GRAVITY PROP AND MODULI SPACES $\mathcal{M}_{g,n}$

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ABSTRACT. Let $\mathcal{M}_{g,m+n}$ be the moduli space of algebraic curves of genus g with $m+n$ marked points decomposed into the disjoint union of two sets of cardinalities m and n, and $H_c^\bullet(\mathcal{M}_{m+n})$ its compactly supported cohomology group. We prove that the collection of $\mathbb{S}_m^{op} \times \mathbb{S}_n$-modules

$$\left\{H_c^{\bullet-m}(\mathcal{M}_{g,m+n})\right\}_{m\geq 1, n\geq 0, 2g+(m+n)\geq 3} =: \mathcal{GR}av,$$

has the structure of a *properad* (called the *gravity properad*) such that it contains the (degree shifted) E. Getzler's *gravity operad* as the sub-collection $\{H_c^{\bullet-1}(\mathcal{M}_{0,1+n})\}_{n\geq 2}$. Moreover, we prove that the generators of the 1-dimensional cohomology groups $H_c^{\bullet-1}(\mathcal{M}_{0,1+2})$, $H_c^{\bullet-2}(\mathcal{M}_{0,2+1})$ and $H_c^{\bullet-3}(\mathcal{M}_{0,3+0})$ satisfy with respect to this properadic structure the relations of the (degree shifted) quasi-Lie bialgebra, a fact making the totality of cohomology groups

$$\prod_{\substack{g,n\geq 0, m\geq 1 \\ 2g+n+m\geq 3}} H_c^\bullet(\mathcal{M}_{g,m+n}) \otimes_{\mathbb{S}_m^{op}\times\mathbb{S}_n} (sgn_m \otimes \mathbb{1}_n)$$

into a complex with the differential fully determined by the just mentioned three cohomology classes (which should be understood in this context as a hyperbolic sphere S^2 with (i) one geodesic boundary and two cusps, (ii) two geodesic boundaries and one cusp, and (iii) three geodesic boundaries, respectively). It is proven that this complex contains infinitely many cohomology classes, all coming from a morphism from M. Kontsevich's odd graph complex GC_{-1}.

The prop structure in $\mathcal{GR}av$ is established with the help of T. Willwacher's twisting endofunctor tw (in the category of properads under the operad of Lie algebras) applied to the properad of ribbon graphs $\mathcal{RG}ra_d$ introduced earlier by T. Willwacher and the author. As a family of complexes, the twisted properad $\mathsf{tw}\mathcal{RG}ra_d = \{\mathsf{tw}\mathcal{RG}ra_d(m,n)\}_{m\geq 1, n\geq 0}$ can in turn be identified with the cell complexes of K. Costello's family of moduli spaces, $\{\mathcal{D}_{g,m,0,n}\}_{m\geq 1, n\geq 0}$ of nodal disks with m marked boundaries and n internal marked points (such that each disk contains at most one internal marked point) making the latter $\mathbb{S}$-bimodule into a dg properad called the *chain gravity properad*. According to K. Costello, $\mathcal{D}_{g,m,0,n}$ is homotopy equivalent to $\mathcal{M}_{g,m+n}$ so that the properadic structure on the above collection $\mathcal{GR}av$ of cohomology groups follows immediately from the explicit purely combinatorial properadic structure in $\mathsf{tw}\mathcal{RG}ra_d$ and K. Costello's homotopy equivalence.

CONTENTS

1. Introduction

Let $\mathcal{M}_{g,n+m}$ be the moduli space of algebraic curves of genus g with $m + n$ marked points decomposed into the disjoint union of two sets of cardinalities m and n, and let $H_c^\bullet(\mathcal{M}_{g,m+n}) \simeq H_{\bullet+6-6g-2(m+n)}(\mathcal{M}_{g,n})$ be its compactly supported cohomology groups. We prove that the collection of $\mathbb{S}_m^{op} \times \mathbb{S}_n$-modules,

$$(1) \qquad \left\{ H_c^{\bullet-m}(\mathcal{M}_{g,m+n}) \right\}_{m \geq 1, n \geq 0, 2g+(m+n) \geq 3} =: \mathcal{GR}av,$$

has the structure of a *properad* (called the *gravity properad*) such that

(i) $\mathcal{GR}av$ contains as a sub-properad the (degree shifted) Ezra Getzler's *gravity operad* [Ge] as the sub-collection of cohomology groups of moduli spaces of genus zero algebraic curves,

$$\{ H_c^{\bullet-1}(\mathcal{M}_{0,1+n}) \}_{n \geq 2} \subset \mathcal{GR}av,$$

(ii) $\mathcal{GR}av$ comes equipped with a morphism from the properad of (degree shifted) Lie quasi-bialgebras[1]

$$(2) \qquad
\begin{array}{rcl}
j: \quad q\mathcal{L}ieb_{-1,0} & \longrightarrow & \mathcal{GR}av \\[6pt]
& \longrightarrow & 1 \in H_c^{\bullet-1}(\mathcal{M}_{0,1+2}) \simeq \mathbb{1}_2[-1] \subset \mathcal{GR}av(1,2) \\[6pt]
& \longrightarrow & 1 \in H_c^{\bullet-2}(\mathcal{M}_{0,2+1}) \simeq sgn_2[-2] \subset \mathcal{GR}av(2,1) \\[6pt]
& \longrightarrow & 1 \in H_c^{\bullet-3}(\mathcal{M}_{0,3+0}) \simeq sgn_3[-3] \subset \mathcal{GR}av(3,0)
\end{array}$$

The images on the r.h.s. should be understood geometrically as hyperbolic spheres S^2 with (i) one geodesic boundary and two cusps, (ii) two geodesic boundaries and one cusp, (iii) and three geodesic boundaries, respectively.

The existence of a prop structure on the family of cohomology groups (1) is established with the help of (a) Thomas Willwacher's twisting endofunctor tw in the category of properads under the operad of Lie algebras [W1] which is applied in this paper to the properad of ribbon graphs $\mathcal{RG}ra_d$ introduced earlier in [MW1] by Thomas Willwacher and the author,

$$\mathsf{tw}: \mathcal{RG}ra_d \longrightarrow \mathsf{tw}\mathcal{RG}ra_d = \{\mathsf{tw}\mathcal{RG}ra_d(m,n)\}_{m \geq 1, n \geq 0},$$

and (b) Kevin Costello's family of moduli spaces $\{\mathcal{D}_{g,m,0,n}\}_{m \geq 1, n \geq 0}$ of nodal disks with m marked boundaries and n internal marked points (such that each disk contains at most one internal marked point) [C1, C2].

Every $\mathbb{S}_m^{op} \times \mathbb{S}_n$-module $\mathsf{tw}\mathcal{RG}ra_d(m,n)$ of the dg properad $\mathsf{tw}\mathcal{RG}ra_d = \{\mathsf{tw}\mathcal{RG}ra_d(m,n)\}$ is generated by oriented ribbon graphs with m labelled boundaries, and with vertices of two types — n labelled white vertices and any number of black (unlabelled) vertices which are assigned the cohomological degree d. The differential and the properadic structure in $\mathsf{tw}\mathcal{RG}ra_d$ are of purely combinatorial nature and can be given by explicit formulae (see §3). Though *orientations* of ribbon graphs depend much on the parity of d we show that, up to a degree shift, all the twisted dg prop(erad)s $\mathsf{tw}\mathcal{RG}ra_d$ for various d are isomorphic to each other so that it is enough to study the case $d = 0$. The dg properad $\mathsf{tw}\mathcal{RG}ra_0$ contains a dg sub-properad $\mathsf{tw}\mathcal{RG}ra_0^{\geq 3}$ spanned by ribbon graphs with all *black* vertices at least trivalent. The inclusion

$$\mathsf{tw}\mathcal{RG}ra_0^{\geq 3} \longrightarrow \mathsf{tw}\mathcal{RG}ra_0$$

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[1]The properad controlling degree shifted quasi-Lie bialgebras with Lie bracket of degree $1 - d$ and the quasi-Lie cobracket of degree $1 - c$ is denoted by $q\mathcal{L}ieb_{c,d}$.

is a quasi-isomorphism modulo the family of polytope-like cohomology classes (for a precise formulation see Proposition **3.5.1**). The main problem at this point is to compute the associated cohomology properad $H^\bullet(\mathsf{tw}\mathcal{R}\mathcal{G}ra_0^{\geq 3})$ which is solved in §3 with the help of the remarkable Kevin Costello's homotopy equivalence [C1, C2] of moduli spaces.

THEOREM A. $H^\bullet(\mathsf{tw}\mathcal{R}\mathcal{G}ra_0^{\geq 3}) = \mathcal{G}\mathcal{R}av.$

This result implies the above claim about the canonical properadic structure on the above family of cohomology groups $H_c^\bullet(\mathcal{M}_{g,m+n})$. Its proof is relatively short and is based on the observation (see §3 for full details) that, as a family of complexes, the twisted properad $\mathsf{tw}\mathcal{R}\mathcal{G}ra_0^{\geq 3} = \{\mathsf{tw}\mathcal{R}\mathcal{G}ra_0^{\geq 3}(m,n)\}_{m\geq 1, n\geq 0}$ can be identified with the cell complexes of the family of moduli spaces $\{\mathcal{D}_{g,m,0,n}\}_{m\geq 1, n\geq 0}$ described explicitly in [C2]. This observation gives us an explicit properadic structure on the $\mathbb{S}$-bimodule

$$\{\mathcal{C}hains(\mathcal{D}_{g,m,0,n})\}_{m\geq 1, n\geq 0, 2g+m_n\geq 3}$$

which is called the *chain gravity properad*. It contains as a suboperad the cell complexes of the topological $\mathbb{S}$-bimodule $\{\mathcal{D}_{0,1,0,n}\}$ whose cohomology can in turn be identified, using the results of [Wa], with the gravity operad $\{H_{\bullet-1}(\mathcal{M}_{0,1+n})\}_{n\geq 2}$ first introduced in [Ge].

According to the results [C1, C2] by Kevin Costello and the uniformization theorem, the moduli spaces $\mathcal{D}_{g,m,0,n}$ are homotopy equivalent to $\mathcal{M}_{g,m+n}$ so that the above Theorem A follows. The claim about the map (2) is proven in §3 by direct calculations using the complex of twisted ribbon graphs. A very useful fact is that the map (2) can be described explicitly in terms of the cocycles representing the cohomology classes on the r.h.s. as follows,

Note that the morphism $j : \mathcal{L}ieb_{-1,0} \to \mathcal{G}\mathcal{R}av$ can *not* be lifted to a morphism from $\mathcal{L}ieb_{-1,0}$ into the chain gravity properad — it holds true only at the cohomology level.

The deformation complex of the above morphism j has, according to the general theory [MeVa] (see also [MW2] for its small but important "plus" adjustment), the following structure

$$(3) \qquad \mathsf{Def}(q\mathcal{L}ieb_{-1,0} \xrightarrow{j} \mathcal{G}\mathcal{R}av) \simeq \prod_{\substack{g,n\geq 0, m\geq 1 \\ 2g+n+m\geq 3}} H^{\bullet-1}(\mathcal{M}_{g,m+n}) \otimes_{\mathbb{S}_m^{op}\times\mathbb{S}_n} (sgn_m \otimes \mathbb{1}_n)$$

The induced differential has three parts corresponding to the above mentioned three incarnations of the pair of pants.

THEOREM B. *The complex (3) has infinitely many cohomology classes.*

Moreover, these particular cohomology classes can described explicitly in terms of twisted ribbon graphs (see §4 below). They all have genus $g = 1$ and there are indications that they all come from $H_c^\bullet(\mathcal{M}_1)$; we conjecture that *the cohomology of the complex (3) is equal to* $\prod_{g\geq 1} H_c^\bullet(\mathcal{M}_g)$, where $\mathcal{M}_g$ is the moduli space of (unpunctured) algebraic curves of genus g (cf. [AWZ]). Moreover, the proof of Theorem B motivates another conjecture: the cohomology of the M. Kontsevich's *odd* graph complex $H^\bullet(\mathsf{GC}_{-1}^{\geq 2})$ injects into the cohomology of the complex (3).

A certain quotient of the chain gravity properad plays an important role in string topology (which will be discussed elsewhere [Me]).

1.1. Notation. The set $\{1, 2, \ldots, n\}$ is abbreviated to $[n]$; its group of automorphisms is denoted by $\mathbb{S}_n$; the trivial one-dimensional representation of $\mathbb{S}_n$ is denoted by $\mathbb{1}_n$, while its one dimensional sign representation is denoted by sgn_n.

The cardinality of a finite set I is denoted by $\#I$.

We work throughout in the category of differential $\mathbb{Z}$-graded vector spaces over a field $\mathbb{K}$ of characteristic zero; all our differentials have degree $+1$ (in particular, the chain complexes of topological spaces and their homology groups are *non-positively* graded). If $V = \oplus_{i \in \mathbb{Z}} V^i$ is a graded vector space, then $V[k]$ stands for the graded vector space with $V[k]^i := V^{i+k}$; for $v \in V^i$ we set $|v| := i$. For a properad $\mathcal{P}$ we denote by $\mathcal{P}\{k\}$ the unique properad which has the following property: for any graded vector space V there is a one-to-one correspondence between representations of $\mathcal{P}\{k\}$ in V and representations of $\mathcal{P}$ in $V[-k]$; in particular, $\mathcal{E}nd_V\{k\} = \mathcal{E}nd_{V[k]}$.

