ON DEFORMATION QUANTIZATION OF QUADRATIC POISSON STRUCTURES

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ABSTRACT. We study the deformation complex of the dg wheeled properad of $\mathbb{Z}$-graded quadratic Poisson structures and prove that it is quasi-isomorphic to the even M. Kontsevich graph complex. As a first application we show that the Grothendieck-Teichmüller group acts on the genus completion of that wheeled properad faithfully and essentially transitively. As a second application we classify all the universal quantizations of $\mathbb{Z}$-graded quadratic Poisson structures (together with the underlying *homogeneous* formality maps).

CONTENTS

1. Introduction

1.1. Deformation quantization. Since the fundamental paper of Maxim Kontsevich [K2] there was a huge progress in our understanding of universal deformation quantizations of *generic* Poisson structures which lead, in particular, to the complete classification such quantizations in terms of Drinfeld associators and, moreover, to the computation of the full cohomology of the deformation complex of any formality map from [K2] in terms of the cohomology of the Kontsevich graph complex GC_2 [K1, W1, Do, AM].

All the above mentioned results become, however, void when applied to the special class of *linear* Poisson structures on a graded vector space V — the deformation quantization of such structures is unique up to homotopy (and is given by the universal enveloping algebra), the action of the Grothendieck-Teichmüller group on linear Poisson structures is hence trivial etc.

The next level of complexity comes with *quadratic* Poisson structures on V. What happens with the above mentioned beautiful general statements when applied to a *generic quadratic* Poisson structure? Do we really need associators to quantize them? How rich is the family of homotopy inequivalent universal quantizations of quadratic Poisson structures?

In this paper we provide answers to both these questions for $\mathbb{Z}$-*graded quadratic Poisson structures* on an arbitrary finite-dimensional $\mathbb{Z}$-graded vector space V; if V is concentrated in degree zero, say $V = \mathbb{R}^N$, this notion reduces precisely to the ordinary notion of quadratic Poisson structure on $\mathbb{R}^N$. The main motivation to study $\mathbb{Z}$-*graded* quadratic Poisson structures comes from a class of so called *homogeneous* Kontsevich formality maps, the ones which respect polynomial degrees of polynomial functions on the underlying $\mathbb{Z}$-graded vector space, and which are discussed in more detail below.

It is worth pointing out right from the start an important conceptual difference between deformation quantizations of generic $\mathbb{Z}$-graded Poisson structures and of generic $\mathbb{Z}$-graded *quadratic* ones. In the former case the deformation quantization produced a *curved* $\mathcal{A}ss_\infty$ algebra structure on the space of formal smooth

functions, the curvature term being unavoidable in general; moreover it is that *curvature terms* which control, rather surprisingly, the homotopy theory of all such universal deformation quantizations and which are more or less directly connected to the Kontsevich graph complex GC_2 [Do, AM]. By contrast, a deformation quantizations of generic $\mathbb{Z}$-graded *quadratic* Poisson structure *always* produces a *flat* (or ordinary) $\mathcal{Ass}_\infty$ algebra structure, with that important for homotopy classifications *the curvature term being equal to zero*. Hence it is not obvious *a priori* that both quantization theories are controlled essentially by one and the same Kontsevich graph complex GC_2. Which is the main claim of this paper.

1.2. **Homogeneous formality maps versus quadratic Poisson structures.** Let V be a finite-dimensional $\mathbb{Z}$-graded vector space, the graded symmetric tensor algebra

$$\mathcal{O}_V = \bigoplus_{k \geqslant 0} \mathcal{O}_V^k, \quad \mathcal{O}_V^k := \odot^k V,$$

can be understood as the algebra of polynomial functions on the linear space V^*; it is well-known that the deformation theory of $\mathcal{O}_V$ as an associative algebra is controlled by the standard Hochschild dg Lie algebra denoted by $\mathsf{Hoch}(\mathcal{O}_V, \mathcal{O}_V) = \oplus_{n \geqslant 0} \mathrm{Hom}(\otimes^n \mathcal{O}_V, \mathcal{O}_V)[1-n]$. A *M. Kontsevich formality map* is a $\mathcal{Lie}_\infty$ quasi-isomorphism of dg lie algebras,

$$(1) \qquad\qquad \mathcal{F} : \mathsf{Poly}(V) \longrightarrow \mathsf{Hoch}(\mathcal{O}_V, \mathcal{O}_V),$$

where $\mathsf{Poly}(V) = \oplus_{k,l \geqslant 0} \mathrm{Hom}(\wedge^l V, \odot^k V)[1-l]$ is the Lie algebra of polyvector fields on the affine space V^* [K2]. If assume that the underlying space V^* is a linear (rather than affine) space, then both sides of the above map come equipped with an extra grading which takes into account the polynomial degrees of formal functions from $\mathcal{O}_V$. We call a quasi-isomorphism above *homogeneous* if it respects that the polynomial degrees. More precisely, we consider the Lie subalgebra of $\mathsf{Poly}(\mathcal{O}_V, \mathcal{O}_V)$

$$\mathsf{Poly}_{(0)}(\mathcal{O}_V, \mathcal{O}_V) = \bigoplus_{k \geqslant 1} \mathrm{Hom}(\wedge^k V, \odot^k V)[1-l],$$

and the Lie subalgebra of $\mathsf{Hoch}(\mathcal{O}_V, \mathcal{O}_V)$,

$$\mathsf{Hoch}_{(0)}(\mathcal{O}_V, \mathcal{O}_V) = \bigoplus_{\substack{n \geqslant 1 \\ k_1, \ldots, k_n \geqslant 1}} \mathrm{Hom}(\mathcal{O}_V^{k_1} \otimes \ldots \otimes \mathcal{O}_V^{k_n}, \mathcal{O}_V^{k_1 + \ldots + k_n})[1-n]$$

which are spanned by polyvector fields and, respectively, polydifferential operators which preserve total polynomial degrees of functions. The Hochschild differential preserves the degree grading and, therefore, the cohomology of the homogeneous Hochschild complex $\mathsf{Hoch}_{(0)}(\mathcal{O}_V, \mathcal{O}_V)$ is equal precisely to $\mathsf{Poly}_{(0)}(\mathcal{O}_V, \mathcal{O}_V)$. Any $\mathcal{Lie}_\infty$ quasi-isomorphism

$$(2) \qquad\qquad \mathcal{F}_{(0)} : \mathsf{Poly}_{(0)}(V) \longrightarrow \mathsf{Hoch}_{(0)}(\mathcal{O}_V, \mathcal{O}_V)$$

is called a *homogeneous* formality map. It is not hard to check that the particular formality map (1) constructed by M. Kontsevich in [K2] does have this property when restricted to the subspace $\mathsf{Poly}_{(0)}(V) \subset \mathsf{Poly}(V)$. So the set of such formality maps is non-empty. We prove in this paper that, up to homotopy equivalence, this set is as large as the set of "full" formality map (1) — it can be identified with the set of Drinfeld's associators.

The Maurer-Cartan elements of the Lie algebra $\mathsf{Poly}(V)$ are called $\mathbb{Z}$-*graded Poisson structures*. They have a decomposition,

$$\pi = \sum_{n,m=0}^{\infty} \pi_n^m, \qquad \pi_n^m \in \mathrm{Hom}(\wedge^n V, \odot^m V)[2-n]$$

and can be identified with representation of a certain dg wheeled properad $\mathcal{H}olieb_{0,1}^{\star\circlearrowleft}$ which has been studied in [AM] where it was proven that its deformation complex is quasi-isomorphic to the Kontsevich graph complex (implying in particular, that the group of homotopy automorphisms of $\mathcal{H}olieb_{0,1}^{\star\circlearrowleft}$ can be identified with the famous and mysterious Grothendieck-teichmüller group GRT).

Similarly, Maurer-Cartan elements $\pi^0 \in \mathsf{Poly}_{(0)}(V)$ are called $\mathbb{Z}$-*graded quadratic Poisson structures* on V^*; they admit a decomposition

$$\nu = \sum_{n \geqslant 1} \nu_n, \qquad \nu^n \in \mathrm{Hom}(\wedge^n V, \odot^n V)[2-n]$$

If the vector space V is concentrated in degree zero, i.e. $V \simeq \mathbb{K}^N$, then only the quadratic term ν^2 : $\mathrm{Hom}(\wedge^2 V, \odot^2 V)$ can be non-zero, i.e. in that case we recover the standard notion of quadratic Poisson structure. Thus theory of homogeneous formality maps is the same as the theory of deformation quantizations of $\mathbb{Z}$-graded quadratic Poisson structures. We will identify each $\mathbb{Z}$-graded quadratic Poisson structure π^0 with a representation in V of a certain dg wheeled properad $\mathcal{H}oqpois_{0,1}^{\circlearrowleft}$ which is introduced and studied in this paper. To understand the homotopy of generic *finite-dimensional* quadratic Poisson structure one has to compute the cohomology of the derivation complex $\mathrm{Der}(\mathcal{H}oqpois_{0,1}^{\circlearrowleft})$ of the genus completion of that wheeled prop (see §2 for its definition and §3 for the computation). The homotopy theory of possibly *infinite-dimensional* quadratic Poisson structures is controlled by a different complex — the derivation complex $\mathrm{Der}(\mathcal{H}oqpois_{0,1})$ of the genus completion of the ordinary (unwheeled) prop $\mathcal{H}oqpois_{0,1} \subsetneqq \mathcal{H}oqpois_{0,1}^{\circlearrowleft}$ of $\mathbb{Z}$-graded quadratic Poisson structures.

The dg prop $\mathcal{H}oqpois_{0,1}$ and its wheeled closure $\mathcal{H}oqpois_{0,1}^{\circlearrowleft}$ come naturally in a *family* of props $\mathcal{H}oqpois_{c,d} \subsetneqq \mathcal{H}oqpois_{c,d}^{\circlearrowleft}$ parameterized by two integers $c, d \in \mathbb{Z}$, the props $\mathcal{H}oqpois_{c,d}$ being certain quotients (see §2 for details) of the family of properads $\mathcal{H}olieb_{c,d}$ whose homotopy theory has been studied in [MW1]; the case $c = 0$, $d = 1$ corresponds to quadratic Poisson structures while the case $c = d = 1$ corresponds to what one might call *quadratic homotopy Lie bialgebras*. At present we are not aware of any applications of the latter case so in applications we are most interested in the case $c = 1, d = 0$. However we solve the problem of computing the cohomology of derivation complexes for arbitrary values of the parameters c and d.

Let FGC_d the Kontsevich graph complex GC_d introduced in [K1] (see also [W1] for a detailed description) spanned by *not* necessary connected graphs with all vertices at least bivalent. This is one of the most important graph complexes in mathematics admitting applications in algebra, geometry, topology and the theory of moduli spaces of algebraic curves (see e.g. ([Me6] for a review). It was proven in [W1] that

$$H^0(\mathsf{FGC}^{\geqslant 2}) = \mathfrak{grt},$$

where $\mathfrak{grt}$ is the Lie algebra of the famous Grothendieck-Teichmüller group GRT introduced by V.Drinfeld in [Dr]. The main result of this paper states the following.

1.2.1. Theorem. (i) *There is a canonical morphism of dg Lie algebras,*

$$(3) \qquad\qquad F^{\circlearrowleft} : \mathsf{FGC}_{c+d+1}^{\geqslant 2} \longrightarrow \mathrm{Der}(\mathcal{H}oqpois_{c,d}^{\circlearrowleft})$$

which is a quasi-isomorphisms. In particular, there is an isomorphism of Lie algebras

$$H^0(\mathrm{Der}(\mathcal{H}oqpois_{0,1}^{\circlearrowleft})) = \mathfrak{grt}$$

that is, the Grothendieck-Teichmüller group GRT acts up to homotopy faithfully (and essentially transitively) on the completion of the wheeled properad $\mathcal{H}oqpois^{\circlearrowleft}$ governing finite-dimensional $\mathbb{Z}$-graded quadratic Poisson *structures. Moreover $H^i(\mathcal{H}oqpois^{\circlearrowleft}) = 0$ for $i \leqslant -1$.*
(ii) *There is an isomorphism of cohomology groups,*

$$H^\bullet(\mathsf{FGC}_{c+d}^{\geqslant 2}) = H^\bullet(\mathrm{Der}(\mathcal{H}oqpois_{c,d}))$$

implying in particular the isomorphism

$$H^0(\mathrm{Der}(\mathcal{H}oqpois_{0,1})) = \mathbb{K}$$

which implies, that (in sharp contrast to the situation (i)), the prop $\mathcal{H}oqpois_{0,1}$ of possibly infinite-dimensional $\mathbb{Z}$-*graded Poisson structures has no homotopy-non-trivial automorphisms other than the standard rescaling of the generators.*

The proof of this theorem is quite different from proofs of analogous theorems in [MW3] and [AM] for the props of Lie bialgebras. It goes through a new *directed weighted* version, dwGC_d of the Kontsevich complex

GC_d (the connected version of FGC_d considered above) which is spanned by directed connected graphs whose vertices v are assigned weights $\mathfrak{w}_v \in \mathbb{N}$ satisfying certain conditions. There is a natural morphism of complex

$$F : \mathsf{GC}_d^{\geqslant 2} \longrightarrow \mathsf{dwGC}_d$$

which sends a graph from GC_d into a sum of graphs of directed weighted graphs by assigning directions to the edges in all possible ways and weights to the vertices in all possible admissible ways. One of the central technical results of this paper is Theorem **3.4** which says that this map is a quasi-isomorphism (up to one rescaling class), hence leads to the conclusions of the above Theorem.

