated with X, their results allow to prove the following result (see [NP09b, Proposition 3.8]). In particular, in such a situation, the set of sufficient conditions (i)-(iii) derived in Theorem VI.1.5, as well as $\phi(n)$, can all be expressed by means of the third, fourth and eight cumulants of F_n.

Theorem VI.1.6. *Let $\{F_n : n \geq 1\}$ be a sequence of random variables such that $F_n = I_2(f_n)$ for some $f_n \in H^{\odot 2}$ for every $n \geq 1$. Assume that* $\mathbf{Var}[F_n] = 1$ *for every $n \geq 1$. Then $F_n \xrightarrow{d} N$ if and only if $\kappa_4(F_n) \to 0$ as $n \to \infty$. In this case, one has that*

$$d_{\mathrm{Kol}}(F_n, N) \leq c\sqrt{\kappa_4(F_n)}$$

for some absolute constant $c > 0$. If in addition, there exists $\alpha \in \mathbb{R}$ such that $\frac{\kappa_3(F_n)}{\sqrt{\kappa_4(F_n)}} \to \alpha$ and $\frac{\kappa_8(F_n)}{\kappa_4(F_n)^2} \to 0$ as $n \to \infty$, then for every $z \in \mathbb{R}$, one has that, as $n \to \infty$,

$$\frac{\mathbb{P}\{F_n \leq z\} - \mathbb{P}\{N \leq z\}}{\sqrt{\kappa_4(F_n)}} \to \frac{\alpha}{3!}(1 - z^2)\frac{e^{-z^2/2}}{\sqrt{2\pi}}.$$

In particular, if $\alpha \neq 0$, then there exists a constant c' and an integer $n_0 \geq 1$ such that for every $n \geq n_0$

$$d_{\mathrm{Kol}}(F_n, N) \geq c'\sqrt{\kappa_4(F_n)}.$$

We finish this section with some bibliographic comments. A multivariate extension of Theorem VI.1.5 for smooth probability distances is established by Campese in [Cam13]. In [NP15, Theorem 1.2], Nourdin and Peccati prove an optimal fourth moment theorem in total variation distance for normalized sequences of chaotic random variables $F_n = I_q(f_n)$ converging in distribution to a standard normal random variable. Therein, they show the existence of absolute constants $0 < c_1 < c_2 < \infty$ such that $c_1\mathbf{M}(F_n) \leq d_{\mathrm{TV}}(F_n, N) \leq c_2\mathbf{M}(F_n)$, where $\mathbf{M}(F_n) := \max(|\kappa_3(F_n)|, |\kappa_4(F_n)|)$. Such a result is to be compared with Theorems 1.9 and 1.11 of [BBNP12], where the authors obtain similar bounds for smooth probability metrics, involving C^2 test functions with bounded derivatives. Finally, in [AP15], Azmoodeh and Peccati study optimal rates in the framework of normal approximations on a Poisson space.

VI.1.4 Gaussian analysis on the second Wiener chaos

Throughout this part, we fix a separable Hilbert space H with orthonormal basis $(e_j : j \geq 1)$ and define an isonormal Gaussian process $X = \{X(h) : h \in H\}$ on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In this section, we present a number of useful properties enjoyed by double Wiener integrals, that is random variables taking the form $F = I_2(f)$ for some $f \in H^{\odot 2}$. In order to do so, we first introduce auxiliary tools from functional analysis associated with a symmetric kernel $f \in H^{\odot 2}$.

Definition VI.1.7. Let $f \in H^{\odot 2}$. We define the Hilbert-Schmidt operator $\mathscr{A}_f : H \to H$ associated with f by $\mathscr{A}_f(g) := f \otimes_1 g$ (where $\otimes_1$ indicates the contraction operator of order 1, defined in (I.1.22)). We write $(\lambda_j(f) : j \geq 1)$ for the eigenvalues of $\mathscr{A}_f$ and assume that $|\lambda_j(f)| \geq |\lambda_{j+1}(f)|$ for every $j \geq 1$; we denote by $(e_j(f) : j \geq 1)$ the eigenvectors associated with $(\lambda_j(f) : j \geq 1)$. Furthermore, we write $N = N(f)$ for the rank of $\mathscr{A}_f$, that is, N is the integer verifying $\lambda_N(f) \neq 0$ and $\lambda_{N+1}(f) = 0$.

The next proposition gives a useful representation of double Wiener integrals as well as a formula for their cumulants (see for instance Propositions 2.7.11 and 2.7.13 in [NP12a]). In particular, from part (i) we deduce that a random variable having the centred chi-square distribution is an element of the second Wiener chaos.

Proposition VI.1.8. *Let $F = I_2(f)$ for some $f \in H^{\odot 2}$.*

(i) We have $F = \sum_{j\geq 1} \lambda_j(f) H_2(N_j)$, both $\mathbb{P}$-almost surely and in $L^2(\mathbb{P})$, where $H_2(x) = x^2 - 1$ is the second Hermite polynomial, $(\lambda_j(f) : j \geq 1)$ are the eigenvalues associated with the Hilbert-Schmidt operator $\mathscr{A}_f$ and $(N_j : j \geq 1)$ is a collection of independent standard Gaussian random variables.

(ii) For every $p \geq 2$, writing $c_p := 2^{p-1}(p-1)!$, the p-th cumulant of F is given by

$$\kappa_p(F) = c_p \sum_{j\geq 1} \lambda_j(f)^p = c_p \mathrm{Tr}(\mathscr{A}_f^p). \tag{VI.1.10}$$

Remark VI.1.9. **(a)** Since the ordering of the eigenvalues $\lambda_j(f)$ has no impact on the distribution of F, the assumption $|\lambda_j(f)| \geq |\lambda_{j+1}(f)|$ stated in Definition VI.1.7 is not restrictive.

(b) The construction in point (i) of Proposition VI.1.8 is a consequence of the multiplication formula for multiple integrals (see (I.1.28)). Indeed, writing $(e_j(f) : j \geq 1)$ for the eigenvectors of $\mathscr{A}_f$ forming an orthonormal system of H, one can expand $f = \sum_{j\geq 1} \lambda_j(f) e_j(f)^{\otimes 2}$ in $H^{\odot 2}$, so that by (I.1.28), one has

$$I_2(f) = \sum_{j\geq 1} \lambda_j(f) I_2(e_j(f)^{\otimes 2}) = \sum_{j\geq 1} \lambda_j(f) \left(I_1(e_j(f))^2 - 1\right) \stackrel{d}{=} \sum_{j\geq 1} \lambda_j(f) \left(N_j^2 - 1\right),$$

where N_j are i.i.d standard Gaussian random variables. The independence of $(N_j : j \geq 1)$ then follows from the fact that $(e_j(f) : j \geq 1)$ is orthonormal in H.

VI.2 Main results: Suboptimal rates on the second Wiener chaos

Let $\{F_n : n \geq 1\}$ be a sequence of centred random variables such that $\mathbf{Var}[F_n] = 2\nu$ for every $n \geq 1$ and $F_n \xrightarrow{d} G(\nu)$ for some $\nu > 0$. In view of the upper bound for the Wasserstein distance in (VI.1.8), our goal of this section is to analyze the following problem.

Problem VI.2.1. *Can we provide a set of sufficient conditions on $\{F_n : n \geq 1\}$ (similar to (i)-(iii) in Theorem VI.1.5) implying that the numerical sequence $\{\phi_\nu(n) : n \geq 1\}$ given by*

$$\phi_\nu(n) := \sqrt{\mathbb{E}\left[\left(2(F_n + \nu) - \langle DF_n, -DL^{-1}F_n\rangle_H\right)^2\right]} \tag{VI.2.1}$$

is an optimal rate of convergence for the Wasserstein distance d_{W} in the sense of Definition VI.1.4?

In order to approach Problem VI.2.1, we follow the route of Nourdin and Peccati [NP09b] for normal approximations. We define the random variable

$$F_n^{(\nu)} := \frac{2(F_n + \nu) - \langle DF_n, -DL^{-1}F_n\rangle_H}{\phi_\nu(n)} \tag{VI.2.2}$$

and are interested in characterizing the joint convergence in distribution of the two-dimensional vector $(F_n, F_n^{(\nu)})$. Our main results show that the answer to Problem VI.2.1 is negative in the case when F_n is a random variable in the second Wiener chaos, in the sense that the rate $\phi_\nu(n)$ is not strongly optimal. In order to state our main results, we write (see Proposition VI.1.8)

$$F_n = I_2(f_n) \stackrel{d}{=} \sum_{j\geq 1} \lambda_{n,j} H_2(N_j), \tag{VI.2.3}$$

where $(\lambda_{n,j} := \lambda_j(f_n), j \geq 1)$ denote the eigenvalues associated with the Hilbert-Schmidt operator $\mathscr{A}_{f_n}$ (see Definition VI.1.7) and $(N_j, j \geq 1)$ are independent standard Gaussian random variables. The following proposition states that the random variable $F_n^{(\nu)}$ in (VI.2.2) is an element of the second Wiener chaos.

Proposition VI.2.2. *Let $\nu > 0$. Assume that $\{F_n : n \geq 1\}$ is a sequence of random variables as in* (VI.2.3) *and such that* $\mathbf{Var}[F_n] = 2\nu$ *for every $n \geq 1$. Then, $F_n^{(\nu)}$ is an element of the second Wiener chaos admitting the $\mathbb{P}$-almost sure and $L^2(\mathbb{P})$ representation*

$$F_n^{(\nu)} = \sum_{j\geq 1} \widetilde{\lambda}_{n,j} H_2(N_j), \tag{VI.2.4}$$

where

$$\widetilde{\lambda}_{n,j} = \frac{\lambda_{n,j} - \lambda_{n,j}^2}{\sqrt{2}\sqrt{\sum_{k\geq 1}[\lambda_{n,k} - \lambda_{n,k}^2]^2}}, \quad j \geq 1. \tag{VI.2.5}$$

In the forthcoming two theorems, we provide a partial characterization of the limiting distribution of the sequence $\left\{F_n^{(\nu)} : n \geq 1\right\}$ as $n \to \infty$, by distinguishing the case where (i) the rank $N = N(f_n)$ of $\mathscr{A}_{f_n}$ is constant for every n (see Theorem VI.2.5) and (ii) the rank $N = N(f_n) \in \mathbb{N} \cup \{\infty\}$ is such that $N(f_n) \to \infty$ as $n \to \infty$ (Theorem VI.2.6).

Notation VI.2.3.

- *For numerical sequences $\{a_n : n \geq 1\}$ and $\{b_n : n \geq 1\}$, we use the symbol $a_n \approx b_n$ to indicate that $a_n - b_n \to 0$ as $n \to \infty$, and $a_n \asymp b_n$ if $0 < \liminf_{n\to\infty} |a_n/b_n| \leq \limsup_{n\to\infty} |a_n/b_n| < \infty$. As usual, $a_n = o(b_n)$ means that $a_n/b_n \to 0$ as $n \to \infty$.*
- *Let $F_n = \sum_{j=1}^N \lambda_{n,j} H_2(N_j)$ be an element of the second Wiener chaos, where $1 \leq N = N(f_n) \leq \infty$ is its rank. For an integer $\nu \geq 1$, we define the following two numerical sequences*

$$\omega(n) := \max\left\{|\lambda_{n,j} - 1| : j = 1, \ldots, \nu\right\}, \quad \vartheta(n) := \sum_{j=\nu+1}^{N} \lambda_{n,j}^2. \tag{VI.2.6}$$

Remark VI.2.4. **(a)** In [AEK18], it is shown that a sequence $F_n = I_2(f_n)$ converges to a centred Gamma distribution $G(\nu)$ with parameter ν if and only if ν is an integer and $\lambda_{n,j} \to \mathbb{1}_{1\leq j\leq \nu}$ as $n \to \infty$. A proof of this fact is presented in Proposition VI.2.11. In the following statements, we therefore implicitly have that ν is an integer.

(b) We shall use the notation in (VI.2.6) independently whether the rank N is finite or infinite. Note that for every $N \in \mathbb{N} \cup \{\infty\}$, we have $\vartheta(n) \leq \sum_{j\geq 1} \lambda_{n,j}^2 = \nu < \infty$ in view of Proposition VI.1.8.

