Insider's problem in the trinomial model: a discrete jump process point of view

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Abstract

In an incomplete market underpinned by the trinomial model, we consider two investors: an ordinary agent whose decisions are driven by public information and an insider who possesses from the beginning a surplus of information encoded through a random variable for which he or she knows the outcome. Through the definition of an auxiliary model based on a marked binomial process, we handle the trinomial model as a volatility one, and use the stochastic analysis and Malliavin calculus toolboxes available in that context. In particular, we connect the information drift, i.e. the drift to eliminate in order to preserve the martingale property within an initial enlargement of filtration in terms of Malliavin's derivative. We solve explicitly the agent and the insider expected logarithmic utility maximization problems and provide a Ocone-Karatzas type formula for replicable claims. We identify insider's expected additional utility with the Shannon entropy of the extra information, and examine then the existence of arbitrage opportunities for the insider.

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1 Introduction

The issue of insider trading in the trinomial model we have chosen to raise lies at the interface of several fundamental problems in finance.

First, it takes place among trading problems where investors act with different levels of information. This is embodied here by an ordinary agent and an insider who has confidential information (not available in the public flow) from which he or she benefits when carrying out financial transactions. Both investors are small enough not to influence market prices. From the point of view of martingale theory we shall take in this paper, this extra information is hidden in a random variable G of which the insider knows the outcome at the beginning of the trading interval, so that the insider's filtration $\mathcal{G}$ is larger than $\mathcal{F}$, the ordinary agent's. This leads back to *enlargement of filtration* whose first investigations in the 1970s targeted several aims: to exhibit the condition(s) under which a $\mathcal{F}$-martingale is a $\mathcal{G}$-semimartingale, to explain the intrinsic relationship between the enlargement of filtration and Girsanov's theorem (see for instance Föllmer and Imkeller [28], Song [61]), and to build a unified methodology to cover most of models involved in the issue. They spawned two main models that have been extensively discussed since in the literature: the *initial enlargement* model under Jacod's hypothesis

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which assumes the equivalence between the conditional laws of G with respect to $\mathcal{F}$ and the law of G (see Jacod [38]), and the *progressive enlargement* model (where G is $\mathbb{Z}_+$-valued and $\mathcal{G}$ is the smallest filtration satisfying usual conditions and making G a stopping time) with honest times (see Barlow [10], Jeulin and Yor [40]). Besides, all related results extend immediately in a discrete time setting as highlighted by Blanchet-Scalliet, Jeanblanc and Romero in [13], most of them simply stemming from Doob's decomposition. For a comprehensive review of the deep results on the enrichment of filtrations the reader can refer to the lecture notes of Mansuy and Yor [44] and with a view to financial purposes in the book of Aksamit and Jeanblanc [5].
Indeed, the theory has enjoyed a significant revival since the 2000s because of its applications in financial mathematics and notably to deal with the insider problem. New questions have more recently emerged in this frame: how to quantify insider's additional expected utility? Does the extra information produce an arbitrage i.e. enables the insider to make profit by deploying well-chosen $\mathcal{G}$-adapted strategies? Following the pioneer work of Pikovsky and Karatzas in [49], J. Amendinger et al. through [7], [6], A. Grorud and M. Pontier in [32] precise criterions for optimization and compute the expected additional utility of the insider in their respective works. In [8] and [9], Imkeller *et al.* exhibit a crucial link between this quantity and the information theory by identifying it with the Shannon entropy of the extra information. Imkeller connects it to Malliavin calculus in [37] by expressing the *information drift* as the logarithmic Malliavin trace of a conditional density characterizing insider's advantage. The existence or not of arbitrage is another core question. An overview of the question of arbitrage and its complementary notion of No Free Lunch (with Vanishing Risk) - NFL(VR) for short - in continuous time is given in the book of Delbaen and Schachermayer [26] and in a discrete setting in Dalang, Morton and Willinger [24]. Amendinger proves in [6] that under Jacod's hypothesis, insider's model is arbitrage free. Many papers deal with arbitrage opportunities under initial or progressive filtration enlargement (see the recent works of Aksamit *et al.* [3], [4], Acciaio *et al.* [1], Choulli *et al.* [2]) and we can cite the work of Chau, Runggaldier and Tankov [20] in an incomplete market. Most of the aforementioned works take shape in continuous-time settings; they are fewer in a discrete one. In [12], Blanchet-Scalliet, Hillairet and Jiao dynamically model insider's extra information flow through a successive enlargement of filtrations and put their working density hypotheses into perspective in regard to Jacod's criterion. In [21] Choulli and Deng set up the necessary and equivalent conditions on a not public information G - incorporated in the market through a progressive enlargement of filtration - to preserve the non-arbitrage market. In the different frame of discrete models with *uncertainty* i.e. without single probability reference measure, the classical notions of non-arbitrage or NFLVR no longer perform. Burzoni, Frittelli and Maggis give in [15] a sense to the notion of arbitrage in that context and construct a universal arbitrage aggregator consisting of all trading strategies which are arbitrages (in the classical sense) with respect to some probability measure in the model.

On another side, insider trading is related to portfolio management. In a *complete* market, all claims are reachable. The books of Shreve [60] and Pascucci and Runggaldier [48] provide a comprehensive overview of the existing results in a discrete setting and notably for the most famous of them, the *Cox-Ross-Rubinstein* (CRR for short), also called *binomial* model and widely investigated for years (see the seminal works of Cox, Ross and Rubinstein [22], Rendleman and Bartter[54]). That latter is a complete market, and so-called Fundamental Asset Pricing Theorem (FAPT) asserts there exists a unique probability measure, equivalent

to the initial under which the discounted price process is a $\mathcal{F}$-martingale (in a discrete setting see Schachermayer [56], Jacod and Shiryaev [39]). In that frame, Privault exhibits an explicit formula of the replicating strategy in terms of the Malliavin derivative (on the Rademacher space) from the application of Clark-Ocone formula (see chapter 1 in Privault [50] or the book [51]).
In contrast, the trinomial model which hosts our topic is an *incomplete* market, and not all claims are redundant. Several routes have been explored for years to get around this problem. A method (see Karatzas *et al.* [41]) consists in introducing additional fictitious stocks so that the optimal portfolio obtained in the so completed market coincides with the optimal portfolio in the original incomplete one. When the incompleteness comes from the existence of transaction costs (or other kind of *friction*), super-replication ensures the full hedging of risk. The question of superhedging is explored in discrete-time markets for instance in Deparis and Martini, [27], Bouchard and Nutz [14], Burzoni *et al.* [16], Obloj *et al.* [17], [47], and Carassus and Vargiolu [18] [19]. From another point of view, there exists on incomplete markets no-redundant claims that carry an *intrinsic risk*; thus, optimizing a portfolio means minimizing this risk as introduced by Föllmer and Sondermann in [30]. In that case the minimizing-risk strategy can be constructed using the Kunita-Watanable projection technique with respect to the initial reference measure (in the martingale case) or to the so-called *minimal martingale measure* (see Föllmer and Schweizer [29], Schweizer [58]). In the discrete setting, Schweizer introduces in [59] the *variance optimal signed martingale measure* which coincides with the (discrete version of) minimal one both in the martingale and deterministic mean-variance tradeoff cases to approximate any claim by the total gain from trade (given in terms of a stochastic integral with respect to the stock process) in $\mathcal{L}^2$. From a slightly different perspective, hedger can aim at maximizing his/her expected utility from the terminal wealth for a given utility function. A very popular method lies on functional analysis tools and the formulation of a dual problem; among the extensive literature on the topic, the reader can refer to the survey paper of Schachermayer [57], or the reference book of Delbaen and Schachermayer [26]. In a discrete setting, Rasonyi and Stettner propose an alternative and directly probabilistic approach [53] to state the existence of optimal strategies and non smooth utility functions for possibly unbounded price processes. More recently, this same question of utility optimisation has been directed at markets with other such as friction (see Bouchard and Nutz [14], Neufeld and Sikic [45]) or with uncertainty (see Nutz [46], Rasonyi and Meireles [52], Obloj and Wiesel [47]).

Our contributions address a number of the issues raised above in the specific and under-researched context of the incomplete and discrete-time market embodied by the trinomial model and where insider's advantage is modelled by an initial enlargement of filtration. Providing as good an approximation to the Black-Scholes model (it converges even faster if the payoff is smooth enough, Heston and Zhou [35], Herath and Kumar [34], Lesne *et al.* [43]) as its famous elder CRR-model, the trinomial model is much less studied and the literature on portfolio management in this frame scarcer. We can cite the books of Delbaen and Schachermayer [26] or Björefeldt *et al.* [11], the survey of Runggaldier [55], the works of Dai and Lyuu [23], Glonti et al. [31]. Our approach differs radically from what has been done before in this context. Indeed, our key starting point is to consider an alternative model called *ternary model*, equivalent to the trinomial, and that does not rely on a $\{-1, 0, 1\}$-valued process but on a discrete compound jump process. Thus we can benefit from the stochastic analysis and Malliavin's toolbox for binomial marked point processes developed in Halconruy [33] of which this paper is the companion. At the very beginning, this formalism was in fact introduced to get around the impossibility to state a Ocone-Karatzas formula from the Clark formula for independent random variables stated by Decreusefond and Halconruy ([25], Theorem 3.3). Indeed, the $\mathcal{F}_k$-measurability of the integrand prevents the definition of a $\mathcal{F}$-predictable drift process. This observation was prone to replace the trinomial model by a so-named *ternary model*, that is a *volatility*-type model based on a jump process and which ultimately proves to be a more suitable setting not only for stating a Ocone-Karatzas type formula but also to address the insider problem. Within it, we have achieved the (main) results of this paper:

- Theorem 4.1 formalises the filtration enlargement tools i.e. the preservation of semi-martingales via the so-called *drift of information* and the conservation of martingales up to a measure change. Both are drawn up in the frame of stochastic analysis for binomial marked point processes. Furthermore the drift of information is connected with Malliavin's calculus in the same vein as Imkeller [36], [37];
- Theorem 4.4 provides a Ocone-Karatzas type formula for replicable claims;
- Theorem 4.7 gives an explicit computation of the expected additional logarithmic utility of the insider. This can be interpreted as the Shannon entropy of the extra information like in the continuous case (see Amendinger *et al.* [8]);
- Proposition 4.11 answers the question of arbitrage-free model addressed in the sense of Blanchet-Scalliet *et al* [13].

The paper is organized as follows. Section 2 is devoted to preliminaries including the presentation of the ternary model as well as reminders of the stochastic analysis and Malliavin's calculus for binomial marked point processes. The following section deals with the enlargement of filtration in this frame. Section 4 is the application of these tools to address insider's problem in the ternary model and gathers all the main results. Most of the proofs are postponed in section 5.

Notations. Here are some notations that will be used throughout the paper. Any interval of $\mathbb{R}$ is as usual denoted by $[a,b]$, for real numbers a,b such that $a \leqslant b$. On $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, we denote $[\![n,m]\!] = \{n,\dots,m\}$ for any $n,m \in \mathbb{Z}_+$ such that $n \leqslant m$. Given $T \in \mathbb{R}_+$, we define $\mathbb{T} = \mathbb{Z}_+ \cap [0,T]$. Denote also $\mathbb{T}^* = \mathbb{T} \setminus \{0\}$, $\mathbb{T}^\circ = \mathbb{Z}_+ \cap [0,T) = [\![0,T-1]\!]$ and $\mathbb{T}^{*,\circ} = \mathbb{T}^* \cap \mathbb{T}^\circ$. We define $\mathbb{X} = \mathbb{T} \times \mathrm{E}$ where $\mathrm{E} = \{-1,1\}$ and for all $n \in \mathbb{N}$, any n-tuple of $\mathbb{X}^n$ can be denoted by bold letters; for instance, $(\mathbf{t}_n, \mathbf{k}_n) = \big((t_1,k_1),\dots,(t_n,k_n)\big)$ where $(t_i,k_i) \in \mathbb{X}$ for all $i \in \{1,\dots,n\}$.
Given a process $(\mathrm{X}_t)_{t\in\mathbb{N}}$ well defined on a probability space $(\Omega,\mathcal{F},\mathbf{P})$, we let $\Delta\mathrm{X}_t = \mathrm{X}_t - \mathrm{X}_{t-1}$ be the increment of X at time t and we set $\Delta\mathrm{X}_0 = \mathrm{X}_0$.
In the following, $\mathbf{E}$ designates the expectation taken under the reference probability measure $\mathbf{P}$. When dealing with other probability measure $\mathbf{Q}$ on $(\Omega,\mathcal{F})$, the expectation with respect to $\mathbf{Q}$ will be denoted $\mathbf{E}_{\mathbf{Q}}$.

2 Preliminaries

2.1 Trinomial and ternary models

To begin with, we recall the main characteristics of the *trinomial model* underpinning our problem. This embodies a simple financial market modelled by two assets i.e. a couple of $\mathbb{R}_+$-

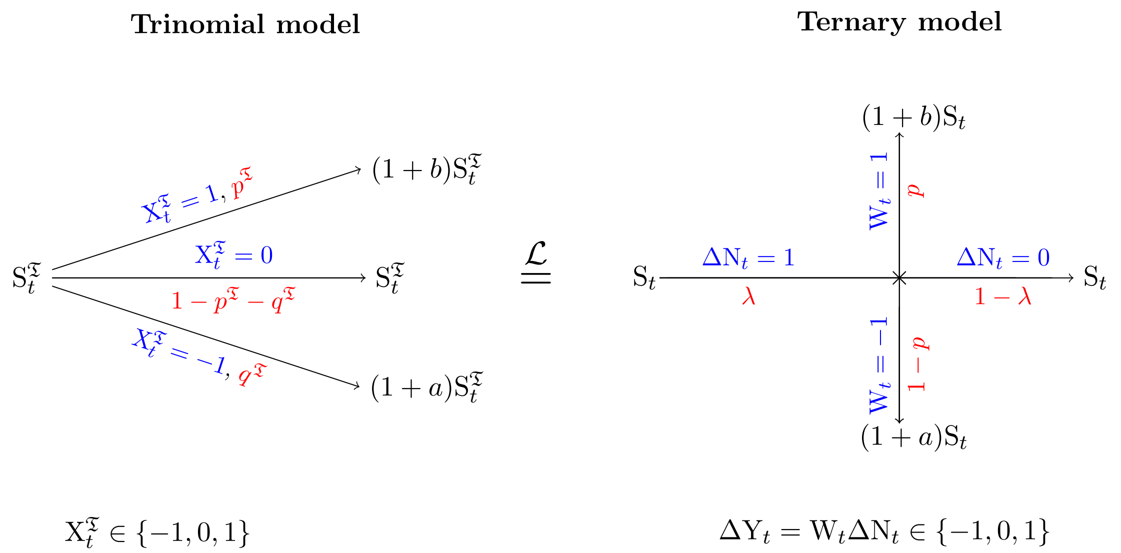
valued processes $(\mathrm{A}_t, \mathrm{S}_t)_{t\in\mathbb{T}}$, defined on the same filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbf{P})$ where $(\mathcal{F}_t)_{t\in\mathbb{T}} =: \mathcal{F}$ is assumed to be generated by the canonical process, $\mathbb{T} := \mathbb{Z}_+ \cap [0, T]$ is the *trading interval* and $T \in \mathbb{N}$ is called the *maturity*. The market model thus defined will be labelled as $(\mathbb{T}, \mathcal{F}, \mathbf{P}, \mathrm{S})$. The *riskless asset* $(\mathrm{A}_t)_{t\in\mathbb{T}}$ is deterministic and is defined for some $r \in \mathbb{R}_+$ (generally smaller than 1) and all $t \in \mathbb{T}$ by

$$\mathrm{A}_t = (1 + r)^t, \tag{2.1}$$

whereas the stock price which models the *risky asset*, is the $\mathcal{F}$-adapted process $(\mathrm{S}^{\mathfrak{T}}_t)_{t\in\mathbb{T}}$ with (deterministic) initial value $\mathrm{S}^{\mathfrak{T}}_0 = 1$ and such that for any $t \in \mathbb{T}^*$,

$$\Delta\mathrm{S}^{\mathfrak{T}}_t = \theta^{\mathfrak{T}}_t\, \mathrm{S}^{\mathfrak{T}}_{t-1}, \tag{2.2}$$

where $\theta^{\mathfrak{T}}_t = b\mathbf{1}_{\{\mathrm{X}^{\mathfrak{T}}_t = 1\}} + a\mathbf{1}_{\{\mathrm{X}^{\mathfrak{T}}_t = -1\}}$, a and b are real numbers such that $-1 < a < r < b$ and $\{\mathrm{X}^{\mathfrak{T}}_t,\, t \in \mathbb{T}^*\}$ is a family of i.i.d. $\{-1, 0, 1\}$-valued random variables such that $\mathbf{P}(\mathrm{X}^{\mathfrak{T}}_t = 1) = \bar{p}$, $\mathbf{P}(\mathrm{X}^{\mathfrak{T}}_t = -1) = \bar{q}$ and $\mathbf{P}(\mathrm{X}^{\mathfrak{T}}_t = 0) = 1 - \bar{p} - \bar{q}$ (with $\bar{p}, \bar{q} \in (0, 1)$). For the technical reasons mentioned in the introduction, we replace the trinomial model with a more computationally amenable one. As suggested in the schema below, this surrogate model is based on a jump process and we name it *ternary model*.