1.3. **Classification of universal quantizations of $\mathbb{Z}$-graded quadratic Poisson structures.** M. Kontsevich *universal* formality map (1) given in [K2] can be equivalently understood as a morphism of dg props [AM]

$$\mathcal{F} : c\mathcal{A}ss_\infty \longrightarrow \mathcal{O}(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circlessthan}).$$

where $c\mathcal{A}ss_\infty$ is the dg free operad of *curved* strongly homotopy associative algebras, and $\widehat{\mathcal{H}olieb}_{0,1}^{\star\circlessthan}$ is the genus completed version of the dg prop of $\mathbb{Z}$-graded (not necessarily quadratic) Poisson structure, and $\mathcal{O}$ is the *polydifferential functor* from the category of dg props to the category of dg operads introduced in [MW2] (see §2 and 4 for detailed definitions and reminders). Reversely, any morphism of dg operads $\mathcal{F}$ as above satisfying certain non-triviality condition gives us a universal formality map (1). The deformation complex of any such a universal formality map has been computed in [AM] where it was identified with the graph complex $\mathsf{FGC}_2^{\geqslant 2}$; the curvature term on the l.h.s. of the map $\mathcal{F}$ played a central role in that classification result.

Similarly, any *universal* homogeneous formality map (2) can be equivalently understood as a morphism of dg operads,

$$\mathcal{F}_{(0)} : \mathcal{A}ss_\infty \longrightarrow \mathcal{O}^{\geqslant 1}(\widehat{\mathcal{H}oqpois}_{0,1}^{\circlessthan}).$$

where $\mathcal{A}ss_\infty$ is dg free operad of ordinary (*flat*) strongly homotopy associative algebras and $\mathcal{O}^{\geqslant 1}$ is a truncation of the functor $\mathcal{O}$ which does not allow curvature terms. We prove in §5 that *the deformation complex of any universal homogeneous formality map $\mathcal{F}_{(0)}$ is quasi-isomorphic (up to degree shift) to the deformation complex* $\mathrm{Der}(\mathcal{H}oqpois_{0,1}^{\circlessthan})$ *studied in Theorem* **1.2.1**. Hence we conclude that there is a canonical morphism of complexes

$$\mathsf{FGC}_2 \longrightarrow \mathsf{Def}\left(\mathcal{A}ss_\infty \overset{\mathcal{F}_{(0)}}{\to} \mathcal{O}(\widehat{\mathcal{H}oqpois}_{0,1}^{\circlessthan}) \right)[1]$$

which is a quasi-isomorphism. This result implies that the space of all infinitesimal homotopy deformations of $\mathcal{F}_{(0)}$ can be identified with the Lie algebra of the Grothendieck-Teichmüller group,

$$H^1 \mathsf{Def}\left(\mathcal{A}ss_\infty \overset{\mathcal{F}_{(0)}}{\to} \mathcal{O}^{\geqslant 1}(\widehat{\mathcal{H}oqpois}_{0,1}^{\circlessthan}) \right) = \mathfrak{grt}.$$

It is not hard to see that every such infinitesimal deformation exponentiates to a genuine deformation of $\mathcal{F}_{(0)}$ implying the second main result of this paper.

1.3.1. Theorem. *The Grothendieck-Teichmüller group GRT acts freely and transitively on the set $\mathcal{S}$ of homotopy classes of universal quantizations of $\mathbb{Z}$-graded quadratic Poisson structures. The set $\mathcal{S}$ itself can be identified with the set of Drinfeld associators.*

1.4. **Structure of the paper.** In §1 we remind a definition of the prop of homotopy Lie bialgebras, introduce the wheeled prop $\mathcal{H}oqpois_{0,1}^{\circlessthan}$ controlling finite-dimensional $\mathbb{Z}$-graded quadratic Poisson structures, define their derivation complexes and remind the basic notions and facts of the theory of graph complexes. In §2 we prove the first main result of this paper, Theorem 1.2.1. In §3 we explain how a universal homogeneous formality map can be understood as a morphism of dg operads, compute the cohomology of any such a morphism, and finally prove the second main Theorem 1.3.1 of this paper.

1.5. Some notation. We work in this paper over field $\mathbb{K}$ of characteristic zero. If $V = \oplus_{i \in \mathbb{Z}} V^i$ is a graded vector space over $\mathbb{K}$, then $V[k]$ stands for the graded vector space with $V[k]^i := V^{i+k}$ and and s^k for the associated isomorphism $V \to V[k]$; for $v \in V^i$ we set $|v| := i$. The set $\{1, 2, \ldots, n\}$ is abbreviated to $[n]$; its group of automorphisms is denoted by $\mathbb{S}_n$; the trivial one-dimensional representation of $\mathbb{S}_n$ is denoted by $\mathbf{1}_n$, while its one dimensional sign representation is denoted by sgn_n. The cardinality of a finite set A is denoted by $\#A$.

2. Derivation complexes of the wheeled prop of $\mathbb{Z}$-graded quadratic Poisson structures

2.1. Reminder on Lie (c, d)-bialgebras. By definition, $\mathcal{L}ieb_{c,d}$ is a quadratic properad given as the quotient,

$$\mathcal{L}ieb_{c,d} := \mathcal{F}ree\langle E\rangle / \langle \mathcal{R}\rangle,$$

of the free properad generated by an $\mathbb{S}$-bimodule $E = \{E(m,n)\}_{m,n\geqslant 1}$ with all $E(m,n) = 0$ except

$$E(2,1) := \mathbf{1}_1 \otimes sgn_2^c[c-1] = \mathrm{span}\left\langle \quad = (-1)^c \quad \right\rangle$$

$$E(1,2) := sgn_2^d \otimes \mathbf{1}_1[d-1] = \mathrm{span}\left\langle \quad = (-1)^d \quad \right\rangle$$

by the ideal generated by the following elements

(4) $\qquad \mathcal{R}: \begin{cases} \quad + \quad + \quad , \quad + \quad + \quad \\ (-1)^{cd+c+d}\quad - \quad + (-1)^d \quad + (-1)^{d+c}\quad + (-1)^c \quad \end{cases}$

The minimal resolution $\mathcal{H}olieb_{c,d}$ of the properad $\mathcal{L}ieb_{c,d}$ was constructed in [K3, MaVo, V]; it is the free properad generated by the following (skew)symmetric (m,n)-corollas of degree $1 + c(1-m) + d(1-n)$

(5) $\qquad$ $= (-1)^{c|\sigma|+d|\tau|}$ $\quad \forall \sigma \in \mathbb{S}_m, \forall \tau \in \mathbb{S}_n, \quad m,n \geqslant 1, m+n \geqslant 3,$

with $m+n \geqslant 3, m, n \geqslant 1$, and equipped with the differential given on the generators by

(6) $\qquad \delta$ $= \displaystyle\sum_{\substack{[1,\ldots,m]=I_1 \sqcup I_2 \\ |I_1|\geqslant 0, |I_2|\geqslant 1}} \sum_{\substack{[1,\ldots,n]=J_1 \sqcup J_2 \\ |J_1|\geqslant 1, |J_2|\geqslant 1}} \pm$

A representation of $\mathcal{H}olieb_{c,d}$ in a graded vector space V can be identified with a degree $c + d + 1$ Maurer-Cartan element π, $\{\pi, \pi\} = 0$, in the completed graded commutative algebra

$$\pi = \sum_{\substack{m,n\geqslant 1 \\ m+n\geqslant 3}} \pi_n^m \in \prod_{m,n\geqslant 1, m+n\geqslant 3} \mathrm{Hom}(\odot^n(V[d]), \odot^m(V[-c])) \subset \prod_{k\geqslant 0} \odot^k(V^*[-d] \oplus V[-c])$$

equipped with the Poisson type Lie bracket $\{\ ,\ \}$, of degree $-c-d$ induced by the natural paring $V \otimes V^* \to \mathbb{K}$. We are mostly interested in this paper in the case $c = 1$, $d = 0$, $V = \mathbb{R}^N$ when π becomes a formal Poisson structure on $\mathbb{R}^N$ which vanishes at $0 \in \mathbb{R}^N$,

(7) $$\pi = \sum_{a,b=1}^N \pi^{a,b}(x)\partial_a \wedge \partial_b, \quad \text{with} \quad \pi^{ab}(x) := \sum_{n\geqslant 1} \frac{1}{n!}\pi^{ab}_{c_1\ldots c_n} x^{c_1}\cdots x^{c_n}$$

for some constants $\pi^{ab}_{c_1\ldots c_n} = -\pi^{ba}_{c_1\ldots c_n} \in \mathbb{R}$.

Consider a differential ideal I in $\mathcal{H}olieb_{c,d}$ generated by graphs which contain at least one (m,n)-corolla with $m \neq n$, and let

$$\mathcal{H}oqpois_{c,d} := \mathcal{H}olieb_{c,d}/I$$

be the quotient dg properad. It is a free properad generated by (m,m)-corollas with $m \geqslant 2$ from the family (5),

$$
\begin{array}{c}
{}^{1\ \ 2}\ \cdots\ {}^{m-1\ m} \\
\times \\
{}_{1\ \ 2}\ \cdots\ {}_{m-1\ m}
\end{array}
\ , \qquad m \geqslant 2
$$

and equipped with differential trivial δ induced from (6); it is easy to see that δ acts trivially on the $(2,2)$-corollas. Let J be the differential closure of the ideal in $\mathcal{H}oqpois_{c,d}$ generated by (m,m)-corollas with $m \geqslant 3$ and set

$$ \mathcal{Q}pois_{c,d} := \mathcal{H}oqpois_{c,d}/J $$

The latter properad is a quadratic properad generated by the $(2,2)$-corolla of degree $1-c-d$,

$$
{}^{1}\!\times\!{}^{2}_{{}_{1}\ \ {}_{2}} = (-1)^c\ {}^{2}\!\times\!{}^{1}_{{}_{1}\ \ {}_{2}} = (-1)^d\ {}^{1}\!\times\!{}^{2}_{{}_{2}\ \ {}_{1}}
$$

modulo the relations

$$
\sum_{\tau,\sigma\in\mathbb{Z}_3}\ {}^{\tau(1)}\!\!\!\underset{\sigma(1)\ \ \sigma(2)}{\overset{\tau(2)\ \ \ \tau(3)}{\times\!\times}}{}_{\sigma(3)} = 0
$$

where $\mathbb{Z}_3$ is the subgroup of $\mathbb{S}_3$ generated by the permutation (123). Thus representations of the properad $\mathcal{Q}pois_{1,0}$ in an arbitrary (not necessarily finite-dimensional) graded vector space V can be identified with quadratic Poisson structures on the vector space V^*; in particular, in the case $V = \mathbb{R}^N$ they are given by (7) with only $\pi^{ab}_{c_1c_2} \neq 0$. It is shown in [KM] that the canonical epimorphism

$$ p : \mathcal{H}oqpois_{c,d} \longrightarrow \mathcal{Q}pois_{c,d} $$

is a quasi-isomorphism implying that the properad $\mathcal{Q}pois_{c,d}$ is Koszul (we do not use this reult in this paper).

The category of wheeled props has been introduced in [Me2] and studied in [MMS]. The properads $\mathcal{H}oqpois_{c,d}$ and $\mathcal{Q}pois_{c,d}$ have very simple wheeled closures denoted by $\mathcal{H}oqpois^{\circlearrowleft}_{c,d}$ and $\mathcal{Q}pois^{\circlearrowleft}_{c,d}$ respectively. They have the same sets of generators, but one is a allowed to build from them (via gluing outgoing legs of one generator to ingoing legs of another or the same generator) graphs which have closed paths of directed edges. For, example the graph

is allowed in both props $\mathcal{H}oqpois^{\circlearrowleft}_{c,d}$ and $\mathcal{Q}pois^{\circlearrowleft}_{c,d}$, and, in the notation used in (7), gets represented by the linear vector field

$$ \sum_{a,b,c} \frac{1}{2}\pi^{ac}_{bc}x^b\partial_a. $$

This example shows a fundamental difference between ordinary props and their wheeled closures: if, for example, the prop $\mathcal{Q}pois_{0,1}$ controls quadratic Poisson structures in both finite- and infinite-dimensional spaces, its wheeled closure $\mathcal{Q}pois^{\circlearrowleft}_{c,d}$ admits representations only in *finite*-dimensional vector spaces as it involves the trace operation $\mathrm{Hom}(V,V) \to \mathbb{K}$! Hence in the context of deformation quantization we must be interested in props $\mathcal{H}oqpois^{\circlearrowleft}_{0,1}$ and $\mathcal{Q}pois^{\circlearrowleft}_{0,1}$ as infinite-dimensional Poisson structures can *not* be quantized in general.

Another important technical point to notice that deformation quantization formulae [K2] of a generic $\mathbb{Z}$-graded (quadratic) Poisson structure involve formal power series in an auxiliary formal parameter $\hbar$ (the "Planck constant"). In terms of props this fact leads us to consider *genus completed* version, $\widehat{\mathcal{H}oqpois}^{\circlearrowleft}_{0,1}$ of the prop $\mathcal{H}oqpois^{\circlearrowleft}_{0,1}$. This is one of the main reasons why we develop below deformation theory of completed wheeled props $\widehat{\mathcal{H}oqpois}^{\circlearrowleft}_{c,d}$ rather than of the ordinary ones.

