

Hessian heat kernel estimates and Calderón-Zygmund inequalities on complete Riemannian manifolds

Jun Cao¹, Li-Juan Cheng^{*2}, and Anton Thalmaier³

¹Department of Applied Mathematics, Zhejiang University of Technology,
Hangzhou 310023, People's Republic of China
caojun1860@zjut.edu.cn

²Department of Applied Mathematics, Zhejiang University of Technology,
Hangzhou 310023, People's Republic of China
chenglj@zjut.edu.cn

³Department of Mathematics, University of Luxembourg, Maison du Nombre,
L-4364 Esch-sur-Alzette, Luxembourg
anton.thalmaier@uni.lu

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Abstract

We address some fundamental questions concerning geometric analysis on Riemannian manifolds. It has been asked whether the L^p -Calderón-Zygmund inequalities extend to a reasonable class of non-compact Riemannian manifolds without the assumption of a positive injectivity radius. In the present paper, we give a positive answer for $1 < p < 2$ under the natural assumption of a lower bound on the Ricci curvature. For $p > 2$, we complement the study in Güneysu-Pigola (2015) and derive sufficient geometric criteria for the validity of the Calderón-Zygmund inequality by adding Kato class bounds on the Riemann curvature tensor and the covariant derivative of Ricci curvature. Probabilistic tools, like Hessian formulas and Bismut type representations for heat semigroups, play a significant role throughout the proofs.

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1 Introduction

Let (M, g) be a complete non-compact connected d -dimensional Riemannian manifold, ∇ the Levi-Civita covariant derivative, $\text{Hess} = \nabla d$ the Hessian operator on functions, and μ the Riemannian volume measure on M . We denote by $|\cdot|$ the norm in the tangent space, and by $\|\cdot\|_p$ the norm in $L^p(M, \mu)$ for $1 \leq p \leq \infty$. Throughout this paper, the Laplace-Beltrami operator Δ acting on functions and forms, is understood as self-adjoint positive operator on $L^2(\mu)$.

The so-called L^p -Calderón-Zygmund inequality, $1 < p < \infty$, is one of the cornerstones in the regularity theory of elliptic equations, see the recent survey [31]. On the flat Euclidean space \mathbb{R}^d such inequalities can be proved via estimates of singular integral operators [6]. Güneysu and Pigola [24] introduce the concept of Calderón-Zygmund inequalities on Riemannian manifolds which is linked to the structure of second order Sobolev function spaces like $H_0^{2,p}(M)$ and $H^{2,p}(M)$, e.g. denseness of $C_0^\infty(M)$ in $H^{2,p}(M)$, see [24, 31] for details.

Definition 1.1. Let (M, g) be a Riemannian manifold. We say that an L^p -Calderón-Zygmund inequality holds on M if there exist two constants $C_1, C_2 > 0$ such that

$$\| |\text{Hess}(\varphi)| \|_{L^p} \leq C_1 \|\varphi\|_{L^p} + C_2 \|\Delta\varphi\|_{L^p}$$

for every function $\varphi \in C_0^\infty(M)$, the set of smooth functions of compact supports. We denote this inequality by $\mathbf{CZ}(p)$.

Recently, Güneysu [24] observed that under $\mathbf{CZ}(p)$, if M is geodesically complete and if it admits a sequence of Laplacian cut-off functions (this is the case e.g. if M has a nonnegative Ricci curvature), then $H_0^{2,p}(M) = H^{2,p}(M)$ holds for all $1 < p < \infty$. For further applications of $\mathbf{CZ}(p)$ the reader is referred to [25]. However, in general, $\mathbf{CZ}(p)$ inequalities may hold or fail on M , depending on the underlying Riemannian geometry, which leads to the question which geometric assumptions on M guarantee $\mathbf{CZ}(p)$ and how the $\mathbf{CZ}(p)$ -constants C_1, C_2 depend on the geometric entities. For an example of a Riemannian manifold of positive sectional curvature where $\mathbf{CZ}(p)$ fails for large values of p , see [30], as well as [28] for a general result in this direction.

For global results, in case of $p = 2$, a bound $\text{Ric} \geq -K$ for some constant K already implies $\mathbf{CZ}(2)$ with constants depending explicitly on the bound K . This is a straightforward consequence of Bochner's identity. We include the argument from [24, 31] for reader's convenience.

1.1 The case $p = 2$

Let M be a complete Riemannian manifold with Ricci curvature tensor bounded from below, that is,

$$\text{Ric} \geq -K \tag{Ric}$$

for some $K \geq 0$. In this case, inequality $\mathbf{CZ}(2)$ is a direct consequence of Bochner's formula. Indeed, for any $u \in C_0^\infty(M)$, Bochner's formula gives

$$-\frac{1}{2}\Delta|\nabla u|^2 = |\text{Hess}(u)|_{\text{HS}}^2 - g(\nabla\Delta u, \nabla u) + \text{Ric}(\nabla u, \nabla u). \tag{1.1}$$

Integration by parts, together with the lower Ricci bound, leads to

$$0 = -\frac{1}{2} \int_M \Delta|\nabla u|^2$$

$$\begin{aligned}
&= \int_M |\text{Hess}(u)|_{\text{HS}}^2 - \int_M g(\nabla \Delta u, \nabla u) + \int_M \text{Ric}(\nabla u, \nabla u) \\
&\geq \int_M |\text{Hess}(u)|_{\text{HS}}^2 - \int_M (\Delta u)^2 - K \int_M |\nabla u|^2 \\
&= \int_M |\text{Hess}(u)|_{\text{HS}}^2 - \int_M (\Delta u)^2 - K \int_M u \Delta u.
\end{aligned}$$

Then, by estimating the last integral via Young's inequality, we obtain for any $\sigma > 0$,

$$\| |\text{Hess}(u)| \|_{L^2}^2 \leq \| |\text{Hess}(u)|_{\text{HS}} \|_{L^2}^2 \leq \frac{K\sigma^2}{2} \|u\|_{L^2}^2 + \left(1 + \frac{K}{2\sigma^2}\right) \|\Delta u\|_{L^2}^2, \quad (1.2)$$

which establishes **CZ(2)**.

Remark 1.2. Inequality (1.2) extends from $u \in C_0^\infty(M)$ to $u \in H^{2,2}(M)$. Thus, in particular, if $u \in L^2(M)$ is a distributional solution to the Poisson equation

$$\Delta u = f \quad \text{on } M \quad (1.3)$$

for some $f \in L^2(M)$, then (1.2) provides an L^2 -Hessian estimate of the solution u .

Remark 1.3. Recall that for L^2 -gradient estimates of u in (1.3), the lower bound on the Ricci tensor is not needed. Indeed, integrating (1.3) by parts and using Young inequality, one obtains directly for every $\sigma > 0$,

$$\int_M |\nabla u|^2 = \int_M f u \leq \frac{\sigma^2}{2} \int_M u^2 + \frac{1}{2\sigma^2} \int_M f^2,$$

which is the L^2 -gradient estimate

$$\|\nabla u\|_{L^2}^2 \leq \frac{\sigma^2}{2} \|u\|_{L^2}^2 + \frac{1}{2\sigma^2} \|f\|_{L^2}^2. \quad (1.4)$$

1.2 The general case $p \neq 2$

The extension of the inequality **CZ(p)** from $p = 2$ to an arbitrary index $1 < p < \infty$ is much more involved and an intriguing problem. If M has bounded Ricci curvature and a positive injectivity radius, Güneysu and Pigola [24] proved that **CZ(p)** holds for all $1 < p < \infty$ with implicit constants depending on p , $\|\text{Ric}\|_\infty$, the dimension of M and the injectivity radius.

In [24] two methods appear for attacking **CZ(p)**: the first one depends on appropriate elliptic estimates under conditions on harmonic bounds of the injectivity radius, while the second one uses boundedness results for the covariant Riesz transform from [35]. Whereas conditions on harmonic bounds of the injectivity radius are usually difficult to verify, the second approach relies on probabilistic covariant derivative formulae for heat semigroups and has the advantage of avoiding assumptions on the injectivity radius.

Let us sketch the main idea of the second method in [24]. Inequality **CZ(p)** is usually reduced to the existence of positive constants C and σ such that

$$\| |\text{Hess}(\Delta + \sigma)^{-1} u| \|_p \leq C \|u\|_p,$$

which is equivalent to

$$\left\| |\nabla(\Delta_1 + \sigma)^{-1/2} \circ d(\Delta_0 + \sigma)^{-1/2} u| \right\|_p \leq C \|u\|_p.$$

The problem is thus reduced to the study of conditions for boundedness of the classical Riesz transform $d(\Delta_0 + \sigma)^{-1/2}$ on functions and boundedness of the covariant Riesz transform $\nabla(\Delta_1 + \sigma)^{-1/2}$ on one-forms. This approach in [24] however is restricted to $p \in (1, 2)$; the constants C_1, C_2 in **CZ**(p) depend on $\dim M, p, \|R\|_\infty, \|\nabla R\|_\infty$ and on the constants D, δ from the following local volume doubling assumption: there are constants $C > 0, 0 \leq \delta < 2$ such that

$$V(x, tr) \leq Ct^d e^{Ctr} V(x, r) \quad (\mathbf{LD})$$

for all $x \in M, r > 0$ and $t \geq 1$, where $V(x, r) := \mu(B(x, r))$. By the Bishop-Gromov comparison theorem and the well-known formula for the volume of balls in the hyperbolic space, **(LD)** holds if $\text{Ric} \geq -K$ for some $K \geq 0$, i.e. under condition **(Ric)**.

For x and $y \in M$, we obviously have $B(y, \sqrt{t}) \subset B(x, \sqrt{t} + \rho(x, y))$. Thus if **(LD)** holds, then

$$V(y, \sqrt{t}) \leq V(x, \sqrt{t} + \rho(x, y)) \leq C \left(1 + \frac{\rho(x, y)}{\sqrt{t}}\right)^d \exp(C(\sqrt{t} + \rho(x, y))) V(x, \sqrt{t}) \quad (1.5)$$

Very recently, Baumgarth, Devyver and Güneysu [4] studied the covariant Riesz transform on j -forms and established a second order Davies-Gaffney estimate for the semigroup $e^{-t\vec{\Delta}_j}$ for small times, where $\vec{\Delta}_j$ is the unique self-adjoint realization of the Hodge-de Rham Laplacian acting on j -forms. These results can be applied to **CZ**(p) when $1 < p < 2$ and allow to establish **CZ**(p) under strong curvature assumptions but without requirements on the injectivity radius.

The observations above raise the following questions:

1. *Without a lower control on the injectivity radius, under which conditions on the manifold M , the inequality **CZ**(p) holds on M for $p > 2$?*
2. *In the case $1 < p < 2$, is it possible to weaken the assumptions on the Riemann curvature tensor $\|R\|_\infty$ and $\|\nabla R\|_\infty$, e.g., replacing them by Ricci curvature bounds?*

In this paper, for $1 < p < 2$, we adopt the direct approach of estimating the L^2 -norm of the Hessian of the heat kernel on functions. This can be done assuming solely a lower bound for the Ricci curvature. Our first main result is the following.

Theorem 1.4. *Let (M, g) be a complete Riemannian manifold satisfying **(Ric)**. Let $1 < p < 2$ be fixed. Then there exists a constant $\sigma > 0$ such that the operator $\text{Hess}(\Delta + \sigma)^{-1}$ is bounded in L^p , i.e. **CZ**(p) holds.*

Comparing Theorem 1.4 with existing results on **CZ**(p), it should be pointed out that first the result is valid without any injectivity radius assumptions and secondly, instead of boundedness of $\|R\|_\infty$ and $\|\nabla R\|_\infty$, only a lower bound of Ric is needed. In view of the argument for $p = 2$, the assumption of Ricci bounded below seems to be quite natural.

Our first contribution is to show that an L^2 -estimate of the Hessian of the heat kernel can be combined with the Hardy-Littlewood maximal function and good- λ inequalities to bound the operators $\text{Hess}(\Delta + \sigma)^{-1}$ in L^p when $1 < p < 2$. This idea is from [12] where the authors study the Riesz transform for $1 < p < 2$, see also [33].

The next natural question is then under which conditions Theorem 1.4 extends to $p > 2$. It is well known [31, 30, 28] that a lower bound on Ric alone is not enough for $\mathbf{CZ}(p)$ to hold when p is large. Hence more geometric information about the manifold is required for the validity of $\mathbf{CZ}(p)$ in this case.

To formulate appropriate conditions, we start with some notations and recall the definition of Kato class. Throughout the paper, we denote by X_t the diffusion process generated by $-\Delta$, which is assumed to be non-explosive (M is stochastically complete). Then, for $f \in \mathcal{B}_b(M)$,

$$\mathbb{E}^x[f(X_t)] = P_t f(x) = (e^{-\Delta t} f)(x).$$

Definition 1.5. A Borel function $K: M \rightarrow \mathbb{R}$ is said to be in the Kato class $\mathcal{K}(M)$ of M , if

$$\limsup_{t \rightarrow 0^+} \sup_{x \in M} \int_0^t \mathbb{E}^x[|K(X_s)|] ds = 0. \quad (\mathbf{K})$$

Obviously, $\mathcal{K}(M)$ is a linear space and $\mathcal{K}(M) \subset L_{\text{loc}}^1(M)$.

The Kato class has been introduced in [29] on Euclidean space for investigating singular potentials, and the Kato condition has been used for decades in order to deal with local singularities of potentials in quantum mechanics, see for instance [1, 34, 20]. For a recent account on Katos class from a geometric perspective, see [32].

Concerning criteria for functions to be in the Kato class the reader may consult [22, 21, 26]. For instance, the following result can be found in [26]. Let $p(t, x, y)$ be the integral kernel of the heat semigroup. By [26, Proposition 3.2], for dimension $d \geq 2$, if there exist constants C and $t_0 > 0$ such that for all $t \in (0, t_0]$ and $x \in M$ the estimate

$$p_t(x, x) \leq C t^{-d/2} \quad (1.6)$$

holds, then $L^p(M) + L^\infty(M) \subset \mathcal{K}(M)$ for $p > d/2$.

