Indeterminacies and models of homotopy limits

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December 12, 2022

Abstract

In [13] we studied the indeterminacy of the value of a derived functor at an object using different definitions of a derived functor and different types of fibrant replacement. In the present work we focus on derived or homotopy limits, which of course depend on the model structure of the diagram category under consideration. The latter is not necessarily unique, which is an additional source of indeterminacy. In the case of homotopy pullbacks, we introduce the concept of full homotopy pullback by identifying the homotopy pullbacks associated with three different model structures of the category of cospan diagrams, thus increasing the number of canonical representatives. Finally, we define generalized representatives or models of homotopy limits and full homotopy pullbacks. The concept of model is a unifying approach that includes the homotopy pullback used in [24] and the homotopy fiber square defined in [21] in right proper model categories. Properties of the latter are generalized to models in any model category.

MSC 2020: 18E35, 18N40, 14A30

Keywords: Localization, model category, homotopy category, derived functor, homotopy limit, homotopy pullback, homotopy fiber square

1 Introduction

Derived functors first appeared in homological algebra. Such right derived functors lead to a natural extension of the left exact sequence induced by applying an additive covariant left exact functor between abelian categories to a left exact sequence in its source category that is assumed to contain enough injectives. These classical derived functors, acting between the same abelian categories as the functor from which they are derived, are special cases of the derived functors between derived categories, constructed as the localization of an induced triangulated functor with respect to an induced null system.

On the topological side, topologists recognized that the category of interest is not the category $\mathtt{Top}$ of topological spaces and continuous maps, but the category in which topological spaces are considered isomorphic if they have approximately the same shape, although they are not necessarily homeomorphic. Hence the need arose to consider weak homotopy equivalences W as isomorphisms, i.e. to introduce the localized category $\mathtt{Top}[W^{-1}]$ in which weak equivalences of the standard model structure on $\mathtt{Top}$ become invertible. More generally, each model category $\mathtt{M}$ can be localized with respect to the class W of its weak equivalences, leading to its homotopy category $\mathtt{Ho(M)} = \mathtt{M}[W^{-1}]$ which is presented by the original category $\mathtt{M}$. In this case, functors between model categories should induce derived functors between the more fundamental homotopy categories or localized categories, provided they send, roughly speaking, weak equivalences to weak equivalences.

As a matter of fact, model categories were introduced by D. Quillen, in particular to unify the homotopy theory of topological spaces and the homology theory of chain complexes of modules. And indeed, for chain complexes of modules, the derived category of the first paragraph agrees with the homotopy category of the second and the 'abelian' derived functor is the same as the 'model-theoretical' one.

Against this background, it is not really surprising that in the literature one can find a number of different definitions of the localization of a category, of a model category, the homotopy category and a derived functor. Furthermore, a chosen definition of the homotopy category and the corresponding localization functor can have different descriptions, and if we also consider the computation of the different types of derived functors, at least four different replacement types are used. Although a certain equivalence of all resulting constructions can be expected and their nature is probably well known to experts, the jungle of these different notions is quite confusing for a beginner and it is not easy to navigate through it. In [13] we unravel this tangle. In fact, to the best of our knowledge, there is no single reference that has carefully examined all of these indeterminacies and compared the resulting concepts.

More precisely, there are four definitions of localization of a category $\mathtt{C}$ at a class of morphisms W. They differ in the strength of the universal property of the pair $(\mathtt{C}[W^{-1}], L)$, where the first element is the localized category and the second is the localization functor $L : \mathtt{C} \to \mathtt{C}[W^{-1}]$ sending morphisms in W to isomorphism. The so-called strong universal property requires that for every pair $(\mathtt{D}, F)$ with the same property as $(\mathtt{C}[W^{-1}], L)$ *there is a unique* functor $\mathrm{Ho}(F) : \mathtt{C}[W^{-1}] \to \mathtt{D}$ which makes the resulting triangle commutative *on the nose*. If in addition the pullback functor $L^\star = - \circ L$ by the localization functor is fully faithful, we call the universal property strict. The faint universal property requires that *there is* a functor $\bar{F} : \mathtt{C}[W^{-1}] \to \mathtt{D}$ that makes the triangle commutative *up to a natural isomorphism* η and that the pair $(\bar{F}, \eta)$ is unique up to a unique natural isomorphism. If the pullback functor $L^\star$ is fully faithful, uniqueness up to a unique natural isomorphism follows, and we refer to the universal property as weak. For instance, the classical Kan homotopy category $\mathtt{Ho}_{\mathrm{K}}(\mathtt{M})$ of a model category $\mathtt{M}$ (see [21, Definition 7.5.8]) and its localization functor $\mathcal{L}_{\mathtt{M}}$ is a weak (hence a faint) localization. The Quillen homotopy category $\mathtt{Ho}_{\mathrm{Q}}(\mathtt{M})$ or just $\mathtt{Ho}(\mathtt{M})$ (see [21, Definition 8.3.2]) and its localization functor $\gamma_{\mathtt{M}}$ is a strict (hence a strong) localization (see also [22, Section 1.2, Paragraph 1]). The concept of faint (resp., strong) localization is used in [23, Definition 7.1.1], [29, Section 2, General Definition] and [27, Definition 2.1] (resp., [11, Definition 6.1], [21, Definition 9.6.1], [22, Lemma 1.2.2], and [36, Chapter 1, Definition 5]).

On the other hand, 'all concepts are Kan extensions' [25]. A right Kan extension operation $L_\star$ along a functor $L : \mathtt{C} \to \mathtt{C}'$ exists if and only if for every functor $F : \mathtt{C} \to \mathtt{D}$ *there is* an 'extension' $L_\star F : \mathtt{C}' \to \mathtt{D}$ which makes the resulting triangle commutative *up to a natural transformation* $\eta : L_\star F \circ L \Rightarrow F$ such that the pair $(L_\star F, \eta)$ is unique up to a unique natural transformation. Note that this definition is similar to the faint localization definition above, except that above we are dealing with natural isomorphisms.

Let now $\mathtt{M}, \mathtt{N}$ be two model categories which admit a functorial factorization system, let $F : \mathtt{M} \to \mathtt{N}$ be a left Quillen functor or any functor which sends weak equivalences between cofibrant objects to weak equivalences, let $i : \mathtt{M}_{\mathtt{c}} \to \mathtt{M}$ be the canonical inclusion functor of the full subcategory of cofibrant objects and let W be the class of weak equivalences of $\mathtt{M}$ or the one of $\mathtt{M}_{\mathtt{c}}$ (the meaning will always be clear from the context).

The authors that use the presentation $(\mathtt{Ho}_{\mathrm{K}}(\mathtt{M}), \mathcal{L}_{\mathtt{M}})$ of the localization $(\mathtt{M}[W^{-1}], L)$ (see [27], [24]), define the total left derived functor of F either as the right Kan extension of $\mathcal{L}_{\mathtt{N}} \circ F \circ i$ along $\mathcal{L}_{\mathtt{M}} \circ i$ (see [26]) or as the faint factorization of $\mathcal{L}_{\mathtt{N}} \circ F \circ i$ (see [24]). It is not too hard to check that the Kan extension derived functor is given by the faint derived functor [13].

Most authors use the presentation $(\mathtt{Ho}(\mathtt{M}), \gamma_{\mathtt{M}})$ of the localization $(\mathtt{M}[W^{-1}], L)$. They define the total left derived functor of F either as the right Kan extension of $\gamma_{\mathtt{N}} \circ F$ along $\gamma_{\mathtt{M}}$ (see [21]) or as the composite of the strong factorization of $\gamma_{\mathtt{N}} \circ F \circ i$ and a quasi-inverse of the strong factorization of $\gamma_{\mathtt{M}} \circ i$ (see [22]). For further details we refer the reader to Section 2. It can be shown that the Kan extension derived functor and the strong derived functor coincide up to a natural isomorphism [13].

In the present work we focus on derived or homotopy limits. Since a derived functor depends on the model structures of the source and target of the original functor, the derived limit functor or homotopy limit functor depends on the model structures of the small diagram category $\mathtt{D} := \mathtt{Fun}(\mathtt{S}, \mathtt{M})$ and of the underlying category $\mathtt{M}$ considered (we use the terminology of [31]). The model structure σ of $\mathtt{D}$ is not necessarily unique, even if the model structure of the ambient category $\mathtt{M}$ remains unchanged and we take into account that the limit functor must be a right Quillen functor with respect to σ . The resulting freedom in choosing σ is an additional source of indeterminacy (see Theorem 4.1).

In the case $\mathtt{S} = \{c \to d \leftarrow b\}$ the diagrams $\mathtt{Fun}(\mathtt{S}, \mathtt{M})$ are the cospan diagrams $C \to D \leftarrow B$ of $\mathtt{M}$ and the homotopy limit is the homotopy pullback. The category of cospan diagrams can be equipped with three Reedy model structures σ_i $(i \in \{1, 2, 3\})$ with respect to which the pullback functor is a right Quillen functor. The homotopy pullbacks of a cospan $C \to D \leftarrow B$ with respect to the different σ_i-s admit as canonical representatives the standard pullbacks of the corresponding σ_i-fibrant replacements of $C \to D \leftarrow B$. We define the full homotopy pullback of $C \to D \leftarrow B$ by identifying its homotopy pullbacks with respect to the σ_i , thus *increasing the number of* canonical *representatives* (see Theorem 5.1).

We further enhance the flexibility of homotopy limits by *defining* generalized *representatives, also referred to as* models. The concept of model is valid in every model category but the model condition simplifies in right proper model categories (see Theorem 6.1 and Theorem 6.2). Models are a unifying approach that captures the notion of homotopy pullback that is used in [24] and the notion of homotopy fiber square that is defined in right proper model categories equipped with a fixed functorial factorization system in [21] (see Corollary 6.1 and Corollary 6.2). Most results of homotopy fiber squares in right proper model categories remain valid for models or model squares in all model categories (see for example Proposition 6.3 and Proposition 6.4).

Although not all of the results of this paper are completely new, a structured rigorous presentation in an appropriate unifying context does not seem to exist. The proven theorems offer guidance in an environment with many indeterminacies and show that the standard concepts concerned have a pretty good stability with regard to all the necessary choices.

Applications can be expected in homotopical algebraic geometry [37, 38, 2, 3, 6] and higher supergeometry [7, 8, 9, 35], as these areas make extensive use of the functor of points and are the contexts from which the need arose to examine the subjects of this paper. We refer the reader to Section 7 for further details.

