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Bismut-Stroock Hessian formulas and local Hessian estimates for heat semigroups and harmonic functions on Riemannian manifolds

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Abstract

In this article, we develop a martingale approach to localized Bismut-type Hessian formulas for heat semigroups on Riemannian manifolds. Our approach extends the Hessian formulas established by Stroock (An estimate on the Hessian of the heat kernel: 355–371, 1996) and removes in particular the compact manifold restriction. To demonstrate the potential of these formulas, we give as application explicit quantitative local estimates for the Hessian of the heat semigroup, as well as for harmonic functions on regular domains in Riemannian manifolds.

Keywords Brownian motion · Heat semigroup · Harmonic function · Hessian estimate

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1 Introduction

Let M be a complete Riemannian manifold of dimension n with Levi-Civita connection ∇. We consider the operator $L = \frac{1}{2}\Delta$ where $\Delta := \operatorname{tr} \nabla \mathbf{d}$ is the Laplace-Beltrami operator on M and $\mathbf{d}$ the exterior differential (note that we use a bold letter for the exterior differential to distinguish it from the Itô differential d appearing later). Denote by $X_{\cdot}(x)$ a Brownian motion on M starting at $x \in M$ with generator L and explosion time $\zeta(x)$. The explosion time is the random time at which the process leaves all compact subsets of M. Furthermore, let B_t be the stochastic anti-development of $X_{\cdot}(x)$ which is a Brownian motion on T_xM, stopped at the lifetime $\zeta(x)$ of $X_{\cdot}(x)$, and denote by $P_t f$ the associated minimal heat semigroup, acting on bounded measurable functions f, which is represented in probabilistic terms by the formula

$$(P_t f)(x) = \mathbb{E}\big[f(X_t(x))\mathbf{1}_{\{t<\zeta(x)\}}\big].$$

Recall that for any $T > 0$ fixed, $(t, x) \mapsto (P_t f)(x)$ is smooth and bounded on $(0, T] \times M$ with $P_0 f = f$.

Let $Q_t : T_xM \to T_{X_t}M$ be defined by

$$DQ_t = -\frac{1}{2}\mathrm{Ric}^\sharp(Q_t)\,dt, \quad Q_0 = \mathrm{id},$$

where $D := /\!/_t \, d \, /\!/_t^{-1}$ and $/\!/_t := /\!/_{0,t} : T_xM \to T_{X_t(x)}M$ denotes parallel transport along $X(x)$. The following formula is taken from [16] and gives a typical probabilistic derivative formula for the semigroup $P_t f$; it extends earlier formulas of Elworthy-Li [8]. Results in this direction are centered around Bismut's integration by parts formula [3]. To be precise, let D be a fixed regular domain in M and τ_D be the first exit time of $X_{\cdot}(x)$ from D when started at $x \in D$. A regular domain is by definition an open, relatively compact connected domain in M with non-empty boundary. Then for $v \in T_xM$,

$$\nabla P_t f(x) = \mathbb{E}\left[f(X_t(x))\mathbf{1}_{\{t<\zeta(x)\}} \int_0^{\tau_D \wedge t} \langle Q_t(\dot{k}(s)v), /\!/_s \, dB_s\rangle\right],$$

where the function $k \in L^{1,2}(\mathbb{R}^+, [0, 1])$ satisfies the property that $k(0) = 0$ and $k(s) = 1$ for all $s \geq \tau_D \wedge t$. An explicit choice of the random function k in the above Bismut formula can be used to derive sharp gradient estimates in various situations, e.g. for heat semigroups, as well as for harmonic functions on regular domains in Riemannian manifolds, see [17, 20] for explicit results. Recently, such formulas have also been applied to quantitative estimates for the derivative of a C^2 function u in terms of local bounds on u and Δu [6].

A similar approach has been used by Arnaudon, Driver and Thalmaier [1] to approach Cheng-Yau type inequalities for the harmonic functions. Actually, the effect of curvature on the behavior of harmonic functions on a Riemannian manifold M is a classical problem (see e.g. [13, 21]), see also [7] for characterizations of gradient estimates in the context of metric measure spaces. In this paper, we aim to clarify in particular the effect of curvature on the Hessian of harmonic functions on regular domains in Riemannian manifolds. To this end, we establish a local version of a Bismut type formula for $\mathrm{Hess}P_t f$ which is consistent with Stroock's formula in [14] while here we follow a martingale approach and allow the manifold to be non-compact.

We start by giving some background on Bismut type formulas for second order derivatives of heat semigroups. A first such formula appeared in Elworthy and Li [8, 9] for a non-compact manifold, however under strong curvature restrictions. An intrinsic formula for $\mathrm{Hess}P_t f$ was given by Stroock [14] for a compact Riemannian manifold, while a localized intrinsic formula was obtained by Arnaudon, Planck and Thalmaier [2] adopting a martingale approach. A localized version of the Hessian formula (still with doubly stochastic damped parallel translations) for the Feynman-Kac semigroup was derived by Thompson [19]. The study of the Hessian of a Feynman-Kac semigroup generated by a Schrödinger operator of the type $\Delta + V$ where V is a potential, has been pushed forward by Li [10, 11]. Very recently, Bismut-type Hessian formulas have been used for new applications, see for instance Cao-Cheng-Thalmaier [4] where L^p Calderón-Zygmund inequalities on Riemannian manifolds have been established under natural geometric assumptions for indices $p > 1$.

The following formula is taken from [2] and gives a localized intrinsic Bismut type formula for the Hessian of the heat semigroup. Define an operator-valued process $W_t : T_xM \otimes T_xM \to T_{X_t}M$ as solution to the covariant Itô equation

$$\begin{aligned} DW_t(v,w) =& R\big(/\!/_t dB_t, Q_t(v)\big)Q_t(w) \\ &- \frac{1}{2}(\mathbf{d}^*R + \nabla \mathrm{Ric})^\sharp(Q_t(v), Q_t(w))\,dt - \frac{1}{2}\mathrm{Ric}^\sharp(W_t(v,w))\,dt \end{aligned}$$

with initial condition $W_0(v,w) = 0$. Here R denotes the Riemann curvature tensor. It is easy to check that

$$\begin{aligned} W_t(v,w) = Q_t \int_0^t Q_r^{-1}\Big(& R\big(/\!/_r dB_r, Q_r(v)\big)Q_r(w) - \frac{1}{2}(\mathbf{d}^*R + \nabla \mathrm{Ric})^\sharp \\ & (Q_r(v), Q_r(w))\,dr\Big). \end{aligned}$$

The operator $\mathbf{d}^*R$ is defined by $\mathbf{d}^*R(v_1)v_2 := -\mathrm{tr}\,\nabla_{\cdot}R(\cdot, v_1)v_2$ and satisfies

$$\langle \mathbf{d}^*R(v_1)v_2, v_3\rangle = \langle (\nabla_{v_3}\mathrm{Ric}^\sharp)(v_1), v_2\rangle - \langle (\nabla_{v_2}\mathrm{Ric}^\sharp)(v_3), v_1\rangle$$

for all $v_1, v_2, v_3 \in T_xM$ and $x \in M$. Let $v, w \in T_xM$ with $x \in D$, $f \in \mathcal{B}_b(M)$ and fix $0 < S < T$. Suppose that D_1 and D_2 are regular domains such that $x \in D_1$ and $\bar{D}_1 \subset D_2 \subset D$. Let σ and τ be two stopping times satisfying $0 < \sigma \le \tau_{D_1} < \tau \le \tau_{D_2}$. Assume k, ℓ are bounded adapted processes with paths in the Cameron-Martin space $L^{1,2}([0,T];[0,1])$ such that

- $k(0) = 1$ and $k(s) = 0$ for $s \ge \sigma \wedge S$;
- $\ell(s) = 1$ for $s \le \sigma \wedge S$ and $\ell(s) = 0$ for $s \ge \tau \wedge T$.

Let Hess $= \nabla \mathbf{d}$. Then for $f \in \mathcal{B}_b(M)$, we have

$$(\mathrm{Hess}_x P_T f)(v, v) = -\mathbb{E}^x \left[f(X_T) \mathbf{1}_{\{T < \zeta(x)\}} \int_0^{T \wedge \tau} \langle W_s(\dot{k}(s)v, v), //_s dB_s \rangle \right]$$
$$+ \mathbb{E}^x \left[f(X_T) \mathbf{1}_{\{T < \zeta(x)\}} \int_S^T \langle Q_s(\dot{\ell}(s)v), //_s dB_s \rangle \int_0^S \langle Q_s(\dot{k}(s)v), //_s dB_s \rangle \right]. \tag{1.1}$$

This formula has the advantage to be concise with only two terms on the right-hand side to characterize the Hessian of semigroup; the martingale approach to this formula is direct and does not need advanced path perturbation theory on Riemannian manifold. However, it depends on the choice of two random test functions k and ℓ. For local estimates the construction of two different random times τ and σ is required, which complicates the calculation of the coefficients in the estimates, see [2].

Let us compare formula (1.1) with the Hessian formula from [14] where the manifold is supposed to be compact. Stroock proved that in this case one deterministic function test function $k \in C^1([0, T], [0, 1])$ satisfying $k(0) = 1$ and $k(t) = 0$ is sufficient in a Bismut-type formula for the Hessian of the semigroup (see [15,Eq. (1.6)] and formula (2.17) below). In what follows, we call it the Bismut-Stroock Hessian formula. Although its proof relies on perturbation theory of Brownian paths and more terms are involved in the formula, the fact that only one test function k enters makes it attractive for applications to the short-time behavior of the heat kernel, see [15]. Recently, Chen, Li and Wu [5] adopted a perturbation of the driving force B_t with a second order term for a new approach to this formula. The formula is derived with localized vector fields and extends to a general (non-compact) complete Riemannian manifold by introducing a family of cut-off processes.

The described works motivate the following two questions:

(i) *Can one extend the Stroock-Bismut Hessian formula* [14] *to a local Bismut-type formula for the Hessian heat semigroup by a direct construction of suitable martingales, and achieve from it explicit local estimates of* $\mathrm{Hess} P_t f$ *by a proper choice of* k*?*

(ii) *Can the localized Bismut-Stroock Hessian formula be transformed into a formula for the Hessian of harmonic functions on regular domains in Riemannian manifolds from where quantitative estimates can be derived?*

In the sequel we answer both questions positively. Note that for the first question, it has been explained in [5,Section 4] that it seems difficult to adopt the perturbation theory of M-valued Brownian motions in order to replace the non-random vector field on path space by a random one; the time reversed field will not be adapted anymore and the Itô integral no longer be well defined. This reason motivates our search for a different stochastic approach to the problem in the form of a direct martingale argument. Actually, establishing localized Bismut-Stroock Hessian formula of semigroup relies on the proper construction of martingales which turns out to be the main difficulty for the first question. As an application of our formula, we can give explicit local Hessian estimates for the heat semigroup: for any regular domain $D \subset M$ and $x \in D$,

$$|\mathrm{Hess} P_t f|(x) \le \inf_{\delta>0}\left\{\left(\frac{t}{2}\sqrt{K_1^2+\frac{K_2^2}{\delta}}+\frac{2}{t}\right)\right.$$
$$\left.\exp\left(t\left(K_0^-+\frac{\delta}{2}+\frac{\pi\sqrt{(n-1)K_0^-}}{2\delta_x}+\frac{\pi^2(n+3)}{4\delta_x^2}\right)\right)\right\}\|f\|_D, \tag{1.2}$$

where $\delta_x = \rho(x, \partial D)$ is the Riemannian distance of x to the boundary of D and constants K_0, K_1 and K_2 are defined as in (2.10)–(2.12) below. Compared with known estimates, e.g. [2], the constants in (1.2) are explicit and concise. The estimate is not essentially more complicated, compared to the local gradient estimate [17].