As a matter of fact, the family $\{\mathrm{X}^{\mathfrak{T}}_t,\, t \in \mathbb{T}^*\}$ is replaced by $\{\Delta\mathrm{Y}_t,\, t \in \mathbb{T}^*\}$ such that for any $t \in \mathbb{T}^*$, $\Delta\mathrm{Y}_t = \mathrm{W}_t\Delta\mathrm{N}_t$, where $\{\mathrm{W}_t,\, t \in \mathbb{T}^*\}$ is a family of i.i.d. $\{-1, 1\}$-Bernoulli random variables of probability success p. They will stand for the direction (positive or negative) of the 1-high jumps of a process constructed as follows. On the one hand, consider a binomial process $(\mathrm{N}_t)_{t\in\mathbb{T}}$ of intensity measure $\mathbf{E}[\mathrm{N}_t] = \lambda t$ ($\lambda \in (0, 1)$) i.e. that can be written as $\mathrm{N}_t = \sum_{s\in\mathbb{T}^*} \mathbf{1}_{\{\mathrm{T}_s \leqslant t\}}$ whose t-th jump time is defined by $\mathrm{T}_t = \sum_{s=1}^{t} \xi_s$, and where the inter-arrival variables $\{\xi_t,\, t \in \mathbb{T}^*\}$ are i.i.d. geometric random variables of parameter λ. On the other hand, let a family of i.i.d. $\{-1, 1\}$-Bernoulli variables $\{\mathrm{V}_t,\, t \in \mathbb{T}^*\}$ that are independent of $(\mathrm{N}_t)_{t\in\mathbb{T}}$, and for any $t \in \mathbb{T}^*$, set $\mathrm{W}_t = \mathrm{V}_{\mathrm{T}_t}$. Thus, the $\Delta\mathrm{Y}_t$ stand for the increments of the process $\mathrm{Y} := (\mathrm{Y}_t)_{t\in\mathbb{T}^*}$ defined by $\mathrm{Y}_0 = 0$,

$$\mathrm{Y}_t = \sum_{s=1}^{t} \Delta \mathrm{Y}_s = \sum_{s=1}^{t} \mathrm{W}_s \Delta \mathrm{N}_s = \sum_{s=1}^{\mathrm{N}_t} \mathrm{V}_s, \tag{2.3}$$

for $t \in \mathbb{T}^*$, and which is a particular case of *compound binomial process*. The corresponding *compensated compound process* denoted $\overline{\mathrm{Y}} := (\overline{\mathrm{Y}}_t)_{t\in\mathbb{T}}$, defined by $\overline{\mathrm{Y}}_0 = 0$ and

$$\overline{\mathrm{Y}}_t = \Big(\sum_{s=1}^{\mathrm{N}_t} \mathrm{V}_s\Big) - \lambda p_k t \ ; \ t \in \mathbb{T}^*, \tag{2.4}$$

where $p_1 := p$, $p_{-1} := 1 - p$ and $\lambda := \mathbf{P}(\{\Delta \mathrm{N}_t = 1\})$ is a $(\mathbf{P}, \mathcal{F})$-martingale. The key fact is that a good choice of the parameter couple (λ, p) makes the trinomial and ternary models equivalent in law. Indeed, by letting $\mathrm{S}_0 = \mathrm{S}_0^{\mathfrak{T}}$, $\lambda \in (0, 1)$, $p^{\mathfrak{T}} = \lambda p$ and $q^{\mathfrak{T}} = \lambda(1 - p)$ such that $1 - p^{\mathfrak{T}} - q^{\mathfrak{T}} = 1 - \lambda$, we get

$$\mathbf{E}\Big[s^{\frac{\mathrm{S}_t}{\mathrm{S}_{t-1}}}\Big] = \mathbf{E}\left[s^{\eta_t \Delta \mathrm{N}_t + 1}\right] = s^{1+b}\,\lambda p + s^{1+a}\,\lambda(1 - p) + s(1 - \lambda) = \mathbf{E}\Big[s^{\frac{\mathrm{T}_t}{\mathrm{T}_{t-1}}}\Big].$$

The stock price of the ternary model is thus the $\mathcal{F}$-adapted process $(\mathrm{S}_t)_{t\in\mathbb{T}}$ with (deterministic) initial value $\mathrm{S}_0 = 1$ and such that for any $t \in \mathbb{T}^*$,

$$\Delta \mathrm{S}_t = \eta_t\, \mathrm{S}_{t-1}\, \Delta \mathrm{N}_t, \tag{2.5}$$

with $\eta_t = b\mathbf{1}_{\{\mathrm{W}_t=1\}} + a\mathbf{1}_{\{\mathrm{W}_t=-1\}}$, where a and b are defined in (2.1). The sequence of discounted prices $\overline{\mathrm{S}} := (\overline{\mathrm{S}}_t)_{t\in\mathbb{T}}$ is defined by $\overline{\mathrm{S}}_t = \mathrm{A}_t^{-1}\mathrm{S}_t$ $(t \in \mathbb{T})$, where $(\mathrm{A}_t)_{t\in\mathbb{T}}$ is given by (2.1). Unless otherwise stated, the results for the insider problem will be established within the framework of the ternary model. All "expected results" will de facto hold in the trinomial model thanks to their equivalence in law, whereas the identities $\{\mathrm{X}_t^{\mathfrak{T}} = 0\} = \{\Delta \mathrm{N}_t = 0\}$ and $\{\mathrm{X}_t^{\mathfrak{T}} = \pm 1\} = \{\Delta \mathrm{N}_t = 1, \mathrm{W}_t = \pm 1\}$ ensure a pathwise correspondence between the two models in stake.

Under this paradigm shift, the ternary model can be interpreted as a *volatility model*. Indeed, the parameter $\lambda = \mathbf{P}(\{\Delta \mathrm{N}_t = 0\}) \in (0, 1)$ can viewed as the *volatility* of the model: The closer λ is to 0, the lower the probability that the stock price process changes and the lower the volatility. On the contrary, λ close to 1 means that the stock market process changes with a high probability, and in the extreme case $\lambda = 1$ the ternary model is no longer equivalent to the trinomial model but coincides with the Cox-Ross-Rubinstein (or binomial). For short, we may and shall write any probability measure on $(\Omega, \mathcal{F})$ by $\mathbf{P}^\alpha = \otimes_{t\in\mathbb{T}^*}\mathbf{P}_t^\alpha$ where $\mathbf{P}_t^\alpha := (\lambda^\alpha, p_t^\alpha, 1 - p_t^\alpha) = \big(\mathbf{P}^\alpha(\{\Delta \mathrm{N}_t = 0\}), \mathbf{P}^\alpha(\{\Delta \mathrm{N}_t = 1\} \cap \{\mathrm{W}_t = 1\}), \mathbf{P}^\alpha(\{\Delta \mathrm{N}_t = 1\} \cap \{\mathrm{W}_t = -1\}\big)$.

Like the trinomial (see Runggaldier [55]), the ternary model stands for an *incomplete* market; the sequence of discounted prices $(\overline{\mathrm{S}}_t)_{t\in\mathbf{T}}$ such that for $t \in \mathbb{T}^*$

$$\Delta \overline{\mathrm{S}}_t = \frac{[b\mathbf{1}_{\{\mathrm{W}_t=1\}} + a\mathbf{1}_{\{\mathrm{W}_t=-1\}}]\ \Delta \mathrm{N}_t - r}{1 + r} \times \overline{\mathrm{S}}_{t-1}, \tag{2.6}$$

is a $(\mathbf{P}^\alpha, \mathcal{F})$-martingale provided the equality $\lambda^\alpha[bp_t^\alpha + a(1 - p_t)^\alpha] = r$ holds for all $t \in \mathbb{T}^*$. In fact for any $t \in \mathbb{T}^*$, there exist infinitely many solutions such that any solution triplet $(\lambda^\alpha, p_t^\alpha, q_t^\alpha) \in (0, 1)^3$ forms a convex set (here a segment) characterized by its extremal points, i.e. the measures (independent from t) $\mathbf{P}^0_t = (1,(r-a)/(b-a),(b-r)/(b-a)) =: \mathbf{P}^0$ and $\mathbf{P}^1_t = (r/b,1,0) =: \mathbf{P}^1$, that are not equivalent to $\mathbf{P}$ but such that any convex combination $\mathbf{P}^\gamma = \gamma\mathbf{P}^0 + (1-\gamma)\mathbf{P}^1$ is. Furthermore, we can characterize $\mathscr{M}^{\mathcal{F}}$ the set of (equivalent) $\mathcal{F}$-*martingale measures*, that consists of probability measures (equivalent to $\mathbf{P}$) with respect to which $\overline{\mathrm{S}}$ is a $\mathcal{F}$-martingale. In fact, there is a bijection between the set $\mathscr{M}^{\mathcal{F}}$ in our model, and its analogue in the trinomial one. Indeed, for $T=1$, for a given measure $\mathbf{P}^\gamma = \gamma\mathbf{P}^0 + (1-\gamma)\mathbf{P}^1 = (\lambda_\gamma, p_\gamma, 1-p_\gamma)$ ($\gamma \in [0,1]$), we can find a probability measure $\mathbf{P}^{\gamma,\mathfrak{T}}$ which coincides with $\mathbf{P}^\gamma$ by letting $p^{\mathfrak{T}}_\gamma := \mathbf{P}^{\gamma,\mathfrak{T}}(\mathrm{X}^{\mathfrak{T}}_t = 1) =$ and $q^{\mathfrak{T}}_\gamma := \mathbf{P}^{\gamma,\mathfrak{T}}(\mathrm{X}^{\mathfrak{T}}_t = -1)$ such that $p^{\mathfrak{T}}_\gamma = p_\gamma\lambda_\gamma$ and $p^{\mathfrak{T}}_\gamma + q^{\mathfrak{T}}_\gamma = \lambda_\gamma$. The set $\mathscr{M}^{\mathcal{F}}$ is the polyhedron which 2^T vertices are depicted by the extremal measure $\mathbf{P}^j$ ($j \in \{1,\dots,2^T\} =: \mathcal{J}$) that be can be written as

$$\mathbf{P}^j = \bigotimes_{s\in\mathbb{T}^*} (\mathbf{P}^0)^{\gamma^j_s} (\mathbf{P}^1)^{1-\gamma^j_s}, \tag{2.7}$$

where $(\gamma^j_s)_{s\in[\![1,t]\!]} \in \{0,1\}^t$. An induction from the case $T=1$ enables to show that the convex set of $\mathcal{F}$-martingale measures equivalent to $\mathbf{P}$ in our model, and the one existing in the trinomial are one-to-one.

2.2 Elements of stochastic analysis for marked binomial processes

Motivated by this paradigm change, we provide here some results of stochastic analysis for marked binomial processes. They can be found in Halconruy [33] and are given here in the particular case of interest where the space mark $\mathrm{E} = \{-1,1\}$. Within the probability space $(\Omega,\mathcal{A},\mathbf{P})$ introduced above, we consider the simple measurable space $(\mathbb{X},\mathcal{X})$ where $\mathbb{X} := \mathbb{T}\times\mathrm{E}$ and $\mathbb{T}$ is defined above. Let $\mathfrak{N}_{\mathbb{X}}$ be the space of simple, integer-valued, finite measures on $\mathbb{X}$ and $\mathcal{N}^{\mathbb{X}}$ be the smallest σ-field of subsets of $\mathfrak{N}_{\mathbb{X}}$ such that the mapping $\chi \in \mathfrak{N}_{\mathbb{X}} \mapsto \chi(\mathrm{A})$ is measurable for all $\mathrm{A} \in \mathcal{X}$. We write η the underlying marked process associated to $(\mathrm{Y}_t)_{t\in\mathbb{T}}$ as the random element of $\mathfrak{N}_{\mathbb{X}}$ such that

$$\eta = \sum_{t\in\mathbb{T}^*} \delta_{(\mathrm{T}_t,\mathrm{V}_t)}, \tag{2.8}$$

where the families $\{\mathrm{V}_t,\, t\in\mathbb{T}^*\}$ and $\{\mathrm{T}_t,\, t\in\mathbb{T}\}$ are defined in the previous subsection. By a slight abuse of notation, we shall write $(t,k)\in\eta$ in order to indicate that the point $(t,k)\in\mathbb{X}$ is charged by the random measure η. Note in particular that $\{\Delta\mathrm{N}_t = 0\} = \{(t,\pm1)\notin\eta\} =: \{(t,1)\notin\eta\}\cap\{(t,-1)\notin\eta\}$ and $\{\Delta\mathrm{N}_t = 1, \mathrm{W}_t = k\} = \{(t,k)\in\eta\}$ for $k\in\mathrm{E}$. From now onwards, we may and will assume that $\mathcal{A} = \sigma(\eta) = \mathcal{F}$ where $\mathcal{F} = (\mathcal{F}_t)_{t\in\mathbb{T}}$ is the canonical filtration defined from η by

$$\mathcal{F}_0 = \{\emptyset,\Omega\} \quad \text{and} \quad \mathcal{F}_t = \sigma\left\{\sum_{(s,k)} \eta(s,k),\ s\leqslant t,\ k\in\mathrm{E}\right\}. \tag{2.9}$$

Let $\mathbf{P}_\eta = \mathbf{P}\circ\eta^{-1}$ be the image measure of $\mathbf{P}$ under η on the space $(\mathfrak{N}_{\mathbb{X}},\mathcal{N}^{\mathbb{X}})$ i.e. the distribution of η; its compensator - the intensity of η - is the measure ν defined on $\mathcal{X}$ by

$$\nu(\mathrm{A}) = \sum_{(t,k)\in\mathrm{A}}\sum_{s\in\mathbb{N}} \Big(\lambda\delta_s(\{t\})\otimes\big(p\delta_1(\{k\}) + (1-p)\delta_{-1}(\{k\})\big)\Big)\ ;\ \mathrm{A}\in\mathcal{X}.$$

Throughout, we denote by $\mathcal{L}^0(\Omega) = \mathcal{L}^0(\Omega, \mathcal{F})$ the class of real-valued measurable functions F on Ω. Since $\mathcal{F} = \sigma(\eta)$, for any $\mathrm{F} \in \mathcal{L}^0(\Omega)$, there exists a real-valued measurable function $\mathfrak{f}$ on $\mathfrak{N}_{\mathbb{X}}$ such that $\mathrm{F} = \mathfrak{f}(\eta)$. The function $\mathfrak{f}$ is called a *representative* of F and is $\mathbf{P} \otimes \eta^{-1}$-a.s. uniquely defined. Similarly, a process $u = (u_{(t,k)})_{(t,k)\in\mathbb{X}}$ is a measurable random variable defined on $(\mathfrak{N}(\mathbb{X})\times\mathbb{X}, \mathcal{F}\otimes\mathcal{X})$ that can be written $u = \sum_{(t,k)\in\mathbb{X}} \mathfrak{u}(\eta, (t,k))\mathbf{1}_{(t,k)}$, where $\{\mathfrak{u}(\eta, (t,k)),\ (t,k) \in \mathbb{X}\}$ is a family of measurable functions from $\mathfrak{N}_{\mathbb{X}} \times \mathbb{X}$ to $\mathbb{R}$ and $\mathfrak{u}$ is called the representative of u. By default, the representative of a random variable or a process will be noted by the corresponding gothic lowercase letter.

2.2.1 Chaotic decomposition

Let $\{\mathbf{P}^j,\ j \in \mathcal{J}\}$ be the set of martingale measures equivalent to $\mathbf{P}$ and denote $\mathbf{P}^j = \otimes_{t\in\mathbb{T}^*}(\lambda^j, p_t^j, 1-p_t^j)$ for all $j \in \mathcal{J}$. The following elaboration is rigorously the same as in Halconruy [33], by taking $\mathrm{E} = \{-1, 1\}$ and $\mathbf{P} = \mathbf{P}^j$ for each $j \in \mathcal{J}$. The detailed proofs can be found in the cited paper, so that we will limit ourselves to giving the most significant elements of the construction for some fixed $j \in \mathcal{J}$. Consider the families $\mathcal{Z}^j := \{\Delta \mathrm{Z}^j_{(t,k)},\ (t,k) \in \mathbb{X}\}$ and $\mathcal{R}^j := \{\Delta \mathrm{R}^j_{(t,k)},\ (t,k) \in \mathbb{X}\}$ respectively defined for all $(t,k) \in \mathbb{X}$ by

$$\Delta \mathrm{Z}^j_{(t,k)} = \mathbf{1}_{\{(t,k)\in\eta\}} - \lambda^j p^j_{t,k}, \quad \Delta \mathrm{R}^j_{(t,1)} = \Delta \mathrm{Z}^j_{(t,1)} \quad \text{and} \quad \Delta \mathrm{R}^j_{(t,-1)} =: \Delta \mathrm{Z}^j_{(t,-1)} + \rho^j \Delta \mathrm{Z}^j_{(t,1)},$$

where $\rho^j := [\lambda^j(1-p_t^j)]/(1-\lambda^j p_t^j)$, $p^j_{t,1} := p^j_t$, $p^j_{t,-1} := 1 - p^j_t$. Thus, $\mathcal{Z}^j$ is the natural family defined from $\overline{\mathrm{Y}}^j := (\overline{\mathrm{Y}}^j_t)_{t\in\mathbb{T}}$, the $\mathbf{P}^j$*-compensated compound process* associated to Y such that $\overline{\mathrm{Y}}^j_0 = 0$, and for all $t \in \mathbb{T}^*$,

$$\overline{\mathrm{Y}}^j_t = \sum_{(s,k)\in[\![0,t]\!]\times\mathrm{E}} \Delta \mathrm{Z}^j_{(s,k)} = \Big(\sum_{s=1}^{\mathrm{N}_t} \mathrm{V}_s\Big) - \lambda p^j_{t,k} t. \tag{2.10}$$

By its very definition, $\overline{\mathrm{Y}}^j$ a $(\mathcal{F}, \mathbf{P}^j)$-martingale. Besides, $\mathcal{R}^j$ is the orthogonal (for the scalar product induced by $\mathbf{P}^j$) family constructed from $\mathcal{Z}^j$ via the Gram-Schmidt process.