We now introduce some geometric quantities; our conditions are then given in terms of Kato bounds on these quantities. Let $R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}$ denote the Riemann curvature tensor and let $R(X, Y, Z, U) := \langle R(X, Y)Z, U \rangle$ where X, Y, Z, U are vector fields on M . We define

$$|R|(x) = \sup \left\{ |R^{\#\#}(v_1, v_2)|_{\text{HS}}(x) : v_1, v_2 \in T_x M, |v_1| \leq 1, |v_2| \leq 1 \right\}, \quad x \in M,$$

where $R^{\#\#}(v_1, v_2) = R(\cdot, v_1, v_2, \cdot)$. Furthermore, let

$$\text{Ric} \in \Gamma(T^* M \otimes T^* M), \quad \text{Ric}(v_1, v_2) = \text{tr} R(\cdot, v_1, v_2, \cdot),$$

be the Ricci curvature tensor. We consider the operator $d^* R$ defined by $d^* R(v_1, v_2) := -\text{tr} \nabla \cdot R(\cdot, v_1) v_2$ which satisfies

$$\langle d^* R(v_1, v_2), v_3 \rangle = \langle (\nabla_{v_3} \text{Ric}^\#)(v_1), v_2 \rangle - \langle (\nabla_{v_2} \text{Ric}^\#)(v_3), v_1 \rangle$$

for all $v_1, v_2, v_3 \in T_x M$ and $x \in M$, where $\text{Ric}^\# \in \Gamma(\text{End } T M)$ is defined in the usual way as $\text{Ric}^\#(v) = \text{Ric}(\cdot, v)^\#$ for $v \in T_x M$.

To deal with the case $p > 2$, we introduce the following condition:

Condition (H) $\text{Ric} \geq -K$ for some $K \geq 0$ and there exist $K_1, K_2 \in \mathcal{K}(M)$ such that

$$|R|^2(x) \leq K_1(x) \quad \text{and} \quad |\nabla \text{Ric}^\# + d^* R|^2(x) \leq K_2(x), \quad x \in M. \quad (\mathbf{H})$$

Note that condition **(Ric)** implies the local diagonal estimates (1.6) for the heat kernel (see also (2.1) below); hence if $d \geq 2$, then under **(Ric)** the inclusion $L^p(M) + L^\infty(M) \subset \mathcal{K}(M)$ holds for $p > d/2$. Let us first explain the motivation behind condition **(H)**. The so-called damped parallel transport $Q_t: T_x M \rightarrow T_{X_t} M$ is defined as the solution, along the paths of X_t , to the covariant ordinary differential equation

$$DQ_t = -\text{Ric}^\sharp Q_t dt, \quad Q_0 = \text{id}_{T_x M}, \quad (1.7)$$

where $DQ_t = //_t d //_t^{-1} Q_t$ and $//_t$ being the parallel transport along the paths of X_t . The derivative formula for $P_t f$ indicates that for $v \in T_x M$, $x \in M$,

$$\langle (\nabla P_t f)(x), v \rangle = \mathbb{E}^x [\langle (\nabla f)(X_t), Q_t v \rangle]. \quad (1.8)$$

Note that the lower bound on Ric allows to estimate $|Q_t| \leq e^{Kt}$ and hence $|\nabla P_t f| \leq e^{Kt} P_t |\nabla f|$. Starting with the representation

$$\nabla(\Delta + \sigma)^{-1/2} = \int_0^\infty \frac{e^{-\sigma t}}{\sqrt{t}} \nabla P_t dt,$$

boundedness of the local Riesz transform has been proved in [13, 14] for $p > 2$ by means of general inequalities of the type

$$|\nabla P_t f| \leq C e^{c_1 t} P_{c_2 t} |\nabla f| \quad (1.9)$$

with positive constants c_1, c_2 and C . Adopting this approach to prove **CZ(p)**, our idea is to start from the observation that

$$\text{Hess}(\Delta + \sigma)^{-1} = \int_0^\infty e^{-\sigma t} \text{Hess} P_t dt.$$

and to replace the gradient formula for $P_t f$ by a Hessian formula for $P_t f$. This requires, under suitable geometric conditions, to estimate $|\text{Hess} P_t f|$ by $|\text{Hess} f|$ in the sense that

$$|\text{Hess} P_t f| \leq C e^{c_1 t} P_{c_2 t} |\text{Hess} f|$$

for some constants c_1 and c_2 . Such a type of second order inequality then may serve as a substitute for condition (1.9) in the method of [13], in order to approach **CZ(p)** when $p > 2$. More precisely, in Section 2 we establish under condition **(H)** the following inequality:

$$|\text{Hess} P_t f|(x) \leq e^{2Kt} P_t |\text{Hess} f| + C e^{(2K+\theta)t} (P_t |df|^2)^{1/2}$$

for some constants $C, \theta > 0$. Since the upper bound not only involves the term $P_t |\text{Hess} f|$ but also the first order term $P_t |df|^2$, an additional treatment of $(P_t |df|^2)^{1/2}$ is required in order to proceed in the way as condition (1.9) is used in [13] to study the Riesz transform for $p > 2$. In Section 3.2, we adopt the local criterion of [3, Theorem 2.4] to study the boundedness of

$$T = \text{Hess}(\Delta + \sigma)^{-1} = \int_0^\infty e^{-\sigma t} \text{Hess} P_t dt. \quad (1.10)$$

Since $|d(\cdot)|$ is of lower order than $|\text{Hess}(\cdot)|$, we observe from [3, Theorem 2.4] that if all additional terms can be controlled in L^p , then one still obtains the boundedness of T under natural conditions.

Our second main result is as follows.

Theorem 1.6. *Let M be a complete Riemannian manifold satisfying **(H)**. Let $p > 2$ be fixed. Then there exists a constant $\sigma > 0$ such that the operator $\text{Hess}(\Delta + \sigma)^{-1}$ is bounded in L^p , i.e. **CZ**(p) holds.*

Theorem 1.6 gives in particular an answer to the open question in [24] about sufficient conditions for **CZ**(p) when $p > 2$ in the absence of control of the injectivity radius. It is worth mentioning that compared with the sufficient conditions even in the case $1 < p < 2$ ([24, Theorem D], [4, Corollary 1.8]), in Theorem 1.6 the geometric quantities $|R|$ and $|\nabla \text{Ric}|$ do not need to be uniformly bounded on M ; it is sufficient to have bounds in the Kato class which is a kind of integral condition. The proof of Theorem 1.6 however is quite involved: it relies on Bismut-type formulas for the Hessian of the heat semigroup (see [2]) and second order derivative formula for heat semigroup, which are used to give the pointwise estimates on the Hessian of the heat semigroup in L^p -sense and the L^p -Gaffney off-diagonal Hessian estimate of heat semigroup. Theorem 1.6 is then established by combining the techniques from [3], in particular, the sharp maximal function and good- λ inequalities.

It should be mentioned that for $p > \dim M/2$ in conjunction with $p \geq 2$ the validity of **CZ**(p) has been announced under strong curvature assumption, namely the uniform boundedness of the Riemann curvature tensor, but without conditions involving the derivative of curvature, see Theorem 5.18 in the survey of S. Pigola [31].

The rest of the paper is organized as follows. In Section 2, we give various forms of estimates on the Hessian of the heat kernel and on the corresponding semigroups, which are used throughout the proof of **CZ**(p). In Sections 3 finally, we present proofs for Theorem 1.4 and 1.6 respectively. A suitable version of the localization techniques of [17] is included.

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2 Hessian heat kernel estimates

This section is divided into two parts: the first part is on L^2 -estimates of the Hessian of heat kernels under the assumptions of a lower Ricci curvature bound; the second part is on the estimates derived under condition **(H)** from the Hessian formula and Bismut type formulas for the Hessian of semigroups, which are established by means of the techniques from stochastic geometry.

2.1 L^2 -estimates for the Hessian of heat kernel

In this section, we assume Ricci curvature to be bounded below, i.e. validity of condition **(Ric)**. Then, in particular, the local doubling assumption **(LD)** with respect to μ holds.

It is well-known that, under a lower Ricci curvature bound, the heat kernel allows an off-diagonal estimate [15]. The following lemma gives a pointwise off-diagonal estimate for the heat kernel and its time derivative.

Lemma 2.1. *Assume that **(Ric)** holds. Then for any $\alpha \in (0, \frac{1}{4})$, there exist constants C and $C_1 > 0$ depending on the dimension d and α such that for all $x, y \in M$ and $t > 0$,*

$$p_t(x, y) + \left| \frac{\partial p_t}{\partial t}(x, y) \right| \leq \frac{C}{V(y, \sqrt{t})} \exp\left(-\alpha \frac{\rho^2(x, y)}{t} + C_1 K t\right). \quad (2.1)$$

Proof. The estimate of $p_t(x, y)$ is an easy consequence of (1.5) and [15, Theorem 2] or [36, Theorem 2.4.4], where it is proved that for all $x, y \in M$ and $t > 0$,

$$p_t(x, y) \leq \frac{C}{V(y, \sqrt{t})} \exp\left(-\alpha \frac{\rho^2(x, y)}{t} + C_1 K t\right). \quad (2.2)$$

The estimate for $|\frac{\partial p_t}{\partial t}(x, y)|$ follows from that of $p_t(x, y)$ and the analytic property of the semigroup (see [16, Theorem 4] or [19, Corollary 3.3]). \square

The following lemma gives weighted L^2 -integral estimates for the heat kernel, its gradient and its Laplacian.

Lemma 2.2. *Assume that (Ric) holds. Let $\alpha \in (0, \frac{1}{4})$ be as in Lemma 2.1. For all $\gamma \in (0, 2\alpha)$, $s > 0$ and $y \in M$,*

$$\int_M \left[|p_s(x, y)|^2 + s |\nabla_x p_s(x, y)|^2 + s^2 |\Delta_x p_s(x, y)|^2 \right] e^{\gamma \frac{\rho^2(x, y)}{s}} \mu(dx) \leq \frac{C_\gamma}{V(y, \sqrt{s})} e^{2C's},$$

where $C_\gamma > 0$ depends on γ and $C' > 0$ on α and K .

Proof. By (LD), it is easy to see that for all $\gamma > 0$, $s, t > 0$ and $y \in M$, there exist two positive constants C_γ (depending on γ and the constants in (LD)) and \tilde{C} such that

$$\begin{aligned} \int_{\rho(x, y) \geq \sqrt{t}} e^{-2\gamma \frac{\rho^2(x, y)}{s}} \mu(dx) &\leq e^{-\gamma t/s} \int_M e^{-\gamma \frac{\rho^2(x, y)}{s}} \mu(dx) \\ &\leq e^{-\gamma t/s} \sum_{i=0}^{\infty} V(y, (i+1)\sqrt{s}) e^{-\gamma i^2} \\ &\leq C e^{-\gamma t/s} V(y, \sqrt{s}) \sum_{i=0}^{\infty} (i+1)^d e^{-\gamma i^2} e^{C(i+1)\sqrt{s}} \\ &\leq C e^{-\gamma t/s} e^{C\sqrt{s}} V(y, \sqrt{s}) \sum_{i=0}^{\infty} (i+1)^d e^{-\gamma i^2} e^{\gamma i^2/2 + C^2 s/2} \\ &\leq C e^{-\gamma t/s} e^{C\sqrt{s} + C^2 s/2} V(y, \sqrt{s}) \sum_{i=0}^{\infty} (i+1)^d e^{-\gamma i^2/2} \\ &\leq C_\gamma V(y, \sqrt{s}) e^{-\gamma t/s} e^{\tilde{C}s}, \end{aligned} \quad (2.3)$$

where the second inequality comes from condition (LD). The remainder of the proof follows from [12, Lemmas 2.1-2.3]. \square

We now turn to the estimates for the Hessian of heat kernel. The following lemma shows that the Hessian of heat semigroup also satisfies an L^2 -Gaffney off-diagonal estimate. Note that in the following discussion the constant C will be different in different lines without confusion.

Lemma 2.3. *Assume that (Ric) holds. There exist constants $C, C_2 > 0$ such that for all $t \in (0, \infty)$, all Borel subsets $E, F \subset M$ with compact closure, and all $f \in L^2(M)$ with $\text{supp } f \subset E$,*

$$\|\mathbb{1}_F t |\text{Hess } P_t f|\|_2 \leq C(1 + \sqrt{t}) \exp\left(-\frac{C_2 \rho^2(E, F)}{t}\right) \|f\|_2.$$

Proof. Recall the following L^2 -Gaffney off-diagonal estimate on one-forms [4]. If **(Ric)** holds, then for $\alpha \in \Gamma_{L^2}(T^*M)$ with support $\text{supp}(\alpha) \subset E$ and any $s \in (0, 1)$, it holds

$$\begin{aligned} \|\mathbb{1}_F \sqrt{s} |\nabla e^{-s\Delta^{(1)}} \alpha|\|_2 &\leq C (1 + \sqrt{s}) \exp\left(-\frac{c_1 \rho^2(E, F)}{s}\right) \|\mathbb{1}_E \alpha\|_2 \\ &\leq 2C \exp\left(-\frac{c_1 \rho^2(E, F)}{s}\right) \|\mathbb{1}_E \alpha\|_2 \end{aligned}$$

for some positive constants C and c_1 . On the other hand, we have Gaffney's off-diagonal estimate for $|\nabla P_{t_2} f|$ (see [3, (3.1)]), i.e. for all $f \in L^2(M)$ with support in E and $u > 0$,

$$\|\mathbb{1}_F \sqrt{u} |\nabla P_u f|\|_2 \leq C \exp\left(-\frac{c_2 \rho^2(E, F)}{u}\right) \|\mathbb{1}_E f\|_2$$

for some positive constants C and c_2 . For $t > 0$, denoting by $\Delta^{(1)}$ the Laplacian on one-forms, we may write

$$\text{Hess } P_t f = \frac{\sqrt{\left(t - \left(\frac{t}{2} \wedge 1\right)\right)} \nabla e^{-\left(t - \left(\frac{t}{2} \wedge 1\right)\right) \Delta^{(1)}} \left(\sqrt{\frac{t}{2} \wedge 1} dP_{\frac{t}{2} \wedge 1} f\right)}{\sqrt{\left(t - \left(\frac{t}{2} \wedge 1\right)\right) \left(\frac{t}{2} \wedge 1\right)}}$$

so that using the composition rule of Gaffney's off-diagonal estimate (see [27, Lemma 2.3]), we obtain

$$\|\mathbb{1}_{E^c} |\text{Hess } P_t f|\|_2 \leq C \frac{t}{\sqrt{\left(t - \left(\frac{t}{2} \wedge 1\right)\right) \left(\frac{t}{2} \wedge 1\right)}} \exp\left(-\frac{c \rho^2(E, F)}{\max\left\{\frac{t}{2} \wedge 1, t - \left(\frac{t}{2} \wedge 1\right)\right\}}\right) \|\mathbb{1}_E f\|_2,$$

for some positive constants C and c . Note that for $0 < t/2 < 1$,

$$\max\left\{\frac{t}{2} \wedge 1, t - \left(\frac{t}{2} \wedge 1\right)\right\} = \max\left\{\frac{t}{2}, t - \frac{t}{2}\right\} = \frac{t}{2}; \quad \frac{t}{\sqrt{\left(t - \left(\frac{t}{2} \wedge 1\right)\right) \left(\frac{t}{2} \wedge 1\right)}} = 2,$$

and that for $t/2 \geq 1$,

$$\max\left\{\frac{t}{2} \wedge 1, t - \left(\frac{t}{2} \wedge 1\right)\right\} = \max\{1, t - 1\} = t - 1 \leq t; \quad \frac{t}{\sqrt{\left(t - \left(\frac{t}{2} \wedge 1\right)\right) \left(\frac{t}{2} \wedge 1\right)}} = \frac{t}{\sqrt{t-1}} \leq \sqrt{2t}.$$

Therefore, we conclude that there exist positive constants C and C_2 such that

$$\|\mathbb{1}_{F^c} |\text{Hess } P_t f|\|_2 \leq C (1 + \sqrt{t}) \exp\left(-\frac{C_2 \rho^2(E, F)}{t}\right) \|\mathbb{1}_E f\|_2. \quad \square$$

The following proposition gives L^2 -weighted estimates for the Hessian of the heat kernel.