(c) Since $\mathbb{E}\left[F_n^2\right] = 2\sum_{j\geq 1} \lambda_{n,j}^2 = 2\nu$, we have the relation

$$0 \leq \sum_{j=1}^{\nu} (1 - \lambda_{n,j})^2 = 2\sum_{j=1}^{\nu} (1 - \lambda_{n,j}) - \sum_{j=\nu+1}^{N} \lambda_{n,j}^2 \leq 2\nu\omega(n) - \vartheta(n), \tag{VI.2.7}$$

showing that $\vartheta(n) \leq 2\nu\omega(n)$. This implies in particular that $\vartheta(n)/\omega(n)$ cannot diverge and thus the only possible cases are $\vartheta(n) \asymp \omega(n)$ and $\vartheta(n) = o(\omega(n))$, as $n \to \infty$.

The following theorem characterizes the limiting distribution of $F_n^{(\nu)}$ in the case of constant rank.

Theorem VI.2.5. *(Case of finite rank) Assume that F_n admits the representation* (VI.2.3) *and is such that $N(f_n) = N < \infty$ for every $n \geq 1$ (that is, the series in* (VI.2.3) *is replaced with a summation from* 1 *to N). Assume that* $\mathbf{Var}[F_n] = 2\nu$ *for every $n \geq 1$ and F_n converges in distribution to a centred Gamma distribution $G(\nu)$ with parameter ν as $n \to \infty$. Then, there exist real numbers $\left\{\ell_j : j = 1, \ldots, N\right\}$ such that $\sum_{j=1}^{\nu} \ell_j = 0$ and, as $n \to \infty$,*

$$F_n^{(\nu)} \xrightarrow{d} \sum_{j=1}^{N} \ell_j H_2(N_j).$$

For the case of infinite rank, we can prove the following partial limit theorem, valid in the case where $\vartheta(n) \asymp \omega(n)$. For the (remaining) case where $\vartheta(n) = o(\omega(n))$, we refer the reader to the subsequent Remark VI.2.7 **(b)**.

Theorem VI.2.6. *(Case of infinite rank) Assume that F_n admits the representation* (VI.2.3) *with rank $N = N(f_n) \in \mathbb{N} \cup \{\infty\}$ such that $N(f_n) \to \infty$ as $n \to \infty$, and is such that* $\mathbf{Var}[F_n] = 2\nu$ *for every $n \geq 1$ and converges in distribution to $G(\nu)$ as $n \to \infty$. If $\vartheta(n) \asymp \omega(n)$ as $n \to \infty$, then $F_n^{(\nu)} \xrightarrow{d} N_0$, where N_0 is a standard Gaussian random variable that is independent of $(N_j : j \geq 1)$.*

Remark VI.2.7. **(a)** We observe that the findings derived in Theorems VI.2.5 and VI.2.6 are both in accordance with Theorem 3.1 of [NP12b] asserting that, if a sequence $G_n = I_2(g_n), n \geq 1$ belonging to the second Wiener chaos (with respect to some isonormal Gaussian process X) converges in distribution to some random variable G_∞, then one has necessarily that $G_\infty \stackrel{d}{=} I_2(g_\infty) + \lambda_0 N_0$, for some $g_\infty \in H^{\odot 2}$ and $\lambda_0 \in \mathbb{R}$, where N_0 is a standard Gaussian random variable that is independent of the underlying isonormal Gaussian process X.

(b) The conclusion of Theorem VI.2.6 is valid under the asymptotic assumption $\vartheta(n) \asymp \omega(n)$. In order to prove such a result, we show that the fourth cumulant of $F_n^{(\nu)}$ converges to zero as $n \to \infty$. We remark that the case $\vartheta(n) = o(\omega(n))$ is open: in this framework, we are able to prove that the fourth cumulant of $F_n^{(\nu)}$ is lower bounded by a strictly positive constant, thus showing that a non-central convergence takes place. Several preliminary examples in this specific case show that the limiting distribution of $F_n^{(\nu)}$ is of the form $\sum_{j\geq 1} \ell_j H_2(N_j) + \lambda_0 N_0$, where ℓ_j are real numbers verifying the condition $\sum_{j=1}^{\nu} \ell_j = 0$, and N_0 is a standard Gaussian independent of $(N_j : j \geq 1)$, thus being of similar nature than the finite rank case.

As a by-product of Theorems VI.2.5 and VI.2.6, we derive the following sub-optimality phenomenon for a large class of random variables living in the second Wiener chaos. For an integer $\nu > 0$, we define the class[5] $\Sigma(\nu)$ to be the collection of all sequences $\{F_n = I_2(f_n), n \geq 1\}$ such that $\mathbf{Var}[F_n] = 2\nu$ for every $n \geq 1, F_n \xrightarrow{d} G(\nu)$ and verifying the conditions and (i) $N(f_n) = N \in \mathbb{N}$ for every $n \geq 1$ or (ii) $N(f_n) \to \infty$ and $\vartheta(n) \asymp \omega(n)$ as $n \to \infty$.

Corollary VI.2.8. *Let $\{F_n : n \geq 1\}$ be an element of $\Sigma(\nu)$. Then,*

(i) For every $n \geq 1$, $d_{\mathrm{W}}(F_n, G(\nu)) \leq \max(1, 2/\nu)\phi_\nu(n)$.

(ii) For every $h \in \mathrm{Lip}(1)$, one has that, as $n \to \infty$

$$\frac{\mathbb{E}[h(F_n)] - \mathbb{E}[h(G(\nu))]}{\phi_\nu(n)} \to 0.$$

[5]Such a class of sequences is formulated in order to contain all the cases covered in Theorems VI.2.5 and VI.2.6.

In particular, the rate of convergence associated with the numerical sequence $\{\phi_\nu(n) : n \geq 1\}$ *is not strongly optimal for the Wasserstein distance.*

Remark VI.2.9. Assuming that the distributional limit theorem for $F_n^{(\nu)}$ conjectured in Remark VI.2.7 **(b)** for the remaining case $\vartheta(n) = o(\omega(n))$ takes place, our arguments used to prove Corollary VI.2.8 would allow to extend the sub-optimality phenomenon to all sequences F_n living in the second Wiener chaos. Indeed, by inspection of our arguments used in the proof of Corollary VI.2.8, the main features determining the sub-optimality phenomenon are the arithmetic condition $\sum_{j=1}^{\nu} \ell_j = 0$ and the independence of N_0 from $(N_j : j \geq 1)$ (which is guaranteed thanks to [NP12b, Theorem 3.1]).

VI.2.1 Further comments and remarks

Whether the phenomenon observed in Corollary VI.2.8 remains valid in the case where F_n is an element of the q-th Wiener chaos with $q \geq 3$ is a natural question to ask. We formulate it in the following conjecture.

Conjecture VI.2.10. *Let* $q \geq 3$ *be an integer. Let* $F_n = I_q(f_n), f_n \in H^{\odot q}$ *be a sequence in the q-th Wiener chaos such that* $\mathbf{Var}[F_n] = 2\nu$ *for every* $n \geq 1$ *and* $F_n \xrightarrow{d} G(\nu)$*. Then, for every* $h \in \mathrm{Lip}(1)$*, one has that, as* $n \to \infty$

$$\frac{\mathbb{E}\left[h(F_n)\right] - \mathbb{E}\left[h(G(\nu))\right]}{\phi_\nu(n)} \to 0.$$

Here below, we make some more comments on our findings and Conjecture VI.2.10.

(a) The statement in Conjecture VI.2.10 is quite strong and might sound far-fetched at first reading. Indeed, our proofs on the second Wiener chaos heavily revolve around the tools from Proposition VI.1.8, whose analog in higher-order chaoses is not available. An explicit example in favour of the conjecture within the fourth Wiener chaos is presented in Section VI.2.3.

(b) The content of Corollary VI.2.8 is of different nature compared to Theorem VI.1.6 for Gaussian approximations. Therein, it becomes clear that the numerical sequence given by $\sqrt{\kappa_4(F_n)}$ provides strongly optimal rates of convergence as soon as α therein is non zero, that is, $\kappa_4(F_n)$ is of the same order as $\kappa_3(F_n)$. Our results show that such a phenomenon is absent for Gamma approximations.

(c) In Proposition VI.2.2 we showed that, if F_n is an element of the second chaos, then so is $F_n^{(\nu)}$. We point out that this phenomenon fails on higher-order chaoses: indeed, an application of the multiplication formulae for multiple integrals (see (I.1.28)) allows to show that, if $F_n = I_q(f_n)$, $(q \geq 2)$ is such that $\mathbf{Var}[F_n] = 2\nu$, then (see for instance [NP09c])

$$\begin{aligned} 2(F_n + \nu) - \frac{1}{q}\|DF_n\|_H^2 &= q(q/2-1)!\binom{q-1}{q/2-1}^2 I_q(c_q f_n - f_n \widetilde{\otimes}_{q/2} f_n) \\ &\quad - q\sum_{r=1, r\neq q/2}^{q-1} (r-1)!\binom{q-1}{r-1}^2 I_{2q-2r}(f_n \widetilde{\otimes}_r f_n), \end{aligned}$$

where

$$c_q := \frac{2}{q(q/2-1)!\binom{q-1}{q/2-1}^2} = \frac{1}{(q/2)!\binom{q-1}{q/2-1}^2}.$$

This shows in particular that the random variable $2(F_n + \nu) - \frac{1}{q}\|DF_n\|_H^2$ lives in a finite sum of Wiener chaoses. By the isometry property of multiple Wiener integrals, one deduces that

$$\begin{aligned}\phi_\nu(n)^2 &= q!q(q/2-1)!\binom{q-1}{q/2-1}^2 \|c_q f_n - f_n \widetilde{\otimes}_{q/2} f_n\|^2_{H^{\otimes q}} \\ &\quad + q\sum_{r=1, r\neq q/2}^{q-1} (2q-2r)!(r-1)!\binom{q-1}{r-1}^2 \|f_n \widetilde{\otimes}_r f_n\|^2_{H^{\otimes(2q-2r)}}. \qquad \text{(VI.2.8)}\end{aligned}$$

Our findings in Theorems VI.2.5 and VI.2.6 on the second Wiener chaos are therefore of independent interest, as their generalization to higher-order chaoses seems to be a rather demanding task in full generality.

(d) In [AEK20, Theorem 1.3], the authors prove that, whenever $F_n = I_2(f_n), f_n \in H^{\odot 2}$ is such that $\mathbf{Var}[F_n] = 2\nu$, then there are constants $0 < c_1 < c_2 < \infty$ such that $c_1 \mathbf{M}(F_n) \leq d_2(F_n, G(\nu)) \leq c_2 \mathbf{M}(F_n)$, where $\mathbf{M}(F_n) := \max(|\kappa_3(F_n) - \kappa_3(G(\nu))|, |\kappa_4(F_n) - \kappa_4(G(\nu))|)$, and d_2 stands for the 2-Wasserstein distance involving C^2 test functions with bounded derivatives, showing in particular that the rate of convergence $\mathbf{M}(F_n)$ is optimal for d_2. We point out that our arguments used in order to prove Theorems VI.2.5 and VI.2.6 are sufficient to directly prove that $\mathbf{M}(F_n) = o(\phi_\nu(n))$, implying that the rate of convergence associated with $\phi_\nu(n)$ is suboptimal for the smooth distance d_2. We believe that our exposition based on the explicit characterization of the limiting distribution of the random variables $F_n^{(\nu)}$ yields a more complete view point on why such a sub-optimality phenomenon occurs.

VI.2.2 Proofs of main results

VI.2.2.1 Preliminary results

The following proposition will be of importance for our results: It tells that if a sequence of double Wiener integrals converges in law to a centred Gamma random variable $G(\nu)$, then necessarily $G(\nu)$ has the chi-squared distribution with ν degrees of freedom. We prove it for completeness.

