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Uncertainty-driven symmetry-breaking and stochastic stability in a generic differential game of lobbying *

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Abstract

We study a 2-players stochastic differential game of lobbying. Players have opposite interests; at any date, each player invests in lobbying activities to alter the legislation, the continuous state variable of the game, in her own benefit. The payoffs are quadratic and uncertainty is driven by a Wiener process. We prove that while a symmetric Markov Perfect Equilibrium (MPE) always exists, (two) asymmetric MPE only emerge when uncertainty is large enough. In the latter case, the legislative state converges to a stationary invariant distribution. We fully characterize existence and stochastic stability of the legislative state for both types of MPE. We finally study the implications for rent dissipation asymptotically. We show in particular that while the average rent dissipation is lower with asymmetric equilibria relative to the symmetric, the former yield larger losses at the most likely asymptotic states for large enough but moderate uncertainty.

Keywords: Political lobbying, symmetric versus asymmetric equilibrium, stochastic differential games, stochastic stability, social cost of lobbying
JEL classification: D72, C73

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1 Introduction

As outlined by Gary Becker in his 1983 seminal contribution, political influence is not uniquely determined by the political process and constitutional rules, it can also be bought through costly campaigning, that's through lobbying. Lobbying is viewed as useful and even necessary in several contexts, in particular in case of distorsionary taxes and subsidies which typically cause deadweight costs, therefore fostering political competition (Tullock, 1967). But lobbying may target many other fields like defense, environmental public policy, health policy etc...In short, all the domains in which public expenditure take place with the associated winners and losers.

Perhaps the most interesting illustration of political lobbying nowadays is the so-called climate politics. Some theory already exists in this area (see for example Yu, 2005, and more recently Prieur and Zou, 2018). Related frameworks study environmental regulation as the equilibrium outcome of political competition between, say, industrialists vs. the environmentalists. As explained in the two papers cited above, the pressure groups can either directly lobby the government to induce a change in the level of the environmental legislation or indirectly by engaging in information campaigns or public persuasion.This paper is pretty much in line with the legislative environmental lobbying literature outlined above. In particular, it shares the dynamic game setting adopted in Prieur and Zou (2018), Yu (2005) being static. Our theory, however, differs from the latter in two essential ways. First, it tackles a generic dynamic legislative lobbying game in the sense that it is not specific to environmental and climate politics. Second, and more importantly, it introduces uncertainty into the game. Uncertainty is undoubtedly a major ingredient of political games, and as such, it's critical to incorporate it into the theoretical analysis.

Uncertainty is meant to cover internal political instability in all its forms, ranging from ordinary changes in the composition of the executive power during the electoral cycle to more drastic legislative moves of a constitutional nature. In both cases, the profitability of legislative lobbying is likely to be significantly altered. External factors may also matter: for example, constraints involved by a country's commitment to enforce international treaties and agreements limit *de facto* the scope for lobbying (see the case of the European Union in Kluver, 2013). External economic shocks may also dramatically change the trade-off inherent in lobbying activities. A typical illustration is given by shocks on commodities' prices in resource-dependent economics.[1]

[1]As reported in Boucekkine and Bouklia-Hassane (2011), the Algerian legislation with respect to foreign investment has been as volatile as the oil price in the three last decades.

Our paper makes three sets of contributions. The first two sets primarily concern the literature on lobbying games, the third one is broader and concerns the literature of symmetry-breaking in games (see a survey in Amir et al., 2010). The first set of contributions is methodological: we target the analytical treatment of continuous time stochastic dynamic games, and more precisely the computation of the corresponding Markov Perfect Equilibria (MPE), when uncertainty is driven by a brownian motion. In particular, in our linear-quadratic framework, we provide with the necessary tools to assess stochastic stability of the computed equilibria. To the best of our knowledge, this is the first work solving a stochastic differential lobbying game, with the exception of a companion paper (Boucekkine et al., 2021). In the latter, and along the lines of the resource-dependent countries story told above, the bargaining power of the players depend on the random size of resource revenues, and a unique (affine) MPE arises, the main objective being the assessment of how volatility affects players' equilibrium lobbying decisions. In this paper, we shut down uncertainty due to commodity prices fluctuations and encompass it in an aggregate (random) state, which also reflects other sources of uncertainty (like political uncertainty). We show that our game gives rise to both symmetric and asymmetric equilibria, whose asymptotic (stochastic) stability is systematically studied. There is no such a stability study in Boucekkine et al. (2021).