2.2. Complexes of derivations of properads. There is a useful endofunctor, $\mathcal{P} \to \mathcal{P}^+$, in the category of (wheeled) dg props introduced in [Me5]. Given any dg (wheeled) prop $\mathcal{P}$, the associated extended dg prop $\mathcal{P}^+$ is obtained from $\mathcal{P}$ by adjoining an extra generator $t \in \mathcal{P}^+(1,1)$ and is uniquely characterized by the property: there is a 1-1 correspondence between representations

$$\rho : \mathcal{P}^+ \longrightarrow \mathcal{E}nd_V$$

of $\mathcal{P}^+$ in a dg vector space (V, d), and representations of $\mathcal{P}$ in the same space V but equipped with a deformed differential $d + d'$, where $d' := \rho(t)$. The complex of derivations of an arbitrary prop $\mathcal{P}$ has been defined in [MW1] as the space of derivations,

$$\mathrm{Der}(\mathcal{P}) := \mathrm{Der}(\mathcal{P}^+ \to \mathcal{P})$$

of the plus-extended prop with values in $\mathcal{P}$ (otherwise we shall loose an important and useful information about homotopy theory of the (wheeled) prop $\mathcal{P}$, cf. [MW1, MW2, GY, MW3]). The dg props $\widehat{\mathcal{H}oqpois}^{\,+}_{c,d}$ an $\mathcal{H}oqpois^{+\circlearrowleft}_{c,d}$ is easy to describe explicitly — both are built from generators (5) and one extra $(1,1)$-generator $\mid$ which is assigned degree $+1$. The differential δ^+ is given on the new generator by the formula

$$\delta^+ \, \mid \, := \, \mid$$

while its value on "old" generators is given by

It is not hard to show that the complexes $\mathrm{Der}(\mathcal{P}^+ \to \mathcal{P})$ and $\mathrm{Der}(\mathcal{P}^+ \to \mathcal{P}^+)$ are quasi-isomorphic so that there is no loss of generality to define a derivation complex of a dg prop $\mathcal{P}$ as $\mathrm{Der}(\mathcal{P}^+ \to \mathcal{P}^+)$ making the Lie algebra structure on it more transparent. Hence it is a matter of taste which definition to use. We prefer the original one from ([MW1]) and set

$$\mathrm{Der}(\mathcal{H}oqpois_{c,d}) := \mathrm{Der}(\widehat{\mathcal{H}oqpois}^{\,+}_{c,d} \to \widehat{\mathcal{H}oqpois}_{c,d}), \quad \mathrm{Der}(\mathcal{H}oqpois^{\circlearrowleft}_{c,d}) := \mathrm{Der}(\widehat{\mathcal{H}oqpois}^{\,+\circlearrowleft}_{c,d} \to \widehat{\mathcal{H}oqpois}^{\circlearrowleft}_{c,d}).$$

As graded vector spaces, they can be identified with spaces generated by certain graphs. For example

$$(8) \quad \mathrm{Der}(\mathcal{H}oqpois^{\circlearrowleft}_{c,d}) = \prod_{m \geqslant 1} \left(\widehat{\mathcal{H}oqpois}^{\circlearrowleft}_{c,d}(m,m) \otimes sgn_m^{\otimes|c|} \otimes sgn_n^{\otimes|d|} \right)^{\mathbb{S}_m \times \mathbb{S}_m} [1 + c(1-m) + d(1-n)],$$

Its elements are directed not necessary connected graphs (possibly, with wheels) which might have incoming and outgoing legs and vertices have the same numbers of incoming and outgoing edges, for example

$$(9) \qquad \qquad \in \mathrm{Der}(\mathcal{H}oqpois^{\circlearrowleft}_{c,d})$$

Note that contrary to the prop $\mathcal{H}olieb^{\circlearrowleft}_{c,d}$, the out- or ingoing legs (if any) of the have their numerical labels are (skew)symmetrized in accordance with the parity of the integer parameters c and d; one can think that they are not labelled at all — just for odd c or d some ordering is chosen (up to an even permutation). The subspace $\mathrm{Der}(\mathcal{H}oqpois_{c,d}) \subset \mathrm{Der}(\mathcal{H}oqpois^{\circlearrowleft}_{c,d})$ is generated by similar graphs but with no wheels (in this case the summand above corresponding to $m = 0$ vanishes).

The value of the differential d on an element $\Gamma \in \mathrm{Der}(\mathcal{H}oqpois^{\circlearrowleft}_{c,d})$ consists of three terms

$$(10) \qquad \qquad d\Gamma = \delta\Gamma \pm \sum_{m \geqslant 2} \quad \pm \sum_{\Gamma}$$

Here

- in the first term the differential δ acts on the vertices of Γ as in the quotient properad $\mathcal{H}oqpois^{\circlearrowleft}_{c,d}$, that is, by formula (6) modulo terms involving (m,n)-corollas with $m \neq n$.
- in the last two terms one attaches (m,m)-corollas, $m \geqslant 2$, to each outgoing leg and each ingoing leg of Γ.

2.2.1. Remark. The dg prop $\mathcal{H}oqpois^{\circlearrowleft}_{c,d} = \{\mathcal{H}oqpois^{\circlearrowleft}_{c,d}(m,n)\}_{m,n\geqslant0}$ contains graphs with no legs at all, i.e. its part $\mathcal{H}oqpois^{\circlearrowleft}_{c,d}(0,0)$ does not vanish. Hence one one might consider an extended deformation theory of that prop in which such graph also "enter the game", and introduce a further extension, $(\widehat{\mathcal{H}oqpois}^{*\circlearrowleft}_{c,d})$ of the above prop $(\widehat{\mathcal{H}oqpois}^{+\circlearrowleft}_{c,d}, \delta^+)$,

$$\widehat{\mathcal{H}oqpois}^{+\circlearrowleft}_{c,d} \subsetneq \widehat{\mathcal{H}oqpois}^{\star\circlearrowleft}_{c,d}$$

by adding to the latter a $(0,0$-generator $\bullet$ in degree $1 + c + d$ and defining the differential δ^* on the new generator by the formula

$$\delta^* \bullet := \; \Big|$$

and setting δ^* to be δ^+ when acting on all other generators. The extended derivation complex of $\mathrm{Der}(\mathcal{H}oqpois^{\circlearrowleft}_{c,d})$ can then be defined as follows (cf. [AM]),

$$\mathrm{Der}^*(\mathcal{H}oqpois^{\circlearrowleft}_{c,d}) := \mathrm{Der}(\widehat{\mathcal{H}oqpois}^{\star\circlearrowleft}_{c,d} \to \widehat{\mathcal{H}oqpois}^{\circlearrowleft}_{c,d})$$

We have obviously a canonical isomorphism of graded vector spaces,

$$(11) \qquad \mathrm{Der}^*(\mathcal{H}oqpois^{\circlearrowleft}_{c,d}) = \prod_{m\geqslant0} \left(\widehat{\mathcal{H}oqpois}^{\circlearrowleft}_{c,d}(m,m) \otimes sgn_m^{\otimes|c|} \otimes sgn_n^{\otimes|d|} \right)^{\mathbb{S}_m \times \mathbb{S}_m} [1 + c(1-m) + d(1-n)],$$

i.e. the only difference from (8) is the presence of the summand with $m = 0$. Contrary to the analogous situation studied in [AM] this extension does *not* give something really new make as the new extended complex is the *direct* sum of already introduced complexes,

$$(12) \qquad \mathrm{Der}^*(\mathcal{H}oqpois^{\circlearrowleft}_{c,d}) = \widehat{\mathcal{H}oqpois}^{\circlearrowleft}_{c,d}(0,0)[1 + c + d] \quad \oplus \quad \mathrm{Der}(\mathcal{H}oqpois^{\circlearrowleft}_{c,d}).$$

Nevertheless it is useful sometimes (see §below) to consider this the extended complex while studying the most interesting for us deformation complex $\mathrm{Der}(\mathcal{H}oqpois^{\circlearrowleft}_{c,d})$.

2.3. Remark. Note that representations of the dg prop $\mathcal{H}olieb_{0,1}$ in a vector space $\mathbb{R}^N$ describe Poisson structures (40) which *vanish* ar $0 \in \mathbb{R}^N$. This is a restriction which is desirable to avoid, and this can be easily done via introducing a new family dg props $\mathcal{H}olieb^{\star}_{c,d}$ as the free prop generated by (m,n)-corollas (5) for all possible values $m, n \geqslant 0$ and equipped with an obvious extension of the differential (6). Representations of the (wheeled closure) of the prop $\mathcal{H}olieb^{\star}_{0,1}$ in a dg (finite-dimensional) vector space V are in 1:1 correspondence with arbitrary $\mathbb{Z}$-graded Poisson structures on V^*.

There is an epimorphism of dg props

$$p : \mathcal{H}olieb^{\star\circlearrowleft}_{c,d} \longrightarrow \mathcal{H}oqpois^{\star\circlearrowleft}_{c,d}$$

so that the above complex $\mathrm{Der}^*(\mathcal{H}oqpois^{\circlearrowleft}_{c,d})$ can be identified with the deformation complex of that epimorphism (up to a degree shift),

$$\mathrm{Der}^*(\mathcal{H}oqpois^{\circlearrowleft}_{c,d}) = \mathrm{Der}(\mathcal{H}olieb^{\star\circlearrowleft}_{c,d} \to \mathcal{H}oqpois^{\star\circlearrowleft}_{c,d}) = \mathsf{Def}(\mathcal{H}olieb^{\star\circlearrowleft}_{c,d} \to \mathcal{H}oqpois^{\star\circlearrowleft}_{c,d})[1].$$

In particular, the epimorphism p induces a morphism of dg Lie algebras

$$\mathrm{Der}(\mathcal{H}olieb^{\star\circlearrowleft}_{c,d}) := \mathrm{Der}(\mathcal{H}olieb^{\star\circlearrowleft}_{c,d} \overset{\mathrm{Id}}{\to} \mathcal{H}olieb^{\star\circlearrowleft}_{c,d}) \longrightarrow \mathrm{Der}^*(\mathcal{H}oqpois^{\circlearrowleft}_{c,d})$$

The derivation complex $\mathrm{Der}(\mathcal{H}olieb^{\star\circlearrowleft}_{c,d})$ has been studied in [AM] where it has been proven that its cohomology is determined (up one rescaling class) by the Kontsevich graph complex $\mathsf{GC}^{\geqslant2}_{c+d+1}$ whose definition is recalled below.

2.4. Directed and undirected graph complexes. A *graph without legs*, or simply *a graph*, Γ is a 1-dimensional CW complex whose 0-cells are called *vertices* and 1-cells are called *edges*; its set of vertices is denoted by $V(\Gamma)$ while the set of edges by $E(\Gamma)$. A graph Γ is called *directed* if each edge e comes equipped with a fixed orientation (one of the two possible on an interval). If a vertex v of a directed graph is said to have type (m, n) if it has $m \geqslant 0$ outgoing edges and $n \geqslant 0$ incoming edges; we write in this case $|v|_{in} = n$, $|v|_{out} = m$; the number $|v| := |v|_{in} + |v|_{out}$ is called the *valency* of v. A $(1, 1)$-vertex is called *passing*.

Given any integer $d \in \mathbb{Z}$, we can associate to any graph Γ with, say, k edges and l vertices,

$$E(\Gamma) = \{e_1, \ldots, e_k\}, \quad V(\Gamma) = \{v_1, \ldots, v_l\}$$

a 1-dimensional vector space

$$\mathbb{K}_\Gamma := \begin{cases} \wedge^{\#E(\Gamma)} \mathbb{K}[E(\Gamma)] & \text{if } d \in 2\mathbb{Z} \\ \wedge^{\#V(\Gamma)} \mathbb{K}[V(\Gamma)] & \text{if } d \in 2\mathbb{Z}+1 \end{cases}$$

where $\mathbb{K}[E(\Gamma)]$ (resp. $\mathbb{K}[V(\Gamma)]$) is the linear span of the set of edges (resp. vertices); there are at most two different bases (which differ by a sign) of this space given by simple vectors of the form

$$e_{i_1} \wedge \ldots \wedge e_{i_k}, \quad \text{respectively}, \quad v_{j_1} \wedge \ldots \wedge v_{j_l}.$$

An *orientation* on Γ is, by definition, a choice of a particular simple basis *or* of $\mathbb{K}_\Gamma$; equivalently, an orientation on Γ is a choice of ordering of edges (resp. vertices) up to an even permutation. If $\#E(\Gamma) \geqslant 2$ for d even or $\#V(\Gamma) \geqslant 2$ for d odd, there are precisely two different orientations, $(or, -or)$, on Γ.

Let dfGC_d be the ("directed full") completed[1] topological vector space generated over a field $\mathbb{K}$ by the set of all pairs (Γ, or) (which is often abbreviated simply by Γ) modulo the equivalence relation,

$$(\Gamma, -or) = -(\Gamma, or).$$

We make dfGC_d into a $\mathbb{Z}$-*graded* vector space by setting the (cohomological) degree of any graph generator Γ to be given by

$$|\Gamma| = d\#V(\Gamma) + (1-d)\#E(\gamma) - d.$$

This graded vector space has a Lie algebra structure with

$$[\Gamma', \Gamma''] := \sum_{v \in V(\Gamma')} \Gamma' \circ_v \Gamma'' - (-1)^{|\Gamma'||\Gamma''|} \Gamma'' \circ_v \Gamma'$$

where $\Gamma' \circ_v \Gamma''$ is defined by substituting the graph Γ'' into the vertex v of Γ' and taking the sum over all possible re-attachments of $|v|_{in} + |v|_{out}$ dangling edges to the vertices of Γ''. It is easy to see that graph $\bullet \!\!\rightarrow\!\! \bullet \in \mathsf{dcGC}_d$ is a Maurer-Cartan element, so that one makes dfGC_d into a complex with the differential

$$\delta := [\bullet \!\!\rightarrow\!\! \bullet, \ ].$$

This dg Lie algebra was introduced and studied in [K1, W1]. The complex dfGC_d contains a subcomplex dfcGC_d spanned by connected graphs. One has an isomorphism of complexes

$$\mathsf{dfGC}_d = \odot^{\bullet \geqslant 1} \left(\mathsf{dfcGC}_d[-d] \right) [d]$$

The complex dfcGC_d contains a subcomplex $\mathsf{dcGC}_d^{\geqslant 2}$ spanned by graphs with all vertices having valency $|v| \geqslant 2$ in turn a subcomplex dcGC_d spanned by graphs with no passing vertices; both monomorphisms,

$$\mathsf{dcGC}_d \hookrightarrow \mathsf{dcGC}_d^{\geqslant 2} \hookrightarrow \mathsf{dfcGC}_d,$$

are quasi-isomorphisms [W1]. The complex dcGC_d contains in turn a subcomplex OGC_d spanned by directed connected graphs with no closed paths of directed edges ("wheels"); such directed graphs are called *oriented*. It was proven in [W2] that

$$H^\bullet(\mathsf{dcGC}_d) = H^\bullet(\mathsf{OGC}_{d+1}).$$

---

[1]We mean the completion with respect to the filtration of dfGC_d by the number of vertices and edges.