The second problem is then to extract Hessian estimates for harmonic functions from the Bismut-Stroock Hessian formula which requires a careful estimate of the terms involving $W_\cdot(v, w)$ and $Q_\cdot(v)$, along with a proper choice of the process k. Let $D \subset M$ be a regular domain. In Section 3, we obtain that for a bounded positive harmonic function u on D,

$$|\mathrm{Hess}\, u|(x) \le \inf_{\substack{0<\delta_1<1/2\\ \delta_2>0}}\left\{\sqrt{(1+2\mathbf{1}_{\{K_0^-\neq 0\}})\left((\delta_1+1)K_1^2+\frac{\delta_2}{2}K_2^2\right)}\right.$$
$$\left.+\sqrt{6}\left(\frac{1}{2\delta_2}+3K_0^-+\frac{\pi\sqrt{(n-1)K_0^-}}{2\delta_x}+\frac{\pi^2(n+3+\delta_1^{-1})}{4\delta_x^2}\right)\right\}\sqrt{\|u\|_D\, u(x)},$$

see also Theorem 3.5.

The paper is organized as follows. In Section 2 we construct several local martingales to achieve local Hessian formulas for the heat semigroup which are consistent with Stroock's construction [15] on compact manifolds with a deterministic function k. In Section 3 we give explicit local Hessian estimates for harmonic functions. For reader's convenience, Section 3.1 provides the method to construct k which follows [17]. Finally, in Section 4, we apply our method to estimate the second derivative of heat semigroups and heat kernels.

2 Bismut-type formulas for the Hessian of $P_t f$

We start by constructing our fundamental martingale which will be the basis for our approach. To this end, for $v, w \in T_xM$ and $k \in L^{1,2}(\mathbb{R}^+, \mathbb{R})$ let

$$W_t^k(v,w) = Q_t\int_0^t Q_r^{-1}R\big(//_r dB_r, Q_r(k(r)v)\big)Q_r(w)$$
$$-\frac{1}{2}Q_t\int_0^t Q_r^{-1}(\mathbf{d}^*R+\nabla\mathrm{Ric})^\sharp\big(Q_r(k(r)v), Q_r(w)\big)\,dr.$$

It is easy to see that W_t^k solves the following equation:

$$\begin{aligned} DW_t^k(v, w) &= R\big(//_t\, dB_t, Q_t(k(t)v)\big)Q_t(w) \\ &\quad - \frac{1}{2}(\mathbf{d}^*R + \nabla\mathrm{Ric})^\sharp\big(Q_t(k(t)v), Q_t(w)\big)\,dt - \frac{1}{2}\mathrm{Ric}^\sharp(W_t^k(v, w))\,dt, \end{aligned}$$

and $W_0^k(v, w) = 0$.

Theorem 2.1 *Let $x \in M$ and D be a relatively compact open domain in M such that $x \in D$. Let τ be a stopping time such that $0 < \tau \le \tau_D$ where τ_D denotes the first exit time of $X(x)$ with starting point $x \in D$. Fix $T > 0$ and suppose that k is a bounded, non-negative and adapted process with paths in the Cameron-Martin space $L^{1,2}([0, T]; \mathbb{R})$ such that $k(s) = 0$ for $s \ge T \wedge \tau$, $k(0) = 1$. Then for $f \in \mathcal{B}_b(M)$ and $v, w \in T_xM$,*

$$\begin{aligned} &(\mathrm{Hess}\,P_{T-t}f)\big(Q_t(k(t)v), Q_t(k(t)v)\big) + (\mathbf{d}P_{T-t}f)(W_t^k(v, k(t)v)) \\ &\quad - 2\mathbf{d}P_{T-t}f(Q_t(k(t)v)) \int_0^t \langle Q_s(\dot{k}(s)v), //_s dB_s\rangle \\ &\quad - P_{T-t}f(X_t) \int_0^t \langle W_s^k(v, \dot{k}(s)v), //_s dB_s\rangle \\ &\quad + P_{T-t}f(X_t)\left(\left(\int_0^t \langle Q_s(\dot{k}(s)v), //_s dB_s\rangle\right)^2 - \int_0^t |Q_s(\dot{k}(s)v)|^2\,ds\right) \end{aligned}$$

is a local martingale on $[0, \tau \wedge T)$.

Proof First of all, by an approximation argument we may assume that $f \in C^\infty(M)$ and is constant outside a compact set so that $|\mathbf{d}f|$ and Δf are bounded.

Let

$$N_t(v, v) = \mathrm{Hess}\,P_{T-t}f(Q_t(v), Q_t(v)) + (\mathbf{d}P_{T-t}f)(W_t(v, v)).$$

Then $N_t(v, v)$ is a local martingale, see for instance the proof of [19,Lemma 2.7] with potential $V \equiv 0$. Furthermore, define

$$N_t^k(v, v) = \mathrm{Hess}\,P_{T-t}f(Q_t(k(t)v), Q_t(v)) + (\mathbf{d}P_{T-t}f)(W_t^k(v, v)).$$

According to the definition of $W_t^k(v, v)$, resp. $W_t(v, v)$, and in view of the fact that $N_t(v, v)$ is a local martingale, it is easy to see that

$$N_t^k(v, v) - \int_0^t (\mathrm{Hess}\,P_{T-s}f)(Q_s(\dot{k}(s)v), Q_s(v))\,ds \tag{2.1}$$

is a local martingale as well. Replacing in $N_t^k(v, v)$ the second argument v by $k(t)v$, we further see that also

$$
\begin{aligned}
N_t^k(v, k(t)v) - &\int_0^t (\mathrm{Hess} P_{T-s} f)(Q_s(\dot{k}(s)v),\, Q_s(k(t)v))\, ds \\
&- \int_0^t \mathrm{Hess} P_{T-s} f(Q_s(k(s)v),\, Q_s(\dot{k}(s)v))\, ds \\
&- \int_0^t (\mathbf{d} P_{T-s} f)(W_s^k(v, \dot{k}(s)v))\, ds \\
&+ \int_0^t \int_0^s (\mathrm{Hess} P_{T-r} f)(Q_r(\dot{k}(r)v),\, Q_r(\dot{k}(s)v))\, dr\, ds
\end{aligned} \tag{2.2}
$$

is a local martingale. Note that $N_t^k(v, k(t)v) = N_t^k(v, v)\, k(t)$. Exchanging the order of integration in the last term shows that

$$
\begin{aligned}
N_t^k(v, k(t)v) &- \int_0^t (\mathrm{Hess} P_{T-s} f)(Q_s(\dot{k}(s)v),\, Q_s(k(t)v))\, ds \\
&- \int_0^t \mathrm{Hess} P_{T-s} f(Q_s(k(s)v),\, Q_s(\dot{k}(s)v))\, ds \\
&- \int_0^t (\mathbf{d} P_{T-s} f)(W_s^k(v, \dot{k}(s)v))\, ds \\
&+ \int_0^t (\mathrm{Hess} P_{T-r} f)(Q_r(\dot{k}(r)v),\, Q_r((k(t) - k(r))v))\, dr \\
= N_t^k&(v, k(t)v) - \int_0^t (\mathbf{d} P_{T-s} f)(W_s^k(v, \dot{k}(s)v))\, ds \\
&- 2 \int_0^t \mathrm{Hess} P_{T-s} f(Q_s(k(s)v),\, Q_s(\dot{k}(s)v))\, ds
\end{aligned} \tag{2.3}
$$

is a local martingale. Moreover, by the formula

$$
P_{T-t} f(X_t) = P_T f(x) + \int_0^t \mathbf{d} P_{T-s} f(//_s dB_s) \tag{2.4}
$$

and integration by parts,

$$
\int_0^t (\mathbf{d} P_{T-s} f)(W_s^k(v, \dot{k}(s)v))\, ds - P_{T-t} f(X_t) \int_0^t \langle W_s^k(v, \dot{k}(s)v),\, //_s\, dB_s \rangle \tag{2.5}
$$

is a local martingale. Similarly, from the formula

$$
\begin{aligned}
\mathbf{d} P_{T-t} f(Q_t(k(t)v)) = \mathbf{d} P_T f(v) &+ \int_0^t (\mathrm{Hess} P_{T-s} f)(//_s dB_s,\, Q_s(k(s)v)) \\
&+ \int_0^t \mathbf{d} P_{T-s} f(Q_s(\dot{k}(s)v))\, ds
\end{aligned}
$$

it follows that

$$\begin{aligned}&\int_0^t (\mathrm{Hess}\, P_{T-s} f)(Q_s(\dot{k}(s)v), Q_s(k(s)v))\, ds\\&\quad - \mathbf{d}P_{T-t} f(Q_t(k(t)v)) \int_0^t \langle Q_s(\dot{k}(s)v), //_s dB_s\rangle\\&\quad + \int_0^t \mathbf{d}P_{T-s} f(Q_s(\dot{k}(s)v))\, ds \int_0^t \langle Q_s(\dot{k}(s)v) //_s dB_s\rangle\end{aligned} \tag{2.6}$$

is also a local martingale. Concerning the last term in (2.6), we note that

$$\begin{aligned}&\int_0^t \mathbf{d}P_{T-s} f(Q_s(\dot{k}(s)v))\, ds \int_0^t \langle Q_s(\dot{k}(s)v), //_s dB_s\rangle\\&\quad - \int_0^t \mathbf{d}P_{T-s} f(Q_s(\dot{k}(s)v)) \int_0^s \langle Q_r(\dot{k}(r)v), //_r dB_r\rangle\, ds\end{aligned}$$

is a local martingale. Combining this with (2.6) we conclude that

$$\begin{aligned}&\int_0^t (\mathrm{Hess}\, P_{T-s} f)(Q_s(\dot{k}(s)v), Q_s(k(s)v))\, ds\\&\quad - \mathbf{d}P_{T-t} f(Q_t(k(t)v)) \int_0^t \langle Q_s(\dot{k}(s)v), //_s dB_s\rangle\\&\quad + \int_0^t \mathbf{d}P_{T-s} f(Q_s(\dot{k}(s)v)) \int_0^s \langle Q_r(\dot{k}(r)v), //_r dB_r\rangle\, ds\end{aligned} \tag{2.7}$$

is a local martingale. Using the local martingales (2.5) and (2.7) to replace the last two terms in (2.3), we conclude that

$$\begin{aligned}&(\mathrm{Hess}\, P_{T-t} f)(Q_t(k(t)v), Q_t(k(t)v)) + (\mathbf{d}P_{T-t} f)(W_t^k(v, k(t)v))\\&\quad - P_{T-t} f(X_t) \int_0^t \langle W_s^k(v, \dot{k}(s)v), //_s dB_s\rangle\\&\quad - 2\mathbf{d}P_{T-t} f(Q_t(k(t)v)) \int_0^t \langle Q_s(\dot{k}(s)v), //_s dB_s\rangle\\&\quad + 2\int_0^t \mathbf{d}P_{T-s} f(Q_s(\dot{k}(s)v)) \int_0^s \langle Q_r(\dot{k}(r)v), //_r dB_r\rangle\, ds\end{aligned} \tag{2.8}$$

is a local martingale as well. On the other hand, by the product rule for martingales, we have

$$\begin{aligned}&\left(\int_0^t \langle Q_s(\dot{k}(s)v), //_s dB_s\rangle\right)^2 - \int_0^t |Q_s(\dot{k}(s)v)|^2\, ds\\&\quad = 2\int_0^t \left(\int_0^s \langle Q_r(\dot{k}(r)v), //_r dB_r\rangle\right) \langle Q_s(\dot{k}(s)v), //_s dB_s\rangle\end{aligned} \tag{2.9}$$