2 Conventions and notations

We assume that the reader is familiar with model categories. Although we use many results of [13], we paid attention to independent readability when writing this work. We adopt the definition of a model category that is used in [21]. More precisely, a model category is a category $\mathtt{M}$ that is equipped with three classes of morphisms called weak equivalences, fibrations and cofibrations. The category $\mathtt{M}$ has all small limits and colimits and the 2-out-of-3 axiom, the retract axiom and the lifting axiom are satisfied. Moreover $\mathtt{M}$ admits a functorial cofibration - trivial fibration factorization system (Cof - TrivFib factorization) and a functorial trivial cofibration - fibration factorization system (TrivCof - Fib factorization). Furthermore, we work with the Quillen homotopy category $\mathtt{Ho(M)}$ of $\mathtt{M}$, which is a strong localization $\mathtt{M}[W^{-1}]$ and even a strict localization $\mathtt{M}[[W^{-1}]]$ of $\mathtt{M}$ at its weak equivalences W with localization functor denoted $\gamma_{\mathtt{M}}$, and we use the Kan extension derived functor operations $\mathbb{L}^{\mathrm{K}}, \mathbb{R}^{\mathrm{K}}$ and the strong derived functor operations $\mathbb{L}^{\mathrm{S}}, \mathbb{R}^{\mathrm{S}}$ in the sense of [13], which we already mentioned above.

More specifically, if we use the same data and notation as above, and in particular consider a functor $F : \mathtt{M} \to \mathtt{N}$ which sends weak equivalences between cofibrant objects to weak equivalences, the total left derived functor of F is defined either as the right Kan extension (see [21])

$$\mathbb{L}^{\mathrm{K}} F := (\gamma_{\mathtt{M}})_{\star}(\gamma_{\mathtt{N}} \circ F) \in \mathtt{Fun}(\mathtt{Ho(M)}, \mathtt{Ho(N)}) \;, \tag{2.1}$$

which comes with a natural transformation

$$\eta : \mathbb{L}^{\mathrm{K}} F \circ \gamma_{\mathtt{M}} \Rightarrow \gamma_{\mathtt{N}} \circ F \;, \tag{2.2}$$

or as the composite

$$\mathbb{L}^{\mathrm{S}}_{\mathcal{I}} F := \mathrm{Ho}(\gamma_{\mathtt{N}} \circ F \circ i) \circ \mathcal{I} \in \mathtt{Fun}(\mathtt{Ho(M)}, \mathtt{Ho(N)}) \tag{2.3}$$

of the strong factorization $\mathrm{Ho}(\gamma_{\mathtt{N}} \circ F \circ i) =: \mathrm{Ho}(F)$ and a quasi-inverse $\mathcal{I}$ of the strong factorization $\mathrm{Ho}(\gamma_{\mathtt{M}} \circ i) =: \mathrm{Ho}(i)$ (here i is the same as $i : \mathtt{M}_c \to \mathtt{M}$ defined in the introduction). Whatever quasi-inverse we choose, we get a representative of the same isomorphism class of functors, so that $\mathbb{L}^{\mathrm{S}} F$ is defined up to a natural isomorphism. It can be checked that every cofibrant replacement functor $Q : \mathtt{M} \to \mathtt{M}_{\rm c}$ induces a quasi-inverse $\mathrm{Ho}(L \circ Q) =: \mathrm{Ho}(Q)$ of $\mathrm{Ho}(i)$, which implies that

$$\mathbb{L}^{\rm S} F \overset{\cong}{\Rightarrow} \mathbb{L}^{\rm S}_Q F = \mathrm{Ho}(F) \circ \mathrm{Ho}(Q) = \mathrm{Ho}(F \circ Q)\ .$$

Moreover, we have the equality (see [22])

$$\mathbb{L}^{\rm S}_Q F \circ \gamma_{\mathtt{M}} = \gamma_{\mathtt{N}} \circ F \circ Q\ . \tag{2.4}$$

Despite the difference between the definitions (2.1) and (2.3) and between the properties (2.2) and (2.4), it can be shown – as mentioned previously – that the Kan extension derived functor and the strong derived functor coincide up to a natural isomorphism [13]. Similar results hold for total right derived functors of functors that send weak equivalences between fibrant objects to weak equivalences (see Theorem 3.1). For more details on the preceding derived functor operations, we refer the reader to Definition 8 and Propositions 5, 11, 12 and 13 in [13].

3 Indeterminacy of a derived functor

Let (α, β) be any functorial TrivCof - Fib factorization system. For every object $X \in \mathtt{M}$, it factors the map $t_X : X \to *$ to the terminal object of $\mathtt{M}$ into a trivial cofibration $r_X := \alpha(t_X)$ followed by a fibration $\beta(t_X)$:

$$t_X : X \overset{\sim}{\rightarrowtail} RX \twoheadrightarrow *\ .$$

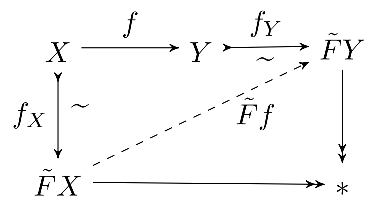
Regardless of the factorization

$$t_X : X \overset{\sim}{\to} FX \twoheadrightarrow *$$

of $t_X : X \to *$ into a weak equivalence f_X followed by a fibration considered, we refer to FX as *a fibrant replacement* of X. The object RX we call a *fibrant C-replacement* of X (or just a fibrant replacement if we do not want to stress that r_X is a cofibration). From the fact that the factorization (α, β) is functorial it follows that R is an endofunctor of $\mathtt{M}$. Moreover $r_X : X \to RX$ is functorial in X: r is a natural transformation $r : \mathrm{id}_{\mathtt{M}} \Rightarrow R$ from the identity functor $\mathrm{id}_{\mathtt{M}}$ to the *fibrant replacement functor* R [22]. Instead of the fibrant C-replacement functor R that is globally defined by the functorial factorization (α, β), we will also use local / object-wise fibrant replacements FX or *local fibrant C-replacements* $\tilde{F}X$ such that the map f_X in the factorization

$$t_X : X \overset{\sim}{\rightarrowtail} \tilde{F}X \twoheadrightarrow *$$

is id_X if X is already fibrant [28]. If for every X we choose such a local fibrant C-replacement and if $f : X \to Y$, there is a lifting $\tilde{F}f : \tilde{F}X \to \tilde{F}Y$, which will play an important role:



(3.1)

From [13] we also know:

Theorem 3.1. *If* $G \in \texttt{Fun}(\texttt{M}, \texttt{N})$ *is a functor between model categories that sends weak equivalences between fibrant objects to weak equivalences, its Kan extension right derived functor*

$$\mathbb{R}^{\mathrm{K}}G \in \texttt{Fun}(\texttt{Ho}(\texttt{M}), \texttt{Ho}(\texttt{N}))$$

and its strongly universal right derived functor

$$\mathbb{R}^{\mathrm{S}}G \in \texttt{Fun}(\texttt{Ho}(\texttt{M}), \texttt{Ho}(\texttt{N}))$$

exist and we have

$$\mathbb{R}^{\mathrm{K}}G \doteq \mathrm{Ho}(\gamma_{\texttt{N}} \circ G \circ \tilde{F}) \doteq \mathbb{R}^{\mathrm{S}}_{R}G := \mathrm{Ho}(\gamma_{\texttt{N}} \circ G \circ R) \overset{\cong}{\Rightarrow} \mathbb{R}^{\mathrm{S}}G \ , \tag{3.2}$$

where $\tilde{F}$ *is a local fibrant C-replacement,* R *is a fibrant C-replacement functor and* Ho *the unique on the nose factorization through* $\texttt{Ho}(\texttt{M})$ *. This implies that*

$$\mathbb{R}^{\mathrm{K}}G \circ \gamma_{\texttt{M}} \doteq \gamma_{\texttt{N}} \circ G \circ \tilde{F} \doteq \mathbb{R}^{\mathrm{S}}_{R}G \circ \gamma_{\texttt{M}} = \gamma_{\texttt{N}} \circ G \circ R \overset{\cong}{\Rightarrow} \mathbb{R}^{\mathrm{S}}G \circ \gamma_{\texttt{M}} \ , \tag{3.3}$$

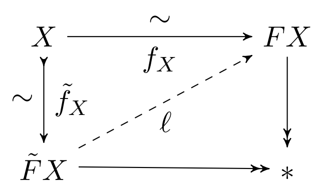
where $\doteq$ *denotes a canonical natural isomorphism and* $\overset{\cong}{\Rightarrow}$ *a not necessarily canonical natural isomorphism.*

Hence, for every $X \in \texttt{M}$, the value of the derived functor at $\gamma_{\texttt{M}}X = X \in \texttt{Ho}(\texttt{M})$ is

$$\mathbb{R}^{\mathrm{K}}G(X) \doteq G(\tilde{F}X) \doteq \mathbb{R}^{\mathrm{S}}_{R}G(X) = G(RX) \cong \mathbb{R}^{\mathrm{S}}G(X) \ . \tag{3.4}$$

Remark 3.1. Since $\mathbb{R}^{\mathrm{K}}G$ (resp., $\mathbb{R}^{\mathrm{S}}G$) is defined up to a canonical natural isomorphism (resp., up to a natural isomorphism) [13], the results of (3.2) are the best possible ones.

The next diagram shows that if FX is any fibrant replacement of X, there is a lifting $\ell : \tilde{F}X \to FX$:



(3.5)