Let $\mathsf{dfGC}_d^{(l)} \subset \mathsf{dfGC}_d$ be the linear subspace spanned by graphs with l edges. The group $\mathbb{Z}_2^{\times l}$ acts naturally on the generators of $\mathsf{dfGC}_d^{(l)}$ by reversing directions on edges. Hence one can define an undirected version of dfGC_d as follows,

$$\mathsf{fGC}_d = \prod_{l \geqslant 0} \mathsf{dfGC}_d^{(l)} \otimes_{\mathbb{Z}_2^{\times l}} sgn_2^{\otimes l|d|}$$

Graphs from fGC_d have no directions on edges for even d, or have a direction given on every edges up a flip and multiplying by -1.

The complex fGC_d contains a subcomplex $\mathsf{fGC}_d^{\geqslant 2}$ spanned by graphs with all vertices having valency at least two, which in turn contains a subcomplex $\mathsf{GC}_d^{\geqslant 2}$ spanned by connected graphs. There is an isomorphism of complexes,

$$(13) \qquad\qquad\qquad \mathsf{fGC}_d^{\geqslant 2} = \odot^{\bullet \geqslant 1}\left(\mathsf{GC}_d^{\geqslant 2}[-d]\right)[d]$$

and a direct splitting,

$$\mathsf{GC}_d^{\geqslant 2} = \mathsf{GC}_d \oplus \mathsf{GC}_d^2$$

where GC_d^2 is the subcomplex spanned by graphs containing at least one bivalent vertex and GC_d is spanned by graphs with all vertices at least trivalent. The cohomology of GC_d^2 has been computed in [W1],

$$H^\bullet(\mathsf{GC}_d^2) = \bigoplus_{\substack{p \geqslant 1 \\ p \equiv 2d+1 \mod 4}} \mathbb{K}[d-p],$$

where the summand $\mathbb{K}[d-p]$ is generated by the polytope with p vertices, that is, a connected graph with p bivalent vertices. On the other hand, the cohomology of GC_d is understood at present only in non-positive degrees [W1]

$$H^{\bullet \leqslant -1}(\mathsf{GC}_2) = 0 \quad\text{and}\quad H^0(\mathsf{GC}_2) = H^0(\mathsf{GC}_2^{\geqslant 2}) = H^0(\mathsf{OGC}_3) = \mathfrak{grt}_1,$$

where $\mathfrak{grt}_1$ is the Lie algebra of the Grothendieck-Teichmüller group GRT_1. Moreover, it was proven in [W1] that there is a quasi-isomorphism of complexes

$$(14) \qquad\qquad\qquad i : \mathsf{GC}_d^{\geqslant 2} \longrightarrow \mathsf{dfcGC}_d^{\geqslant 2}$$

which sends an undirected graph into a sum of directed graphs by assigning directions to its edges in all possible ways.

In this paper we have to work with the "full" Grothendieck-Teichmüller group GRT rather than with its subgroup GRT_1. Hence we need a slight extension [MW2] of the complex $\mathsf{GC}_d^{\geqslant 2}$,

$$\mathsf{GC}_d^{\geqslant 2} \longrightarrow \mathsf{GC}_d^{\geqslant 2} \oplus \mathbb{K}$$

where the generator of the 1-dimensional summand $\mathbb{K}$ is a cycle, "a graph with no vertices and edges", and has Lie bracket with any graph Γ from $\mathsf{GC}_d^{\geqslant 2}$ equal to $2\ell\Gamma$, where ℓ is the number of its loops. In our story this class takes care about the obvious rescaling automorphism of the prop $\mathcal{H}oqpois_{c,d}^{\circ}$. Hence we define the *full graph complexes* of not necessarily connected graphs as the completed graded symmetric tensor algebra

$$(15) \qquad\qquad\qquad \mathsf{FGC}_d^{\geqslant 2} := \widehat{\odot}^\bullet\left((\mathsf{GC}_d^{\geqslant 2} \oplus \mathbb{K})[-d]\right)[d],$$

It is then easy to see that

$$H^0(\mathsf{FGC}_2^{\geqslant 2}) = \mathfrak{grt},$$

where $\mathfrak{grt}$ is the Lie algebra of GRT.

2.5. A quotient complex of dcGC_d. The complex dfcGC_d contains a subcomplex $\mathsf{dcGC}_d^{\neq}$ spanned by graphs with at least one vertex v satisfying $|v|_{in} \neq |v|_{out}$. The quotient complex $\mathsf{dcGC}_d^{=}$,

$$\text{(16)} \qquad 0 \longrightarrow \mathsf{dcGC}_d^{\neq} \longrightarrow \mathsf{dfcGC}_d \longrightarrow \mathsf{dcGC}_d^{=} \longrightarrow 0,$$

is spanned by graphs with all vertices satisfying $|v|_{in} = |v|_{out}$. It is precisely the connected part of the direct summand $\widehat{\mathcal{H}oqpois}_{c,d}^{\circlearrowleft}(0,0)[1+c+d]$ in the "full" derivation complex (12).

Any graph Γ in $\mathsf{dcGC}_d^{=}$ has vertices $v \in V(\Gamma)$ of type (l_v, l_v) for some $l_v \geqslant 2$; hence its cohomological degree is given by

$$|\Gamma| = d(\#V(\Gamma) - 1) + (1-d) \sum_{v \in V(\Gamma)} l_v.$$

Hence for $d \geqslant 2$ we obtain

$$|\Gamma| \leqslant d(\#V(\Gamma) - 1) + (1-d) \sum_{v \in V(\Gamma)} 2 = (2-d)\#V(\Gamma) - d$$

and conclude that for $d = 2$ the quotient complex $\mathsf{dcGC}_2^{=}$ is concentrated in degrees $\leqslant -2$. As $H^\bullet(\mathsf{dcGC}_2)$ is concentrated in non-negative degrees [W1], we also conclude that the canonical projection at the cohomology level

$$H^\bullet(\mathsf{dcGC}_2) \longrightarrow H^\bullet(\mathsf{dcGC}_2^{=})$$

is equal to the zero map (hence the map $H^\bullet(\mathsf{dcGC}_d) \longrightarrow H^\bullet(\mathsf{dcGC}_d^{=})$ is also equal to zero for any *even d*). Therefore in the most important for us in this paper case $d = 2$ the long exact sequence of cohomology groups associated with (16) decomposes into a collection of short exact sequences,

$$\text{(17)} \qquad 0 \longrightarrow H^\bullet(\mathsf{dcGC}_2^{=}) \xrightarrow{\ b\ } H^{\bullet+1}(\mathsf{dcGC}_2^{\neq}) \longrightarrow H^{\bullet+1}(\mathsf{dcGC}_2) \longrightarrow 0$$

The subcomplex $\widehat{\mathcal{H}oqpois}_{0,1}^{\circlearrowleft}(0,0)[2] = \odot^{\geqslant 1}(\mathsf{dcGC}_2^{=}[-2])[2]$ in the "full" derivation complex (12) does not interact with the second summand in (12) at all; it the second summand $\mathrm{Der}(\mathcal{H}oqpois_{c,d}^{\circlearrowleft})$ which plays the key role in classification of the universal quantizations of $\mathbb{Z}$-graded Poisson structures. We shall study its cohomology in §4 below.

3. Cohomology of the deformation complex of $\mathcal{H}oqpois_{c,d}^{\circlearrowleft}$

3.1. Graph complexes versus derivation complexes. The differentials in the dg props $\mathcal{H}olieb_{c,d}$, $\mathcal{H}olieb_{c,d}^{\star}$ and $\mathcal{H}oqpois_{c,d}$ (and of their wheeled closures) are given by connected graphs (see (6) so the subspaces of all these props spanned by *connected* graphs form *dg* properads which we denote by the same symbols. However we use a slightly different symbol der to denote the Lie algebras of derivations of the properads as opposed to the symbol Der is reserved for derivations of the associated props. Both types of dg Lie algebras are related to each other by an exact symmetric tensor algebra functor as follows,

$$\begin{aligned} \mathrm{Der}(\mathcal{H}olieb_{c,d}) &= \odot^{\bullet \geqslant 1}(\mathrm{der}(\mathcal{H}olieb_{c,d})[-1-c-d])[1+c+d], \\ \mathrm{Der}(\mathcal{H}oqpois_{c,d}^{\circlearrowleft}) &= \odot^{\bullet \geqslant 1}(\mathrm{der}(\mathcal{H}oqpois_{c,d}^{\circlearrowleft})[-1-c-d])[1+c+d] \end{aligned}$$

Thus it is enough to study the cohomology of the complex der of *properadic* derivations in each case. There is a morphism of dg Lie algebras [MW1]

$$\text{(18)} \qquad \begin{aligned} F: \quad \mathsf{OGC}_{c+d+1} &\rightarrow \mathrm{der}(\mathcal{H}olieb_{c,d}) \\ \Gamma &\rightarrow F(\Gamma) \end{aligned}$$

where the derivation $F(\Gamma)$ acts (from the right) on the generators completed properad $\widehat{\mathcal{H}olieb}_{c,d}$ as follows

$$\text{(19)} \qquad \left(\vcenter{\hbox{}} \right) \cdot F(\Gamma) = \sum_{\substack{s:[n] \to V(\Gamma) \\ \hat{s}:[m] \to V(\Gamma)}} \vcenter{\hbox{}}$$

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where the sum is taken over all ways of attaching the incoming and outgoing legs to the graph Γ, and one sets to zero every resulting graph if it contains (i) a vertex with valency < 3, or (ii) or a vertex with no incoming edge(s)/leg(s), or (iii) a vertex with no outgoing edge(s)/leg(s). Moreover, it was proven in [MW1] that map F is a quasi-isomorphism up to one class in the complex $\mathrm{Der}(\mathcal{H}olieb_{c,d})$ given explicitly by the cycle,

$$(20) \qquad\qquad r = \sum_{m,n}(n-1)\, \overset{\overbrace{\quad}^{m\times}}{\underset{\underbrace{\quad}_{n\times}}{\times}} .$$

Note that coefficient $(n-1)$ above can be replaced by $(m-1)$ or by $(m+n-2)$ — all the corresponding cycles represent the same cohomology class.

A similar morphism of dg Lie algebras

$$F^{\circlearrowleft} : \mathsf{dcGC}_{c+d+1} \longrightarrow \mathsf{der}(\mathcal{H}olieb_{c,d}^{\star\circlearrowleft})$$

for the wheeled closure $\mathcal{H}olieb_{c,d}^{\star\circlearrowleft}$ of the "full" ordinary dg properad has been studied in [AM] where it has been proven that $F^{\circlearrowleft}$ is also a quasi-isomorphism up to the same rescaling class r; that proof is in fact much easier and shorter than the proof of the quasi-isomorphism (18)) in the unwheeled case.

The first (resp. second) result has been used in [MW3] (resp. in [AM]) to classify (up to homotopy) all universal deformation quantizations of Lie bialgebras (resp., of generic Poisson structures).

In this section we address a similar problem for the derivation complex of the wheeled properad of homotopy quadratic Poisson structures. Its solution given below will lead us in §5 to the classification of all homotopy classes of universal quantization of $\mathbb{Z}$-graded quadratic Poisson structures.

3.2. **Marked graph complex.** A *weight function* on a directed graph $\Gamma \in \mathsf{dfcGC}_d$ is, by definition, a map

$$\begin{aligned} \mathfrak{w}: \quad V(\Gamma) &\longrightarrow \mathbb{N}_{\geqslant 1} \\ v &\longrightarrow \mathfrak{w}_v \end{aligned}$$

satisfying the condition

$$(21) \qquad\qquad \mathfrak{w}_v \geqslant \max\{1, |v|_{in} - 1, |v|_{out} - 1\}$$

The weight function $\mathfrak{s}$ given by the equality

$$\mathfrak{s}_v := \max\{1, |v|_{in} - 1, |v|_{out} - 1\} =: \mathfrak{s}_v$$

is called *canonical*. Let $\mathfrak{W}(\Gamma)$ stand for the set of all possible weight functions on a given graph Γ.

A *weighted* graph is, by definition, a pair $\Gamma^{\mathfrak{w}} := (\Gamma, \mathfrak{w})$ where $\Gamma \in \mathsf{dfcGC}_d$ and $\mathfrak{w}$ is a weighted function on Γ. The integer $\mathfrak{w}_v$ is called the *weight* of the vertex v of a weighted graph. The natural number $\mathfrak{w}(\Gamma) := \sum_{v \in V(\Gamma)} \mathfrak{w}_v$ is called the *(total) weight* of the weighted graph $\Gamma^{\mathfrak{w}}$. The cohomological degree of $\Gamma^{\mathfrak{w}}$ is defined by the standard formula

$$(22) \qquad\qquad |\Gamma^{\mathfrak{w}}| := d(\#V(\Gamma) - 1) - (d - 1)\#E(\Gamma)$$

i.e. $|\Gamma^{\mathfrak{w}}| = |\Gamma|$, the degree of the underlying graph.