Contrary to the general framework investigated in Halconruy [33], the set $\mathbb{X}$ is here finite. So are all the random measures on $\mathbb{X}$ we consider, and there is no question of integrability: all functionals in stake are p-integrable (for any $p \in \mathbb{N}$) on Ω with respect to $\mathbf{P}^j$. For any function f_0 defined on $\mathbb{X}$, we set $\mathrm{J}^j_0(f_0) = f_0$. For any $n \in \mathbb{T}^*$ we denote respectively by $\mathscr{F}(\mathbb{X})^n$ and $\mathscr{F}(\mathbb{X})^{\circ n}$, the space of functions defined on $\mathbb{X}^n$ and the subspace of $\mathscr{F}(\mathbb{X})^n$ composed of functions that are symmetric in their n variables, i.e. such that for any permutation τ of $\{1, \ldots, n\}$, $f_n\big((t_{\tau(1)}, k_{\tau(1)}), \ldots, (t_{\tau(n)}, k_{\tau(n)})\big) = f_n\big((t_1, k_1), \ldots, (t_n, k_n)\big)$, for all $(t_1, k_1), \ldots, (t_n, k_n) \in \mathbb{X}$. The $\mathcal{R}^j$-stochastic integral of order $n \in \mathbb{T}^*$ is the application defined for any function $f_n \in \mathscr{F}(\mathbb{X})^{\circ n}$ by

$$\mathrm{J}^j_n(f_n) = n \sum_{(t,k)\in\mathbb{X}} \mathrm{J}^j_{n-1}(f_n(\star, (t,k)))\, \Delta \mathrm{R}^j_{(t,k)} = n! \sum_{(\mathbf{t}_n, \mathbf{k}_n)\in\mathbb{X}^n} f_n(\mathbf{t}_n, \mathbf{k}_n) \prod_{i=1}^n \Delta \mathrm{R}^j_{(t_i,k_i)}, \tag{2.11}$$

where " $\star$ " denotes the first $n-1$ variables of $f_n((t_1, k_1), \ldots, (t_n, k_n))$. Set $\mathcal{H}^j_0 = \mathbb{R}$ and for any $n \in \mathbb{T}^*$, let $\mathcal{H}^j_n$ be the subspace of $\mathcal{L}^0(\mathbf{P})$ made of integrals of order $n \in \mathbb{T}^*$ given by

$$\mathcal{H}^j_n = \left\{\mathrm{J}^j_n(f_n)\ ;\ f_n \in \mathscr{F}(\mathbb{X})^{\circ n}\right\},$$

and called $\mathbf{P}^j$-*chaos of order* n. By replacing $\mathbf{P}$ by $\mathbf{P}^j$ and each $\mathcal{H}_n$ by $\mathcal{H}_n^j$ in Halconruy ([33], theorem 2.11) we can state $\mathcal{L}^0(\Omega) = \bigoplus_{n\in\mathbb{T}} \mathcal{H}_n^j$. In other words, any *marked binomial functional*, that is any random variable of the form

$$\mathrm{F} = f_0 \mathbf{1}_{\{\eta(\mathbb{X})=0\}} + \sum_{n\in\mathbb{N}} \sum_{(\mathbf{t}_n,\mathbf{k}_n)\in\mathbb{X}^n} \mathbf{1}_{\{\eta(\mathbb{X})=n\}} f_n(\mathbf{t}_n,\mathbf{k}_n) \prod_{i=1}^{n} \mathbf{1}_{\{(t_i,k_i)\in\eta\}}, \tag{2.12}$$

can be expanded in a unique way as

$$\mathrm{F} = \mathbf{E}_{\mathbf{P}^j}[\mathrm{F}] + \sum_{n\in\mathbb{T}^*} \mathrm{J}_n^j(f_n). \tag{2.13}$$

In the following we will refer to this expansion as the $\mathbf{P}^j$-*chaotic decomposition* of F.

2.2.2 Clark-Ocone formula

As a reminiscence of the Malliavin operator on the Poisson space, the *add-one cost operator* or *Malliavin's derivative* D is defined for any $\mathrm{F} \in \mathcal{L}^0(\Omega)$ by

$$\mathrm{D}_{(t,k)}\mathrm{F} := \mathfrak{f}(\pi_t(\eta) + \delta_{(t,k)}) - \mathfrak{f}(\pi_t(\eta)), \tag{2.14}$$

where the application $\pi_t : \mathfrak{N}_{\mathbb{X}} \to \mathfrak{N}_{\mathbb{X}}$ is the restriction of η to $\mathcal{F}_t := \sigma\big\{ \sum_{(s,k)\in(\mathbb{T}\setminus\{t\}\times \mathrm{E})} \eta(s,k)\big\}$, i.e.

$$\pi_t(\eta) = \sum_{s\neq t} \sum_{k\in \mathrm{E}} \eta(s,k). \tag{2.15}$$

By rewriting Proposition 4.4 of Halconruy [33] with respect to the $\mathbf{P}^j$-decomposition, we get the analogue of the Clark-Ocone formula: for any $\mathrm{F} \in \mathcal{L}^0(\Omega)$,

$$\mathrm{F} = \mathbf{E}_{\mathbf{P}^j}[\mathrm{F}] + \sum_{(t,k)\in\mathbb{X}} \mathbf{E}_{\mathbf{P}^j}\left[\mathrm{D}_{(t,k)}\mathrm{F} \,|\, \mathcal{F}_{t-1}\right] \Delta \mathrm{R}^j_{(t,k)}. \tag{2.16}$$

As a corollary, if $(\mathrm{L}_t)_{t\in\mathbb{T}}$ is a $(\mathcal{F}, \mathbf{P}^j)$-martingale, for any $(s,t) \in \mathbb{T}^2$, $s < t$,

$$\mathrm{L}_t = \mathrm{L}_s + \sum_{r=s+1}^{t} \sum_{k\in\mathrm{E}} \mathbf{E}_{\mathbf{P}^j}\left[\mathrm{D}_{(r,k)}\mathrm{L}_t \,|\, \mathcal{F}_{r-1}\right] \Delta \mathrm{R}^j_{(r,k)}. \tag{2.17}$$

3 Enlargement of filtration in a discrete setting

On the ternary model defined above, we introduce two agents with different levels of information; the first one, called *insider*, possesses from the beginning extra information whereas the second one, the *ordinary agent*, bases his/her investment decisions on the public flow. This difference translates mathematically into the introduction of two distinct filtrations: the ordinary agent information level corresponds to the initial filtration $\mathcal{F}$ (i.e. his/her knowledge at time $t \in \mathbb{T}$ is given by $\mathcal{F}_t$) whereas the insider disposes at time t an information given by the σ-algebra $\mathcal{G}_t$ defined via the initial enlargement

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\mathrm{G}),$$

where G is a $\mathcal{F}_T$-measurable random variable with values in a finite space $(\Gamma, \mathscr{G})$ that encodes the information overload enjoyed by the insider. His/her filtration is then denoted by $\mathcal{G} := (\mathcal{G}_t)_{t\in\mathbb{T}}$. The crucial point in order to make computations from insider's point of view, is the study of the preservation of (semi)martingales within this information enrichment.
Throughout this section and with no loss of generality, we consider a measure $\mathbf{P}^j$ $(j \in \mathcal{J})$ in the non-void (see subsection 2.1) set of martingale measures $\mathscr{M}^{\mathcal{F}}$. All forthcoming results hold for any $j \in \mathcal{J}$. The *enlarged market model* where both the ordinary agent (with information flow given by $\mathcal{F}$) and the insider (with information flow given by $\mathcal{G}$) trade will be referred to as $(\mathbb{T}, \mathcal{F}, \mathcal{G}, \mathbf{P}, \mathrm{S})$.

3.1 From $\mathcal{F}$-martingales to $\mathcal{G}$-semimartingales via Girsanov's transformation

In the continuous case, *Jacod's condition* indicates that the absolutely continuity of the conditional laws of G with respect to its law is a sufficient criterion for the preservation of semimartingales. In a discrete setting, no such assumption is required and any $(\mathbf{P}, \mathcal{F})$-martingale is a $(\mathbf{P}, \mathcal{G})$-semimartingale. Since the set Γ is finite, the conditional distributions of G for all $t \in \mathbb{T}^\circ$ and the law of G are even equivalent. Indeed, any set $\mathrm{C} \in \mathscr{G}$ is of the form $\mathrm{C} = \bigcup_{\mathrm{c}\in\mathrm{C}}\{\mathrm{G} = \mathrm{c}\}$ and, for any $t \in \mathbb{T}^\circ$,

$$\mathbf{P}(\{\mathrm{G} \in \mathrm{C}\} \,|\, \mathcal{F}_t) = \sum_{\mathrm{c}\in\mathrm{C}} \mathbf{P}(\{\mathrm{G} = \mathrm{c}\} \,|\, \mathcal{F}_t) = \sum_{\mathrm{c}\in\mathrm{C}} \frac{\mathbf{P}(\{\mathrm{G} = \mathrm{c}\} \,|\, \mathcal{F}_t)}{\mathbf{P}(\{\mathrm{G} = \mathrm{c}\})} \mathbf{P}(\{\mathrm{G} = \mathrm{c}\}) = \mathbf{E}\left[p_t^{\mathrm{G}} \mathbf{1}_{\mathrm{C}}\right],$$

where the random variable p_t^{G} is defined for any $\omega \in \Omega, \mathrm{c} \in \Gamma$ by

$$p_t^{\mathrm{G}}(\omega) = \frac{\mathbf{P}(\{\mathrm{G} \in \cdot\}|\mathcal{F}_t)(\omega)}{\mathbf{P}(\{\mathrm{G} \in \cdot\})} \quad \text{such that} \quad p_t^{\mathrm{c}}(\omega) = \frac{\mathbf{P}(\{\mathrm{G} = \mathrm{c}\} \,|\mathcal{F}_t)(\omega)}{\mathbf{P}(\{\mathrm{G} = \mathrm{c}\})}, \tag{3.1}$$

has an expectation equal to 1. Note however that there is no equivalence on $\mathcal{F}_T$, since $\mathbf{P}(\{\mathrm{G} \in \cdot\}|\mathcal{F}_T) = \mathbf{1}_{\{\mathrm{G}\in\cdot\}}$ is zero with positive probability (unless G is constant). Considering two $(\mathbf{P}, \mathcal{F})$-martingales X and Z, we denote by $\langle \cdot, \cdot \rangle^{\mathbf{P}}$ the angle bracket i.e. the $\mathcal{F}$-adapted process such that $(\mathrm{X}_t\mathrm{Z}_t - \langle \mathrm{X}_t, \mathrm{Z}_t\rangle^{\mathbf{P}})_{t\in\mathbb{T}}$ is a $(\mathbf{P}, \mathcal{F})$-martingale. The difficulties inherent in the preservation of semimartingales are directly lifted as a consequence of Doob's decomposition. Indeed it is clear (see Blanchet-Scalliet, Jeanblanc and Romero [13]) that any integrable process is a special semimartingale in any filtration with respect to which it is adapted. Furthermore, the hypotheses of section 2.2 in Blanchet-Scalliet *et al* (see [13]) are fulfilled in our context; thus, for some $j \in \mathcal{J}$ and a given $(\mathbf{P}^j, \mathcal{F})$-martingale X^j, the process $(\mathrm{X}_t^{\mathcal{G},j})_{t\in\mathbb{T}}$ defined by

$$\mathrm{X}_t^{\mathcal{G},j} = \mathrm{X}_t^j - \sum_{s=1}^{t} \frac{\langle \mathrm{X}^j, p_t^{\mathrm{c}}\rangle_s^{\mathbf{P}^j}\big|_{\mathrm{c}=\mathrm{G}}}{p_{s-1}^{\mathrm{G}}} =: \mathrm{X}_t^j - \mu_t^{\mathcal{G},j} \tag{3.2}$$

is a $(\mathbf{P}^j, \mathcal{G})$-martingale. Consequently any $(\mathbf{P}^j, \mathcal{F})$-martingale X^j is a $(\mathbf{P}^j, \mathcal{G})$-special semimartingale, i.e. a $\mathcal{G}$-adapted process which can be decomposed as $\mathrm{X}^j = \mathrm{M}^j + \mathrm{V}^j$ where M^j is a $(\mathbf{P}^j, \mathcal{G})$-martingale and V^j a $\mathcal{G}$-predictable process. In particular, we can define the process $\overline{\mathrm{Y}}^{\mathcal{G},j} := (\overline{\mathrm{Y}}_t^{\mathcal{G},j})_{t\in\mathbb{T}}$, such that for any $t \in \mathbb{T}$, $k \in \mathrm{E}$,

$$\overline{\mathrm{Y}}_t^{\mathcal{G},j} = \overline{\mathrm{Y}}_t^j - \mu_t^{\mathcal{G},j},$$

where $\mu_t^{\mathcal{G},j}$ is defined by (3.2) by replacing X_t^j by $\overline{\mathrm{Y}}_t^j$ (given by (2.10)). Thus $\overline{\mathrm{Y}}^{\mathcal{G},j}$ is a $(\mathbf{P}^j, \mathcal{G})$-martingale.

3.2 Martingale conservation via a particular measure

In this section, we focus on the conservation of martingale property up to a change of the underlying measure. The main result of this part is that for all $j \in \mathcal{J}$, $t \in \mathbb{T}^\circ$, any $(\mathbf{P}^j, \mathcal{F})$-martingale (where $\mathbf{P}^j$ is defined in (2.7)) is a $(\mathbf{Q}^j_t, \mathcal{G})$-martingale on $[\![0,t]\!]$ for some particular measure $\mathbf{Q}^j_t$ which definition is widely inspired from the works of Amendinger *et al.*, [6] and [8]. Let $(\mathrm{L}^j_t)_{t\in\mathbb{T}}$ be the density process of $\mathbf{P}^j$ with respect to $\mathbf{P}$ i.e. such that $\mathrm{L}^j_t = (\mathrm{d}\mathbf{P}^j/\mathrm{d}\mathbf{P})|\mathcal{F}_t$, and introduce the $\mathcal{F}$-adapted process $u^{\mathrm{G},j} = \mathrm{L}^j/p^{\mathrm{G}}$, of key importance afterwards. This is well defined; indeed, the finiteness of Γ implies that for any $(t, \mathrm{c}) \in \mathbb{T}^\circ \times \Gamma$, the random variable p^{c}_t is not null $\mathbf{P}$-a.s. Note that for any $t \in \mathbb{T}$, the σ-algebra $\mathcal{G}_t$ is generated by the set

$$\{\mathrm{B} \cap \mathrm{C}\ ;\ \mathrm{B} \in \mathcal{F}_t,\ \mathrm{C} \in \mathscr{G}\}.$$

Proposition 3.1. *1. For all $j \in \mathcal{J}$, the process $u^{\mathrm{G},j}$ is a $(\mathbf{P}, \mathcal{G})$-martingale on $\mathbb{T}^\circ$.*

2. For any $t \in \mathbb{T}^\circ$, $j \in \mathcal{J}$, the σ-algebras $\mathcal{F}_t$ and $\sigma(\mathrm{G})$ are independent under the probability measure $\mathbf{Q}^j_t$ defined for any $\mathrm{A}_t \in \mathcal{G}_t$ by

$$\mathbf{Q}^j_t(\mathrm{A}_t) = \mathbf{E}\big[u^{\mathrm{G},j}_t \mathbf{1}_{\mathrm{A}_t}\big] = \mathbf{E}_{\mathbf{P}^j}\big[(p^{\mathrm{G}}_t)^{-1}\mathbf{1}_{\mathrm{A}_t}\big]. \tag{3.3}$$

3. For any $t \in \mathbb{T}^\circ$, $j \in \mathcal{J}$, the probability measure $\mathbf{Q}^j_t$ coincides with $\mathbf{P}^j$ on $(\Omega, \mathcal{F}_t)$ and with $\mathbf{P}$ on $(\Omega, \sigma(\mathrm{G}))$, so that for $\mathrm{B}_t c \in \mathcal{F}_t$ and $\mathrm{C} \in \mathscr{G}$,

$$\mathbf{Q}^j_t(\mathrm{B}_t \cap \{\mathrm{G} \in \mathrm{C}\}) = \mathbf{P}^j(\mathrm{B}_t)\mathbf{P}(\{\mathrm{G} \in \mathrm{C}\}) = \mathbf{Q}^j_t(\mathrm{B}_t)\mathbf{Q}^j_t(\{\mathrm{G} \in \mathrm{C}\}). \tag{3.4}$$

For any $t \in \mathbb{T}^\circ$, $j \in \mathcal{J}$, the measure $\mathbf{Q}^j_t$ thus defined is a *martingale preserving measure*, what can be justified by the following proposition.