Proposition VI.2.11. *Let $\{F_n : n \geq 1\}$ be a sequence of random variables such that $F_n = I_2(f_n)$ for some $f_n \in H^{\odot 2}$ and $\mathbf{Var}[F_n] = 2\nu$ for every $n \geq 1$. Fix $\nu > 0$. Then, as $n \to \infty$, we have that*

$$\left(F_n \xrightarrow{d} G(\nu)\right) \iff \left(\nu \in \mathbb{N} \quad \text{and} \quad \lambda_j(f_n) \to \mathbb{1}_{1\leq j \leq \nu},\ \forall j \geq 1\right).$$

Proof. Assume first that $F_n \xrightarrow{d} G(\nu)$ for some $\nu > 0$ as $n \to \infty$. Then, we have that $\kappa_p(F_n) \to \kappa_p(G(\nu))$ for every $p \geq 2$. The cumulants of $G(\nu)$ can be computed as $\kappa_1(G(\nu)) = 0$ and $\kappa_p(G(\nu)) = 2^{p-1}(p-1)!\nu$, for $p \geq 2$. In particular, by Proposition VI.1.8 (ii), we deduce that $\sum_{j\geq 1} \lambda_{n,j}^p \to \nu$ for every $p \geq 2$. In view of [NP09a, Theorem 1.2], we moreover have that $\mathbf{Var}[\|DF_n\|_H - 2F_n] \to 0$. This implies that for every $j \geq 1$,

$$\lambda_{n,j}^2(\lambda_{n,j} - 1)^2 \leq \sum_{j\geq 1} \lambda_{n,j}^2(\lambda_{n,j} - 1)^2 = \frac{1}{8}\mathbf{Var}[\|DF_n\|_H - 2F_n] \to 0.$$

This shows that accumulation points of $\lambda_{n,j}$ are 0 and 1 for every $j \geq 1$. We first consider $\lambda_{n,1}$. Assume that there is a subsequence $\{n_k : k \geq 1\}$ such that $\lambda_{n_k,1} \to 0$ as $k \to \infty$. Then, by the ordering assumption of the eigenvalues (see Definition VI.1.7), we have that

$$\sum_{j\geq 1} \lambda_{n_k,j}^4 \leq \lambda_{n_k,1}^2 \sum_{j\geq 1} \lambda_{n_k,j}^2 = \lambda_{n_k,1}^2 \mathbf{Var}[F_{n_k}] \to 0,$$

which contradicts the fact that $\kappa_4(F_{n_k}) \to \kappa_4(G(\nu))$. We deduce that $\lambda_{n,1} \to 1$, implying in turn that $\sum_{j\geq 2} \lambda_{n,j}^p \to \nu - 1$ for every $p \geq 2$. Repeating this process inductively shows that $\nu \in \mathbb{N}$ and $\lambda_{n,j} \to 1$ for every $j = 1, \ldots, \nu$. Therefore, we are left with $\sum_{j\geq \nu+1} \lambda_{n,j}^p \to 0$ for every $p \geq 2$. In particular, we have for every $j \geq \nu + 1$

$$\lambda_{n,j}^2 \leq \sum_{j\geq \nu+1} \lambda_{n,j}^2 \to 0,$$

so that $\lambda_{n,j} \to 0$ for every $j \geq \nu + 1$, as $n \to \infty$. Let us now prove the reverse direction: assume that $\nu \in \mathbb{N}$ and $\lambda_{n,j} \to \mathbb{1}_{1\leq j\leq \nu}$ for every $j \geq 1$, as $n \to \infty$. Since $\nu \in \mathbb{N}$, we have that $G(\nu) \overset{d}{=} \sum_{j=1}^{\nu} H_2(N_i)$ where $(N_1, \ldots, N_\nu) \sim \mathcal{N}_\nu(0, \mathbf{I}_\nu)$. In particular, using Proposition VI.1.8 (i) yields that

$$F_n - G(\nu) \overset{d}{=} \sum_{j\geq 1} \left(\lambda_{n,j} - \mathbb{1}_{1\leq j\leq \nu}\right) H_2(N_j),$$

so that, the orthogonality relation for Hermite polynomials in Proposition I.1.22 gives

$$\mathbb{E}\left[(F_n - G(\nu))^2\right] = 2\sum_{j\geq 1} \left(\lambda_{n,j} - \mathbb{1}_{1\leq j\leq \nu}\right)^2 = 2\sum_{1\leq j\leq \nu} [\lambda_{n,j} - 1]^2 + 2\sum_{j\geq \nu+1} \lambda_j(f_n)^2.$$

The first sum converges to zero by assumption. For the second sum, we write

$$\sum_{j\geq \nu+1} \lambda_{n,j}^2 = \sum_{j\geq 1} \lambda_{n,j}^2 - \sum_{1\leq j\leq \nu} \lambda_{n,j}^2 \to \nu - \nu = 0,$$

where we used the fact that $\mathbf{Var}[F_n] = 2\nu$. This shows that F_n converges to $G(\nu)$ in $L^2(\mathbb{P})$ and therefore in distribution. □

We now prove Proposition VI.2.2.

Proof of Proposition VI.2.2. Since F_n is an element of the second Wiener chaos, we have that $\langle DF_n, -DL^{-1}F_n\rangle_H = \frac{1}{2}\|DF_n\|_H^2$. Writing $F_n = I_2(f_n) = \sum_{j\geq 1} \lambda_{n,j} I_2(e_{n,j}^{\otimes 2})$, where $e_{n,j} := e_j(f_n)$ denote the eigenvectors associated with $\lambda_{n,j}$, a direct computation gives

$$DF_n = \sum_{j\geq 1} \lambda_{n,j} DI_2(e_{n,j}^{\otimes 2}) = 2\sum_{j\geq 1} \lambda_{n,j} I_1(e_{n,j}) e_{n,j}$$

so that

$$\begin{aligned}\frac{1}{2}\|DF_n\|_H^2 &= 2\sum_{j,k\geq 1} \lambda_{n,j}\lambda_{n,k} I_1(e_{n,j}) I_1(e_{n,k}) \langle e_{n,j}, e_{n,k}\rangle_H \\ &= 2\sum_{j\geq 1} \lambda_{n,j}^2 I_1(e_{n,j})^2 = 2\sum_{j\geq 1} \lambda_{n,j}^2 H_2(I_1(e_{n,j})) + 2\nu,\end{aligned}$$

where we used the fact that $2\sum_{j\geq 1} \lambda_{n,j}^2 = \kappa_2(F_n) = \mathbf{Var}[F_n] = 2\nu$ in view of Proposition VI.1.8. Therefore, we have that

$$2(F_n + \nu) - \frac{1}{2}\|DF_n\|_H^2 \overset{d}{=} 2\sum_{j\geq 1} [\lambda_{n,j} - \lambda_{n,j}^2] H_2(N_j). \tag{VI.2.9}$$

Using the orthogonality relation of Hermite polynomials in Proposition I.1.22, it follows that

$$\phi_\nu(n) = \mathbb{E}\left[\left(2(F_n + \nu) - \frac{1}{2}\|DF_n\|_H^2\right)^2\right] = 8\sum_{j\geq 1} [\lambda_{n,j} - \lambda_{n,j}^2]^2.$$

Dividing the R.H.S in (VI.2.9) by the expression of $\phi_\nu(n)$ then yields the desired conclusion. □

VI.2.2.2 Proofs of Theorem and VI.2.5 and Theorem VI.2.6

Proof of Theorem VI.2.5. Since $F_n \xrightarrow{d} G(\nu)$ as $n \to \infty$, it follows from Proposition VI.2.11 that $\nu \in \{1, \ldots, N\}$ is an integer and $\lambda_{n,j} \to 1$ for every $j = 1, \ldots, \nu$ and $\lambda_{n,j} \to 0$ for every $j = \nu+1, \ldots, N$. Since $\mathbf{Var}\left[F_n^{(\nu)}\right] = 1$ for every $n \geq 1$, the sequence $\left\{F_n^{(\nu)} : n \geq 1\right\}$ is tight and thus there exists a subsequence $\{n_k : k \geq 1\}$ and a random variable $F_\infty^{(\nu)}$ such that $F_{n_k}^{(\nu)} \xrightarrow{d} F_\infty^{(\nu)}$ as $k \to \infty$. Without loss of generality and for notational reasons, we will assume that $F_n^{(\nu)} \xrightarrow{d} F_\infty^{(\nu)}$ as $n \to \infty$. For $j = 1, \ldots, N$, we write $\ell_j := \lim_{n\to\infty} \widetilde{\lambda}_{n,j}$, where $\widetilde{\lambda}_{n,j}$ is as in (VI.2.5) and where the series over $j \geq 1$ is replaced with a finite sum over $j = 1, \ldots, N$. We now show that $F_n^{(\nu)}$ converges to $F_\infty^{(\nu)} \overset{d}{=} \sum_{j=1}^N \ell_j H_2(N_j)$ in $L^2(\mathbb{P})$ as $n \to \infty$. Indeed, we have that

$$\mathbb{E}\left[\left(F_n^{(\nu)} - F_\infty^{(\nu)}\right)^2\right] = \mathbb{E}\left[\left(\sum_{i,j=1}^N (\widetilde{\lambda}_{n,i} - \ell_i)(\widetilde{\lambda}_{n,j} - \ell_j) H_2(N_i) H_2(N_j)\right)\right] = 2\sum_{j=1}^N \left(\widetilde{\lambda}_{n,j} - \ell_j\right)^2$$

which converges to zero by definition of ℓ_j. It remains to show that $\sum_{j=1}^\nu \ell_j = 0$. In view of Proposition VI.2.11, the assumption that $F_n \xrightarrow{d} G(\nu)$ implies that both $\omega(n)$ and $\vartheta(n)$ in (VI.2.6) converge to zero as $n \to \infty$. In order to prove the claim, we distinguish two cases: (i) $\nu \in \{2, \ldots, N-1\}$ and (ii) $\nu = 1$. We start with the case (i). For notational convenience, we assume that $\omega(n) = |1 - \lambda_{n,1}|$. The arguments work similarly if $\omega(n) = |1 - \lambda_{n,k}|$ for some $k = 2, \ldots, N$. In view of Remark VI.2.7 **(a)**, it suffices to treat the two cases $\vartheta(n) = o(\omega(n))$ and $\vartheta(n) \asymp \omega(n)$. In the case $\vartheta(n) = o(\omega(n))$, we write

$$\left|\sum_{j=1}^\nu \widetilde{\lambda}_{n,j}\right| = \frac{1}{\sqrt{2}} \left| \frac{1 + \sum_{j=2}^\nu \frac{\lambda_{n,j}(1-\lambda_{n,j})}{\lambda_{n,1}(1-\lambda_{n,1})}}{\sqrt{1 + \sum_{k=2}^\nu \frac{\lambda_{n,k}^2(1-\lambda_{n,k})^2}{\lambda_{n,1}^2(1-\lambda_{n,1})^2} + \sum_{k=\nu+1}^N \frac{\lambda_{n,k}^2(1-\lambda_{n,k})^2}{\lambda_{n,1}^2(1-\lambda_{n,1})^2}}} \right| \leq \frac{1}{\sqrt{2}} \left| 1 + \sum_{j=2}^\nu \frac{\lambda_{n,j}(1-\lambda_{n,j})}{\lambda_{n,1}(1-\lambda_{n,1})} \right|.$$

Using the fact that $\lambda_{n,1} \to 1$ as $n \to \infty$, we have that

$$\begin{aligned} 1 + \sum_{j=2}^\nu \frac{\lambda_{n,j}(1-\lambda_{n,j})}{\lambda_{n,1}(1-\lambda_{n,1})} &\approx 1 + \sum_{j=2}^\nu \frac{(1-\lambda_{n,j})}{(1-\lambda_{n,1})} \sim 1 + \frac{1}{2}\frac{1}{1-\lambda_{n,1}} \sum_{j=2}^\nu (1 - \lambda_{n,j}^2) \\ &= 1 + \frac{1}{2}\frac{1}{1-\lambda_{n,1}}\left(\nu - 1 - \sum_{j=2}^\nu \lambda_{n,j}^2\right) = 1 + \frac{1}{2}\frac{\lambda_{n,1}^2 - 1 + \sum_{j=\nu+1}^N \lambda_{n,j}^2}{1-\lambda_{n,1}}. \end{aligned}$$

Now, we observe that, in view of the asymptotic assumption $\vartheta(n) = o(\omega(n))$ and the triangle inequality,

$$\left|\frac{\lambda_{n,1}^2 - 1 + \sum_{j=\nu+1}^N \lambda_{n,j}^2}{1-\lambda_{n,1}} + 2\right| \leq \left|\frac{\lambda_{n,1}^2 - 1}{1-\lambda_{n,1}} + 2\right| + \frac{\vartheta(n)}{\omega(n)} \to 0.$$

This shows that $\sum_{j=1}^\nu \ell_j = 0$. In the case where $\vartheta(n) \asymp \omega(n)$, we write

$$\left|\sum_{j=1}^\nu \widetilde{\lambda}_{n,j}\right| \leq \frac{1}{\sqrt{2}} \left| \frac{1 + \sum_{j=2}^\nu \frac{\lambda_{n,j}(1-\lambda_{n,j})}{\lambda_{n,1}(1-\lambda_{n,1})}}{\sqrt{1 + \sum_{k=\nu+1}^N \frac{\lambda_{n,k}^2(1-\lambda_{n,k})^2}{\lambda_{n,1}^2(1-\lambda_{n,1})^2}}} \right|.$$

Reasoning similarly as above and exploiting the asymptotic assumption $\vartheta(n) \asymp \omega(n)$, shows that the numerator of the R.H.S is asymptotically bounded by $\vartheta(n)/\omega(n) \leq 2\nu$ (see also Remark VI.2.4 **(c)**).