Proposition 2.4. *Assume that **(Ric)** holds. Fix $\alpha \in (0, \frac{1}{4})$ as in Lemma 2.1. Then for all $\gamma \in (0, 2\alpha)$, $s > 0$ and $y \in M$, there exists a constant $C > 0$ such that*

$$\int_M |\text{Hess}_x p_s(x, y)|^2 \exp\left(\gamma \frac{\rho^2(x, y)}{s}\right) \mu(dx) \leq \frac{C(1 + Ks) e^{2C's}}{s^2 V(y, \sqrt{s})}$$

where $C' > 0$ is the same constant as in Lemma 2.2.

Proof. We begin by integrating Bochner's identity (1.1) to obtain

$$\begin{aligned} \frac{1}{2} \int |\nabla p_s|^2 \Delta e^{\gamma \rho^2(x,y)/s} \mu(dx) &= \frac{1}{2} \int_M (\Delta |\nabla p_s|^2) e^{\gamma \rho^2(x,y)/s} \mu(dx) \\ &= \int_M \left(-|\text{Hess}(p_s)|_{\text{HS}}^2 + g(\nabla \Delta p_s, \nabla p_s) - \text{Ric}(\nabla p_s, \nabla p_s) \right) e^{\gamma \rho^2(x,y)/s} \mu(dx), \end{aligned}$$

which then implies

$$\begin{aligned} &\int_M |\text{Hess}(p_s)|_{\text{HS}}^2 e^{\gamma \rho^2(x,y)/s} \mu(dx) \tag{2.4} \\ &= -\frac{1}{2} \int_M |\nabla p_s|^2 \Delta e^{\gamma \rho^2(x,y)/s} \mu(dx) + \int_M g(\nabla \Delta p_s, \nabla p_s) e^{\gamma \rho^2(x,y)/s} \mu(dx) \\ &\quad - \int_M \text{Ric}(\nabla p_s, \nabla p_s) e^{\gamma \rho^2(x,y)/s} \mu(dx) \\ &= -\frac{\gamma}{s} \int_M |\nabla p_s|^2 \text{div} \left(e^{\gamma \rho^2(x,y)/s} \rho(x,y) \nabla \rho(\cdot, y)(x) \right) \mu(dx) + \int \Delta p_s \text{div} \left(e^{\gamma \rho^2(x,y)/s} \nabla p_s \right) \mu(dx) \\ &\quad - \int \text{Ric}(\nabla p_s, \nabla p_s) e^{\gamma \rho^2(x,y)/s} \mu(dx) =: I_1 + I_2 + I_3 \end{aligned}$$

(with the sign convention $\Delta = \text{div } \nabla$ for the divergence). Now for any $\beta \in (0, 2\alpha)$, let

$$E(s, y, \beta) := \int_M \left[|p_s(x, y)|^2 + s |\nabla_x p_s(x, y)|^2 + s^2 |\Delta_x p_s(x, y)|^2 \right] e^{\beta \rho^2(x,y)/s} \mu(dx).$$

Then by condition **(Ric)** and Lemma 2.2, we have

$$I_3 \leq K \int_M |\nabla p_s|^2 e^{\gamma \rho^2(x,y)/s} \mu(dx) \leq \frac{K}{s} E(s, y, \gamma) \leq \frac{K}{s} \frac{C_\gamma}{V(y, \sqrt{s})} e^{2C' s}. \tag{2.5}$$

For I_2 , using the fact $|\nabla \rho| \leq 1$, we deduce from Lemma 2.2 again that

$$\begin{aligned} I_2 &= \int_M (\Delta p_s)^2 e^{\gamma \rho^2(x,y)/s} \mu(dx) - \frac{2\gamma}{s} \int_M (\Delta p_s) e^{\gamma \rho^2(x,y)/s} \rho(x,y) \langle \nabla \rho, \nabla p_s \rangle \mu(dx) \\ &\leq 2 \int_M (\Delta p_s)^2 e^{\gamma \rho^2(x,y)/s} \mu(dx) + \frac{\gamma^2}{s^2} \int_M e^{\gamma \rho^2(x,y)/s} \rho^2(x,y) |\nabla p_s|^2 \mu(dx) \\ &\leq \frac{C_\gamma}{s^2} E(s, y, \gamma') \leq \frac{1}{s^2} \frac{C_{\gamma'}}{V(y, \sqrt{s})} e^{2C' s}, \end{aligned} \tag{2.6}$$

where $\gamma < \gamma' < 2\alpha$ with $\gamma' - \gamma$ sufficiently small.

To bound I_1 , we write

$$\begin{aligned} \frac{s}{\gamma} I_1 &= \int_M |\nabla p_s|^2 e^{\gamma \rho^2(x,y)/s} \mu(dx) + 2 \int_M |\nabla p_s|^2 e^{\gamma \rho^2(x,y)/s} \frac{\gamma \rho^2(x,y)}{s} \mu(dx) \\ &\quad - \int_M |\nabla p_s|^2 e^{\gamma \rho^2(x,y)/s} \rho(x,y) \Delta \rho(\cdot, y)(x) \mu(dx). \end{aligned} \tag{2.7}$$

By the Laplacian comparison theorem (see e.g. [8, p185]), we have

$$-\Delta \rho(\cdot, y)(x) \leq \frac{d-1}{\rho(x,y)} + K \rho(x,y)$$

outside of the cut-locus. This yields in particular

$$\begin{aligned}
& -\frac{\gamma}{s} \int_M |\nabla p_s|^2 e^{\gamma \rho^2(x,y)/s} \rho(x,y) \Delta \rho(\cdot, y)(x) \mu(dx) \\
& \leq \frac{(d-1)\gamma}{s} \int_M |\nabla p_s|^2 e^{\gamma \rho^2(x,y)/s} \mu(dx) + K \int_M |\nabla p_s|^2 e^{\gamma \rho^2(x,y)/s} \frac{\gamma \rho^2(x,y)}{s} \mu(dx) \\
& \leq \frac{(d-1)\gamma}{s} \int_M |\nabla p_s|^2 e^{\gamma \rho^2(x,y)/s} \mu(dx) + K \int_M |\nabla p_s|^2 e^{\frac{\gamma' \rho^2(x,y)}{s}} \mu(dx) \\
& \leq \frac{C_\gamma(1+Ks)}{s^2} E(s, y, \gamma')
\end{aligned}$$

with $\gamma < \gamma' < 2\alpha$ as in (2.6). Combining this estimate with (2.7), it follows

$$I_1 \leq C_\gamma \frac{(1+Ks)}{s^2} E(s, y, \gamma') \leq \frac{c(1+Ks)}{s^2 \sqrt{V(y, \sqrt{s})}} e^{2C's}. \quad (2.8)$$

Altogether (2.4) through (2.8), we conclude that

$$\int_M |\text{Hess}(p_s)|_{\text{HS}}^2 e^{\gamma \rho^2(x,y)/s} \mu(dx) = I_1 + I_2 + I_3 \leq c \frac{(1+Ks)}{s^2 \sqrt{V(y, \sqrt{s})}} e^{2C's}.$$

which completes the proof of Proposition 2.4 by using the fact

$$|\text{Hess}(p_s)|^2 \leq |\text{Hess}(p_s)|_{\text{HS}}^2. \quad \square$$

Using Proposition 2.4, we obtain the following L^2 -integral estimates for Hessian of heat kernel.

Corollary 2.5. *Assume (Ric) holds. Fix $\alpha \in (0, \frac{1}{4})$ as in Lemma 2.1. There exist $0 < \beta < \alpha$ and $C'' > C' > 0$ with C' as in Lemma 2.2 such that*

$$\int_{\rho(x,y) \geq t^{1/2}} |\text{Hess}_x p_s(x, y)| \mu(dx) \leq C(1 + \sqrt{s}) e^{C's} e^{-\beta t/s} s^{-1}$$

for all $y \in M$ and $s, t > 0$.

Proof. Let $0 < \beta < \alpha$. By Cauchy's inequality we obtain

$$\begin{aligned}
& \int_{\rho(x,y) \geq t^{1/2}} |\text{Hess}_x p_s(x, y)| \mu(dx) \\
& \leq \left(\int_M |\text{Hess}_x p_s(x, y)|^2 e^{2\beta \rho^2(x,y)/s} \mu(dx) \right)^{1/2} \left(\int_{\rho(x,y) \geq t^{1/2}} e^{-2\beta \rho^2(x,y)/s} \mu(dx) \right)^{1/2} \\
& \leq \frac{C e^{C's} (1 + \sqrt{s})}{s \sqrt{V(y, \sqrt{s})}} \sqrt{V(y, \sqrt{s})} e^{-\beta t/s} e^{\tilde{C}s} \\
& = \frac{C(1 + \sqrt{s})}{s} e^{C's - \beta t/s} e^{\tilde{C}s} \leq \frac{C(1 + \sqrt{s})}{s} e^{C''s - \beta t/s}
\end{aligned}$$

where the second inequality follows from Proposition 2.4 and inequality (2.3). This finishes the proof. \square

2.2 Stochastic Hessian formulas and L^p -estimates for Hessian of heat kernel

In this subsection, we establish some pointwise and L^p -integral estimates for the Hessian of heat kernel when $p > 2$. To this end, let us first introduce some necessary notations. For fixed $x \in M$, let B_t be the stochastic anti-development of $X(x)$ which is a Brownian motion in $T_x M$. Let $//_t: T_x M \rightarrow T_{X_t(x)} M$ be the parallel transport and Q_t be the damped parallel transport along $X(x)$ as defined in (1.7). For each $w \in T_x M$ define an operator-valued process $W_t(\cdot, w): T_x M \rightarrow T_{X_t} M$ by

$$W_t(\cdot, w) = Q_t \int_0^t Q_r^{-1} R(//_r dB_r, Q_r(\cdot)) Q_r(w) - Q_t \int_0^t Q_r^{-1} (\nabla \text{Ric}^\sharp + d^* R)(Q_r(\cdot), Q_r(w)) dr.$$

This means that the process $W_t(\cdot, w)$ is the solution to the following covariant Itô equation

$$\begin{cases} DW_t(\cdot, w) = R(//_t dB_t, Q_t(\cdot)) Q_t(w) - (d^* R + \nabla \text{Ric}^\sharp)(Q_t(\cdot), Q_t(w)) dt - \text{Ric}^\sharp(W_t(\cdot, w)) dt, \\ W_0(\cdot, w) = 0. \end{cases}$$

First, we collect some easy curvature estimates for Riemannian manifolds M satisfying condition **(H)**.

Lemma 2.6. *Assume that **(H)** holds. There exist constants $C > 0$, $\theta > 0$ such that for any $t > 0$,*

$$\sup_{x \in M} \mathbb{E}^x \left[\int_0^t |R|^2(X_s) ds + \int_0^t |\nabla \text{Ric}^\sharp + d^* R|^2(X_s) ds \right] \leq C e^{2\theta t}. \quad (2.9)$$

Proof. By means of the trivial inequality $s \leq e^s$ for $s \geq 0$, we see that

$$\begin{aligned} & \sup_{x \in M} \mathbb{E}^x \left[\int_0^t |R|^2(X_s) ds + \int_0^t |\nabla \text{Ric}^\sharp + d^* R|^2(X_s) ds \right] \\ & \leq \sup_{x \in M} \mathbb{E}^x \left[\int_0^t K_1(X_s) ds + \int_0^t K_2(X_s) ds \right] \\ & \leq \sup_{x \in M} \mathbb{E}^x \left[\exp \left(\int_0^t K_1(X_s) ds \right) \right] + \sup_{x \in M} \mathbb{E}^x \left[\exp \left(\int_0^t K_2(X_s) ds \right) \right]. \end{aligned}$$

Recall in [23, Lemma 3.9] that for any $K \in \mathcal{K}(M)$, there exist positive constants c and C such

$$\sup_{x \in M} \mathbb{E}^x \left[\exp \left(\int_0^t |K(X_s)| ds \right) \right] \leq C e^{ct}.$$

This immediately completes the proof by substituting K_1 and K_2 as K . \square

By this lemma and Bismut-type Hessian formula, we obtain the following Hessian estimates for heat semigroup.