Since ℓ is a weak $\mathtt{M}$-equivalence between fibrant objects, its image $G(\ell)$ is a weak $\mathtt{N}$-equivalence

$$G(\tilde F X) \xrightarrow{\sim} G(FX) \tag{3.6}$$

and the image $\gamma_{\mathtt{N}}(G(\ell))$ is a $\mathtt{Ho}(\mathtt{N})$-isomorphism

$$G(\tilde F X) \cong G(FX)\ . \tag{3.7}$$

Proposition 3.1. *The isomorphism* (3.7) *is canonical:*

$$G(\tilde F X) \doteq G(FX)\ . \tag{3.8}$$

Proof. Take two different $\mathtt{M}$-morphisms $\ell_i : \tilde F X \to FX$ $(i \in \{1,2\})$ that render the upper triangle in Diagram 3.5 commutative, so that $\ell_1 \circ \tilde f_X = \ell_2 \circ \tilde f_X$. Since $Y := FX$ is fibrant and $\tilde f_X : X \overset{\sim}{\rightarrowtail} \tilde F X$ is a trivial cofibration, right composition by $\tilde f_X$ induces a 1:1 correspondence between right homotopy classes of morphisms (the result is well known; we gave the proof of its dual in [13]), we get $\ell_1 \simeq^r \ell_2$. This means that $\ell_1 \times \ell_2 : \tilde F X \to Y \times Y$ factors through a path object $\mathrm{Path}(Y)$ of Y , i.e., that there is a factorization

$$p_1 \circ w := \psi_1 \circ p \circ w = \mathrm{id}_Y \quad \text{and} \quad p_2 \circ w := \psi_2 \circ p \circ w = \mathrm{id}_Y\ , \tag{3.9}$$

where $\psi_1, \psi_2 : Y \times Y \to Y$, $w : Y \xrightarrow{\sim} \mathrm{Path}(Y)$ and $p : \mathrm{Path}(Y) \twoheadrightarrow Y \times Y$, and a factorization

$$p_1 \circ K = \ell_1 \quad \text{and} \quad p_2 \circ K = \ell_2\ , \tag{3.10}$$

where $K : \tilde F X \to \mathrm{Path}(Y)$. From (3.9) it follows that $p_i : \mathrm{Path}(Y) \to Y$ is a weak equivalence between fibrant objects (indeed, since fibrations are closed under pullbacks and compositions, the product of fibrant objects and the path object of a fibrant object are fibrant). If we apply $\gamma_{\mathtt{N}} \circ G$ to (3.9) and remember that $\gamma_{\mathtt{N}}(G(p_i))$ is an isomorphism in view of the assumption on G , we see that $\gamma_{\mathtt{N}}(G(w))$ is the inverse isomorphism and that $\gamma_{\mathtt{N}}(G(p_1)) = \gamma_{\mathtt{N}}(G(p_2))$. It now follows from (3.10) that

$$\gamma_{\mathtt{N}}(G(\ell_1)) = \gamma_{\mathtt{N}}(G(\ell_2))\ , \tag{3.11}$$

so that the $\mathtt{Ho}(\mathtt{N})$-isomorphism $\gamma_{\mathtt{N}}(G(\ell))$ is canonically implemented by the replacements $\tilde F X$ and FX . □

Notice now that if $X, Y \in \mathtt{M}$ are related by a zigzag of weak $\mathtt{M}$-equivalences it suffices to apply the localization functor $\gamma_{\mathtt{M}}$ to see that X and Y are isomorphic in $\mathtt{Ho}(\mathtt{M})$. It is well known that the converse is also true:

Proposition 3.2. *Two objects of a model category* $\mathtt{M}$ *are isomorphic as objects of* $\mathtt{Ho}(\mathtt{M})$ *if and only if they are related by a zigzag of weak equivalences of* $\mathtt{M}$.

The previous observations clarify the *indeterminacy of a value of a derived functor*:

Conclusion 3.1. In view of (3.4) and (3.8) the value of a derived functor at an object is well defined only up to isomorphism of the target homotopy category. The isomorphism class is independent of the type of derived functor considered, Kan extension or strongly universal, as well as independent of the type of fibrant C-replacement considered, local or global. Also the choice of another local or another global replacement does not change the isomorphism class. If we compute the value of the derived functor using a local fibrant replacement that is not necessarily a C-replacement, we get again the same class. Finally, the three representatives considered of the value of the derived functor are related by canonical isomorphisms when viewed as objects of $\mathtt{Ho}(\mathtt{N})$ and by zigzags of weak equivalences when viewed as objects of $\mathtt{N}$. This zigzag is the first source of ambiguity or indeterminacy in the values of a derived functor.

Remark 3.2. In the following we write $X \approx Y$ if X and Y are related by a zigzag of weak equivalences and we write $X \overset{\sim}{\rightleftarrows} Y$ if there is a weak equivalence from X to Y *and* a weak equivalence from Y to X.

If we use the notation of Remark 3.2, Equation (3.4) and Equation (3.6) imply that if $G \in \mathtt{Fun}(\mathtt{M}, \mathtt{N})$ is a functor between model categories that sends weak $\mathtt{M}$-equivalences between fibrant objects to weak $\mathtt{N}$-equivalences, then, for every $X \in \mathtt{M}$, we have

$$\mathbb{R}^{\mathrm{K}} G(X) \approx \mathbb{R}^{\mathrm{S}} G(X) \approx G(RX) \overset{\sim}{\rightleftarrows} G(\tilde{F}X) \overset{\sim}{\to} G(FX)\,. \tag{3.12}$$

We get the weak equivalences between $G(RX)$ and $G(\tilde{F}X)$ just as we got the one from $G(\tilde{F}X)$ to $G(FX)$.

The dual versions of the results in this section for left derived functors are also true.

4 Indeterminacy of a homotopy limit

If $\mathtt{S}$ is a small category and $\mathtt{M}$ a model category, the functor category $\mathtt{Fun}(\mathtt{S}, \mathtt{M})$ admits under mild conditions on the target category $\mathtt{M}$ an injective (resp., projective) model structure. The injective weak equivalences and cofibrations are defined as object-wise weak $\mathtt{M}$-equivalences and $\mathtt{M}$-cofibrations. The resulting classes of weak equivalences, cofibrations and fibrations satisfy the model category axioms, if $\mathtt{M}$ is a combinatorial model category. In this case, we refer to the model structure defined in this way as the **injective model structure**. The **projective model** structure is defined dually. A sufficient condition of existence is that the target category $\mathtt{M}$ is cofibrantly generated. Details can be found for instance in [10, Chapter III], [21, Section 11.6] and [22, Chapter 2].

Remark 4.1. Note that besides the injective and projective model structures - if $\mathtt{M}$ is combinatorial, the functor category considered also admits a Reedy model structure - if $\mathtt{S}$ is a Reedy category (see Equation 4.8).

The constant functor $-^* : \mathtt{M} \to \mathtt{Fun}(\mathtt{S}, \mathtt{M})$ is the left adjoint to the limit functor:

$$-^* : \mathtt{M} \rightleftarrows \mathtt{Fun}(\mathtt{S}, \mathtt{M}) : \mathrm{Lim} \tag{4.1}$$

If the injective model structure of $\mathtt{Fun}(\mathtt{S}, \mathtt{M})$ exists, the constant functor respects cofibrations and trivial cofibrations and the adjunction is therefore a Quillen adjunction. More generally, let σ be any model structure on $\mathtt{Fun}(\mathtt{S}, \mathtt{M})$ such that (4.1) is a Quillen adjunction $-^* \dashv \mathrm{Lim}$. It follows from Brown's lemma [22, Lemma 1.1.12] that a right (resp., left) Quillen functor sends weak equivalences between fibrant (resp., cofibrant) objects to weak equivalences, so that its right (resp., left) derived functor exists (see Theorem 3.1). The left and right derived functors induced between the homotopy categories by adjoint Quillen functors are themselves adjoint functors. The Quillen adjunction (4.1) induces therefore the adjunction

$$\mathbb{L}_\sigma(-^*) : \mathtt{Ho}(\mathtt{M}) \rightleftarrows \mathtt{Ho}(\mathtt{Fun}_\sigma(\mathtt{S}, \mathtt{M})) : \mathbb{R}_\sigma\mathrm{Lim} \ .$$

This holds in both the case of K and S derived functors [13].

Definition 4.1. *The derived functor $\mathbb{R}_\sigma\mathrm{Lim}$ is referred to as the* **homotopy limit functor** *with respect to the model structure σ on diagrams.*

From Equation (3.12) follows the

Theorem 4.1. *Let $\mathtt{S}$ be a small category, let $\mathtt{M}$ be a model category and let σ be a model structure on the category $\mathtt{Fun}(\mathtt{S}, \mathtt{M})$ of $\mathtt{S}$-shaped diagrams in $\mathtt{M}$ such that the adjunction (4.1) is a Quillen adjunction $-^* \dashv \mathrm{Lim}$. If $X \in \mathtt{Fun}(\mathtt{S}, \mathtt{M})$ its homotopy limit with respect to σ is given as an object of $\mathtt{M}$ by*

$$\mathbb{R}_\sigma\mathrm{Lim}(X) \approx \mathrm{Lim}(R_\sigma X) \overset{\sim}{\rightleftarrows} \mathrm{Lim}(\tilde{F}_\sigma X) \overset{\sim}{\to} \mathrm{Lim}(F_\sigma X) \ , \tag{4.2}$$

where $R_\sigma, \tilde{F}_\sigma, F_\sigma$ are a fibrant C-replacement functor, a local fibrant C-replacement and a local fibrant replacement, respectively, in the model structure σ on $\mathtt{Fun}(\mathtt{S},\mathtt{M})$. The weak equivalence $\xrightarrow{\sim}$ is the universal morphism

$$\mathrm{Lim}(\ell_\sigma) : \mathrm{Lim}(\tilde{F}_\sigma X) \xrightarrow{\sim} \mathrm{Lim}(F_\sigma X) \tag{4.3}$$

that is induced by a lifting $\ell_\sigma : \tilde{F}_\sigma X \Rightarrow F_\sigma X$ and its image $\gamma_{\mathtt{M}}(\mathrm{Lim}(\ell_\sigma))$ in homotopy is independent of the lifting considered (see (3.11)*). A similar remark holds for the weak equivalences $\overset{\sim}{\rightleftarrows}$.*

In particular, if the injective model structure exists, then for the homotopy limit functor we have:

$$\mathbb{R}_{\mathrm{inj}}\mathrm{Lim}(X) \approx \mathrm{Lim}(R_{\mathrm{inj}}X) \overset{\sim}{\rightleftarrows} \mathrm{Lim}(\tilde{F}_{\mathrm{inj}}X) \xrightarrow{\sim} \mathrm{Lim}(F_{\mathrm{inj}}X)\ . \tag{4.4}$$

If the projective model structure exists, then the dual result holds for the homotopy colimit functor:

$$\mathbb{L}_{\mathrm{proj}}\mathrm{Colim}(X) \approx \mathrm{Colim}(Q_{\mathrm{proj}}X) \overset{\sim}{\rightleftarrows} \mathrm{Colim}(\tilde{C}_{\mathrm{proj}}X) \xleftarrow{\sim} \mathrm{Colim}(C_{\mathrm{proj}}X)\ , \tag{4.5}$$

where $Q_{\mathrm{proj}}, \tilde{C}_{\mathrm{proj}}$ and C_{proj} are cofibrant replacements.

Remark 4.2. Equation (4.4) clarifies the indeterminacy of a small homotopy limit $\mathbb{R}_{\mathrm{inj}}\mathrm{Lim}(X)$ viewed as an object of the underlying model category in relation to the chosen definition of derived functors and the chosen replacement of the diagram under consideration. However, if the index category $\mathtt{S}$ is an appropriate Reedy category $\mathtt{R}$, the limit functor is also a right Quillen functor if the diagram category is equipped with its Reedy model structure. This leads to a homotopy limit $\mathbb{R}_{\mathrm{Ree}}\mathrm{Lim}(X)$ with respect to the Reedy structure and thus to another possible indeterminacy.