Let $\mathsf{dwGC}_d^{\star}$ be the completed (by the number of vertices and edges) graded vector space over a field $\mathbb{K}$ which is generated by weighted directed connected graphs. Its generators $\Gamma^{\mathfrak{w}}$ can be represented pictorially as follows,

 $\in \mathsf{dwGC}_d^{\star}$,
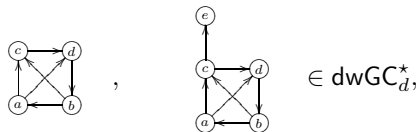
where the natural numbers $a, b, d, c, e \in \mathbb{N}_{\geqslant 1}$ stand for weights $\mathfrak{w}_v$ of the corresponding vertices.

Recall that the standard differential δ in dfcGC_d is given by the formula (23)

$$(23) \qquad \delta\Gamma := [\, \overset{\bullet}{\underset{\bullet}{\mathsf{I}}}\, , \Gamma] = \delta_0\Gamma \pm \sum_{v \in V(\Gamma)} \overset{\bullet}{\underset{\Gamma}{\mathsf{I}}} \pm \sum_{v \in V(\Gamma)} \overset{\Gamma}{\underset{\bullet}{\mathsf{I}}}$$

where

- the first summand $\delta_0\Gamma$ is equal to $\pm\sum_{v \in V(\Gamma)} \delta_v\Gamma^{\mathfrak{w}}$ with $\delta_v\Gamma^{\mathfrak{w}}$ obtained from $\Gamma^{\mathfrak{w}}$ by replacing the vertex v with the graph $\overset{\bullet}{\underset{\bullet}{\mathsf{I}}}$ and taking the sum over all possible ways to reattach the dangling edges (connected earlier to v) to the two newly created vertices.
- the second (resp., the third) summand is obtained by attaching the ingoing (resp., outgoing) leg of $\bullet\!\!\rightarrow\!\!\bullet$ to a vertex $v \in V(\Gamma)$.

We make dwGC_d into a complex by defining a differential d in a close analogy to the above formula,

$$(24) \qquad d\Gamma^{\mathfrak{w}} = d_0\Gamma^{\mathfrak{w}} \pm \sum_{c \in \mathbb{N}_{\geqslant 1}} \sum_{\substack{v \in V(\Gamma) \\ \mathfrak{w}_v > |v|_{out}-1}} \overset{\textcircled{c}}{\underset{\Gamma^{\mathfrak{w}}}{\mathsf{I}}} \pm \sum_{c \in \mathbb{N}_{\geqslant 1}} \sum_{\substack{v \in V(\Gamma) \\ \mathfrak{w}_v > |v|_{in}-1}} \overset{\Gamma^{\mathfrak{w}}}{\underset{\textcircled{c}}{\mathsf{I}}}$$

where

- the first summand $d_0\Gamma^{\mathfrak{w}}$ is equal to $\pm\sum_{v \in V(\Gamma)} \delta_v\Gamma^{\mathfrak{w}}$ with $\delta_v\Gamma^{\mathfrak{w}}$ obtained from $\Gamma^{\mathfrak{w}}$ by replacing the vertex v with weight, say, $a \in \mathbb{N}_{\geqslant 1}$ by the sum of graphs

$$(25) \qquad \textcircled{a} \longrightarrow \sum_{a=a'+a''} \overset{\textcircled{a'}}{\underset{\textcircled{a''}}{\mathsf{I}}}$$

  and taking one more sum over all possible way to reattach the dangling edges to the newly created vertices $\textcircled{a'}$ and $\textcircled{a''}$ in such a way that the condition (21) is respected; for example, if v has weight 1, then $\delta_v\Gamma^{\mathfrak{w}} = 0$;
- the second summand consists of graphs obtained by attaching the ingoing leg of the graph $\overset{\textcircled{c}}{\underset{}{\mathsf{I}}}$ to each vertex v of $\Gamma^{\mathfrak{w}}$ whose weight satisfies the strict inequality $\mathfrak{w}_v > \#|v|_{out} - 1$ (so that the condition (21) is not violated);
- the third summand is an obvious "outgoing" analogue of the second one.

The rule of signs in (24) is identical to the rule of signs in (23) as weights on vertices do not influence the cohomological degrees of weighted graphs.

This complex splits as a direct sum (cf. (12))

$$(26) \qquad \mathsf{dwGC}_d^\star = \mathsf{dwGC}_d^{\overline{=}} \oplus \mathsf{dwGC}_d$$

where the summand $\mathsf{dwGC}_d^{\overline{=}}$ is spanned by graphs Γ such that for each $v \in V(\Gamma)$ one has $\mathfrak{w}_v = |v|_{in} - 1 = |v|_{out} - 1$ (implying $|v|_{in} = |v|_{out} \geqslant 2$); the differential in this summand is given by the first term d_0 only in the above general formula (24) for d. It is not hard to check that $\mathsf{dwGC}_d^{\overline{=}}$ is identical to the quotient complex $\mathsf{dcGC}_d^{\overline{=}}$ of dcGC_d introduced in §**2.5**. This observation tells us that the similarity between formulae (23) and (24) can be very deceiving — the complex $\mathsf{dcGC}_d^{\overline{=}}$ is *not* a direct summand of dcGC_d.

The summand dwGC_d is spanned by graphs with at least one vertex v having weight $\mathfrak{w}_v \neq |v|_{in} - 1$ or $\mathfrak{w}_v \neq |v|_{out} - 1$.

3.2.1. Lemma. *There are canonical isomorphisms of complexes,*

$$\mathsf{der}^\star(\mathcal{H}oqpois_{c,d}^{\circlearrowleft}) \simeq \mathsf{dwGC}_{c+d+1}^\star, \qquad \mathsf{der}(\mathcal{H}oqpois_{c,d}^{\circlearrowleft}) \simeq \mathsf{dwGC}_{c+d+1}.$$

Proof. Consider a map

$$\begin{array}{cccc} w: & \mathsf{der}^\star(\mathcal{H}oqpois_{c,d}^{\circlearrowleft}) & \longrightarrow & \mathsf{dwGC}_{c+d+1}^\star \\ & \gamma & \longrightarrow & \gamma^w \end{array}$$

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defined as follows: γ^w is obtained from $\gamma \in \mathrm{Der}^*(\mathcal{H}oqpois^{\circlearrowleft}_{c,d})$ by decorating each (m,m)-vertex v of γ, $m \geqslant 2$, with the weight $\mathfrak{w}_v := m - 1$ and removing all ingoing and outgoing legs; the condition (21). As the original graph γ can be reconstructed from the resulting graph γ^w uniquely, this map is an isomorphism of vector spaces.

Let us check next that the map w respects the $\mathbb{Z}$-grading of both sides. On the one hand we have (cf. (22)

$$|\gamma^w| = (c + d + 1)(\#V(\gamma) - 1) - (c + d)\#E(\gamma).$$

On the other hand, the isomorphism (8) implies that each vertex $v \in V(\gamma)$, that is, each $(\mathfrak{w}_v + 1, \mathfrak{w}_v + 1)$-corolla in γ contributes the degree

$$1 + c + d - (c + d)(a_v + 1) = 1 - (c + d)\mathfrak{w}_v$$

Shifting the resulting total degree by $(1 + c + d - (c + d)\frac{1}{2}L(\gamma))$, where $L(\gamma)$ is the total number of ingoing and outgoing legs of γ, we obtain

$$
\begin{aligned}
|\gamma| &= \sum_{v \in V(\gamma)} (1 - (c + d)\mathfrak{w}_v) - \left(1 + c + d - (c + d)\frac{1}{2}L(\gamma)\right) \\
&= \#V(\Gamma) - (c + d)\mathfrak{w}(\gamma^w) - \left(1 + c + d - (c + d)\frac{1}{2}L(\gamma)\right).
\end{aligned}
$$

The total number of half-edges in Γ is equal to $2\sum_{v \in V(\gamma)}(1 + \mathfrak{w}_v) = 2\#V(\gamma) + 2\mathfrak{w}(\gamma^w)$, out of this number $2\#E(\gamma)$ are paired into internal edges, so that the remaining number of legs is given by

$$L(\gamma) = 2\#V(\gamma) + 2\mathfrak{w}(\gamma) - 2\#E(\gamma)$$

and we obtain finally,

$$
\begin{aligned}
|\gamma| &= \#V(\gamma) - (c + d)\mathfrak{w}(\gamma^w) - (1 + c + d - (c + d)(\#V(\gamma) + \mathfrak{w}(\gamma^w) - \#E(\gamma))) \\
&= (c + d + 1)(\#V(\gamma) - 1) - (c + d)\#E(\gamma).
\end{aligned}
$$

Hence $|\gamma^w| = |\gamma|$ so that the map w does respect the degrees.

The compatibility of the map w with the differentials follows from the very definition (24) of the differential in dwGC_{c+d+1} — the latter was essentially copied from the differential (10) in $\mathrm{Der}^\star(\mathcal{H}oqpois^{\circlearrowleft}_{c,d})$ via the isomorphism w. $\qquad\square$

It is easy to check that

$$(27) \qquad\qquad r := \sum_{a \geqslant 1}(a - 1)\,\textcircled{\scriptsize a}$$

is a non-trivial cohomology class in dwGC_d. It is the image of the rescaling class (20) under the composition of the projection $\mathsf{der}(\widehat{\mathcal{H}olieb}_{c,d}) \to \mathsf{der}(\mathcal{H}oqpois^{\circlearrowleft}_{c,d})$ with the above isomorphism.

There is a monomorphism of complexes

$$(28) \qquad
\begin{array}{ccccc}
\mathcal{F}^{\circlearrowleft}: & \mathsf{dcGC}_d & \longrightarrow & & \mathsf{dwGC}_d \\
& \Gamma & \longrightarrow & & \displaystyle\sum_{\mathfrak{w}:V(\Gamma)\to\mathbb{N}_{\geqslant 1}} \Gamma^{\mathfrak{w}}
\end{array}
$$

where $\Gamma^{\mathfrak{w}}$ is the graph obtained from Γ by assigning to each vertex v the weight $a_v := \mathfrak{w}(v)$, and setting $\Gamma^{\mathfrak{w}}$ to zero if for at least one vertex the condition (21) is violated.

3.3. An auxiliary complex.
Consider an auxiliary graph complex,

$$(29) \qquad\qquad C = \bigoplus_{n \geqslant 1} C_n.$$

where C_n is spanned by graphs of the form

$$(30) \qquad\qquad \textcircled{\scriptsize a_1}\!\longrightarrow\!\textcircled{\scriptsize a_2}\!\longrightarrow \cdots \longrightarrow\!\textcircled{\scriptsize a_n}\!\longrightarrow \quad,$$

with $a_1, \ldots, a_n \in \mathbb{N}_{\geqslant 1}$. It is useful to interpret such graphs as elements of a free properad generated by weighted corollas of type $(1,1)$ and $(1,0)$ because the differential can defined by its action on generators as follows

$$(31) \qquad d \;\, \boxed{a}\!\!\rightarrow \;=\; \sum_{a=b+c} \, \boxed{b}\!\rightarrow\!\boxed{c}\!\rightarrow \;, \qquad d \;\rightarrow\!\boxed{a}\!\rightarrow \;=\; \sum_{a=b+c} \;\rightarrow\!\boxed{b}\!\rightarrow\!\boxed{c}\!\rightarrow \;,$$

3.3.1. Lemma [CMW]. $H^\bullet(C) = \mathbb{K}[-1] \simeq \mathrm{span}\langle\; \boxed{1}\!\rightarrow \;\rangle.$

As cohomology of C is and one-dimensional and is concentrated in degree 1, we notice for the future use that $H^\bullet(\odot^{k \geqslant 2} C) = 0$.

3.4. Theorem. *The morphism*

$$(32) \qquad \begin{aligned} \mathcal{F}^{\circlearrowleft} : \quad \mathsf{dcGC}_d^{\geqslant 2} &\longrightarrow \quad \mathsf{dwGC}_d \\ \Gamma &\longrightarrow \sum_{\mathfrak{w}:V(\Gamma)\to\mathbb{N}_{\geqslant 1}} \Gamma^{\mathfrak{w}} \end{aligned}$$

is a quasi-isomorphism up to one rescaling class (27).

Proof. We shall study the full graph complex $\mathsf{dwGC}_d^\star$ and, using certain spectral sequences, show the quasi-isomorphism of complexes

$$\mathsf{dwGC}_d^\star \equiv \mathsf{dwGC}_d^= \oplus \mathsf{dwGC}_d \simeq \mathsf{dcGC}^= \oplus \mathcal{F}^{\circlearrowleft}(\mathsf{dcGC}_d^{\geqslant 2}) \oplus \mathrm{span}\langle r \rangle.$$

which implies the theorem.

Let us call (weighted or un-weighted) univalent vertices of our graphs with precisely *one* outgoing (resp., ingoing) edge *univalent in-vertices* (resp. *univalent out-vertices*); such vertices have the form $\boxed{a}\!\rightarrow$ or $\bullet\!\!\rightarrow$ (resp. $\boxed{a}\!\!\leftarrow$ or $\bullet\!\!\leftarrow$).