Proposition 2.7. *Assume that **(H)** holds. Then, for any $p > 2$, there exist positive constants C and θ as in Lemma 2.6 such that, for every $f \in \mathcal{B}_b(M)$, $t > 0$,*

$$t |\text{Hess } P_t f|(x) \leq C(1 + \sqrt{t}) e^{(2K+\theta)t} (P_t |f|^p)^{1/p}(x); \quad (2.10)$$

$$t \|\text{Hess } P_t f\|_p \leq C(1 + \sqrt{t}) e^{(2K+\theta)t} \|f\|_p. \quad (2.11)$$

Proof. We start by recalling the following Bismut type formula for Hess P_t (see [2, 18, 9]). Let $x \in D$ with $v, w \in T_x M$, $f \in \mathcal{B}_b(M)$ and $0 < s < t$. Suppose that D_1 and D_2 are regular domains such that $x \in D_1$ and $\bar{D}_1 \subset D_2 \subset D$, and denote by σ, τ the exit times of $X_\cdot(x)$ from D_1 and D_2 respectively. Assume that k, ℓ are bounded adapted processes with paths in the Cameron-Martin space $L^2([0, t]; [0, 1])$ such that $k_r = 0$ for $r \geq \sigma \wedge s$, $k_0 = 1$, $\ell_r = 1$ for $r \leq \sigma \wedge s$, $\ell_s = 0$ for $r \geq \tau \wedge t$. Then, for $f \in \mathcal{B}_b(M)$, we have

$$\begin{aligned} (\text{Hess } P_t f)(v, w) &= -\mathbb{E}^x \left[f(X_t) \int_0^t \langle W_s(\dot{k}_s v, w), \parallel_s dB_s \rangle \right] \\ &\quad + \mathbb{E} \left[f(X_t) \int_s^t \langle Q_r(\dot{\ell}_r w), \parallel_r dB_r \rangle \int_0^s \langle Q_r(\dot{k}_r v), \parallel_r dB_r \rangle \right]. \end{aligned} \quad (2.12)$$

Note that under condition **(H)** one has $|Q_t|^2 \leq e^{2Kt}$ and according to the Itô formula, then for $t < \tau < \tau_D$ and $0 < \delta < 2K$,

$$\begin{aligned} d|W_s(v, w)|^2 &= 2 \langle R(\parallel_s dB_s, Q_s(v)) Q_s(w), W_s(v, w) \rangle + 2|R^{\#\#}(Q_s(v), Q_s(w))|_{\text{HS}}^2 ds \\ &\quad - 2 \langle (d^* R + \nabla \text{Ric}^{\#\#})(Q_s(v), Q_s(w)), W_s(v, w) \rangle ds \\ &\quad - 2 \text{Ric}(W_s(v, w), W_s(v, w)) ds \\ &\leq 2 \langle R(\parallel_s dB_s, Q_s(v)) Q_s(w), W_s(v, w) \rangle + 2e^{4Ks} K_1(X_s) ds \\ &\quad + 2\sqrt{K_2(X_s)} e^{2Ks} |W_s(v, w)| ds + 2K|W_s(v, w)|^2 ds \\ &\leq 2 \langle R(\parallel_s dB_s, Q_s(v)) Q_s(w), W_s(v, w) \rangle + 2e^{4Ks} K_1(X_s) ds \\ &\quad + (2\delta)^{-1} e^{4Ks} K_2(X_s) ds + (2K + \delta) |W_s(v, w)|^2 ds. \end{aligned}$$

From this and Lemma 2.6 there exist constants $C > 0$ and $\theta > 0$ such that

$$\begin{aligned} \mathbb{E}|W_{t \wedge \tau_D}(v, w)|^2 &\leq 2e^{(2K+\delta)t} \left(\mathbb{E} \left[\int_0^t e^{(2K-\delta)s} K_1(X_s) ds \right] + (4\delta)^{-1} \mathbb{E} \left[\int_0^t e^{(2K-\delta)s} K_2(X_s) ds \right] \right) \\ &\leq 2 \max\{1, (4\delta)^{-1}\} e^{4Kt} \mathbb{E} \left[\int_0^t K_1(X_s) ds + \int_0^t K_2(X_s) ds \right] \\ &\leq C e^{(4K+2\theta)t}. \end{aligned} \quad (2.13)$$

Since the upper of $\mathbb{E}|W_{t \wedge \tau_D}(v, w)|^2$ is independent of the domain D , taking a sequence of domains D_n increasing to M , we conclude that $\mathbb{E}|W_t(v, w)|^2 \leq C e^{(4K+2\theta)t}$. All these estimates allow to verify that formula (2.12) holds for the deterministic functions $k_s = \frac{t-2s}{t} \vee 0$ and $\ell_s = 1 \wedge \frac{2(t-s)}{t}$ as well. Then, by estimate (2.13) and $|Q_s| \leq e^{Ks}$, we get for any $p > 2$,

$$\begin{aligned} |\text{Hess } P_t f| &\leq \frac{2}{t} (P_t |f|^p)^{1/p} \left[\int_0^{t/2} \mathbb{E}|W_s(v, w)|^2 ds \right]^{1/2} \\ &\quad + \frac{4}{t^2} (P_t |f|^p)^{1/p} \mathbb{E} \left[\left(\int_0^{t/2} \langle Q_s(w), \parallel_s dB_s \rangle \right)^2 \mathbb{E} \left[\left(\int_{t/2}^t \langle Q_s(v), \parallel_s dB_s \rangle \right)^2 \middle| \mathcal{F}_{t/2} \right] \right]^{1/2} \\ &\leq \frac{2}{t} (P_t |f|^p)^{1/p} \left(\int_0^{t/2} e^{(4K+2\theta)s} ds \right)^{1/2} \\ &\quad + \frac{4}{t^2} (P_t |f|^p)^{1/p} \mathbb{E} \left[\int_0^{t/2} e^{2Ks} ds \right]^{1/2} \left(\int_{t/2}^t e^{2Ks} ds \right)^{1/2} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{C}{\sqrt{t}} e^{(2K+\theta)t} (P_t|f|^p)^{1/p} + \frac{C}{t} e^{2Kt} (P_t|f|^p)^{1/p} \\
&\leq \frac{C(1+\sqrt{t})}{t} e^{(2K+\theta)t} (P_t|f|^p)^{1/p}
\end{aligned} \tag{2.14}$$

which proves (2.10). Moreover, integration with respect to the measure μ yields

$$\mu(|\text{Hess } P_t f|^p)^{1/p} \leq \frac{C(1+\sqrt{t})}{t} e^{(2K+\theta)t} \mu(|f|^p)^{1/p}.$$

This proves (2.11) and finishes the proof. \square

The following proposition gives a pointwise estimate for the Hessian of heat semigroup.

Proposition 2.8. *Assume that (H) holds. Then there exist constants $C, \theta > 0$ such that*

$$|\text{Hess } P_t f| \leq e^{2Kt} (P_t |\text{Hess } f|^2)^{1/2} + C e^{(2K+\theta)t} (P_t |df|^2)^{1/2} \tag{2.15}$$

for $t > 0$ and $f \in C_0^\infty(M)$.

Proof. Recall the the following relation

$$d\Delta f = \text{tr Hess}(df) - df(\text{Ric}^\sharp). \tag{2.16}$$

Moreover, using the differential Bianchi identity [10, p. 185], we have

$$\text{Hess}(\Delta f) = \text{tr Hess}(\text{Hess } f) - 2(\text{Hess } f)(\text{Ric}^\sharp \otimes \text{id} - R^{\sharp, \sharp}) - df(d^*R + \nabla \text{Ric}^\sharp). \tag{2.17}$$

Now, for $s > 0$ and $v, w \in T_x M$, let

$$N_s(v, w) := (\text{Hess } P_{t-s} f)(Q_s(v), Q_s(w))(X_s) + (dP_{t-s} f)(W_s(v, w))(X_s).$$

We claim that N_s is a local martingale. Combining (2.16), (2.17) and applying Itô's formula, we obtain

$$\begin{aligned}
dN_s(v, w) &= (\nabla_{\parallel_s dB_s} \text{Hess } P_{t-s} f)(Q_s(v), Q_s(w)) + (\text{Hess } P_{t-s} f)\left(\frac{D}{ds} Q_s(v), Q_s(w)\right) ds \\
&\quad + (\text{Hess } P_{t-s} f)(Q_s(v), \frac{D}{ds} Q_s(w)) ds + \partial_s (\text{Hess } P_{t-s} f)(Q_s(v), Q_s(w)) ds \\
&\quad + \text{tr Hess}(\text{Hess } P_{t-s} f)(Q_s(v), Q_s(w)) ds + (\nabla_{\parallel_s dB_s} dP_{t-s} f)(W_s(v, w)) \\
&\quad + (dP_{t-s} f)(DW_s(v, w)) + \langle d(dP_{t-s} f), DW_s(v, w) \rangle + \partial_s (dP_{t-s} f)(W_s(v, w)) ds \\
&\quad + \text{tr Hess}(dP_{t-s} f)(W_s(v, w)) ds \stackrel{\text{m}}{=} 0,
\end{aligned}$$

where $\stackrel{\text{m}}{=}$ denotes equality modulo the differential of a local martingale, and here we used for the quadratic covariation the formula

$$[d(dP_{t-s} f), DW_s(v, w)]_s = (\text{Hess } P_{t-s} f)(R^{\sharp, \sharp}(Q_s(v), Q_s(w))) ds.$$

Therefore, N_s is a local martingale. Recall that condition (H) implies

$$|Q_t|^2 \leq e^{2Kt} \quad \text{and} \quad \mathbb{E}|W_t(v, w)|^2 \leq C e^{(4K+2\theta)t}$$

for some positive constants. In view of estimate (2.14), $|\text{Hess } P_{t-s}f|$ is bounded on $[0, t - \varepsilon] \times M$, and

$$|\nabla P_{t-s}f| \leq e^{K(t-s)} \|\nabla f\|_\infty$$

as a consequence of the derivative formula (1.8). Thus N_s is a martingale on the time interval $[0, t - \varepsilon]$, and by taking expectation at time 0 and $t - \varepsilon$, we arrive at

$$(\text{Hess } P_t f)(v, w) = \mathbb{E}\left[(\text{Hess } P_\varepsilon f)(Q_{t-\varepsilon}(v), Q_{t-\varepsilon}(w))(X_t) + dP_\varepsilon f(W_{t-\varepsilon}(v, w))(X_{t-\varepsilon})\right]. \quad (2.18)$$

Letting ε tend to 0, it then follows that

$$\begin{aligned} |\text{Hess } P_t f(v, w)| &\leq \left| \mathbb{E}\left[\text{Hess } f(Q_t(v), Q_t(w))(X_t)\right] \right| + \left| \mathbb{E}\left[df(W_t(v, w))(X_t)\right] \right| \\ &\leq P_t\left[|\text{Hess } f| |Q_t(v)| |Q_t(w)|\right] + (P_t |df|^2)^{1/2} \mathbb{E}\left[|W_t(v, w)|^2\right]^{1/2} \\ &\leq (P_t |\text{Hess } f|^2)^{1/2} e^{2Kt} + C e^{(2K+\theta)t} (P_t |df|^2)^{1/2}. \quad \square \end{aligned}$$

Remark 2.9. Let

$$\|R\|_\infty := \sup_{x \in M} |R|(x) \quad \text{and} \quad \|\nabla \text{Ric}^\sharp + d^*R\|_\infty := \sup_{x \in M} |\nabla \text{Ric}^\sharp + d^*R|(x).$$

If assumption **(H)** is replaced by **(Ric)** with $K \geq 0$, $\|R\|_\infty < \infty$ and $\|\nabla \text{Ric}^\sharp + d^*R\|_\infty < \infty$, then it is straightforward to see from (2.13) that for $0 < \delta < 2K$,

$$\mathbb{E}|W_t(v, w)|^2 \leq e^{(2K+\delta)t} \mathbb{E}\left[\int_0^t e^{(2K-\delta)s} \|R\|_\infty^2 ds + \int_0^t e^{(2K-\delta)s} \|\nabla \text{Ric}^\sharp + d^*R\|_\infty^2 ds\right] \leq C e^{4Kt}$$

for some explicit constant $C > 0$. Proceeding as in the proof of Lemma 2.8 above, we obtain in this case

$$|\text{Hess } P_t f| \leq e^{2Kt} (P_t |\text{Hess } f|^2)^{1/2} + C \sqrt{t} e^{2Kt} (P_t |df|^2)^{1/2}.$$

Proposition 2.8 is used to establish pointwise Gaussian upper bounds for the Hessian of the heat kernel via inequality (2.15). Note that Carron [7] applied pointwise Gaussian upper bounds for the gradient of the heat kernel to the Riesz transform on complete manifolds whose Ricci curvature satisfies a quadratic decay control. Here, we shall use the following Hessian heat kernel estimate in the next section to establish the Calderón-Zygmund inequality.

Proposition 2.10. *Assume that **(H)** holds. Fix $\alpha \in (0, 1/4)$ as in Lemma 2.1 and let $\theta > 0$ be as in (2.9). Then there exist $0 < \beta < 2\alpha$ and $C_3 > 0$ such that*

$$|\text{Hess}_x p_t(x, y)| \leq \frac{C(1 + \sqrt{t})}{tV(y, \sqrt{t})} \exp\left(-\beta \frac{\rho^2(x, y)}{t} + \frac{1}{2}(C_3 + \theta)t\right).$$

for any $t > 0$ and $x, y \in M$.

Proof. For $t > 0$ and $x, y \in M$, write

$$\text{Hess}_x p_{2t}(x, y) = \text{Hess } P_t(p_t(\cdot, y))(x)$$

Applying Proposition 2.8 and taking $\gamma \in (0, 2\alpha)$, we have

$$\begin{aligned} |\text{Hess } P_t(p_t(\cdot, y))(x)| &\leq e^{2Kt} \left(\int |\text{Hess}_z p_t(z, y)|^2 e^{\gamma\rho^2(z,y)/t} e^{-\gamma\rho^2(z,y)/t} |p_t(x, z)| \mu(dz) \right)^{1/2} \\ &\quad + Ce^{(2K+\theta)t} \left(\int |\nabla_z p_t(z, y)|^2 e^{\gamma\rho^2(z,y)/t} e^{-\gamma\rho^2(z,y)/t} |p_t(x, z)| \mu(dz) \right)^{1/2}. \end{aligned}$$

Moreover, by Lemma 2.2 and Proposition 2.4, we know that there exists $C' > 0$ such that

$$\int_M (|\nabla_z p_t(z, y)|^2 e^{\gamma\rho^2(z,y)/t} + |\text{Hess}_z p_t(z, y)|^2 e^{\gamma\rho^2(z,y)/t}) \mu(dz) \leq \frac{C(1+Kt)}{t^2 V(y, \sqrt{t})} e^{2C't}.$$

This further implies that there exists a constant $C_3 > 0$ such that

$$\begin{aligned} |\text{Hess } P_t(p_t(\cdot, y))(x)| &\leq \frac{C(1+\sqrt{t})}{t\sqrt{V(y, \sqrt{t})}} e^{((C'+2K)+\theta)t} \sup_{z \in M} \left\{ e^{-\gamma\rho^2(z,y)/t} |p_t(x, z)| \right\}^{1/2} \\ &\leq \frac{C(1+\sqrt{t})}{tV(y, \sqrt{t})} e^{(C_3+\theta)t} e^{-\beta\rho^2(x,y)/t} \end{aligned}$$

for β small enough, where in the last inequality we have used the estimate

$$\sup_{z \in M} \left\{ e^{-\gamma\rho^2(z,y)/t} |p_t(x, z)| \right\} \leq \frac{e^{C_1 Kt}}{V(y, \sqrt{t})} e^{-2\beta\rho^2(x,y)/t},$$

which can be deduced from Lemma 2.1. Thus, we conclude

$$|\text{Hess}_x p_{2t}(x, y)| \leq \frac{C(1+\sqrt{t})}{tV(y, \sqrt{t})} e^{(C_3+\theta)t} e^{-\beta\rho^2(x,y)/t}$$

which finishes the proof of Proposition 2.10. \square

By interpolating the L^p -Hessian inequality with the L^2 -Gaffney off-diagonal estimates, one obtains L^p -Gaffney off-diagonal estimates for any $p > 2$ as follows.