Proposition 3.2. *Let some $t \in \mathbb{T}^\circ$ and $j \in \mathcal{J}$. Any $(\mathbf{P}^j, \mathcal{F})$-martingale is a $(\mathbf{Q}^j_t, \mathcal{G})$-martingale on $[\![0,t]\!]$. Moreover the set of $(\mathbf{P}^j, \mathcal{F})$-martingales and the set of $(\mathbf{Q}^j_t, \mathcal{F})$-martingales all on $[\![0,t]\!]$ are equal.*

As evoked above, p^{G}_T is null (except if G is constant) with positive probability and the regular conditional laws of G are not equivalent to the law of G. In some way, the analogue of Assumption 2.1 in Amendinger [6] in our context is not satisfied at time T so that the previous result does not hold on $\mathbb{T}$.

4 Application to the insider's problem in the ternary model

4.1 Information drift and Malliavin derivative

We saw in subsection 3.1 that martingales with respect to the initial filtration become semi-martingales by moving to the enlarged one. This transfer is encoded by a particular process $\mu^{\mathcal{G}}$, called the *information drift* and defined by (3.2). In the same vein as Imkeller [37], we can traduct its connection to the random variable G thanks to the Malliavin derivative D.

Theorem 4.1. *Let $j \in \mathcal{J}$. The information drift $\mu^{\mathcal{G},j}$ defined in (3.2) for the $\mathbf{P}^j$-compensated compound process $\overline{\mathrm{Y}}^j$ (given in (2.10)) can be written as*

$$\mu^{\mathcal{G},j}_t = \sum_{k\in\mathrm{E}}\sum_{\ell\in\mathrm{E}} \frac{a_{k,\ell}\ \mathbf{E}\big[\mathrm{D}_{(t,\ell)}\, p^{\mathrm{c},j}_t\big]|_{\mathrm{c}=\mathrm{G}}}{p^{\mathrm{G},j}_{t-1}}$$

for any $t \in \mathbb{T}^*$, *where the family* $\{a^j_{t,k,\ell},\ t \in \mathbb{T}^*,\ (k,\ell) \in \mathrm{E}^2\}$ *is defined for* $t \in \mathbb{T}^*$, $(k,\ell) \in \mathrm{E}^2$ *by* $a^j_{t,k,\ell} = \mathbf{E}_{\mathbf{P}^j}[\Delta \mathrm{Z}^j_{(t,k)} \Delta \mathrm{R}^j_{(t,\ell)}]$, *i.e.*

$$a^j_{t,1,1} = \lambda^j p^j_t (1 - \lambda p^j_t), \quad a^j_{t,1,-1} = 0, \quad a^j_{t,-1,1} = (\lambda^j)^2 p^j_t (1 - p^j_t) \ \ \textit{and} \ \ a^j_{t,-1,-1} = \frac{\lambda^j (1 - \lambda^j)(1 - p^j_t)}{1 - \lambda^j p^j_t}.$$

This result is the discrete analogue of the formula (17) in Imkeller [37]. Classical Malliavin's derivative (in the Wiener space) enjoys the chain rule, so that the formula exhibited by Imkeller elegantly reduces in the continuous case (with the corresponding notations) to $\mu^{\mathcal{G}}_t = \nabla_t \log(p_t(\cdot, \mathrm{c}))|_{\mathrm{c}=\mathrm{G}}$.

4.2 Portfolio optimization an additional expected utility of the insider

We consider an economic agent and an insider both disposing of $x \in \mathbb{R}^*_+$ euros at date $t = 0$ (initial budget constraint), for whom we want first to determine the maximal expected logarithmic utility from terminal wealth. Let $\mathcal{H}$ be some filtration on $(\Omega, \mathbf{P})$, that will be replaced by $\mathcal{F}$ or $\mathcal{G}$ later on. As a reminder, the value of a $\mathcal{H}$-portfolio at time $t \in \mathbb{T}$ is given by the random variable

$$\mathrm{V}_t(\psi) = \alpha_t \, \mathrm{A}_t + \varphi_t \, \mathrm{S}_t,$$

where the so-called $\mathcal{H}$-*strategy* $\psi = (\alpha_t, \varphi_t)_{t \in \mathbb{T}}$ is a couple of $\mathcal{H}$-predictable processes modelling respectively the amounts of riskless and risky assets held in the portfolio. A $\mathcal{H}$-strategy $\psi = (\alpha, \varphi)$ is said to be *self-financed* if it verifies the condition:

$$\mathrm{A}_t \, (\alpha_{t+1} - \alpha_t) + \mathrm{S}_t \, (\varphi_{t+1} - \varphi_t) = 0, \tag{4.1}$$

for any $t \in \mathbb{T}^\circ$. The discounted value of the $\mathcal{H}$-portfolio at time $t \in \mathbb{T}$ is given by $\overline{\mathrm{V}}_t(\psi) = \mathrm{V}_t(\psi)/\mathrm{A}_t$. We designate by $\mathrm{V}_{x,t}(\psi)$ (respectively $\overline{\mathrm{V}}_{x,t}(\psi)$) the value at time $t \in \mathbb{T}$ of a portfolio (respectively a discounted portfolio) of initial value $\mathrm{V}_0(\psi) = x$ (respectively $\overline{\mathrm{V}}_0(\psi) = \mathrm{V}_0(\psi) = x$) and strategy ψ. Before going on, we can assert two straightforward and well-known facts. First, for any given $\mathcal{H}$-predictable process φ there exists a unique $\mathcal{H}$-predictable process α such that $\psi = (\alpha, \varphi)$ is a self-financing process (see for instance Lamberton and Lapeyre [42], proposition 1.1.3). On the other hand, the quantity of riskless asset indicated by the process $(\alpha_t)_{t \in \mathbb{T}}$ does not change the discounted value of the portfolio namely $(\overline{\mathrm{V}}_t(\psi))_{t \in \mathbb{T}}$ since, by its very definition the discounted version of the asset $(\mathrm{A}_t)_{t \in \mathbb{T}}$ is deterministic constant equal to 1. As a consequence, the knowledge of the initial investment $\alpha_0 = x$ and the process of risky asset amount $(\varphi_t)_{t \in \mathbb{T}}$ is enough to compute the value of the (discounted) portfolio. From now on, we identify, with a slight abuse of notation, $\psi = (\alpha, \varphi)$ with (α_0, φ). A nonnegative $\mathcal{H}_T$-measurable random variable F (called *claim*) is *replicable* or *reachable* if there exists an $\mathcal{H}$-predictable self-financed strategy $\psi = (\alpha_0, \varphi)$ which corresponding portfolio value satisfies $\alpha_0 = \mathrm{V}_0(\psi) > 0$, $\mathrm{V}_t(\psi) \geqslant 0$ for all $t \in \mathbb{T}^\circ$, and $\mathrm{V}_T(\psi) = \mathrm{F}$. We denote by $\mathscr{S}_{\mathcal{H}}(x,t)$ the class of $\mathcal{H}$-*eligible strategies* up to time $t \in \mathbb{T}^*$ by

$$\mathscr{S}_{\mathcal{H}}(x,t) = \{\psi \,|\, \psi \text{ is } \mathcal{H}\text{-predictable and } \mathbf{E}[\log(\mathrm{V}_{x,t}(\psi)) < \infty]\}.$$

4.3 Portfolio optimization in the ternary model

In this subsection, we are led to consider the optimization problem at any time $t \in \mathbb{T}^*$ from the agent's point of view

$$\Phi^{\mathcal{F}}_t(x) = \sup_{\psi \in \mathscr{S}_{\mathcal{F}}(x,t)} \mathbf{E}\left[u(\mathrm{V}_{x,t}(\psi)\right] = \sup_{\varphi \in \mathscr{S}_{\mathcal{F}}(x,t)} \mathbf{E}\left[u((1+r)^t \overline{\mathrm{V}}_{x,t}(\varphi))\right], \tag{4.2}$$

and from the insider's

$$\Phi_t^{\mathcal{G}}(x) = \sup_{\psi \in \mathscr{S}_{\mathcal{G}}(x,t)} \mathbf{E}\left[u(\mathrm{V}_{x,t}(\psi))\right] = \sup_{\varphi \in \mathscr{S}_{\mathcal{G}}(x,t)} \mathbf{E}\left[u((1+r)^t \overline{\mathrm{V}}_{t,x}(\varphi))\right], \tag{4.3}$$

where u is a *utility function*, strictly increasing and strictly concave on $\mathbb{R}$ or $\mathbb{R}_+^*$. Throughout, we consider $u = \log$.

4.3.1 Link with portfolio optimization in the binomial model

A very substantiated expression and comprehensive look at arbitrage-related issues can be found in the book of Delbaen and Schachermayer [26]; our results obtained in the ternary model are linked to theirs found in the binary one. To that end, let us first introduce, the "CRR-embedded" processes $(\mathrm{X}_t^{\mathfrak{B}})_{t\in\mathbb{T}}$ and $(\mathrm{S}_t^{\mathfrak{B}})_{t\in\mathbb{T}}$ by $\mathrm{S}_0^{\mathfrak{B}} = 1$ and

$$(\mathcal{S}^{\mathfrak{B}}) : \begin{cases} \mathrm{X}_{t+1}^{\mathfrak{B}} = 0 & \text{and} \quad \mathrm{S}_{t+1}^{\mathfrak{B}} = \mathrm{S}_t^{\mathfrak{B}} & \text{if} \quad (1,\pm 1) \notin \eta \\ \mathrm{X}_{t+1}^{\mathfrak{B}} = 1 & \text{and} \quad \mathrm{S}_{t+1}^{\mathfrak{B}} = (1+b)\,\mathrm{S}_t^{\mathfrak{B}} & \text{if} \quad (1,1) \in \eta \\ \mathrm{X}_{t+1}^{\mathfrak{B}} = -1 & \text{and} \quad \mathrm{S}_{t+1}^{\mathfrak{B}} = (1+a)\,\mathrm{S}_t^{\mathfrak{B}} & \text{if} \quad (1,-1) \in \eta \end{cases},$$

with $\lambda^{\mathfrak{B}} := \mathbf{P}(\{(1,\pm 1) \in \eta\}) = 1$ and $p^{\mathfrak{B}} := \mathbf{P}(\{(1,1) \in \eta\}) = p$, which means $\mathbf{P}(\{\mathrm{X}_{t+1}^{\mathfrak{B}} = 0\}) = 0$, and then $\mathrm{X}_{t+1}^{\mathfrak{B}} = \pm 1$ $\mathbf{P}$-almost surely. The law of $\mathrm{S}_t^{\mathfrak{B}}$ is given by $\mathbf{P}^{\mathfrak{B}} = (1, p^{\mathfrak{B}}, 1 - p^{\mathfrak{B}}) = (1, p, 1-p)$ so that $(\mathrm{S}_t^{\mathfrak{B}})_{t\in\mathbb{T}}$ is identically distributed to the CRR-stock price process and stands for its "embedding" into our framework.

Until the end of the subsection, let $T = 1$. As proved in Proposition 3.3.2 of Delbaen and Schachermayer [26], there exists a unique optimizing strategy $\widehat{\varphi}^{\mathfrak{B}}$ such that

$$\Phi^{\mathfrak{B}}(x) = \log(1+r) + \sup_{\psi \in \mathscr{S}_{\mathcal{F}}(x,T)} \mathbf{E}_{\mathbf{P}^{\mathfrak{B}}}\left[\log(\overline{\mathrm{V}}_{x,T}^{\mathfrak{B}}(\psi))\right] = \log(1+r) + \mathbf{E}_{\mathbf{P}^{\mathfrak{B}}}\left[\log(\overline{\mathrm{V}}_{x,T}^{\mathfrak{B}}(\widehat{\psi}^{\mathfrak{B}}))\right], \tag{4.4}$$

where $\mathrm{V}_{x,T}^{\mathfrak{B}}(\psi)$ is the terminal value of the portfolio of initial value x and defined from the CRR-stock price i.e.

$$\overline{\mathrm{V}}_{x,t}^{\mathfrak{B}}(\psi) = x + \varphi_t \Delta\overline{\mathrm{S}}_t^{\mathfrak{B}}, \ t \in \mathbb{T}.$$

Besides, any portfolio based on the stock price of the ternary model, of initial value x and $\mathcal{F}$-eligible strategy ψ has a terminal value $\overline{\mathrm{V}}_{x,T}(\psi) = x + \varphi_T \Delta\overline{\mathrm{S}}_T$, and we have

$$\begin{aligned} \mathbf{E}\left[\log(\overline{\mathrm{V}}_{x,T}(\psi))\right] &= (1-\lambda)\mathbf{E}\left[\log(x + \varphi_T\Delta\overline{\mathrm{S}}_T) \,|\, \Delta\mathrm{N}_T = 0\right] + \lambda\mathbf{E}\left[\log(x + \varphi_T\Delta\overline{\mathrm{S}}_T) \,|\, \Delta\mathrm{N}_T = 1\right] \\ &= (1-\lambda)\log(x) + \lambda\mathbf{E}_{\mathbf{P}^{\mathfrak{B}}}\left[\log(x + \varphi_T\Delta\overline{\mathrm{S}}_T^{\mathfrak{B}})\right] \\ &< (1-\lambda)\log(x) + \lambda\mathbf{E}_{\mathbf{P}^{\mathfrak{B}}}\left[\log(\overline{\mathrm{V}}_{x,T}^{\mathfrak{B}}(\widehat{\psi}^{\mathfrak{B}}))\right] = \mathbf{E}\left[\log(\overline{\mathrm{V}}_{x,T}(\widehat{\psi}^{\mathfrak{B}}))\right], \end{aligned}$$

so that the optimizing strategy obtained for the ternary model coincides with the one computed in the binary model, and we finally get

$$\Phi(x) = (1-\lambda)\log((1+r)x) + \lambda\Phi^{\mathfrak{B}}(x). \tag{4.5}$$

4.3.2 Agent's portfolio optimization

Let us start with the case $T = 1$. As suggested in the previous subsection, we can deduce the solution of the optimization problem (4.2) from that of (4.4) where $\mathrm{S}^{\mathfrak{B}}$ is the CRR-price sequence embedded in the ternary model via $(\mathcal{S}^{\mathfrak{B}})$. Our result directly lies on martingale and duality methods usually used to deal with utility optimization in incomplete markets (see Karatzas *et al.* [41]). A simple translation of the results of Delbaen and Schachermayer (see [26], example 3.3.2) into our frame leads to $\Phi^{\mathfrak{B}}(x) = \log((1+r)x) - \mathbf{E}[\log(\mathbf{P}^0/\mathbf{P}^{\mathfrak{B}})] =: \log((1+r)x) + c_{\mathfrak{B}}$ since $\mathbf{P}^0 = (1, (r-a)/(b-a), (b-r)/(b-a)) =: (1, p^0, 1-p^0)$ stands for the unique CRR risk-neutral measure written in the pattern of the ternary model. Then the solution of (4.2) is given by $\Phi^{\mathfrak{F}}(x) =: \log((1+r)x)+c_{\mathfrak{F}}$ where, by using (4.5), $\log((1+r)x)+c_{\mathfrak{F}} = (1-\lambda)\log((1+r)x) + \lambda\Phi_{\mathfrak{B}} = \log((1+r)x) + \lambda c_{\mathfrak{B}}$, so that $c_{\mathfrak{F}} = \lambda c_{\mathfrak{B}}$. Then, using that $\mathbf{P}^{\mathfrak{B}} = (1, p, 1-p)$,

$$\begin{aligned}
-\lambda c_{\mathfrak{B}} = \mathbf{E}\left[\log\left(\frac{\mathrm{d}\mathbf{P}^0}{\mathrm{d}\mathbf{P}^{\mathfrak{B}}}\right)\right] &= (1-\lambda)\log\left(\frac{1}{1}\right) + \lambda p \log\left(\frac{p^0}{p}\right) + \lambda(1-p)\log\left(\frac{1-p^0}{1-p}\right) \\
&= (1-\lambda)\log\left(\frac{1-\lambda}{1-\lambda}\right) + \lambda p \log\left(\frac{\lambda p^0}{\lambda p}\right) + \lambda(1-p)\log\left(\frac{\lambda(1-p^0)}{\lambda(1-p)}\right) \\
&= \mathbf{E}\left[\log\left(\frac{\mathrm{d}\widehat{\mathbf{P}}_1^{\mathfrak{F}}}{\mathrm{d}\mathbf{P}}\right)\right],
\end{aligned}$$

where $\widehat{\mathbf{P}}_1^{\mathfrak{F}} = (1-\lambda, \lambda p^0, \lambda(1-p^0))$. As proved earlier, the optimal strategy obtained for the ternary model is that found in the binomial frame. This case (with a logarithmic utility) is completely solved through in Delbaen and Schachermayer ([26], example 3.3.2) and we won't give here a detailed proof. The results related to agent's portfolio optimization are summed up in the following proposition.