Step 1: skeleton filtration. Delete from a graph $\Gamma^{\mathfrak{w}} \in \mathsf{dwGC}_d^\star$ recursively all univalent in-vertices. The result is a weighted graph $\Gamma_{sk}^{\mathfrak{w}}$ called the *skeleton graph* of $\Gamma^{\mathfrak{w}}$; the vertices of $\Gamma_{sk}^{\mathfrak{w}}$ are called the *skeleton vertices* of $\Gamma^{\mathfrak{w}}$. If we forget the induced weight function on $\Gamma_{sk}^{\mathfrak{w}}$ we get a graph Γ_{sk} which belongs to a subspace $\mathsf{dfcGC}_d' \subset \mathsf{dfcGC}_d$ of the full directed graph complex spanned by graphs with no univalent in-vertices. For a vertex $r \in \Gamma_{sk}$ set

$$\mathfrak{s}_r^{sk} := \max\{1, |r|_{in}^{sk} - 1, |r|_{out}^{sk} - 1\},$$

where $|r|_{in}^{sk}$ (resp., $|r|_{out}^{sk}$) is the number of incoming (resp., outgoing) edges attached to the skeleton vertex r *in the skeleton graph* Γ_{sk} (*not* in the original graph $\Gamma^{\mathfrak{w}}$).

Consider a filtration of the complex dwGC_d by the number of skeleton vertices (which can not decrease under the action of the differential), and let $\{E_k \mathsf{dwGC}_d, d_k\}_{k \geqslant 0}$ be the associated spectral sequence.

Step 2: Computation of $E_1 \mathsf{dwGC}_d = H^\bullet(E_0 \mathsf{dwGC}_d, d_0)$. Let $\mathsf{tGC}_d \subset \mathsf{dfcGC}_d$ be the subspace spanned by directed trees of the form

$$(33) \qquad T_r = $$

that is, by genus zero directed graphs whose every vertex (except the special one called *the root vertex* and denoted by r) has precisely one outgoing edge; the root vertex has no outgoing edges. Note that T_r may consist solely of the root vertex, i.e. in general $\#V(T_r) \geqslant 1$.

The complex $E_0 \mathsf{dwGC}_d$ decomposes into a product

$$\prod_{\Gamma_{sk} \in \mathsf{dfcGC}_d'} \bigotimes_{r \in V(\Gamma_{sk})} C_r$$

where

$$C_r := \prod_{T_r \in \mathsf{tGC}_d} \prod_{\mathfrak{w}:V(T)\to\mathbb{N}_{\geqslant 1}} T_r^{\mathfrak{w}}$$

where the second product is taken over all weight functions $\mathfrak{w}$ satisfying the standard condition $\mathfrak{w}(v) \geqslant \max\{1, |v|_{in} - 1\}$ for every non-root vertex $v \neq r$ of a tree T, and

$$\mathfrak{w}(r) \geqslant \max\{1, |r|_{in}^{T} + |r|_{in}^{sk} - 1, |r|_{out}^{sk} - 1\}$$

where $|r|_{in}^{T}$ (resp., $|r|_{in}^{sk}$) counts the number of incoming edges to r in the tree T (resp., in the skeleton graph Γ_{sk}).

The induced differential is given by

(34)
$$d_0 T^{\mathfrak{w}} = \delta_0 T^{\mathfrak{w}} \pm \sum_{\substack{v \in V(\Gamma) \\ \mathfrak{w}(v) > \#In_v - 1}} \sum_{c \geqslant 1} \quad \overset{T^{\mathfrak{w}}}{\underset{\textcircled{c}}{\uparrow}}$$

where

- the first term $\delta_0 \Gamma^{\mathfrak{w}}$ is equal to the sum $\pm \sum_{v \in V(\Gamma)} \delta_{0,v} T^{\mathfrak{w}}$ over all vertices of T (including the root vertex r) with $\delta_{0,v} \Gamma^{\mathfrak{w}}$ obtained from $T^{\mathfrak{w}}$ by replacing the vertex v with weight, say, $a \in \mathbb{N}_{\geqslant 1}$ by the sum of graphs

$$\textcircled{a} \quad \longrightarrow \quad \sum_{a = a' + a''} \quad \overset{\textcircled{a'}}{\underset{\textcircled{a''}}{\uparrow}}$$

and taking one more sum over all possible way to reattach the dangling edges to the newly created vertices $\textcircled{a'}$ and $\textcircled{a''}$ in such a way that the condition (21) is respected and no new skeleton vertices appear.
- the second summand is a linear combination of trees obtained by attaching an in-vertex $\textcircled{c}\!\!\rightarrow$ to each vertex v of $T^{\mathfrak{w}}$ whose weight satisfies the strict inequality $\mathfrak{w}(v) > |v|_{in}^{T} - 1$ if $v \neq r$ and $\mathfrak{w}(r) > |v|_{in}^{T} + |v|_{in}^{sk} - 1$ if v is the root vertex r.

Each complex C_r has at least one cohomology class spanned by the root vertex assigned the following sum of weights,

(35)
$$O_r := \sum_{a \geqslant \mathfrak{s}_r^{sk}} \textcircled{a}$$

If the root vertex r has skeleton valency satisfying $|r|_{out}^{sk} \leqslant |r|_{in}^{sk}$ and $|r|_{in}^{sk} \geqslant 2$, then there is one more cohomology class

$$o_r := \textcircled{$\mathfrak{s}_r^{sk}$}$$

spanned by the root vertex equipped with the minimal possible weight, that is with the skeleton weight. Let us show that there are no other cohomology classes in the complexes C_r. Call a maximal connected subgraph of a tree $T_r^{\mathfrak{w}}$ a *string* if it consists of passing vertices and a univalent in-vertex (so the root vertex is never a part of some string); every string looks like an element (30) of the auxiliary complex C for some $n \geqslant 1$. If $\#V(T^{\mathfrak{w}}) \geqslant 2$, then $T^{\mathfrak{w}}$ contains at least one string. Call vertices of $T^{\mathfrak{w}}$ which belong to a string *stringy vertices* and the remaining vertices *core* ones. The set of core vertices is non-empty as it always contains the root vertex. Consider a filtration of C_r by the number of core vertices, and then a filtration by the total weight of the stringy vertices. The induced differential acts only on stringy vertices in exactly the same way as on the vertices of the auxiliary complex C (see (31) above) so that, by Lemma **3.3.1**, each string (if any) has length 1 and is spanned by the single in-vertex of weight 1. If the number of core vertices is greater than or equal to 2, then at least one core vertex contains at least two strings and hence does not contribute to the cohomology as $H^{\bullet}(\odot^{\geqslant 2} C) = 0$. Hence it remains to consider the case when the number of core vertices is equal to 1, i.e. the only core vertex is the root vertex. The next page of the spectral sequence by the total weight filtration of stringy vertices gives us a complex spanned by elements of two types — by the single root vertex (equipped with an arbitrary weight $a \geqslant \mathfrak{s}_r^{sk}$) and by the root vertex together with one univalent in-vertex of weight 1 attached,

$$\text{span} \left\langle \quad \textcircled{a} \quad , \quad \overset{\textcircled{b}}{\underset{\textcircled{1}}{\uparrow}} \quad \right\rangle$$

where
$$b \geqslant \begin{cases} \mathfrak{s}_r^{sk} & \text{if } |r|_{out}^{sk} > |r|_{in}^{sk} \text{ or } \left(|r|_{in}^{sk}=1, |r|_{out}^{sk} \leqslant 1\right) \\ \mathfrak{s}_r^{sk}+1 & \text{if } |r|_{out}^{sk} < |r|_{in}^{sk} \text{ or } |r|_{in}^{sk}=|r|_{out}^{sk} \geqslant 2 \end{cases}$$

The induced differential acts trivially on the second generator and it acts on the first generator as follows

$$\text{\textcircled{a}} \longrightarrow \begin{cases} -\;\text{\textcircled{a}}\!\leftarrow\!\text{\textcircled{1}} & \text{if } \left(|r|_{out}^{sk} > |r|_{in}^{sk} \text{ or } \left(|r|_{in}^{sk}=1, |r|_{out}^{sk} \leqslant 1\right)\right) \text{ and } a=\mathfrak{s}_r^{sk} \\ \text{\textcircled{a-1}}\!\leftarrow\!\text{\textcircled{1}}\;-\;\text{\textcircled{a}}\!\leftarrow\!\text{\textcircled{1}} & \text{if } |r|_{out}^{sk} > |r|_{in}^{sk} \text{ or } |r|_{in}^{sk}=|r|_{out}^{sk}=1 \ a>\mathfrak{s}_r^{sk} \\ 0 & \text{if } |r|_{out}^{sk} < |r|_{in}^{sk} \text{ or } |r|_{in}^{sk}=|r|_{out}^{sk} \geqslant 2 \text{ and } a=\mathfrak{s}_r^{sk} \\ -\;\text{\textcircled{a}}\!\leftarrow\!\text{\textcircled{1}} & \text{if } |r|_{out}^{sk} < |r|_{in}^{sk} \text{ or } |r|_{in}^{sk}=|r|_{out}^{sk} \geqslant 2 \text{ and } a=\mathfrak{s}_r^{sk}+1 \\ \text{\textcircled{a-1}}\!\leftarrow\!\text{\textcircled{1}}\;-\;\text{\textcircled{a}}\!\leftarrow\!\text{\textcircled{1}} & \text{if } |r|_{out}^{sk} < |r|_{in}^{sk} \text{ or } |r|_{in}^{sk}=|r|_{out}^{sk} \geqslant 2 \text{ and } a>\mathfrak{s}_r^{sk}+1 \end{cases}$$

It is now almost immediate to see that the cohomology is generated by the class O_r and, in the case when $|r|_{out}^{sk} \leqslant |r|_{in}^{sk}$ and $|r|_{in}^{sk} \geqslant 2$, by one extra the class o_r, i.e. that it is at most 2-dimensional.

We conclude that the next page $E_1 \mathsf{dwGC}_d = H^\bullet(E_0\mathsf{dwGC}_d, d_0)$ is generated by weighted graphs $\Gamma^{\mathfrak{w}}$ with no univalent in-vertices and with every vertex v either assigned the sum of weights as in (35) (we call such vertices the *hairy* ones and denote them by double circles $\text{\textcircled{\textcircled{}}}$) or, if $|v|_{out} \leqslant |v|_{in}$ and $|v|_{in} \geqslant 2$, the weight $\mathfrak{s}_v^{sk}$; we call the latter vertices the *bold* ones.

Step 3: Study of the complex $(E_1\mathsf{dwGC}_d, d_1)$. The induced differential d_1 is given by

$$(36) \qquad d_1\Gamma = \sum_{v\in V(\Gamma)} d_1^v\Gamma \pm \sum_{\substack{v \text{ is bold and} \\ |r|_{out}^{sk} < |r|_{in}^{sk}}} \overset{\text{\textcircled{\textcircled{}}}}{\underset{\Gamma}{\uparrow}} \pm \sum_{v \text{ is hairy}} \overset{\text{\textcircled{\textcircled{}}}}{\underset{\Gamma}{\uparrow}}$$

where the operator d_1^v splits a bold vertex v into a pair bold vertices as in (25) and a hairy vertex v into a pair of hairy vertices as follows

$$\text{\textcircled{\textcircled{}}} \longrightarrow \overset{\text{\textcircled{\textcircled{}}}}{\underset{\text{\textcircled{\textcircled{}}}}{\uparrow}}$$

Two other summands stand for the sums over attaching a new hairy univalent-out vertex to bold vertices v with $|r|_{out}^{sk} < |r|_{in}^{sk}$ and to hairy vertices.

Note that passing and univalent (out-)vertices (if any) in a graph Γ from $E_1\mathsf{dwGC}_d$ must be hairy.

Deleting from a graph $\Gamma \in E_1\mathsf{dwGC}_d^\star$ recursively all univalent (hairy) out-vertices, we obtain a graph Γ' called the *prime graph* of Γ (cf. Step 1 above). Consider next a filtration by the number of vertices in the prime graphs, and let $(E_{r,1}\mathsf{dwGC}_d, d_{r,1})$ be the associated spectral sequence. The associated graded complex $(E_{0,1}\mathsf{dwGC}_d, d_{0,1})$ is the tensor product of tree type complexes spanned by elements of the form (33) with arrows reversed on every edge, and with every vertex except the root being a hairy vertex. The root vertex r can be a hairy vertex or a bold vertex satisfying the conditions $|r|_{out}^{prime} < |r|_{in}^{prime}$ and $|r|_{in}^{prime} \geqslant 2$; here $|r|_{in}^{prime}$ and $|r|_{out}^{prime}$ stand for the valencies of the root vertex in the *prime graph* Γ' (not in the original graph Γ). An analysis similar to the one discussed above leads us easily to the conclusion that the next page

$$E_{1,1}\mathsf{dwGC}_d \simeq H^\bullet((E_{0,1}\mathsf{dwGC}_d, d_{0,1})$$

is spanned by graphs with no univalent vertices and with no bold vertices v such that $|v|_{in} \neq |v|_{out}$. The induced differential $d_{1,1}$ splits a bold vertex into the pair of bold vertices, and a hairy vertex into a pair of hairy vertices. This complex decomposes naturally into a direct sum,

$$E_{1,1}\mathsf{dwGC}_d = E_{1,1}\mathsf{dwGC}_d^{bold} \oplus E_{1,1}\mathsf{dwGC}_d^{hairy} \oplus E_{1,1}\mathsf{dwGC}_d^{mixed}$$

where

- the summand $E_{1,1}\mathsf{dwGC}_d^{bold}$ is spanned by graphs whose every vertex v is bold and hence satisfies $|v|_{in} = |v|_{out}$. This subcomplex is precisely the summand $\mathsf{dwGC}_d^{\overline{=}} \simeq \mathrm{Der}^=(\mathcal{H}oqpois_{c,d}^{\circ})$ in the decomposition (26).
- The summand $E_{1,1}\mathsf{dwGC}_d^{hairy}$ is exactly the image, $\mathcal{F}^{\circlearrowright}(\mathsf{dcGC}_d^{\geqslant 2})$, of the injective morphism of complexes (28).
- The summand $E_{1,1}\mathsf{dwGC}_d^{mixed}$ is spanned by graphs with at least one vertex bold and at least one hairy vertex.