Proposition 2.11. *Assume that (H) hold. Then for $p > 2$ there exists constants $C, C_4 > 0$ such that for all $t \in (0, \infty)$, all Borel subsets $E, F \subset M$ with compact closure, and all $f \in L^p(M)$ with $\text{supp } f \subset E$,*

$$\|\mathbb{1}_F |\text{Hess } P_t f|\|_p \leq C(1+\sqrt{t}) e^{(2K+\theta)t} e^{-C_4\rho^2(E,F)/t} \|f\|_p.$$

Proof. Let $p > 2$ and $t > 0$. For E, F and f as above, by inequality (2.10), there exist positive constants C and C_2 such that

$$\begin{aligned} \int_F t^p |\text{Hess } P_t f|^p(x) \mu(dx) &\leq Ce^{(2K+\theta)pt} (1+\sqrt{t})^p \int_F (P_t |f|^{p/2})^2(x) \mu(dx) \\ &\leq Ce^{-2C_2\rho^2(E,F)/t} e^{(2K+\theta)pt} (1+\sqrt{t})^p \int_E |f|^p \mu(dx) \end{aligned}$$

where the last inequality follows from the L^2 -Gaffney off-diagonal estimates for $P_t f$ (Lemma 2.3), i.e. the existence of positive constants C and C_2 such that

$$\|\mathbb{1}_F |P_t f|^{p/2}\|_2 \leq C e^{-C_2\rho^2(E,F)/t} \|\mathbb{1}_E |f|^{p/2}\|_2$$

for all $f \in L^p(M)$ with $\text{supp } f \subset E$. \square

3 Calderón-Zygmund inequality for $p \neq 2$

In this section, we prove the main results of this paper. We first prove Theorem 1.4 in Section 3.1, namely **CZ**(p) for $p \in (1, 2)$ under the condition **(Ric)**; then in Section 3.2, we prove Theorem 1.6, namely, **CZ**(p) for $p > 2$ under the condition **(H)**. Recall that **CZ**(2) always holds under condition **(Ric)**.

3.1 The case $p \in (1, 2)$

Let M be a complete Riemannian manifold satisfying **(Ric)**. In this subsection, we prove Theorem 1.4 and show that the Calderón-Zygmund inequality holds for $p \in (1, 2)$. To be precise, let

$$T = \text{Hess}(\Delta + \sigma)^{-1} = \int_0^\infty e^{-\sigma t} \text{Hess} P_t dt. \quad (3.1)$$

We show that T is bounded on $L^p(M)$ for any $p \in (1, 2)$, provided $\sigma > 0$ is large enough. Since T is already bounded on $L^2(M)$, using interpolation, it suffices to prove that T is of weak type $(1, 1)$, that is for some constant C ,

$$\mu(\{x \in M : |Tf(x)| > \lambda\}) \leq C \frac{\|f\|_1}{\lambda} \quad (3.2)$$

for all $\lambda > 0$ and $f \in L^1(M)$. To this end, we need the following technical lemma from [3, Section 4] on the finite overlap property of M .

Lemma 3.1. [3] *Assume that **(LD)** holds. There exists a countable subset $C = \{x_j\}_{j \in \Lambda} \subset M$ such that*

- (i) $M = \cup_{j \in \Lambda} B(x_j, 1)$;
- (ii) $\{B(x_j, 1/2)\}_{j \in \Lambda}$ are disjoint;
- (iii) there exists $N_0 \in \mathbb{N}$ such that for any $x \in M$, there are at most N_0 balls $B(x_j, 4)$ containing x ;
- (iv) for any $c_0 \geq 1$, there exists $C > 0$ such that for any $j \in \Lambda$, $x \in B(x_j, c_0)$ and $r \in (0, \infty)$,

$$\mu(B(x, 2r) \cap B(x_j, c_0)) \leq C \mu(B(x, r) \cap B(x_j, c_0))$$

and

$$\mu(B(x, r)) \leq C \mu(B(x, r) \cap B(x_j, c_0))$$

for any $x \in B(x_j, c_0)$ and $r \in (0, 2c_0]$.

The following lemma provides the localization argument to prove (3.2).

Lemma 3.2. *Assume that **(Ric)** holds. Let $C = \{x_j\}_{j \in \Lambda}$ be a countable subset of M having finite overlap property as in Lemma 3.1. Let $\sigma > C''$ where C'' is as in Corollary 2.5. Suppose that there exists a positive constant C such that*

$$\mu\left(\{x : \mathbb{1}_{B(x_j, 2)} |Tf(x)| > \lambda\}\right) \leq \frac{C}{\lambda} \|f\|_1 \quad (3.3)$$

for any $j \in \Lambda$, $\lambda \in (0, \infty)$ and $f \in C_0^\infty(B(x_j, 1))$. Then (3.2) holds for any $f \in C_0^\infty(M)$.

Proof. For $j \in \Lambda$, let $B_j := B(x_j, 1)$ and let $\{\varphi_j\}_{j \in \Lambda}$ be a C^∞ -partition of unity such that $0 \leq \varphi_j \leq 1$ and is supported in B_j . Then, for any $f \in C^\infty(M)$ and $x \in M$, write

$$Tf(x) = \sum_{j \in \Lambda} \mathbb{1}_{2B_j} T(f\varphi_j)(x) + \sum_{j \in \Lambda} (1 - \mathbb{1}_{2B_j}) T(f\varphi_j)(x),$$

which yields that for any $\lambda > 0$,

$$\begin{aligned} \mu(\{x: |Tf(x)| > \lambda\}) &\leq \mu\left(\left\{x: \sum_{j \in \Lambda} \mathbb{1}_{2B_j} |T(f\varphi_j)(x)| > \frac{\lambda}{2}\right\}\right) + \mu\left(\left\{x: \sum_{j \in \Lambda} (1 - \mathbb{1}_{2B_j}) |T(f\varphi_j)(x)| > \frac{\lambda}{2}\right\}\right) \\ &=: I_1 + I_2. \end{aligned}$$

For I_1 , by Lemma 3.1(iii) and (3.3), we have

$$I_1 \leq \sum_{j \in \Lambda} \mu\left(\left\{x: \mathbb{1}_{2B_j} |T(f\varphi_j)(x)| > \frac{\lambda}{2N_0}\right\}\right) \lesssim \frac{1}{\lambda} \|f\|_1 \quad (3.4)$$

as desired, where the notation $a \lesssim b$ means $a \leq Cb$ for some constant C .

To bound I_2 , again by Lemma 3.1(iii), since φ_j is supported in B_j , it is easy to see that

$$\sum_{j \in \Lambda} |(1 - \mathbb{1}_{2B_j})(x)\varphi_j(y)| \leq N_0 \mathbb{1}_{\{\rho(x,y) \geq 1\}}.$$

Hence, according to the definition of T in (3.1) and Corollary 2.5, we get

$$\begin{aligned} I_2 &\leq \frac{2}{\lambda} \sum_{j \in \Lambda} \left\| \left| (1 - \mathbb{1}_{2B_j}) T(f\varphi_j) \right| \right\|_1 \\ &\lesssim \frac{1}{\lambda} \int_M \left(\int_0^\infty e^{-\sigma t} \int_M |\text{Hess}_x p_t(x, y)| \sum_{j \in \Lambda} |(1 - \mathbb{1}_{2B_j})(x)\varphi_j(y)| |f(y)| \mu(dy) dt \right) \mu(dx) \\ &\lesssim \frac{1}{\lambda} \int_M \int_0^\infty e^{-\sigma t} \left(\int_{\rho(x,y) \geq 1} |\text{Hess}_x p_t(x, y)| \mu(dx) \right) dt |f(y)| \mu(dy) \\ &\leq \frac{1}{\lambda} \int_M |f(y)| \mu(dy) \int_0^\infty e^{-\sigma t} (1 + \sqrt{t}) e^{C''t} e^{-\beta/t} t^{-1} dt, \end{aligned}$$

where $\beta \in (0, \alpha)$. Thus, since $\sigma > C''$, we obtain

$$I_2 \lesssim \frac{1}{\lambda} \int_0^\infty e^{(C'' - \sigma)t - \beta/t} \frac{(1 + \sqrt{t})}{t} dt \|f\|_1 \lesssim \frac{1}{\lambda} \|f\|_1,$$

which combined with the estimate about I_1 in (3.4) finishes the proof of Lemma 3.2. \square

To prove property (3.3), we remove the subscript j and simply write B for each $B_j := B(x_j, 1)$. Let $c_0 \geq 1$. By Lemma 3.1(iv), we have that $(c_0 B, \mu, \rho)$ is a metric measure subspace satisfying the *volume doubling property* that there exists $C_D \geq 1$ such that

$$\mu(B(x, 2r) \cap c_0 B) \leq C_D \mu(B(x, r) \cap c_0 B) \quad (\mathbf{D})$$

for all $x \in c_0 B$ and $r > 0$.

We also need the following Calderón-Zygmund decomposition from [11].

Lemma 3.3 ([11]). *Let (X, ν, ρ) be a metric measure space satisfying **(D)** with c_0B replaced by X . Let $f \in L^1(X)$ and $\lambda \in (0, \infty)$. Assume $\|f\|_{L^1} < \lambda\nu(X)$. Then f has a decomposition of the form*

$$f = g + b = g + \sum_i b_i$$

such that

(a) $g(x) \leq C\lambda$ for almost all $x \in M$;

(b) there exists a sequence of balls $B_i = B(x_i, r_i)$ so that the support of each b_i is contained in B_i :

$$\int_X |b_i(x)| \nu(dx) \leq C\lambda\nu(B_i) \quad \text{and} \quad \int_X b_i(x) \nu(dx) = 0;$$

(c) $\sum_i \nu(B_i) \leq \frac{C}{\lambda} \int_X |f(x)| \nu(dx)$;

(d) there exists $k_0 \in \mathbb{N}^*$ such that each point of M is contained in at most k_0 balls B_i .

Lemma 3.4. *Assume that **(Ric)** holds. Let $\lambda \in (0, \infty)$ and $f \in L^1(B)$ be as in Lemma 3.3. Assume that $\{b_i\}$ is the sequence of bad functions as in Lemma 3.3 and $\{P_t^\sigma\}_{t \geq 0}$ the heat semigroup associated to $-(\Delta + \sigma)$ with $\sigma > 0$. Then there exist $\sigma > 0$ and $C > 0$ independent of f such that*

$$\left\| \sum_i P_{t_i}^\sigma b_i \right\|_2^2 \leq C\lambda \|f\|_1$$

where $t_i = r_i^2$ with r_i denoting the radius of the ball B_i as in Lemma 3.3(b).

Proof. Recall that $\text{supp } b_i \subset B(x_i, \sqrt{t_i})$. Using the upper bound of the heat kernel in Lemma 2.1 and Lemma 3.3, we have for $x \in M$,

$$\begin{aligned} |P_{t_i}^\sigma b_i(x)| &\leq \int_M \frac{e^{-\sigma' t_i - \alpha \frac{\rho^2(x,y)}{t_i}}}{V(x, \sqrt{t_i})} |b_i(y)| \mu(dy) \\ &\leq \frac{C}{V(x, \sqrt{t_i})} e^{-\sigma' t_i - \alpha' \frac{\rho^2(x,x_i)}{t_i}} \int_{B_i} |b_i(y)| \mu(dy) \\ &\leq C_2 \lambda \int_M \frac{e^{-\sigma' t_i - \alpha'' \frac{\rho^2(x,y)}{t_i}}}{V(x, \sqrt{t_i})} \mathbb{1}_{B_i}(y) \mu(dy), \end{aligned}$$

where $\sigma' = \sigma - C_1 K > 0$ and $0 < \alpha'' < \alpha' < \alpha$. It is therefore sufficient to verify that

$$\left\| \sum_i \int_M \frac{e^{-\sigma' t_i - \alpha'' \frac{\rho^2(x,y)}{t_i}}}{V(\cdot, \sqrt{t_i})} \mathbb{1}_{B_i}(y) \mu(dy) \right\|_2 \lesssim \left\| \sum_i \mathbb{1}_{B_i} \right\|_2, \quad (3.5)$$

since from this and Lemma 3.3 we obtain as consequence, as desired,

$$\left\| \sum_i P_{t_i}^\sigma b_i \right\|_2^2 \lesssim \lambda^2 \left\| \sum_i \mathbb{1}_{B_i} \right\|_2^2 \lesssim \lambda^2 \sum_i \mu(B_i) \lesssim \lambda \|f\|_1.$$

In order to prove (3.5), we write by duality

$$\begin{aligned}
& \left\| \sum_i \int_M \frac{e^{-\sigma' t_i - \alpha'' \frac{\rho^2(x,y)}{t_i}}}{V(\cdot, \sqrt{t_i})} \mathbb{1}_{B_i}(y) \mu(dy) \right\|_2 \\
&= \sup_{\|u\|_2=1} \left| \int_M \left(\sum_i \int_M \frac{e^{-\sigma' t_i - \alpha'' \frac{\rho^2(x,y)}{t_i}}}{V(x, \sqrt{t_i})} \mathbb{1}_{B_i}(y) \mu(dy) \right) u(x) \mu(dx) \right| \\
&\leq \sup_{\|u\|_2=1} \int_M \sum_i \left(\int_M \frac{e^{-\sigma' t_i - \alpha'' \frac{\rho^2(x,y)}{t_i}}}{V(x, \sqrt{t_i})} |u(x)| \mu(dx) \right) \mathbb{1}_{B_i}(y) \mu(dy).
\end{aligned} \tag{3.6}$$

By (1.5), we have for any $x \in M$ and $y \in B_i$,

$$V(y, \sqrt{t_i}) \leq C \left(1 + \frac{\rho(x, y)}{\sqrt{t_i}} \right)^d e^{C(t_i^{1/2} + \rho(x, y))} V(x, \sqrt{t_i})$$

with $\delta \in [0, 2)$. From this, we obtain that there exist $0 < \tilde{\alpha} < \alpha''' < \alpha''$ such that

$$\begin{aligned}
& \int_M \frac{e^{-\sigma' t_i - \alpha'' \frac{\rho^2(x,y)}{t_i}}}{V(x, \sqrt{t_i})} |u(x)| \mu(dx) \\
&\lesssim \frac{e^{-\frac{1}{2}\sigma' t_i}}{V(y, \sqrt{t_i})} \int_M e^{-\alpha''' \frac{\rho^2(x,y)}{t_i}} |u(x)| \mu(dx) \\
&\lesssim \frac{1}{V(y, \sqrt{t_i})} \left(\int_{\rho(x,y) < \sqrt{t_i}} |u(x)| \mu(dx) + \sum_{k=0}^{\infty} \int_{2^k \sqrt{t_i} \leq \rho(x,y) < 2^{k+1} \sqrt{t_i}} e^{-\alpha''' \frac{\rho^2(x,y)}{t_i}} |u(x)| \mu(dx) \right) \\
&\leq \frac{1}{V(y, \sqrt{t_i})} \left(\int_{B(y, \sqrt{t_i})} |u(x)| \mu(dx) + \sum_{k=0}^{\infty} e^{-\alpha''' 2^{2k}} \int_{B(y, 2^{k+1} \sqrt{t_i})} |u(x)| \mu(dx) \right) \\
&= \left(1 + \sum_{k=0}^{\infty} \frac{V(y, 2^{k+1} \sqrt{t_i})}{V(y, \sqrt{t_i})} e^{-\alpha''' 2^{2k}} \right) (\mathcal{M}u)(y) \\
&\leq \left(1 + C \sum_{k=0}^{\infty} 2^{(k+1)d} e^{C 2^{k+1} \sqrt{t_i}} e^{-\tilde{\alpha} 2^{2k}} \right) (\mathcal{M}u)(y) \\
&\leq \left(1 + C \sum_{k=0}^{\infty} 2^{(k+1)d} e^{C 2^{k+1}} e^{-\tilde{\alpha} 2^{2k}} \right) (\mathcal{M}u)(y) \lesssim (\mathcal{M}u)(y),
\end{aligned}$$

where

$$(\mathcal{M}u)(y) := \sup_{r>0} \frac{1}{V(y, r)} \int_{B(y, r)} |u(x)| \mu(dx)$$

denotes the Hardy-Littlewood maximal function of u . This together with (3.6) and the L^2 -boundedness of \mathcal{M} gives

$$\left\| \sum_i \int_M \frac{e^{-\alpha'' \frac{\rho^2(x,y)}{t_i}}}{V(\cdot, \sqrt{t_i})} \mathbb{1}_{B_i}(y) \mu(dy) \right\|_2 \lesssim \sup_{\|u\|_2=1} \int_M (\mathcal{M}u)(y) \sum_i \mathbb{1}_{B_i}(y) \mu(dy) \lesssim \left\| \sum_i \mathbb{1}_{B_i} \right\|_2,$$

which shows that (3.5) holds true and finishes the proof of Lemma 3.4. \square

With the help of Lemmas 3.2 through 3.4, we are now in position to the proof of Theorem 1.4.