which along with (2.4) implies that

$$P_{T-t}f(X_t)\left(\left(\int_0^t \langle Q_s(\dot k(s)v), //_s dB_s\rangle\right)^2 - \int_0^t |Q_s(\dot k(s)v)|^2\,ds\right)$$
$$-2\int_0^t \mathbf{d}P_{T-s}f(Q_s(\dot k(s)v))\int_0^s \langle Q_s(\dot k(s)v), //_s dB_s\rangle\,ds$$

is a local martingale. Applying this observation to Eq. (2.8), we finally see that

$$\begin{aligned}&(\mathrm{Hess}P_{T-t}f)\big(Q_t(k(t)v), Q_t(k(t)v)\big) + (\mathbf{d}P_{T-t}f)(W_t^k(v, k(t)v))\\&-2\mathbf{d}P_{T-t}f(Q_t(k(t)v))\int_0^t \langle Q_s(\dot k(s)v), //_s dB_s\rangle\\&-P_{T-t}f(X_t)\int_0^t \langle W_s^k(v, \dot k(s)v), //_s dB_s\rangle\\&+P_{T-t}f(X_t)\left(\left(\int_0^t \langle Q_s(\dot k(s)v), //_s dB_s\rangle\right)^2 - \int_0^t |Q_s(\dot k(s)v)|^2\,ds\right)\end{aligned}$$

is a local martingale. This completes the proof. □

In order to formulate explicit Hessian estimates we need to introduce some geometric bounds. Let $D \subset M$ be a regular domain and τ_D be the first exit time of $X(x)$ from D. We consider the following constants:

$$K_0 := \inf\left\{\mathrm{Ric}(v, v) \colon y \in D,\ v \in T_yM,\ |v| = 1\right\}; \tag{2.10}$$
$$K_1 := \sup\{|R|(y) \colon y \in D\}; \tag{2.11}$$
$$K_2 := \sup\left\{|(\mathbf{d}^*R + \nabla\mathrm{Ric})^\sharp(v, v)|(y) \colon y \in D,\ v \in T_yM, |v| = 1\right\}. \tag{2.12}$$

For $x \in M$ and $v, w \in T_xM$, we remark that

$$|R^{\sharp,\sharp}(v, v)|_{\mathrm{HS}}(x) \le |R|(x)|v|^2$$

where

$$|R^{\sharp,\sharp}(v, v)|_{\mathrm{HS}} := \sqrt{\sum_{i,j=1}^n R(e_i, v, v, e_j)^2}$$

and $\{e_i\}_{i=1}^n$ denotes an orthonormal base of T_xM.

We shall need the following lemma.

Lemma 2.2 *Keep the assumptions and the definition of k as in Theorem 2.1, and assume that k is bounded by 1. If $\int_0^T |\dot k(t)|^{2q}\,dt \in L^1$ for some $q \in (1, +\infty]$ and*

$v \in T_x M$, then

$$\mathbb{E}\left[\int_0^T |W_t^k(v, \dot{k}(t)v)|^2\, dt\right] \le \left(\frac{(2p-1)^p K_1^{2p}}{2^2\delta^{p-1}} + \frac{K_2^{2p}}{2^2\delta^{2p-1}}\right)^{1/p}$$
$$e^{(2K_0^- + \delta)T}\, T^{1/p}\, \left(\mathbb{E}\left[\int_0^T |\dot{k}(t)|^{2q}\, dt\right]\right)^{1/q} < \infty$$

for $\delta > 0$ and $1/p + 1/q = 1$. In particular, if $K_2 \equiv 0$ and $k \in C^1([0, T], [0, 1])$ a deterministic function, then

$$\mathbb{E}\left[\int_0^T |W_t^k(v, \dot{k}(t)v)|^2\, dt\right] \le K_1^2 T\, e^{2K_0^- T} \int_0^T |\dot{k}(t)|^2\, dt < \infty.$$

Proof From the definition of the operator Q_s and the constant K_0, we know that $|Q_s| \le e^{-K_0 s/2}$. According to Itô's formula and the definitions of K_1, K_2, we have for $v \in T_x D$ and $s < \tau \le \tau_D$,

$$\begin{aligned}
d|W_s^k(v, v)|^{2p} &= p\,(|W_s^k(v, v)|^2)^{p-1}\Big[2\Big\langle R\big(//_s dB_s, Q_s(k(s)v)\big)Q_s(v), W_s^k(v, v)\Big\rangle \\
&\quad + |R^{\sharp,\sharp}\big(Q_s(k(s)v), Q_s(v)\big)|_{\mathrm{HS}}^2\, ds \\
&\quad - \Big\langle(\mathbf{d}^* R + \nabla \mathrm{Ric}^\sharp)(Q_s(k(s)v), Q_s(v)), W_s^k(v, v)\Big\rangle\, ds \\
&\quad - \mathrm{Ric}(W_s^k(v, v), W_s^k(v, v))\, ds\Big] \\
&\quad + 2p(p-1)\,(|W_s^k(v, v)|^2)^{p-1}|R^{\sharp,\sharp}\big(Q_s(k(s)v), Q_s(v)\big)|_{\mathrm{HS}}^2\, ds \\
&\le p\,(|W_s^k(v, v)|^2)^{p-1}\Big[2\Big\langle R\big(//_s dB_s, Q_s(k(s)v)\big)Q_s(v), W_s^k(v, v)\Big\rangle \\
&\quad + K_2\, e^{-K_0 s}\, |v|^2\, |W_s^k(v, v)|\, ds - K_0\, |W_s^k(v, v)|^2\, ds\Big] \\
&\quad + p(2p-1)\, K_1^2\, e^{-2K_0 s}\, |v|^4\, (|W_s^k(v, v)|^2)^{p-1}\, ds \\
&\le 2p\,(|W_s^k(v, v)|^2)^{p-1}\Big\langle R\big(//_s dB_s, Q_s(k(s)v)\big)Q_s(v), W_s^k(v, v)\Big\rangle \\
&\quad + p(2p-1)\, K_1^2\, e^{-2K_0 s}(|W_s^k(v, v)|^2)^{p-1}|v|^4\, ds \\
&\quad + K_2 p\, e^{-2K_0 s}\, |v|^2(|W_s^k(v, v)|)^{2p-1}\, ds - K_0 p\, |W_s^k(v, v)|^{2p}\, ds.
\end{aligned} \tag{2.13}$$

Taking into account that for $\delta > 0$,

$$\begin{aligned}
&p(2p-1)K_1^2\, e^{-2K_0 s}(|W_s^k(v, v)|^2)^{p-1}|v|^4 \\
&\quad \le \frac{\delta p}{2}|W_t^k(v, v)|^{2p} + \frac{(2p-1)^p}{2\delta^{p-1}}\, e^{-2pK_0 s}\, K_1^{2p}|v|^{4p};
\end{aligned}$$

$$\begin{aligned}
&K_2 p\, e^{-2K_0 s}\, |v|^2 (|W_s^k(v,v)|)^{2p-1} \\
&\quad \le \frac{\delta p}{2} |W_t^k(v,v)|^{2p} + \frac{1}{2\delta^{2p-1}}\, e^{-2pK_0 s}\, K_2^{2p} |v|^{4p},
\end{aligned}$$

we thus have

$$\begin{aligned}
&d\, e^{(K_0 p-\delta p)s}\, |W_s^k(v,v)|^{2p} \\
&\quad = 2p\, e^{(K_0 p-\delta p)s}\, |W_s^k(v,v)|^{2(p-1)} \Big\langle R\big(//_s dB_s, Q_s(k(s)v)\big) Q_s(v), W_s^k(v,v)\Big\rangle \\
&\qquad + e^{-(K_0 p+\delta p)s} \left(\frac{(2p-1)^p K_1^{2p}}{2\delta^{p-1}} + \frac{K_2^{2p}}{2\delta^{2p-1}} \right) |v|^{4p}\, ds.
\end{aligned}$$

Integrating this inequality from 0 to $t \wedge \tau$ and taking expectation yields

$$\begin{aligned}
&\mathbb{E}\left[e^{(K_0-\delta)p(t\wedge\tau)}\, |W_{t\wedge\tau}^k(v,v)|^{2p} \right] \\
&\quad \le \left(\frac{(2p-1)^p K_1^{2p}}{2\delta^{p-1}} + \frac{K_2^{2p}}{2\delta^{2p-1}} \right) \mathbb{E}\left[\int_0^{t\wedge\tau} e^{-(K_0+\delta)ps}\, |v|^{4p}\, ds \right].
\end{aligned}$$

If $|v| = 1$, we then arrive at

$$\begin{aligned}
&\mathbb{E}\left[e^{(K_0-\delta)p(t\wedge\tau)}\, |W_{t\wedge\tau}^k(v,v)|^{2p} \right] \\
&\quad \le \left(\frac{(2p-1)^p K_1^{2p}}{2\delta^{p-1}} + \frac{K_2^{2p}}{2\delta^{2p-1}} \right) \mathbb{E}\left[\int_0^{t\wedge\tau} e^{-(K_0+\delta)ps}\ ds \right],
\end{aligned}$$

from where we conclude that

$$\begin{aligned}
\mathbb{E}\left[|W_t^k(v,v)|^{2p} \right] &\le \left(\frac{(2p-1)^p K_1^{2p}}{2\delta^{p-1}} + \frac{K_2^{2p}}{2\delta^{2p-1}} \right) e^{(K_0^-+\delta)pt} \left[\int_0^t e^{-(K_0+\delta)ps}\ ds \right] \\
&\le \left(\frac{(2p-1)^p K_1^{2p}}{2\delta^{p-1}} + \frac{K_2^{2p}}{2\delta^{2p-1}} \right) e^{\delta pt}\, e^{K_0^- pt} \int_0^t e^{K_0^- ps}\ ds. \qquad (2.14)
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T |W_t^k(v, \dot{k}(t)v)|^2\, dt \right] \\
&\quad = \mathbb{E}\left[\int_0^T \dot{k}(t)^2 |W_t^k(v,v)|^2\, dt \right] \\
&\quad \le \mathbb{E}\left[\left(\int_0^T \dot{k}(t)^{2q}\, dt \right)^{1/q} \left(\int_0^T |W_t^k(v,v)|^{2p}\, dt \right)^{1/p} \right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\mathbb{E}\left[\int_0^T \dot{k}(t)^{2q}\,dt\right]\right)^{1/q} \left(\mathbb{E}\left[\int_0^T |W_t^k(v,v)|^{2p}\,dt\right]\right)^{1/p}\\
&\leq \left(\mathbb{E}\left[\int_0^T \dot{k}(t)^{2q}\,dt\right]\right)^{1/q} \left(\frac{(2p-1)^p K_1^{2p}}{2\delta^{p-1}} + \frac{K_2^{2p}}{2\delta^{2p-1}}\right)^{1/p} \mathrm{e}^{\delta T}\\
&\quad\times \left(\int_0^T \mathrm{e}^{K_0^- pt} \int_0^t \mathrm{e}^{K_0^- ps}\,ds dt\right)^{1/p}\\
&\leq \left(\mathbb{E}\left[\int_0^T \dot{k}(t)^{2q}\,dt\right]\right)^{1/q} \left(\frac{(2p-1)^p K_1^{2p}}{2^2\delta^{p-1}} + \frac{K_2^{2p}}{2^2\delta^{2p-1}}\right)^{1/p} \mathrm{e}^{\delta T + 2K_0^- T}\, T^{1/p}
\end{aligned}
$$

which completes the proof of first inequality.