Of course, there are a bunch of papers working out stochastic dynamic games. In Jorgensen and Yeung (1996) for example, a stochastic dynamic game model of common property fishery is tackled. However, just like Boucekkine et al. (2021), while the corresponding stationary distribution is explicitly computed by the authors, there is no related stochastic stability result. Clearly, our linear-quadratic setting allows to do so more easily but it's far nontrivial. Even more importantly, and differently from the latter fishing game, our players are not identical in the sense that their actions do not affect the state variable in the same way in contrast to the fishing game. The mechanisms behind the emergence of equilibria are therefore totally different.[2]

The second set of contributions has to do with the rent dissipation problem, central in the theory of lobbying games. Rent dissipation refers to the social cost of lobbying, first invoked by Tullock (1967). A large part of the lobbying games literature is devoted to uncovering conditions and frames under which the social cost is alleviated or enhanced. An overwhelming part of this literature tackles static games.[3] Two excellent systematic studies of the rent-

[2]Also we do allow for the state variable to go to zero, the center of the political distribution in our case, while the counterpart-stock of fishes equal to zero- is dismissed in Jorgensen and Yeung, who consequently only focus on a unique symmetric equilibrium.

[3]The simplest rent-seeking games are typically modeled into the form of competition involving a given number of players whose object is a given prize. Each agent makes a bet and his probability of wining the prize is an

seeking games literature can be found in Pérez-Castrillo and Verdier (1992) and Treich (2010). In both studies, one can see the great diversity of results obtained concerning the two key questions outlined above depending on the rent-seeking technology, the type of competition and strategic interactions between players, and behavioral characteristics towards risk. Very few studies consider dynamic settings. We single out here two types of dynamic lobbying games. One is the repeated sequential game proposed and studied by Leininger and Yang (1994), the second is the differential game considered by Wirl (1994). The former is a repeated game version of the Tullock model while the latter is a quite generic lobbying differential game: players do not compete for a given prize but invest in lobbying activities to alter the legislation advantageously in continuous time. As a result, the legislative state may change over time, which paves the way for a natural dynamic formulation of the lobbying decisions. While quite different, the two models yield the same kind of conclusion as to rent dissipation: dynamic games would lead to lower rent dissipation compared to the static counterparts because of the threat of retaliation. Both papers use deterministic environments. In this paper, we introduce uncertainty in the generic Wirl's differential game and we show that uncertainty increases the set of MPE by giving rise to asymmetric ones, in addition to the symmetric MPE "inherited" from the deterministic counterpart.

We then explore the asymptotic implications of these equilibria in terms of rent dissipation. Our results are quite rich and involved in this respect compared to the earlier deterministic literature. We first show that upon existence of the asymmetric equilibria (that's for uncertainty large enough), the average rent dissipation is lower with these equilibria relative to the symmetric ones. However, this result should be qualified on two grounds. First, because asymmetric MPE lead to convergence to invariant stationary distributions, they may yield large losses with positive probability. Indeed, when we compare rent dissipation at the most likely asymptotic states, we further show that the asymptotic implications for rent dissipation depend pretty much on the amount of uncertainty.

Finally the third set of contributions of our work concerns symmetry-breaking. In our setting, two asymmetric MPE are shown to emerge in addition to a symmetric MPE. The emphasis here is not on the fact that a priori symmetric (lobbying) game can give rise to asymmetric equilibria.[4] Our point is finer: we do prove that the asymmetric equilibria only emerge when the level of uncertainty is large enough, in contrast to the symmetric equilibrium which always exists with or without uncertainty and whatever the amount of uncertainty. This is a major difference with respect to the standard symmetry-breaking literature (see Flaherty, 1980, and

increasing function of the bet. More refined versions endogenize the number of players often through free entry.

[4]This is a property which is referred to even in the early (static) rent-seeking literature (see Tullock, 1985).