In the remainder of this section, we recall the results on Reedy categories and Reedy model structures (see [10, Chapter III], [21, Chapter 15] or [22, Chapter 5]) that we need in the next section to explore the indeterminacy just mentioned. The exact understanding of all the indeterminacies is a prerequisite for the new approach to model categorical homotopy fiber sequences that we detail in [14].

If $\mathtt{R}$ is a Reedy category and $\mathtt{M}$ *any* model category, the functor category $\mathtt{Fun}(\mathtt{R},\mathtt{M})$ can be equipped with a **Reedy model structure**.

Reedy categories are defined using direct and inverse categories which are particularly simple examples of Reedy categories [10]. A **direct category** (resp., an **inverse category**) is a small category that comes with a map deg from objects to ordinals such that every non-identity morphism $r \to s$ raises (resp., lowers) the degree: $\deg r < \deg s$ (resp., $\deg r > \deg s$). A **Reedy category** is a small category $\mathtt{R}$ together with two subcategories $\mathtt{R}_+$ and $\mathtt{R}_-$ which contain all the objects, and a map deg from objects to ordinals such that:

1. every $\mathtt{R}$-morphism factors uniquely into an $\mathtt{R}_-$-morphism and an $\mathtt{R}_+$-morphism,
2. every non-identity $\mathtt{R}_-$-morphism lowers the degree,
3. every non-identity $\mathtt{R}_+$-morphism raises the degree.

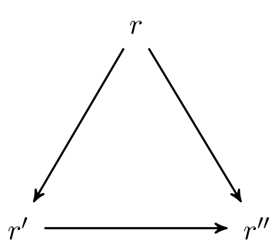
Example 4.1. *The category* $\mathtt{I} := \{c \to d \leftarrow b\}$ *is a direct category when equipped with degree map* $\deg_1$ *defined by* $\{0 \to 2 \leftarrow 1\}$, *it is an inverse category for the degree map* $\deg_2$ *defined by* $\{1 \to 0 \leftarrow 2\}$ *and it is a non-trivial Reedy category for* $\deg_3$ *and* $\deg_4$ *given by* $\{0 \to 1 \leftarrow 2\}$ *and* $\{2 \to 1 \leftarrow 0\}$, *respectively.*

For every $X \in \mathtt{Fun}(\mathtt{R}, \mathtt{M})$ and every $r \in \mathtt{R}$ one defines the **matching object** $M_r X$ of X at r as the limit

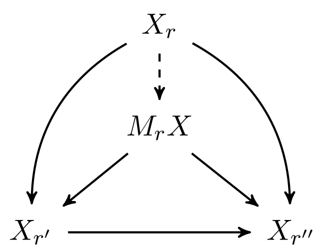
$$M_r X := \mathrm{Lim}(\mathtt{R}^r_- \overset{\mathrm{For}}{\to} \mathtt{R} \overset{X}{\to} \mathtt{M}) = \mathrm{Lim}(\mathrm{For}^* X) \in \mathtt{M} ,$$

where $\mathtt{R}^r_-$ is the full subcategory of the under-category $r \downarrow \mathtt{R}_-$ that contains all the objects except the identity of r and where For is the forgetful functor.

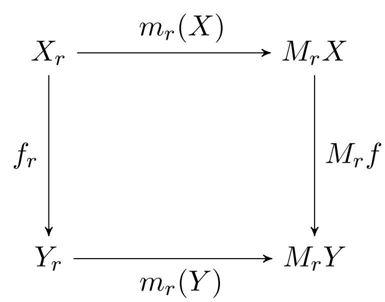
For instance, let $r \to r'$ and $r \to r''$ be two non-identity morphisms of $\mathtt{R}_-$ and let $r' \to r''$ be a morphism of $\mathtt{R}_-$ that makes the resulting triangle (4.5.1) commutative:



(4.5.1)



(4.5.2)



(4.5.3)

The functor For sends this morphism of $\mathtt{R}^r_-$ to $r' \to r''$ and the functor X sends the latter to $X_{r'} \to X_{r''}$: see Diagram (4.5.2). One can prove that the matching objects $M_r X$ ($X \in \mathtt{Fun}(\mathtt{R}, \mathtt{M})$) can be extended into a **matching functor** M_r and that the universal **matching morphisms** $m_r(X) : X_r \dashrightarrow M_r X$ define a **matching natural transformation** m_r:

$$M_r = \mathrm{Lim} \circ \mathrm{For}^* \in \mathtt{Fun}(\mathtt{Fun}(\mathtt{R}, \mathtt{M}), \mathtt{M}) \quad \text{and} \quad m_r : -_r \Rightarrow M_r , \tag{4.6}$$

where $-_r : \mathtt{Fun}(\mathtt{R}, \mathtt{M}) \to M$ is the evaluation functor. Hence, if $X, Y \in \mathtt{Fun}(\mathtt{R}, \mathtt{M})$ and $f : X \Rightarrow Y$, the square (4.5.3) commutes.

The **Reedy weak equivalences** are defined object-wise and are therefore the same as for the projective and injective model structures. The **Reedy fibrations** are the natural transformations $f : X \Rightarrow Y$ such that the induced universal $\mathtt{M}$-morphism

$$X_r \to Y_r \times_{M_r Y} M_r X \tag{4.7}$$

is a fibration for every $r \in \mathtt{R}$. The **Reedy cofibrations** are defined dually. For more details we refer the reader to [10].

If the Reedy category $\mathtt{R}$ is a direct category, its subcategory $\mathtt{R}_+$ is the full category $\mathtt{R}$ and its subcategory $\mathtt{R}_-$ is the discrete category that contains all the objects $r \in \mathtt{R}$. In this case the full subcategory $\mathtt{R}_-^r$ is the empty category, the functor $\mathrm{For}^* X$ is the empty diagram and $M_r X$ is the terminal object $*$ of $\mathtt{M}$ for all X . It follows from (4.7) that the Reedy fibrations are exactly the object-wise fibrations. Therefore the Reedy model structure of $\mathtt{Fun}(\mathtt{R}, \mathtt{M})$ is the projective model structure. The dual result is also true:

Remark 4.3. If the Reedy category $\mathtt{R}$ is a direct (resp., an inverse) category, the Reedy model structure of $\mathtt{Fun}(\mathtt{R}, \mathtt{M})$ is the projective (resp., the injective) model structure.

If $\mathtt{R}$ is any Reedy category and $\mathtt{M}$ **is a combinatorial model category**, so that all three model structures exist, the identity functor id of $\mathtt{Fun}(\mathtt{R}, \mathtt{M})$ is a left Quillen equivalence from the projective model structure to the Reedy model structure and from the Reedy structure to the injective one and a right Quillen equivalence in the other direction [24, A.2.9, paragraph 1 and A.2.9.23]:

$$\mathrm{id} : \mathtt{Fun}_{\mathrm{proj}}(\mathtt{R}, \mathtt{M}) \rightleftarrows \mathtt{Fun}_{\mathrm{Reedy}}(\mathtt{R}, \mathtt{M}) \rightleftarrows \mathtt{Fun}_{\mathrm{inj}}(\mathtt{R}, \mathtt{M}) : \mathrm{id} \ . \tag{4.8}$$

5 Indeterminacy of a homotopy pullback

In this section we examine the additional indeterminacy of a homotopy pullback, which we already addressed in Remark 4.2, namely the ambiguity caused by the choice of the model structure on the functor, diagram or here the cospan category. As stated in Remark 4.1, the model structures to be considered are the injective, the projective and the various Reedy structures, where the projective model structure is used to compute homotopy colimits (see

Equation 4.5) and in particular homotopy pushouts and is therefore not of interest in our case of homopoty pullbacks.

Let $\mathtt{M}$ now be any model category and let $\mathtt{R}$ be the inverse Reedy category $\mathtt{I}_2$ whose underlying category is $\mathtt{I} := \{c \to d \leftarrow b\}$ and whose degree map is the above-mentioned map $\deg_2$ defined by $\{1 \to 0 \leftarrow 2\}$ (see Example 4.1). The objects X of the functor category

$$\mathtt{M}^{\mathtt{I}} := \mathtt{Fun}(\mathtt{I}, \mathtt{M})$$

are the $\mathtt{M}$-cospans $C \to D \leftarrow B$ and its morphisms $f : X \Rightarrow Y$ are the corresponding adjacent commutative squares

$$\begin{array}{ccccc} C & \longrightarrow & D & \longleftarrow & B \\ {\scriptstyle f_c}\downarrow & & {\scriptstyle f_d}\downarrow & & \downarrow{\scriptstyle f_b} \\ C' & \longrightarrow & D' & \longleftarrow & B' \end{array} \tag{5.1}$$

In view of Remark 4.3 the Reedy model structure on $\mathtt{M}^{\mathtt{I}_2}$ is the injective model structure of $\mathtt{M}^{\mathtt{I}}$. Further, a natural transformation $f : X \Rightarrow Y$ is an injective fibration if and only if Condition (4.7) is satisfied. It follows from the definition of matching objects of objects $X \in \mathtt{M}^{\mathtt{I}_2}$ that $M_b X = D$, $M_c X = D$ and $M_d X = *$, so that f is an injective fibration if and only if the induced universal $\mathtt{M}$-morphisms are fibrations:

$$B \twoheadrightarrow B' \times_{D'} D\,, \quad C \twoheadrightarrow C' \times_{D'} D\,, \quad D \twoheadrightarrow D'\,.$$

In particular:

Proposition 5.1. *For any model category* $\mathtt{M}$, *the injective model structure on the category of* $\mathtt{M}$-*cospans exists. Moreover, an* $\mathtt{M}$-*cospan* $C \to D \leftarrow B$ *is injectively fibrant if and only if* D *is a fibrant object of* $\mathtt{M}$ *and both arrows are fibrations of* $\mathtt{M}$:

$$C \twoheadrightarrow D_{\mathrm{f}} \twoheadleftarrow B\,. \tag{5.2}$$

If $\mathtt{I}_3$ is the Reedy category $\mathtt{I} = \{c \to d \leftarrow b\}$ with degree map $\deg_3$ defined by $\{0 \to 1 \leftarrow 2\}$ (see Example 4.1), the computation of the Reedy fibrations is the same as in the case of $\mathtt{I}_2$, except that $M_c X = *$, so that f is a fibration of the Reedy model structure $\mathrm{Ree}_{\mathtt{I}}$ defined by the increasing labelling $\{0 \to 1 \leftarrow 2\}$ if and only if

$$B \twoheadrightarrow B' \times_{D'} D\,, \quad C \twoheadrightarrow C'\,, \quad D \twoheadrightarrow D'\,.$$