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2. On twisting of properads under $\mathcal{L}ie_d$ (after Thomas Willwacher [W1])

2.1. Operad of degree shifted Lie algebras. For any $d \in \mathbb{Z}$ the operad of degree shifted Lie algebras is defined as a quotient,

$$\mathcal{L}ie_d := \mathcal{F}ree\langle E \rangle / \langle \mathcal{R} \rangle,$$

of the free prop generated by an $\mathbb{S}$-module $E = \{e(n)\}_{n \geq 2}$ with all $e(n) = 0$ except[2]

$$E(2) := sgn_2^d \otimes \mathbb{1}_1[d-1] = \text{span}\left\langle \quad \underset{1 \quad 2}{\bigwedge} \quad = (-1)^d \underset{2 \quad 1}{\bigwedge} \quad \right\rangle$$

by the ideal generated by the following relation[3]

$$\oint_{123} \underset{1 \quad 2}{\bigwedge}{}_{3} = 0$$

Its representations in a dg vector space V are in one-to-one correspondence with Lie algebra structures on V which have Lie brackets,

$$[\,,\,] : \odot^2(V[d]) \to V[1+d]$$

of degree $1 - d$; the case $d = 1$ corresponds to the ordinary Lie algebra structure in V.

2.2. Twisting of (prop)operads under $\mathcal{L}ie_d$. Let $\mathcal{P} = \{\mathcal{P}(m, n)\}_{m,n \geq 0}$ be a dg properad with the differential denoted by d. We represent its generic elements pictorially as (m, n)-corollas

(4)
$$\underset{1 \;\; 2 \;\; \cdots \;\; n}{\overset{1 \;\; 2 \;\; \cdots \;\; m}{\bigtimes}}$$

whose white vertex is decorated by an element of $\mathcal{P}(m, n)$. Properadic compositions in $\mathcal{P}$ are represented pictorially by gluing out-legs of such decorated corollas to in-legs of another decorated corollas.

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[2]When representing elements of operads and props as (decorated) graphs we tacitly assume that all edges and legs are *directed* along the flow going from the bottom of the graph to the top.

[3]Given any module V of the $\mathbb{K}$-algebra $\mathbb{K}[\mathbb{S}_3]$, the action of the element $\sum_{k=0}^{2}(123)^k \in \mathbb{K}[\mathbb{S}_3]$ on any $a \in V$ is denoted by $\oint_{123} a$.

Assume that $\mathcal{P}$ comes equipped with a morphism of properads

$$\imath: \quad (\mathcal{L}ie_d, 0) \quad \longrightarrow \quad (\mathcal{P}, \mathsf{d})$$

where $\mathcal{L}ie_d$ is understood as a differential operad with trivial differential. The image of the generator of $\mathcal{L}ie_d$ is a special element in $\mathcal{P}$ represented as a $(1,2)$-corolla with the vertex denoted by $\odot$.

Following [W1], one defines a twisted dg properad $(\mathsf{tw}\mathcal{P}, \mathsf{d}_{\bullet})$ as the unique properad freely generated by $\mathcal{P}$ and an extra $(1,0)$ generator $\blacksquare$ of cohomological degree d. The differential $\mathsf{d}_{\bullet}$ in $\mathsf{tw}\mathcal{P}$ is given explicitly on the elements of $\mathcal{P}$ by

(5)

and on the extra generator (called the *MC generator*) by

(6)

The twisted properad comes equipped with a natural epimorphism of dg properads

$$(\mathsf{tw}\mathcal{P}, \mathsf{d}_{\bullet}) \longrightarrow (\mathcal{P}, \mathsf{d})$$

which sends the MC generator to zero.

It is easy to check using Jacobi identity in $\mathcal{L}ie_d$ that the element remains a cocycle even after the twisting of the original differential, $\mathsf{d} \to \mathsf{d}_{\bullet}$, in $\mathcal{P}$,

Hence the original morphism i extends to the twisted version by the same formula,

(7)
$$\imath: \quad (\mathcal{L}ie_d, 0) \quad \longrightarrow \quad (\mathsf{tw}\mathcal{P}, \mathsf{d}_{\bullet})$$

Given a representation,

$$\rho : \mathcal{P} \to \mathcal{E}nd_V,$$

of a prop(erad) $\mathcal{P}$ in a dg vector space V, then composing ρ with the map i we get a dg Lie algebra structure $[\,,\,]$ in V. Assume γ is a Maurer-Cartan element of that dg Lie algebra, that is a degree d element $\gamma \in V$ satisfying the equation

$$d\gamma + \frac{1}{2}[\gamma, \gamma] = 0.$$

Then the pair (ρ, γ) gives rise to a representation of $\mathsf{tw}\mathcal{P}$ in V which is given on the elements from $\mathcal{P}$ by the map ρ, and on the extra MC generator by the association,

$$\blacksquare \longrightarrow \gamma,$$

which explains the main idea of the twisting endofunctor and the terminology.

2.3. Twisted operad of ribbon trees and the gravity operad. Let $\mathcal{M}_{0,1+n}$, $n \geq 2$, be the moduli space of rational curves with $1 + n$ distinct points marked by the set $[n^+] := \{0, 1, \ldots, n\}$, that is, the space of injections of $[n^+]$ into the complex projective line,

$$\mathcal{M}_{0,1+n} := \frac{\{[n^+] \hookrightarrow \mathbb{CP}^1\}}{PGL(2,\mathbb{C})},$$

modulo the action of the group $PGL(2,\mathbb{C}) = SL(2,\mathbb{C})/\{\mathrm{Id}, -\mathrm{Id}\}$. E. Getzler has shown in [Ge] that the $\mathbb{S}$-module

$$\mathcal{G}ravity := \{H_c^{\bullet-1}(\mathcal{M}_{0,1+n}) \simeq H_{\bullet-2n+3}(\mathcal{M}_{0,1+n})\}_{n \geq 2}$$

has the structure of an operad. Its representation in a graded vector space V is given by a collection of linear maps,

$$\{ , \ldots, \}_n : \odot^n V \longrightarrow V[2n-3], \quad n \geq 2,$$

satisfying the relations

$$\{\{v_1, \ldots, v_k\}, v_{k+1}, \ldots, v_{k+l}\} = \sum_{1 \leq i < j \leq k} (-1)^{\varepsilon(i,j)}\{\{v_i, v_j\}, v_1, \ldots, \widehat{v_i}, \ldots, \widehat{v_j}, \ldots, v_k, v_{k+1}, \ldots, v_{k+l}\}$$

for any $v_1, \ldots, v_{k+l} \in V$; here $(-1)^{\varepsilon(i,j)}$ is the standard Koszul sign. We refer to [KMS] for a further study of the gravity operad, its Koszul dual operad, and its relation to the operad of Gerstenhaber algebras.

There is a nice chain representation of the gravity operad given in terms of twisted ribbon trees which was constructed in [Wa]. Let $\mathcal{RG}ra_d$, $d \in \mathbb{Z}$, be the properad of ribbon graphs introduced in [MW1] (see §**3.2** below for a short reminder of its definition and main properties), and let $\mathcal{RT}ree_d$ be its sub-operad generated by ribbon trees, that is, by ribbon graphs of genus zero with precisely one boundary, e.g.
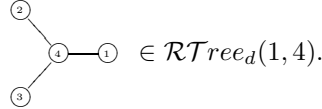
$$\in \mathcal{RT}ree_d(1,4).$$

This operad comes equipped with a morphism

and hence can be twisted using T. Willwacher's endofunctor tw. The associated dg operad $\mathsf{twRT}ree_d$ is generated by $\mathcal{RT}ree_d$ and an extra element of cohomological degree d with no inputs and one boundary which we identify with the ribbon graph $\bullet$ consisting of one black vertex and one boundary; composing that boundary with, for example, the input vertex 4 of the above graph in $\mathcal{RT}ree_d(1,4)$ one gets a ribbon graph

$$\in \mathsf{twRT}ree_d(1,3).$$

Thus $\mathsf{twRT}ree_d(1,3)$ is generated by ribbon trees with vertices of two types, white ones which are labelled and black ones which are unlabelled and have the cohomological degree d assigned. According to the general formula (5), the induced differential in $\mathsf{twRT}ree_d(1,3)$ is given by

$$(8) \qquad d.\Gamma := \; {\overset{\bullet}{\underset{\textcircled{\scriptsize 1}}{}}} \; \circ_1 \Gamma \;\; - \;\; (-1)^{|\Gamma|} \sum_{v \in V_\circ(\Gamma)} \Gamma \circ_v \; {\overset{\bullet}{\underset{\textcircled{\scriptsize 1}}{}}} \;\; - (-1)^{|\Gamma|} \frac{1}{2} \sum_{v \in V_\bullet(\Gamma)} \Gamma \circ_v (\bullet\!\!-\!\!\bullet)$$

where the symbol $\gamma_1 \circ_v \gamma_2$ means substitution of the ribbon tree γ_2 into the vertex v of the ribbon tree γ_1 and re-distributing edges attached to v among all (i.e. both white and black (if any)) vertices of γ_2 in all possible ways while respecting their cyclic order while moving along its unique boundary.

It has been proven in [Wa] that the cohomology of this twisted operad is precisely the degree shifted gravity operad,

$$H^\bullet(\mathsf{twRT}ree_d) = \mathcal{G}ravity\{d\}.$$

The elements of $\mathcal{G}ravity\{d\}$ in this chain complex get represented as equivalence classes of twisted ribbon trees with all white vertices being univalent and all black vertices (if any) being trivalent. It is worth noting that $\mathsf{tw}\mathcal{RT}ree_d$ is quasi-isomorphic as a dg operad to its suboperad generated by ribbon trees with all black vertices being at least trivalent (cf. Proposition **3.5.1** below).

3. Twisted properad of ribbon graphs and Kevin Costello's moduli spaces

3.1. Degree shifted quasi-Lie bialgebras and their deformation complex. Recall [MW2] that the prop(erad) of degree shifted Lie bialgebras is defined, for any pair of integer $c, d \in \mathbb{Z}$, as the quotient

$$\mathcal{L}ieb_{c,d} := \mathcal{F}ree\langle E_0\rangle/\langle\mathcal{R}\rangle,$$

of the free prop(erad) generated by an $\mathbb{S}$-bimodule $E = \{E(m,n)\}_{m,n \geq 0}$ with all $E_0(m,n) = 0$ except

$$E_0(2,1) := \mathbb{1}_1 \otimes (sgn_2)^{\otimes|c|}[c-1] = \mathrm{span}\left\langle\ \overset{2}{\curlyvee}\ = (-1)^c\ \overset{2\quad1}{\curlyvee}\ \right\rangle$$

$$E_0(1,2) := (sgn_2)^{\otimes|d|} \otimes \mathbb{1}_1[d-1] = \mathrm{span}\left\langle\ \underset{1\quad2}{\curlywedge}\ = (-1)^d\ \underset{2\quad1}{\curlywedge}\ \right\rangle$$

by the ideal generated by the following relations

$$(9)\qquad \mathcal{R}:\begin{cases} \displaystyle\oint_{123}\ \overset{1\ \ 2}{\underset{3}{\curlyvee}}\ = 0\ ,\quad \displaystyle\oint_{123}\ \underset{1\ \ 2}{\overset{3}{\curlywedge}}\ = 0 \\[2em] (-1)^{c+d}\ \underset{1\ \ 2}{\overset{1\ \ 2}{\bowtie}}\ + (-1)^{cd}\left(\ \overset{2}{\underset{1\ \ 2}{\curlywedge}}\ + (-1)^d\ \overset{2}{\underset{2\ \ 1}{\curlywedge}}\ + (-1)^{d+c}\ \overset{2\ \ 1}{\underset{2}{\curlywedge}}\ + (-1)^c\ \overset{2\ \ 1}{\underset{1}{\curlywedge}}\ \right) = 0. \end{cases}$$

where the vertices are assumed to be ordered in such a way that the ones on the top come first. Its arbitrary representation,

$$\rho : \mathcal{L}ieb_{c,d} \longrightarrow \mathcal{E}nd_V,$$

in a dg vector space V is uniquely determined by the values of ρ on the generators,

$$\rho\left(\curlyvee\right) : V[-c] \to \odot^2(V[-c])[1], \quad \left(\curlywedge\right) : \odot^2(V[d]) \to V[1+d],$$

which make V into a (degree shifted) dg Lie algebra and dg Lie coalgebra, with Lie bracket and Lie cobracket satisfying a compatibility condition. This notion for $c = d = 1$ was introduced by V. Drinfeld in the context of the deformation theory of the universal enveloping algebras in the category of quantum groups [Dr1]. The homotopy theory of Lie (c,d)-bialgebras is highly non-trivial — it is controlled by M. Kontsevich's graph complex GC_{c+d} (see [MW2] for the proof); in particular, for $c = d = 1$ the famous and mysterious Grothendieck-Teichmüller group GRT [Dr3] acts on the genus completed version of the properad $\mathcal{L}ieb_{1,1}$ as homotopy non-trivial automorphisms. The minimal resolution of $\mathcal{L}ieb_{c,d}$ is denoted by $\mathcal{H}olieb_{c,d}$.