Proof of Theorem 1.4. Recall that $T = \text{Hess}(\Delta + \sigma)^{-1}$. We choose σ big enough such that $\sigma > \max\{C'', C_1 K\}$, where $\tilde{\alpha}$ is as in proof of Lemma 3.4. By Lemma 3.2, it suffices to prove

$$\mu(\{x \in 2B : |Tf(x)| > \lambda\}) \lesssim \frac{\|f\|_1}{\lambda} \quad (3.7)$$

for all $f \in C_0^\infty(B)$. By means of Lemma 3.3 with $\mathcal{X} = B$, we deduce that f has a decomposition

$$f = g + b = g + \sum_i b_i$$

which implies

$$\begin{aligned} \mu(\{x \in 2B : |Tf(x)| > \lambda\}) &\leq \mu\left(\left\{x \in 2B : |Tg(x)| > \frac{\lambda}{2}\right\}\right) + \mu\left(\left\{x \in 2B : |Tb(x)| > \frac{\lambda}{2}\right\}\right) \\ &=: I_1 + I_2. \end{aligned} \quad (3.8)$$

Using the facts that T is bounded on $L^2(M)$ and that $|g(x)| \leq C\lambda$, we obtain as desired

$$I_1 \lesssim \lambda^{-2} \|Tg\|_2^2 \lesssim \lambda^{-2} \|g\|_2^2 \lesssim \lambda^{-1} \|g\|_1 \lesssim \lambda^{-1} \|f\|_1. \quad (3.9)$$

We now turn to the estimate of I_2 . Recall that $\{P_t^\sigma\}_{t \geq 0}$ is the heat semigroup generated by $-(\Delta + \sigma)$, that is $P_t^\sigma = e^{-t\sigma} P_t$. We write

$$Tb_i = TP_{t_i}^\sigma b_i + T(I - P_{t_i}^\sigma)b_i,$$

where $t_i = r_i^2$ with r_i the radius of B_i . By Lemma 3.4, we have

$$\left\| \sum_i P_{t_i}^\sigma b_i \right\|_2^2 \lesssim \lambda \|f\|_1.$$

This combined with the L^2 -boundedness of T yields

$$\mu\left(\left\{x \in 2B : \left|T\left(\sum_i P_{t_i}^\sigma b_i\right)(x)\right| > \frac{\lambda}{2}\right\}\right) \lesssim \frac{1}{\lambda} \|f\|_1 \quad (3.10)$$

as desired. Consider now the term $T \sum_i (I - P_{t_i}^\sigma)b_i$. We write

$$\begin{aligned} &\mu\left(\left\{x \in 2B : \left|T\left(\sum_i (I - P_{t_i}^\sigma)b_i\right)(x)\right| > \frac{\lambda}{2}\right\}\right) \\ &\leq \sum_i \mu(2B_i) + \mu\left(\left\{x \in 2B \setminus \cup_i 2B_i : \left|T\left(\sum_i (I - P_{t_i}^\sigma)b_i\right)(x)\right| > \frac{\lambda}{2}\right\}\right). \end{aligned} \quad (3.11)$$

From Lemma 3.3, it follows that

$$\sum_i \mu(2B_i) \lesssim \frac{\|f\|_1}{\lambda} \quad (3.12)$$

as desired. To estimate the second term, let $k_{t_i}^\sigma(x, y)$ denote the integral kernel of the operator $T(I - P_{t_i}^\sigma)$. Note that

$$(\Delta + \sigma)^{-1}(I - P_{t_i}^\sigma) = \int_0^{+\infty} (P_s^\sigma - P_{t_i+s}^\sigma) ds = \int_0^{t_i} P_s^\sigma ds$$

and

$$T(I - P_{t_i}^\sigma) = \text{Hess}(\Delta + \sigma)^{-1}(I - P_{t_i}^\sigma) = \int_0^{t_i} \text{Hess} P_s^\sigma ds.$$

Therefore,

$$k_{t_i}^\sigma(x, y) = \int_0^{t_i} \text{Hess}_x p_s^\sigma(x, y) ds, \quad (3.13)$$

where $p_s^\sigma = e^{-\sigma s} p_s$ is the kernel of P_s^σ with respect to μ . Since b_i is supported in B_i , we have

$$\begin{aligned} \int_{2B \setminus (2B_i)} \left| T((I - P_{t_i}^\sigma)b_i)(x) \right| \mu(dx) &\leq \int_{2B \setminus (2B_i)} \left(\int_{B_i} |k_{t_i}^\sigma(x, y)| |b_i(y)| \mu(dy) \right) \mu(dx) \\ &\leq \int_{B_i} \left(\int_{\rho(x, y) \geq t_i^{1/2}} |k_{t_i}^\sigma(x, y)| \mu(dx) \right) |b_i(y)| \mu(dy). \end{aligned} \quad (3.14)$$

Now by means of (3.13) and Corollary 2.5, we get

$$\begin{aligned} \int_{\rho(x, y) \geq t_i^{1/2}} |k_{t_i}^\sigma(x, y)| \mu(dx) &\leq \int_0^{t_i} \left(\int_{\rho(x, y) \geq t_i^{1/2}} |\text{Hess}_x p_s(x, y)| \mu(dx) \right) e^{-s\sigma} ds \\ &\leq C \int_0^{t_i} e^{-\beta t_i/s} s^{-1} e^{C''s} e^{-s\sigma} (1 + \sqrt{s}) ds \\ &\leq C \int_0^1 \frac{e^{-\beta/u}}{u} du < \infty, \end{aligned}$$

where for the last inequality we use the fact that

$$e^{s(C'' - \sigma)}(1 + \sqrt{s}) < \infty, \quad s \in (0, \infty).$$

The estimate above together with (3.14) and Lemma 3.3 implies that

$$\mu \left(\left\{ x \in 2B \setminus \cup_i 2B_i : \left| T \left(\sum_i (I - P_{t_i}^\sigma) b_i \right) (x) \right| > \frac{\lambda}{2} \right\} \right) \lesssim \frac{\|f\|_1}{\lambda}. \quad (3.15)$$

Altogether, combining (3.8) through (3.10), (3.12) and (3.15), we conclude that (3.7) holds which completes the proof of Theorem 1.4. \square

3.2 The case $p \in (2, \infty)$

Let M be a complete Riemannian manifold satisfying **(H)**. In this subsection, we prove Theorem 1.6 and show that the Calderón-Zygmund inequality **CZ**(p) holds for all $p \in (2, \infty)$, that is,

$$\|Tf\|_p \lesssim \|f\|_p \quad (3.16)$$

holds for any $f \in L^p(M)$ with T as in (3.1) and $\sigma \in (2K + \theta, \infty)$ where K, θ are respectively in **(Ric)** and (2.9).

To this end, let w be a C^∞ function on $[0, \infty)$ satisfying $0 \leq w \leq 1$ and

$$w(t) = \begin{cases} 1 & \text{on } [0, 3/4], \\ 0 & \text{on } [1, \infty), \end{cases}$$

and let \tilde{T} be an operator defined by

$$\tilde{T}f := \int_0^\infty v(t) \text{Hess } P_t f dt \quad (3.17)$$

with $v(t) := w(t)e^{-\sigma t}$. We need the following lemma, which reduces (3.16) to a time and spatial localized version.

Lemma 3.5. *Assume that **(H)** holds. Let $p \in (2, \infty)$ and $\{x_j\}_{j \in \Lambda}$ be a countable subset of M having the finite overlap property as in Lemma 3.1. If there exists a positive constant C such that*

$$\| |\tilde{T}(f)| \|_{L^p(B(x_j, 4))} \leq C \|f\|_{L^p(B(x_j, 1))} \quad (3.18)$$

holds for any $j \in \Lambda$ and $f \in C_0^\infty(B(x_j, 1))$ with \tilde{T} defined as in (3.17), then (3.16) holds.

Proof. By the fact that $w \equiv 1$ on $[0, 3/4]$, we obtain from Proposition 2.7 that if $\sigma > 2K + \theta$, then for any $g \in L^p(M)$,

$$\left\| \int_0^\infty (1 - w(t)) e^{-\sigma t} |\text{Hess } P_t g| dt \right\|_p \lesssim \int_{3/4}^\infty e^{(2K + \theta - \sigma)t} \frac{1 + \sqrt{t}}{t} dt \|g\|_p \lesssim \|g\|_p.$$

This and (3.17) imply that to prove (3.16), it suffices to show that

$$\| |\tilde{T}(g)| \|_p \lesssim \|g\|_p \quad (3.19)$$

for any $g \in C_0^\infty(M)$.

Let $\{x_j\}_{j \in \Lambda}$ be a countable subset of M having the finite overlap property as in Lemma 3.1. Let $\{\varphi_j\}$ be a corresponding C^∞ partition of unity such that $0 \leq \varphi_j \leq 1$ and φ_j is supported in $B_j := B(x_j, 1)$. Let χ_j be the characteristic function of the ball $4B_j$. For any $g \in C_0^\infty(M)$ and $x \in M$, write

$$\tilde{T}g(x) \leq \sum_{j \in \Lambda} \chi_j \tilde{T}(g\varphi_j)(x) + \sum_{j \in \Lambda} (1 - \chi_j) \tilde{T}(g\varphi_j)(x) =: \text{I}(x) + \text{II}(x). \quad (3.20)$$

By Lemma 3.1, we know

$$\sum_{j \in \Lambda} |(1 - \chi_j)(x)\varphi_j(y)| \leq N_0 \mathbb{1}_{\{\rho(x,y) \geq 3\}}.$$

Hence, by Hölder's inequality, we have

$$\text{II}(x) \leq \int_0^1 \int_M |\text{Hess}_x p_t(x, y)| \left(\sum_{j \in \Lambda} |(1 - \chi_j)(x)\varphi_j(y)| \right) |g(y)| \mu(dy) dt$$

$$\begin{aligned}
&\leq N_0 \int_0^1 \int_{\rho(x,y) \geq 3} |\text{Hess}_x p_t(x,y)| |g(y)| \mu(dy) dt \\
&\lesssim \int_0^1 \left(\int_M |t \text{Hess}_x p_t(x,y)|^p e^{\gamma \rho^2(x,y)/t} (V(y, \sqrt{t}))^{p/p'} |g(y)|^p \mu(dy) \right)^{1/p} \frac{e^{-c/t}}{t} dt,
\end{aligned}$$

where $c = \gamma p'/p$ and γ is a positive constant small enough so that in view of Proposition 2.10 and an argument similar to the proof of (2.3), it implies that for any $t \in (0, 1)$,

$$\int_M |t \text{Hess}_x p_t(x,y)|^p e^{\gamma \rho^2(x,y)/t} \mu(dx) \lesssim \frac{1}{(V(y, \sqrt{t}))^{p-1}}.$$

This immediately implies

$$\begin{aligned}
&\int_M |\text{II}(x)|^p \mu(dx) \tag{3.21} \\
&\lesssim \int_M \left[\int_0^1 \left(\int_M |t \text{Hess}_x p_t(x,y)|^p e^{\gamma \rho^2(x,y)/t} V(y, \sqrt{t})^{p/p'} |f(y)|^p \mu(dy) \right)^{1/p} \frac{e^{-c/t}}{t} dt \right]^p \mu(dx) \\
&\lesssim \int_M \int_0^1 \left(\int_M |t \text{Hess}_x p_t(x,y)|^p e^{\gamma \rho^2(x,y)/t} V(y, \sqrt{t})^{p/p'} |f(y)|^p \mu(dy) \right) dt \mu(dx) \\
&\quad \times \left(\int_0^1 \frac{e^{-c p'/t}}{t^{p'}} dt \right)^{p/p'} \\
&\lesssim \int_0^1 \left[\int_M [V(y, \sqrt{t})]^{p/p'} |f(y)|^p \left(\int_M |t \text{Hess}_x p_t(x,y)|^p e^{\gamma \rho^2(x,y)/t} \mu(dx) \right) \mu(dy) \right] dt \\
&\lesssim \int_M |f(y)|^p \mu(dy)
\end{aligned}$$

as desired.