Therefore in order to conclude, it suffices to show that the denominator diverges to infinity as $n \to \infty$. Using that $\lambda_{n,1} \to 1$ and $\lambda_{n,k} \to 0$ for every $k = \nu + 1, \dots, N$, yields that, as $n \to \infty$

$$\sum_{k=\nu+1}^{N} \frac{\lambda_{n,k}^2 (1-\lambda_{n,k})^2}{\lambda_{n,1}^2 (1-\lambda_{n,1})^2} \approx \sum_{k=\nu+1}^{N} \frac{\lambda_{n,k}^2}{(1-\lambda_{n,1})^2} = \frac{1}{|1-\lambda_{n,1}|} \times \frac{\vartheta(n)}{\omega(n)} \to +\infty \qquad \text{(VI.2.10)}$$

since $\vartheta(n) \asymp \omega(n)$, which suffices. The remaining case (ii) where $\nu = 1$ can be dealt with similarly. □

Proof of Theorem VI.2.6. As in the beginning of the proof of Theorem VI.2.5, we assume that $F_n^{(\nu)} \xrightarrow{d} F_\infty^{(\nu)}$ as $n \to \infty$ for some random variable $F_\infty^{(\nu)}$. Since for every $n \geq 1$, $F_n^{(\nu)}$ is an element of the second Wiener chaos, in order to show that $F_n^{(\nu)} \xrightarrow{d} N_0$, it suffices to show that $\kappa_4(F_n^{(\nu)}) \to 0$ as $n \to \infty$ in view of Theorem I.1.30. Using the explicit form of $F_n^{(\nu)}$ in (VI.2.4) together with formula (VI.1.10) with $p = 4$, we have that

$$\begin{aligned}\kappa_4(F_n^{(\nu)}) &= 48 \sum_{j\geq 1} \widetilde{\lambda}_{n,j}^4 = 12 \frac{\sum_{j\geq 1} \lambda_{n,j}^4 (1-\lambda_{n,j})^4}{\left(\sum_{j\geq 1} \lambda_{n,j}^2 (1-\lambda_{n,j})^2\right)^2} \\ &= 12 \frac{\sum_{j=1}^{\nu} \lambda_{n,j}^4 (1-\lambda_{n,j})^4}{\left(\sum_{j\geq 1} \lambda_{n,j}^2 (1-\lambda_{n,j})^2\right)^2} + 12 \frac{\sum_{j\geq \nu+1} \lambda_{n,j}^4 (1-\lambda_{n,j})^4}{\left(\sum_{j\geq 1} \lambda_{n,j}^2 (1-\lambda_{n,j})^2\right)^2} =: 12[a(n)+b(n)].\end{aligned}$$

We show that $a(n)$ and $b(n)$ both converge to zero as $n \to \infty$. Without loss of generality, we assume that $\omega(n) = |1-\lambda_{n,1}|$. For $a(n)$, we have that

$$a(n) = \frac{\sum_{j=1}^{\nu} \lambda_{n,j}^4 (1-\lambda_{n,j})^4}{\left(\sum_{j\geq 1} \lambda_{n,j}^2 (1-\lambda_{n,j})^2\right)^2} \leq \frac{1 + \sum_{j=2}^{\nu} \frac{\lambda_{n,j}^4}{\lambda_{n,1}^4} \frac{(1-\lambda_{n,j})^4}{(1-\lambda_{n,1})^4}}{\left(1 + \sum_{j\geq \nu+1} \frac{\lambda_{n,j}^2}{\lambda_{n,1}^2} \frac{(1-\lambda_{n,j})^2}{(1-\lambda_{n,1})^2}\right)^2}.$$

For the numerator in the R.H.S, we use that $\lambda_{n,j}^4 \leq \lambda_{n,1}^4$ for $j = 2, \dots, \nu$ and $(1-\lambda_{n,j})^4 \leq \omega(n)^4 = (1-\lambda_{n,1})^4$, to obtain that

$$1 + \sum_{j=2}^{\nu} \frac{\lambda_{n,j}^4}{\lambda_{n,1}^4} \frac{(1-\lambda_{n,j})^4}{(1-\lambda_{n,1})^4} \leq 1 + (\nu - 1) \frac{\omega(n)^4}{(1-\lambda_{n,1})^4} = \nu.$$

For the denominator, we have for every integer $N \geq \nu + 1$

$$\sum_{j\geq \nu+1} \frac{\lambda_{n,j}^2}{\lambda_{n,1}^2} \frac{(1-\lambda_{n,j})^2}{(1-\lambda_{n,1})^2} \geq \sum_{j=\nu+1}^{N} \frac{\lambda_{n,j}^2}{\lambda_{n,1}^2} \frac{(1-\lambda_{n,j})^2}{(1-\lambda_{n,1})^2} \to \infty$$

as $n \to \infty$ in view of the finite-rank case obtained in (VI.2.10) and the asymptotic relation $\vartheta(n) \asymp \omega(n)$. This shows that $a(n) \to 0$ as $n \to \infty$. For $b(n)$, we write

$$b(n) = \frac{\sum_{j\geq \nu+1} \lambda_{n,j}^4 (1-\lambda_{n,j})^4}{\left(\sum_{j\geq 1} \lambda_{n,j}^2 (1-\lambda_{n,j})^2\right)^2} \leq \frac{(1-\lambda_{n,1})^4}{\lambda_{n,1}^4 (1-\lambda_{n,1})^4} \frac{\sum_{j\geq \nu+2} \lambda_{n,j}^4}{\left(1 + \sum_{j\geq \nu+2} \frac{\lambda_{n,j}^2}{\lambda_{n,1}^2} \frac{(1-\lambda_{n,j})^2}{(1-\lambda_{n,1})^2}\right)^2} \leq \frac{1}{\lambda_{n,1}^4} \sum_{j\geq \nu+2} \lambda_{n,j}^4$$

which converges to zero as $n \to \infty$, thanks to Proposition VI.1.8. This proves that $\kappa_4(F_n^{(\nu)}) \to 0$ as $n \to \infty$ and thus $F_n^{(\nu)}$ converges in distribution to a standard Gaussian random variable N_0. The fact that N_0 is independent of $(N_j : j \geq 1)$ directly follows from the characterization of limiting distributions within the second chaos (see [NP12b, Theorem 3.1] and also Remark VI.2.7 **(a)**).

□

VI.2.2.3 Proof of Corollary VI.2.8

Proof of Corollary VI.2.8. Part (i) is exactly (VI.1.8) proven in [DP18]. We prove part (ii). Denote by $F_\infty^{(\nu)}$ the limiting distribution of $F_n^{(\nu)}$. The desired conclusion follows the following two facts:

$$(\mathbf{I}) : \left(F_n, F_n^{(\nu)}\right) \xrightarrow{d} \left(G(\nu), F_\infty^{(\nu)}\right), \quad n \to \infty, \qquad (\mathbf{II}) : \mathbb{E}\left[f_h'(G(\nu))F_\infty^{(\nu)}\right] = 0.$$

Indeed, assume that (**I**) and (**II**) hold. Then, since f_h' is bounded and $F_n^{(\nu)}$ has variance 1, we have that the sequence $\left\{f_h'(F_n)F_n^{(\nu)} : n \geq 1\right\}$ is uniformly integrable, so that $\mathbb{E}\left[f_h'(F_n)F_n^{(\nu)}\right] \to \mathbb{E}\left[f_h'(G(\nu))F_\infty^{(\nu)}\right]$ as $n \to \infty$, in view of (**I**). Combining the Stein equation (VI.1.5) and the integration by part formula (I.1.31), one therefore obtains that for every function $h : \mathbb{R} \to \mathbb{R}$,

$$\begin{aligned}\frac{\mathbb{E}[h(F_n)] - \mathbb{E}[h(G(\nu))]}{\phi_\nu(n)} &= \mathbb{E}\left[f_h'(F_n) \cdot \frac{2(F_n + \nu) - \langle DF_n, -DL^{-1}F_n\rangle_H}{\phi_\nu(n)}\right] \\ &= \mathbb{E}\left[f_h'(F_n)F_n^{(\nu)}\right] \to \mathbb{E}\left[f_h'(G(\nu))F_\infty^{(\nu)}\right] = 0\end{aligned}$$

by (**II**). We now prove that (**I**) and (**II**) hold in the cases of finite and infinite rank. If $N(f_n) = N < \infty$ for every $n \geq 1$, we have $F_\infty^{(\nu)} \stackrel{d}{=} \sum_{j=1}^N \ell_j H_2(N_j)$ with $\sum_{j=1}^\nu \ell_j = 0$, in view of Theorem VI.2.5. Thus, for every $s, t \in \mathbb{R}$, using dominated convergence together with the fact that $\lambda_{n,j} \to \mathbb{1}_{1 \leq j \leq \nu}$ and $\widetilde{\lambda}_{n,j} \to \ell_j$, for $j = 1, \dots, N$, shows that

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}\left[e^{i(tF_n + sF_n^{(\nu)})}\right] &= \lim_{n\to\infty} \mathbb{E}\left[\exp\left\{i\sum_{j=1}^N (t\lambda_{n,j} + s\widetilde{\lambda}_{n,j})H_2(N_j)\right\}\right] \\ = \mathbb{E}\left[\exp\left\{it\sum_{j=1}^\nu H_2(N_j) + is\sum_{j=1}^N \ell_j H_2(N_j)\right\}\right] &= \mathbb{E}\left[e^{i(tG(\nu) + sF_\infty^{(\nu)})}\right],\end{aligned}$$

which implies the desired conclusion. Let us now deal with the case of infinite rank and assume that $\vartheta(n) \asymp \omega(n)$, as $n \to \infty$. In view of Theorem VI.2.6, we have that $F_\infty^{(\nu)} \stackrel{d}{=} N_0 \sim \mathcal{N}(0,1)$, where N_0 is independent of $(N_j : j \geq 1)$. We can use [NR14, Theorem 4.7] (in the case $r = s = 1$), according to which it is sufficient to prove that $\mathbb{E}\left[F_n F_n^{(\nu)}\right] \to 0$ as $n \to \infty$. After straightforward simplifications based on the explicit forms of F_n and $F_n^{(\nu)}$, one has that

$$\mathbb{E}\left[F_n F_n^{(\nu)}\right] = \sqrt{2}\frac{\sum_{j=1}^\nu \lambda_{n,j}^2(1 - \lambda_{n,j}) + \sum_{j \geq \nu+1} \lambda_{n,j}^2(1 - \lambda_{n,j})}{\sqrt{\sum_{j\geq 1} \lambda_{n,j}^2(1 - \lambda_{n,j})^2}} =: \sqrt{2}[a(n) + b(n)].$$