Proposition 4.2. *For $T = 1$, the maximal expected logarithmic utility up to expiry for the agent initially having x euros is given by*

$$\Phi_T^{\mathfrak{F}}(x) = \log((1+r)x) - \mathbf{E}\left[\log\left(\frac{\mathrm{d}\widehat{\mathbf{P}}_1^{\mathfrak{F}}}{\mathrm{d}\mathbf{P}}\right)\right], \tag{4.6}$$

where the density of $\widehat{\mathbf{P}}_1^{\mathfrak{F}}$ with respect to $\mathbf{P}$ is defined by

$$\frac{\mathrm{d}\widehat{\mathbf{P}}_1^{\mathfrak{F}}}{\mathrm{d}\mathbf{P}} = \mathbf{1}_{\{(1,\pm1)\notin\eta\}} + \frac{p^0}{p}\mathbf{1}_{\{(1,1)\in\eta\}} + \frac{1-p^0}{1-p}\mathbf{1}_{\{(1,-1)\in\eta\}}.$$

This is reached for a unique strategy $\widehat{\psi}^{\mathfrak{F},T}$ of discounted terminal value is given by $\widehat{\mathrm{V}}_{x,T}^{\mathfrak{F}} = \overline{\mathrm{V}}_{x,T}(\widehat{\psi}^{\mathfrak{F}}) = x + \widehat{\varphi}_T^{\mathfrak{F}}\Delta\overline{\mathrm{S}}_T$ where

$$\widehat{\mathrm{V}}_{x,T}^{\mathfrak{F}} = x \cdot \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\widehat{\mathbf{P}}_1^{\mathfrak{F}}} \quad \text{and} \quad \widehat{\varphi}_T^{\mathfrak{F}} = \frac{x(1+r)[p(1-p^0) + (1-p)p^0]}{(b-a)p^0(1-p^0)}.$$

As suggested in Delbaen and Schachermayer ([26], example 3.3.5), the optimization problem can be solved at expiry by extending the previous results via the principle of dynamic programming, that sounds as an induction from the case $T = 1$. The choice of this procedure can be justified by the independence of the increments of the underlying jump process $(\mathrm{N}_t)_{t\in\mathbb{T}}$. Let us consider some $T \in \mathbb{N}$ such that $T \geqslant 2$.

Proposition 4.3. *For some $T \in \mathbb{N}$, $T \geqslant 2$, the maximal expected logarithmic utility up to time $t \in \mathbb{T}^*$ for the agent initially having x euros is given by*

$$\Phi_t^{\mathcal{F}}(x) = \log((1+r)^t x) - \mathbf{E}\left[\log\left(\frac{\mathrm{d}\widehat{\mathbf{P}}^{\mathcal{F}}}{\mathrm{d}\mathbf{P}}\bigg|_{\mathcal{F}_t}\right)\right],$$

where the density of $\widehat{\mathbf{P}}^{\mathcal{F}}$ with respect to $\mathbf{P}$ is defined on $\mathcal{F}_t$ by

$$\frac{\mathrm{d}\widehat{\mathbf{P}}^{\mathcal{F}}}{\mathrm{d}\mathbf{P}}\bigg|_{\mathcal{F}_t} = \prod_{s=1}^{t}\left(\mathbf{1}_{\{(s,\pm 1)\notin\eta\}} + \frac{p^0}{p}\mathbf{1}_{\{(s,1)\in\eta\}} + \frac{1-p^0}{1-p}\mathbf{1}_{\{(s,-1)\in\eta\}}\right).$$

This is reached for a unique strategy which discounted value at time t is $\widehat{\mathrm{V}}_{x,t}^{\mathcal{F}} = \overline{\mathrm{V}}_{x,t}(\widehat{\psi}^{\mathcal{F}}) = x + \sum_{s\in[\![1,t]\!]}\widehat{\varphi}_s^{\mathcal{F}}\Delta\overline{\mathrm{S}}_s$ such that

$$\widehat{\mathrm{V}}_{x,t}^{\mathcal{F}} = x\cdot\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\widehat{\mathbf{P}}^{\mathcal{F}}}\bigg|_{\mathcal{F}_t}.$$

The following proposition provides an expression of the hedging strategy for replicable claims and in terms of Malliavin derivative.

Theorem 4.4 (Hedging formula for replicable claims)**.** *Let a reachable claim $\mathrm{F}\in\mathcal{L}^0(\Omega)$ and $j\in\mathcal{J}$. The $(\mathbf{P}^j,\mathcal{F})$-strategy $\psi^j = (\alpha^j,\varphi^j)$ defined by $\varphi_0^j = 0$ and*

$$\varphi_t^j = (1+r)^{-T+t}\frac{\sum_k \beta_k^j \mathbf{E}_{\mathbf{P}^j}\left[\mathrm{D}_{(t,k)}\mathrm{F}\,|\,\mathcal{F}_{t-1}\right]}{\mathrm{S}_{t-1}},$$

where $\beta_1^j := 2^{-1}(b-a\rho^j)^{-1}$, $\beta_{-1}^j := (2a)^{-1}$ and, on the other one, $\alpha_0^j = (1+r)^{-T}\mathbf{E}_{\mathbf{P}^j}[\mathrm{F}]$ and for any $t\in\mathbb{T}^*$,

$$\alpha_t^j = \alpha_{t-1}^j - \frac{(\varphi_t^j - \varphi_{t-1}^j)\mathrm{S}_{t-1}}{\mathrm{A}_{t-1}},$$

is a $\mathcal{F}$-predictable self-financed strategy that replicates F.

The application of the latter theorem to $\mathrm{F} = (1+r)^T\widehat{\mathrm{V}}_{x,T}^{\mathcal{F}}$ gives an expression of the optimizing strategy for the agent

$$\widehat{\varphi}_t = (1+r)^t\frac{\sum_k \widehat{\beta}_k \mathbf{E}_{\widehat{\mathbf{P}}^{\mathcal{F}}}\left[\mathrm{D}_{(t,k)}\widehat{\mathrm{V}}_{x,T}^{\mathcal{F}}\,|\,\mathcal{F}_{t-1}\right]}{\mathrm{S}_{t-1}}, \tag{4.7}$$

with $\widehat{\beta}_1 := 2^{-1}(b-a\widehat{\rho})^{-1}$, $\widehat{\beta}_{-1} := (2a)^{-1}$ where $\widehat{\rho} := [\lambda(1-p^0)]/(1-\lambda p^0)$, and

$$\begin{aligned}&\mathbf{E}_{\widehat{\mathbf{P}}^{\mathcal{F}}}\left[\mathrm{D}_{(t,k)}\widehat{\mathrm{V}}_{x,T}^{\mathcal{F}}\,|\,\mathcal{F}_{t-1}\right]\\ &\quad= x\Big(\frac{p}{p_0}\mathbf{1}_{\{k=1\}} + \frac{1-p}{1-p_0}\mathbf{1}_{\{k=-1\}}\Big)\prod_{s=1}^{t-1}\left(\mathbf{1}_{\{(s,\pm 1)\notin\eta\}} + \frac{p}{p^0}\mathbf{1}_{\{(s,1)\in\eta\}} + \frac{1-p}{1-p^0}\mathbf{1}_{\{(s,-1)\in\eta\}}\right).\end{aligned}$$

The expression (4.7) of the amount of risky asset that is needed to replicate F is the transposition into our frame of the Ocone-Karatzas formula stated in Privault in the binomial model ([50], proposition 1.14.4). The two expressions are closely resembling and differ only in the different expression of the gradient in each context.

4.3.3 Insider's portfolio optimization

Let us first tackle the problem (4.3) for some $t \in \mathbb{T}^{*,\circ}$. Insider's portfolio optimization can be performed by repeating *mutatis mutandis* the solution of agent's problem. This means replacing the underlying filtration $\mathcal{F}$ by $\mathcal{G}$, and identifying the set $\mathscr{S}_{\mathcal{G}}(x,t)$. As seen in subsection 3.2, for all $j \in \mathcal{J}$, any $(\mathbf{P}^j, \mathcal{F})$-martingale is a $(\mathbf{Q}^j_t, \mathcal{G})$-martingale on $[\![0,t]\!]$ where $\mathbf{Q}^j_t$ is defined by (3.3). In particular, the discounted prices process $(\overline{\mathrm{S}}_t)_{t\in\mathbb{T}^\circ}$ is a $(\mathbf{Q}^j_{T-1}, \mathcal{G})$-martingale on $\mathbb{T}^\circ$; so does $\mathrm{V}^{\mathcal{G}}_{x,\cdot}(\psi)$ viewed as the $\mathcal{G}$-martingale transform and such that for any $t \in \mathbb{T}^\circ$,

$$\overline{\mathrm{V}}^{\mathcal{G}}_{x,t}(\psi) = x + \sum_{s=1}^{t} \varphi^{\mathcal{G}}_s \,\Delta\overline{\mathrm{S}}_s,$$

where $(x, \varphi^{\mathcal{G}})$ is a $\mathcal{G}$-eligible strategy. Moreover, for $T=1$, the set of $\mathcal{G}$-martingale measures consists of the convex combinations of $\mathbf{Q}^{\mathrm{G},0}$ and $\mathbf{Q}^{\mathrm{G},1}$. In particular we have

$$\mathrm{d}\mathbf{Q}^{\mathrm{G},0} = u_1^{\mathrm{G},0}\mathrm{d}\mathbf{P} = \frac{\mathrm{L}^0_1}{p^{\mathrm{G}}_1}\mathrm{d}\mathbf{P} = \frac{1}{p^{\mathrm{G}}_1}\mathrm{d}\mathbf{P}^0.$$

The transposition of Proposition 4.3 into insider's paradigm gives the following result.

Proposition 4.5. *The maximal expected logarithmic utility up to time $t \in \mathbb{T}^{*,\circ}$ for the insider initially having x euros is given by*

$$\Phi^{\mathcal{G}}_t(x) = \log((1+r)^t x) - \mathbf{E}\left[\log\left(\frac{\mathrm{d}\widehat{\mathbf{Q}}^{\mathcal{G}}}{\mathrm{d}\mathbf{P}}\bigg|_{\mathcal{G}_t}\right)\right]$$

where

$$\frac{\mathrm{d}\widehat{\mathbf{Q}}^{\mathcal{G}}}{\mathrm{d}\mathbf{P}}\bigg|_{\mathcal{G}_t} := \prod_{s=1}^{t}\frac{1}{p^{\mathrm{G}}_s}\left(\mathbf{1}_{\{(s,\pm1)\notin\eta\}} + \frac{p^0}{p}\mathbf{1}_{\{(s,1)\in\eta\}} + \frac{1-p^0}{1-p}\mathbf{1}_{\{(s,-1)\in\eta\}}\right) = \frac{\mathrm{d}\widehat{\mathbf{P}}^{\mathcal{F}}}{\mathrm{d}\mathbf{P}}\bigg|_{\mathcal{F}_t}\prod_{s=1}^{t}\frac{1}{p^{\mathrm{G}}_s}. \tag{4.8}$$

This is reached for a unique strategy which discounted value at time t is given $\widehat{\mathrm{V}}^{\mathcal{G}}_{x,t} = \overline{\mathrm{V}}_{x,t}(\widehat{\psi}^{\mathcal{G}}) = x + \sum_{s\in[\![1,t]\!]}\widehat{\varphi}^{\mathcal{G}}_s\Delta\overline{\mathrm{S}}_s$ where

$$\widehat{\mathrm{V}}^{\mathcal{G}}_{x,t} = x \cdot \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\widehat{\mathbf{Q}}^{\mathcal{G}}}\bigg|_{\mathcal{G}_t}.$$

Proof. For any $t \in \mathbb{T}^{*,\circ}$, $(\Delta\overline{\mathrm{S}}_s)_{s\in[\![0,t]\!]}$ is a $(\mathbf{Q}^j_t, \mathcal{G})$-martingale. Then the result follows simply by replacing in Proposition 4.3 $\mathcal{F}$ and $\widehat{\mathbf{P}}^{\mathcal{F}}_t$ respectively by $\mathcal{G}$ and $\widehat{\mathbf{P}}^{\mathcal{G}}_t$. □

Since there is absolutely no reason why a $(\mathbf{P}, \mathcal{F})$-martingale should be a $(\mathbf{Q}^j_T, \mathcal{G})$-martingale on $\mathbb{T}$, we need to address the problem at the deadline T with another kind of argument. For some $x \in \mathbb{R}^*_+$, let $\Phi^{\mathcal{G}}_1(x)$ be the solution of (4.3) when $T=1$. For all $s,t\in\mathbb{T}$ such that $s<t$, we define $\mathcal{H}^{s,t} = (\mathcal{F}_r)_{s+1\leqslant r\leqslant t}$. The class of $\mathcal{H}^{s,t}$-eligible strategies up to time $r \in [\![s+1,t]\!]$ is defined via (4.2) by taking $\mathcal{H} = \mathcal{H}^{s,t}$ and denoted by $\mathscr{S}_{\mathcal{H}^{s,t}}(x,r)$. We can state the following result.

Proposition 4.6. *For any $x \in \mathbb{R}^*_+$ define $\Phi^{\mathcal{G}}_T(x)$ by considering (4.3) at the deadline T. Then,*

$$\Phi^{\mathcal{G}}_T(x) = \sup_{\psi\in\mathscr{S}_{\mathcal{H}^{T-1,T}}(x,T)} \mathbf{E}\left[\log\left(\mathrm{V}^{\mathcal{G}}_{\tilde{x},T}(\psi)\right)\right] = \sup_{\psi\in\mathscr{S}_{\mathcal{H}^{0,1}}(x,1)} \mathbf{E}\left[\log\left(\mathrm{V}^{\mathcal{G}}_{\tilde{x},1}(\psi)\right)\right] = \Phi^{\mathcal{G}}_1(\tilde{x}),$$

where $\tilde{x} = \Phi^{\mathcal{G}}_{T-1}(x)$.

4.4 Additional expected utility of the insider in the ternary model

Insider's additional expected logarithmic utility up to time $t \in \mathbb{T}^\circ$ is defined by

$$\mathcal{U}_t(x) = \sup_{\psi \in \mathscr{S}_{\mathcal{G}}(x,t)} \mathbf{E}\left[u(\mathrm{V}_{x,t}(\psi))\right] - \sup_{\psi \in \mathscr{S}_{\mathcal{F}}(x,t)} \mathbf{E}\left[u(\mathrm{V}_{x,t}(\psi))\right].$$

Given two probability measures defined on the same mesurable space $(\Omega, \mathcal{F})$, $\mathfrak{D}_{\mathcal{F}}(\mathbf{P}||\mathbf{Q})$ designates the relative entropy of $\mathbf{P}$ with respect to $\mathbf{Q}$ on $\mathcal{F}$ and is defined by

$$\mathfrak{D}_{\mathcal{F}}(\mathbf{P}||\mathbf{Q}) = \begin{cases} \mathbf{E}\left[\log\left(\left.\dfrac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{Q}}\right|_{\mathcal{F}}\right)\right] & \text{if} \quad \mathbf{P} \ll \mathbf{Q} \text{ on } \mathcal{F}, \\ +\infty & \text{otherwise.} \end{cases}$$

Define also for any $t \in \mathbb{T}^\circ$, $\mathrm{Ent}(\mathrm{G})$ and $\mathrm{Ent}(\mathrm{G}\,|\,\mathcal{F}_t)$ by

$$\mathrm{Ent}(\mathrm{G}) = -\sum_{\mathrm{c} \in \Gamma} \log\big(\mathbf{P}(\mathrm{G} = \mathrm{c})\big)\,\mathbf{P}(\mathrm{G} = \mathrm{c}),$$

and

$$\mathrm{Ent}(\mathrm{G}\,|\,\mathcal{F}_t) = -\mathbf{E}\left[\sum_{\mathrm{c} \in \Gamma} \log\big(\mathbf{P}(\mathrm{G} = \mathrm{c}\,|\,\mathcal{F}_t)\big)\,\mathbf{P}(\mathrm{G} = \mathrm{c}\,|\,\mathcal{F}_t)\right],$$

that respectively stand for the *entropy* and the *conditional entropy* of the random variable G. Here stands our third main result.