V. Drinfeld introduced also in the same context of quantum groups the notion of *quasi-Lie bialgebra* or *Lie quasi-bialgebra* [Dr2]. We construct in this paper a new example of this structure in the context of the theory of cohomology groups of moduli spaces $\mathcal{M}_{g,n}$. The prop(erad) of degree shifted quasi-Lie bialgebras can be defined, for any pair of integer $c, d \in \mathbb{Z}$, as the quotient

$$\mathcal{L}ieb_{c,d} := \mathcal{F}ree\langle E_q\rangle/\langle\mathcal{R}_q\rangle,$$

of the free prop(erad) generated by an $\mathbb{S}$-bimodule $E_q = \{E_q(m,n)\}_{m,n \geq 0}$ with all $E_q(m,n) = 0$ except

$$E_q(2,1) \quad := \quad \mathbb{1}_1 \otimes sgn_2^c[c-1] = \mathrm{span}\left\langle\ \overset{1\quad2}{\curlyvee}\ = (-1)^c\ \overset{2\quad1}{\curlyvee}\ \right\rangle,$$

$$E_q(1,2) \quad := \quad sgn_2^d \otimes \mathbb{1}_1[d-1] = \mathrm{span}\left\langle\ \underset{1\quad2}{\curlywedge}\ = (-1)^d\ \underset{2\quad1}{\curlywedge}\ \right\rangle,$$

$$E_q(3,0) \quad := \quad (sgn_3)^{\otimes|c|}[1-2c+d] = \mathrm{span}\left\langle\ \overset{1\ \ 2\ \ 3}{\curlyvee}\ = (-1)^{c|\sigma|}\ \overset{\sigma(1)\ \sigma(2)\ \sigma(3)}{\curlyvee}\ \quad \forall\sigma \in \mathbb{S}_3\right\rangle,$$

modulo the ideal generated by the following relations

$$\mathcal{R}_q : \begin{cases} \oint_{123}\left(\vcenter{\hbox{\includegraphics{}}} + \vcenter{\hbox{\includegraphics{}}} \right) = 0 \;, \quad \oint_{123} \vcenter{\hbox{\includegraphics{}}} = 0 \;, \quad \oint_{123}\left(\vcenter{\hbox{\includegraphics{}}} + (-1)^c \vcenter{\hbox{\includegraphics{}}} \right), = 0 \\[2em] (-1)^{c+d}\vcenter{\hbox{\includegraphics{}}} + (-1)^{cd}\left(\vcenter{\hbox{\includegraphics{}}} + (-1)^d \vcenter{\hbox{\includegraphics{}}} + (-1)^{d+c}\vcenter{\hbox{\includegraphics{}}} + (-1)^c \vcenter{\hbox{\includegraphics{}}} \right) = 0. \end{cases}$$

To understand the deformation theory of properads under $q\mathcal{L}ieb_{c,d}$ one needs its minimal resolution, $\mathcal{H}oqlieb_{c,d}$, which is easy to construct using the same ideas which work fine for ordinary Lie bialgebras. It is a dg free properad,

$$\mathcal{H}oqlieb_{c,d} := \mathcal{F}ree\,\langle E\rangle$$

generated by an $\mathbb{S}$-bimodule

$$E = \{E(m,n)\}_{m\geq 1, n\geq 0, m+n\geq 3}$$

given by

$$E(m,n) := sgn_m^{\otimes|c|} \otimes sgn_n^{|d|}[cm+dn-1-c-d] \equiv \text{span}\left\langle \vcenter{\hbox{\includegraphics{}}} = (-1)^{c|\sigma|+d|\tau|} \vcenter{\hbox{\includegraphics{}}} \right\rangle_{\substack{\forall\sigma\in\mathbb{S}_m \\ \forall\tau\in\mathbb{S}_n}}.$$

for $m+n \geq 3$, $m \geq 1$, $n \geq 0$. The differential in $\mathcal{H}oqlieb_{c,d}$ is given on the generators by

(10) $$\delta \vcenter{\hbox{\includegraphics{}}} = \sum_{\substack{[m]=I_1\sqcup I_2 \\ |I_1|\geq 0, |I_2|\geq 1}} \sum_{\substack{[n]=J_1\sqcup J_2 \\ |J_1|,|J_2|\geq 0}} \pm \vcenter{\hbox{\includegraphics{}}}$$

If $c,d \in 2\mathbb{Z}$, all the signs in the r.h.s. are equal to -1. Taking the quotient of $\mathcal{H}oqlieb_{c,d}$ by the ideal generated by all $(m,0)$-corollas, $m \geq 3$, gives the minimal model $\mathcal{H}olieb_{c,d}$ of the Lie bialgebras properad $\mathcal{L}ieb_{c,d}$.

3.2. Reminder on the properad of ribbon graphs (after [MW1]). A ribbon graph is a graph in which the set of half-edges attached to each vertex is equipped with a cyclic ordering. More precisely, a ribbon graph is a triple $\Gamma = (H(\Gamma), \tau, \sigma)$ consisting of (i) a finite set $H(\Gamma)$ called the set of half-edges, (ii) a fixed point free involution $\tau : H(\Gamma) \to H(\Gamma)$ whose set of orbits is denoted by $E(\Gamma)$ and is called the set of edges, and (iii) a permutation $\sigma : H(\Gamma) \to H(\Gamma)$ whose set of orbits is denoted by $V(\Gamma)$ and is is called the set of vertices of the ribbon graph Γ. The permutation σ can be uniquely represented as the product of cycles, $\sigma = \prod_{v\in V(\Gamma)} \sigma_v$; the elements $h_{i_1}, \ldots h_{i_v} \in H(\Gamma)$ constituting the cycle $\sigma_v = (h_{i_1} \cdots h_{i_v})$ are called half-edges attached to the vertex v, they come equipped with a cyclic ordering for each $v \in V(\Gamma)$. Next, the orbits of the permutation $\sigma^{-1} \circ \tau$ are called *boundaries* of the ribbon graph Γ; they are better understood as real circle-like boundaries of the 2-dimensional surface obtained from Γ by thickening of edges into 2-dimensional strips. The set of boundaries of Γ is denoted by $B(\Gamma)$. For example, the graph $\vcenter{\hbox{\includegraphics{}}}$ has two vertices and three boundaries, while the ribbon graph $\vcenter{\hbox{\includegraphics{}}}$ has two vertices and one boundary.

The integer

$$g = 1 + \frac{1}{2}\left(\#E(\Gamma) - \#V(\Gamma) - \#B(\Gamma)\right)$$

is called the *genus* of Γ. This is the genus of a closed surface obtained from Γ by first thickening edges into strips and then gluing disks into each S^1-like boundary of Γ. A ribbon graph Γ is called *directed* if its every edge is equipped with the fixed total ordering of its half-edges.

Let $RG_{m,n;l}$ be the set of (isomorphism classes of) directed ribbon graphs Γ whose edges, vertices and boundaries are distinguished, say, enumerated via some bijections $E(\Gamma) \to [l]$, $V(\Gamma) \to [n] = \{1, 2, \ldots, n\}$ and $B(\Gamma) \to [\bar{m}] = \{\bar{1}, \ldots, \bar{m}\}$. The group $\mathbb{S}_l \ltimes \mathbb{S}_l^2 \times \mathbb{S}_m \times \mathbb{S}_n$ acts on elements of $RG_{m,n;l}$ by relabelling the

edges, vertices and boundaries, and by flipping directions on edges. For any integer integer $d \in \mathbb{Z}$ we define an $\mathbb{S}_m^{op} \times \mathbb{S}_n$-module

$$\mathcal{R}\mathcal{G}ra_d(m,n) := \prod_{l \geq 0} \mathbb{K}\langle RG_{m,n:l}\rangle \otimes_{\mathbb{S}_l \ltimes (\mathbb{S}_2)^l} sgn_l^{|d-1|} \otimes sgn_2^{\otimes l|d|}[l(d-1)]$$

Thus a ribbon graph Γ of $\mathcal{R}\mathcal{G}ra_d(m,n)$ has distinguished vertices and distinguished boundaries and is equipped with an *orientation* which depends on the parity of d:

- for even d the orientation of Γ is defined as the choice of ordering of the set of edges $E(\Gamma)$ up to a sign action of $\mathbb{S}_l$,
- for odd d the orientation of Γ is defined as the choice of a direction on each edge up to a sign action of $\mathbb{S}_2$.

Note that every ribbon graph Γ has precisely two possible orientations, or and or^{opp}, and Γ vanishes if it admits an automorphism which changes the orientation. The group $\mathbb{S}_m^{op} \times \mathbb{S}_n$ acts on $\mathcal{R}\mathcal{G}ra_d(m,n)$ by relabelling the vertices and the boundaries.

The $\mathbb{S}$-bimodule $\mathcal{R}\mathcal{G}ra_d = \{\mathcal{R}\mathcal{G}ra_d(m,n)\}_{m,n \geq 1}$ has a natural structure of a prop(erad) described in full details in [MW1]; horizontal compositions are given by disjoint unions of ribbon graphs while the vertical compositions are given by substituting boundaries $b \in B(\Gamma_1)$ of one graph into (blown into disks) vertices $v \in V(\Gamma_2)$ of another graph and then re-distributing edges attached earlier to v among the vertices belonging to the closed oriented path b in all possible ways while respecting the cyclic orderings. This composition can also be understood geometrically in terms of the cells of Kevin Costello's moduli spaces $\mathcal{D}_{g,m,0,n}$ of nodal disks with m marked boundaries and n internal marked points (such that each disk contains at most one internal marked point) [C2], as we shall see below.

The main point of introducing the prop $\mathcal{R}\mathcal{G}ra_d$ is that it comes equipped with several remarkable morphisms of properads [MW1], and that these morphisms possess in turn quite interesting deformation complexes. The first important morphism is the following one,

$$i : \mathcal{L}ieb_{d,d} \longrightarrow \mathcal{R}\mathcal{G}ra_d$$

which is given on generators by,

(11)
$$i : \quad \vee^{\bar{1} \quad \bar{2}}_{1} \quad \longrightarrow \quad \overset{\bar{2}}{\underset{\bar{1}}{\bigcirc}}_{\textcircled{1}} \quad , \quad i : \quad \overset{\bar{1}}{\underset{1 \quad 2}{\wedge}} \quad \longrightarrow \quad \textcircled{1}\!-\!\!\textcircled{2}^{\bar{1}} \; .$$

For d even we assume (implicitly) that a particular arrow on the unique edge of graphs in the r.h.s. of the map i is chosen up to a flip and multiplying by -1 (we often omit showing too many details in our pictures such as arrows on edges and labels of vertices or boundaries).

Another important morphism is given by Theorem 4.3.3 in [MW1] which says that there is a properad map

(12) $$s : \mathcal{H}olieb_{0,1} \longrightarrow \mathcal{R}\mathcal{G}ra_1,$$

defined on generators by

$$s : \left(\overset{1 \; 2 \; \ldots \; m-1 \; m}{\underset{1 \; 2 \; \ldots \; n-1 \; n}{\bowtie}} \right) = \begin{cases} \dfrac{1}{m}\sum_{\sigma \in \mathbb{S}_m} \underbrace{\bullet\!\!\overset{\cdots}{\overgroup{}}\!\!\bullet}_{m \text{ edges}} & \text{if } n = 2 \text{ and } m \text{ odd} \\[2em] 0 & \text{otherwise} \end{cases} \; ,$$

where the symbol $\sum_{\sigma \in \mathbb{S}_m}$ means the summation over all ways of numbering the m boundaries on r.h.s. More examples of morphisms and their deformation complexes can be found in [MW1].

3.2.1. On graph complexes associated with the properad $\mathcal{R}\mathcal{G}ra_d$. The deformation complex of the morphism i in (11) is given by,

$$(\mathsf{RGC}_d, \delta + \Delta) := \mathsf{Def}(\mathcal{L}ieb_{d,d} \xrightarrow{i} \mathcal{R}gra_d) \simeq \bigoplus_{m,n \geq 1} \left(\mathcal{R}\mathcal{G}ra_d(m,n) \otimes_{\mathbb{S}_m \times \mathbb{S}_n} (sgn_m^{|d|} \otimes sgn_n^{|d|}) \right) [d(2-m-n)],$$

It is spanned by ribbon graphs with unidentifiable boundaries and vertices and its differential consists of two operations δ and Δ which are governed by the values of i on the Lie and, respectively, coLie generators of $\mathcal{L}ieb_{c,d}$. The Jacobi and co-Jacoby relations in the ideal (9) assure that these operations are differentials each,

$$\delta^2 = 0, \quad \Delta^2 = 0,$$

while the V. Drinfeld's relation in (9) guarantees their compatibility,

$$\delta\Delta + \Delta\delta = 0.$$

The differential δ splits vertices of ribbon graphs Γ into two new vertices connected by an edge, while the differential Δ attaches a new edge to Γ in such a way that the number of boundaries is increased by 1 but no new loop-like boundaries (i.e. the ones consisting of single edges only) are created. The reduced complex

$$(\mathsf{RGC}_d, \delta)$$

(more precisely, its subcomplex $\mathsf{RGC}_{\bar{d}}^{\geq 3}$ spanned by at least trivalent graphs) is identical to the ribbon graph complex studied earlier by Robert Penner [Pe] and Maxim Kontsevich [K1]; it computes the compactly supported cohomology groups of moduli spaces of algebraic curves of arbitrary genus with skew-symmetrized marked points,

$$H^\bullet(\mathsf{RGC}_{\bar{d}}^{\geq 3}, \delta) = \prod_{g,m} H_c^{\bullet+2dg-m}(\mathcal{M}_{g,m}) \otimes_{\mathbb{S}_m} sgn_m.$$

The full complex $(\mathsf{RGC}_d, \delta + \Delta)$ computes conjecturally the totality $\prod_{g\geq 1} H_c^\bullet(\mathcal{M}_g)$ of moduli spaces of algebraic curves of arbitrary genus without punctures [AWZ]. It was proven in [MW1] that there is an explicit morphism of cohomology groups

$$f : H^\bullet(\mathsf{GC}_2) \longrightarrow H^{\bullet+1}(\mathsf{RGC}_1, \delta + \Delta)$$

which is conjecturally an injection; here GC_2 is the famous graph complex introduced by Maxim Kontsevich in [K2] in the context of the deformation quantization of Poisson structures (we remind its definition in §**4.3** below); it has a remarkable property established by Thomas Willwacher in [W1] that

$$H^0(\mathsf{GC}_2) = \mathfrak{grt}_1,$$

the Lie algebra of the mysterious Grothendieck-Teichmüller group GRT_1. It has been proven recently by Melody Chan, Soren Galatius and Sam Payne in [CGP] that there is an injection of the cohomology groups

$$H^\bullet(\mathsf{GC}_2) \longrightarrow \prod_g H_c^{\bullet+2g}(\mathcal{M}_g; \mathbb{K}).$$

Conjecturally, this injection coincides with the purely combinatorial morphism of cohomology groups f considered just above.