Hence to complete the proof of the Theorem it remains to show that the subcomplex $E_{1,1}\mathsf{dwGC}_d^{mixed}$ is acyclic. For this let us call a maximal connected path of consisting of passing vertices (if any) a *weighted edge* (recall that every passing vertex must be a hairy vertex). One can understand such a weighted edge as an ordinary edge together with a positive number (its *weight*) assigned to it,

$$\xrightarrow{k}$$

where k counts the number of passing vertices, e.g.

$$\xrightarrow{2} := \ \longrightarrow\!\!\!\odot\!\!\longrightarrow\!\!\!\odot\!\!\longrightarrow$$

We shall allow next weighted edges equipped with the zero weight by identifying them with ordinary edges,

$$\xrightarrow{0} := \longrightarrow \ .$$

Any graph Γ from $E_1\mathsf{dwGC}_d^{mixed}$ can be understood as a graph $\Gamma^{reduced}$ with no passing vertices but with weighted edges. Every such a graph containing at least one weighted wedge connecting bold and hairy vertices; the number of such edges is preserved under the differential $d_{1,1}$ so that we can assume without loss of generality that such *mixed* edges are distinguished, e.g. marked by positive integers starting with 1. Consider a filtration by the number of vertices in $\Gamma^{reduced}$ plus the total weight of mixed edges with labels $\geqslant 2$. The induced differential acts only on the mixed edge labelled by 1 increasing its weight from $k \geqslant 0$ to $k+1$. Hence the complex $E_1\mathsf{dwGC}_d^{mixed}$ is acyclic indeed, and the Theorem is proven. $\qquad\square$

The above Theorem together with the isomorphism

$$\mathrm{Der}(\mathcal{H}oqpois_{c,d}^{\,\circ}) = \odot^{\bullet\geqslant 1}(\mathsf{dwGC}_{c+d+1}[-1-c-d])[1+c+d]$$

immediately imply Theorem **1.2.1** in the Introduction.

An immediate corollary to Theorem **1.2.1** is that the canonical morphism of dg Lie algebras

$$\mathrm{Der}(\mathcal{H}olieb_{c,d}^{\star\circ})) \longrightarrow \mathrm{Der}(\mathcal{H}oqpois_{c,d}^{\star\circ})$$

is a quasi-isomorphism.

Considering an oriented version of the weighted graph complex dwGC_d and Using arguments almost identical to the ones used in the proof of Theorem **3.4** leads us to the the following result about the derivation complex of the ordinary properad of $\mathbb{Z}$-graded Poisson structures.

3.4.1. Theorem. *There is a canonical morphism of dg Lie algebras,*

$$\mathsf{OGC}_{c+d+1}^{\geqslant 2} \longrightarrow \mathrm{Der}(\mathcal{H}oqpois_{c,d})$$

which is a quasi-isomorphism up to one rescaling class r. In particular, there is an isomorphism of Lie algebras

$$H^0(\mathrm{Der}(\mathcal{H}oqpois_{c,d})) = \mathbb{K}$$

that is, the only homotopy non-trivial automorphism of the properad $\mathcal{H}oqpois_{c,d}$ is given by the rescaling of the generators as follows

$$\longrightarrow \lambda^{n-1} \qquad\qquad , \quad \forall \lambda \in \mathbb{K} \setminus 0, \ \forall\, n \geqslant 2.$$

The second claim follows from the well-known equality of cohomology groups [W2]

$$H^\bullet(\mathsf{OGC}_2) = H^\bullet(\mathsf{GC}_1^{\geqslant 2})$$

and the fact that $H^0(\mathsf{GC}_1^{\geqslant 2}) = 0$.

4. Classification of universal quantizations of quadratic Poisson structures

4.1. Kontsevich formality map as a morphism of dg operads. Let V be a finite-dimensional $\mathbb{Z}$-graded vector space, and $V^* := \mathrm{Hom}(V, \mathbb{K})$ be its dial vector space. Let us view V^* as a pointed manifold and view the symmetric tensor algebra

$$\mathcal{O}_V = \bigoplus_{k \geqslant 0} \mathcal{O}_V^k, \quad \mathcal{O}_V^k := \odot^k V,$$

as the algebra of formal smooth functions on V^*. Deformations of the standard graded commutative product in $\mathcal{O}_V$ is controlled by the standard dg Lie algebra, the so called *Hochschild complex*,

$$\mathsf{Hoch}(\mathcal{O}_V, \mathcal{O}_V) := \bigoplus_{n \geqslant 0} \mathsf{Hoch}^n(\mathcal{O}_V, \mathcal{O}_V), \quad \mathsf{Hoch}^n(\mathcal{O}_V, \mathcal{O}_V) := \mathrm{Hom}(\otimes^n \mathcal{O}_V, \mathcal{O}_V)[1 - n]$$

of polydifferential operators (see [K2] for the explicit formulae of the differential in $\mathsf{Hoch}(\mathcal{O}_V, \mathcal{O}_V)$ and the Lie brackets).

The space of tangent vector fields on V^* can be identified with the Lie algebra of derivations of $\mathcal{O}_V$ which are uniquely determined by its values on the generators, that is by the values on the elements of V,

$$\mathrm{Der}(\mathcal{O}_V) \simeq \mathrm{Hom}(V, \odot^\bullet V)$$

The Lie algebra of polyvector fields on V^* can be identified as a vector space with

$$\mathsf{Poly}(V) := \wedge^\bullet_{\mathcal{O}_V} \mathrm{Der}(\mathcal{O}_V) \simeq \bigoplus_{k \geqslant 0} \mathsf{Poly}^k(V), \quad \mathsf{Poly}^k(V) := \bigoplus_l \mathrm{Hom}(\wedge^l V, \odot^k V)[1 - l]$$

The explicit formula for the Lie bracket in $\mathsf{Poly}(V)$ can be found, for example, in [K2]. The differential in $\mathsf{Poly}(V)$ is assumed to be zero.

Let $\hbar$ be a formal parameter (of coh0mological degree zero), and let, for a vector space A, the symbol $A[[\hbar]]$ stand for the vector space of formal power series in $\hbar$ with coefficients in A. Maxim Kontsevich formality map is a quasi-isomorphism of dg Lie algebras

$$(37) \qquad \mathsf{F}^K : \mathsf{Poly}(V)[[\hbar]] \longrightarrow \mathsf{Hoch}(\mathcal{O}_V, \mathcal{O}_V)[[\hbar]],$$

An explicit formula for such a quasi-isomorphism for $\mathbb{K} = \mathbb{R}$ was constructed in [K2] (implying its existence over any field $\mathbb{K}$ of characteristic zero). That formula gives a *universal* formality morphism which is applicable to an arbitrary finite-dimensional grade vector space V, and hence can be reformulated without any reverence to any particular V. Such a reformulation was given in [AM] as a morphism of dg operads

$$(38) \qquad \mathcal{F} : c\mathcal{A}ss_\infty \longrightarrow \mathcal{O}(\mathcal{H}olieb_{0,1}^{\star\circlearrowleft})$$

satisfying a certain non-triviality condition. Here $c\mathcal{A}ss_\infty$ is a dg operad of *curved A_∞-algebras* which is, by definition, is the free operad generated by the $\mathbb{S}$-module

$$(39) \qquad E(n) := \mathbb{K}[\mathbb{S}_n][n - 2] = \mathrm{span}\left(\underset{\sigma(1)\ \sigma(2)\quad\ \sigma(n)}{\overbrace{\bigwedge\cdots}} \right)_{\sigma \in \mathbb{S}_n}, \quad \forall\, n \geqslant 0$$

where $\mathbb{K}[\mathbb{S}_n]$ is the group algebra of the permutation group. It has a canonical basis is given by permutations $\sigma \in \mathbb{S}_n$ which we identify with planar corollas with one output leg and n input legs (every such a corolla stands, roughly speaking, for a generic lineal map $\otimes^n V \longrightarrow V[2 - n]$). The differential in $c\mathcal{A}ss_\infty$ is given on the generators by the formula
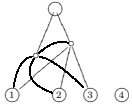
$$\delta \underset{1\ \cdots\ n}{\bigwedge} = \sum_{k=0}^{n} \sum_{l=0}^{n-k} (-1)^{k+l(n-k-l)+1} \underset{\substack{1\ \cdots\ k \quad (k+l+1)\ \cdots\ n \\ k+1\ \cdots\ k+l}}{\bigwedge} .$$

The properad $\mathcal{H}olieb_{0,1}^{\star\circlearrowleft}$ has been defined in §2. The symbol $\mathcal{O}$ stands for a polydifferential functor

$$\mathcal{O} : \text{Category of dg (wheeled) props} \longrightarrow \text{Category of dg operads}$$

introduced in [MW2]. Elements of $\mathcal{O}(\mathcal{H}olieb_{0,1}^{\star\circlearrowleft})$ are generated by graphs from $\mathcal{H}olieb_{0,1}^{\star\circlearrowleft}$ whose out legs are symmetrized and attached to the new white out-vertex, while in-legs are partitions into disjoint union of some subsets, legs in each subset are symmetrized and are attached to a new in-vertex which is labelled

by an integer. For example [AM], For example, an element $e = $ $\in \widehat{\mathcal{H}olieb}_{0,1}^{\star\circ}(2,6)$ can generate the

following element in the operad $\mathcal{O}(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circ})(4)$.

It was shown in [AM] that for any universal formality morphism

$$\mathcal{F} : c\mathcal{A}ss_\infty \longrightarrow \mathcal{O}(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circ})$$

there is a canonically associated morphism of complexes

$$f_\mathcal{F} : \mathsf{FGC}_2^{\geqslant 2} \longrightarrow \mathsf{Def}\left(c\mathcal{A}ss_\infty \xrightarrow{\mathcal{F}} \mathcal{O}(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circ})\right)[1]$$

which is a quasi-isomorphism, where the symbol Def stands for the standard deformation complex of the particular morphism $\mathcal{F}$ constructed along the recipe given in [MV]. This results gives us (almost immediately) the classification of all universal formality morphisms up to homotopy equivalence — the set of such morphisms can be identified with set of Drinfeld associators. In the next subsection we obtain a similar conclusion for the set of homotopy classes of *homogeneous* formality morphisms.

4.2. Kontsevich formality map applied to $\mathbb{Z}$-graded quadratic Poisson structures. Let

$$\mathsf{Hoch}_{(0)}(\mathcal{O}_V, \mathcal{O}_V) = \bigoplus_{n \geqslant 0} \mathsf{Hoch}_{(0)}^n(\mathcal{O}_V, \mathcal{O}_V), \quad \mathsf{Hoch}_{(0)}^n(\mathcal{O}_V, \mathcal{O}_V) := \mathsf{Hom}(\mathcal{O}_V^{k_1} \otimes \ldots \otimes \mathcal{O}_V^{k_n}, \mathcal{O}_V^{k_1+\ldots+k_n})[1-n]$$

be the subspace of $\mathsf{Hoch}(\mathcal{O}_V, \mathcal{O}_V)$ spanned by homogeneous polydifferential operators, that is, the ones which preserve the total polynomial degree of formal functions. It is a dg Lie subalgebra.

Similarly, let

$$\mathsf{Poly}_{(0)}(V) := \wedge \mathsf{Der}_{(0)}(\mathcal{O}_V) \simeq \bigoplus_k \mathsf{Hom}(\wedge^k V, \odot^k V)[1-k]$$

be the Lie subalgebra of $\mathsf{Poly}(V)$ spanned by homogeneous polyvector fields. The Maurer-Cartan elements of the Lie algebra $\mathsf{Poly}_{(0)}(V)$ are called $\mathbb{Z}$-*graded quadratic Poisson structures*. They have a decomposition,

$$(40) \qquad \pi = \sum_{n=0}^\infty \pi_n, \qquad \pi_n \in Hom(\wedge^n V, \odot^n V)[2-n]$$

If the vector space V is concentrated in degree zero, then only the term with $n = 2$ survives giving the notion of ordinary quadratic Poisson structure. The term π_0 is just a constant playing no role in the deformation quantization theory; hence it can be ignored.

Using properties P1-P5 (see §7 in [K2]) of the Kontsevich formality $\mathcal{F}^K$, it is easy to see that the restriction of the map $\mathsf{F}^K : \mathsf{Poly}(V)[[\hbar]] \longrightarrow \mathsf{Hoch}(\mathcal{O}_V, \mathcal{O}_V)[[\hbar]]$ to the subcomplex

$$\mathsf{Poly}(V)_{(0)}^{\geqslant 1} := \bigoplus_{k \geqslant 1} \mathsf{Poly}(V)_{(0)}^k \subset \mathsf{Poly}(V)$$

takes values in

$$\mathsf{Hoch}_{(0)}^{\geqslant 1}(\mathcal{O}_V, \mathcal{O}_V)[[\hbar]] = \bigoplus_{k \geqslant 1} \mathsf{Hoch}_{(0)}^k(\mathcal{O}_V, \mathcal{O}_V)[[\hbar]].$$

Put another way, the Kontsevich map gives us a quasi-isomorphism of reduced complexes,

$$(41) \qquad \mathsf{F}^K : \mathsf{Poly}(V)_{(0)}^{\geqslant 1} \longrightarrow \mathsf{Hoch}_{(0)}^{\geqslant 1}(\mathcal{O}_V, \mathcal{O}_V)[[\hbar]]$$

Note that such a reduction is not possible in the case (37), with the summand $mathsf Hoch^0(\mathcal{O}_V, \mathcal{O}_V)[[\hbar]]$ in the r.h.s. playing a key role of the classification of homotopy inequivalent formality maps [Do, AM]. Let us call any strongly homotopy quasi-isomorphism of Lie algebras as in (41) a *homogeneous* formality map.