Amir et al., 2010, for comprehensive related studies) in which only asymmetric equilibria (in pure Nash strategy) exist or are the unique to be locally stable.

It's of course granted that introducing uncertainty may increase the set of strategic equilibria (see for example, Amir, 1986) together with the emergence of new uncertainty-driven strategic mechanisms. A key peculiarity of our setting is that such mechanisms do not lead to (stochastically stable) Markovian equilibria below a certain level of uncertainty. We provide with an interpretation of the emergence of these equilibria combining the retaliation mechanism outlined in the preexisiting (deterministic) dynamic lobbying games and a pure uncertainty-driven mechanism. Indeed under uncertainty, because by definition players are unsure about the evolution of the situation, there is an incentive for any given player to do more efforts (than under certainty) in order to protect himself against future bad realizations, which may further distort the original retaliation strategies.

The paper is organized as follows. Section 2 describes the stochastic legislative lobbying game and introduces some technical concepts. Section 3 deals with the symmetric MPE. Section 4 highlights the properties of the asymmetric MPE, discusses the economic mechanisms involved, and provides a full analysis of stochastic stability. Section 5 delivers the implications of our game for the rent dissipation problem. Section 6 concludes.

2 Framework

2.1 The stochastic dynamic lobbying game

We consider two players, indexed by $i = 1, 2$, who have opposite interests on how legislation, as measured by the state variable z, should evolve. We postulate that a rising z is favorable to player 1 and harmful for player 2. Each player i invests x_i to push variable z in the most preferred direction. Compared to Wirl (1994), the novelty is to account for the uncertainty that surrounds the legislative process. Indeed, for internal or external reasons and through direct or indirect mechanisms, the legislative state z is essentially a stochastic variable from the point of view of lobbyists, and should be treated as such. That is why we assume that the economy's state of legislation is governed by the following stochastic equation:

$$dz = [x_1 - x_2]dt + \sigma z \, dW, \tag{1}$$

where $W = (W_t)_{t \geq 0}$ is a standard Wiener process, and $z(0) = z_0$ is given. Parameter σ measures the volatility of z, that may originate in the wide set of internal and external factors discussed in the Introduction. It's important to notice that as the state equation (1) is specified, the two players have the same ex ante lobbying power: an increase in the lobbying effort by the same amount would lead the two players to have opposite but equal (in absolute value) impacts on the state of legislation.[5]

Players' payoffs have two components. They earn a direct benefit from the level of legislation, $\omega_i(z)$, but also have to incur a cost of lobbying, $\beta(x_i)$. Their objective is to maximize the present value of benefit from their efforts of liberalization minus the associated cost:

$$V_i(z) = \max_{x_i} \mathbb{E} \int_0^\infty e^{-rt} \left[\omega_i(z) - \beta(x_i)\right] dt, \tag{2}$$

with $\mathbb{E}$ the mathematical expectation operator, $r > 0$ the rate of time preference, taking as given the state constraint (1), and the lobbying strategy of the competitor.

To keep things as simple as possible, we take a linear-quadratic (LQ) specification example, i.e., benefit and cost have (affine-)quadratic forms:

$$\begin{aligned} &\omega_1(z) = a_0 + a_1 z + \tfrac{a_2}{2} z^2, \quad \omega_2(z) = a_0 - a_1 z + \tfrac{a_2}{2} z^2, \\ &\beta(x_i) = \tfrac{b}{2} x_i^2, \end{aligned} \tag{3}$$

with $a_0, a_1 > 0$, $a_2 < 0$ and $b > 0$. What is important to note is the opposite sign of the term in z in the benefit. This reflects players' opposite interests with respect to the legislation. By convention, player 1 is the one pushing for a large z, i.e, we put a $+$ in front of a_1.