Dually, a natural transformation f is a cofibration of the Reedy model structure $\mathrm{Ree}_{\mathtt{I}}$ if and only if

$$B \rightarrowtail B' , \quad C \rightarrowtail C' , \quad D \amalg_C C' \rightarrowtail D' . \tag{5.3}$$

In particular:

Proposition 5.2. *For any model category* $\mathtt{M}$, *an* $\mathtt{M}$*-cospan* $C \to D \leftarrow B$ *is fibrant for the Reedy model structure* $\mathrm{Ree}_{\mathtt{I}}$ *defined by the increasing labelling* $\{0 \to 1 \leftarrow 2\}$ *if and only if* C *and* D *are fibrant objects of* $\mathtt{M}$ *and the second arrow is a fibration of* $\mathtt{M}$ *:*

$$C_{\mathrm{f}} \to D_{\mathrm{f}} \twoheadleftarrow B . \tag{5.4}$$

If $\mathtt{M}^{\mathtt{I}}$ is equipped with its Reedy structure $\mathrm{Ree}_{\mathtt{I}}$, the constant functor

$$-^* : \mathtt{M} \rightleftarrows \mathtt{Fun}_{\mathrm{Ree}_{\mathtt{I}}}(\mathtt{I}, \mathtt{M}) : \mathrm{Lim} \tag{5.5}$$

preserves cofibrations. Indeed, the image by $-^*$ of an $\mathtt{M}$-morphism $m : E \to E'$ is the commutative diagram

$$\begin{array}{ccccc} E & \xrightarrow{\mathrm{id}_E} & E & \xleftarrow{\mathrm{id}_E} & E \\ {\scriptstyle m}\downarrow & & {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m} \\ E' & \xrightarrow{\mathrm{id}_{E'}} & E' & \xleftarrow{\mathrm{id}_{E'}} & E' \end{array} \tag{5.6}$$

and if $m : E \rightarrowtail E'$, this diagram is a cofibration of $\mathrm{Ree}_{\mathtt{I}}$ if and only if the conditions (5.3) are satisfied, i.e., if and only if the universal morphism $u : E \amalg_E E' \to E'$ induced by m is a cofibration in $\mathtt{M}$. One easily sees that $u = \mathrm{id}_{E'}$, so the previous diagram is indeed a cofibration of $\mathrm{Ree}_{\mathtt{I}}$. As weak equivalences are defined object-wise in any Reedy structure, the constant functor preserves also trivial cofibrations and therefore the adjunction (5.5) is a Quillen adjunction. The right adjoint functor of the resulting adjunction in homotopy

$$\mathbb{L}_{\mathrm{Ree}_{\mathtt{I}}}(-^*) : \mathtt{Ho}(\mathtt{M}) \rightleftarrows \mathtt{Ho}(\mathtt{Fun}_{\mathrm{Ree}_{\mathtt{I}}}(\mathtt{I}, \mathtt{M})) : \mathbb{R}_{\mathrm{Ree}_{\mathtt{I}}}\mathrm{Lim}$$

is the K or the S homotopy limit functor with respect to the Reedy model structure $\mathrm{Ree}_{\mathtt{I}}$ (see Definition 4.1).

Similarly, if $\mathtt{I}_4$ is the Reedy category $\mathtt{I} = \{c \to d \leftarrow b\}$ with degree map $\deg_4$ defined by the decreasing labelling $\{2 \to 1 \leftarrow 0\}$ (see Example 4.1), we get the

Proposition 5.3. *For any model category* $\mathtt{M}$, *an* $\mathtt{M}$*-cospan* $C \to D \leftarrow B$ *is fibrant for the Reedy model structure* $\mathrm{Ree}_{\mathrm{D}}$ *defined by the decreasing labelling* $\{2 \to 1 \leftarrow 0\}$ *if and only if* D *and* B *are fibrant objects of* $\mathtt{M}$ *and the first arrow is a fibration of* $\mathtt{M}$:

$$C \twoheadrightarrow D_{\mathrm{f}} \leftarrow B_{\mathrm{f}} \ . \tag{5.7}$$

Moreover, just as in the case of $\mathrm{Ree}_{\mathrm{I}}$, there is a K and an S homotopy limit functor $\mathbb{R}_{\mathrm{Ree}_{\mathrm{D}}}\mathrm{Lim}$ with respect to $\mathrm{Ree}_{\mathrm{D}}$.

Remark 5.1. With regard to Remark 4.3, the Reedy model structure that is induced on cospans by the direct categorical structure, which in turn is defined by the degree map $\deg_1$ of example 4.1 given by $0 \to 2 \leftarrow 1$, is the projective model structure and is therefore not relevant for our purpose here - see first paragraph of Section 5. The same applies to the degree map $1 \to 2 \leftarrow 0$. The degree map $2 \to 0 \leftarrow 1$ defines an inverse categorical structure so that the induced Reedy structure on cospans is the injective model structure and the situation is the same as for $\deg_2$ defined by $1 \to 0 \leftarrow 2$. Therefore, the only relevant model structures of the category of cospans are the model structures $\sigma \in \{\mathrm{inj}, \mathrm{Ree}_{\mathrm{I}}, \mathrm{Ree}_{\mathrm{D}}\}$, which are implemented by the degree maps $1 \to 0 \leftarrow 2$, $0 \to 1 \leftarrow 2$ and $2 \to 1 \leftarrow 0$ and which we explored in detail above.

Definition 5.1. *Let* $\mathtt{I}$ *be the category* $\{c \to d \leftarrow b\}$, *let* $\mathtt{M}$ *be a model category and let* σ *be a model structure on the category* $\mathtt{Fun}(\mathtt{I},\mathtt{M})$ *of cospans of* $\mathtt{M}$ *such that the adjunction* (4.1) *is a Quillen adjunction* $-^* \dashv \mathrm{Lim}$. *From what we said earlier, these model structures are precisely the structures* $\sigma \in \{\mathrm{inj}, \mathrm{Ree}_{\mathrm{I}}, \mathrm{Ree}_{\mathrm{D}}\}$. *For every* $\mathtt{M}$*-cospan* $X = \{C \to D \leftarrow B\}$, *its homotopy limit with respect to* σ

$$\mathbb{R}_\sigma \mathrm{Lim}(X) \approx \mathrm{Lim}(R_\sigma X) \overset{\sim}{\rightleftarrows} \mathrm{Lim}(\tilde{F}_\sigma X) \overset{\sim}{\to} \mathrm{Lim}(F_\sigma X) \tag{5.8}$$

(Theorem 4.1) is referred to as the **homotopy pullback** *of* X *with respect to* σ *and it is denoted*

$$B \times_D^{h_\sigma} C := \mathbb{R}_\sigma \mathrm{Lim}(C \to D \leftarrow B) \ .$$

If $F_{\sigma 1}X$ and $F_{\sigma 2}X$ are two fibrant replacements of X in σ, it follows from (5.8) that there is a span of weak equivalences

$$\mathrm{Lim}(F_{\sigma 1}X) \overset{\sim}{\leftarrow} \cdot \overset{\sim}{\to} \mathrm{Lim}(F_{\sigma 2}X) \ .$$

If $\sigma = \mathrm{Ree}_{\mathrm{I}}$, we get in particular

$$B \times_D^{h_{\mathrm{Ree}_{\mathrm{I}}}} C \approx \mathrm{Lim}(G \twoheadrightarrow H_{\mathrm{f}} \twoheadleftarrow E) \overset{\sim}{\leftarrow} \cdot \overset{\sim}{\to} \mathrm{Lim}(L_{\mathrm{f}} \to M_{\mathrm{f}} \twoheadleftarrow K) \ , \tag{5.9}$$

for every cospans to which X is weakly equivalent. Similarly, if $\sigma = \mathrm{Ree}_{\mathrm{D}}$, we obtain

$$B \times_D^{h_{\mathrm{Ree}_{\mathrm{D}}}} C \approx \mathrm{Lim}(G \twoheadrightarrow H_{\mathrm{f}} \twoheadleftarrow E) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \mathrm{Lim}(P \twoheadrightarrow S_{\mathrm{f}} \leftarrow N_{\mathrm{f}}) \ , \tag{5.10}$$

whenever the cospans considered are replacements of X . Since the first replacement in the last two equations is also fibrant if $\sigma = \mathrm{inj}$, we have

$$B \times_D^{h_{\mathrm{inj}}} C \approx \mathrm{Lim}(G \twoheadrightarrow H_{\mathrm{f}} \twoheadleftarrow E) \ . \tag{5.11}$$

Of course, the standard limit of a cospan is its standard pullback.

Conclusion 5.1. In every model category $\mathtt{M}$ the homotopy pullback of a cospan with respect to $\sigma \in \{\mathrm{inj}, \mathrm{Ree}_{\mathrm{I}}, \mathrm{Ree}_{\mathrm{D}}\}$ is well defined as an isomorphism class of objects of $\mathtt{Ho}(\mathtt{M})$, but is only defined up to a zigzag of weak equivalences if it is viewed as an object of $\mathtt{M}$. All types of fibrant replacement (fibrant C-replacement functor, local fibrant C-replacement, or just any fibrant replacement) provide representatives of the σ-homotopy pullback considered, and this for both interpretations of the homotopy pullback (Kan extension derived functor or strongly homotopy derived functor). In addition, *we can regard* the representatives of a homotopy pullback for the three model structures on cospans (injective model structure, Reedy model structure defined by the increasing labelling, or Reedy model structure defined by the decreasing labelling) as being the same. *In this sense* the homotopy pullback is independent of the model structure on cospans.

What we said above leads to the next theorem which deals with all of the possible indeterminacies in homotopy pullbacks (see (5.9), (5.10), (5.11)).

Theorem 5.1. *The homotopy pullback of a cospan in a model category is independent of the type of derived functor and of the model structure*

$$\sigma \in \{\mathrm{inj}, \mathrm{Ree}_{\mathrm{I}}, \mathrm{Ree}_{\mathrm{D}}\}$$

on cospan diagrams considered. We get canonical representatives of the homotopy pullback from the standard pullback of weakly equivalent cospans with three fibrant objects and at least one morphism that is a fibration: more precisely, if in the adjacent commutative squares

$$\begin{array}{ccccc} C & \longrightarrow & D & \longleftarrow & B \\ \sim\downarrow & & \sim\downarrow & & \downarrow\sim \\ C' & \longrightarrow & D' & \longleftarrow & B' \end{array} \tag{5.12}$$

all vertical arrows are weak equivalences, all bottom nods are fibrant objects and at least one of the bottom arrows is a fibration, we have

$$B \times^h_D C \approx B' \times_{D'} C' \ . \tag{5.13}$$

In other words, we consider the **full homotopy pullback** *(or simply the homotopy pullback)* $B \times^h_D C$ *, whose* **canonical representatives** *are the standard pullbacks of weakly equivalent cospans whose three nods are fibrant and at least one of whose arrows is a fibration.*

6 Models of a homotopy pullback

In this section we generalize the concept of representative of a homotopy limit under the name of homotopy limit model and apply the model notion in particular to homotopy pullbacks. More precisely, the canonical representatives of a full homotopy pullback are the standard pullback of an appropriate weakly equivalent cospan, so that they complete this equivalent cospan into a commutative square. Generalized representatives or models of a full homotopy pullback will be defined as specific objects that complete the original cospan into a commutative diagram.