The deformation complex of another important morphism (12),

$$\left(\mathsf{RGC}_{0,1}, \delta + \sum_{i\geq 1} \Delta_{2i+1}\right) := \mathsf{Def}(\mathcal{H}olieb_{0,1} \xrightarrow{s} \mathcal{R}gra_1) \simeq \bigoplus_{m,n\geq 1} (\mathcal{RGra}_1(m,n) \otimes_{\mathbb{S}_m\times\mathbb{S}_n} (\mathbb{1}_m \otimes sgn_n)) [(1-n)]$$

has a differential which is also a deformation of the standard differential δ, but this time the deformed part is an infinite series $\sum_{i\geq 1} \Delta_{2i+1}$ of operators, each increasing the number of boundaries by $2i$ as $i \to \infty$. The cohomology of this complex comes equipped with a morphism from the cohomology of M. Kontsevich's *odd* graph complex,

$$H^\bullet(\mathsf{GC}_1) \longrightarrow H^{\bullet+1}\left(\mathsf{RGC}_{0,1}, \delta + \sum_{i\geq 1} \Delta_{2i+1}\right)$$

which is non-trivial on infinitely many polytope classes. Probably this complex $\mathsf{RGC}_{0,1}$ from [MW1] and the complex (3) with 3-terms differential introduced in this paper are related to each other.

3.3. Twisted properad of ribbon graphs. Let tw be Thomas Willwacher's twisting endofunctor [W1] in the category of dg properads under $\mathcal{L}ie_d$, and let $\mathsf{tw}\mathcal{R}\mathcal{G}ra_d = \{\mathsf{tw}\mathcal{R}\mathcal{G}ra_d(m,n)\}_{m \geq 1, n \geq 0}$ be the twisted version of the above properad $\mathcal{R}\mathcal{G}ra_d$ viewed just as a properad under $\mathcal{L}ie_d$. The properad $\mathsf{tw}\mathcal{R}\mathcal{G}ra_d$ is generated by elements of $\mathcal{R}\mathcal{G}ra_d$ and one extra MC generator with one output (i.e. one boundary) and no inputs which we represent as a ribbon graph consisting of just one black vertex $\bullet$ of cohomological degree d (cf. §**2.3**); it is not allowed (by its very definition) to substitute boundaries of other graphs into that vertex, but it is allowed to substitute its unique boundary into the vertices of ribbon graphs from $\mathcal{R}\mathcal{G}ra_d$. Thus $\mathsf{tw}\mathcal{R}\mathcal{G}ra_d(m,n)$ is generated by ribbon graphs with m labelled boundaries, n labelled white vertices and any number of black vertices which are assigned the cohomological degree d, for example

$$\begin{array}{cccc} \bigcirc_1 \!\!\! & \in \mathsf{tw}\mathcal{R}\mathcal{G}ra_d(3,1), & \bigcirc_1 \!\!\! \in \mathsf{tw}\mathcal{R}\mathcal{G}ra_d(1,1), & \underset{\textcircled{\scriptsize 1}}{\overset{\bar{1}}{\textcircled{}}} \!\!-\!\! \textcircled{\scriptsize 2} \in \mathsf{tw}\mathcal{R}\mathcal{G}ra_d(1,2), & \overset{\bar{2}}{\underset{\textcircled{\scriptsize 1}}{\overset{\bar{1}}{\circ}}} \in \mathsf{tw}\mathcal{R}\mathcal{G}ra_d(2,1), \end{array}$$

the first two graphs having degree $3(1-d)+d = 3-2d$, the last two graph having degree $1-d$. We denote the set of white (resp. black) vertices of a ribbon graph $\Gamma \in \mathsf{tw}\mathcal{R}\mathcal{G}ra_d$ is denote by $V_\circ(\Gamma)$ (resp. $V_\bullet(\Gamma)$). Note that all boundaries of these graphs are labelled but we omit often their labellings in our pictures. The orientation of graphs Γ from $\mathsf{tw}\mathcal{R}\mathcal{G}ra_d$ is defined as a choice of one of the two unit vectors in the 1-dimensional real vector space $Or_d(\Gamma)$ (equipped with the standard Euclidean metric) which depends on the parity of d,

- for even d, $Or_d(\Gamma) := \det(E(\Gamma))$,
- for odd d, $Or_d(\Gamma) := \det(V_\bullet(\Gamma)) \bigotimes_{e \in E(\Gamma)} \det(H(e))$ where $H(e)$ is the set of two half-edges constituting the edge e.

Here, for a finite set I, we denote $\det(I) := \wedge^{\#I}(\mathbb{R}[I])$ with $\mathbb{R}[I]$ being the linear span of I over the field $\mathbb{R}$ of real numbers.

The properadic compositions in $\mathsf{tw}\mathcal{R}\mathcal{G}ra_d$ are given by substituting boundaries $b \in B(\Gamma')$ into white vertices $v \in V_\circ(\Gamma'')$ using exactly same rules as in the case of $\mathcal{G}\mathcal{R}ra_d$; it is worth noting that, when distributing dangling half-edges attached to v to the vertices belonging to the boundary b, we allow their reconnections to vertices of both types, white and black ones.

Note that $\mathcal{R}\mathcal{G}ra_d$ has, by its very definition, the trivial differential so that the induced differential $\mathsf{d}_\centerdot$ in $\mathsf{tw}\mathcal{R}\mathcal{G}ra_d$ comes solely from the twisting procedure and is given, on an arbitrary ribbon graph $\Gamma \in \mathsf{tw}\mathcal{R}\mathcal{G}ra_d(m,n)$, by the formula[4]

$$(13) \qquad \mathsf{d}_\centerdot\Gamma := \sum_{i=1}^{m} \underset{\textcircled{\scriptsize 1}}{\overset{}{\vert}}\; {}_1\circ_j \Gamma \;\; - \;\; (-1)^{|\Gamma|} \sum_{j=1}^{n} \Gamma\; {}_j\circ_1 \underset{\textcircled{\scriptsize 1}}{\overset{}{\vert}} \;\; - (-1)^{|\Gamma|} \frac{1}{2} \sum_{v \in V_\bullet(\Gamma)} \Gamma \circ_v (\bullet\!\!-\!\!\bullet)$$

where the symbol $\Gamma \circ_v (\bullet\!\!-\!\!\bullet)$ means substituting the graph $\bullet\!\!-\!\!\bullet$ into the black vertex v of the graph Γ and then taking the sum over all possible re-attachments of the half-edges attached earlier to v among the the two black vertices in a way which respects their cyclic ordering. Note that for almost all graphs the new univalent black vertices arising in the first part of $\mathsf{d}_\centerdot$ cancel out the new univalent black vertices arising in the second and the third part of that differential.

3.4. On the role of the integer parameter d. It is clear from the very definition that dg props $\mathsf{tw}\mathcal{R}\mathcal{G}ra_d$ with d of the same *parity* are isomorphic to each other (up to degree shifts). However it is not immediately evident that dg props $\mathsf{tw}\mathcal{R}\mathcal{G}ra_d$ for d with opposite parity control essentially one and the same mathematics because the definitions of *orientation* of ribbon graphs for d even and d odd look quite different.

Let Γ be any connected ribbon graph and Σ_Γ be the associated connected compact oriented Riemann surface canonically associated to by Γ by thickening edges of Γ into strips and filling every boundary with a disk. The ribbon graph gives us a cell decomposition of Σ_Γ and hence the associated cell complex,

$$C_2(\Sigma_\Gamma) \xrightarrow{\partial_2} C_1(\Sigma_\Gamma) \xrightarrow{\partial_1} C_0(\Sigma_\Gamma)$$

---

[4]Given a prop $\mathcal{P} = \{\mathcal{P}(m,n)\}$, we denote by ${}_i\circ_j$ the partial vertical composition $\mathcal{P}(m_1,n_1) \otimes \mathcal{P}(m_2,n_2) \to \mathcal{P}(m_1 + m_2 - 1, n_1 + n_2 - 1)$ given, for any $i \in [n_1]$ and $j \in [m_2]$, by inserting j-th output of $\mathcal{P}(m_2,n_2)$ into i-th input of $\mathcal{P}(m_1,n_1)$.

computing the singular homology $H_\bullet(\Sigma_\Gamma)$ with coefficients in the field $\mathbb{K}$. We can identify $C_2(\Sigma_\Gamma)$ with the vector space $\mathbb{K}[B(\Gamma)]$ generated by the set of boundaries of Γ, $C_0(\Sigma_\Gamma)$ with the vector space $\mathbb{K}[V(\Gamma)]$ generated by its set of vertices, and

$$C_1(\Sigma_\Gamma) = \bigoplus_{e \in E(\Gamma)} \mathbb{K}[\det(H(e))]$$

Then the three exact sequences,

$$0 \longrightarrow H_2(\Sigma_\Gamma) \longrightarrow C_2(\Sigma_\Gamma) \longrightarrow \mathsf{Im}\,\partial_2 \longrightarrow 0$$

$$0 \longrightarrow \mathsf{Im}\,\partial_2 \longrightarrow \mathsf{Ker}\,\partial_1 \longrightarrow H_1(\Sigma_\Gamma) \longrightarrow 0$$

$$0 \longrightarrow \mathsf{Ker}\,\partial_1 \longrightarrow C_1(\Sigma_\Gamma) \longrightarrow C_0(\Sigma_\Gamma) \longrightarrow H_0(S_\Gamma) \longrightarrow 0$$

imply the following canonical isomorphisms

$$\det(B(\Gamma)) = \det(H_2(\Sigma_\Gamma)) \otimes \det(\mathsf{Im}\,\partial_2), \qquad \det(\mathsf{Ker}\,\partial_1) = \det(\mathsf{Im}\,\partial_2) \otimes \det(H_1(\Sigma_\Gamma)),$$

$$\det(\mathsf{Ker}\,\partial_1) \otimes \det(V(\Gamma)) = \det(E(\Gamma)) \otimes \bigotimes_{e \in E(\Gamma)} \det(H(e)) \otimes \det(H_0(S_\Gamma)).$$

As spaces $H_2(\Sigma_\Gamma)$ and $H_0(\Sigma_\Gamma)$ are 1-dimensional and come equipped with canonical bases, they can be omitted in the above equalities. The vector space $H_1(\Sigma_\Gamma)$ can be equipped with a so called *canonical basis of cycles* $\{a_i, b_i\}_{i \in [g]}$ characterized by the standard intersection properties, $a_i \cdot a_j = 0$, $b_i \cdot b_j = 0$, $a_i \cdot b_j = \delta_{ij}$. This basis is not defined uniquely, but up to a multiplication by a matrix from the symplectic group $Sp(2g, \mathbb{Z})$. Therefore such a non-unique canonical basis of cycles gives us a well-defined distinguished basis vector of $\det(H_1(\Sigma_\Gamma))$ so that this tensor factor can also be omitted in the above equalities. The remaining equalities assemble into the canonical isomorphism

$$\det(E(\Gamma)) \otimes \det(V(\Gamma)) \otimes \det(B(\Gamma)) \equiv \bigotimes_{e \in E(\Gamma)} \det(H(e)).$$

or, equivalently,

$$\det(E(\Gamma)) \otimes \det(V_\circ(\Gamma)) \otimes \det(V_\bullet(\Gamma)) \otimes \det(B(\Gamma)) \equiv \bigotimes_{e \in E(\Gamma)} \det(H(e)).$$

As boundaries and white vertices of Γ are labelled, the vector spaces $\det(V_\circ(\Gamma))$ and $\det(B(\Gamma))$ can be identified canonically with $\mathbb{K}$ so that the above identity simplifies to the following one,

$$\det(E(\Gamma)) = \det(V_\bullet(\Gamma)) \bigotimes_{e \in E(\Gamma)} \det(H(e))$$

which identifies orientation spaces Or_d for any d. Thus we have proved the following

3.4.1. Lemma. *Up to degree shifts, the dg properads* $\mathsf{tw}\mathcal{RGra}_d$ *for different d are isomorphic to each other.*

Let $\mathsf{tw}\mathcal{RGra}_d(m, n; g) \subset \mathsf{tw}\mathcal{RGra}_d(m, n)$ be the subcomplex spanned by twisted ribbon graphs of genus g. Then above mentioned degree shift is given explicitly by

(14) $$\mathsf{tw}\mathcal{RGra}_d(m, n; g) = \mathsf{tw}\mathcal{RGra}_0(m, n; g)[d(2g - 2 + m + n)], \quad \forall d \in \mathbb{Z},$$

that is, an oriented genus g ribbon graph Γ of cohomological degree k in $\mathsf{tw}\mathcal{RGra}_0(m, n)$ will have degree $k - d(2g - 2 + m + n)$ when considered as an oriented graph in $\mathsf{tw}\mathcal{RGra}_d(m, n)$.