Several remarks are worth formulating at this stage. First, just like in the deterministic differential game of Jun and Vives (2004), we could have explored some of the properties of the Markovian equilibria with more general payoff functions before getting to the linear-quadratic case and ultimately extracting closed-form solutions as in the latter paper. However, as outlined in the introduction, it's far nontrivial to get analytical results regarding stochastic stability with general payoff functions. That's why we stick to the quadratic payoff specification from the beginning. By doing so, we are able to point out the main economic mechanisms at work and provide with a comprehensive nontrivial investigation of stochastic stability, at the same time. Second, corruption motives and office rents are left aside in our analysis. This allows us to focus on a game where the players are entirely devoted to push the legislation in the direction they wish, which is the essence of lobbying. Third, players' lobbying efforts have the same marginal impact on z in absolute value (that's efforts x_1 and x_2 enter the state equation with opposite coefficients). In other words, we assume that they have identical lobbying powers. This needs not always be the case. For example, one may expect that the closer the lobbyists to the ideological line of the dominating party, the larger their lobbying power. Cultural aspects may

[5] We come back to this point in the series of remarks following the model presentation.

matter too: clearly it is easier nowadays to lobby for a more environment-friendly legislation in the Netherlands than in Russia. But we prefer to keep our game as symmetric as possible. This will be a means to emphasize how uncertainty, in such a neutral framework, affects the number and nature – symmetric vs. asymmetric – of equilibria.

Finally, it is worth summarizing the differences between our framework and the ones considered in the related literature. Our model is similar to Wirl (1994), who does not take uncertainty into account. Boucekkine et al. (2014) do examine a stochastic lobbying problem but they choose a very different approach by assuming that a_1 is a discrete random variable that can take two values, with given probabilities. This is their unique source of uncertainty in the model. Unfortunately, the latter modeling (usually referred to as piecewise deterministic game) turns out quite ineffective in the analysis of the stochastic stability of the induced equilibria. We show here below that moving to a continuous Wiener process modeling allows us to make a decisive move towards such an analysis.

Importantly enough, one can readily see that our linear-quadratic modeling works for both positive and negative x_i. One can interpret $x_i > 0$ as the instantaneous investment by player i to push the legislation in a favorable direction while $x_i < 0$ can be understood as disinvestment by player i leading to a legislative move contrary to his preferred direction. For example, investment would consist in building a propaganda platform (website, radio, newspaper...), and disinvestment would correspond to shutting down the propaganda platform. Clearly, in the face of unexpected adverse shocks, which is inherent in stochastic models, disinvestment may result optimal. Hereafter, though keeping using the traditional expression "lobbying effort", we will not impose any positivity constraints in the mathematical treatment.[6]

2.2 Symmetry, stability and optimality concepts

The LQ stochastic game, characterized by the objective (2) and the constraint (1), can be solved by using Markov perfect Nash equilibrium (MPE) as the solution concept. Three additional concepts will be central in the coming resolution.

The first concept refers to the **symmetric** vs. **asymmetric** nature of the equilibrium. We define the symmetric MPE as follows:

Definition 1. *An MPE is said symmetric if the corresponding state z converges almost surely to zero. Otherwise, the MPE is said asymmetric.*

[6] If such a constraint is imposed, then the invariant distribution of the legislative state would be eventually truncated resulting in severe computational complications.

Definition 1 is a direct extension of Wirl's definition to a stochastic environment. In his deterministic game, he shows that there exists a unique symmetric MPE, i.e., the state variable converges to the neutral (or central) level $z = 0$ along the MPE. In other words, $z = 0$ is asymptotically stable along the MPE, which incidentally can only hold if the lobbying efforts are equal asymptotically (by the deterministic counterpart of equation (1)). In sum, the lobbying strategies lead to the center of the political spectrum asymptotically. Such an equilibrium is particularly natural when lobbying powers are equal. In the coming analysis, we will show, among others, that considering identical lobbying powers is not sufficient to rule out asymmetric MPE in a stochastic framework. Indeed, as we show in this paper, even for stochastically stable asymmetric MPE, on average, the lobbying powers still become equal asymptotically. (See Proposition 4.) However, for asymmetric MPE, it calls for an appropriate notion of stochastic stability.

The second important ingredient is the **stochastic stability** of the equilibrium. Here, we adopt Merton (1975)'s definition of the stability of stochastic dynamic processes:

Definition 2. *A stochastic process $z(t)$ is stable if there is stationary time invariant distribution of $z(t)$ for $t \to \infty$.*[7]

So according to Definition 2, the z-process is said to be stable if and only if there is a unique distribution which is time and initial condition independent, and toward which the stochastic process tends. A major contribution of this paper will be to show that while we also get a unique symmetric MPE,[8] there exist new asymmetric MPE which differ in many respects from the symmetric one. In particular, we will emphasize its different asymptotic behavior and examine the related economic implications.