In particular, if $k \in C^1([0,T])$, then

$$
\mathbb{E}\left[\int_0^T |W_t^k(v, \dot{k}(t)v)|^2\,dt\right] = \left[\int_0^T \dot{k}(t)^2\,\mathbb{E}(|W_t^k(v,v)|^2)\,dt\right]. \tag{2.15}
$$

Moreover, from the calculation in (2.13) with $K_2 \equiv 0$ we see that

$$
\begin{aligned}
d|W_s^k(v,v)|^2 = 2\Big\langle R\big(//_s dB_s, Q_s(k(s)v)\big)Q_s(v), W_s^k(v,v)\Big\rangle\\
+ K_1^2\,\mathrm{e}^{-2K_0 s}\,|v|^4\,ds - K_0|W_s^k(v,v)|^2\,ds
\end{aligned}
$$

which implies

$$
\mathbb{E}|W_t^k(v,v)|^2 \leq K_1^2\,\mathrm{e}^{2K_0^- T}\,T. \tag{2.16}
$$

Combining (2.15) and (2.16) we obtain

$$
\mathbb{E}\left[\int_0^T |W_t^k(v, \dot{k}(t)v)|^2\,dt\right] \leq K_1^2\,\mathrm{e}^{2K_0^- T}\,T\left[\int_0^T \dot{k}(t)^2\,dt\right]
$$

which gives the additional claim. □

By means of Theorem 2.1 and Lemma 2.2 we are now in position to formulate the localized Bismut-Stroock Hessian formula as follows.

Theorem 2.3 *Let D be a relatively compact open domain such that $x \in D$. Let τ be a stopping time such that $0 < \tau \leq \tau_D$. Suppose that k is a bounded, non-negative and adapted process with paths in the Cameron-Martin space $L^{1,2q}([0,T];\mathbb{R})$ such that $k(t) = 0$ for $t \geq T \wedge \tau$, $k(0) = 1$ and $\int_0^{T\wedge\tau} |\dot{k}(t)|^{2q}\,dt \in L^1$ for some constant $q > 1$. Then for $f \in \mathcal{B}_b(M)$ and $v \in T_xM$, we have*

$$
\begin{aligned}
&(\mathrm{Hess}\,P_T f)(v,v)\\
&\quad= -\mathbb{E}^x\left[f(X_T)\mathbf{1}_{\{T<\zeta(x)\}}\int_0^{T\wedge\tau}\left\langle Q_s\int_0^s Q_r^{-1}R(//_r dB_r, Q_r(k(r)v))Q_r(\dot{k}(s)v), //_s dB_s\right\rangle\right]
\end{aligned}
$$

$$- \mathbb{E}^x \left[f(X_T) \mathbf{1}_{\{T<\zeta(x)\}} \int_0^{T\wedge\tau} \left\langle Q_s \int_0^s Q_r^{-1} (\mathbf{d}^* R + \nabla \mathrm{Ric})^\sharp (Q_r(k(r)v), \right.\right.$$
$$\left.\left. \times\, Q_r(\dot{k}(s)v))\, dr, //_s dB_s \right\rangle \right]$$
$$+ \mathbb{E}\left[f(X_T) \mathbf{1}_{\{T<\zeta(x)\}} \left(\left(\int_0^{T\wedge\tau} \langle Q_s(\dot{k}(s)v), //_s dB_s \rangle \right)^2 - \int_0^{T\wedge\tau} |Q_s(\dot{k}(s)v)|^2\, ds \right) \right]. \tag{2.17}$$

Proof For $\varepsilon > 0$ small, let $k^\varepsilon(t)$ be a bounded, non-negative and adapted process with path in the Cameron-Martin space $L^{1,2q}([0,T];\mathbb{R})$ such that $k^\varepsilon(t) = 0$ for $t \geq (T-\varepsilon)\wedge\tau$, $k^\varepsilon(0) = 1$. We know that

$$|Q_{t\wedge\tau}| \leq \mathrm{e}^{-\frac{1}{2}K_0(t\wedge\tau)} \quad \text{and} \quad \mathbb{E}\left[\int_0^T |W_t^{k^\varepsilon}(v, \dot{k}^\varepsilon(t)v)|^2\, dt \right] < \infty.$$

By the boundedness of $P_t f$ on $[0,T]\times D$ and the boundedness of $|\mathbf{d}P_t f|$ and $|\mathrm{Hess} P_t f|$ on $[\varepsilon, T]\times D$ since $P_\cdot f$ is smooth on $(0,T]\times M$, it follows first that the local martingale in Theorem 2.1 is a true martingale for the chosen k^ε. By taking expectations, along with the strong Markov property, we get the formula

$$(\mathrm{Hess} P_T f)(v,v) = -\mathbb{E}^x \left[f(X_T)\mathbf{1}_{\{T<\zeta(x)\}} \int_0^{T\wedge\tau} \langle W_s^{k^\varepsilon}(v, \dot{k}^\varepsilon(s)v), //_s dB_s \rangle \right]$$
$$+ \mathbb{E}^x \left[f(X_T)\mathbf{1}_{\{T<\zeta(x)\}} \left(\left(\int_0^{T\wedge\tau} \langle Q_s(\dot{k}^\varepsilon(s)v), //_s dB_s \rangle \right)^2 - \int_0^{T\wedge\tau} |Q_s(\dot{k}^\varepsilon(s)v)|^2\, ds \right) \right].$$

Finally, through approximating the given k by an appropriate sequence of k^ε as above and letting $\varepsilon \downarrow 0$, we finish the proof. □

Remark 2.4 (i) In view of Eq. (2.9), formula (2.17) also can be written as

$$(\mathrm{Hess} P_T f)(v,v) = -\mathbb{E}^x \left[f(X_T)\mathbf{1}_{\{T<\zeta(x)\}} \int_0^{T\wedge\tau} \langle W_s^k(v, \dot{k}(s)v), //_s dB_s \rangle \right]$$
$$+ 2\mathbb{E}^x \left[f(X_T)\mathbf{1}_{\{T<\zeta(x)\}} \int_0^{T\wedge\tau} \left(\int_0^s \langle Q_r(\dot{k}(r)v), //_r dB_r \rangle \right) \langle Q_s(\dot{k}(s)v), //_s dB_s \rangle \right]. \tag{2.18}$$

(ii) Since $(\mathrm{Hess} P_T f)$ is a symmetric form, it is straightforward from Theorem 2.3 that

$$(\mathrm{Hess} P_T f)(v,w) = -\frac{1}{2}\mathbb{E}^x \left[f(X_T)\mathbf{1}_{\{T<\zeta(x)\}} \int_0^{T\wedge\tau} \langle W_s^k(v, \dot{k}(s)w), //_s dB_s \rangle \right]$$
$$- \frac{1}{2}\mathbb{E}^x \left[f(X_T)\mathbf{1}_{\{T<\zeta(x)\}} \int_0^{T\wedge\tau} \langle W_s^k(w, \dot{k}(s)v), //_s dB_s \rangle \right]$$

$$
\begin{aligned}
&+\mathbb{E}^x\left[f(X_T)\mathbf{1}_{\{T<\zeta(x)\}}\left(\int_0^{T\wedge\tau}\langle Q_s(\dot k(s)v),//_s dB_s\rangle\right)\left(\int_0^{T\wedge\tau}\langle Q_s(\dot k(s)w),//_s dB_s\rangle\right)\right]\\
&-\mathbb{E}^x\left[f(X_T)\mathbf{1}_{\{T<\zeta(x)\}}\int_0^{T\wedge\tau}\langle Q_s(\dot k(s)v),Q_s(\dot k(s)w)\rangle\,ds\right] \qquad (2.19)
\end{aligned}
$$

for $v, w \in T_xM$ and $x \in D$.

Denoting by $p_t(x, y)$ the transition density of the diffusion with generator L, using Theorem 2.3, we obtain the following Bismut-type Hessian formula for the logarithmic density.

Corollary 2.5 *We keep the assumptions of Theorem 2.3. Then*

$$
\begin{aligned}
&(\mathrm{Hess}_x \log p_T(x,y))(v,v)\\
&= -\mathbb{E}^x\left[\int_0^{T\wedge\zeta(x)}\langle W_s^k(v,\dot k(s)v),//_s dB_s\rangle \,\middle|\, X_T=y\right]\\
&\quad+\mathbb{E}^x\left[\left(\int_0^{T\wedge\zeta(x)}\langle Q_s(\dot k(s)v),//_s dB_s\rangle\right)^2 \,\middle|\, X_T=y\right]\\
&\quad-\mathbb{E}^x\left[\int_0^{T\wedge\zeta(x)}|Q_s(\dot k(s)v)|^2\,ds \,\middle|\, X_T=y\right]\\
&\quad-\left(\mathbb{E}^x\left[\int_0^{T\wedge\zeta(x)}\langle Q_s(\dot k(s)v),//_s dB_s\rangle \,\middle|\, X_T=y\right]\right)^2.
\end{aligned}
$$

Remark 2.6 Corollary 2.5 is well adapted to determine the short-time behavior of the Hessian of the heat kernel on a non-compact manifold (see the procedure in [5,Section 5]).

3 Hessian estimates for Harmonic functions

We now adjust Theorem 2.3 to the case of harmonic functions on regular domains. Recall that by a regular domain we mean a connected relatively compact open subset of M with non-empty boundary. For $D \subset M$ a regular domain and $f \in C^2(\bar D)$, let

$$
|\mathrm{Hess} f|(x) = \sup\left\{|(\mathrm{Hess} f)(v,w)|\colon |v|\le 1, |w|\le 1,\ v,w\in T_xM\right\},\quad x\in D.
$$

Since $\mathrm{Hess} f$ is symmetric, it is easy to see that

$$
|\mathrm{Hess} f|(x) = \sup\left\{|(\mathrm{Hess} f)(v,v)|\colon |v|\le 1,\ v\in T_xM\right\},\quad x\in D.
$$

Theorem 3.1 *Let $D \subset M$ be a regular domain, $u \in C^2(\bar D)$ be harmonic on D, further $x \in D$ and $v \in T_xM$. Then for any bounded adapted process k with sample paths in $L^{1,2q}(\mathbb{R}^+;\mathbb{R})$ for some $q > 1$ such that*

$$\int_0^{\tau_D} |\dot{k}(s)|^{2q}\, ds \in L^1,$$

and the property that $k(0) = 0$ *and* $k(s) = 1$ *for all* $s \geq \tau_D$*, the following formula holds:*

$$\begin{aligned}(\mathrm{Hess}\, u)_x(v, v) \\ &= -\mathbb{E}^x \left[u(X_{\tau_D}(x)) \int_0^{\tau_D} \langle W_s^k(v, \dot{k}(s)v), //_s dB_s \rangle \right] \\ &+ \mathbb{E}^x \left[u(X_{\tau_D}(x)) \left\{ \left(\int_0^{\tau_D} \langle Q_s(\dot{k}(s)v), //_s dB_s \rangle \right)^2 - \int_0^{\tau_D} |Q_s(\dot{k}(s)v)|^2\, ds \right\} \right].\end{aligned}$$

Proof Let $(P_t^D u)(x) := \mathbb{E}[u(X_{t\wedge\tau_D}(x))]$. Since u is harmonic, we have $P_t^D u = u$ for $t \geq 0$, and then analogously to Theorem 2.3,

$$\begin{aligned}(\mathrm{Hess}\, u)_x(v, v) &= (\mathrm{Hess}\, P_t^D u)_x(v, v) \\ &= -\mathbb{E}^x \left[u(X_{t\wedge\tau_D}(x)) \int_0^{t\wedge\tau_D} \langle W_s^k(v, \dot{k}(s)v), //_s dB_s \rangle \right] \\ &\quad + \mathbb{E}^x \left[u(X_{t\wedge\tau_D}(x)) \left(\int_0^{t\wedge\tau_D} \langle Q_s(\dot{k}(s)v), //_s dB_s \rangle \right)^2 \right] \\ &\quad - \mathbb{E}^x \left[u(X_{t\wedge\tau_D}(x)) \int_0^{t\wedge\tau_D} |Q_s(\dot{k}(s)v)|^2\, ds \right].\end{aligned}$$