We conclude that it is enough to understand the meaning of the dg prop $\mathsf{tw}\mathcal{RGra}_0$ (though in applications, especially in string topology [Me], it is often useful to work with its degree shifted version $\mathsf{tw}\mathcal{RGra}_d$ for a suitable d, as d is determined by the dimension of a manifold under study).

3.5. The (chain) gravity properad. It is easy to see that the subspace $\mathsf{tw}\mathcal{R}\mathcal{G}ra_d^{\geq 3}$ of $\mathsf{tw}\mathcal{R}\mathcal{G}ra_d$ spanned by ribbon graphs *with every black vertex at least trivalent* is a dg sub-properad which is called the *chain gravity properad*, while its cohomology

$$\mathcal{G}\mathcal{R}av_d := H^\bullet\left(\mathsf{tw}\mathcal{R}\mathcal{G}ra_d^{\geq 3}\right)$$

is called the *gravity properad* (for the reasons explained below).

The properad $\mathcal{G}\mathcal{R}av_0$ is abbreviated by $\mathcal{G}\mathcal{R}av$.

3.5.1. Proposition. *The inclusion*

$$\mathsf{tw}\mathcal{R}\mathcal{G}ra_d^{\geq 3}(m,n) \longrightarrow \mathsf{tw}\mathcal{R}\mathcal{G}ra_d(m,n)$$

is a quasi-isomorphism for any $m,n \geq 1$. For $n = 0$ on has

$$H^\bullet(\mathsf{tw}\mathcal{R}\mathcal{G}ra_d(m,0)) = H^\bullet\left(\mathsf{tw}\mathcal{R}\mathcal{G}ra_d^{\geq 3}(m,n)\right) \oplus \bigoplus_{\substack{p \geq 1 \\ p \equiv 2d+1 \bmod 4}} \mathbb{K}[-p]$$

where the summand $\mathbb{K}[-p]$ is generated by the polytope with p edges and p bivalent vertices which are all black.

Proof. The argument is the same as in the case of ordinary (i.e. non-ribbon) graph complexes. We shall sketch it only following [W1].

Let $\mathsf{tw}\mathcal{R}\mathcal{G}ra_d(m,n)^1 \subset \mathsf{tw}\mathcal{R}\mathcal{G}ra_d(m,n)$ be the subcomplex spanned by ribbon graphs with at least one univalent black vertex. The differential preserves the total number of univalent black vertices so we can assume without loss of generality that they are distinguished, say, enumerated. Call *antenna* the maximal subgraph of a ribbon graph from $\mathsf{tw}\mathcal{R}\mathcal{G}ra_d(m,n)^1$ consisting of the univalent black vertex labelled by the minimal integer and the maximal connected chain of bivalent black vertices attached to it; denote such a chain with $n \geq 0$ bivalent black vertices by A_n. Consider a filtration of $\mathsf{tw}\mathcal{R}\mathcal{G}ra_d(m,n)^1$ by the total number of non-antenna vertices. The induced differential acts only on A_n antennas as follows,

$$d : A_{2n} \to A_{2n+1}, \quad d : A_{2n+1} \to 0.$$

Hence the associated graded complex is acyclic implying acyclicity of $\mathsf{tw}\mathcal{R}\mathcal{G}ra_d(m,n)^1$.

Let $\mathsf{tw}\mathcal{R}\mathcal{G}ra_d(m,n)^2 \subset \mathsf{tw}\mathcal{R}\mathcal{G}ra_d(m,n)$ be the subcomplex spanned by ribbon graphs with at least one bivalent black vertex. It contains in turn a subcomplex spanned by graphs with at least one trivalent black vertex or at least one white vertex. It is easy to see that this subcomplex is acyclic (see the proof of Proposition 3.4 in [W1]). It remains to consider a subcomplex of $\mathsf{tw}\mathcal{R}\mathcal{G}ra_d(m,n)$ spanned by ribbon graphs whose all vertices are bivalent and black. Its cohomology is given by the polytopes described in the Proposition. $\qquad\square$

3.6. What does the properad $\mathcal{G}\mathcal{R}av$ measure and control? An answer about the part, $\mathcal{G}\mathcal{R}av(m,0)$, of the gravity properad with no white vertices is immediate — the complex $\mathsf{tw}\mathcal{R}\mathcal{G}ra^{\geq 3}(m,0)$ is precisely the (degree shifted) R. Penner's ribbon graph complex [Pe, K1] with marked boundaries, so that its cohomology is given by

$$\mathcal{G}\mathcal{R}av(m,0) = \prod_{2g+m \geq 3} H_c^{\bullet-m}(\mathcal{M}_{g,m})$$

where $\mathcal{M}_{g,m}$ is the moduli spaces of genus g algebraic curves with m marked points, and $H_c^\bullet$ stands for the compactly supported cohomology functor. Our main purpose in this section is to understand the algebro-geometric meaning of $\mathcal{G}\mathcal{R}av(m,n)$ for $n \geq 1$.

Let $\mathcal{M}_{g,m+n}$ the moduli space of genus g algebraic curves with the set S of distinct marked points decomposed into the disjoint union

$$S = S^{out} \sqcup S_{in}, \quad \#S^{out} = m \geq 1, \ \#S_{in} = n \geq 0.$$

The permutation group $\mathbb{S}_m^{op} \times \mathbb{S}_n$ acts naturally on $\mathcal{M}_{g,m+n}$ by relabelling (separately) the out- and in-points. Note that $\mathcal{M}_{g,m_1+n_1}$ and $\mathcal{M}_{g,m_2+n_2}$ with the same total number of marked points, $n_1+m_1 = n_2+m_2$, have to be considered as different topological $\mathbb{S}$-bimodules if $m_1 \neq m_2$.

3.6.1. Theorem. *For any $m \geq 1, n \geq 0$ there is an isomorphism of $\mathbb{S}_m^{op} \times \mathbb{S}_n$-modules.*

$$H^\bullet(\mathcal{GR}av(m,n)) = \prod_{\substack{2g+m+n \geq 3 \\ m \geq 1, n \geq 0}} H_c^{\bullet-m}(\mathcal{M}_{g,n+m})$$

For $n = 0$ this result says nothing new. For $n \geq 1$ it is less obvious; the proof is based on the theory of K. Costello's moduli spaces of nodal disks [C1, C2] and his wonderful homotopy equivalence; it is presented in the subsection §**3.8** below.

The chain gravity properad contains the dg operad of twisted ribbon trees from §**2.3** as a dg sub-operad. Thus the gravity operad is a sub-operad of $\mathcal{GR}av$.

3.7. Quasi-Lie bialgebra sub-properad of the gravity properad. Before proving the above Theorem, let us study three simplest non-trivial cohomology classes of the chain gravity properad, and their inter-relations.

It is easy to check that the graph ①—② is a cycle in $\mathsf{tw}\mathcal{RGra}_0^{\geq 3}$ so that there is a morphism of dg properads

$$\imath: \quad \mathcal{Lie}_0 \quad \longrightarrow \quad \mathsf{tw}\mathcal{RGra}_0$$

given by the same formula as in the case of $\mathcal{RGra}_0$. Moreover, this class is not a coboundary so that it induces a non-trivial map of properads

$$\mathcal{Lie}_0 \longrightarrow \mathcal{GR}av.$$

Another important graph in $\mathsf{tw}\mathcal{RGra}_0$ is *not* a cocycle,

so that the above map $\imath$ does *not* extend to a morphism $\mathcal{Lieb}_{0,0} \to \mathsf{tw}\mathcal{RGra}_0$. It is easy to see that the graph

is a cocycle in $\mathsf{tw}\mathcal{RGra}_0^{\geq 3}$ which is not a coboundary. Moreover, at the cohomology level it defines a cohomology class in $\mathcal{GR}av(2,1)$ of degree 2 which is, by the above formula, skew-symmetric with respect to the out-labels $\bar{1}$ and $\bar{2}$.

3.7.1. Proposition. *There is a morphism of properads*

$$j: q\mathcal{Lieb}_{-1,0} \longrightarrow \mathcal{GR}av$$

given on the generators of $\mathcal{Lieb}_{-1,0}$ by

$$(15) \quad j: \quad $$

Proof. Let us lift the above values of the map j to the gravity chain complex by the same formulae as above except the first one

$$j: \quad $$

and check that all the relations $\mathcal{R}_q$ in $q\mathcal{L}ieb_{c,d}$ are respected by the map j modulo $\mathsf{d}_{\bullet}$-exact terms. The element $\overset{\bar{1}\quad\bar{2}}{\curlyvee}$ is a cycle in $\mathsf{twRGra}_0^{\geq 3}$.

We assume tacitly in all our pictures that the orientation of any twisted ribbon graph is given by *ordering the edges from the bottom to the top, and, if on the same level, from left to right*. This rule explains the signs in the formulae below.

STEP 1: first we check the Drinfeld compatibility condition, that is, the equality

$$j\left(\overset{1\;\;2}{\underset{1\;\;2}{\textstyle\bigtimes}} \;-\; \overset{2}{\underset{1}{\textstyle\bigvee}}\;\; -\; \overset{2}{\underset{2}{\textstyle\bigvee}}\;\; +\; \overset{1}{\underset{2}{\textstyle\bigvee}}\;\; +\; \overset{1}{\underset{1}{\textstyle\bigvee}}\;\; \right) = \mathsf{d}_{\bullet}(\text{something})$$

Remarkably, it holds true at the level of cocycles, i.e. we can write zero on the r.h.s. The rules for the properadic composition in $\mathsf{twRGra}_0^{\geq 3}$ give us the following formulae,

$$\overset{\bar{1}\;\;\bar{2}}{\underset{1\;\;2}{\textstyle\bigvee}} \;=\; \begin{matrix}\bar{2}\\\bar{1}\\2\\1\end{matrix} \;+\; \begin{matrix}\bar{2}\\\bar{1}\\1\\2\end{matrix} \;-\; \begin{matrix}\bar{1}\\\bar{2}\\2\\1\end{matrix} \;-\; \begin{matrix}\bar{1}\\\bar{2}\\2\\1\end{matrix}$$

and

$$\overset{\bar{2}}{\underset{1}{\overset{\bar{1}}{\textstyle\bigvee}}}\,2 \;=\; \begin{matrix}\bar{2}\\\bar{1}\\1\\2\end{matrix} \;-\; \overset{\bar{2}}{\underset{1\;\;2}{\overset{\bar{1}}{\textstyle\bigvee}}} \;+\; \overset{\bar{2}}{\underset{2\;\;1}{\overset{\bar{1}}{\textstyle\bigvee}}}$$

The symmetrization over (12) kills the last two terms in the last formula so that the Drinfeld compatibility condition follows.

STEP 2: we check next the claim

$$\oint_{123} j\left(\overset{1\;\;2}{\underset{3}{\textstyle\bigvee}} \;+\; \overset{3}{\underset{1}{\underset{1\;\;2}{\textstyle\bigvee}}} \right) \text{ is } \mathsf{d}_{\bullet}\text{-exact.}$$

The properadic composition in $\mathsf{twRGra}_0^{\geq 3}$ gives the following result

$$\overset{\bar{2}\quad\bar{3}}{\underset{1}{\overset{\bar{1}}{\textstyle\bigvee}}} \;=\; \frac{1}{4}\,(\mathrm{Id} - (\bar{2}\bar{3}))\left(-\,\begin{matrix}\bar{2}\\\bar{3}\\\bar{1}\\1\end{matrix} \;+\; \begin{matrix}\bar{2}\\\bar{1}\\3\\1\end{matrix} \;+\; \begin{matrix}\bar{2}\\3\\\bar{1}\\1\end{matrix} \;-\; \begin{matrix}\bar{1}\\3\\1\\\bar{2}\end{matrix} \right)$$

One has

$$\oint_{123}\left(-\,\begin{matrix}\bar{1}\\3\\1\\\bar{2}\end{matrix} \;+\; \begin{matrix}\bar{1}\\2\\1\\\bar{3}\end{matrix} \right) = 0$$

because the linear combination of graphs on the l.h.s. admits an automorphism which reverses the orientation. Hence one can ignore the last summand in the formula for $\overset{\bar{2}\quad\bar{3}}{\underset{1}{\overset{\bar{1}}{\textstyle\bigvee}}}$ above.

One has

$$\mathsf{d}_\bullet \; \cdot \; = \; \cdot \; + \; \cdot \; + \; \cdot \; + \; \cdot$$

and

$$\mathsf{d}_\bullet \left(\; \cdot \; - \; \cdot \; \right) = \; \cdot \; + \; \cdot \; - \; \cdot \; - \; \cdot \; + 2 \left(\; \cdot \; - \; \cdot \; \right)$$

These results imply after a straightforward calculation that

$$\oint_{123} \left(\; \cdot \; + \frac{1}{2} \; \cdot \; - \frac{1}{2} \; \cdot \; \right) = \mathsf{d}_\bullet \oint \left(\frac{1}{4} \; \cdot \; - \frac{1}{4} \; \cdot \; - \frac{1}{2} \; \cdot \; + \frac{1}{2} \; \cdot \; \right)$$

Finally one checks the equality

$$\oint_{123} \frac{1}{2} \left(\; \cdot \; - \; \cdot \; \right) = \oint_{123} \; \cdot$$

where on the r.h.s. we use the notation

$$\cdot \; := \frac{1}{2} \left(- \; \cdot \; + \; \cdot \; \right)$$

and the properadic composition in $\mathsf{tw}\mathcal{RG}ra_0^{\geq 3}$. Hence the claim in STEP 2 follows.