Let $\mathtt{S}$ be a small category, let $\mathtt{M}$ be a model category and let σ be a model structure on the category $\mathtt{Fun}(\mathtt{S},\mathtt{M})$ such that the adjunction

$$-^* : \mathtt{M} \rightleftarrows \mathtt{Fun}_\sigma(\mathtt{S},\mathtt{M}) : \mathrm{Lim}$$

is a Quillen adjunction $-^* \dashv \mathrm{Lim}$. If $X \in \mathtt{Fun}(\mathtt{S},\mathtt{M})$ and $F_\sigma X$ is a fibrant replacement

$$t_{F_\sigma X} \circ f_X : X \xrightarrow{\sim} F_\sigma X \twoheadrightarrow *$$

of X , the universal morphism

$$\mathrm{Lim}(f_X) : \mathrm{Lim}\, X \to \mathrm{Lim}(F_\sigma X)$$

from the limit $\mathrm{Lim}\, X$ of X to the representative $\mathrm{Lim}(F_\sigma X)$ of the homotopy limit $\mathbb{R}_\sigma \mathrm{Lim}(X)$ of X is usually not a weak equivalence.

Definition 6.1. *Let* $\mathtt{S}$, $\mathtt{M}$ *and* σ *be as above, and let* $A \in \mathtt{M}$, $X \in \mathtt{Fun}(\mathtt{S},\mathtt{M})$ *and*

$$\alpha \in \mathrm{Hom}_{\mathtt{Fun}(\mathtt{S},\mathtt{M})}(A^*, X) \cong \mathrm{Hom}_{\mathtt{M}}(A, \mathrm{Lim}\, X) \ni \mathrm{Lim}\, \alpha \ .$$

We say that A is a **generalized representative** *of the σ*-**homotopy limit** *of X or is a σ-homotopy limit* **model** *of X, if there exists a fibrant replacement $F_\sigma X$ of X such that the composite of universal morphisms*

$$\mathrm{Lim}(f_X) \circ \mathrm{Lim}\,\alpha : A \to \mathrm{Lim}\,X \to \mathrm{Lim}(F_\sigma X)$$

is a weak equivalence.

Proposition 6.1. *If the condition in Definition 6.1 is satisfied for one fibrant replacement, it holds also for every other fibrant replacement.*

Proof. Let $F'_\sigma X$ be another fibrant replacement of X and let $\tilde{F}_\sigma X$ be a fibrant C-replacement:

$$\begin{array}{ccccc} & & A^* & & \\ & & \downarrow{\scriptstyle \alpha} & & \\ & & X & & \\ & {\scriptstyle f_X}\swarrow & \downarrow{\scriptstyle \tilde{f}_X} & \searrow{\scriptstyle f'_X} & \\ F_\sigma X & \xleftarrow[\ell_\sigma]{} & \tilde{F}_\sigma X & \xrightarrow[\ell'_\sigma]{} & F'_\sigma X \end{array} \tag{6.1}$$

Recall that the liftings ℓ_σ and ℓ'_σ in the previous commutative triangles are weak equivalences since $f_X, \tilde{f}_X$ and f'_X are (see (3.5)). As

$$\mathrm{Lim}(f_X) \circ \mathrm{Lim}\,\alpha = \mathrm{Lim}(f_X \circ \alpha) = \mathrm{Lim}(\ell_\sigma \circ \tilde{f}_X \circ \alpha) = \mathrm{Lim}(\ell_\sigma) \circ \mathrm{Lim}(\tilde{f}_X \circ \alpha)\ , \tag{6.2}$$

it follows from (4.3) that $\mathrm{Lim}(\tilde{f}_X \circ \alpha)$ is a weak equivalence, and it follows from (6.2) written for f'_X and ℓ'_σ and from (4.3) that $\mathrm{Lim}(f'_X) \circ \mathrm{Lim}\,\alpha$ is a weak equivalence. □

In the special case of the homotopy pullback the category $\mathtt{S}$ is $\mathtt{I} = \{c \to d \leftarrow b\}$ and X is a cospan $\{C \to D \leftarrow B\}$. The natural transformation α is made of adjacent commutative squares whose top row $A \to A \leftarrow A$ contains two copies of id_A, or, better, is made of a single commutative square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array} \tag{6.3}$$

and $\operatorname{Lim}\alpha$ is the universal morphism $A \to B \times_D C$. The replacement $F_\sigma X$ is a fibrant cospan $C' \to D' \leftarrow B'$ to which $C \to D \leftarrow B$ is weakly equivalent; its pullback $B' \times_{D'} C'$ is a representative of $B \times_D^{h_\sigma} C$. The composite $\operatorname{Lim}(f_X) \circ \operatorname{Lim}\alpha$ of universal morphisms is the universal morphism $A \to B \times_D C \to B' \times_{D'} C'$ from A to the representative of $B \times_D^{h_\sigma} C$ considered. Hence, the definition (6.1) becomes:

Definition 6.2. *Let* $\mathtt{M}$ *be a model category and let* σ *be a model structure on the category of cospans of* $\mathtt{M}$ *such that* $-^* \dashv \operatorname{Lim}$ *is a Quillen adjunction. The vertex* A *of a commutative square* (6.3) *is a* **model or generalized representative** *of the* σ**-homotopy pullback** $B \times_D^{h_\sigma} C$ *if there exists a fibrant replacement* $C' \to D' \leftarrow B'$ *of* $C \to D \leftarrow B$ *in* σ *such that the universal morphism* $A \to B' \times_{D'} C'$ *from* A *to the representative of* $B \times_D^{h_\sigma} C$ *considered, is a weak equivalence.*

The condition in Definition 6.2 is satisfied for every fibrant replacement in σ if it is satisfied for one of them. As mentioned in the proof of Proposition 6.1, this independence of the replacement is due to (4.3), therefore it is a consequence of the fact that the limit functor preserves weak equivalences between fibrant objects; so it is ultimately a consequence of the assumption that $-^* \dashv \operatorname{Lim}$ is a Quillen adjunction.

Given the remark 5.1, we can restrict ourselves to the model structures $\sigma \in \{\mathrm{inj}, \mathrm{Ree_I}, \mathrm{Ree_D}\}$, so that the definition is not only independent of the replacement, but also of the model structure in which this replacement is chosen:

Theorem 6.1. *The vertex* A *of a commutative square* (6.3) *in a model category is a* **model** *of the* **full homotopy pullback** $B \times_D^h C$ *if the universal morphism from* A *to a* canonical representative *of* $B \times_D^h C$ *is a weak equivalence. In other words, there must exist a cospan* $C' \to D' \leftarrow B'$ *to which* $C \to D \leftarrow B$ *is weakly equivalent, whose three nodes are fibrant objects and at least one of whose morphisms is a fibration, such that the universal morphism* $A \to B' \times_{D'} C'$ *is a weak equivalence.*

Proof. If the condition is satisfied for a fibrant replacement in one of the three model structures, it is satisfied for all the fibrant replacements in this model structure and in particular for the replacements of the type $G \twoheadrightarrow H_\mathrm{f} \twoheadleftarrow E$. Hence, it is also satisfied for all the fibrant replacements in any of the other two model structures (see (5.9),(5.10),(5.11)). □

Remark 6.1. We just showed that if the condition in Theorem 6.1 is satisfied for one replacement with three fibrant nodes and at least one fibration, it is satisfied for all replacements of this type. In other words, the concept of model is compatible with our identification of the homotopy pullbacks with respect to $\sigma \in \{\mathrm{inj}, \mathrm{Ree_I}, \mathrm{Ree_D}\}$.

Remark 6.2. The definition of homotopy pullbacks varies from author to author. For example, in [11], the authors define a homotopy pullback as the value of the right derived limit functor for the *injective* model structure on the category of cospan diagrams. In [22] the author *fixes* functorial TrivCof-Fib and Cof-TrivFib factorization systems (α, β) and (α', β'), respectively, and uses *framings* in dealing with homotopy limits. In [21] the author works in a *right proper* model category, *fixes* a factorization system (α, β) and defines the homotopy pullback of a cospan $C \to D \leftarrow B$ with mappings g and f as the standard pullback of the cospan $\beta(g)$ and $\beta(f)$. Note that the latter cospan is *not a fibrant replacement* of the original cospan, neither in the injective model structure on cospans, nor for $\mathrm{Ree_I}$ or $\mathrm{Ree_D}$. For this specific definition, the standard pullback of $C \to D \leftarrow B$ is also its homotopy pullback if either arrow g or f is a fibration. Note that none of the three objects need be fibrant. In [24] the homotopy pullback is computed using a replacement of the considered cospan whose central object is fibrant and whose two morphisms are fibrations. A generalization of homotopy pullbacks is defined under the name of homotopy pullback square. The claim is made that a standard pullback square is also a homotopy pullback square if one of the morphisms of the cospan is a fibration and either all three of its objects are fibrant or the underlying model category is right proper. No proof is provided and no mention is made of the model structure on cospans considered. In [30] the 'global' definition of [11] is juxtaposed with a 'local' definition which is equivalent only if all three objects of the cospan are fibrant. The same two sufficient conditions for a standard pullback to be a homotopy pullback are stated as in [24] and the one valid in a right proper model category is proved independently in a rather involved way. This sufficient condition is intuitively justified by referring to the possibility of using *the* Reedy model structure on cospans to compute the homotopy pullback. However, if the limit functor's source category model structure is changed, we change its right derived functor, i.e. in our case we change the homotopy pullback functor.

In the present paper, we consider the *full* homotopy pullback and its *generalized representatives or models* - thus including the injective and the *two* relevant Reedy model structures on cospans - and prove that the known results are valid. In particular, the one that holds in right proper model categories does not require an independent complex proof, but is merely a consequence of the general result valid in each model category. Hence, in what follows, the reader will not find anything really new – what he does find is a new, structured, linear presentation that rigorously embeds all the different possible choices and all the known, sometimes somewhat handswavingly accepted outcomes into a homogeneous, compact and (hopefully) clear explanatory text.