We prove that both $a(n)$ and $b(n)$ converge to zero as $n \to \infty$. This can be achieved by the same techniques used in Theorem VI.2.6 for proving that $\kappa_4(F_n^{(\nu)}) \to 0$, as well as exploiting the asymptotic assumption $\vartheta(n) \asymp \omega(n)$. We omit the details. We thus conclude that (**I**) holds. Let us now prove the validity of (**II**). If the finite rank case, we have that $F_\infty^{(\nu)} \stackrel{d}{=} \sum_{j=1}^N \ell_j H_2(N_j)$ with $\sum_{j=1}^\nu \ell_j = 0$, so that

$$\begin{aligned}\mathbb{E}\left[f_h'(G(\nu))F_\infty^{(\nu)}\right] &= \sum_{j=1}^N \ell_j \mathbb{E}\left[f_h'\left(\sum_{i=1}^\nu H_2(N_i)\right)H_2(N_j)\right] = \sum_{j=1}^\nu \ell_j \mathbb{E}\left[f_h'\left(\sum_{i=1}^\nu H_2(N_i)\right)H_2(N_j)\right] \\ &= \mathbb{E}\left[f_h'\left(\sum_{i=1}^\nu H_2(N_i)\right)H_2(N_1)\right]\sum_{j=1}^\nu \ell_j = 0,\end{aligned}$$

where we used the fact that $N_1, \dots, N_\nu$ are i.i.d $\mathcal{N}(0,1)$ and $\sum_{j=1}^{\nu} \ell_j = 0$. This implies that $\mathbb{E}\left[f_h'(G(\nu))F_\infty^{(\nu)}\right] = 0$, since by Cauchy-Schwarz and (VI.1.7),

$$\mathbb{E}\left[f_h'\left(\sum_{i=1}^{\nu} H_2(N_i)\right) H_2(N_1)\right] \le \sqrt{\max(1, 2/\nu)}\,\sqrt{\mathbb{E}\left[H_2(N_1)^2\right]} < \infty.$$

In the case of infinite rank and the assumption $\vartheta(n) \asymp \omega(n)$, we have $F_\infty^{(\nu)} \stackrel{d}{=} N_0$ where $N_0 \sim \mathcal{N}(0,1)$ is independent of $(N_j : j \ge 1)$, so that $\mathbb{E}\left[f_h'(G(\nu))N_0\right] = \mathbb{E}\left[f_h'(G(\nu))\right]\mathbb{E}[N_0] = 0$, which suffices. □

VI.2.3 Suboptimal rates of convergence on the fourth Wiener chaos: an example

In this section, we provide a concrete example of a sequence in the fourth Wiener chaos (obtained by squaring elements of the second Wiener chaos) for which the sequence $\{\phi_\nu(n) : n \ge 1\}$ in (VI.2.1) leads to non-strongly optimal bounds (see also Conjecture VI.2.10).

Let H be a separable Hilbert space with orthonormal system $(e_i : i \ge 1)$ and consider a sequence of i.i.d standard Gaussian random variables $\{N_i = I_1(e_i) : i \ge 1\}$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define the random variable

$$G_n := \frac{1}{\sqrt{2n}} \sum_{i \in [n]} H_2(N_i) = I_2\left(\frac{1}{\sqrt{2n}} \sum_{i \in [n]} e_i \otimes e_i\right) =: I_2(g_n), \tag{VI.2.11}$$

where, as usual, $[n] := \{1, \dots, n\}$. By the classical Central Limit Theorem, we have that $G_n \xrightarrow{d} \mathcal{N}(0,1)$ as $n \to \infty$. Squaring G_n and using the identity $H_2(x)^2 = H_4(x) + 4H_2(x) + 2$, gives

$$\begin{aligned} F_n &:= G_n^2 - 1 = \frac{1}{2n} \sum_{i,j \in [n]} H_2(N_i)H_2(N_j) - 1 \\ &= \frac{1}{2n} \sum_{i \in [n]} H_4(N_i) + \frac{1}{2n} \sum_{i \neq j \in [n]} H_2(N_i)H_2(N_j) + \frac{4}{2n} \sum_{i \in [n]} H_2(N_i) \\ &=: \Pi_4(F_n) + \Pi_2(F_n) = I_4(g_n \widetilde{\otimes} g_n) + 4I_2(g_n \widetilde{\otimes}_1 g_n), \end{aligned}$$

where $\Pi_k(\bullet)$ stands for the projection on the k-th Wiener chaos. In view of the Fourth Moment Theorem I.1.30 applied to G_n, we have that $\|g_n \widetilde{\otimes}_1 g_n\|_{H^{\odot 2}}^2 \to 0$ as $n \to \infty$, implying that

$$\mathbb{E}\left[(F_n - \Pi_4(F_n))^2\right] = 32\|g_n \widetilde{\otimes}_1 g_n\|_{H^{\otimes 2}}^2 \to 0,$$

that is, F_n is dominated by its projection on the fourth Wiener chaos. In view of this, we set

$$\widetilde{F_n} := \frac{\sqrt{2}}{\sqrt{\mathbf{Var}[\Pi_4(F_n)]}} \Pi_4(F_n) = \frac{\sqrt{2}}{\sqrt{\mathbf{Var}[\Pi_4(F_n)]}} I_4(g_n \widetilde{\otimes} g_n), \tag{VI.2.12}$$

so that $\widetilde{F_n}$ satisfies $\mathbb{E}\left[\widetilde{F_n}\right] = 0$, $\mathbf{Var}\left[\widetilde{F_n}\right] = 2$, and $\widetilde{F_n} \xrightarrow{d} G(1)$, where $G(1)$ denotes a centred chi-square with one degree of freedom. Let $v_n := \mathbf{Var}[\Pi_4(F_n)] \to 2$. We compute the limiting distribution of the random variable

$$\widetilde{F_n}^{(1)} := \frac{2(\widetilde{F_n} + 1) - 4^{-1}\|D\widetilde{F_n}\|_H^2}{\phi_1(n)}, \tag{VI.2.13}$$

where $\phi_1(n) := \sqrt{\mathbf{Var}\left[2(\widetilde{F_n} + 1) - 4^{-1}\|D\widetilde{F_n}\|_H^2\right]}$.

The following theorem characterizes the asymptotic joint law of the vector $(\widetilde{F_n}, \widetilde{F_n}^{(1)})$, and in particular shows that the rate of convergence $\phi_1(n)$ is not strongly optimal.

Theorem VI.2.12. *Let $\widetilde{F_n}$ and $\widetilde{F_n}^{(1)}$ be as in* (VI.2.12) *and* (VI.2.13)*, respectively. As $n \to \infty$,*

$$\left(\widetilde{F_n}, \widetilde{F_n}^{(1)}\right) \xrightarrow{d} \left(Z^2 - 1, -\frac{2\sqrt{2}}{\sqrt{120}} Z^3\right),$$

where $Z \sim \mathcal{N}(0,1)$. In particular, for every $h \in \mathrm{Lip}(1)$, we have that, as $n \to \infty$,

$$\frac{\mathbb{E}\left[h(\widetilde{F_n})\right] - \mathbb{E}\left[h(G(1))\right]}{\phi_1(n)} \to 0.$$

VI.2.3.1 Proof of Theorem VI.2.12

The following lemma provides asymptotics for $\phi_1(n)$.

Lemma VI.2.13. *As $n \to \infty$, we have that $\phi_1(n) \sim c_1 n^{-1/2}$ for some absolute constant $c_1 > 0$.*

Proof. By definition of $\widetilde{F_n}$ in (VI.2.12) and (VI.2.8), it follows that $\phi_1(n)$ is a linear combination of the contraction norms $\|\frac{1}{18} f_n - f_n \widetilde{\otimes}_2 f_n\|_{H^{\otimes 4}}$, $\|f_n \widetilde{\otimes}_1 f_n\|_{H^{\otimes 6}}$ and $\|f_n \widetilde{\otimes}_3 f_n\|_{H^{\otimes 2}}$, where $f_n := g_n \widetilde{\otimes} g_n$ and g_n is as in (VI.2.11). A standard computation shows that

$$\begin{aligned} f_n = g_n \widetilde{\otimes} g_n = \frac{1}{2n} \sum_{i,j \in [n]} &[(e_i \otimes e_i \otimes e_j \otimes e_j) + (e_i \otimes e_j \otimes e_i \otimes e_j) + (e_i \otimes e_j \otimes e_j \otimes e_i) \\ &+ (e_j \otimes e_j \otimes e_i \otimes e_i) + (e_j \otimes e_i \otimes e_j \otimes e_i) + (e_j \otimes e_i \otimes e_i \otimes e_j)] \end{aligned}$$

We now compute $\|f_n \widetilde{\otimes}_1 f_n\|^2_{H^{\otimes 6}}$. By the above, it follows that $f_n \otimes_1 f_n$ is a finite sum of terms of the type

$$\frac{1}{4n^2} \sum_{i,j,l,k} (e_i \otimes e_i \otimes e_j \otimes e_j) \otimes_1 (e_l \otimes e_l \otimes e_k \otimes e_k) = \frac{1}{4n^2} \sum_{i,j,k} (e_i \otimes e_j \otimes e_j \otimes e_i \otimes e_k \otimes e_k).$$

Therefore, $\|f_n \widetilde{\otimes}_1 f_n\|^2_{H^{\otimes 6}} = \langle f_n \otimes_1 f_n, f_n \widetilde{\otimes}_1 f_n \rangle_{H^{\otimes 6}}$ is a finite sum of terms of the type

$$\begin{aligned} &\frac{1}{16n^4} \sum_{i,j,k} \sum_{r,s,t} \langle e_i \otimes e_j \otimes e_j \otimes e_i \otimes e_k \otimes e_k, e_r \otimes e_s \otimes e_s \otimes e_r \otimes e_t \otimes e_t \rangle_{H^{\otimes 6}} \\ &\qquad + \langle e_i \otimes e_j \otimes e_j \otimes e_i \otimes e_k \otimes e_k, e_r \otimes e_r \otimes e_s \otimes e_s \otimes e_t \otimes e_t \rangle_{H^{\otimes 6}} \\ &\qquad + \langle e_i \otimes e_j \otimes e_j \otimes e_i \otimes e_k \otimes e_k, e_t \otimes e_s \otimes e_r \otimes e_r \otimes e_s \otimes e_t \rangle_{H^{\otimes 6}} \\ = \;& c_1 n^{-1} + c_2 n^{-2} + c_3 n^{-3} \sim c_1 n^{-1}, \end{aligned}$$

where c_1, c_2, c_3 are absolute constants. This shows that $\|f_n \widetilde{\otimes}_1 f_n\|^2_{H^{\otimes 6}} = \|f_n \widetilde{\otimes}_3 f_n\|^2_{H^{\otimes 2}} \sim c_1 n^{-1}$. The proof showing that $\|\frac{1}{18} f_n - f_n \widetilde{\otimes}_2 f_n\|^2_{H^{\otimes 4}} = O(n^{-1})$ is obtained similarly by writing $\|\frac{1}{18} f_n - f_n \widetilde{\otimes}_2 f_n\|^2_{H^{\otimes 4}} = \|f_n \widetilde{\otimes}_2 f_n\|^2_{H^{\otimes 4}} - 2/18 \langle f_n, f_n \widetilde{\otimes}_2 f_n \rangle_{H^{\otimes 4}} + (1/18)^2 \|f_n\|^2_{H^{\otimes 4}}$. □

We are now in the position to prove Theorem VI.2.12.