Out purpose is to classify all *universal* homotopy inequivalent homogeneous formalities. To give a precise definition of what *universal* means to shall use again the theory of operads. Let $\mathcal{A}ss_\infty$ be the standard

operad of (flat) strongly homotopy associative algebras. It is a free operad generators by the $\mathbb{S}$ module (39) with $n \geqslant 2$ only. The canonical epimorphism of dg props

$$\widehat{\mathcal{H}olieb}_{0,1}^{\star\circ} \longrightarrow \widehat{\mathcal{H}oqpois}_{0,1}^{\star\circ}$$

induces an epimorphism of dg operads

$$\mathcal{O}\left(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circ}\right) \longrightarrow \mathcal{O}\left(\widehat{\mathcal{H}oqpois}_{0,1}^{\star\circ}\right).$$

Let $\mathcal{O}^{\geqslant 1}\left(\widehat{\mathcal{H}oqpois}_{0,1}^{\star\circ}\right)$ be a subspace of $\mathcal{O}\left(\widehat{\mathcal{H}oqpois}_{0,1}^{\star\circ}\right)$ spanned by graphs with at least one incoming white vertex. Then we have an obvious direct sum decomposition of dg operads

$$\mathcal{O}\left(\widehat{\mathcal{H}oqpois}_{0,1}^{\star\circ}\right) = \mathsf{dcGC}_2^{=} \oplus \mathcal{O}^{\geqslant 1}\left(\widehat{\mathcal{H}oqpois}_{0,1}^{\star\circ}\right)$$

The Kontsevich formality map (41) gives us a morphism of dg operads

(42)
$$\mathcal{F}^K : \mathcal{A}ss_\infty \longrightarrow \mathcal{O}^{\geqslant 1}\left(\widehat{\mathcal{H}oqpois}_{0,1}^{\star\circ}\right)$$

satisfying the following boundary conditions, $O(2)$)

(43)
$$\mathcal{F}^K \left(\begin{array}{c} \text{(graph)} \\ 1 \ \cdots \ n \end{array} \right) = \begin{cases} \text{(graph)} + \sum_{p \geqslant 0} \frac{1}{2} \text{(graph)} + O(2) & \text{if } n = 2 \\[2em] \frac{1}{n!} \text{(graph)} + O(2) & \text{for } n \geqslant 3 \end{cases}$$

where $O(2)$ stand for the terms with the number of internal vertices $\geqslant 2$. Any morphism $\mathcal{F}_0$ of dg operads as in (42) satisfying the above boundary condition (which guarantees its non-triviality) is called a *universal homogeneous formality map*.

4.3. **Theorem (Classification of homogeneous formality maps).** *For any homogeneous formality morphism*

$$\mathcal{F}_0 : \mathcal{A}ss_\infty \longrightarrow \mathcal{O}^{\geqslant 1}(\widehat{\mathcal{H}oqpois}_{0,1}^{\star\circ})$$

there is a canonically associated morphism of complexes

$$f_0 : \mathsf{FGC}_2^{\geqslant 2} \longrightarrow \mathsf{Def}\left(\mathcal{A}ss_\infty \xrightarrow{\mathcal{F}_0} \mathcal{O}^{\geqslant 1}(\widehat{\mathcal{H}oqpois}_{0,1}^{\star\circ})\right)[1]$$

which is a quasi-isomorphism.

Proof. The proof of this Theorem is a straightforward adoption of the arguments used in the proof of Proposition 5.4.1 in [MW3], and is based essentially on the contractibility of the permutahedra polytopes. Let us first explain the structure of the induced morphism f_0. Any derivation of the dg wheeled prop $\widehat{\mathcal{H}oqpois}_{0,1}^{\star\circ}$ gives us an infinitesimal deformation of the identity automorphism of $\widehat{\mathcal{H}oqpois}_{0,1}^{\star\circ}$ which in turn induces an infinitesimal deformation of the identity automorphism of the dg operad $\mathcal{O}^{\geqslant 1}(\widehat{\mathcal{H}oqpois}_{0,1}^{\star\circ})$ which in its turn induces an infinitesimal deformation of the homogeneous formality map $\mathcal{F}$ via the action on the r.h.s. Therefore we have a sequence of canonical morphisms of complexes

$$\mathrm{Der}(\widehat{\mathcal{H}oqpois}_{0,1}^{\star\circ}) \xrightarrow{\simeq} \mathsf{Def}\left(\widehat{\mathcal{H}oqpois}_{0,1}^{\star\circ} \xrightarrow{\mathrm{Id}} \widehat{\mathcal{H}olieb}_{0,1}^{\star\circ}\right)[1] \xrightarrow{\simeq} \mathsf{Def}\left(\mathcal{O}^{\geqslant 1}(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circ}) \xrightarrow{\mathrm{Id}} \mathcal{O}^{\geqslant 1}(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circ})\right)[1]$$

$$\downarrow$$

$$\mathsf{Def}\left(\mathcal{A}ss_\infty \xrightarrow{\mathcal{F}_0} \mathcal{O}^{\geqslant 1}(\widehat{\mathcal{H}oqpois}_{0,1}^{\star\circ})\right)[1]$$

The first arrow above is a quasi-isomorphism because both complexes are identical to each up to degree shift. The second arrow is a quasi-isomorphism because the functor $\mathcal{O}$ is exact [MW2]. We shall show below that the third arrow is also a quasi-isomorphism. Combining this latest claim with the quasi-isomorphism (3) proves the Theorem.

Thus to prove the Theorem it is enough to show that the composition

(44)
$$c : \mathrm{Der}(\widehat{\mathcal{H}oqpois}_{0,1}^{\star\circ}) \longrightarrow \mathsf{Def}\left(\mathcal{A}ss_\infty \xrightarrow{\mathcal{F}_0} \mathcal{O}^{\geqslant 1}(\widehat{\mathcal{H}oqpois}_{0,1}^{\star\circ})\right)[1]$$

Both complexes in (44) admits filtrations by the number of edges which is preserved by the map c. Hence that map induces a morphism of the associated spectral sequences,

$$c^r : (\mathcal{E}_r \mathrm{Der}(\widehat{\mathcal{Hoqpois}}_{0,1}^{\star\circ}), d_r) \longrightarrow \left(\mathcal{E}_r \mathsf{Def} \left(\mathcal{Ass}_\infty \xrightarrow{\mathcal{F}} \mathcal{O}^{\geqslant 1}(\widehat{\mathcal{Hoqpois}}_{0,1}^{\star\circ}) \right) [1], \delta_r \right).$$

The induced differential d_0 on the initial page of the spectral sequence of the l.h.s. is trivial, $d_0 = 0$. The induced differential δ_0 on the initial page of the spectral sequence of the r.h.s. is not trivial and is determined by the following summand in $\mathcal{F}_0$ (see the boundary condition (43) for $\mathcal{F}_0$),

$$\circ$$

$$①\quad②$$

Hence the differential δ_0 acts only on big input white vertices of graphs by splitting each such big white vertex ⓥ into two big white vertices ⓥ' ⓥ" and redistributing all edges (if any) attached to v in all possible ways among the new vertices v' and v''. The cohomology

$$\mathcal{E}_1 \mathsf{Def} \left(c\mathcal{Ass}_\infty \xrightarrow{\mathcal{F}} \mathcal{O}(\widehat{\mathcal{Holieb}}_{0,1}^{\star\circ}) \right) [1] = H \left(\mathcal{E}_0 \mathsf{Def} \left(c\mathcal{Ass}_\infty \xrightarrow{\mathcal{F}} \mathcal{O}(\widehat{\mathcal{Holieb}}_{0,1}^{\star\circ}) \right) [1], \delta_0 \right)$$

is spanned by graphs all of whose white vertices are precisely univalent and skew symmetrized (see, e.g., Theorem 3.2.4 in [Me4] where this result is obtained from the cell complexes of permutahedra, or Appendix A in [W1] for another purely algebraic argument) and hence is isomorphic (after erasing these no more needed big white vertices) to $\mathrm{Der}(\mathcal{Holieb}_{0,1}^{\star\circ})$ as a graded vector space. The boundary condition (43) says that the induced differential δ_1 in the complex $\mathcal{E}_1 \mathsf{Def} \left(\mathcal{Ass}_\infty \xrightarrow{\mathcal{F}} \mathcal{O}^{\geqslant 1}(\widehat{\mathcal{Holieb}}_{0,1}^{\star\circ}) \right) [1]$ agrees precisely with the induced differential d_1 in $\mathcal{E}_1 \mathrm{Der}(\widehat{\mathcal{Hoqpois}}_{0,1}^{\star\circ}) = \mathrm{Der}(\widehat{\mathcal{Hoqpois}}_{0,1}^{\star\circ})$ so that the morphism of the next pages of the spectral sequences,

$$c_\mathcal{F}^1 : (\mathcal{E}_1 \mathrm{Der}(\widehat{\mathcal{Hoqpois}}_{0,1}^{\star\circ}), d_1) \longrightarrow \left(\mathcal{E}_1 \mathsf{Def}(\mathcal{Ass}_\infty \xrightarrow{\mathcal{F}} \mathcal{O}^{\geqslant 1}(\widehat{\mathcal{Hoqpois}}_{0,1}^{\star\circ}), \delta_1 \right)$$

is simply an isomorphism. By the comparison theorem, the morphism $c_\mathcal{F}$ is a quasi-isomorphism. □

4.4. Proof of Theorem 1.3.1. . The above theorem implies

$$H^{i+1} \left(\mathsf{Def} \left(\mathcal{Ass}_\infty \xrightarrow{\mathcal{F}_0} \mathcal{O}^{\geqslant 1}(\widehat{\mathcal{Hoqpois}}_{0,1}^{\star\circ}) \right) \right) = H^i(\mathsf{fGC}_2^{\geqslant 2}), \quad \forall\, i \in \mathbb{Z}.$$

In particular,

$$H^1 \left(\mathsf{Def} \left(\mathcal{Ass}_\infty \xrightarrow{\mathcal{F}} \mathcal{O}(\widehat{\mathcal{Hoqpois}}_{0,1}^{\star\circ}) \right) \right) = H^0(\mathsf{fGC}_2^{\geqslant 2}) = \mathfrak{grt}$$

To complete the proof of Theorem **1.3.1** it remains to show that every infinitesimal deformation $\eta \in H^1 \left(\mathsf{Def} \left(\mathcal{Ass}_\infty \xrightarrow{\mathcal{F}_0} \mathcal{O}(\widehat{\mathcal{Holieb}}_{0,1}^{\star\circ}) \right) \right)$ of any given homogeneous formality map $\mathcal{F}_0$ exponentiates to a genuine homogeneous formality map $\mathcal{F}_0^\eta$, and then apply Lemma 3 from Thomas Willwacher paper [W3]. The argument is standard and is based on the remarkable fact that the two dg Lie algebras

$$\mathrm{Der}(\widehat{\mathcal{Hoqpois}}_{0,1}^{\star\circ}) \ textand \ \mathsf{Def}(\widehat{\mathcal{Hoqpois}}_{c,d}^{\star\circ} \xrightarrow{\mathrm{Id}} \widehat{\mathcal{Hoqpois}}_{c,d}^{\star\circ})$$

are identical – after degree shift — as complexes

$$\mathrm{Der}(\widehat{\mathcal{Hoqpois}}_{0,1}^{\star\circ}) = \mathsf{Def}(\widehat{\mathcal{Hoqpois}}_{c,d}^{\star\circ} \xrightarrow{\mathrm{Id}} \widehat{\mathcal{Hoqpois}}_{c,d}^{\star\circ})[1]$$

but have really different Lie algebra structures (even of different degrees). The element η (more precisely its arbitrary lift to a cycle) has degree zero when understood as an element of the dg Lie algebra $\mathrm{Der}(\widehat{\mathcal{Hoqpois}}_{0,1}^{\star\circ})$. The latter is positively graded so it makes sense to consider its exponent $exp(\eta)$ as an element of the group $Aut(\widehat{\mathcal{Hoqpois}}_{0,1}^{\star\circ})$. Hence $exp(\eta)$ gives us a non-trivial Maurer-Cartan element of the second dg Lie algebra

$$\mathsf{Def}(\widehat{\mathcal{Hoqpois}}_{c,d}^{\star\circ} \xrightarrow{\mathrm{Id}} \widehat{\mathcal{Hoqpois}}_{c,d}^{\star\circ})$$

which is mapped into a non-trivial Maurer-Cartan element of the dg Lie algebra

$$\mathsf{Def} \left(\mathcal{Ass}_\infty \xrightarrow{\mathcal{F}} \mathcal{O}(\widehat{\mathcal{Hoqpois}}_{0,1}^{\star\circ}) \right)$$

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giving is the required exponentiation of the infinitesimal deformation η of $\mathcal{F}_0$ to a genuine formality morphism

$$\mathcal{F}_0^\eta : \mathcal{A}ss_\infty \longrightarrow \mathcal{O}(\widehat{\mathcal{H}oqpois}_{0,1}^{\star\circ})$$

satisfying the boundary condition (43) and which differs at the infinitesimal level from $\mathcal{F}_0$ precisely by the cycle η. The proof is completed.

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