Note that we have worked in *any* model category so far. If the model category is *right proper*, the model condition of Definition 6.1 simplifies and we recover the well-known

Theorem 6.2. *The vertex A of a commutative square* (6.3) *in a* right proper *model category is a model (in the sense of Definition 6.1) of the full homotopy pullback $B \times^h_D C$ if there exists a cospan $C' \to D' \leftarrow B'$ to which $C \to D \leftarrow B$ is weakly equivalent and at least one of whose morphisms is a fibration, such that the universal morphism $A \to B' \times_{D'} C'$ is a weak equivalence.*

Lemma 6.1. *Let* $\mathtt{M}$ *be a* right proper *model category and let $f : A \to D$, $g : B \to C$ and $h : C \to D$ be morphisms in* $\mathtt{M}$. *The pullbacks $A \times_D B$ and $A \times_D C$ exist and there is a universal morphism $u : A \times_D B \to A \times_D C$. If $f : A \twoheadrightarrow D$ and $g : B \overset{\sim}{\to} C$, we have $u : A \times_D B \overset{\sim}{\dashrightarrow} A \times_D C$:*

$$\begin{array}{ccccc} A \times_D B & \overset{\sim \quad u}{\dashrightarrow} & A \times_D C & \longrightarrow & A \\ \downarrow & & \downarrow k & & \downarrow f \\ B & \xrightarrow{\sim \quad g} & C & \xrightarrow{h} & D \end{array} \tag{6.4}$$

Proof. This lemma is well known and will not be proved again here. □

Proof of Theorem 6.2. Assume that $C' \overset{g}{\to} D' \overset{f}{\leftarrow} B'$ is a replacement of $C \to D \leftarrow B$ and that one of its morphisms is a fibration, for instance the *second* one. If we apply a fibrant C-replacement functor R to $C' \overset{g}{\to} D' \overset{f}{\twoheadleftarrow} B'$ and decompose the *first* arrow $RC' \overset{Rg}{\to} RD'$ into $RC' \overset{\sim}{\to} F(Rg) \twoheadrightarrow RD'$, we get the commutative diagram

$$\begin{array}{ccccc} & & A & & \\ & \swarrow & & \searrow & \\ C & \longrightarrow & D & \longleftarrow & B \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ C' & \xrightarrow{g} & D' & \overset{f}{\twoheadleftarrow} & B' \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ RC' & \xrightarrow{Rg} & RD' & \xleftarrow{Rf} & RB' \\ \downarrow \sim & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} \\ F(Rg) & \twoheadrightarrow & RD' & \xleftarrow{Rf} & RB' \end{array} \tag{6.5}$$

From the 2-out-of-3 axiom it follows that there is a universal weak equivalence

$$C' \overset{\sim}{\dashrightarrow} D' \times_{RD'} F(Rg) ,$$

as the model category is right proper. In view of Lemma 6.1, we have now universal weak equivalences

$$B' \times_{D'} C' \overset{\sim}{\dashrightarrow} B' \times_{D'} (D' \times_{RD'} F(Rg)) \cong B' \times_{RD'} F(Rg) \overset{\sim}{\dashrightarrow} RB' \times_{RD'} F(Rg) .$$

Hence, the universal morphism

$$A \dashrightarrow B' \times_{D'} C' \tag{6.6}$$

is a weak equivalence if and only if the universal morphism

$$A \dashrightarrow RB' \times_{RD'} F(Rg) \tag{6.7}$$

is a weak equivalence. Since the cospan $F(Rg) \twoheadrightarrow RD' \leftarrow RB'$ is weakly equivalent to $C \to D \leftarrow B$, has three fibrant nodes and at least one of its morphisms is a fibration, the vertex A of the square (6.3) is a model of $B \times^h_D C$, if (6.6) is a weak equivalence. □

Proposition 6.2. *If the condition in Theorem 6.2 is satisfied for one replacement with one fibration it is satisfied for all replacements of this type.*

Proof. We see from Equations (6.6) and (6.7) that the condition is satisfied for a given replacement with one fibration if and only if it is satisfied for an associated replacement with three fibrant nodes and one fibration. However, from Remark 6.1 we know that if the condition is satisfied for one replacement of the latter type it is satisfied for all replacements of this type. □

Remark 6.3. Theorem 6.2 shows that our definition of a homotopy pullback model generalizes the definition in a right proper model category.

The following corollary is stated without proof in [24]:

Corollary 6.1. *In a model category the standard pullback $B \times_D C$ of a cospan $C \xrightarrow{g} D \xleftarrow{f} B$ is a homotopy pullback if at least one of the morphisms f or g is a fibration and either all three objects B, C, D are fibrant or the model category is* right proper*.*

Proof. Under the stated conditions $B \times_D C$ is a model of $B \times^h_D C$. Indeed, if the model category is right proper (resp., B, C and D are fibrant), the cospan $C \to D \leftarrow B$ is a replacement of itself, one of its morphisms is a fibration (resp., and all its nodes are fibrant), and the universal morphism $\mathrm{id} : B \times_D C \dashrightarrow B \times_D C$ is a weak equivalence. The conclusion now follows from Theorem 6.2 (resp., Theorem 6.1). □

Remark 6.4. The concept of model of a homotopy pullback is actually a unifying approach that captures not only the notion of homotopy pullback that is used in [24] (Corollary 6.1) but also the notion of homotopy fiber square that is defined in right proper model categories equipped with a fixed functorial factorization system in [21] (Corollary 6.2).

Let (α, β) be a **f**ixed **f**unctorial trivial cofibration - fibration **f**actorization system (FFF for short) of a model category and let $C \xrightarrow{g} D \xleftarrow{f} B$ be a cospan. The system considered provides decompositions

$$C \overset{\sim}{\rightarrowtail} \Xi(g) \twoheadrightarrow D \twoheadleftarrow \Xi(f) \overset{\sim}{\leftarrowtail} B$$

[21, Definition 13.3.12].

Definition 6.3. *Let* $\mathtt{M}$ *be a* right proper *model category with an FFF. A commutative square*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow f \\ C & \xrightarrow{g} & D \end{array} \tag{6.8}$$

is a **homotopy fiber square** *if the universal morphism* $A \dashrightarrow \Xi(f) \times_D \Xi(g)$ *is a weak equivalence.*

Corollary 6.2. *In a* right proper *model category with an FFF, the vertex* A *of a commutative square* (6.8) *is a model of the homotopy pullback* $B \times^h_D C$ *if and only if it is a homotopy fiber square.*

Proof. In view of Proposition 6.2, since the second row in the commutative diagram

$$\begin{array}{ccccc} & & A & & \\ & \swarrow & & \searrow & \\ C & \xrightarrow{g} & D & \xleftarrow{f} & B \\ \downarrow \sim & & \downarrow \mathrm{id} & & \downarrow \sim \\ \Xi(g) & \twoheadrightarrow & D & \twoheadleftarrow & \Xi(f) \end{array} \tag{6.9}$$

is a replacement of the first row with at least one fibration, the vertex A of the commutative triangle or square is a model of the homotopy pullback $B \times^h_D C$ if and only if the universal morphism $A \dashrightarrow \Xi(f) \times_D \Xi(g)$ is a weak equivalence, i.e., if and only if the square is a homotopy fiber square. □

Remark 6.5. Our philosophy has been to refer to the upper left vertex of a commutative square as a model for the homotopy pullback of the square's cospan when the universal morphism from it to a canonical representative of the homotopy pullback is a weak equivalence. In view of Corollary 6.2 and Definition 6.3 it makes therefore sense to regard the standard pullback $\Xi(f)\times_D \Xi(g)$ as a representative of $B\times^h_D C$, provided the underlying model category is right proper and equipped with an FFF. Actually the homotopy pullback $B\times^h_D C$ is defined in [21] as being this representative. If the lower right vertex of the square is fibrant, the homotopy pullback of [21], which is well defined as an object of the model category, is a canonical representative of our full homotopy pullback, which is only defined up to a zigzag of weak equivalences.

Next we prove that there is a **pasting law for model squares** in any model category. This result generalizes Proposition 13.3.15 of [21].

Proposition 6.3. *Let*

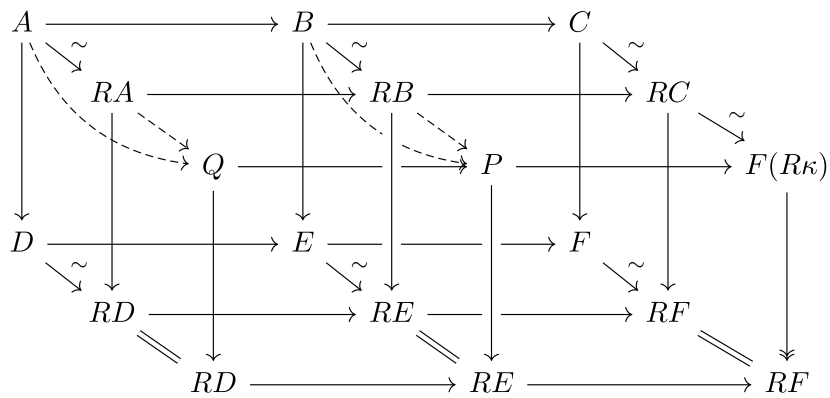
$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ D & \longrightarrow & E & \longrightarrow & F \end{array} \tag{6.10}$$

be a commutative diagram in a model category. If the right square is a model square, i.e., if B is a model of the homotopy pullback $C\times^h_F E$, then the left square is a model square if and only if the total square is a model square.