Proof of Theorem VI.2.12. Exploiting the chain rule of the Malliavin derivative (I.1.23) as well as the identity $H_2(x)H_1(x) = H_3(x) + 2H_1(x)$, we compute

$$\begin{aligned} D\widetilde{F_n} &= \frac{\sqrt{2}}{\sqrt{v_n}} D\Pi_4(F_n) = \frac{\sqrt{2}}{\sqrt{v_n}} D \left\{ \frac{1}{2n} \sum_{i \in [n]} H_4(N_i) + \frac{1}{2n} \sum_{i \neq j \in [n]} H_2(N_i) H_2(N_j) \right\} \\ &= \frac{\sqrt{2}}{\sqrt{v_n}} \frac{4}{2n} \sum_{i \in [n]} H_3(N_i) e_i + \frac{\sqrt{2}}{\sqrt{v_n}} \frac{4}{2n} \sum_{i \neq j \in [n]} N_i e_i H_2(N_j) =: \frac{\sqrt{2}}{\sqrt{v_n}} (A(n) + B(n)), \end{aligned}$$

where

$$A(n) := \frac{4}{2n}\sum_{i\in[n]} H_3(N_i)e_i\,, \quad B(n) := \frac{4}{2n}\sum_{i\neq j\in[n]} N_i e_i H_2(N_j).$$

Therefore, we have that $4^{-1}\|D\widetilde{F_n}\|_H^2 = \frac{2}{4v_n}\langle A(n)+B(n), A(n)+B(n)\rangle_H$. Elementary computations based on expressing products of Hermite polynomials in the basis of Hermite polynomials, yield the following identities (after simplifications)

$$\begin{aligned}
\langle A(n), A(n)\rangle_H &= \frac{4}{n^2}\sum_{i\in[n]} H_6(N_i) + \frac{36}{n^2}\sum_{i\in[n]} H_4(N_i) + \frac{72}{n^2}\sum_{i\in[n]} H_2(N_i) + \frac{24}{n}\\
\langle A(n), B(n)\rangle_H &= \frac{4}{n^2}\sum_{i\neq j} H_2(N_j)H_4(N_i) + \frac{12}{n^2}\sum_{i\neq j} H_2(N_i)H_2(N_j)\\
\langle B(n), B(n)\rangle_H &= \frac{4}{n^2}\sum_{i\neq j} H_4(N_i)H_2(N_j) + \frac{16}{n^2}\sum_{i\neq j} H_2(N_i)H_2(N_j) + \frac{4}{n^2}\sum_{i\neq j\neq l} H_2(N_i)H_2(N_j)H_2(N_l)\\
&\quad +4\frac{n-1}{n^2}\sum_{i\in[n]} H_4(N_i) + \frac{24(n-1)}{n^2}\sum_{i\in[n]} H_2(N_i) + \frac{8n(n-1)}{n^2}\\
&\quad +4\frac{n-2}{n^2}\sum_{i\neq j} H_2(N_i)H_2(N_j).
\end{aligned}$$

In view of (VI.2.13) and the fact that, by Lemma VI.2.13, $\phi_1(n) \sim c_1 n^{-1/2}$, we have

$$\sqrt{n}[2(\widetilde{F_n}+1)] = \frac{\sqrt{2}}{\sqrt{v_n}}\frac{1}{\sqrt{n}}\sum_{i\in[n]} H_4(N_i) + \frac{\sqrt{2}}{\sqrt{v_n}}\frac{1}{\sqrt{n}}\sum_{i\neq j} H_2(N_i)H_2(N_j) + 2\sqrt{n}.$$

Using orthogonality relations of Hermite polynomials, it is easy to verify the following asymptotic relations as $n \to \infty$

$$\begin{aligned}
\sqrt{n}\langle A(n), A(n)\rangle_H &= \frac{4}{n\sqrt{n}}\sum_{i\in[n]}[H_6(N_i) + 9H_4(N_i) + 18H_2(N_i) + 6] \xrightarrow{L^2(\mathbb{P})} 0,\\
\sqrt{n}\langle A(n), B(n)\rangle_H &= \frac{4}{n\sqrt{n}}\sum_{i\neq j} H_2(N_i)H_4(N_j) + \frac{12}{n\sqrt{n}}\sum_{i\neq j} H_2(N_i)H_2(N_j) \xrightarrow{L^2(\mathbb{P})} 0,\\
\sqrt{n}\langle B(n), B(n)\rangle_H &= \frac{4}{n\sqrt{n}}\sum_{i\neq j\neq l} H_2(N_i)H_2(N_j)H_2(N_l) + 4\frac{n-1}{n\sqrt{n}}\sum_{i\in[n]} H_4(N_i)\\
&\quad +\frac{24(n-1)}{n\sqrt{n}}\sum_{i\in[n]} H_2(N_i) + \frac{8n(n-1)}{n\sqrt{n}} + 4\frac{n-2}{n\sqrt{n}}\sum_{i\neq j} H_2(N_i)H_2(N_j) + o_{L^2(\mathbb{P})}(1),
\end{aligned}$$

where $o_{L^2(\mathbb{P})}(1)$ denotes a sequence of random variables converging to zero in $L^2(\mathbb{P})$. Therefore, we have that

$$\begin{aligned}
H_n &:= \frac{2(\widetilde{F_n}+1) - 4^{-1}\|D\widetilde{F_n}\|_H^2}{1/\sqrt{n}}\\
&\overset{d}{=} \frac{\sqrt{2}}{\sqrt{v_n}}\frac{1}{\sqrt{n}}\sum_{i\in[n]} H_4(N_i) + \frac{\sqrt{2}}{\sqrt{v_n}}\frac{1}{\sqrt{n}}\sum_{i\neq j} H_2(N_i)H_2(N_j) + 2\sqrt{n}\\
&\quad -\frac{2}{v_n}\frac{1}{n\sqrt{n}}\sum_{i\neq j\neq l} H_2(N_i)H_2(N_j)H_2(N_l) - \frac{2}{v_n}\frac{n-1}{n\sqrt{n}}\sum_{i\in[n]} H_4(N_i)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{2}{v_n}\frac{6(n-1)}{n\sqrt{n}}\sum_{i\in[n]}H_2(N_i)-\frac{2}{v_n}\frac{2n(n-1)}{n\sqrt{n}}-\frac{2}{v_n}\frac{n-2}{n\sqrt{n}}\sum_{i\neq j}H_2(N_i)H_2(N_j)+o_{L^2(\mathbb{P})}(1)\\
=\;&\underbrace{\left[\frac{\sqrt{2}}{\sqrt{v_n}}\frac{1}{\sqrt{n}}-\frac{2}{v_n}\frac{n-1}{n\sqrt{n}}\right]}_{\alpha(n)}\sum_{i\in[n]}H_4(N_i)-\frac{2}{v_n}\frac{6(n-1)}{n\sqrt{n}}\sum_{i\in[n]}H_2(N_i)\\
&+\underbrace{\left[\frac{\sqrt{2}}{\sqrt{v_n}}\frac{1}{\sqrt{n}}-\frac{2}{v_n}\frac{n-2}{n\sqrt{n}}\right]}_{:=\beta(n)}\sum_{i\neq j}H_2(N_i)H_2(N_j)\\
&-\frac{2}{v_n}\frac{1}{n\sqrt{n}}\sum_{i\neq j\neq l}H_2(N_i)H_2(N_j)H_2(N_l)+\underbrace{\left[2\sqrt{n}-\frac{2}{v_n}\frac{2n(n-1)}{n\sqrt{n}}\right]}_{\gamma(n)}+o_{L^2(\mathbb{P})}(1).
\end{aligned}
$$

Now, using the fact that $v_n \to 2$, and the asymptotic relations $\alpha(n) \sim \frac{1}{n\sqrt{n}}$, $\beta(n) \sim \frac{2}{n\sqrt{n}}$, $\gamma(n) \to 0$ as $n \to \infty$, we deduce that

$$
H_n \overset{d}{=} -\frac{6(n-1)}{n\sqrt{n}}\sum_{i\in[n]}H_2(N_i)-\frac{1}{n\sqrt{n}}\sum_{i\neq j\neq l}H_2(N_i)H_2(N_j)H_2(N_l)+o_{L^2(\mathbb{P})}(1). \tag{VI.2.14}
$$

Straightforward simplifications allows us to write

$$
\frac{1}{n\sqrt{n}}\sum_{i\neq j\neq l}H_2(N_i)H_2(N_j)H_2(N_l)=\left(\frac{1}{\sqrt{n}}\sum_{i\in[n]}H_2(N_i)\right)^3-\frac{6(n-1)}{n\sqrt{n}}\sum_{j\in[n]}H_2(N_j)+o_{L^2(\mathbb{P})}(1),
$$

so that by (VI.2.14), we deduce

$$
H_n \overset{d}{=} -\left(\frac{1}{\sqrt{n}}\sum_{i\in[n]}H_2(N_i)\right)^3+o_{L^2(\mathbb{P})}(1) \overset{d}{=} -2\sqrt{2}\left(\frac{1}{\sqrt{2n}}\sum_{i\in[n]}H_2(N_i)\right)^3+o_{L^2(\mathbb{P})}(1) \xrightarrow{d} -2\sqrt{2}Z^3,
$$

where $Z \sim \mathcal{N}(0,1)$. The limiting variance is $\mathbf{Var}\left[-2\sqrt{2}Z^3\right] = 8\mathbb{E}\left[Z^6\right] = 120$. From this it follows that $\widetilde{F_n}^{(1)} \xrightarrow{d} -\frac{2\sqrt{2}}{\sqrt{120}}Z^3$, as $n \to \infty$. Furthermore, it is readily checked that the following joint convergence takes place

$$
\left(\widetilde{F_n},\widetilde{F_n}^{(1)}\right) \xrightarrow{d} \left(Z^2-1,-\frac{2\sqrt{2}}{\sqrt{120}}Z^3\right).
$$

Arguing similarly as in the previous section, we therefore conclude that for every $h \in \mathrm{Lip}(1)$,

$$
\frac{\mathbb{E}\left[h(F_n)\right]-\mathbb{E}\left[h(G(1))\right]}{\phi_1(n)} = \mathbb{E}\left[f_h'(\widetilde{F_n})\widetilde{F_n}^{(1)}\right] \to -\frac{2\sqrt{2}}{\sqrt{120}}\mathbb{E}\left[f_h'(Z^2-1)Z^3\right] = 0,
$$

as $n \to \infty$, where we used the fact that Z and $-Z$ have the same distribution. □

Bibliography

[ACP14] Ehsan Azmoodeh, Simon Campese, and Guillaume Poly. Fourth Moment Theorems for Markov diffusion generators. *J. Funct. Anal.*, 266(4):2341–2359, 2014.

[AEK18] Ehsan Azmoodeh, Peter Eichelsbacher, and Lukas Knichel. On the rate of convergence to a gamma distribution on wiener space. *arXiv preprint arXiv:1806.03878*, 2018.

[AEK20] Ehsan Azmoodeh, Peter Eichelsbacher, and Lukas Knichel. Optimal gamma approximation on Wiener space. *ALEA Lat. Am. J. Probab. Math. Stat.*, 17(1):101–132, 2020.

[Ale96] Kenneth S. Alexander. Boundedness of level lines for two-dimensional random fields. *Ann. Probab.*, 24(4):1653–1674, 1996.

[AMMP16] Ehsan Azmoodeh, Dominique Malicet, Guillaume Mijoule, and Guillaume Poly. Generalization of the Nualart-Peccati criterion. *Ann. Probab.*, 44(2):924–954, 2016.

[AP15] Ehsan Azmoodeh and Giovanni Peccati. Optimal berry-esseen bounds on the poisson space, 2015.

[APP18] Jürgen Angst, Viet-Hung Pham, and Guillaume Poly. Universality of the nodal length of bivariate random trigonometric polynomials. *Trans. Amer. Math. Soc.*, 370(12):8331–8357, 2018.

[AS92] Milton Abramowitz and Irene A. Stegun, editors. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. Dover Publications, Inc., New York, 1992. Reprint of the 1972 edition.

[AT07] Robert J. Adler and Jonathan E. Taylor. *Random fields and geometry*. Springer Monographs in Mathematics. Springer, New York, 2007.

[AW09] Jean-Marc Azaïs and Mario Wschebor. *Level sets and extrema of random processes and fields*. John Wiley & Sons, Inc., Hoboken, NJ, 2009.

[BBNP12] Hermine Biermé, Aline Bonami, Ivan Nourdin, and Giovanni Peccati. Optimal Berry-Esseen rates on the Wiener space: the barrier of third and fourth cumulants. *ALEA Lat. Am. J. Probab. Math. Stat.*, 9(2):473–500, 2012.

[BCW17] Dmitry Beliaev, Valentina Cammarota, and Igor Wigman. Two Point Function for Critical Points of a Random Plane Wave. *International Mathematics Research Notices*, 2019(9):2661–2689, 08 2017.

[Bel66] Ju. K. Beljaev. On the number of intersections of a level by a Gaussian stochastic process. *Teor. Verojatnost. i Primenen*, 11:120–128, 1966.

[Ber77] Michael V. Berry. Regular and irregular semiclassical wavefunctions. *J. Phys. A*, 10(12):2083–2091, 1977.

[Ber02] Michael V. Berry. Statistics of nodal lines and points in chaotic quantum billiards: perimeter corrections, fluctuations, curvature. *J. Phys. A*, 35(13):3025–3038, 2002.