Now we estimate $\text{I}(x)$. Using Lemma 3.1, we know that the balls $\{4B_j\}_{j \in \Lambda}$ are of uniform overlap and hence

$$\sum_j \|h \chi_j\|_{p'}^{p'} \lesssim \|h\|_{p'}^{p'},$$

for all $h \in C_0^\infty(M)$. Since $g\varphi_j \in C_0^\infty(B(x_j, 1))$ and using (3.18), we conclude that

$$\begin{aligned}
\left| \int_M h(x) \text{I}(x) \mu(dx) \right| &\leq \int_M |h(x)| \left| \sum_j \chi_j \tilde{T}(g\varphi_j)(x) \right| \mu(dx) \\
&\lesssim \sum_j \|g\varphi_j\|_p \|h \chi_j\|_{p'} \lesssim \|g\|_p \|h\|_{p'},
\end{aligned}$$

which together with (3.20) and (3.21) implies (3.16), and hence finishes the proof of Lemma 3.5. \square

From Lemma 3.5, it is easy to see that to prove T is of strong type (p, p) for $p > 2$, it suffices to show that \tilde{T} is bounded from $L^p(B_j, \mu)$ to $L^p(4B_j, \mu)$ for each $j \in \Lambda$ as in (3.18). To this end, we need an L^p local bounded criterion from [3] via maximal functions. Recall that the *local maximal function* by

$$(\mathcal{M}_{\text{loc}} f)(x) := \sup_{\substack{B \ni x \\ r(B) \leq 32}} \frac{1}{\mu(B)} \int_B |f| d\mu, \quad x \in M, \tag{3.22}$$

for any locally integrable function f on M . From **(LD)**, it follows that \mathcal{M}_{loc} is bounded on $L^p(M)$ for all $1 < p \leq \infty$. For a measurable subset $E \subset M$, the *maximal function relative to E* is defined by

$$(\mathcal{M}_E f)(x) := \sup_{B \text{ ball in } M, B \ni x} \frac{1}{\mu(B \cap E)} \int_{B \cap E} |f| d\mu, \quad x \in E, \quad (3.23)$$

for any locally integrable function f on M . If in particular E is a ball with radius r , it is enough to consider balls B with radii not exceeding $2r$. It is also easy to see \mathcal{M}_E is of weak type $(1, 1)$ and $L^p(M)$ -bounded for $1 < p \leq \infty$ if E satisfies the *relative doubling property*, namely, if there exists a constant C_E such that for all $x \in E$ and $r > 0$,

$$\mu(B(x, 2r) \cap E) \leq C_E \mu(B(x, r) \cap E). \quad (3.24)$$

Note that in Lemma 3.1 (iv), for any $j \in \Lambda$, the subsets $4B_j$ satisfy the relative doubling property (3.24) with a constant independent of j .

The following theorem is essential to the proof of Theorem 1.6.

Lemma 3.6. *Let $p \in (2, \infty)$ and assume that **(LD)** holds. For any ball B of M centered in $4B_j$ with radius less than 8 , assume that*

- (i) *there exists an integer n depending only on condition **(LD)** such that the map $f \rightarrow \mathcal{M}_{4B_j, \tilde{T}, n}^\# f$ is bounded from $L^p(B_j, \mu)$ to $L^p(4B_j, \mu)$ with operator norm independent of j , where*

$$\mathcal{M}_{4B_j, \tilde{T}, n}^\# f(x) := \sup_{B \text{ ball in } M, B \ni x} \left(\frac{1}{\mu(B \cap 4B_j)} \int_{B \cap 4B_j} |\tilde{T}(I - P_{r^2})^n f(y)|^2 \mu(dy) \right)^{1/2};$$

- (ii) *for all $k \in \{1, 2, \dots, n\}$, and all $f \in L^2(M, \mu)$ supported in B_j , there exists a sublinear operator S_j bounded from $L^p(B_j, \mu)$ to $L^p(4B_j, \mu)$ with operator norm independent of j such that for $x \in B \cap 4B_j$,*

$$\left(\frac{1}{\mu(B \cap 4B_j)} \int_{B \cap 4B_j} |\tilde{T} P_{kr^2} f|^p d\mu \right)^{1/p} \leq C \left(\mathcal{M}_{4B_j}(|\tilde{T} f|^2) + (S_j f)^2 \right)^{1/2}(x) \quad (3.25)$$

where \mathcal{M}_{4B_j} is as in (3.23).

Then \tilde{T} is bounded from $L^p(B_j, \mu)$ to $L^p(4B_j, \mu)$, that is (3.18) holds with a constant depending on p , the doubling constant C_D in **(D)**, the operator norm of \tilde{T} on $L^2(M, \mu)$, the operator norms of $\mathcal{M}_{4B_j, \tilde{T}, n}^\#$ and S_j on L^p , and the constant in (3.25).

Proof. We use the following result [3, Theorem 2.4]:

Let $p_0 \in (2, \infty]$ and (M, μ, ρ) be a measured metric space. Suppose that T is a bounded sublinear operator which is bounded on $L^2(M, \mu)$, and let $\{A_r\}_{r>0}$ be a family of linear operators acting on $L^2(M, \mu)$. Let E_1 and E_2 be two subsets of M such that E_2 has the relative doubling property, $\mu(E_2) < \infty$ and $E_1 \subset E_2$. Assume that

- (i) *the sharp maximal functional $\mathcal{M}_{E_2, T, A}^\#$ is bounded from $L^p(E_1, \mu)$ into $L^p(E_2, \mu)$ for all $p \in (2, p_0)$, where*

$$(\mathcal{M}_{E_2, T, A}^\# f)^2(x) = \sup_{B \text{ ball in } M, B \ni x} \frac{1}{\mu(B \cap E_2)} \int_{B \cap E_2} |T(I - A_{r(B)})f|^2 d\mu$$

for $x \in E_2$;

(ii) for some sublinear operator S bounded from $L^p(E_1, \mu)$ into $L^p(E_2, \mu)$ for all $p \in (2, p_0)$,

$$\left(\frac{1}{\mu(B \cap E_2)} \int_{B \cap E_2} |TA_{r(B)}f|^{p_0} d\mu \right)^{1/p_0} \leq C \left(\mathcal{M}_{E_2}(|Tf|^2) + (S(f)^2)^{1/2} \right)(x), \quad (3.26)$$

for all $f \in L^2(M, \mu)$ supported in E_1 , all balls B in M and all $x \in B \cap E_2$, where $r(B)$ is the radius of B .

If $2 < p < p_0$ and $Tf \in L^p(E_2, \mu)$ whenever $f \in L^p(E_1, \mu)$, then T is bounded from $L^p(E_1, \mu)$ into $L^p(E_2, \mu)$ and its operator norm is bounded by a constant depending only on the operator norm of T on $L^2(M, \mu)$, C_{E_2} (see in (3.24) for E_2), p and p_0 , the operator norms of $\mathcal{M}_{4B_j, \bar{T}, n}^\#$ and S on L^p , and the constant in (3.26).

Here we may take E_1 and E_2 as B_j and $4B_j$ respectively, as the sets B_j and $4B_j$ have the relatively volume doubling property as in (3.24) with the constant C_E independent of j (see Lemma 3.1). Moreover, taking the operators $\{A_r\}_{r>0}$ such that

$$I - A_r = (I - P_{r^2})^n$$

for some integer n sufficiently large, then the result follows directly. \square

Now it suffices to check (i) and (ii) of Lemma 3.6. We establish two technical lemmas which verify (i) and (ii) respectively. To this end, observe that **(LD)** implies: for all $r_0 > 0$ there exists C_{r_0} such that for all $x \in M$, $r \in (0, r_0)$,

$$V(x, 2r) \leq C_{r_0} V(x, r).$$

An easy consequence of the definition is that for all $y \in M$, $0 < r < 8$ and $s \geq 1$ satisfying $sr < 32$,

$$V(y, sr) \leq C s^{D_L} V(y, r), \quad (3.27)$$

for some constants C and $D_L > 0$. The following lemma plays an important role, when checking (i) of Lemma 3.6.

Lemma 3.7. *Assume that **(H)** hold. Then there exists an integer n such that the inequality*

$$\sup_{B \text{ ball in } M, B \ni x} \left(\frac{1}{\mu(B \cap 4B_j)} \int_{B \cap 4B_j} |\bar{T}(I - P_{r^2})^n f(y)|^2 \mu(dy) \right)^{1/2} \leq C \left(\mathcal{M}_{\text{loc}}(|f|^2)(x) \right)^{1/2} \quad (3.28)$$

holds for any $x \in 4B_j$, $f \in L^2(4B_j)$ satisfying $\text{supp } f \subset 4B_j$, where \mathcal{M}_{loc} is defined by (3.22).

Proof. Viewing the left-hand side of (3.28) as the maximal function relative to $4B_j$, since the radius of $4B_j$ is 4, it is enough to consider balls B of radii not exceeding 8. Let $B = B(x_0, r)$ be an arbitrary ball containing x and satisfying $x_0 \in 4B_j$ and $r \in (0, 8)$. By Lemma 3.1, we know that

$$\mu(B) \lesssim \mu(B \cap 4B_j) \quad (3.29)$$

with an implicit constant independent of B and j . Hence, it is easy to see

$$\frac{1}{\mu(B \cap 4B_j)} \int_{B \cap 4B_j} |\bar{T}(I - P_{r^2})^n f|^2 d\mu \lesssim \frac{1}{\mu(B)} \int_B |\bar{T}(I - P_{r^2})^n f|^2 d\mu.$$

Thus, we only need to show that

$$\sup_{B \text{ ball in } M, B \ni x} \left(\frac{1}{\mu(B)} \int_B |\widetilde{T}(I - P_{r^2})^n f(y)|^2 \mu(dy) \right)^{1/2} \lesssim (\mathcal{M}_{\text{loc}}(|f|^2)(x))^{1/2}, \quad (3.30)$$

for any $x \in 4B_j$. Since $r < 8$, we choose $i_r \in \mathbb{Z}_+$ satisfying

$$2^{i_r} r \leq 8 < 2^{i_r+1} r. \quad (3.31)$$

Denote by

$$C_i \text{ the annulus } 2^{i+1} B \setminus (2^i B) \text{ if } i \geq 2 \text{ and } C_1 = 4B. \quad (3.32)$$

Using the fact $\text{supp } f \subset 4B_j \subset 2^i B$ when $i > i_r$, we find that

$$f = \sum_{i=1}^{i_r} f \mathbb{1}_{C_i} =: \sum_{i=1}^{i_r} f_i$$

which then implies

$$\|\widetilde{T}(I - P_{r^2})^n f\|_{L^2(B)} \leq \sum_{i=1}^{i_r} \|\widetilde{T}(I - P_{r^2})^n f_i\|_{L^2(B)}. \quad (3.33)$$

For $i = 1$ we use the L^2 -boundedness of $\widetilde{T}(I - P_{r^2})^n$ to obtain

$$\|\widetilde{T}(I - P_{r^2})^n f_1\|_{L^2(B)} \leq \|f\|_{L^2(4B)} \leq \mu(4B)^{1/2} (\mathcal{M}_{\text{loc}}(|f|^2)(x))^{1/2} \quad (3.34)$$

as desired. For $i \geq 2$, we infer from (3.17) that

$$\begin{aligned} \widetilde{T}(I - P_{r^2})^n f_i &= \int_0^\infty v(t) \text{Hess}(P_t(I - P_{r^2})^n f_i) dt \\ &= \int_0^\infty v(t) \sum_{k=0}^n \binom{n}{k} (-1)^k \text{Hess } P_{t+kr^2}(f_i) dt \\ &= \int_0^\infty \left(\sum_{k=0}^n \binom{n}{k} (-1)^k \mathbf{1}_{\{t > kr^2\}} v(t - kr^2) \right) \text{Hess } P_t(f_i) dt \\ &=: \int_0^\infty g_r(t) \text{Hess } P_t f_i dt. \end{aligned}$$

For g_r , according to the definition $v(t) = w(t)e^{-\sigma t}$ and an elementary calculation, we observe that

$$\begin{cases} |g_r(t)| \lesssim 1 & \text{for } 0 < t \leq (1 + nr^2) \wedge (1 + n)r^2, \\ |g_r(t)| \lesssim r^{2n} & \text{for } (1 + nr^2) \wedge (1 + n)r^2 < t \leq 1 + nr^2, \\ g_r(t) = 0 & \text{for } t > 1 + nr^2, \end{cases}$$

where the second estimate comes from $w^{(i)}(t) \lesssim 1$ for $t \in [0, 1]$ and $i \in \{1, \dots, n\}$, along with

$$\left| \sum_{k=0}^n \binom{n}{k} (-1)^k w(t - kr^2) \right| \leq C_n \sup_{u \geq \frac{t}{n+1}} |(w(u) e^{-\sigma u})^{(n)}| r^{2n} \lesssim r^{2n}$$

for all $t \geq (1+n)r^2$ and some constant C_n . Combined with Lemma 2.3, this gives

$$\left\| |\widetilde{T}(I - P_{r^2})^n f_i| \right\|_{L^2(B)} \lesssim \left(\int_0^\infty |g_r(t)| (1 + \sqrt{t}) e^{-\alpha' 4^i r^2 / t} \frac{dt}{t} \right) \|f_i\|_{L^2(C_i)}$$

where using the fact that $0 > r < 8$, we know

$$\begin{aligned} & \int_0^\infty (1 + \sqrt{t}) |g_r(t)| e^{-\frac{\alpha' 4^i r^2}{t}} \frac{dt}{t} \\ & \lesssim \int_0^{(1+nr^2) \wedge (1+n)r^2} (1 + \sqrt{t}) e^{-\frac{\alpha' 4^i r^2}{t}} \frac{dt}{t} + \int_{(1+nr^2) \wedge (1+n)r^2}^{1+nr^2} (1 + \sqrt{t}) r^{2n} e^{-\frac{\alpha' 4^i r^2}{t}} \frac{dt}{t} \\ & \leq C_n \left(\int_0^{(n+1)r^2} e^{-\frac{\alpha' 4^i r^2}{t}} \frac{dt}{t} + \int_{(1+nr^2) \wedge (1+n)r^2}^{1+nr^2} (1 + \sqrt{t}) r^{2n} \frac{t^{n-1}}{4^{in} r^{2n}} dt \right) \\ & \leq C_n \left(4^{-in} + 4^{-in} r^{2n} (1 + \sqrt{r}) \right) \leq C'_n 4^{-in}. \end{aligned}$$

Now an easy consequence of the local doubling (3.27), since $r(2^i B) \leq 8$ when $1 \leq i \leq i_r$, is that

$$\mu(2^{i+1} B) \leq C 2^{(i+1)D_L} \mu(B),$$

with constants C and D_L independent of B and i . Therefore, as $C_i \subset 2^{i+1} B$,

$$\|f\|_{L^2(C_i)} \leq \mu(2^{i+1} B)^{1/2} (\mathcal{M}_{\text{loc}}(|f|^2)(x))^{1/2} \leq C 2^{iD_L/2} \mu(B)^{1/2} (\mathcal{M}_{\text{loc}}(|f|^2)(x))^{1/2}.$$

Using the definition of i_r , $r \leq 8$, and by choosing $2n > D_L/2$, we finally obtain

$$\left\| |\widetilde{T}(I - P_{r^2})^n f| \right\|_{L^2(B)} \leq C' \left(\sum_{i=1}^{i_r} 2^{i(D_L/2 - 2n)} \right) \mu(B)^{1/2} (\mathcal{M}_{\text{loc}}(|f|^2)(x))^{1/2},$$

which proves Lemma 2.7. \square

The following lemma is essential to the proof of (ii) of Lemma 3.6.