STEP 3: we check the last non-obvious claim that the twisted ribbon graph

$$\oint_{123} j \left(\; \cdot \; - \; \cdot \; \right) \equiv \left(\mathrm{Id} - \zeta + \zeta^2 - \zeta^3 - (23) - (24) \right) j \left(\; \cdot \; \right) \quad \text{is } \mathsf{d}_\bullet\text{-exact,}$$

where $\zeta := (1234) \in \mathbb{S}_4$. Rules for the properadic composition in $\mathsf{tw}\mathcal{RG}ra_0^{\geq 3}$ give

$$\cdot \; = \frac{1}{2} \left(- \; \cdot \; + \; \cdot \; + \; \cdot \; - \; \cdot \; \right)$$

A straightforward but tedious calculation shows that the above claim is true indeed with the $\mathsf{d}_\bullet$-exact term given by

$$\oint_{123} j \left(\; \cdot \; - \; \cdot \; \right) =$$

$$= \frac{1}{2} \left(\mathrm{Id} - \zeta + \zeta^2 - \zeta^3 - (23) - (24) \right) \mathsf{d}_\bullet \left(- \; \cdot \; + \; \cdot \; + \; \cdot \; - \; \cdot \; \right)$$

The proof is completed. $\qquad\square$

3.8. K. Costello's moduli spaces as a dg properad. For integers $g, n, r \geq 0$ and $m > 1$, let $\mathcal{N}_{g,m,r,n}$ be the moduli space of connected stable Riemann surfaces Σ of genus g with m *marked* boundary components, n *marked* points in the interior $\Sigma \setminus \partial\Sigma$ and with r *marked* points on the boundary $\partial\Sigma$. It is assumed that $\mathcal{N}_{g,m,r,n} = \emptyset$ if $g = 0$, $m = 1$ and $r + 2n < 3$ or $g = 0$, $m = 2$, $r = n = 0$.

Kevin Costello introduces in [C1, C2] a certain partial compactification $\overline{\mathcal{N}}_{g,m,r,n}$ of $\mathcal{N}_{g,m,r,n}$ by allowing stable Riemann surfaces Σ with nodes *on the boundary*; the marked points are required to be distinct from the nodes and each other, and the stability condition requires that the normalization — cutting each node into two separate "half-nodes" — of each singular surface Σ gives a disconnected surface whose every connected component is stable in the above sense. The moduli space $\overline{\mathcal{N}}_{g,m,r,n}$ is a $(6g - 6 + 3m + r + 2n)$-dimensional orbifold with corners so that the inclusion $\mathcal{N}_{g,m,r,n} \hookrightarrow \overline{\mathcal{N}}_{g,m,r,n}$ is a homotopy equivalence. The boundary $\partial\overline{\mathcal{N}}_{g,m,r,n}$ consists of singular Riemann surfaces, the ones with nodes. It is proven in [C2] that the inclusion $\partial\overline{\mathcal{N}}_{g,m,r,n} \hookrightarrow \overline{\mathcal{N}}_{g,m,r,n}$ is also a homotopy equivalence.

Let $\mathcal{D}_{g,m,r,n} \subset \overline{\mathcal{N}}_{g,m,r,n}$ be the locus where all irreducible components of surfaces are discs (with *at most one* internal marked point). A remarkable theorem by K. Costello [C2] says that this inclusion is a homotopy equivalence so that

(16)
$$H_\bullet(\mathcal{D}_{g,m,r,n}) = H_\bullet(\mathcal{N}_{g,m,r,n})$$

Moreover, the compact orbi-space $\mathcal{D}_{g,m,r,n}$ of disks admits a decomposition into orbi-cells which can be parameterized by ribbon graphs. We are interested in this paper in the case $r = 0$ only; the general case $r \geq 1$ employs twisted ribbon graphs *with hairs* which we do not need in this paper.

The space $\mathcal{D}_{g,m,0,n}$ is decomposed into orbi-cells via an equivalence relation, $\Sigma \sim \Sigma'$, if there exists a homeomorphism $\Sigma \to \Sigma'$ which preserves the marked points and the orientation. Each orbi-cell O is uniquely determined by [C2]

- the set $V_\bullet$ of its irreducible components (disks) with no marked points inside,
- the set $V_\circ$ of its irreducible components (disks) with precisely one marked point inside; this set comes equipped with a fixed isomorphism $V_\circ \to [n]$ which associates to an irreducible component the marking of its unique internal point;
- the set of "half-nodes" $H(v)$ on each irreducible component $v \in V_\circ \sqcup V_\bullet$; the stability condition requires $\#H(v) \geq 3$ for any $v \in V_\bullet$ and $\#H(v) \geq 1$ for any $v \in V_\circ$; each such "half-node" is a point on the boundary $(\simeq S^1)$ of the irreducible component v; hence each set $H(v)$ comes equipped with a fixed cyclic ordering;
- the fixed point free involution τ on the disjoint union

$$H := \coprod_{v \in V_\circ \sqcup V_\bullet} H(v)$$

  controlling which "half-nodes" in the boundaries of disks are glued into genuine nodes; the set of orbits of this involution, i.e. the set of nodes, is denoted by E; for each $e \in E$ the associated 2-element orbit of τ is denoted by $H(e)$;
- the orientation of cell O can be encoded, according to [C2], into a total ordering of the set

$$V_\bullet \sqcup H$$

  defined up to an even permutation.

As the involution τ decomposes the set H into the disjoint union of its orbits,

$$H = \coprod_{e \in E} H(e).$$

an orientation of the cell O can be equivalently encoded into the choice of one of the two possible unit vectors of the 1-dimensional Euclidean space (cf. §**3.4**)

$$\det(V_\bullet(\Gamma)) \bigotimes_{e \in E(\Gamma)} \det(H(e))$$

Thus each orbi-cell $O \subset \mathcal{D}_{g,m,0,n}$ gives us a uniquely defined oriented ribbon graph Γ such that

(i) its set of vertices comes decomposed into the disjoint union of black and white vertices,

$$V_\circ(\Gamma) \sqcup V_\bullet(\Gamma)$$

and a marking isomorphism $V_\circ(\Gamma) \to [n]$ is fixed; each vertex v has $H(v)$ half-edges attached with $\#H(v) \geq 3$ for $v \in V_\bullet(\Gamma)$ and $\#H(v) \geq 1$ for $v \in V_\circ(\Gamma)$;

(ii) it has m labelled boundaries;

(iii) its set of edges $E(\Gamma)$ is in one-to-one correspondence with the set of nodes E of surfaces in the cell O, or, equivalently, with the set of orbits of the involution τ on H;

(iv) it is assigned the cohomological degree

$$|\Gamma|_{geom} := - \dim O = 3\#V_\bullet(\Gamma) + \#V_\circ(\Gamma) - 2\#E(\Gamma)$$

(v) it is equipped with an orientation in exactly the same sense as graphs from $\mathsf{tw}\mathcal{RG}ra_d(m,n)$ for d odd (see §**3.3**);

(vi) the equality

$$g = 1 + \frac{1}{2}\left(\#E(\Gamma) - \#V_\bullet(\Gamma) - n - m\right)$$

holds true.

The correspondence between the set of cells O of $\mathcal{D}_{g,m,0,n}$ and the set oriented ribbon graphs Γ satisfying the conditions (i)-(iv) above is one-to-one. Given such a graph Γ, the associated cell O_Γ is given by

$$O_\Gamma = \left(\prod_{v \in V_\bullet(\Gamma)} \frac{\{H(v) \hookrightarrow S^1\}}{SL(2, \mathbb{R})} \times \prod_{v \in V_\circ(\Gamma)} \frac{\{H(v) \hookrightarrow S^1\}}{S^1}\right)_{Aut\Gamma}$$

where $Aut\Gamma$ is the automorphism groups of Γ. This formula explains the above formula for the geometric degree $|\Gamma| = - \dim O_\Gamma = 3\#V_\bullet(\Gamma) - 2\#E(\Gamma) + n$. On the other hand, a ribbon graph in $\mathsf{tw}\mathcal{RG}ra_3$ with exactly the same characteristics would have the cohomological degree $3\#V_\bullet(\Gamma) - 2\#E(\Gamma)$. Therefore we observe a one-to-one correspondence between the set of generators (=cells) of the K. Costello's cell complex, $\mathcal{C}hains(\mathcal{D}_{g,m,0,n})$, and the set of ribbon graph generators of the complex $\mathsf{tw}\mathcal{RG}ra_3(m,n;g)[-n]$ introduced in §**3.3** implying the canonical isomorphism,

$$\mathcal{C}hains(\mathcal{D}_{g,m,0,n}) = \mathsf{tw}\mathcal{RG}ra_3^{\geq 3}(m,n;g)[-n]$$

of graded vector spaces. It is immediate to see that the differentials of both sides are also identical to each other so that, using (14), we get a canonical isomorphism of complexes (in fact, of dg $\mathbb{S}_m^{op} \times \mathbb{S}_n$-bimodules),

$$\mathcal{C}hains(\mathcal{D}_{g,m,0,n}) = \mathsf{tw}\mathcal{RG}ra_3^{\geq 3}(m,n;g)[-n] = \mathsf{tw}\mathcal{RG}ra_0^{\geq 3}(m,n;g)[6g-6+3m+2n]$$

or, equivalently,

$$\mathsf{tw}\mathcal{RG}ra_0^{\geq 3}(m,n;g) = \mathcal{C}hains(\mathcal{D}_{g,m,0,n})[6-6g-3m-2n]$$

Thus we proved the following

3.8.1. Proposition-definition. *The $\mathbb{S}$-bimodule*

$$\{\mathcal{C}hains(\mathcal{D}_{g,m,0,n})[6-6g-3m-2n]\}_{m\geq 1, n\geq 0, 2g+m+n\geq 3}$$

has a canonical structure of dg properad with properadic compositions given by the explicit combinatorial rules for $\mathsf{tw}\mathcal{RG}ra_0^{\geq 3}$.

K. Costello's isomorphism (16) and the Poincare duality imply

$$H_\bullet(\mathcal{D}_{g,m,0,n}) = H_\bullet(\mathcal{N}_{g,m,0,n}) = H_c^{\bullet+6g-6+3m+2n}(\mathcal{N}_{g,m,0,n})$$

On the other hand, the uniformization theorem gives us an equality of compactly supported cohomology groups

$$H_c^\bullet(\mathcal{N}_{g,m,0,n}) = H_c^\bullet(\mathcal{M}_{g,m+n} \times \mathbb{R}_{>0}^m) = H_c^{\bullet-m}(\mathcal{M}_{g,m+n})$$
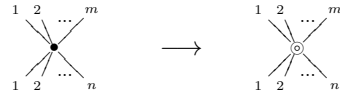
Combining all the above isomorphisms we finally obtain

$$H^\bullet(\mathsf{tw}\mathcal{RG}ra_0^{\geq 3}(m,n;g)) = H_\bullet(\mathcal{D}_{g,m,0,n})[6-6g-3m-2n] = H_c^\bullet(\mathcal{N}_{g,m,0,n}) = H_c^{\bullet-m}(\mathcal{M}_{g,m+n})$$

which proves finally Theorem **3.6.1** and hence Theorem A in the Introduction.

4. Totality of cohomology groups of $\mathcal{M}_{g,m+n}$ as a complex

4.1. Deformation complex of properads under $q\mathcal{L}ieb_{c,d}$. Let $(\mathcal{P},\mathsf{d})$ be a dg properad. We represent its elements as decorated (m,n)-corollas, see (4). Assume $\mathcal{P}$ comes equipped with a morphism of properads

$$i: \quad (\mathcal{H}oqlieb_{c,d},\delta) \quad \longrightarrow \quad (\mathcal{P},\mathsf{d})$$

The images of the generators of $\mathcal{H}oqlieb_{c,d}$ are special elements in $\mathcal{P}$ and hence are distinguished by a special notation $\odot$ for the vertices.

According to the general theory developed in [MeVa], the deformation complex of the morphism i is given by the following $\mathbb{Z}$-graded vector space,

$$\mathsf{Def}(\mathcal{H}oqlieb_{c,d} \xrightarrow{i} \mathcal{P}) = \prod_{m \geq 1, n \geq 0} \mathcal{P}(m,n) \otimes_{\mathbb{S}_m \times \mathbb{S}_m} \left(sgn_m^{\otimes|c|} \otimes sgn_n^{\otimes|d|} \right) [c(1-m) + d(1-n)]$$

Its elements can also be represented as corollas whose vertices are decorated by elements of $\mathcal{P}$ and whose out- and in-going legs (if any) are (skew)symmetrized so that their labellings can be omitted. To distinguish these elements from elements of $\mathcal{P}$ and $\mathcal{H}oqlieb_{c,d}$, we denote vertices of such corollas by $\circledast$, e.g. the corolla

can stand for an element in the deformation complex of the form

 $\in \mathsf{Def}(\mathcal{H}oqlieb_{c,d} \to \mathcal{P}).$

which should be understood as the decoration of the $\circledast$-vertex of the above corolla. Then the differential in the deformation complex δ is given by

(17) $$\delta: \quad \text{}$$

where the rule of signs depends on d and is read from (10).