By letting t tend to infinity, we obtain the claim. □

Remark 3.2 Taking Eq. (2.9) into account, the above formula can be rewritten as follows:

$$\begin{aligned}(\mathrm{Hess}\, u)_x(v, v) &= -\mathbb{E}^x \left[u(X_{\tau_D}(x)) \int_0^{\tau_D} \langle W_s^k(v, \dot{k}(s)v), //_s dB_s \rangle \right] \\ &+ 2\mathbb{E}^x \left[u(X_{\tau_D}(x)) \int_0^{\tau_D} \left(\int_0^s \langle Q_r(\dot{k}(r)v), //_r dB_r \rangle \right) \langle Q_s(\dot{k}(s)v), //_s dB_s \rangle \right].\end{aligned} \tag{3.1}$$

3.1 Construction of the random function *k*

We are now going to briefly sketch the method from [17, 18] to construct the function k. We restrict ourselves to Brownian motion on D with lifetime τ_D (exit time from D). Let f be a continuous function defined on $\bar{D}$ which is strictly positive on D. For $x \in D$, we consider the strictly increasing process

$$T(t) = \int_0^t f^{-2}(X_s(x))\, ds, \quad t \leq \tau_D, \tag{3.2}$$

and let

$$\tau(t) = \inf\{s \geq 0 : T(s) \geq t\}. \tag{3.3}$$

Obviously $T(\tau(t)) = t$ since $\tau_D < \infty$, and $\tau(T(t)) = t$ for $t \leq \tau_D$. Note that since X is a diffusion with generator $\frac{1}{2}\Delta$, the time-changed diffusion $X'_t := X_{\tau(t)}$ has generator $L' = \frac{1}{2} f^2 \Delta$. In particular, the lifetime $T(\tau_D)$ of X' is infinite, see [17].

Fix $t > 0$ and let

$$h_0(s) = \int_0^s f^{-2}(X_r(x)) 1_{\{r<\tau(t)\}}\, dr, \quad s \geq 0.$$

Then for $s \geq \tau(t)$,

$$h_0(s) = h_0(\tau(t)) = \int_0^{\tau(t)} f^{-2}(X_r(x))\, dr = t.$$

Next, let $h_1 \in C^1([0,t], \mathbb{R})$ with $h_1(0) = 0$ and $h_1(t) = 1$. We consider $k(s) = 1 - (h_1 \circ h_0)(s)$. Then k is an adapted bounded process with $k(0) = 1$ and $k(s) = 0$ for $s \geq \tau(t)$.

Lemma 3.3 *Suppose $f \in C^2(\bar{D})$ with $0 < f \leq 1$ on D, $f(x) = 1$ and $f|_{\partial D} = 0$. For $\delta_1, \delta_2 > 0$ and $q > 1$ set*

$$c_1(f) := \sup_D \left\{ \left((2\delta_2)^{-1} + K_0^-\right) f^2 + (3 + \delta_1^{-1})|\nabla f|^2 - f\Delta f \right\}; \tag{3.4}$$

$$c_2(f) := \sup_D \left\{ K_0^- f^2 + 5|\nabla f|^2 - f\Delta f \right\}; \tag{3.5}$$

$$c_3(f) := \sup_D \left\{ K_0^- f^2 + 3|\nabla f|^2 - f\Delta f \right\}; \tag{3.6}$$

$$\tilde{c}_q(f) := \sup_D \left\{ (2q+1)|\nabla f|^2 - f\Delta f \right\} \tag{3.7}$$

where K_0 is defined as in (2.10). Then there exists f satisfying conditions with,

$$c_1(f) \leq \frac{1}{2\delta_2} + K_0^- + \frac{\pi\sqrt{(n-1)K_0^-}}{2\delta_x} + \frac{\pi^2(n+3+\delta_1^{-1})}{4\delta_x^2} := c_1,$$

$$c_2(f) \leq K_0^- + \frac{\pi\sqrt{(n-1)K_0^-}}{2\delta_x} + \frac{\pi^2(n+5)}{4\delta_x^2} := c_2,$$

$$c_3(f) \leq K_0^- + \frac{\pi\sqrt{(n-1)K_0^-}}{2\delta_x} + \frac{\pi^2(n+3)}{4\delta_x^2} := c_3,$$

$$\tilde{c}_q(f) \leq \frac{\pi\sqrt{(n-1)K_0^-}}{2\delta_x} + \frac{\pi^2(n+2q+1)}{4\delta_x^2} := \tilde{c}_q,$$

where $\delta_x := \rho(x, \partial D)$ denotes the Riemannian distance of x to the boundary ∂D.

Proof Take

$$f(y) = \sin\left(\frac{\pi\rho(y, \partial D)}{2}\right), \quad y \in B(x, \delta_x).$$

This choice of f clearly satisfies the conditions and furthermore,

$$|\nabla f| \leq \frac{\pi}{2\delta_x}. \tag{3.8}$$

By the Laplacian comparison theorem, we have

$$-\Delta f \leq \frac{\pi}{2\delta_x}\sqrt{(n-1)K_0^-} + \frac{\pi^2 n}{4\delta_x^2}$$

which together with (3.8) gives the claimed estimates on $\tilde{c}_q(f), c_1(f), c_2(f)$ and $c_3(f)$. □

The following lemma is taken from [6,Lemma 2.2] or [17] with a trivial modification.

Lemma 3.4 *Suppose $f \in C^2(\bar{D})$ with $f > 0$ and $f \leq 1$ on D, $f(x) = 1$, $f|_{\partial D} = 0$ and set $\tilde{c}_q(f)$ as in (3.7) for $q \geq 1$. Then there exists a bounded adapted process k with paths in the Cameron-Martin space $L^{1,2q}([0,t];\mathbb{R})$ for some $q \in [1,\infty)$ such that $k(0) = 1$, $k(s) = 0$ for $s \geq t \wedge \tau_D$ and*

$$\mathbb{E}\left[\int_0^{t\wedge\tau_D} |\dot{k}(s)|^{2q}\, ds\right] \leq \frac{\tilde{c}_q(f)}{1 - \mathrm{e}^{-\tilde{c}_q(f)t}},$$

where $\tilde{c}_q$ is defined in Lemma 3.3.

3.2 Estimate for the Hessian of a harmonic function

In this subsection we focus on explicit Hessian estimates for harmonic functions defined on regular domains in Riemannian manifolds.

Theorem 3.5 *Let $D \subset M$ be a regular domain. For $x \in D$, let $\delta_x = \rho(x, \partial D)$ be the Riemannian distance of x to the boundary of D. For a bounded positive harmonic function u on D, one has*

$$|\text{Hess}\, u|(x) \le \inf_{\substack{0<\delta_1<1/2\\ \delta_2>0}} \Bigg\{ \sqrt{(1+2\mathbf{1}_{\{K_0^-\neq 0\}})\left((\delta_1+1)K_1^2+\frac{\delta_2}{2}K_2^2\right)} + \sqrt{6}\left(\frac{1}{2\delta_2}+3K_0^- + \frac{\pi\sqrt{(n-1)K_0^-}}{2\delta_x}+\frac{\pi^2(n+3+\delta_1^{-1})}{4\delta_x^2}\right)\Bigg\}\sqrt{\|u\|_D\, u(x)}; \tag{3.9}$$

$$|\text{Hess}\, u|(x) \le \inf_{\substack{\delta_1>0\\ \delta_2>0}} \Bigg\{ \sqrt{(1+2\mathbf{1}_{\{K_0^-\neq 0\}})\left((\delta_1+1)K_1^2+\frac{\delta_2}{2}K_2^2\right)} + \left(\frac{1}{\delta_2}+6K_0^- + \frac{\pi\sqrt{(n-1)K_0^-}}{\delta_x}+\frac{\pi^2(n+3+\delta_1^{-1})}{2\delta_x^2}\right)\Bigg\}\|u\|_D, \tag{3.10}$$

where K_0, K_1 and K_2 are defined as in (2.10)–(2.12).

Proof of Theorem 3.5 (Part I) We fix $x \in D$ and denote by $B = B(x, \delta_x)$ the open ball about x of radius δ_x. Then Theorem 3.1 applies with B now playing the role of D. First, by formula (3.1) and the Cauchy inequality, we get

$$\begin{aligned}
\big|(\text{Hess}\, u)_x(v,v)\big| &= \lim_{t\to\infty}\big|(\text{Hess}\, P_t^D u)_x(v,v)\big| \\
&\le \left|\mathbb{E}^x\left[u(X_{\tau_D}(x))\int_0^{\tau_D}\langle W_s^k(v,\dot k(s)v), //_s dB_s\rangle\right]\right| \\
&\quad + 2\left|\mathbb{E}^x\left[u(X_{\tau_D}(x))\int_0^{\tau_D}\left(\int_0^s\langle Q_r(\dot k(r)v), //_r dB_r\rangle\right)\langle Q_s(\dot k(s)v), //_s dB_s\rangle\right]\right| \\
&\le \sqrt{\|u\|_D\, u(x)}\sqrt{\lim_{t\to\infty}\mathbb{E}^x\left[\int_0^{\tau(t)}|W_s^k(v,v)|^2\dot k(s)^2\,ds\right]} \\
&\quad + 2\sqrt{\|u\|_D\, u(x)}\sqrt{\lim_{t\to\infty}\mathbb{E}^x\left[\int_0^{\tau(t)}\left(\int_0^s\langle Q_r(\dot k(r)v), //_r dB_r\rangle\right)^2|Q_s(\dot k(s)v)|^2\,ds\right]} := \mathrm{I}+\mathrm{II},
\end{aligned} \tag{3.11}$$

where the stopping time $\tau(t)$ is defined by (3.3). For the first term I, according to the relation between $\tau(t)$ and f, we can estimate as

$$\begin{aligned}
\mathbb{E}^x\left[\int_0^{\tau(t)}|W_s^k(v,v)|^2\dot k(s)^2\,ds\right] &\le \mathbb{E}^x\left[\int_0^{\tau(t)}|W_s^k(v,v)|^2\dot k(s)^2\,ds\right] \\
&\le \mathbb{E}^x\left[\int_0^{\tau(t)}|W_s^k(v,v)|^2\dot h_1(h_0(s))^2 f^{-4}(X_s)\,ds\right] \\
&= \mathbb{E}^x\left[\int_0^{t}|W_{\tau(s)}^k(v,v)|^2\dot h_1(s)^2 f^{-4}(X_{\tau(s)})\,d\tau(s)\right] \\
&= \int_0^t \dot h_1(s)^2\mathbb{E}^x\big[|W_{\tau(s)}^k(v,v)|^2 f^{-2}(X_{\tau(s)})\big]\,ds.
\end{aligned} \tag{3.12}$$

For the second term II, we find

$$\mathbb{E}^x\left[\int_0^{\tau(t)}\left(\int_0^s \langle Q_r(\dot{k}(r)v),\,//_r dB_r\rangle\right)^2 |Q_s(\dot{k}(s)v)|^2\,ds\right]$$
$$\leq \mathbb{E}^x\left[\int_0^t \dot{h}_1(s)^2 f^{-2}(X_{\tau(s)})\,\mathrm{e}^{-K_0\tau(s)}\left(\int_0^{\tau(s)}\langle Q_r(\dot{k}(r)v),\,//_r dB_r\rangle\right)^2 ds\right].$$