Theorem 4.7. *The insider's additional expected logarithmic utility up to time $t \in \mathbb{T}^\circ$ is given by*

$$\mathcal{U}_t = \mathfrak{D}_{\mathcal{G}_t}(\widehat{\mathbf{P}}^{\mathcal{F}}||\widehat{\mathbf{Q}}^{\mathcal{G}}_t) = \mathrm{Ent}(\mathrm{G}) - \mathrm{Ent}(\mathrm{G}\,|\,\mathcal{F}_t). \tag{4.9}$$

Thus, we get an identity akin to the one established by J. Amendinger, P. Imkeller and M. Schweizer [8] in the Black-Scholes model; the additional expected logarithmic utility of the insider can be expressed in terms of relative entropy. The presence of $\widehat{\mathbf{P}}^{\mathcal{F}}$ instead of $\mathbf{P}$ in the continuous original result can be justified by the fact the price process is not a $(\mathbf{P}, \mathcal{F})$-martingale in our frame (unlike that of [8]). Under an initial enlargement (continuous) setting, Ankirchner *et al.* ([9], theorem 5.12) provide an expression of the additional utility of the insider in terms of the *relative difference* of the enlarged filtration with respect to the initial one, and that also coincides with the Shannon entropy between (with the corresponding notations) G and $\mathrm{Id}_{\mathcal{F}^0_T}$ where $\mathrm{Id}_{\mathcal{A}} : \omega \in (\Omega, \mathcal{A}) \mapsto \omega \in (\Omega, \mathcal{F})$ for any sub-algebra $\mathcal{A} \subset \mathcal{F}$. Moreover, in the continuous case, the result still holds at the deadline T by taking the limit when t goes to T. Here, we need to appeal to Proposition 4.6 to state the following result.

Corollary 4.8. *The insider's additional expected logarithmic utility at expiry is given by*

$$\mathcal{U}_T(x) = \Phi^{\mathcal{G}}_1(\tilde{x}^{\mathcal{G}}) - \Phi^{\mathcal{F}}_1(\tilde{x}^{\mathcal{F}}), \tag{4.10}$$

where $\tilde{x}^{\mathcal{G}} = \Phi^{\mathcal{G}}_{T-1}(x)$ and $\tilde{x}^{\mathcal{F}} = \Phi^{\mathcal{F}}_{T-1}(x)$ are respectively given by Proposition 4.5 and Proposition 4.3.

Proof. The identity $\Phi^{\mathcal{F}}_1(\tilde{x}^{\mathcal{F}}) = \Phi^{\mathcal{F}}_T(\tilde{x})$ can be easily stated by adapting Proposition 4.6 to agent's optimization problem. The result follows. □

4.5 Arbitrages

An important question that arises is whether arbitrage is produced by the enlargement of insider's filtration. Roughly speaking, an investor has an *arbitrage opportunity* also called *free lunch* if he or she can hope to make profit without taking some risk. Several equivalent definitions coexist and we recall here the definition used by Dalang, Morton and Willinger in [24].

Definition 4.9 (Arbitrage, see Dalang *et al.* [24])**.** In a market model $(\mathbb{T}, \mathcal{F}, \mathbf{P}, \mathrm{S})$, an *arbitrage opportunity* is a $\mathcal{F}$-predictable self-financed trading strategy ψ such that $\mathrm{V}_0(\psi) = 0$, $\mathrm{V}_T(\psi) \geqslant 0$ $\mathbf{P}$-a.s. and $\mathrm{V}_T(\psi) > 0$ with positive probability. The market model $(\mathbb{T}, \mathcal{F}, \mathbf{P}, \mathrm{S})$ has *no-arbitrage* or is said to be *arbitrage-free* if it contains no arbitrage opportunities, i.e. if for all $\mathcal{F}$-predictable self-financed trading strategies ψ with $\mathrm{V}_0(\psi) = 0$ and $\mathrm{V}_T(\psi) = 0$ $\mathbf{P}$-a.s., we have $\mathrm{V}_T(\psi) = 0$ almost surely.

Furthermore, it can be shown (see again Dalang *et al.* [24]) that $(\mathbb{T}, \mathcal{F}, \mathbf{P}, \mathrm{S})$ is arbitrage-free is there exists a positive $(\mathbf{P}, \mathcal{F})$-martingale M, with $\mathrm{M}_0 = 1$ such that SM is a $(\mathbf{P}, \mathcal{F})$-martingale. From that starting point, Blanchet-Scalliet *et al.* design the notion of *model free* in the setting of enlargement of filtrations. We adapt it slightly to support our coming result.

Definition 4.10 (Model free, see Blanchet *et al.* [13])**.** Given two filtrations $\mathcal{F}$ and $\mathcal{G}$ such that $\mathcal{F} \subset \mathcal{G}$ and $\mathbb{I}$ a subset of $\mathbb{T}$, the enlarged model $(\mathbb{T}, \mathcal{F}, \mathcal{G}, \mathbf{P}, \mathrm{S})$ is *arbitrage-free* on the time horizon $\mathbb{I}$ if there exists a positive $(\mathbf{P}, \mathcal{G})$-martingale $\mathrm{M} = (\mathrm{M}_t)_{t\in\mathbb{I}}$, called $(\mathbb{I}, \mathcal{F}, \mathcal{G})$-*deflator*, such that $(\mathrm{S}_t\mathrm{M}_t)_{t\in\mathbb{I}}$ is a $(\mathbf{P}, \mathcal{G})$-martingale.

Proposition 4.11. *The model* $(\mathbb{T}, \mathcal{F}, \mathcal{G}, \mathbf{P}, \mathrm{S})$ *is not free on the time horizon* $\mathbb{T}$*. It is however on time horizon* $\mathbb{T}^\circ$*, and for all* $j \in \mathcal{J}$*,* $u^{\mathrm{G},j}$ *is a* $(\mathbb{T}^\circ, \mathcal{F}, \mathcal{G})$*-deflator.*

4.6 Computations in the case $\mathrm{G} = \mathbf{1}_{\{\overline{\mathrm{S}}_T \leqslant \mathrm{S}_0\}}$

In that case, the insider knows from the start whether it is worth investing in the risky asset; this one appears in fact riskless for him or her since he or she knows the outcome. One of two things must be true: either $\mathrm{G} = 1$ namely the discounted stock price does not increase and it may be better to invest the entire capital in the asset A, or $\mathrm{G} = 0$ and investing in S is more profitable. As a reminder the budget constraint can be written as

$$x = \mathrm{V}_0(\psi) = \alpha_0^{\mathcal{G}} + \varphi_0^{\mathcal{G}}\mathrm{S}_0,$$

and be transposed at time $T = 1$ to $\alpha_1^{\mathcal{G}} + \varphi_1^{\mathcal{G}}\mathrm{S}_0 = x$, by readjusting the portfolio under the self-financing condition. We get clearly the $\mathcal{G}$-eligible optimal strategy $\widehat{\psi} = (\widehat{\alpha}, \widehat{\varphi})$

$$\widehat{\alpha}_1^{\mathcal{G}} = x\mathbf{1}_{\{\mathrm{G}=1\}}, \quad \widehat{\varphi}_1^{\mathcal{G}} = x\mathrm{S}_0^{-1}\mathbf{1}_{\{\mathrm{G}=0\}} \quad \text{and} \quad \mathrm{V}_1(\widehat{\psi}) = x\Big[(1+r)\mathbf{1}_{\{\mathrm{G}=1\}} + \mathrm{S}_1\mathrm{S}_0^{-1}\mathbf{1}_{\{\mathrm{G}=0\}}\Big].$$

Besides, the maximal expected logarithmic utility of the agent is provided at any time $t \in \mathbb{T}$ by (4.6). That of the insider is given in Proposition 4.5 where $\widehat{\mathbf{Q}}^{\mathcal{G}}|\mathcal{G}_t$ is defined by (4.8) and $(p_t^{\mathrm{G}})_{t\in\mathbb{T}^\circ}$ is defined by $p_t^1 = \mathbf{P}(\mathrm{G} = 1 \,|\, \mathcal{F}_t)/\mathbf{P}(\mathrm{G} = 1)$, $p_t^0 = \mathbf{P}(\mathrm{G} = 0 \,|\, \mathcal{F}_t)/\mathbf{P}(\mathrm{G} = 0)$ with

$$\begin{aligned} p_t^1 &= \mathbf{P}(\{\overline{\mathrm{S}}_T \leqslant \mathrm{S}_0\} \,|\, \mathcal{F}_t) \\ &= \mathbf{P}\big(\{(\overline{\mathrm{S}}_t)^{-1}\overline{\mathrm{S}}_T \leqslant \mathrm{c}^{-1}\mathrm{S}_0\} \,\big|\, \mathcal{F}_t\big)\big|_{\mathrm{c}=\overline{\mathrm{S}}_t} = \mathbf{P}\big(\{\mathrm{S}_{T-t} \leqslant \mathrm{c}^{-1}\mathrm{S}_0(1+r)^{T-t}\} \,\big|\, \mathcal{F}_t\big)\big|_{\mathrm{c}=\overline{\mathrm{S}}_t}, \end{aligned}$$

where we have used that the variable $\mathrm{S}_T/\mathrm{S}_t$ has the same law as S_{T-t}. Moreover we can write $\mathrm{S}_t = \mathrm{S}_0(1+b)^{(\zeta_1^t-\zeta_{-1}^t)_+}(1+a)^{(\zeta_{-1}^t-\zeta_1^t)_+}$ where $\zeta_{\pm 1}^t = \sum_{s\in[\![1,t]\!]} \mathbf{1}_{\{(s,\pm1)\in\eta\}}$ for all $t \in \mathbb{T}$. Then for all $t \in \mathbb{T}$, $(\zeta_1^t, \zeta_{-1}^t, t-(\zeta_1^t+\zeta_{-1}^t))$ follows a trinomial law of parameters t (number of trials), $\mathbf{P}(\zeta_1^1) = \lambda p$ and $\mathbf{P}(\zeta_{-1}^1) = \lambda(1-p)$. Let $\mathfrak{n}$ be the maximal integer such that $(1+b)^{\mathfrak{n}} \leqslant (1+r)^T$. Last, we get an explicit expression insider's additional expected logarithmic utility at expiry.

Proposition 4.12. *In the case* $\mathrm{G} = \mathbf{1}_{\{\overline{\mathrm{S}}_T \leqslant \mathrm{S}_0\}}$*, the additional expected logarithmic utility of a insider having initially* $x \in \mathbb{R}_+^*$ *is given at expiry* T *by*

$$\begin{aligned}\mathcal{U}_T(x) &= \Phi_1^{\mathcal{G}}(\tilde{x}^{\mathcal{G}}) - \Phi_1^{\mathcal{G}}(\tilde{x}^{\mathcal{F}})\\ &= \mathbf{E}\big[\mathbf{P}(\mathrm{G}=1\,|\,\mathcal{F}_{T-1})\log\big(\mathbf{P}(\mathrm{G}=1\,|\,\mathcal{F}_{T-1})\big) + \mathbf{P}(\mathrm{G}=0\,|\,\mathcal{F}_{T-1})\log\big(\mathbf{P}(\mathrm{G}=0\,|\,\mathcal{F}_{T-1})\big)\big]\\ &\quad + \mathbf{P}(\mathrm{G}=1)\log\big(\mathbf{P}(\mathrm{G}=1)\big) + \mathbf{P}(\mathrm{G}=0)\log\big(\mathbf{P}(\mathrm{G}=0)\big)\\ &\quad + \lambda p \log\left[\frac{(1+b)p_0}{(1+r)p}\right] + \lambda(1-p)\log\left(\frac{1-p_0}{1-p}\right),\end{aligned}$$

with

$$\mathbf{P}(\mathrm{G}=1) = \mathbf{P}(\chi_+^T \leqslant T, \chi_-^T \leqslant \mathfrak{n}) = 1 - \mathbf{P}(\mathrm{G}=0),$$

and

$$\mathbf{P}(\mathrm{G}=1\,|\,\mathcal{F}_{T-1}) = \mathbf{P}(\chi_+^T - \chi_+\zeta^{T-1} \leqslant 1, \chi_-^T - \chi_-^{T-1} \leqslant \mathfrak{n}-\mathfrak{n}^-)|_{\mathrm{A}_{T-1}},$$

where we have defined for $t \in \{T-1, T\}$*,* $\chi_+^t = \zeta_1^t + \zeta_{-1}^t$*,* $\chi_-^t = \zeta_1^t - \zeta_{-1}^t$*, as well as* $\mathrm{A}_{T-1} = \chi_+^{T-1} = \mathfrak{n}^+, \chi_-^{T-1} = \mathfrak{n}^-\}$*, and the couple* $(\mathfrak{n}^+, \mathfrak{n}^-) \in (\mathbb{Z}_+)^2$ *satisfies* $\mathfrak{n}^+ + \mathfrak{n}^- \leqslant T-1$ *and* $\mathfrak{n}^+ - \mathfrak{n}^- \leqslant \mathfrak{n}+1$.

This explicit example highlights that insider's additional expected logarithmic utility depends on the parameter λ. As was anticipated, the bigger $\lambda \in (0,1)$, the more volatile the model and the greater the benefit of the information surplus.

5 Proofs

5.1 Proofs of Section 3

Proof of Proposition 3.1. The proof follows closely the one of Proposition 2.3 in Amendinger, Imkeller and Schweizer [8]. For any $t \in \mathbb{T}^\circ$, $\mathrm{B}_t \in \mathcal{F}_t$, $\mathrm{C} \in \mathscr{G}$, and some $j \in \mathcal{J}$,

$$\mathbf{E}\left[\mathbf{1}_{\mathrm{B}_t\cap\{\mathrm{G}\in\mathrm{C}\}}\frac{\mathrm{L}_t^j}{p_t^{\mathrm{G}}}\right] = \mathbf{E}\left[\mathrm{L}_t^j\mathbf{1}_{\mathrm{B}_t}\mathbf{E}\left[\mathbf{1}_{\{\mathrm{G}\in\mathrm{C}\}}\frac{1}{p_t^{\mathrm{G}}}\,\middle|\,\mathcal{F}_t\right]\right] = \mathbf{E}\left[\mathrm{L}_t^j\mathbf{1}_{\mathrm{B}_t}\right]\mathbf{P}(\{\mathrm{G}\in\mathrm{C}\}) = \mathbf{P}^j(\mathrm{B}_t)\mathbf{P}(\{\mathrm{G}\in\mathrm{C}\}), \tag{5.1}$$

where we have used: $\mathbf{E}\big[\mathbf{1}_{\{\mathrm{G}\in\mathrm{C}\}}/p_t^{\mathrm{G}}\,\big|\,\mathcal{F}_t\big] = \sum_{\mathrm{c}\,\in\,\Gamma\cap\mathrm{C}} \frac{1}{p_t^{\mathrm{c}}(\omega)} \cdot p_t^{\mathrm{c}}(\omega) \cdot \mathbf{P}(\{\mathrm{G}=\mathrm{c}\}) = \mathbf{P}(\{\mathrm{G}\in\mathrm{C}\})$.