Proof. We apply a fibrant C-replacement functor R to the commutative diagram (6.10) and factor the morphism

$$R(C \xrightarrow{\kappa} F) = RC \xrightarrow{R\kappa} RF = RC \xrightarrow{\sim} F(R\kappa) \twoheadrightarrow RF$$

into a weak equivalence followed by a fibration. Moreover, we set $P := F(R\kappa)\times_{RF} RE$ and $Q := P\times_{RE} RD$ and thus get the following commutative diagram:



(6.11)

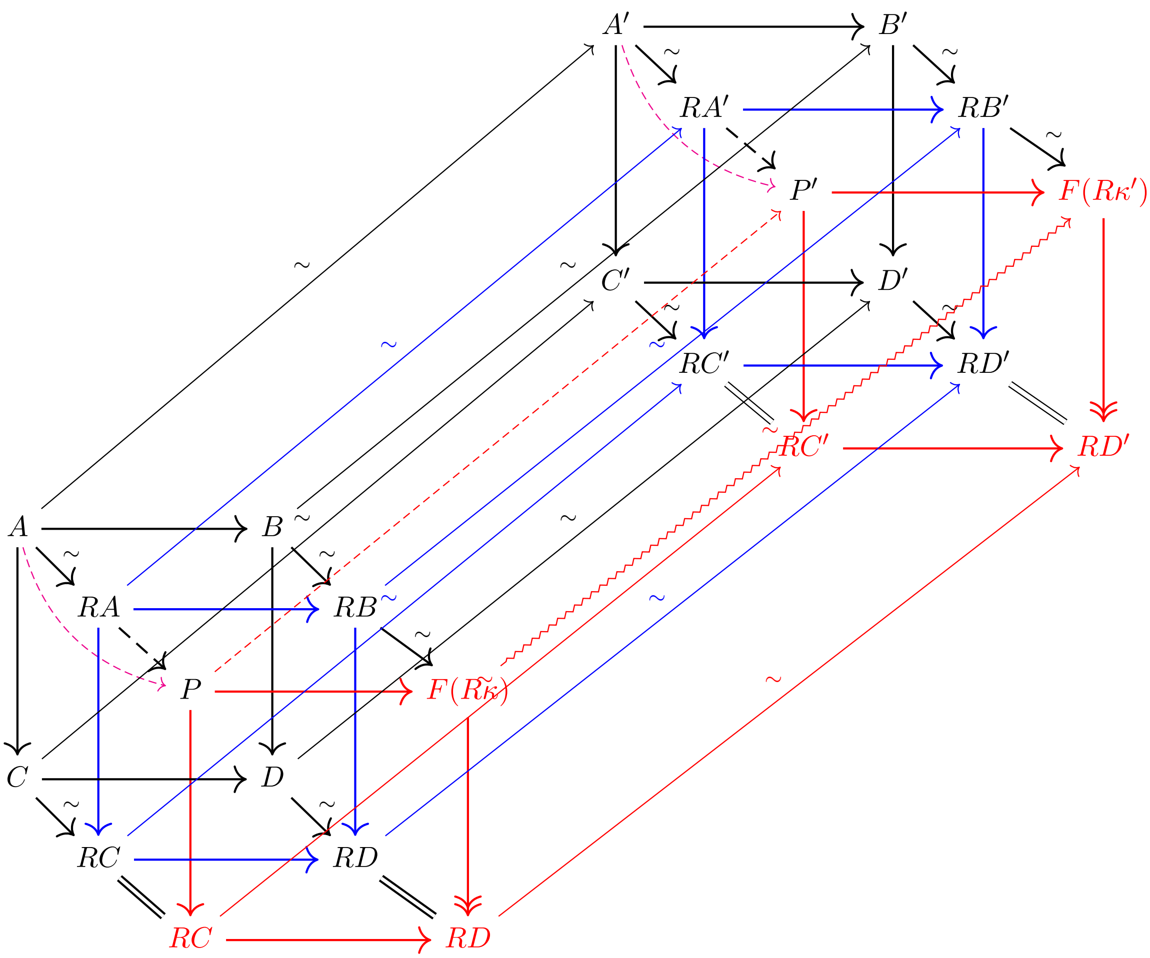
As the universal arrow $B \dashrightarrow P$ (resp., $A \dashrightarrow Q$) is the unique arrow $B \to P$ (resp., $A \to Q$) that makes the corresponding triangles commutative, this arrow coincides with the composite $B \overset{\sim}{\to} RB \dashrightarrow P$ (resp., $A \overset{\sim}{\to} RA \dashrightarrow Q$). Since the right square of (6.10) is a model square, the universal arrow $B \dashrightarrow P$ is a weak equivalence in view of Remark 6.1, and therefore the universal arrow $RB \dashrightarrow P$ is a weak equivalence. In view of closeness of fibrations under pullbacks the arrow $P \to RE$ is a fibration. From here it follows that the left square in (6.10) is a model square if and only if

$$A \dashrightarrow Q = P \times_{RE} RD \cong F(R\kappa) \times_{RF} RD$$

is a weak equivalence, which is the case if and only if the total square of (6.10) is a model square. □

The next result is valid for homotopy fiber squares [21, Proposition 13.3.14] in a right proper model category with an FFF. We prove that it holds also for model squares in an arbitrary model category.

Proposition 6.4. *Let $ABCD$ and $A'B'C'D'$ be two commutative squares in a model category* $\mathtt{M}$*. If there exist four* $\mathtt{M}$*-morphisms from the vertices of the first square to the corresponding vertex of the second such that the four resulting squares commute and if these* $\mathtt{M}$*-morphisms are weak equivalences, then the first square is a model square if and only if the second is.*



(6.12)

Proof. First we apply a fibrant C-replacement functor R to the commutative parallelepiped, which is described in the statement of Proposition 6.4. Then we factor the morphism

$$R(B \xrightarrow{\kappa} D) = RB \xrightarrow{R\kappa} RD = RB \xrightarrow{\sim} F(R\kappa) \twoheadrightarrow RD$$

into a weak equivalence followed by a fibration and proceed analogously for $R(B' \xrightarrow{\kappa'} D')$, using a *functorial factorization system*. We also set $P := F(R\kappa) \times_{RD} RC$ and $P' := F(R\kappa') \times_{RD'} RC'$. Since the factorization system used is functorial, we get an arrow $F(R\kappa) \overset{\sim}{\rightsquigarrow} F(R\kappa')$, and thus a universal arrow $P \dashrightarrow P'$. Finally, we have the commutative diagram (6.12) (see above).

The two commutative parallelograms with four red vertices in (6.12) are a weak equivalence from the $\mathrm{Ree}_{\mathrm{I}}$-fibrant cospan $RC \to RD \twoheadleftarrow F(R\kappa)$ to the $\mathrm{Ree}_{\mathrm{I}}$-fibrant cospan $RC' \to RD' \twoheadleftarrow F(R\kappa')$. Since the limit or pullback functor transforms weak equivalences between $\mathrm{Ree}_{\mathrm{I}}$-fibrant cospans into weak equivalences, the universal arrow $P \dashrightarrow P'$ is a weak equivalence. As the square $ABCD$ (resp., $A'B'C'D'$) is a model square if and only if the universal arrow $A \dashrightarrow P$ (resp., $A' \dashrightarrow P'$) is a weak equivalence, it follows that $ABCD$ is a model square if and only if $A'B'C'D'$ is a model square. □

Remark 6.6. Proposition 6.3, Proposition 6.4 and the concept of model square are indispensable building blocks of our papers [14] and [15], which in turn are part of a larger project on PDEs and their symmetries (for more details, see Section 7).

7 Concluding remarks

Building on ideas from works by Beilinson, Costello, Drinfeld, Gwilliam, Schreiber, Paugam, Toën, Vezzosi, and Vinogradov [1, 4, 32, 33, 37, 38, 39], Di Brino and two of the authors of the present paper have introduced derived algebraic geometry over the ring $\mathcal{D}$ of differential operators of an underlying affine scheme, as a suitable framework for investigating the solution space of a system of partial differential equations up to symmetries [5, 6, 34]. The implementation of the associated research program requires in particular that the tuple

$$(\mathtt{DG}\mathcal{D}\mathtt{M}, \mathtt{DG}\mathcal{D}\mathtt{M}, \mathtt{DG}\mathcal{D}\mathtt{A}, \tau, \mathbf{P})$$

be a homotopical algebraic geometric context (HAGC) in the sense of [38], where $\mathtt{DG}\mathcal{D}\mathtt{M}$ is the symmetric monoidal model category of differential graded $\mathcal{D}$-modules, the subcategory $\mathtt{DG}\mathcal{D}\mathtt{A}$ is the model category of differential graded $\mathcal{D}$-algebras, τ is an appropriate model pretopology on the opposite category of $\mathtt{DG}\mathcal{D}\mathtt{A}$ and $\mathbf{P}$ is a compatible class of morphisms. The (really) challenging proof of this 'HAGC theorem' is based on a new simplified perspective on the concept of homotopy fiber sequence [36] and a generalization of the long exact sequence of Puppe. Using the notion of model, model square or homotopy fiber square in any model category, which we have introduced and studied in the present work so that it now stands on a solid mathematical basis, we were able to develop a novel approach to model categorical homotopy fiber sequences and to generalize Puppe's sequence [14].

We now give some details on this application of models of homotopy pullbacks. In [14] we work in a general pointed model category $(\mathtt{M}, 0)$, we define a loop space functor Ω from an arbitrary 'dual cone functor' and define homotopy fiber sequences as commutative $\mathtt{M}$–squares

(A, B, C, D) such that A is a model of the homotopy pullback of $C \to D \leftarrow B$ (in the sense of the present work) and the map $C \to 0$ is a weak equivalence. Further, for every morphism $f : F \to \mathcal{F}$ between fibrant objects we define its homotopy fiber K_f such that $K_f \to F \to \mathcal{F}$ is a homotopy fiber sequence (in the sense of [14]). We get a universal connecting morphism $\Omega\mathcal{F} \to K_f$ such that $\Omega\mathcal{F} \to K_f \to F$ is also a homotopy fiber sequence. It turns out that Quillen's loop space functor Ω^Q (see [36]) is a loop-space functor Ω in our sense. Furthermore, an objectwise fibrant homotopy fiber sequence in our sense is a fibration sequence in the sense of Quillen (see [36]) and our universal connecting morphism is the same as Quillen's connecting morphism (see again [36]) induced by an action of the group object $\Omega^Q\mathcal{F}$ on K_f. Although all of this shows that the two theories are closely related, the new approach to homotopy fiber sequences or fibration sequences does not rely on the additional structure of an action. The point is that we use the homotopy theory of the category $\mathtt{M}^{\to}$ of $\mathtt{M}$–morphisms, which contains all relevant information about homotopy fiber sequences of $\mathtt{M}$.

It follows that it is much easier to apply the new concept of homotopy fiber sequence. For example, the proof of the 'HAGC theorem' mentioned above involves proving that in our homotopical $\mathcal{D}$-geometric environment, flat (resp., étale) morphisms are the same as strongly flat (resp., strongly étale) ones. This proof not only requires a handy concept of homotopy fiber sequence, but in addition it requires that Quillen's Tor spectral sequence – which connects the graded Tor functor in homology with the homology of the derived tensor product of two differential graded $\mathcal{D}$-modules over a differential graded $\mathcal{D}$-algebra – is valid in the derived $\mathcal{D}$-geometric world. These partly subtle results were proved in [15] (the first part of which is already available online). We expect being able to combine all the mentioned results to complete the proof of the 'HAGC theorem', to prove that solid concepts of derived stack and geometric derived stack do exist in homotopical $\mathcal{D}$-Geometry, and thus to make an important step towards the full implementation of the mentioned 'PDEs and Symmetries program'. Furthermore, we are convinced that our approach to model squares and homotopy fiber sequences can explain the '(non-)functoriality of the cone' in triangulated categories resulting from model categories, without resorting to the theory of derivators [17, 19, 20, 18].

Acknowledgements

The authors cordially thank the anonymous referee for valuable remarks and suggestions, which have served to improve this article.

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