Substituting these estimates back into (3.11) yields

$$\begin{aligned}\left|(\mathrm{Hess}\,u)_x(v,v)\right| &\leq \sqrt{\|u\|_D\,u(x)}\sqrt{\lim_{t\to\infty}\int_0^t \dot{h}_1(s)^2\,\mathbb{E}^x\big[|W^k_{\tau(s)}(v,v)|^2 f^{-2}(X_{\tau(s)})\big]\,ds}\\ &+2\sqrt{\|u\|_D\,u(x)}\sqrt{\lim_{t\to\infty}\int_0^t \dot{h}_1(s)^2\,\mathbb{E}^x\left[f^{-2}(X_{\tau(s)})\,\mathrm{e}^{-K_0\tau(s)}\left(\int_0^{\tau(s)}\langle Q_r(\dot{k}(r)v),\,//_r dB_r\rangle\right)^2\right]ds}.\end{aligned} \tag{3.13}$$

Let

$$\begin{aligned}\Phi_1(s) &:= \mathbb{E}^x\big[|W^k_{\tau(s)}(v,v)|^2 f^{-2}(X_{\tau(s)})\big];\\ \Phi_2(s) &:= \mathbb{E}^x\left[f^{-2}(X_{\tau(s)})\,\mathrm{e}^{-K_0\tau(s)}\left(\int_0^{\tau(s)}\langle Q_r(\dot{k}(r)v),\,//_r dB_r\rangle\right)^2\right].\end{aligned}$$

Itô’s formula gives

$$\begin{aligned}&d\left(f^{-2}(X_{\tau(s)})|W^k_{\tau(s)}(v,v)|^2\right)\\ &\quad= -2f^{-2}(X_{\tau(s)})|W^k_{\tau(s)}(v,v)|^2\langle\nabla f(X_{\tau(s)}),\,//_{\tau(s)}\,dB_s\rangle\\ &\qquad+(3|\nabla f|^2 - f\Delta f)f^{-2}(X_{\tau(s)})|W^k_{\tau(s)}(v,v)|^2\,ds\\ &\qquad+f^{-2}(X_{\tau(s)})\,d|W^k_{\tau(s)}(v,v)|^2\\ &\qquad+2f^{-1}(X_{\tau(s)})\,\langle R(\nabla f,\,Q_{\tau(s)}(k(\tau(s))v))Q_{\tau(s)}(v),\,W^k_{\tau(s)}(v,v)\rangle\,ds.\end{aligned}$$

According to the definition of W^k_s, we thus obtain

$$\begin{aligned}&d\left(f^{-2}(X_{\tau(s)})|W^k_{\tau(s)}(v,v)|^2\right)\\ &\quad= -2f^{-2}(X_{\tau(s)})|W^k_{\tau(s)}(v,v)|^2\langle\nabla f(X_{\tau(s)}),\,//_{\tau(s)}dB_s\rangle\\ &\qquad+2f^{-1}(X_{\tau(s)})\langle R(//_{\tau(s)}\,dB_s,\,Q_{\tau(s)}(k(\tau(s))v))Q_{\tau(s)}(v),\,W^k_{\tau(s)}(v,v)\rangle\\ &\qquad+(3|\nabla f|^2 - f\Delta f)(X_{\tau(s)})f^{-2}(X_{\tau(s)})|W^k_{\tau(s)}(v,v)|^2\,ds\\ &\qquad+2f^{-1}(X_{\tau(s)})\langle R(\nabla f,\,Q_{\tau(s)}(k(\tau(s))v))Q_{\tau(s)}(v),\,W^k_{\tau(s)}(v,v)\rangle\,ds\end{aligned}$$

$$
\begin{aligned}
&+ |R^{\sharp,\sharp}(Q_{\tau(s)}(k(\tau(s))v),\, Q_{\tau(s)}v)|^2_{\mathrm{HS}}\, ds\\
&- \langle (\mathbf{d}^*R + \nabla \mathrm{Ric}^\sharp)(Q_{\tau(s)}(k(\tau(s))v),\, Q_{\tau(s)}(v)),\, W^k_{\tau(s)}(v,v)\rangle\, ds\\
&- \mathrm{Ric}(W^k_{\tau(s)}(v,v),\, W^k_{\tau(s)}(v,v))\, ds.
\end{aligned}
$$

Since all geometric quantities are bounded on D and $|Q_{\tau(s)}| \leq \exp\big(-\frac{1}{2}K_0\tau(s)\big)$, we have

$$
\begin{aligned}
&d\left(f^{-2}(X_{\tau(s)})|W^k_{\tau(s)}(v,v)|^2\right)\\
&\overset{\mathrm{m}}{\leq} \big(3|\nabla f|^2 - f\Delta f\big)(X_{\tau(s)})f^{-2}(X_{\tau(s)})|W^k_{\tau(s)}(v,v)|^2\, ds\\
&\quad + 2|R|(X_{\tau(s)})\, \mathrm{e}^{-K_0\tau(s)}\, |W^k_{\tau(s)}(v,v)|\, |\nabla f|(X_{\tau(s)})f^{-1}(X_{\tau(s)})\, ds\\
&\quad + |R|^2(X_{\tau(s)})\, \mathrm{e}^{-2K_0\tau(s)}\, ds\\
&\quad + K_2\, \mathrm{e}^{-K_0\tau(s)}\, |W^k_{\tau(s)}(v,v)|\, ds - K_0|W^k_{\tau(s)}(v,v)|^2\, ds.
\end{aligned}
$$

For any $\delta_1 > 0$ and $\delta_2 > 0$, it thus follows that

$$
\begin{aligned}
&d\left(f^{-2}(X_{\tau(s)})|W^k_{\tau(s)}(v,v)|^2\right)\\
&\quad\overset{\mathrm{m}}{\leq} \left(3|\nabla f|^2 - f\Delta f\right)(X_{\tau(s)})f^{-2}(X_{\tau(s)})|W^k_{\tau(s)}(v,v)|^2\, ds\\
&\qquad + (\delta_1+1)K_1^2\, \mathrm{e}^{-2K_0\tau(s)}\, ds + \frac{1}{\delta_1}|W^k_{\tau(s)}(v,v)|^2\, |\nabla f|^2(X_{\tau(s)})f^{-2}(X_{\tau(s)})\, ds\\
&\qquad + \frac{\delta_2}{2}K_2^2\, \mathrm{e}^{-2K_0\tau(s)}\, ds + \left(\frac{1}{2\delta_2} - K_0\right)|W^k_{\tau(s)}(v,v)|^2\, ds\\
&\quad= \left[\left(\frac{1}{2\delta_2} - K_0\right)f^2 + \left(3 + \frac{1}{\delta_1}\right)|\nabla f|^2 - f\Delta f\right]f^{-2}(X_{\tau(s)})|W^k_{\tau(s)}(v,v)|^2\, ds\\
&\qquad + \left((\delta_1+1)K_1^2 + \frac{\delta_2}{2}K_2^2\right)\mathrm{e}^{-2K_0\tau(s)}\, ds\\
&\quad\leq c_1(f)f^{-2}(X_{\tau(s)})|W^k_{\tau(s)}(v,v)|^2\, ds + \left((\delta_1+1)K_1^2 + \frac{\delta_2}{2}K_2^2\right)\mathrm{e}^{-2K_0\tau(s)}\, ds,
\end{aligned}
$$

where

$$
c_1(f) := \sup_D\left\{\left(\frac{1}{2\delta_2} + K_0^-\right)f^2 + \left(3 + \frac{1}{\delta_1}\right)|\nabla f|^2 - (f\Delta f)\right\}.
$$

This implies

$$
\begin{aligned}
&d\left(\mathrm{e}^{-c_1(f)s}\, f^{-2}(X_{\tau(s)})|W^k_{\tau(s)}(v,v)|^2\right)\\
&\overset{\mathrm{m}}{\leq} \left((\delta_1+1)K_1^2 + \frac{\delta_2}{2}K_2^2\right)\mathrm{e}^{-c_1(f)s}\, \mathrm{e}^{-2K_0\tau(s)}\, ds\\
&\leq \left((\delta_1+1)K_1^2 + \frac{\delta_2}{2}K_2^2\right)\mathrm{e}^{(2K_0^- - c_1(f))s}\, ds.
\end{aligned}
$$

By integrating from 0 to s and taking expectations, we then arrive at

$$\begin{aligned}\Phi_1(s) &= \mathbb{E}^x\left[f^{-2}(X_{\tau(s)})|W^k_{\tau(s)}(v,v)|^2\right]\\ &\le \left((\delta_1+1)K_1^2+\frac{\delta_2}{2}K_2^2\right)\mathrm{e}^{c_1(f)s}\int_0^s \mathrm{e}^{(2K_0^- - c_1(f))r}\,dr. \qquad (3.14)\end{aligned}$$

On the other hand, concerning Φ_2, we first observe

$$\begin{aligned}
& d\left(f^{-2}(X_{\tau(s)})\,\mathrm{e}^{-K_0\tau(s)}\left(\int_0^{\tau(s)}\langle Q_u(\dot{k}(u)v), //_u dB_u\rangle\right)^2\right)\\
&= d\Big[f^{-2}(X_{\tau(s)})\,\mathrm{e}^{-K_0\tau(s)}\left(\int_0^s \dot{h}_1(u)f^{-1}(X_{\tau(u)})\langle Q_{\tau(u)}(v), //_{\tau(u)}dB_u\rangle\right)^2\Big]\\
&\overset{\mathrm{m}}{=} \mathrm{e}^{-K_0\tau(s)}\,\dot{h}_1(s)^2 f^{-4}(X_{\tau(s)})\langle Q_{\tau(s)}(v), Q_{\tau(s)}(v)\rangle ds\\
&- K_0\,\mathrm{e}^{-K_0\tau(s)}\left(\int_0^s \dot{h}_1(u)f^{-1}(X_{\tau(u)})\langle Q_{\tau(u)}(v), //_{\tau(u)}dB_u\rangle\right)^2 ds\\
&- 4\,\mathrm{e}^{-K_0\tau(s)}\,f^{-3}(X_{\tau(s)})\int_0^s \dot{h}_1(u)f^{-1}(X_{\tau(u)})\langle Q_{\tau(u)}(v), //_{\tau(u)}dB_u\rangle\,\dot{h}_1(s)\\
&\times\langle\nabla f(X_{\tau(s)}), Q_{\tau(s)}(v)\rangle ds + (3|\nabla f|^2 - f\Delta f)(X_{\tau(s)})f^{-2}(X_{\tau(s)})\,\mathrm{e}^{-K_0\tau(s)}\\
&\times\left(\int_0^s \dot{h}_1(u)f^{-1}(X_{\tau(u)})\langle Q_{\tau(u)}(v), //_{\tau(u)}dB_u\rangle\right)^2 ds\\
&\le \mathrm{e}^{-2K_0\tau(s)}\,\dot{h}_1(s)^2 f^{-4}(X_{\tau(s)})\,ds\\
&- K_0\,\mathrm{e}^{-K_0\tau(s)}\left(\int_0^s \dot{h}_1(u)f^{-1}(X_{\tau(u)})\langle Q_{\tau(u)}(v), //_{\tau(u)}dB_u\rangle\right)^2 ds\\
&+ 2f^{-2}(X_{\tau(s)})|\nabla f|^2(X_{\tau(s)})\,\mathrm{e}^{-K_0\tau(s)}\left(\int_0^s \dot{h}_1(u)f^{-1}(X_{\tau(u)})\langle Q_{\tau(u)}(v), //_{\tau(u)}dB_u\rangle\right)^2 ds\\
&+ 2\,\mathrm{e}^{-2K_0\tau(s)}\,\dot{h}_1(s)^2 f^{-4}(X_{\tau(s)})ds + (3|\nabla f|^2 - f\Delta f)(X_{\tau(s)})f^{-2}(X_{\tau(s)})\,\mathrm{e}^{-K_0\tau(s)}\\
&\times\left(\int_0^s \dot{h}_1(u)f^{-1}(X_{\tau(u)})\langle Q_{\tau(u)}(v), //_{\tau(u)}dB_u\rangle\right)^2 ds
\end{aligned}$$