This yields by the definition of $\mathbf{Q}_t^j$ given in the theorem,

$$\mathbf{Q}_t^j(\mathrm{B}_t \cap \{\mathrm{G}\in\mathrm{C}\}) = \mathbf{P}^j(\mathrm{B}_t)\mathbf{P}(\mathrm{G}\in\mathrm{C}).$$

Taking $\mathrm{B}_t = \Omega$, then $\mathrm{C} = \Gamma$ provides

$$\mathbf{Q}_t^j(\mathrm{B}_t \cap \{\mathrm{G}\in\mathrm{C}\}) = \mathbf{Q}_t^j(\mathrm{B}_t)\mathbf{Q}_t^j(\mathrm{G}\in\mathrm{C}),$$

and enables to establish 2. and 3. Let $s \in [\![0, t-1]\!]$, $\mathrm{B}_s \in \mathcal{F}_s$ and $\mathrm{A}_s = \mathrm{B}_s \cap \{\mathrm{G} \in \mathrm{C}\}$ an element of $\mathcal{G}_s$. Then, identical computations as in (5.1) but by conditioning with respect to $\mathcal{F}_s$ (in the first equality) lead to

$$\mathbf{E}\left[\mathbf{1}_{\mathrm{A}_s}\frac{\mathrm{L}^j_t}{p^{\mathrm{G}}_t}\right] = \mathbf{P}^j(\mathrm{B}_s)\mathbf{P}(\mathrm{G}\in\mathrm{C}) = \mathbf{Q}^j_t(\mathrm{B}_s \cap \{\mathrm{G}\in\mathrm{C}\}) = \mathbf{E}\left[\mathbf{1}_{\mathrm{A}_s}\frac{\mathrm{L}^j_s}{p^{\mathrm{G}}_s}\right],$$

so that the process $u^{\mathrm{G},j} = \mathrm{L}^j/p^{\mathrm{G}}$ is a $(\mathbf{P}, \mathcal{G})$-martingale on $\mathbb{T}^\circ$. Hence the result. □

Proof of Proposition 3.2. Let $t \in \mathbb{T}^\circ$ and $(\mathrm{M}_s)_{0\leqslant s\leqslant t}$ a $(\mathbf{P}^j, \mathcal{F})$-martingale on $[\![0, t]\!]$. For $r \in [\![0, t-1]\!]$ and $s \in [\![r+1, t]\!]$, let $\mathrm{B}_r \in \mathcal{F}_r$, $\mathrm{C} \in \mathscr{G}$ and $\mathrm{A}_r = \mathrm{B}_r \cap \{\mathrm{G}\in\mathrm{C}\}$ an element of $\mathcal{G}_r$. Let $\mathbf{E}_{\mathbf{Q}^j_t}$ denote the expectation taken with respect to $\mathbf{Q}^j_t$.

$$\begin{aligned}
\mathbf{E}_{\mathbf{Q}^j_t}\left[\mathbf{1}_{\mathrm{A}_r}\mathrm{M}_s\right] &= \mathbf{E}_{\mathbf{Q}^j_t}\left[\mathbf{1}_{\mathrm{B}_r}\mathrm{M}_s\right]\mathbf{E}_{\mathbf{Q}^j_t}\left[\mathbf{1}_{\{\mathrm{G}\in\mathrm{C}\}}\right]\\
&= \mathbf{E}_{\mathbf{P}^j}\left[\mathbf{1}_{\mathrm{B}_r}\mathrm{M}_s\right]\mathbf{E}_{\mathbf{Q}^j_t}\left[\mathbf{1}_{\{\mathrm{G}\in\mathrm{C}\}}\right]\\
&= \mathbf{E}_{\mathbf{P}^j}\left[\mathbf{1}_{\mathrm{B}_r}\mathbf{E}\left[\mathrm{M}_s|\mathcal{F}_r\right]\right]\mathbf{E}_{\mathbf{Q}^j_t}\left[\mathbf{1}_{\{\mathrm{G}\in\mathrm{C}\}}\right]\\
&= \mathbf{E}_{\mathbf{P}^j}\left[\mathbf{1}_{\mathrm{B}_r}\mathrm{M}_r\right]\mathbf{E}_{\mathbf{Q}^j_t}\left[\mathbf{1}_{\{\mathrm{G}\in\mathrm{C}\}}\right]\\
&= \mathbf{E}_{\mathbf{Q}^j_t}\left[\mathbf{1}_{\mathrm{B}_r}\mathrm{M}_r\right]\mathbf{E}_{\mathbf{Q}^j_t}\left[\mathbf{1}_{\{\mathrm{G}\in\mathrm{C}\}}\right] = \mathbf{E}_{\mathbf{Q}^j_t}\left[\mathbf{1}_{\mathrm{A}_r}\mathrm{M}_r\right],
\end{aligned}$$

where we have used that the σ-algebras $\mathcal{F}_s \subset \mathcal{F}_t$ and $\sigma(\mathrm{G})$ are independent under $\mathbf{Q}^j_t$ in the first line, that $\mathbf{P}^j$ coincides with $\mathbf{Q}^j_t$ on $\mathcal{F}_t$ in the second one, and that $(\mathrm{M}_s)_{1\leqslant s\leqslant t}$ is a $(\mathbf{P}^j, \mathcal{F})$-martingale in the fourth one. Then, $(\mathrm{M}_s)_{1\leqslant s\leqslant t}$ is a $(\mathbf{P}^j, \mathcal{G})$-martingale on $[\![0, t]\!]$. Since $\mathbf{P}^j = \mathbf{Q}^j_t$ on $(\Omega, \mathcal{F}_t)$, the sets of $(\mathbf{P}^j, \mathcal{F})$- and $(\mathbf{Q}^j_t, \mathcal{F})$-martingales on $[\![0, t]\!]$ are equal. The proof is complete. □

5.2 Proofs of Section 4

Proof of Theorem 4.1. Fix $j \in \mathcal{J}$ and consider the process $\mu^{\mathcal{G},j}$ defined in (3.2) by taking $\mathrm{X}^j_t = \overline{\mathrm{Y}}^j_t$. The proof directly derives from the Clark-Ocone formula (2.17) applied to the $(\mathbf{P}^j, \mathcal{F})$-martingale $u^{\mathrm{G},j}$. Taking $s = t-1$ provides

$$\Delta p^{\mathrm{G}}_t = p^{\mathrm{G}}_t - p^{\mathrm{G}}_{t-1} = \sum_{\ell\in\mathrm{E}} \mathbf{E}_{\mathbf{P}^j}[\mathrm{D}_{(t,\ell)}\, p^{\mathrm{G}}_t \,|\, \mathcal{F}_{t-1}]\, \Delta\mathrm{R}^j_{(t,\ell)}.$$

As stated in Lemma 1.4 of Blanchet *et al.* [13], for two $\mathcal{F}$-adapted processes U and K, $\langle \mathrm{U}, \mathrm{K}\rangle^{\mathbf{P}}_0 = 0$ and $\Delta\langle \mathrm{U}, \mathrm{K}\rangle^{\mathbf{P}}_t = \mathbf{E}_{\mathbf{P}}[\Delta\mathrm{U}_t\Delta\mathrm{K}_t \,|\, \mathcal{F}_{t-1}]$ for all $t \in \mathbb{T}^*$. Then we get, for any $\mathrm{c} \in \Gamma$,

$$\begin{aligned}
\Delta\langle \overline{\mathrm{Y}}^j, p^{\mathrm{c}}\rangle^{\mathbf{P}^j}_t &= \mathbf{E}_{\mathbf{P}^j}\left[\sum_{k\in\mathrm{E}}\Delta\mathrm{Z}^j_{(t,k)}\sum_{\ell\in\mathrm{E}}\mathbf{E}_{\mathbf{P}^j}[\mathrm{D}_{(t,\ell)}\, p^{\mathrm{c}}_t|\mathcal{F}_{t-1}]\,\Delta\mathrm{R}^j_{(t,\ell)}\middle|\mathcal{F}_{t-1}\right]\\
&= \sum_{k\in\mathrm{E}}\sum_{\ell\in\mathrm{E}}\mathbf{E}_{\mathbf{P}^j}[\mathrm{D}_{(t,\ell)}\, p^{\mathrm{c}}_t|\mathcal{F}_{t-1}]\mathbf{E}_{\mathbf{P}^j}\left[\Delta\mathrm{Z}^j_{(t,k)}\Delta\mathrm{R}^j_{(t,\ell)}\right] = \sum_{k\in\mathrm{E}}\sum_{\ell\in\mathrm{E}} a_{t,k,\ell}\mathbf{E}_{\mathbf{P}^j}[\mathrm{D}_{(t,\ell)}\, p^{\mathrm{c}}_t],
\end{aligned}$$

where we have got the second line by conditioning with respect to $\mathcal{F}_{t-1}$ and by defining the family $\{a^j_{t,k,\ell},\ (k,\ell)\in\mathrm{E}^2\}$ by $a^j_{t,k,\ell} = \mathbf{E}_{\mathbf{P}^j}[\Delta\mathrm{Z}^j_{(t,k)}\Delta\mathrm{R}^j_{(t,\ell)}]$, i.e.

$$a^j_{t,1,1} = \lambda^j p^j_t(1-\lambda p^j_t),\ \ a^j_{t,1,-1} = 0,\ \ a^j_{t,-1,1} = (\lambda^j)^2 p^j_t(1-p^j_t)\ \text{ and }\ a^j_{t,-1,-1} = \frac{\lambda^j(1-p^j_t)\,(1-\lambda^j)}{1-\lambda^j p^j_t}.$$

Hence the result. □

Proof of Propostion 4.3. We define for all $s, t \in \mathbb{T}$ such that $s < t$, $\mathcal{H}^{s,t} = (\mathcal{F}_r)_{s+1 \leqslant r \leqslant t}$. Let $x \in \mathbb{R}_+^*$. The expression of $\Phi_T^{\mathcal{F}}(x)$ can be deduced from the identity $\Theta_t^{\mathcal{F}}(x) = \Phi_{T-t}^{\mathcal{F}}(x)$, together with the solution of the following induction system

$$\begin{cases} \Theta_T^{\mathcal{F}}(x) & = & \log(x) \\ \Theta_{t-1}^{\mathcal{F}}(x) & = & \sup\limits_{\psi \in \mathscr{S}_{\mathcal{H}^{t-1,t}}(x,t-1)} \mathbf{E}\left[\Theta_t^{\mathcal{F}}(x + \varphi \Delta \mathrm{S}_t)\right] \ ; \ t \in \mathbb{T}^*, \end{cases}$$

where each ψ is as usual identified as $\psi = (x, \varphi)$. For $t = T-1$, since the $\Delta \mathrm{S}_t$ are independent and identically distributed,

$$\begin{aligned} \Theta_{T-1}^{\mathcal{F}}(x) &= \sup_{\psi \in \mathscr{S}_{\mathcal{H}^{T-1,T}}(x,T)} \mathbf{E}\left[u\left(x + \varphi\, \Delta \mathrm{S}_T\right) \,\Big|\, \mathcal{F}_{T-1}\right] \\ &= \sup_{\psi \in \mathscr{S}_{\mathcal{H}^{0,1}}(x,1)} \mathbf{E}\left[u\left(\tilde{x} + \varphi\, \Delta \mathrm{S}_1\right)\right] \big|_{(\tilde{x} = x + \varphi_{T-1}\, \Delta \mathrm{S}_{T-1})}, \end{aligned}$$

and so on by downward induction. The iteration of (4.6) provides

$$\Theta_t^{\mathcal{F}}(x) = \log(x(1+r)^{T-t}) - \mathbf{E}\left[\log\left(\frac{\mathrm{d}\widehat{\mathbf{P}}^{\mathcal{F}}}{\mathrm{d}\mathbf{P}}\Big|_{\mathcal{F}_{T-t}}\right)\right],$$

so that by letting $t = T - s$ with $s \in \mathbb{T}^*$,

$$\Phi_s^{\mathcal{F}}(x) = \log(x(1+r)^s) - \mathbf{E}\left[\log\left(\frac{\mathrm{d}\widehat{\mathbf{P}}^{\mathcal{F}}}{\mathrm{d}\mathbf{P}}\Big|_{\mathcal{F}_s}\right)\right]$$

where

$$\frac{\mathrm{d}\widehat{\mathbf{P}}^{\mathcal{F}}}{\mathrm{d}\mathbf{P}}\Big|_{\mathcal{F}_s} := \prod_{r=1}^{s}\left(\mathbf{1}_{\{(r,\pm 1) \notin \eta\}} + \frac{p}{p^0}\mathbf{1}_{\{(r,1) \in \eta\}} + \frac{1-p}{1-p^0}\mathbf{1}_{\{(r,-1) \in \eta\}}\right).$$

Moreover,

$$\widehat{\mathrm{V}}_s^{\mathcal{F}} = x \cdot \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\widehat{\mathbf{P}}^{\mathcal{F}}}\Big|_{\mathcal{F}_s},$$

holds for any $s \in \mathbb{T}^*$ so that the maximal expected utility and portfolio value are obtained the by taking $s = T$. □

Proof of Theorem 4.4. Let $j \in \mathcal{J}$. As a reminder, the strategy $\psi^j = (x, \varphi^j)$ is self-financed and only if the condition (4.1) is satisfied for all $t \in \mathbb{T}^*$ so that $\mathrm{V}_{t-1}(\psi^j) = \alpha_t^j \mathrm{A}_{t-1} + \varphi_t^j \mathrm{S}_{t-1}$. Let $\alpha_0^j = x$ and $\varphi_0^j = 0$. Assume the existence of a $\mathcal{F}$-eligible strategy ψ^j such that $\mathrm{V}_0(\psi^j) = x$ and which final value satisfies

$$\mathrm{V}_T(\psi^j) = \alpha_T^j\, \mathrm{A}_T + \varphi_T^j\, \mathrm{S}_T = \mathrm{F}.$$

Let π^j be the $\mathcal{F}$-predictable process such that $\pi^j_t = \dfrac{\varphi^j_t\,\mathrm{S}_{t-1}}{\mathrm{V}_{t-1}(\psi^j)}$ for any $t \in \mathbb{T}^*$. By definition,

$$\begin{aligned}
\Delta\mathrm{V}_t(\psi^j) &= \alpha^j_t \Delta\mathrm{A}_t + \varphi^j_t \Delta\mathrm{S}_t \\
&= \mathrm{V}_{t-1}(\psi^j)\left[\frac{\alpha^j_t\,\Delta\mathrm{A}_t}{\mathrm{V}_{t-1}(\psi^j)} + \pi^j_t\Big(b\mathbf{1}_{\{(\Delta\mathrm{N}_t,\mathrm{W}_t)=(1,1)\}} + a\mathbf{1}_{\{(\Delta\mathrm{N}_t,\mathrm{W}_t)=(1,-1)\}}\Big)\right] \\
&= \mathrm{V}_{t-1}(\psi^j)\Big(\frac{\alpha^j_t\, r\,\mathrm{A}_{t-1}}{\mathrm{V}_{t-1}(\psi^j)} + \pi^j_t\big(b\,\Delta\mathrm{Z}^j_{(t,1)} + a\,\Delta\mathrm{Z}^j_{(t,-1)} + r\big)\Big) \\
&= \mathrm{V}_{(t-1)}(\psi^j)\big(r(1-\pi^j_t) + \pi^j_t\big(b\,\Delta\mathrm{Z}^j_{(t,1)} + a\,\Delta\mathrm{Z}^j_{(t,-1)} + r\big)\big) \\
&= r\mathrm{V}_{t-1}(\psi^j) + \mathrm{V}_{t-1}(\psi^j)\pi^j_t\big((b - a\rho^j)\,\Delta\mathrm{R}^j_{(t,1)} + a\,\Delta\mathrm{R}^j_{(t,-1)}\big),
\end{aligned}$$

where we used that $\lambda^j[bp^j + a(1-p^j)] = r$ since $\overline{\mathrm{S}}$ is a $(\mathbf{P}^j, \mathcal{F})$-martingale in the second line and that $\dfrac{\alpha^j_t\,\mathrm{A}_{t-1}}{\mathrm{V}_{t-1}(\psi^j)} + \pi^j_t = 1$ in the third one. Then,

$$\Delta\overline{\mathrm{V}}_t(\psi^j) = \frac{\overline{\mathrm{V}}_{t-1}(\psi^j)\,\pi^j_t}{1+r}\big((b - a\rho^j)\,\Delta\mathrm{R}^j_{(t,1)} + a\,\Delta\mathrm{R}^j_{(t,-1)}\big),$$

so that

$$\overline{\mathrm{V}}_T(\psi^j) = \mathrm{V}_0(\psi^j) + \sum_{t=1}^{T}\frac{\overline{\mathrm{V}}_{t-1}(\psi^j)\,\pi^j_t}{1+r}\big((b - a\rho^j)\,\Delta\mathrm{R}^j_{(t,1)} + a\,\Delta\mathrm{R}^j_{(t,-1)}\big).$$

Since we have supposed that $\mathrm{F} = \mathrm{V}_T(\psi^j) = (1+r)^T\overline{\mathrm{V}}_T(\psi^j)$, by uniqueness of the Clark formula (2.16), we get $\mathrm{V}_0(\psi^j) = (1+r)^{-T}\mathbf{E}_{\mathbf{P}^j}[\mathrm{F}]$,

$$\mathbf{E}_{\mathbf{P}^j}\big[\mathrm{D}_{(t,1)}\mathrm{F}\,|\,\mathcal{F}_{t-1}\big] = (1+r)^T\mathbf{E}_{\mathbf{P}^j}\big[\mathrm{D}_{(t,1)}\overline{\mathrm{V}}_T(\psi^j)\,|\,\mathcal{F}_{t-1}\big] = (1+r)^{T-1}(b-a\rho^j)\overline{\mathrm{V}}_{t-1}(\psi^j)\,\pi^j_t,$$

and

$$\mathbf{E}_{\mathbf{P}^j}\big[\mathrm{D}_{(t,-1)}\mathrm{F}\,|\,\mathcal{F}_{t-1}\big] = (1+r)^T\mathbf{E}_{\mathbf{P}^j}\big[\mathrm{D}_{(t,-1)}\overline{\mathrm{V}}_T(\psi^j)\,|\,\mathcal{F}_{t-1}\big] = (1+r)^{T-1}a\overline{\mathrm{V}}_{t-1}(\psi^j)\,\pi^j_t.$$

This entails

$$\frac{1}{2}\mathbf{E}_{\mathbf{P}^j}\big[(b-a\rho)^{-1}\mathrm{D}_{(t,1)}\mathrm{F} + a^{-1}\mathrm{D}_{(t,-1)}\mathrm{F}\,|\,\mathcal{F}_{t-1}\big] = (1+r)^{T-1}\pi^j_t\,\overline{\mathrm{V}}_{t-1}(\psi^j) = (1+r)^{T-t}\pi^j_t\,\mathrm{V}_{t-1}(\psi^j).$$