Third, the objective (2) involves an integral of unbounded instantaneous utility function over an unbounded domain. Therefore, justification is needed for the derived control path to be optimal in some sense. There are a number of criteria for optimality. The one particularly suitable for differential games in infinite horizon is the so called **catching up optimality** (cf. Dockner *et al.*, 2000). We first recall the basic definitions.

Definition 3. *For one-player optimal control problem,*

$$J(x(\cdot)) = \max_{x(\cdot)} \mathbb{E}_{x(\cdot)} \int_0^\infty e^{-rt} F(z(t), x(t), t)\, dt$$

subject to the stochastic differential equation

$$dz = f(z(t), x(t), t)\, dt + \sigma(z(t), x(t), t)\, dW,$$

[7] If the density distribution degenerates into a Dirac function, then the stochastic process converges to a point.

[8] Therefore we properly generalize Wirl's finding to a stochastic environment.

a feasible control path $x(\cdot)$ is called catching up optimal if the inequality

$$\liminf_{T\to\infty}\left\{\mathbb{E}_{x(\cdot)}\int_0^T e^{-rt}F\left(z\left(t\right),x\left(t\right),t\right)dt-\mathbb{E}_{\tilde{x}(\cdot)}\int_0^T e^{-rt}F\left(z\left(t\right),\tilde{x}\left(t\right),t\right)dt\right\}\geq 0$$

holds for any feasible control path $\tilde{x}(\cdot)$.

The definition for multiple-player differential games is similar. In particular, for two-player games with the objectives

$$J_i\left(x_1\left(\cdot\right),x_2\left(\cdot\right)\right)=\max_{x_i(\cdot)}\mathbb{E}_{(x_1(\cdot),x_2(\cdot))}\int_0^\infty e^{-rt}F_i\left(z\left(t\right),x_1\left(\cdot\right),x_2\left(t\right),t\right)dt,\quad i=1,2,$$

subject to the same stochastic equation for $z(t)$, a feasible control path $(x_1(\cdot),x_2(\cdot))$ is catching up optimal if

$$\begin{aligned}\liminf_{T\to\infty}\left\{J_{1,T}\left(x_1\left(\cdot\right),x_2\left(\cdot\right)\right)-J_{1,T}\left(\tilde{x}_1\left(\cdot\right),x_2\left(\cdot\right)\right)\right\}&\geq 0,\\ \liminf_{T\to\infty}\left\{J_{2,T}\left(x_1\left(\cdot\right),x_2\left(\cdot\right)\right)-J_{2,T}\left(x_1\left(\cdot\right),\tilde{x}_2\left(\cdot\right)\right)\right\}&\geq 0,\end{aligned}$$

for any feasible control paths $\tilde{x}_1(\cdot)$ and $\tilde{x}_2(\cdot)$, where

$$J_{i,T}\left(x_1\left(\cdot\right),x_2\left(\cdot\right)\right)=\mathbb{E}_{(x_1(\cdot),x_2(\cdot))}\int_0^T e^{-rt}F_i\left(z\left(t\right),x_1\left(\cdot\right),x_2\left(t\right),t\right)dt,\quad i=1,2.$$

(See Sections 3.6 and 8.2 in Dockner et al.*, 2000).*

As we shall show in this paper, the feedback rules we derive based on the HJB equations give rise to catching up optimal control paths.

The next sections are devoted to the analysis of the possible outcomes of our stochastic game of lobbying. All the proofs are relegated to the Appendix.