and conclude

$$\begin{aligned}
& d\left(f^{-2}(X_{\tau(s)})\,\mathrm{e}^{-K_0\tau(s)}\left(\int_0^s \dot{h}_1(u)f^{-1}(X_{\tau(u)})\langle Q_{\tau(u)}(v), //_{\tau(u)}dB_u\rangle\right)^2\right)\\
&\overset{\mathrm{m}}{\le} 3\,\mathrm{e}^{-2K_0\tau(s)}\,\dot{h}_1(s)^2 f^{-4}(X_{\tau(s)})\,ds\\
&+ \left(-K_0 f^2 + 5|\nabla f|^2 - f\Delta f\right)(X_{\tau(s)})f^{-2}(X_{\tau(s)})\,\mathrm{e}^{-K_0\tau(s)}\\
&\left(\int_0^s \dot{h}_1(u)f^{-1}(X_{\tau(u)})\langle Q_{\tau(u)}(v), //_{\tau(u)}dB_u\rangle\right)^2 ds\\
&\le 3\,\mathrm{e}^{-2K_0\tau(s)}\,\dot{h}_1(s)^2 f^{-4}(X_{\tau(s)})\,ds\\
&+ c_2(f)f^{-2}(X_{\tau(s)})\,\mathrm{e}^{-K_0\tau(s)}\left(\int_0^s \dot{h}_1(u)f^{-1}(X_{\tau(u)})\langle Q_{\tau(u)}(v), //_{\tau(u)}dB_u\rangle\right)^2 ds,
\end{aligned}$$

where $c_2(f)$ is defined in (3.5). Integrating the above inequality from 0 to s then yields

$$\mathbb{E}^x\left[f^{-2}(X_{\tau(s)})\,\mathrm{e}^{-K_0\tau(s)}\left(\int_0^s \dot h_1(u) f^{-1}(X_{\tau(u)})\langle Q_{\tau(u)}(v), //_{\tau(u)}dB_u\rangle\right)^2\right] \\ \le 3\,\mathrm{e}^{c_2(f)s}\int_0^s \mathrm{e}^{-c_2(f)u}\,\dot h_1(u)^2\mathbb{E}^x\left[\mathrm{e}^{-2K_0\tau(u)}\,f^{-4}(X_{\tau(u)})\right]du. \tag{3.15}$$

Hence it remains to estimate the term $\mathbb{E}^x\left[\mathrm{e}^{-2K_0\tau(u)}\,f^{-4}(X_{\tau(u)})\right]$. By Itô's formula, we have

$$d\left(\mathrm{e}^{-2K_0\tau(u)}\,f^{-4}(X_{\tau(u)})\right) \\ \overset{m}{=} -2K_0\,\mathrm{e}^{-2K_0\tau(u)}\,f^{-2}(X_{\tau(u)})\,du + (-2f\Delta f + 10|\nabla f|^2)(X_{\tau(u)})\,\mathrm{e}^{-2K_0\tau(u)}\,f^{-4}(X_{\tau(u)})\,du \\ \le 2c_2(f)\,\mathrm{e}^{-2K_0\tau(u)}\,f^{-4}(X_{\tau(u)})\,du$$

which implies

$$\mathbb{E}^x\left[\mathrm{e}^{-2K_0\tau(u)}\,f^{-4}(X_{\tau(u)})\right] \le f^{-4}(x)\,\mathrm{e}^{2c_2(f)u}\,.$$

Substituting this estimate into (3.15) leads to

$$\Phi_2(s) = \mathbb{E}^x\left[f^{-2}(X_{\tau(s)})\,\mathrm{e}^{-K_0\tau(s)}\left(\int_0^s \dot h_1(u) f^{-1}(X_{\tau(u)})\langle Q_{\tau(u)}(v), //_{\tau(u)}dB_u\rangle\right)^2\right] \\ \le 3f^{-4}(x)\,\mathrm{e}^{c_2(f)s}\int_0^s \dot h_1(u)^2\,\mathrm{e}^{-c_2(f)u}\,\mathrm{e}^{2c_2(f)u}\;du \\ \le 3f^{-4}(x)\,\mathrm{e}^{c_2(f)s}\int_0^s \dot h_1(u)^2\,\mathrm{e}^{c_2(f)u}\;du.$$

By means of this inequality we can estimate the second integral on the right-hand side of (3.13) to obtain

$$\int_0^t \dot h_1(s)^2\mathbb{E}^x\left[f^{-2}(X_{\tau(s)})\,\mathrm{e}^{-K_0\tau(s)}\left(\int_0^s \dot h_1(u) f^{-1}(X_{\tau(u)})\langle Q_{\tau(u)}(v), //_{\tau(u)}dB_u\rangle\right)^2\right]ds \\ \le 3f^{-4}(x)\int_0^t \dot h_1(s)^2\,\mathrm{e}^{c_2(f)s}\int_0^s \dot h_1(u)^2\,\mathrm{e}^{c_2(f)u}\;duds \\ = \frac{3}{2}f^{-4}(x)\left(\int_0^t \dot h_1(s)^2\,\mathrm{e}^{c_2(f)s}\;ds\right)^2.$$

Combining this estimate and inequality (3.14) with inequality (3.13) yields

$$\left|(\mathrm{Hess}\,u)_x(v,v)\right| \\ \le \sqrt{\|u\|_D\,u(x)}\sqrt{\left((\delta_1+1)K_1^2+\frac{\delta_2}{2}K_2^2\right)\lim_{t\to\infty}\int_0^t \dot h_1(s)^2\,\mathrm{e}^{c_1(f)s}\int_0^s \mathrm{e}^{(2K_0^- - c_1(f))r}\;dr\,ds}$$

$$
\begin{aligned}
&+2f^{-2}(x)\sqrt{\|u\|_D\,u(x)}\sqrt{3\lim_{t\to\infty}\int_0^t \dot h_1(s)^2\,\mathrm{e}^{c_2(f)s}\int_0^s \dot h_1(u)^2\,\mathrm{e}^{c_2(f)u}\;du\,ds}\\
&\le \sqrt{\|u\|_D\,u(x)}\sqrt{\left((\delta_1+1)K_1^2+\frac{\delta_2}{2}K_2^2\right)\lim_{t\to\infty}\int_0^t \dot h_1(s)^2\,\mathrm{e}^{c_1(f)s}\int_0^s \mathrm{e}^{(2K_0^- - c_1(f))r}\;dr\,ds}\\
&+\sqrt{6}f^{-2}(x)\sqrt{\|u\|_D\,u(x)}\lim_{t\to\infty}\left(\int_0^t \dot h_1(s)^2\,\mathrm{e}^{c_2(f)s}\;ds\right).
\end{aligned}
$$

It now suffices to choose a suitable function h_1. By Lemma 3.3, we have

$$
\begin{aligned}
&\mathrm{e}^{c_1(f)s}\int_0^s \mathrm{e}^{(2K_0^- - c_1(f))r}\;dr \le \mathrm{e}^{c_1 s}\int_0^s \mathrm{e}^{(2K_0^- - c_1)r}\;dr \le \mathrm{e}^{(c_1+2K_0^-)s}\int_0^s \mathrm{e}^{-c_1 r}\;dr\\
&\le \frac{\mathrm{e}^{(c_1+2K_0^-)s}}{c_1}.
\end{aligned}
$$

We define

$$
h_1(s)=\frac{\int_0^s \mathrm{e}^{-(c_1+2K_0^-)r}\;dr}{\int_0^t \mathrm{e}^{-(c_1+2K_0^-)r}\;dr}.
$$

If $\delta_1\le\frac{1}{2}$, then it is easy to see that $c_2(f)\le c_2\le c_1$ and $c_1\ge K_0^-$,

$$
\begin{aligned}
&|(\mathrm{Hess}\,u)_x(v,v)|\le\sqrt{\|u\|_D\,u(x)}\sqrt{\left((\delta_1+1)K_1^2+\frac{\delta_2}{2}K_2^2\right)\frac{c_1+2K_0^-}{c_1}}\\
&+f^{-2}(x)\sqrt{6\|u\|_D\,u(x)}(c_1+2K_0^-)\\
&\le\sqrt{\|u\|_D\,u(x)}\sqrt{(1+21_{K_0\ne 0})\left((\delta_1+1)K_1^2+\frac{\delta_2}{2}K_2^2\right)}\\
&+\sqrt{6\|u\|_D\,u(x)}\left((2\delta_2)^{-1}+3K_0^-+\frac{\pi\sqrt{(n-1)K_0^-}}{2\delta_x}+\frac{\pi^2(n+3+\delta_1^{-1})}{4\delta_x^2}\right)
\end{aligned}
$$

for $0<\delta_1\le\frac{1}{2}$ and $\delta_2>0$. We complete the proof of inequality (3.9) by taking the infimum with respect to δ_1 and δ_2. □

Proof of Theorem 3.5 (Part II) Theorem 3.1 allows to control $|\mathrm{Hess}\,u(v,v)|$ directly in terms of $\|u\|_D$ as follows:

$$
\begin{aligned}
|(\mathrm{Hess}\,u)_x(v,v)| &= \lim_{t\to\infty}|(\mathrm{Hess}\,P_t^D u)_x(v,v)|\\
&\le \|u\|_D\sqrt{\mathbb{E}^x\left[\lim_{t\to\infty}\int_0^{\tau(t)}|W_s^k(v,v)|^2\,\dot k(s)^2\,ds\right]}
\end{aligned}
$$

$$+ 2\|u\|_D\,\mathbb{E}^x\left[\lim_{t\to\infty}\int_0^{\tau(t)} \|Q_s\|^2\,\dot{k}(s)^2\,ds\right]$$

$$\le \|u\|_D \sqrt{\mathbb{E}^x\left[\lim_{t\to\infty}\int_0^{\tau(t)} |W_s^k(v,v)|^2\,\dot{k}(s)^2\,ds\right]}$$

$$+ 2\|u\|_D\,\mathbb{E}^x\left[\lim_{t\to\infty}\int_0^{\tau(t)} \mathrm{e}^{-K_0 s}\,\dot{k}(s)^2\,ds\right] = \mathrm{I} + \mathrm{II}. \tag{3.16}$$

For the second term II we observe

$$\mathbb{E}^x\left[\int_0^{\tau(t)} \mathrm{e}^{-K_0 s}\,\dot{k}(s)^2\,ds\right] = \int_0^t \dot{h}_1(s)^2 \mathbb{E}^x\big[\mathrm{e}^{-K_0\tau(s)}\,f^{-2}(X_{\tau(s)})\big]\,ds;$$

the first term has been dealt with in (3.12) above. We conclude from (3.16) that

$$\big|(\mathrm{Hess}\,u)_x(v,v)\big| \le \|u\|_D \sqrt{\lim_{t\to\infty}\int_0^t \dot{h}_1(s)^2\,\mathbb{E}^x\left[|W^k_{\tau(s)}(v,v)|^2 f^{-2}(X_{\tau(s)})\right]ds}$$

$$+ 2\|u\|_D \lim_{t\to\infty}\int_0^t \dot{h}_1(s)^2\,\mathbb{E}^x\big[\mathrm{e}^{-K_0\tau(s)}\,f^{-2}(X_{\tau(s)})\big]\,ds.$$