Then, by letting on the one hand $\varphi^j_0 = 0$ and

$$\varphi^j_t = \frac{\mathrm{V}_{t-1}(\psi^j)\,\pi^j_t}{\mathrm{S}_{t-1}} = (1+r)^{-T+t}\frac{\sum_{k\in\mathrm{E}}\beta^j_k\mathbf{E}_{\mathbf{P}^j}\big[\mathrm{D}_{(t,k)}\mathrm{F}\,|\,\mathcal{F}_{t-1}\big]}{\mathrm{S}_{t-1}},$$

where $\beta^j_1 := 2^{-1}(b-a\rho^j)^{-1}$ and $\beta^j_{-1} := (2a)^{-1}$ and, on the other hand, $\alpha_0 = (1+r)^{-T}\mathbf{E}_{\mathbf{P}^j}[\mathrm{F}]$ and for any $t \in \mathbb{T}^*$,

$$\alpha^j_t = \alpha^j_{t-1} - \frac{(\varphi^j_t - \varphi^j_{t-1})\mathrm{S}_{t-1}}{\mathrm{A}_{t-1}},$$

we get a couple of $\mathcal{F}$-predictable processes $\psi^j = (\alpha^j, \varphi^j)$ that satisfies the self-financing condition and of terminal value F. Hence the result. □

Proof of Proposition 4.6. As a reminder, we have defined for all $s, t \in \mathbb{T}$ such that $s < t$, $\mathcal{H}^{s,t} = (\mathcal{F}_r)_{s+1 \leqslant r \leqslant t}$. By the very definition (4.3) of Φ, for any $\mathcal{G}$-eligible strategy ξ, $\Phi^{\mathcal{G}}_{T-1}(x) \geqslant \mathbf{E}[u(\mathrm{V}_{x,T-1}(\xi))]$. Since it is obvious that the greater the initial investment, the greater the expected utility, we get

$$\Phi^{\mathcal{G}}_1(\widetilde{x}) \geqslant \Phi^{\mathcal{G}}_1(x_\xi),$$

where $\widetilde{x} = \Phi^{\mathcal{G}}_{T-1}(x)$ and $x_\xi = \mathbf{E}[u(\mathrm{V}_{x,T-1}(\xi))]$. This holds for any $\mathcal{G}$-eligible strategy ξ. Let $\varepsilon \in \mathbb{R}^*_+$. By definition, there exists a $\mathcal{G}$-eligible strategy $\widehat{\xi}$ such that

$$\mathbf{E}[u(\mathrm{V}_{x,T}(\widehat{\xi}))] > \Phi^{\mathcal{G}}_T(x) - \frac{\varepsilon}{2},$$

and let $x_{\widehat{\xi}} = \mathbf{E}[u(\mathrm{V}_{x,T-1}(\widehat{\xi}))]$. Then,

$$\begin{aligned}
\Phi^{\mathcal{G}}_T(x) - \frac{\varepsilon}{2} &< \mathbf{E}[u(\mathrm{V}_{x,T}(\widehat{\xi}))] \\
&\leqslant \sup_{\psi \in \mathcal{H}^{T-1,T}-\text{portfolio}} \mathbf{E}[u(\mathrm{V}_{x_{\widehat{\xi}},T}(\psi))] \\
&= \sup_{\psi \in \mathcal{H}^{0,1}-\text{portfolio}} \mathbf{E}[u(\mathrm{V}_{x_{\widehat{\xi}},1}(\psi))] \\
&= \Phi^{\mathcal{G}}_1(x_{\widehat{\xi}}) < \mathbf{E}[u(\mathrm{V}_{x,T}(\zeta))] + \frac{\varepsilon}{2},
\end{aligned}$$

where ζ be the $\mathcal{G}$-eligible strategy defined by $\zeta_t = \widehat{\xi}_t$ for all $t \in \mathbb{T}^\circ$ and $\zeta_T = \varphi^{\widehat{\xi}}$ where, since the $\Delta\mathrm{S}_t$ are identically distributed, $\psi^{x_{\widehat{\xi}},\widehat{\xi}} = (x_{\widehat{\xi}}, \varphi^{\widehat{\xi}})$ is an element of $\mathscr{S}_{\mathcal{G}}(x_{\widehat{\xi}}, T)$ with $T = 1$ and such that $\mathbf{E}[u(x_{\widehat{\xi}} + \varphi^{\widehat{\xi}}\Delta\mathrm{S}_T)] > \Phi^{\mathcal{G}}_1(x_{\widehat{\xi}}) - \varepsilon/2$. The definition of $\Phi^{\mathcal{G}}_T(x)$ ensures that $\mathbf{E}[u(\mathrm{V}_{x,T}(\zeta))] \leqslant \Phi^{\mathcal{G}}_T(x)$ and provides the result. □

Proof of Proposition 4.7. Follows from Proposition 4.5 together with the definition (4.8) of $\widehat{\mathbf{Q}}^{\mathcal{G}}$, that for any $t \in \mathbb{T}^\circ$,

$$\widehat{\mathrm{V}}^{\mathcal{G}}_{x,t} = x \cdot \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\widehat{\mathbf{Q}}^{\mathcal{G}}}\bigg|_{\mathcal{G}_t} = x \cdot \frac{1}{p^{\mathrm{G}}_t} \cdot \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\widehat{\mathbf{P}}^{\mathcal{F}}}\bigg|_{\mathcal{F}_t}.$$

Then,

$$\begin{aligned}
\mathcal{U}_t(x) &= \log(x(1+r)^t) + \mathbf{E}\left[\log\left(\frac{1}{p^{\mathrm{G}}_t} \cdot \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\widehat{\mathbf{P}}^{\mathcal{F}}}\bigg|_{\mathcal{F}_t}\right)\right] - \mathbf{E}\left[\log(x(1+r)^t) - \log\left(\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\widehat{\mathbf{P}}^{\mathcal{F}}}\bigg|_{\mathcal{F}_t}\right)\right] \\
&= \mathbf{E}\left[\log(p^{\mathrm{G}}_t)\right] = \mathfrak{D}_{\mathcal{G}_t}(\widehat{\mathbf{P}}^{\mathcal{F}} || \widehat{\mathbf{Q}}^{\mathcal{G}}_t)
\end{aligned}$$

Moreover, since Γ is finite,

$$\begin{aligned}
\mathcal{U}_t(x) &= \mathbf{E}\left[\log(p^{\mathrm{G}}_t)\right] = \mathbf{E}\left[\sum_{\mathrm{c}\in\Gamma} \log(p^{\mathrm{c}}_t)\, \mathbf{P}(\mathrm{G} = \mathrm{c} \,|\, \mathcal{F}_t)\right] \\
&= \mathbf{E}\left[\sum_{\mathrm{c}\in\Gamma} \log\big(\mathbf{P}(\mathrm{G} = \mathrm{c} \,|\, \mathcal{F}_t)\big)\, \mathbf{P}(\mathrm{G} = \mathrm{c} \,|\, \mathcal{F}_t)\right] - \sum_{\mathrm{c}\in\Gamma} \log\big(\mathbf{P}(\mathrm{G} = \mathrm{c})\big) \mathbf{E}\left[\mathbf{E}\left[\mathbf{1}_{(\mathrm{G}=\mathrm{c})} \,|\, \mathcal{F}_t\right]\right] \\
&= \mathbf{E}\left[\sum_{\mathrm{c}\in\Gamma} \log\big(\mathbf{P}(\mathrm{G} = \mathrm{c} \,|\, \mathcal{F}_t)\big) \mathbf{P}(\mathrm{G} = \mathrm{c} | \mathcal{F}_t)\right] - \sum_{\mathrm{c}\in\Gamma} \log\big(\mathbf{P}(\mathrm{G} = \mathrm{c})\big)\, \mathbf{P}(\mathrm{G} = \mathrm{c}) \\
&= \mathrm{Ent}(\mathrm{G}) - \mathrm{Ent}(\mathrm{G} \,|\, \mathcal{F}_t),
\end{aligned}$$

where we get the second equality by conditioning on $\mathcal{F}_t$. Hence the result. □

Proof of Proposition 4.11. To prove that $(\mathbb{T}, \mathcal{F}, \mathcal{G}, \mathbf{P}, \mathrm{S})$ is not a free model on $\mathbb{T}$, we proceed as in Blanchet-Scalliet *et al.* (see [13], lemma 2.3). For any $t \in \mathbb{T}$, let $\mathrm{U}_t = \mathbf{E}[\mathrm{G}|\mathcal{F}_t]$. If a $(\mathbb{T}, \mathcal{F}, \mathcal{G})$-deflator M exists, the process UM would be a $(\mathbf{P}, \mathcal{G})$-martingale and then $\mathrm{U}_t\mathrm{M}_t = \mathbf{E}[\mathrm{U}_T\mathrm{M}_T|\mathcal{G}_t]$. Since $\mathrm{G} \in \mathcal{G}_t$ for all $t \in \mathbb{T}$ and M is a $(\mathbf{P}, \mathcal{G})$-martingale, we have $\mathrm{U}_T \in \mathcal{G}_t$ and $\mathbf{E}[\mathrm{U}_T\mathrm{M}_T|\mathcal{G}_t] = \mathrm{U}_T\mathrm{M}_t$ that leads by taking $t = 0$ to $\mathrm{U}_T\mathrm{M}_0 = \mathrm{U}_0\mathrm{M}_0$ and contradicts $\mathrm{U}_T = \mathrm{G} \notin \mathcal{F}_0$.
Let us show that for a given $j \in \mathcal{J}$, $u^{\mathrm{G},j}$ defines a $(\mathbb{T}^\circ, \mathcal{F}, \mathcal{G})$-deflator on $\mathbb{I} = \mathbb{T}^\circ$. For any $t \in \mathbb{T}^\circ$,

$$\mathbf{E}[u^{\mathrm{G},j}\Delta\overline{\mathrm{S}}_t|\mathcal{G}_{t-1}] = \mathbf{E}_{\mathbf{Q}_t^j}[\Delta\overline{\mathrm{S}}_t|\mathcal{G}_{t-1}] = 0,$$

since $\overline{\mathrm{S}}$ is a $(\mathbf{P}^j, \mathcal{F})$-martingale on $\mathbb{T}^\circ$ and then a $(\mathbf{Q}_t^j, \mathcal{G})$-martingale on any $[\![0, t]\!]$ for $t \in \mathbb{T}^\circ$ by Proposition 3.1. Then by letting $\mathrm{M} := u^{\mathrm{G},j}$, $(\mathrm{SM})_{t\in\mathbb{T}^\circ}$ is a $(\mathbf{P}, \mathcal{G})$-martingale on $\mathbb{T}^\circ$ so that it defines a $(\mathbb{T}^\circ, \mathcal{F}, \mathcal{G})$-deflator. □

Proof of Proposition 4.12. Follows from Corollary 4.8 together with Propositions 4.3, 4.5 and the solutions of agent's and insider's optimization problems in the case $T = 1$ that

$$\begin{aligned}
\mathcal{U}_T(x) &= \Phi_1^{\mathcal{G}}(\tilde{x}^{\mathcal{G}}) - \Phi_1^{\mathcal{G}}(\tilde{x}^{\mathcal{F}}) \\
&= \mathbf{E}\Big[\log(\Phi_{T-1}^{\mathcal{G}}(x)) + \log\Big[(1+r)\mathbf{1}_{\{\overline{\mathrm{S}}_T \leqslant \overline{\mathrm{S}}_{T-1}\}} + \mathrm{S}_T\overline{\mathrm{S}}_{T-1}^{-1}\mathbf{1}_{\{\overline{\mathrm{S}}_T > \overline{\mathrm{S}}_{T-1}\}}\Big]\Big] \\
&\quad - \mathbf{E}\Big[\log(\Phi_{T-1}^{\mathcal{F}}(x)) + \log(1+r) - \log\Big(\frac{\mathrm{d}\widehat{\mathbf{P}}_T^{\mathcal{H}}}{\mathrm{d}\mathbf{P}}\Big)\Big] \\
&= \mathbf{E}\Big[\log(\Phi_{T-1}^{\mathcal{G}}(x)) + \log\Big[(1+r)\mathbf{1}_{\{\overline{\mathrm{S}}_1 \leqslant \mathrm{S}_0\}} + \mathrm{S}_1\mathrm{S}_0^{-1}\mathbf{1}_{\{\overline{\mathrm{S}}_1 > \mathrm{S}_0\}}\Big]\Big] \\
&\quad - \mathbf{E}\Big[\log(\Phi_{T-1}^{\mathcal{F}}(x)) + \log(1+r) - \log\Big(\frac{\mathrm{d}\widehat{\mathbf{P}}_1^{\mathcal{F}}}{\mathrm{d}\mathbf{P}}\Big)\Big] \\
&= \mathrm{Ent}(\mathrm{G}) - \mathrm{Ent}(\mathrm{G}\,|\,\mathcal{F}_{T-1}) + \mathrm{rem},
\end{aligned}$$

where the density of $\widehat{\mathbf{P}}_T^{\mathcal{H}}$ with respect to $\mathbf{P}$ is given by

$$\frac{\mathrm{d}\widehat{\mathbf{P}}_T^{\mathcal{H}}}{\mathrm{d}\mathbf{P}} = \mathbf{1}_{\{(T,\pm1)\notin\eta\}} + \frac{p^0}{p}\mathbf{1}_{\{(T,1)\in\eta\}} + \frac{1-p^0}{1-p}\mathbf{1}_{\{(T,-1)\in\eta\}}$$

and where we have used that $\mathrm{S}_0 = 1$, the definition (2.5) of S and got the third equality thanks to the i.i.d. property of the $\Delta\mathrm{S}_t$. Besides,

$$\begin{aligned}
\mathrm{rem} &= \mathbf{E}\Big[\log\Big[(1+r)\mathbf{1}_{\{(1,\pm1)\notin\eta\}} + \mathrm{S}_1\mathbf{1}_{\{(1,1)\in\eta\}} + (1+r)\mathbf{1}_{\{(1,-1)\in\eta\}}\Big]\Big] \\
&\quad - \mathbf{E}\Big[\log\Big((1+r)\mathbf{1}_{\{(1,\pm1)\notin\eta\}} + (1+r)\frac{p}{p_0}\mathbf{1}_{\{(1,1)\in\eta\}} + (1+r)\frac{1-p}{1-p_0}\mathbf{1}_{\{(1,-1)\in\eta\}}\Big)\Big] \\
&= \lambda p \log\Big[\frac{(1+b)p_0}{(1+r)p}\Big] + \lambda(1-p)\log\Big(\frac{1-p_0}{1-p}\Big).
\end{aligned}$$

Let $\mathfrak{n}$ be the maximal integer such that $(1+b)^{\mathfrak{n}} \leqslant (1+r)^T$. Then

$$\mathrm{Ent}(\mathrm{G}) = -\mathbf{P}(\mathrm{G}=1)\log[\mathbf{P}(\mathrm{G}=1)] - \mathbf{P}(\mathrm{G}=0)\log[\mathbf{P}(\mathrm{G}=0)].$$

Define for $t \in \{T-1, T\}$, $\chi_+^t = \zeta_1^t + \zeta_{-1}^t$ and $\chi_-^t = \zeta_1^t - \zeta_{-1}^t$. Then we have $\mathbf{P}(\mathrm{G} = 1) = \mathbf{P}(\chi_+^T \leqslant T, \chi_-^T \leqslant \mathfrak{n}) = 1 - \mathbf{P}(\mathrm{G} = 0)$ as well as

$$\begin{aligned}\mathrm{Ent}(\mathrm{G} \,|\, \mathcal{F}_{T-1}) = -\mathbf{E}\big[&\mathbf{P}(\mathrm{G} = 1 \,|\, \mathcal{F}_{T-1}) \log(\mathbf{P}(\mathrm{G} = 1 \,|\, \mathcal{F}_{T-1})) \\ &- \mathbf{P}(\mathrm{G} = 0 \,|\, \mathcal{F}_{T-1}) \log(\mathbf{P}(\mathrm{G} = 0 \,|\, \mathcal{F}_{T-1}))\big],\end{aligned}$$

with $\mathbf{P}(\mathrm{G} = 0 \,|\, \mathcal{F}_{T-1}) = 1 - \mathbf{P}(\mathrm{G} = 1 \,|\, \mathcal{F}_{T-1})$ and

$$\begin{aligned}\mathbf{P}(\mathrm{G} = 1 \,|\, \mathcal{F}_{T-1}) &= \mathbf{P}(\chi_+^T \leqslant T, \chi_-^T \leqslant \mathfrak{n} \,|\, \mathcal{F}_{T-1}) \\ &= \mathbf{P}(\chi_+^T - \chi_+^{T-1} \leqslant T - \mathfrak{n}^+, \chi_-^T - \chi_-^{T-1} \leqslant \mathfrak{n} - \mathfrak{n}^-)|_{\mathrm{A}_{T-1}},\end{aligned}$$

by letting $\mathrm{A}_{T-1} = \{\mathfrak{n}^+ = \zeta_1^{T-1} + \zeta_{-1}^{T-1}, \mathfrak{n}^- = \zeta_1^{T-1} - \zeta_{-1}^{T-1}\}$, where the integers $\mathfrak{n}^+$ and $\mathfrak{n}^-$ satisfy $\mathfrak{n}^+ + \mathfrak{n}^- \leqslant T - 1$ and $\mathfrak{n}^+ - \mathfrak{n}^- \leqslant \mathfrak{n} + 1$. The proof is complete. □

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