Lemma 3.8. *Let $p \in (2, \infty)$. Assume that **(H)** hold. For a ball B with radius $r \in (0, 8)$, let i_r be the integer such that (3.31) holds. Then the following estimate hold: For any $i \leq i_r$ and any C^2 -function f supported in C_i as in (3.32), and every $k \in \{1, \dots, n\}$, where $n \in \mathbb{N}$ is chosen according to Lemma 3.7, one has*

$$\begin{aligned} & \left(\frac{1}{\mu(B)} \int_B |\text{Hess } P_{kr^2} f|^p d\mu \right)^{1/p} \\ & \leq C e^{-\alpha_1 4^i} \left[\left(\frac{1}{\mu(2^{i+1} B)} \int_{C_i} |\nabla f|^2 d\mu \right)^{1/2} + \left(\frac{1}{\mu(2^{i+1} B)} \int_{C_i} |\text{Hess } f|^2 d\mu \right)^{1/2} \right] \end{aligned} \quad (3.35)$$

for some positive constants C and α_1 depending on K, θ, p and D_L as in (3.27).

Proof. We first observe that Lemma 2.8 yields

$$\left(\int_B |\text{Hess } P_t f|^p d\mu \right)^{1/p} \leq e^{2Kt} \left(\int_B (P_t |\text{Hess } f|^2)^{p/2}(x) \mu(dx) \right)^{1/p}$$

$$+ Ce^{(2K+\theta)t} \left(\int_B (P_t |df|^2)^{p/2}(x) \mu(dx) \right)^{1/p}. \quad (3.36)$$

We substitute $t = kr^2$ in (3.36). By Lemma 2.1, one has the upper bound of $p_t(x, y)$,

$$p_t(x, y) \leq \frac{C_\alpha}{V(y, \sqrt{t})} \exp\left(-\alpha \frac{\rho^2(x, y)}{t} + C_1 Kt\right)$$

for all $x, y \in M$. Since $r \leq 8$, it follows from the above estimate that for all $x \in B$,

$$\begin{aligned} |P_{kr^2}(|df|^2)(x)| &\leq C \int_{C_i} V(y, \sqrt{kr})^{-1} \exp\left(-\alpha \frac{\rho^2(x, y)}{kr^2} + C_1 Kkr^2\right) |df|^2(y) \mu(dy) \\ &\leq Ce^{-\alpha 4^i/k} \int_{C_i} V(y, \sqrt{kr})^{-1} |df|^2(y) \mu(dy). \end{aligned}$$

Moreover, since $y \in C_i$, we have $2^{i+1}B \subset B(y, 2^{i+2}r)$, and since $2^{i+2}r \leq 2^{i+2}r < 32$, we know by (3.27) that

$$\frac{1}{V(y, \sqrt{kr})} \leq \frac{C2^{(i+2)D_L}}{V(y, 2^{i+2}r)} \leq \frac{C2^{(i+2)D_L}}{\mu(2^{i+1}B)}.$$

It then follows that

$$|P_{kr^2}|df|^2(x)| \leq Ce^{-c4^i} 2^{(i+2)D_L} \left(\frac{1}{\mu(2^{i+1}B)} \int_{C_i} |\nabla f|^2 d\mu \right)$$

for all $x \in B$, and there exists $\alpha_1 < c$ such that

$$\left(\frac{1}{\mu(B)} \int_B (P_{kr^2} |\nabla f|^2)^{p/2} d\mu \right)^{1/p} \leq Ce^{-\alpha_1 4^i} \left(\frac{1}{\mu(2^{i+1}B)} \int_{C_i} |\nabla f|^2 d\mu \right)^{1/2}.$$

With similar arguments, we obtain

$$\left(\frac{1}{\mu(B)} \int_B (P_{kr^2} |\text{Hess } f|^2)^{p/2} d\mu \right)^{1/p} \leq Ce^{-\alpha_1 4^i} \left(\frac{1}{\mu(2^{i+1}B)} \int_{C_i} |\text{Hess } f|^2 d\mu \right)^{1/2}.$$

Altogether these yields

$$\begin{aligned} &\left(\frac{1}{\mu(B)} \int_B |\text{Hess } P_{kr^2} f|^p d\mu \right)^{1/p} \\ &\leq Ce^{-\alpha_1 4^i} \left(\left(\frac{1}{\mu(2^{i+1}B)} \int_{C_i} |\nabla f|^2 d\mu \right)^{1/2} + \left(\frac{1}{\mu(2^{i+1}B)} \int_{C_i} |\text{Hess } f|^2 d\mu \right)^{1/2} \right), \end{aligned}$$

which completes the proof. \square

With the help of Lemma 3.6 through Lemmas 3.7 and 3.8, we are now in a position to the proof of Theorem 1.6.

Proof of Theorem 1.6. By Lemma 3.6, we only need to show (i) and (ii) of Lemma 3.6 hold true under our condition **(H)**. We first verify (i) of Lemma 3.6. Observe from Lemma 3.7 that there is an integer n depending only on D_L as in (3.27) such that for all $f \in L^p(B_j)$ and $x \in 4B_j$,

$$\sup_{B \text{ ball in } M, B \ni x} \frac{1}{\mu(B \cap 4B_j)} \int_{B \cap 4B_j} |\tilde{T}(I - P_{r^2})^n f|^2 d\mu \leq C \mathcal{M}_{\text{loc}}(|f|^2)(x).$$

Recall that \mathcal{M}_{loc} is bounded on $L^p(M)$ for $1 < p \leq \infty$; thus for all $p > 2$, the operator $\mathcal{M}_{4B_j, \tilde{T}, n}^\#$ is bounded from $L^p(B_j, \mu)$ to $L^p(4B_j, \mu)$ uniformly in j , i.e. assertion (i) is proved.

Next, we prove (ii) of Lemma 3.6. Assume that $f \in L^2(B_j)$ and let $h = \int_0^\infty v(t)P_t f dt$ with v defined as in (3.17). Since $\tilde{T}f = \text{Hess } h$ and the inequality (3.29) for $B \cap 4B_j$, we have

$$\begin{aligned} \left(\frac{1}{\mu(B \cap 4B_j)} \int_{B \cap 4B_j} |\tilde{T}P_{kr^2}f|^p d\mu \right)^{1/p} &= \left(\frac{1}{\mu(B \cap 4B_j)} \int_{B \cap 4B_j} |\text{Hess } P_{kr^2}h|^p d\mu \right)^{1/p} \\ &\lesssim \left(\frac{1}{\mu(B)} \int_B |\text{Hess } P_{kr^2}h|^p d\mu \right)^{1/p}. \end{aligned}$$

We write

$$\text{Hess } P_{kr^2}h = \sum_{i=1}^{\infty} \text{Hess } P_{kr^2}(h \mathbb{1}_{C_i}) = \sum_{i=1}^{\infty} \text{Hess } P_{kr^2}g_i,$$

where $g_i = h \mathbb{1}_{C_i}$ and C_i is as in (3.32). Next, we distinguish the two regimes $i \leq i_r$ and $i > i_r$ where i_r is defined as in (3.31). In the regime $i \leq i_r$, by Lemma 3.8, we have

$$\begin{aligned} &\left(\frac{1}{\mu(B)} \int_B |\text{Hess } P_{kr^2}g_i|^p d\mu \right)^{1/p} \\ &\leq C e^{-\alpha_1 4^i} \left(\left(\frac{1}{\mu(2^{i+1}B)} \int_{C_i} |\nabla h|^2 d\mu \right)^{1/2} + \left(\frac{1}{\mu(2^{i+1}B)} \int_{C_i} |\text{Hess } h|^2 d\mu \right)^{1/2} \right) \\ &\leq C e^{-\alpha_1 4^i} \left(\left(\mathcal{M}_{\text{loc}}(|\nabla h|^2)(x) \right)^{1/2} + \left(\frac{1}{\mu(2^{i+1}B)} \int_{C_i} |\text{Hess } h|^2 d\mu \right)^{1/2} \right). \end{aligned}$$

On the other hand, since $C_i \subset 2^{i+1}B$, we know

$$\begin{aligned} &\frac{1}{\mu(2^{i+1}B)} \int_{C_i} |\text{Hess } h|^2 d\mu \\ &\leq \frac{1}{\mu(2^{i+1}B \cap 4B_j)} \int_{(2^{i+1}B) \cap 4B_j} |\text{Hess } h|^2 d\mu + \frac{1}{\mu(2^{i+1}B)} \int_{2^{i+1}B} \mathbb{1}_{M \setminus 4B_j} |\text{Hess } h|^2 d\mu \\ &\leq \mathcal{M}_{4B_j}(|\text{Hess } h|^2)(x) + \mathcal{M}_{\text{loc}}(|\text{Hess } h|^2 \mathbb{1}_{M \setminus 4B_j})(x), \end{aligned}$$

for any $x \in B \cap 4B_j$. Hence in this case

$$\begin{aligned} &\left(\frac{1}{\mu(B)} \int_B \left| \sum_{i=1}^{i_r} \text{Hess } P_{kr^2}g_i \right|^p d\mu \right)^{1/p} \\ &\leq \sum_{i=1}^{i_r} C e^{-\alpha_1 4^i} \left(\mathcal{M}_{\text{loc}}(|\nabla h|^2) + \mathcal{M}_{4B_j}(|\text{Hess } h|^2) + \mathcal{M}_{\text{loc}}(|\text{Hess } h|^2 \mathbb{1}_{M \setminus 4B_j}) \right)^{1/2}(x). \end{aligned} \tag{3.37}$$

For the second regime $i > i_r$, we proceed with the L^p -Gaffney off-diagonal estimate (Proposition 2.11) together with the assumptions $r < 8$ and $k \leq n$ that there exist positive constants c_1 and c_2 such that

$$\left(\frac{1}{\mu(B)} \int_B |\text{Hess } P_{kr^2} g_i|^p d\mu \right)^{1/p} \leq \frac{c_1 e^{-c_2 4^i}}{r^2} \left(\frac{1}{\mu(2^{i+1}B)} \int_{C_i} |h|^2 d\mu \right)^{1/2}. \quad (3.38)$$

On the other hand, since $i > i_r$, it is easy to see $4B_j \subset 2^{i+1}B$ and $\mu(C_i) \lesssim \mu(4B_j)$; thus

$$\left(\frac{1}{\mu(2^{i+1}B)} \int_{C_i} |h|^2 d\mu \right)^{1/2} \leq C \left(\frac{1}{\mu(4B_j)} \int_{4B_j} |f|^2 d\mu \right)^{1/2} \leq C (\mathcal{M}_{4B_j}(|f|^2)(x))^{1/2}. \quad (3.39)$$

The contribution of the terms in the second regime $i > i_r$ is bounded by combining (3.38) and (3.39),

$$\sum_{i > i_r} \left(\frac{1}{\mu(B)} \int_B |\text{Hess } P_{kr^2} g_i|^p d\mu \right)^{1/p} \leq \sum_{i > i_r} \frac{c_1 e^{-c_2 4^i}}{r^2} (\mathcal{M}_{4B_j}(|f|^2)(x))^{1/2} \quad (3.40)$$

and it remains to recall that $1/r^2 \leq 4^i/8$ and $1/r \leq 2^i/(2\sqrt{2})$ when $i > i_r$.

We then conclude from (3.37) and (3.40) that for any $p > 2$ and $k \in \{1, 2, \dots, n\}$, there exists a constant C independent of j such that

$$\left(\frac{1}{\mu(B \cap 4B_j)} \int_{B \cap 4B_j} |\widetilde{T} P_{kr^2} f|^p d\mu \right)^{1/p} \leq C (\mathcal{M}_{4B_j}(|\widetilde{T} f|^2) + (S_j f)^2)^{1/2}(x)$$

all $f \in L^2(M, \mu)$ supported in B_j , all balls B in M and all $x \in B \cap 4B_j$, where the radius r of B is less than 8, and where

$$(S_j f)^2 := \mathcal{M}_{\text{loc}}(|\widetilde{T} f|^2 \mathbb{1}_{M \setminus 4B_j}) + \mathcal{M}_{\text{loc}}(|\nabla h|^2) + \mathcal{M}_{4B_j}(|f|^2). \quad (3.41)$$

Our last step is to show that the operator S_j defined in (3.41) is bounded from $L^p(B_j)$ to $L^p(4B_j)$ for any $p \in (2, \infty)$ with operator norm independent of j . By (3.41), we only need to show that the operators

$$(\mathcal{M}_{\text{loc}}(|\widetilde{T} f|^2 \mathbb{1}_{M \setminus 4B_j}))^{1/2}, (\mathcal{M}_{\text{loc}}(|\nabla h|^2))^{1/2} \text{ and } (\mathcal{M}_{4B_j}(|f|^2))^{1/2}$$

are respectively bounded from $L^p(B_j)$ to $L^p(4B_j)$. Indeed, for any $f \in L^p(4B_j)$, by Lemma 3.1 we know that $4B_j$ satisfies the doubling property **(D)**, which combined with $p > 2$ implies that $(\mathcal{M}_{4B_j}(|f|^2))^{1/2}$ is bounded from $L^p(B_j)$ to $L^p(4B_j)$ by a constant depending only on the doubling property **(D)**. On the other hand, using the fact from Bismut's formula [5] that

$$\|\sqrt{t} \nabla P_t f\|_p \lesssim e^{Kt} \|f\|_p,$$

we deduce from $p > 2$ and $\sigma > 2K + \theta$ that

$$\|(\mathcal{M}_{\text{loc}}(|\nabla h|^2))^{1/2}\|_p \lesssim \|\nabla h\|_p \leq \left(\int_0^\infty \frac{v(t) e^{Kt}}{\sqrt{t}} dt \right) \|f\|_{L^p(B_j)} \leq C \|f\|_{L^p(B_j)}.$$

Finally, the L^p -boundedness of

$$(\mathcal{M}_{\text{loc}}(|\widetilde{T} f|^2 \mathbb{1}_{M \setminus 4B_j}))^{1/2}$$

follows from the $L^{p/2}$ -boundedness of $(\mathcal{M}_{\text{loc}}(|\widetilde{T} f|^2))^{1/2}$ and an argument similar to (3.21). This implies that the operator S_j is bounded from $L^p(B_j)$ to $L^p(4B_j)$ with an upper bound independent of j .

We conclude that the requirements (i) and (ii) in Lemma 3.6 both hold under the condition **(H)**. Thus, the operator \widetilde{T} is bounded from $L^p(B_j, \mu)$ to $L^p(4B_j, \mu)$ for $p > 2$ with a constant independent of j . Therefore, by Lemma 3.5, the operator T is strong type (p, p) for $p > 2$. This proves Theorem 1.6. \square

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