[BG17] Vincent Beffara and Damien Gayet. Percolation of random nodal lines. *Publ. Math. Inst. Hautes Études Sci.*, 126:131–176, 2017.

[Bil99] Patrick Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition, 1999. A Wiley-Interscience Publication.

[BM18] Dmitry Beliaev and Stephen Muirhead. Discretisation schemes for level sets of planar Gaussian fields. *Comm. Math. Phys.*, 359(3):869–913, 2018.

[BM19] Jacques Benatar and Riccardo W. Maffucci. Random waves on $\mathbb{T}^3$: nodal area variance and lattice point correlations. *Int. Math. Res. Not. IMRN*, (10):3032–3075, 2019.

[BMW20] Jacques Benatar, Domenico Marinucci, and Igor Wigman. Planck-scale distribution of nodal length of arithmetic random waves. *J. Anal. Math.*, 141(2):707–749, 2020.

[Boc33] Salomon Bochner. Monotone Funktionen, Stieltjessche Integrale und harmonische Analyse. *Math. Ann.*, 108(1):378–410, 1933.

[BS17] Jeremiah Buckley and Mikhail Sodin. Fluctuations of the increment of the argument for the Gaussian entire function. *J. Stat. Phys.*, 168(2):300–330, 2017.

[Cam13] Simon Campese. Optimal convergence rates and one-term edge-worth expansions for multidimensional functionals of Gaussian fields. *ALEA Lat. Am. J. Probab. Math. Stat.*, 10(2):881–919, 2013.

[Cam19] Valentina Cammarota. Nodal area distribution for arithmetic random waves. *Trans. Amer. Math. Soc.*, 372(5):3539–3564, 2019.

[CGS11] Louis H. Y. Chen, Larry Goldstein, and Qi-Man Shao. *Normal approximation by Stein's method*. Probability and its Applications (New York). Springer, Heidelberg, 2011.

[CH20] Yaiza Canzani and Boris Hanin. Local universality for zeros and critical points of monochromatic random waves. *Comm. Math. Phys.*, 378(3):1677–1712, 2020.

[Che76] Shiu Yuen Cheng. Eigenfunctions and nodal sets. *Comment. Math. Helv.*, 51(1):43–55, 1976.

[Chi92] Yasuko Chikuse. Properties of Hermite and Laguerre polynomials in matrix argument and their applications. *Linear Algebra Appl.*, 176:237–260, 1992.

[Chi03] Yasuko Chikuse. *Statistics on special manifolds*, volume 174 of *Lecture Notes in Statistics*. Springer-Verlag, New York, 2003.

[CL65] Harald Cramér and M. R. Leadbetter. The moments of the number of crossings of a level by a stationary normal process. *Ann. Math. Statist.*, 36:1656–1663, 1965.

[CMR20] Valentina Cammarota, Domenico Marinucci, and Maurizia Rossi. Lipschitz-Killing curvatures for arithmetic random waves. *arXiv preprint arXiv:2010.14165*, 2020.

[CMW16a] Valentina Cammarota, Domenico Marinucci, and Igor Wigman. Fluctuations of the Euler-Poincaré characteristic for random spherical harmonics. *Proc. Amer. Math. Soc.*, 144(11):4759–4775, 2016.

[CMW16b] Valentina Cammarota, Domenico Marinucci, and Igor Wigman. On the distribution of the critical values of random spherical harmonics. *J. Geom. Anal.*, 26(4):3252–3324, 2016.

[CS19] Yaiza Canzani and Peter Sarnak. Topology and nesting of the zero set components of monochromatic random waves. *Comm. Pure Appl. Math.*, 72(2):343–374, 2019.

[DE01] Persi Diaconis and Steven N. Evans. Linear functionals of eigenvalues of random matrices. *Trans. Amer. Math. Soc.*, 353(7):2615–2633, 2001.

[DEL21] Federico Dalmao, Anne Estrade, and José León. On 3-dimensional Berry's model. *ALEA, Lat. Am. J. Probab. Math. Stat.*, 2021.

[Den01] Mark Richard Dennis. *Topological singularities in wave fields*. PhD thesis, University of Bristol, 2001.

[DF88] Harold Donnelly and Charles Fefferman. Nodal sets of eigenfunctions on Riemannian manifolds. *Invent. Math.*, 93(1):161–183, 1988.

[DNPR19] Federico Dalmao, Ivan Nourdin, Giovanni Peccati, and Maurizia Rossi. Phase singularities in complex arithmetic random waves. *Electron. J. Probab.*, 24:Paper No. 71, 45, 2019.

[DNPR20] Gauthier Dierickx, Ivan Nourdin, Giovanni Peccati, and Maurizia Rossi. Small scale CLTs for the nodal length of monochromatic waves. *arXiv preprint arXiv:2005.06577*, 2020.

[Dow72] Thomas D. Downs. Orientation statistics. *Biometrika*, 59:665–676, 1972.

[DP12] Guido De Philippis. Weak notions of Jacobian determinant and relaxation. *ESAIM Control Optim. Calc. Var.*, 18(1):181–207, 2012.

[DP18] Christian Döbler and Giovanni Peccati. The gamma Stein equation and noncentral de Jong theorems. *Bernoulli*, 24(4B):3384–3421, 2018.

[DSP90] William Duke and Rainer Schulze-Pillot. Representation of integers by positive ternary quadratic forms and equidistribution of lattice points on ellipsoids. *Invent. Math.*, 99(1):49–57, 1990.

[DT89] Herold Dehling and Murad S. Taqqu. The empirical process of some long-range dependent sequences with an application to U-statistics. *The Annals of Statistics*, pages 1767–1783, 1989.

[Dud02] Richard M. Dudley. *Real analysis and probability*, volume 74 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2002. Revised reprint of the 1989 original.

[Duk88] William Duke. Hyperbolic distribution problems and half-integral weight maass forms. *Inventiones mathematicae*, 92(1):73–90, 1988.

[DZ08] Youri Davydov and Ričardas Zitikis. On weak convergence of random fields. *Annals of the Institute of Statistical Mathematics*, 60(2):345–365, 2008.

[Edw94] C. H. Edwards, Jr. *Advanced calculus of several variables*. Dover Publications, Inc., New York, 1994. Corrected reprint of the 1973 original.

[EF16] Anne Estrade and Julie Fournier. Number of critical points of a Gaussian random field: condition for a finite variance. *Statist. Probab. Lett.*, 118:94–99, 2016.

[EH99] Paul Erdős and Richard R. Hall. On the angular distribution of Gaussian integers with fixed norm. volume 200, pages 87–94. 1999. Paul Erdős memorial collection.

[EL16] Anne Estrade and José R. León. A central limit theorem for the Euler characteristic of a Gaussian excursion set. *Ann. Probab.*, 44(6):3849–3878, 2016.

[Fed59] Herbert Federer. Curvature measures. *Trans. Amer. Math. Soc.*, 93:418–491, 1959.

[FFM04] Irene Fonseca, Nicola Fusco, and Paolo Marcellini. On the total variation of the Jacobian. *J. Funct. Anal.*, 207(1):1–32, 2004.

[GK12] Martin J. Gander and Felix Kwok. Chladni figures and the Tacoma bridge: motivating PDE eigenvalue problems via vibrating plates. *SIAM Rev.*, 54(3):573–596, 2012.

[GN00] A. K. Gupta and D. K. Nagar. *Matrix variate distributions*, volume 104 of *Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics*. Chapman & Hall/CRC, Boca Raton, FL, 2000.

[GR14] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products*. Academic press, 2014.

[Gro85] Emil Grosswald. Representations as sums of three squares. In *Representations of Integers as Sums of Squares*, pages 38–65. Springer, 1985.

[GW11] Damien Gayet and Jean-Yves Welschinger. Exponential rarefaction of real curves with many components. *Publ. Math. Inst. Hautes Études Sci.*, (113):69–96, 2011.

[GW17] Andrew Granville and Igor Wigman. Planck-scale mass equidistribution of toral Laplace eigenfunctions. *Comm. Math. Phys.*, 355(2):767–802, 2017.

[Hay69] Takesi Hayakawa. On the distribution of the latent roots of a positive definite random symmetric matrix. I. *Ann. Inst. Statist. Math.*, 21:1–21, 1969.

[HJ20] Jeffrey Humpherys and Tyler J. Jarvis. *Foundations of applied mathematics. Vol. 2*. Society for Industrial and Applied Mathematics, Philadelphia, PA, [2020] ©2020. Algorithms, approximation, optimization.

[HNY08] C. P. Hughes, A. Nikeghbali, and M. Yor. An arithmetic model for the total disorder process. *Probab. Theory Related Fields*, 141(1-2):47–59, 2008.

[Ito] Kiyoshi Itô. The expected number of zeros of continuous stationary Gaussian processes. *J. Math. Kyoto Univ.*, 3:207–216, 1963/64.

[Jam61] Alan T. James. Zonal polynomials of the real positive definite symmetric matrices. *Ann. of Math. (2)*, 74:456–469, 1961.

[Kho91] A. G. Khovanskiĭ. *Fewnomials*, volume 88 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1991. Translated from the Russian by Smilka Zdravkovska.

[KKW13] Manjunath Krishnapur, Pär Kurlberg, and Igor Wigman. Nodal length fluctuations for arithmetic random waves. *Ann. of Math. (2)*, 177(2):699–737, 2013.

[KL01] Marie F. Kratz and José R. León. Central limit theorems for level functionals of stationary Gaussian processes and fields. *J. Theoret. Probab.*, 14(3):639–672, 2001.

[Koc96] E. Kochneff. Rotational symmetry of the Hermite projection operators. *Proc. Amer. Math. Soc.*, 124(5):1539–1547, 1996.

[Kra14] Ilia Krasikov. Approximations for the Bessel and Airy functions with an explicit error term. *LMS J. Comput. Math.*, 17(1):209–225, 2014.

[KW17] Pär Kurlberg and Igor Wigman. On probability measures arising from lattice points on circles. *Math. Ann.*, 367(3-4):1057–1098, 2017.

[KW18] Pär Kurlberg and Igor Wigman. Variation of the Nazarov-Sodin constant for random plane waves and arithmetic random waves. *Adv. Math.*, 330:516–552, 2018.

[Leb83] J. L. Lebowitz. Charge fluctuations in coulomb systems. *Physical Review A*, 27(3):1491, 1983.

[Led12] Michel Ledoux. Chaos of a Markov operator and the fourth moment condition. *Ann. Probab.*, 40(6):2439–2459, 2012.

[Lee97] John M. Lee. *Riemannian manifolds*, volume 176 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1997. An introduction to curvature.

[Lin68] Yu. V. Linnik. *Ergodic properties of algebraic fields*. Translated from the Russian by M. S. Keane. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 45. Springer-Verlag New York Inc., New York, 1968.

[Log18] Alexander Logunov. Nodal sets of Laplace eigenfunctions: proof of Nadirashvili's conjecture and of the lower bound in Yau's conjecture. *Ann. of Math. (2)*, 187(1):241–262, 2018.

[Luk94] Ho Ming Luk. *Stein's method for the Gamma distribution and related statistical applications*. ProQuest LLC, Ann Arbor, MI, 1994. Thesis (Ph.D.)–University of Southern California.

[Maf17] Riccardo W. Maffucci. Nodal intersections of random eigenfunctions against a segment on the 2-dimensional torus. *Monatsh. Math.*, 183(2):311–328, 2017.

[Maf20] Riccardo W. Maffucci. Restriction of 3D arithmetic Laplace eigenfunctions to a plane. *Electron. J. Probab.*, 25:Paper No. 60, 17, 2020.

[Mal78] Paul Malliavin. Stochastic calculus of variation and hypoelliptic operators. In *Proceedings of the International Symposium on Stochastic Differential Equations (Res. Inst. Math. Sci., Kyoto Univ., Kyoto, 1976)*, pages 195–263. Wiley, New York-Chichester-Brisbane, 1978.

[MP11] Domenico Marinucci and Giovanni Peccati. *Random fields on the sphere*, volume 389 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2011. Representation, limit theorems and cosmological applications.