In practice we work in this paper with properads under $q\mathcal{L}ieb_{c,d}$ rather than under $\mathcal{H}oqlieb_{c,d}$. The above machinery applies to such cases via the composition

$$i: \mathcal{H}oqlieb_{c,d} \twoheadrightarrow \mathcal{L}ieb_{c,d} \xrightarrow{\imath} \mathcal{P}$$

i.e. again we work, by definition, with the same graded vector space

$$\mathsf{Def}(q\mathcal{L}ieb_{c,d} \xrightarrow{i} \mathcal{P}) = \prod_{m \geq 1, n \geq 0} \mathcal{P}(m,n) \otimes_{\mathbb{S}_m \times \mathbb{S}_m} \left(sgn_m^{\otimes|c|} \otimes sgn_n^{\otimes|d|} \right) [c(1-m) + d(1-n)]$$

but the differential in (17) simplifies — all the terms in the r.h.s. vanish except the ones containing images of the generators of $q\mathcal{L}ieb_{c,d}$,

(18)

$$\delta: \quad \text{[corolla diagram]} \quad \longrightarrow d \quad \text{[corolla diagram]} \quad + \sum \left(\pm \text{[diagram]} \pm \text{[diagram]} \pm \text{[diagram]} \pm \text{[diagram]} \pm \text{[diagram]} \right)$$

that is, the following corollas

$$\text{[corolla]} \;\; := \imath \left(\text{[corolla]} \right), \qquad \text{[corolla]} \;\; := \imath \left(\text{[corolla]} \right), \qquad \text{[corolla]} \;\; := \imath \left(\text{[corolla]} \right).$$

Hence the differential in the deformation complex spits into four parts,

(19) $$\delta = \mathsf{d} + \delta_1 + \delta_2 + \delta_3$$

where d is the differential in $\mathcal{P}$, δ_1 (resp., δ_2 and δ_3) is given by all terms involving [corolla] (resp., [corolla]

and [corolla]). It is easy to see that they satisfy the relations

$$\delta_1^2 = 0, \quad \delta_3^2 = 0, \quad \delta_2^2 + \delta_1\delta_3 + \delta_3\delta_1 = 0, \quad \delta_2\delta_3 + \delta_3\delta_2 = 0, \quad \delta_1\delta_2 + \delta_2\delta_1 = 0.$$

All these relations are just incarnations of the generating relations in the ideal $\mathcal{R}_q$ used to define $q\mathcal{L}ieb_{c,d}$.

4.2. An application to the gravity properad. The above Proposition and Theorem **3.6.1** imply that the totality of compactly supported cohomology groups of moduli spaces $\mathcal{M}_{g,m+n}$ with (skew)symmetrized punctures can be made into a complex. Indeed, the deformation complex of the above morphism j has the following structure

(20) $$\mathsf{Def}(q\mathcal{L}ieb_{-1,0} \xrightarrow{j} \mathcal{GR}av) \simeq \prod_{\substack{g,n\geq 0, m\geq 1 \\ 2g+n+m\geq 3}} H^{\bullet-1}(\mathcal{M}_{g,m+n}) \otimes_{\mathbb{S}_m^{op}\times\mathbb{S}_n} (sgn_m \otimes \mathbb{1}_n)$$

The induced differential has three parts (cf. (19)),

$$\delta = \delta' + \delta'' + \delta''',$$

$$\delta' : H^{\bullet-1}(\mathcal{M}_{g,m+n}) \otimes_{\mathbb{S}_m^{op}\times\mathbb{S}_n} (sgn_m \otimes \mathbb{1}_n) \longrightarrow H^{\bullet}(\mathcal{M}_{g,m+(1+n)}) \otimes_{\mathbb{S}_m^{op}\times\mathbb{S}_{n+1}} (sgn_m \otimes \mathbb{1}_{n+1})$$

$$\delta'' : H^{\bullet-1}(\mathcal{M}_{g,m+n}) \otimes_{\mathbb{S}_m^{op}\times\mathbb{S}_n} (sgn_m \otimes \mathbb{1}_n) \longrightarrow H^{\bullet}(\mathcal{M}_{g,(m+1)+n}) \otimes_{\mathbb{S}_{m+1}^{op}\times\mathbb{S}_n} (sgn_{m+1} \otimes \mathbb{1}_n)$$

$$\delta''' : H^{\bullet-1}(\mathcal{M}_{g,m+n}) \otimes_{\mathbb{S}_m^{op}\times\mathbb{S}_n} (sgn_m \otimes \mathbb{1}_n) \longrightarrow H^{\bullet}(\mathcal{M}_{g,(m+2)+(n-1)}) \otimes_{\mathbb{S}_{m+2}^{op}\times\mathbb{S}_{n-1}} (sgn_{m+2} \otimes \mathbb{1}_{n-1})$$

with each part having a geometric interpretation as the attachment of the pair of pants but in different hyperbolic incarnations — the first part employs the pair of pants as the hyperbolic sphere with one geodesic boundary and two cusps, the second one as the hyperbolic sphere with two geodesic boundaries and one cusp, and the third one as $\mathbb{S}^2$ with three geodesic boundaries and no cusps. The differential preserves the genus parameter so this complex is a product of complexes, one for each fixed g.

4.2.1. Theorem B from the Introduction. *The complex (20) has infinitely many cohomology classes.*

These cohomology classes can described explicitly in terms of twisted ribbon graphs (see below). There are strong indications that they all come from $H_c^\bullet(\mathcal{M}_1)$ so we conjecture that *the cohomology of the complex (20) is equal to* $\prod_{g\geq 1} H_c^\bullet(\mathcal{M}_g)$, where $\mathcal{M}_g$ is the moduli space of (unpunctured) algebraic curves of genus g.

To proof this Theorem we have to understand better the deformation complex of the dg properad $\mathcal{H}oqlieb_{c,d}$ and its relation to the M. Kontsevich graph complexes and their oriented versions.

4.3. Reminder on Maxim Kontsevich's graph complexes (after [K2, W1]). An (ordinary, i.e. non-ribbon) *graph* Γ can be understood as a 1-dimensional *CW* complex whose 0-cells are called *vertices* and 1-cells are called *edges*; the set of vertices of Γ is denoted by $V(\Gamma)$ and the set of edges by $E(\Gamma)$. A graph Γ is called *directed* if each edge $e \in E(\Gamma)$ comes equipped with a fixed orientation; we show it in pictures as an arrow. If a vertex v of a directed graph Γ has $m \geq 0$ outgoing edges and $n \geq 0$ incoming edges, then we say that v is an (m,n)-*vertex*. A $(1,1)$-vertex is called *passing*. Here are a few examples of directed graphs

$$\longrightarrow \; , \qquad \triangleright \; , \qquad \infty \; .$$

Let $DG_{n,l}$ be the set of directed graphs Γ with n vertices and l edges such that some bijections $V(\Gamma) \to [n]$ and $E(\Gamma) \to [l]$ are fixed, i.e. every edge and every vertex of Γ is marked. The permutation group $\mathbb{S}_l$ acts on $DG_{n,l}$ by relabeling the edges. Given any integer $d \in \mathbb{Z}$, we consider a collection of $\mathbb{S}_n$-modules,

$$\mathcal{DGra}_d = \left\{ \mathcal{DGra}_d(n) := \prod_{l \geq 0} \mathbb{K}\langle DG_{n,l} \rangle \otimes_{\mathbb{S}_l} sgn_l^{\otimes |d-1|}[l(d-1)] \right\}_{n \geq 1}$$

where the group $\mathbb{S}_n$ acts on $\mathcal{DGra}_d(n)$ by relabeling the vertices. Each generator Γ of this $\mathbb{Z}$-graded vector space is assigned the degree $(1-d)\#E(\Gamma)$ and, if d is even, it is also assumed that Γ is equipped with a choice of ordering of edges up to an even permutation (an odd permutation acts as the multiplication by -1). The latter condition kills some generators from $DG_{n,l}$; for example, the graph $\infty \in \mathcal{DGra}_{d \in 2\mathbb{Z}}(2)$ vanishes identically as it admits an automorphism which changed the ordering of edges by an odd permutation, i.e. it is equal to minus itself.

The $\mathbb{S}$-module $\mathcal{DGra}_d$ has an operad structure with the operadic composition,

$$\circ_i : \quad \mathcal{DGra}_d(n) \times \mathcal{DGra}_d(m) \quad \longrightarrow \quad \mathcal{DGra}_d(m+n-1), \quad \forall \, i \in [n]$$
$$(\Gamma_1, \Gamma_2) \qquad \longrightarrow \qquad \Gamma_1 \circ_i \Gamma_2,$$

given the substitution of the graph Γ_2 into the i-labeled vertex of Γ_1 and taking a sum over re-attachments of dangling edges (attached earlier to v_i) to the vertices of Γ_2 in all possible ways. The ordering of edges $\Gamma_1 \circ_i \Gamma_2$ is uniquely determined by the given of orderings of edges in Γ_1 followed by the given ordering of edges in Γ_2. For example,
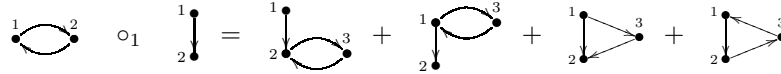
One often works with an "undirected" version, $\mathcal{Gra}_d$, of the above operad by taking coinvariants with respect of the action of the group $\mathbb{S}_l \ltimes (\mathbb{S}_2)^l$ on the linear space $\mathbb{K}\langle DG_{n,l} \rangle$ given by relabelling the edges and reversing their directions,

$$\mathcal{Gra}_d := \left\{ \mathcal{Gra}_d(n) := \prod_{l \geq 0} \mathbb{K}\langle G_{n,l} \rangle \otimes_{\mathbb{S}_l \ltimes (\mathbb{S}_2)^l} sgn_l^{\otimes |d-1|} \otimes sgn_2^{\otimes l|d|}[l(d-1)] \right\}_{n \geq 1}$$

For d even elements $\mathcal{Gra}_d(n)$ can be understood as genuine undirected graphs, e.g.

$$\underset{1}{\bullet}\!-\!\!\underset{2}{\bullet} \; \in \mathcal{Gra}_{d \in 2\mathbb{Z}}(2), \qquad \triangleright \; \in \mathcal{Gra}_{d \in 2\mathbb{Z}}(3)$$

while for d odd the directions on edges are defined up to a flip (and multiplication by -1),

$$\underset{1}{\bullet}\!\longrightarrow\!\underset{2}{\bullet} = - \; \underset{2}{\bullet}\!\longrightarrow\!\underset{1}{\bullet} \; , \qquad \infty = - \; \infty \quad \text{in} \;\; \mathcal{Gra}_{d \in 2\mathbb{Z}+1}(2).$$

There is an obvious epimorphism of operads,

$$\mathcal{DGra}_d \longrightarrow \mathcal{Gra}_d.$$

and a less obvious (but easy to check) morphism of operads

$$i: \quad \mathcal{Lie}_d \quad \longrightarrow \qquad \mathcal{DGra}_d$$
$$\bigwedge_{1 \; 2} \quad \longrightarrow \quad \tfrac{1}{2}\left(\underset{1}{\bullet}\!\longrightarrow\!\underset{2}{\bullet} + (-1)^d \; \underset{2}{\bullet}\!\longrightarrow\!\underset{1}{\bullet} \right)$$

Hence both operads of graphs $\mathcal{D}\mathcal{G}ra_d$ and $\mathcal{G}ra_d$ can be twisted with the help of T. Willwacher's endofunctor producing new important dg operads [K3, W1]. Another very useful corollary of this fact is that one can introduce dg Lie algebras, or *graph complexes*,

$$\mathsf{dfGC} := \mathsf{Def}(\mathcal{L}ie_d \to \mathcal{D}\mathcal{G}ra_d) \quad \text{and} \quad \mathsf{fGC}_d := \mathsf{Def}(\mathcal{L}ie_d \to \mathcal{G}ra_d)$$

controlling deformations of the above map i which are given explicitly by

$$\mathsf{dfGC}_d = \prod_{l \geq 0} \prod_{n \geq 1} \mathbb{K}\langle DG_{n,l} \rangle \otimes_{\mathbb{S}_n \times \mathbb{S}_l} \left(sgn_n^{\otimes |d|} \otimes sgn_l^{|d-1|} \right) [d(1-n) + l(d-1)]$$

$$\mathsf{fGC}_d = \prod_{l \geq 0} \prod_{n \geq 1} \mathbb{K}\langle DG_{n,l} \rangle \otimes_{\mathbb{S}_n \times (\mathbb{S}_l \ltimes (\mathbb{S}_2)^l)} \left(sgn_n^{\otimes |d|} \otimes (sgn_l^{|d-1|} \otimes (sgn_2^{|d|})^{\otimes l}) \right) [d(1-n) + l(d-1)]$$

the letter f standing for *full*. Thus numerical labels of vertices of graphs get (skew)symmerized and hence can be forgotten. Graphs Γ from dfGC_d or fGC_d are assigned the cohomological degree,

$$|\Gamma| = d(\#V(\Gamma) - 1) + (1-d)\#E(\Gamma).$$

The differential is determined by the image of the Lie generator in the corresponding operad of graphs, and is given in, say, fGC_d explicitly by

$$\delta\Gamma := \sum_{v \in V(\Gamma)} \left(\delta_v' \Gamma - (-1)^{|\Gamma|} \delta_v'' \Gamma \right)$$

where δ_v' substitutes into the vertex v the graph $\bullet\!\!-\!\!\!\bullet$ and redistributes the attached to v half-edges (whose set is denoted by $H(v)$) among the two new vertices in all possible ways,

while δ_v'' attaches to v a new univalent vertex

If a vertex $v \in V(\Gamma)$ has at least one half-edge, then the term with the new univalent vertex in the sum $\delta_v'\Gamma$ cancels out with the similar term in $\delta_v''\Gamma$. The graph complexes dfGC_d and, resp., fGC_d contain subcomplexes dGC_d and $\mathsf{GC}_d^{\geq 2}$ spanned by *connected* graphs with all vertices at least bivalent and with no passing vertices; the inclusions

$$\mathsf{dGC}_d \longrightarrow \mathsf{dfGC}_d, \quad \mathsf{GC}_d^{\geq 2} \longrightarrow \mathsf{fGC}_d$$

are quasi-isomorphisms [W1]. There is also a quasi-isomorphism of dg Lie algebras

(21) $$\mathsf{GC}_d^{\geq 2} \longrightarrow \mathsf{dGC}_d,$$

which sends a graph with no direction on edges into a sum of graphs with all possible directions on edges [W1]. The complex dGC_d contains a subcomplex OGC_d spanned by directed graphs with no closed paths of directed edges. It was proven in [W2] that

$$H^\bullet(\mathsf{GC}_d^{\geq 2}) = H^\bullet(\mathsf{GC}_{d+1}^{or})$$

and that for $d = 2$

$$H^0(\mathsf{dGC}_2) = H^0(\mathsf{GC}_2^{\geq 2}) = H^0(\mathsf{GC}_3^{or}) = \mathfrak{grt}_1,$$

where $\mathfrak{grt}_1$ is the Lie algebra of the Grothendieck-Teichmüller group GRT_1 introduced in [Dr3].