Recall that by Lemma 3.3,

$$c_3(f) = \sup_D \left\{K_0^- f^2 + 3|\nabla f|^2 - (f\Delta f)\right\}$$

$$\le K_0^- + \frac{\pi\sqrt{(n-1)K_0^-}}{2\delta_x} + \frac{\pi^2(n+3)}{4\delta_x^2} =: c_3 < c_1.$$

Thus

$$\mathbb{E}^x\big[f^{-2}(X_{\tau(s)})\,\mathrm{e}^{-K_0\tau(s)}\big] \le f^{-2}(x)\,\mathrm{e}^{c_3(f)s} \le \mathrm{e}^{c_3 s},$$

and hence

$$\big|(\mathrm{Hess}\,u)_x(v,v)\big| \le \sqrt{\left((\delta_1+1)K_1^2 + \frac{\delta_2}{2}K_2^2\right)\left(\int_0^t \dot{h}_1(s)^2\,\mathrm{e}^{c_1 s}\int_0^s \mathrm{e}^{(2K_0^- - c_1)r}\,dr\,ds\right)}\,\|u\|_D$$

$$+ 2\left(\int_0^t \dot{h}_1(s)^2\,\mathrm{e}^{c_3 s}\,ds\right)\|u\|_D. \tag{3.17}$$

Define

$$h_1(s) = \frac{\int_0^s \mathrm{e}^{-(c_1+2K_0^-)r}\,dr}{\int_0^t \mathrm{e}^{-(c_1+2K_0^-)r}\,dr},$$

so that

$$\dot{h}_1(s) = \frac{e^{-(c_1+2K_0^-)s}}{\int_0^t e^{-(c_1+2K_0^-)r}\,dr}.$$

The estimate

$$e^{c_1 s}\int_0^s e^{(2K_0^- - c_1)r}\,dr \le e^{(c_1+2K_0^-)s}\int_0^s e^{-c_1 r}\,dr \le \frac{e^{(c_1+2K_0^-)s}}{c_1}$$

is immediate. Since $c_3(f) \le c_3 \le c_1$, we have

$$\begin{aligned}\int_0^t \frac{e^{-2(c_1+2K_0^-)s}\,e^{c_3 s}}{\left(\int_0^t e^{-(c_1+2K_0^-)r}\,dr\right)^2}\,ds &\le \int_0^t \frac{e^{-(c_1+2K_0^-)s}}{\left(\int_0^t e^{-(c_1+2K_0^-)r}\,dr\right)^2}\,ds\\ &\le \frac{1}{\int_0^t e^{-(c_1+2K_0^-)r}\,dr} = \frac{c_1+2K_0^-}{1-e^{-(c_1+2K_0^-)t}}.\end{aligned}$$

Using these estimates to bound (3.17) from above and letting t tend to ∞, we deduce

$$\begin{aligned}|(\mathrm{Hess}\,u)_x(v,v)| &\le \sqrt{\left((\delta_1+1)K_1^2+\frac{\delta_2}{2}K_2^2\right)\left(\frac{c_1+2K_0^-}{c_1}\right)}\,\|u\|_D + 2\left(c_1+2K_0^-\right)\|u\|_D\\ &\le \sqrt{(1+2\mathbf{1}_{\{K_0^-\neq 0\}})\left((\delta_1+1)K_1^2+\frac{\delta_2}{2}K_2^2\right)}\,\|u\|_D\\ &\quad + 2\left((2\delta_2)^{-1}+3K_0^-+\frac{\pi\sqrt{(n-1)K_0^-}}{2\delta_x}+\frac{\pi^2(n+3+\delta_1^{-1})}{4\delta_x^2}\right)\|u\|_D\end{aligned}$$

for $\delta_1 > 0$ and $\delta_2 > 0$. This completes the proof of inequality (3.10). □

4 Estimate of the Hessian of the semigroup

We are now going to apply our results to give explicit local Hessian estimates for the heat semigroup on a Riemannian manifold.

Remark 4.1 Local Hessian estimates for the semigroup have been derived in [12,Section 4.2] by using the Hessian formula in [2]. We can improve these results significantly by clarifying the coefficients.

Theorem 4.2 *Let $x \in M$ and D be a relatively compact open domain such that $x \in D$. Then for $f \in \mathcal{B}_b(M)$,*

$$|\mathrm{Hess} P_T f|(x) \leq \inf_{\delta>0} \left\{ \left(\frac{T}{2}\sqrt{K_1^2 + \frac{K_2^2}{\delta}} + \frac{2}{T} \right) \exp\left(T\left(K_0^- + \frac{\delta}{2} + \frac{\pi\sqrt{(n-1)K_0^-}}{2\delta_x} + \frac{\pi^2(n+3)}{4\delta_x^2} \right)\right) \right\} \|f\|_D.$$

Proof By Theorem 2.3 and Cauchy's inequality, we have

$$\begin{aligned}
\left|(\mathrm{Hess} P_T f)_x(v,v)\right| &\leq \|f\|_D \left(\mathbb{E}^x \left[\int_0^T |W_s^k(v, \dot{k}(s)v)|^2\, ds \right] \right)^{1/2} \\
&\quad + 2\|f\|_D \left(\mathbb{E}^x \int_0^T |Q_s(\dot{k}(s)v)|^2\, ds \right) \\
&\leq \|f\|_D \sqrt{\mathbb{E}^x \left[\int_0^{\tau(T)} |W_s^k(v,v)|^2\, \dot{k}(s)^2\, ds \right]} \\
&\quad + 2\|f\|_D\, \mathbb{E}^x \left[\int_0^{\tau(T)} |Q_s|^2\, \dot{k}(s)^2\, ds \right] = \mathrm{I} + \mathrm{II}. \qquad (4.1)
\end{aligned}$$

Using a similar argument as in the proof of Theorem 3.5 (Part II) and Lemma 2.2, we obtain

$$\begin{aligned}
&\left|(\mathrm{Hess} P_T f)_x(v,v)\right| \\
&\leq \|f\|_D \left(\frac{(2p-1)^p K_1^{2p}}{2\delta^{p-1}} + \frac{K_2^{2p}}{2\delta^{2p-1}} \right)^{1/(2p)} \mathrm{e}^{(K_0^- + \frac{1}{2}\delta)T}\, T^{1/p} \left[\mathbb{E}^x \int_0^T |\dot{k}(t)|^{2q}\, dt \right]^{1/(2q)} \\
&\quad + 2\|f\|_D\, \mathbb{E}^x \left[\int_0^T \mathrm{e}^{-K_0 s}\, \dot{k}(s)^2\, ds \right] \\
&\leq \|f\|_D \left(\frac{(2p-1)^p K_1^{2p}}{2^2\delta^{p-1}} + \frac{K_2^{2p}}{2^2\delta^{2p-1}} \right)^{1/(2p)} \mathrm{e}^{\left(K_0^- + \frac{1}{2}\delta + \frac{\tilde{c}_q}{2q}\right)T}\, T^{3/(2p)-1/2} \\
&\quad + \|f\|_D\, \mathrm{e}^{(K_0^- + \tilde{c}_1)T}\, \frac{2}{T} \\
&\leq \|f\|_D \left(\frac{(2p-1)^p K_1^{2p}}{2^2\delta^{p-1}} + \frac{K_2^{2p}}{2^2\delta^{2p-1}} \right)^{1/(2p)} \mathrm{e}^{\left(K_0^- + \frac{1}{2}\delta + \frac{\tilde{c}_1}{2}\right)T}\, T^{(3-p)/2p} \\
&\quad + \|f\|_D\, \mathrm{e}^{(K_0^- + \tilde{c}_1)T}\, \frac{2}{T}
\end{aligned}$$

for any $\delta > 0$. We complete the proof by substituting $\tilde{c}_1$ defined in Lemma 2.2 into the above inequality and letting $p \to 1$. □

For global results (i.e. the case $D = M$), if M is compact, Theorem 2.3 holds with $\tau \equiv \infty$ and the Cameron-Martin valued process k can be chosen deterministic and linear, which leads immediately to straightforward estimates. If the manifold is non-compact, $\mathrm{Ric} \geq K_0$, $\|R\|_\infty := \sup_{x\in M} |R|(x) < \infty$ and

$$K_2 := \sup\left\{|(\mathbf{d}^* R + \nabla \mathrm{Ric})^\sharp(v,v)|:\ \in T_x M,\ x \in M,\ |v| = 1\right\} < \infty,$$

then Theorem 4.2 still holds when replacing D by M and letting $\delta_x \to \infty$. In the following, we use a different argument by choosing $k \in C^1([0, T])$ such that $k(0) = 0$ and $k(T) = 1$ and estimating $\mathbb{E}^x|W_s^k(v, v)|^2$.

Corollary 4.3 *Let M be a complete Riemannian manifold. Assume* $\mathrm{Ric} \geq K_0$ *and* $K_1 := \|R\|_\infty < \infty$,

$$K_2 = \sup\left\{|(\mathbf{d}^* R + \nabla \mathrm{Ric})^\sharp(v,v)|:\ \in T_x M,\ |v| = 1\right\} < \infty.$$

Then,

$$|\mathrm{Hess} P_T f| \leq \left(\left(\frac{K_1^2}{2} + \frac{K_2^2}{2\delta}\right)^{1/2} + \frac{2}{T}\right) \exp\left(\left(K_0^- + \frac{1}{2}\delta\right) T\right) \|f\|_\infty \tag{4.2}$$

for $\delta > 0$. *Moreover, if the manifold M is Ricci parallel, i.e.* $\nabla \mathrm{Ric} = 0$, *then*

$$|\mathrm{Hess} P_T f| \leq \left(\frac{K_1}{\sqrt{2}} + \frac{2}{T}\right) \mathrm{e}^{K_0^- T} \|f\|_\infty. \tag{4.3}$$

Proof From (4.1), we know that for $k \in C^1([0, T])$ such that $k(0) = 0$ and $k(T) = 1$,

$$\begin{aligned} \left|(\mathrm{Hess} P_T f)_x(v, v)\right| &\leq \|f\|_\infty \left(\int_0^T \mathbb{E}^x |W_s^k(v,v)|^2\, \dot{k}(s)^2\, ds\right)^{1/2} \\ &\quad + 2\|f\|_\infty\, \mathbb{E}^x \left[\int_0^T |Q_s|^2\, \dot{k}(s)^2\, ds\right]. \end{aligned}$$

If we choose $k(s) = (T - s)/T$, then it follows from (2.14) that

$$\begin{aligned} \left|(\mathrm{Hess} P_T f)_x(v, v)\right| &\leq \left(\frac{K_1^2}{2} + \frac{K_2^2}{2\delta}\right)^{1/2} \mathrm{e}^{(K_0^- + \frac{\delta}{2})T} \|f\|_\infty + \frac{2}{T^2}\, \mathbb{E}^x \left[\int_0^T |Q_s|^2\, ds\right] \|f\|_\infty \\ &\leq \left(\left(\frac{K_1^2}{2} + \frac{K_2^2}{2\delta}\right)^{1/2} + \frac{2}{T}\right) \exp\left(\left(K_0^- + \frac{1}{2}\delta\right) T\right) \|f\|_\infty. \end{aligned}$$

If the manifold is Ricci parallel, then $\mathbf{d}^* R + \nabla \mathrm{Ric} = 0$ and δ is not needed in the estimate of $\mathbb{E}^x|W_t(v, v)|^2$, see inequality (2.13). Thus, in this case, estimate (4.2) reduces to (4.3). ∎

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