[MPH95] A. M. Mathai, Serge B. Provost, and Takesi Hayakawa. *Bilinear forms and zonal polynomials*, volume 102 of *Lecture Notes in Statistics*. Springer-Verlag, New York, 1995.

[MPRW16] Domenico Marinucci, Giovanni Peccati, Maurizia Rossi, and Igor Wigman. Non-universality of nodal length distribution for arithmetic random waves. *Geom. Funct. Anal.*, 26(3):926–960, 2016.

[MR15] Domenico Marinucci and Maurizia Rossi. Stein-Malliavin approximations for nonlinear functionals of random eigenfunctions on $\mathbb{S}^d$. *J. Funct. Anal.*, 268(8):2379–2420, 2015.

[MR21] Domenico Marinucci and Maurizia Rossi. On the correlation between nodal and nonzero level sets for random spherical harmonics. In *Annales Henri Poincaré*, volume 22, pages 275–307. Springer, 2021.

[MRW20] Domenico Marinucci, Maurizia Rossi, and Igor Wigman. The asymptotic equivalence of the sample trispectrum and the nodal length for random spherical harmonics. *Ann. Inst. Henri Poincaré Probab. Stat.*, 56(1):374–390, 2020.

[Mui82] Robb J. Muirhead. *Aspects of multivariate statistical theory*. John Wiley & Sons, Inc., New York, 1982. Wiley Series in Probability and Mathematical Statistics.

[MW11] Domenico Marinucci and Igor Wigman. On the area of excursion sets of spherical gaussian eigenfunctions. *Journal of Mathematical Physics*, 52(9):093301, 2011.

[MW14] Domenico Marinucci and Igor Wigman. On nonlinear functionals of random spherical eigenfunctions. *Comm. Math. Phys.*, 327(3):849–872, 2014.

[Neu71] Georg Neuhaus. On weak convergence of stochastic processes with multidimensional time parameter. *Ann. Math. Statist.*, 42:1285–1295, 1971.

[NOL08] David Nualart and Salvador Ortiz-Latorre. Central limit theorems for multiple stochastic integrals and Malliavin calculus. *Stochastic Process. Appl.*, 118(4):614–628, 2008.

[Not21] Massimo Notarnicola. Fluctuations of nodal sets on the 3-torus and general cancellation phenomena. *ALEA, Lat. Am. J. Probab. Math. Stat.*, 18:1127–1194, 2021.

[NP05] David Nualart and Giovanni Peccati. Central limit theorems for sequences of multiple stochastic integrals. *Ann. Probab.*, 33(1):177–193, 2005.

[NP09a] Ivan Nourdin and Giovanni Peccati. Noncentral convergence of multiple integrals. *Ann. Probab.*, 37(4):1412–1426, 2009.

[NP09b] Ivan Nourdin and Giovanni Peccati. Stein's method and exact Berry-Esseen asymptotics for functionals of Gaussian fields. *Ann. Probab.*, 37(6):2231–2261, 2009.

[NP09c] Ivan Nourdin and Giovanni Peccati. Stein's method on Wiener chaos. *Probab. Theory Related Fields*, 145(1-2):75–118, 2009.

[NP12a] Ivan Nourdin and Giovanni Peccati. *Normal approximations with Malliavin calculus*, volume 192 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2012. From Stein's method to universality.

[NP12b] Ivan Nourdin and Guillaume Poly. Convergence in law in the second Wiener/Wigner chaos. *Electron. Commun. Probab.*, 17:no. 36, 12, 2012.

[NP15] Ivan Nourdin and Giovanni Peccati. The optimal fourth moment theorem. *Proc. Amer. Math. Soc.*, 143(7):3123–3133, 2015.

[NPR19] Ivan Nourdin, Giovanni Peccati, and Maurizia Rossi. Nodal statistics of planar random waves. *Comm. Math. Phys.*, 369(1):99–151, 2019.

[NR14] Ivan Nourdin and Jan Rosiński. Asymptotic independence of multiple Wiener-Itô integrals and the resulting limit laws. *Ann. Probab.*, 42(2):497–526, 2014.

[NS09] Fedor Nazarov and Mikhail Sodin. On the number of nodal domains of random spherical harmonics. *Amer. J. Math.*, 131(5):1337–1357, 2009.

[NS16] Fedor Nazarov and Mikhail Sodin. Asymptotic laws for the spatial distribution and the number of connected components of zero sets of Gaussian random functions. *Zh. Mat. Fiz. Anal. Geom.*, 12(3):205–278, 2016.

[Nua95] David Nualart. *The Malliavin calculus and related topics*. Probability and its Applications (New York). Springer-Verlag, New York, 1995.

[ORW08] Ferenc Oravecz, Zeév Rudnick, and Igor Wigman. The Leray measure of nodal sets for random eigenfunctions on the torus. *Ann. Inst. Fourier (Grenoble)*, 58(1):299–335, 2008.

[PP73] S. R. Paranjape and C. Park. Distribution of the supremum of the two-parameter Yeh-Wiener process on the boundary. *J. Appl. Probability*, 10:875–880, 1973.

[PR18] Giovanni Peccati and Maurizia Rossi. Quantitative limit theorems for local functionals of arithmetic random waves. In *Computation and combinatorics in dynamics, stochastics and control*, volume 13 of *Abel Symp.*, pages 659–689. Springer, Cham, 2018.

[Pri20] Lakshmi Priya. Overcrowding estimates for zero count and nodal length of stationary gaussian processes. *arXiv preprint arXiv:2012.10857*, 2020.

[PT05] Giovanni Peccati and Ciprian A. Tudor. Gaussian limits for vector-valued multiple stochastic integrals. In *Séminaire de Probabilités XXXVIII*, volume 1857 of *Lecture Notes in Math.*, pages 247–262. Springer, Berlin, 2005.

[PT11] Giovanni Peccati and Murad S. Taqqu. *Wiener chaos: moments, cumulants and diagrams*, volume 1 of *Bocconi & Springer Series*. Springer, Milan; Bocconi University Press, Milan, 2011. A survey with computer implementation, Supplementary material available online.

[PV20] Giovanni Peccati and Anna Vidotto. Gaussian random measures generated by Berry's nodal sets. *J. Stat. Phys.*, 178(4):996–1027, 2020.

[Ros11] Nathan Ross. Fundamentals of Stein's method. *Probab. Surv.*, 8:210–293, 2011.

[Ros16] Maurizia Rossi. The geometry of spherical random fields. *arXiv preprint arXiv:1603.07575*, 2016.

[Roz17] Yoni Rozenshein. The number of nodal components of arithmetic random waves. *Int. Math. Res. Not. IMRN*, (22):6990–7027, 2017.

[Rud87] Walter Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.

[RW08] Zeév Rudnick and Igor Wigman. On the volume of nodal sets for eigenfunctions of the Laplacian on the torus. *Ann. Henri Poincaré*, 9(1):109–130, 2008.

[RW16] Zeév Rudnick and Igor Wigman. Nodal intersections for random eigenfunctions on the torus. *Amer. J. Math.*, 138(6):1605–1644, 2016.

[RW18] Maurizia Rossi and Igor Wigman. Asymptotic distribution of nodal intersections for arithmetic random waves. *Nonlinearity*, 31(10):4472–4516, 2018.

[RWH17] Michal Różański, Roman Witula, and Edyta Hetmaniok. More subtle versions of the Hadamard inequality. *Linear Algebra Appl.*, 532:500–511, 2017.

[RWY16] Zeév Rudnick, Igor Wigman, and Nadav Yesha. Nodal intersections for random waves on the 3-dimensional torus. *Ann. Inst. Fourier (Grenoble)*, 66(6):2455–2484, 2016.

[RY99] Daniel Revuz and Marc Yor. *Continuous Martingales and Brownian Motion*. Springer-Verlag, 1999.

[Sar42] Arthur Sard. The measure of the critical values of differentiable maps. *Bull. Amer. Math. Soc.*, 48:883–890, 1942.

[Sel46] Atle Selberg. Contributions to the theory of the Riemann zeta-function. *Arch. Math. Naturvid.*, 48(5):89–155, 1946.

[Sel92] Atle Selberg. Old and new conjectures and results about a class of Dirichlet series. In *Proceedings of the Amalfi Conference on Analytic Number Theory (Maiori, 1989)*, pages 367–385. Univ. Salerno, Salerno, 1992.

[Sha14] Anant R. Shastri. *Basic algebraic topology*. CRC Press, Boca Raton, FL, 2014. With a foreword by Peter Wong.

[Ste72] Charles Stein. A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. II: Probability theory*, pages 583–602, 1972.

[Ste86] Charles Stein. *Approximate computation of expectations*, volume 7 of *Institute of Mathematical Statistics Lecture Notes—Monograph Series*. Institute of Mathematical Statistics, Hayward, CA, 1986.

[SW08] Rolf Schneider and Wolfgang Weil. *Stochastic and integral geometry*. Probability and its Applications (New York). Springer-Verlag, Berlin, 2008.

[SW19] Peter Sarnak and Igor Wigman. Topologies of nodal sets of random band-limited functions. *Comm. Pure Appl. Math.*, 72(2):275–342, 2019.

[Sze39] Gábor Szego. *Orthogonal polynomials*, volume 23. American Mathematical Soc., 1939.

[Tha93] Sundaram Thangavelu. Hermite expansions on $\mathbf{R}^n$ for radial functions. *Proc. Amer. Math. Soc.*, 118(4):1097–1102, 1993.

[Tod19] Anna Paola Todino. A quantitative central limit theorem for the excursion area of random spherical harmonics over subdomains of $\mathbb{S}^2$. *J. Math. Phys.*, 60(2):023505, 33, 2019.

[Tod20] Anna Paola Todino. Nodal lengths in shrinking domains for random eigenfunctions on $\mathbb{S}^2$. *Bernoulli*, 26(4):3081–3110, 2020.

[Vid21] Anna Vidotto. A note on the reduction principle for the nodal length of planar random waves. *Statist. Probab. Lett.*, 174:109090, 2021.

[Vit91] Richard Vitale. Expected absolute random determinants and zonoids. *The Annals of Applied Probability*, 1(2):293–300, 1991.

[Wic69] Michael J. Wichura. Inequalities with applications to the weak convergence of random processes with multi-dimensional time parameters. *Ann. Math. Statist.*, 40:681–687, 1969.

[Wie02] Kelly Wieand. Eigenvalue distributions of random unitary matrices. *Probab. Theory Related Fields*, 123(2):202–224, 2002.

[Wig10] Igor Wigman. Fluctuations of the nodal length of random spherical harmonics. *Comm. Math. Phys.*, 298(3):787–831, 2010.

[Wig12] Igor Wigman. On the nodal lines of random and deterministic Laplace eigenfunctions. In *Spectral geometry*, volume 84 of *Proc. Sympos. Pure Math.*, pages 285–297. Amer. Math. Soc., Providence, RI, 2012.

[Wsc82] Mario Wschebor. Formule de Rice en dimension d. *Z. Wahrsch. Verw. Gebiete*, 60(3):393–401, 1982.

[Wsc83] Mario Wschebor. On crossings of Gaussian fields. *Stochastic Process. Appl.*, 14(2):147–155, 1983.

[WY19] Igor Wigman and Nadav Yesha. Central limit theorem for Planck-scale mass distribution of toral Laplace eigenfunctions. *Mathematika*, 65(3):643–676, 2019.

[Yau82] Shing Tung Yau. Survey on partial differential equations in differential geometry. In *Seminar on Differential Geometry*, volume 102 of *Ann. of Math. Stud.*, pages 3–71. Princeton Univ. Press, Princeton, N.J., 1982.

[Yau93] Shing-Tung Yau. Open problems in geometry. In *Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990)*, volume 54 of *Proc. Sympos. Pure Math.*, pages 1–28. Amer. Math. Soc., Providence, RI, 1993.

[Zel09] Steve Zelditch. Real and complex zeros of Riemannian random waves. In *Spectral analysis in geometry and number theory*, volume 484 of *Contemp. Math.*, pages 321–342. Amer. Math. Soc., Providence, RI, 2009.

[ZK12] Dmitry. N. Zaporozhets and Zakhar Kabluchko. Random determinants, mixed volumes of ellipsoids, and zeros of Gaussian random fields. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 408(Veroyatnostcprime i Statistika. 18):187–196, 327, 2012.