The complex $\mathsf{GC}_d^{\geq 2}$ decomposes into a direct sum of complexes,

$$\mathsf{GC}_d^{\geq 2} = \mathsf{GC}_d \oplus \mathsf{GC}_d^2,$$

where GC_d is generated by graphs with each vertex at least trivalent, and GC_d^2 is generated by graphs containing at least one bivalent vertex. The cohomology of the latter has been computed in [W1],

$$H^\bullet(\mathsf{GC}_d^2) = \bigoplus_{\substack{j \geq 1 \\ j \equiv 2d+1 \mod 4}} \mathbb{K}[d-j],$$

where the summand $\mathbb{K}[d-j]$ is generated by the polytope with j bivalent vertices and j edhes. For example, for d odd, the graph,

contributes into that cohomology, while for d even the graph

is a cohomology class.

The complexes GC_d for d of the same parity are isomorphic to each other (up to degree shift) so that there are essentially two different complexes, one for $d \in 2\mathbb{Z}$ and one for $d \in 2\mathbb{Z}+1$. The even complex GC_2 is very useful because its degree zero cohomology is equal to $\mathfrak{grt}_1$. The odd complex GC_3 has also very nice properties: its cohomology is concentrated in degrees ≤ -3; its degree -3 cohomology is generated by trivalent graphs and has the structure of a graded commutative algebra which is conjecturally generated by formal variables t, ω_0, ω_1, ... subject to the relations $\omega_p \omega_q = \omega_0 \omega_{p+q}$ [Kn, V, KWZ].

Unfortunately, there is no direct relation between the complexes $\mathsf{GC}_d^{\geq 2}$ and OGC_{d+1} (however there is a beautiful explicit quasi-isomorphism between their dual subcomplexes [Z] spanned by graphs containing at least one trivalent vertex). It is not hard too see, however, what happens to the polytope classes in GC_d in their oriented incarnation in OGC_{d+1}: the polytope cohomology class with j vertices in $\mathsf{GC}_d^{\geq 2}$ gets associated a polytope cohomology class with $j+1$ non-passing vertices in OGC_{d+1} [MW1],

$$H^\bullet(\mathsf{GC}_{odd}^{\geq 2}) \ni \qquad \rightsquigarrow \qquad \in H^\bullet(\mathsf{OGC}_{even}),$$

$$H^\bullet(\mathsf{GC}_{even}^{\geq 2}) \ni \qquad \rightsquigarrow \qquad \in H^\bullet(\mathsf{OGC}_{odd}).$$

We shall use this correspondence below.

4.4. Graph complexes and deformations of quasi-Lie bialgebras. Let I_s be the differential ideal in $\mathcal{H}oqlieb_{c,d}$ generated by $(m,0)$-corollas with $m \geq 3$ (we call such corollas *sources*). The associated short exact sequence of dg properads,

$$0 \longrightarrow I_s \longrightarrow \mathcal{H}oqlieb_{c,d} \longrightarrow \mathcal{H}olieb_{c,d} \longrightarrow 0$$

implies an associated short exact sequence of dg Lie algebras,

$$0 \longrightarrow \mathsf{Ker}\,\pi \longrightarrow \mathsf{Def}(\mathcal{H}oqlieb_{c,d} \overset{\mathrm{Id}}{\to} \mathcal{H}oqlieb_{c,d}) \overset{\pi}{\longrightarrow} \mathsf{Def}(\mathcal{H}oqlieb_{c,d} \overset{\mathrm{Id}}{\to} \mathcal{H}olieb_{c,d}) \longrightarrow 0$$

where $\mathsf{Ker}\,\pi$ is the subcomplex of $\mathsf{Def}(\mathcal{H}oqlieb_{c,d} \to \mathcal{H}oqlieb_{c,d})$ spanned by graphs with at least one source. As

$$\mathsf{Def}(\mathcal{H}oqlieb_{c,d} \overset{\mathrm{Id}}{\to} \mathcal{H}olieb_{c,d}) \simeq \mathsf{Def}(\mathcal{H}olieb_{c,d} \overset{\mathrm{Id}}{\to} \mathcal{H}olieb_{c,d})$$

we have a canonical morphism of cohomology groups

$$\pi : H^\bullet\left(\mathsf{Def}(\mathcal{H}oqlieb_{c,d} \overset{\mathrm{Id}}{\to} \mathcal{H}oqlieb_{c,d})\right) \longrightarrow H^\bullet\left(\mathsf{Def}(\mathcal{H}olieb_{c,d} \overset{\mathrm{Id}}{\to} \mathcal{H}olieb_{c,d})\right).$$

Let $\widehat{\mathcal{H}oqlieb}_{c,d}$ (resp., $\widehat{\mathcal{H}olieb}_{c,d}$) be the genus completed version of the properad $\mathcal{H}oqlieb_{c,d}$ (resp., $\widehat{\mathcal{H}olieb}_{c,d}$) and let $\mathrm{Der}(\widehat{\mathcal{H}oqlieb}_{c,d})$ (resp. $\mathrm{Der}(\widehat{\mathcal{H}oqlieb}_{c,d})$) be the dg Lie algebra of its derivations. There is a canonical isomorphism of complexes (but not of Lie algebras),

$$\mathrm{Der}(\widehat{\mathcal{H}oqlieb}_{c,d}) \simeq \mathsf{Def}(\mathcal{H}oqlieb_{c,d} \overset{\mathrm{Id}}{\to} \mathcal{H}olieb_{c,d})[1]$$

and similarly for $\mathrm{Der}(\widehat{\mathcal{H}olieb}_{c,d})$. One of the main results in [MW2] is a construction of an explicit quasi-isomorphism (up to one rescaling class) of dg Lie algebras

(22)
$$F: \quad \begin{matrix} \mathsf{OGC}_{c+d+1} & \to & \mathrm{Der}(\widehat{\mathcal{H}olieb}_{c,d}) \\ \Gamma & \to & F(\Gamma) \end{matrix}$$

where the derivation $F(\Gamma)$ has, by definition, the following values on the generators of the completed properad $\widehat{\mathcal{H}olieb}_{c,d}$

(23)
$$\left(\overset{1\ 2\ \cdots\ m}{\underset{1\ 2\ \cdots\ n}{\times}} \right) \cdot F(\Gamma) = \sum_{\substack{s:[n]\to V(\Gamma) \\ \hat{s}:[m]\to V(\Gamma)}} \overset{1\ 2\ \cdots\ m}{\underset{1\ 2\ \quad n}{\Gamma}} \qquad \forall\, m,n \geq 1, m+n \geq 3,$$

where the sum is taken over all ways of attaching the incoming and outgoing legs to the graph Γ, and setting to zero every graph which has at east one vertex of valency ≤ 2 or at least one vertex of type $(0,n)$ or $(n,0)$ with $n \geq 3$.

It is easy to see that the above map F extends to a morphism of dg Lie algebras,

$$F_q : \mathsf{OGC}_{c+d+1} \longrightarrow \mathrm{Der}(\widehat{\mathcal{H}oqlieb}_{c,d})$$

which is given by the same formula as in (22) as above except that now $m \geq 1, n \geq 0, m+n \geq 3$ and we set to zero on the r.h.s. every graph which has at east one vertex of valency ≤ 2 or at least one vertex of type $(0,n)$ with $n \geq 3$. Therefore we get a commutative diagram,

$$\begin{array}{ccc} H^{\bullet}(\mathsf{OGC}_{c+d+1}) & \overset{F_q}{\longrightarrow} & H^{\bullet+1}\left(\mathsf{Def}(\mathcal{H}oqlieb_{c,d} \overset{\mathrm{Id}}{\to} \mathcal{H}oqlieb_{c,d}) \right) \\ & \overset{F}{\searrow} & \downarrow \pi \\ & & H^{\bullet+1}\left(\mathsf{Def}(\mathcal{H}olieb_{c,d} \overset{\mathrm{Id}}{\to} \mathcal{H}olieb_{c,d}) \right) \end{array}$$

where the diagonal map is an isomorphism (up to one rescaling class). Therefore there is an *injection* of cohomology groups,

$$H^{\bullet}(\mathsf{GC}_{c+d}^{\geq 2}) \simeq H^{\bullet}(\mathsf{OGC}_{c+d+1}) \overset{F_q}{\longrightarrow} H^{\bullet+1}\left(\mathsf{Def}(\mathcal{H}oqlieb_{c,d} \overset{\mathrm{Id}}{\to} \mathcal{H}oqlieb_{c,d}) \right).$$

Given a dg properad $\mathcal{P}$ under $\mathcal{H}oqlieb_{c,d}$,

$$i : \mathcal{H}oqlieb_{c,d} \longrightarrow \mathcal{P},$$

the map i induces a morphism of complexes

$$\mathsf{Def}(\mathcal{H}oqlieb_{c,d} \overset{\mathrm{Id}}{\to} \mathcal{H}oqlieb_{c,d}) \longrightarrow \mathsf{Def}(\mathcal{H}oqlieb_{c,d} \overset{i}{\to} \mathcal{P})$$

and hence a morphism of cohomology groups,

$$H^{\bullet}(\mathsf{OGC}_{c+d+1}) \overset{F_q}{\longrightarrow} H^{\bullet+1}\left(\mathsf{Def}(\mathcal{H}oqlieb_{c,d} \overset{i}{\to} \mathcal{P}) \right).$$

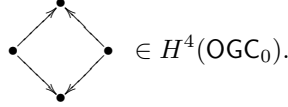which sometimes can be checked by explicit computations.

4.5. Proof of Theorem B from the Introduction. Let us apply the above machinery to the morphism

$$j : q\mathcal{L}ieb_{-1,0} \longrightarrow H^\bullet(\mathcal{GR}av)$$
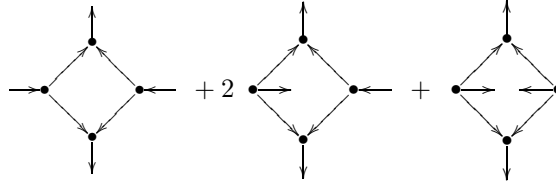
and check what happens to the polytope cohomology classes in $H^\bullet(\mathsf{OGC}_0)$ under the canonical morphism

$$H^\bullet(\mathsf{OGC}_0) \longrightarrow H^{\bullet+1}\left(\mathsf{Def}(q\mathcal{L}ieb_{-1,0} \overset{j}{\to} \mathcal{GR}av)\right).$$

The simplest polytope class is given by the graph

 $\in H^4(\mathsf{OGC}_0).$

The map F_q sends it into the following degree 5 cohomology class in $\mathsf{Def}(q\mathcal{L}ieb_{-1,0} \overset{\mathrm{Id}}{\to} q\mathcal{L}ieb_{-1,0})$,

which under the explicit map j given by (15) goes into a genus 1 cycle in the complex (20) represented as a linear combination of twisted graphs from $\mathsf{tw}\mathcal{RGR}a_0^{\geq 3}$. That linear combination contains a summand with no white vertices at all which, in this particular case, comes from the last term in the above sum, and is given by

From this explicit representation of that summand we conclude that the cocycle can not be a coboundary; a similar observation holds true for any other polytope class from GC_{-1}^2 — the summands with no white vertices are all of the similar type and hence can not coboundaries with respect to δ_3 (cf. with §7.4 in [MW1]). This completes the proof of Theorem B.

All these polytope cohomology classes in the complex (20) have genus 1 and come conjecturally from the cusp series of cohomology classes[5] in $H_c^\bullet(\mathcal{M}_1)$ (cf. [B, T]). Thus one can make two conjectures about the complex (20): the cohomology of that complex is equal to (a) $\prod_{g\geq 1} H_c^\bullet(\mathcal{M}_g)$, and (b) it contains the cohomology of M. Kontsevich's *odd* graph complex $H^\bullet(\mathsf{GC}_{-1}^{\geq 2})$.

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