3 Symmetric equilibrium

Two types of – symmetric vs. asymmetric – equilibria can arise as outcomes of our lobbying game. We start by considering the symmetric MPE as it is the one that is the closest to the solution of the baseline deterministic case. This allows us to emphasize the role of uncertainty. The Proposition below displays the features of the symmetric MPE:

Proposition 1. *The stochastic game of lobbying admits a unique symmetric MPE.*

(i) Players' lobbying strategies are given by the following linear feedback rules:

$$x_1(\cdot) = \frac{B + Cz(\cdot)}{b}, \quad x_2(\cdot) = \frac{B - Cz(\cdot)}{b} \tag{4}$$

with

$$C = \frac{-b(\sigma^2 - r) - \sqrt{b^2(\sigma^2 - r)^2 - 12ba_2}}{6} < 0 \text{ and } B = \frac{ba_1}{br - C} > 0. \tag{5}$$

(ii) The stochastic process $z(t)$, whose dynamic behavior is given by

$$dz(\cdot) = \frac{2C}{b} z(\cdot)\, dt + \sigma z(\cdot)\, dW, \tag{6}$$

almost surely converges to the steady state $z_\infty = 0$

(iii) The control path based on the feedback rules in (4),

$$(x_1(\cdot), x_2(\cdot)) = \left(\frac{B + Cz(\cdot)}{b}, \frac{B - Cz(\cdot)}{b} \right),$$

is catching up optimal.

Let us examine the shape and determinants of the strategies. An initial step to this end is to highlight the impact of strategic interaction by neutralizing the role of uncertainty. Taking $\sigma = 0$ boils down to assessing players' reactions to a change in z. Since C is always negative (for any value of $\sigma \geq 0$), player 1's feedback rule is decreasing in z whereas player's 2 feedback is increasing in the state ($\frac{\partial x_1}{\partial z} < 0$, $\frac{\partial x_2}{\partial z} > 0$). The reason why player 1 behaves this way while she is interested in large values of z is the fear that player 2 would exert an opposite lobbying effort in retaliation. So one gets the retaliation motive invoked by Wirl (1994) to argue that the social cost of lobbying is likely to be low.[9] A similar argument can be found in Leininger and Yang (1994).

Going back to the stochastic environment, one can solve equation (6) to obtain:

$$z(t) = z_0 e^{\left(\frac{2C}{b} - \frac{\sigma^2}{2}\right)t + \sigma W(t)}.$$

Stochastic convergence to $z = 0$ holds if and only if $\frac{2C}{b} - \frac{\sigma^2}{2} < 0$.[10] At the symmetric MPE (S-MPE hereafter), this inequality is fulfilled for all σ, because $C < 0$ and $b > 0$. So the

[9]The same logic is at work for player 2. Actually, with $\sigma = 0$, one exactly recovers the solution of the deterministic counterpart of the problem.

[10]See Boucekkine et al. (2018), for a simple mathematical exposition.

system stochastically converges to the central position of the legislation, $z = 0$, as in Wirl.[11] So, uncertainty seems to be strongly stabilizing along the S-MPE. In fact, it turns out that the retaliation motive, which arises in the deterministic setting and is itself stabilizing, is further reinforced under uncertainty.

A natural way to look at the stabilization role of uncertainty is to precisely assess its impact on the lobbying efforts at the S-MPE. Paying attention to long-term strategy, it is easy to show, from (4) and (7), that the larger the uncertainty, the lower the effort exerted by lobbyists at the stochastic steady state. This means that more uncertainty is always associated with lower lobbying effort in the long-run. That is to say, stationary rent dissipation due to lobbying is decreasing with uncertainty at the symmetric MPE. Admittedly, this is a well-known result in the traditional rent-seeking literature under uncertainty.[12] Even if we depart from the standard formulation of static rent-seeking games,[13] the intuition basically remains the same: a higher volatility surrounding the evolution of z makes the returns to lobbying more uncertain. This, in turn, is an incentive, for risk-averse players, to devote less resources to this activity.

In the short-term, things get more complicated because two additional effects are at work: uncertainty not only affects the players' reaction to a change in z, but also changes the state itself.[14] At least, we can get some insight into the impact of uncertainty in the neighborhood of the origin, $t = 0$. Indeed, assuming that $W(0) = 0$ and $z(0) = z_0$ is large enough in absolute value, we obtain that the player who is adversely affected by the state of the system initially responds to a higher level of uncertainty by increasing her lobbying efforts. Interestingly, uncertainty does not always induce lobbyists to decrease their efforts in the short run. This result differs from the general conclusion drawn in the static lobbying game literature and from what

[11] Uncertainty only affects the speed of convergence to the steady state. Stochastic convergence is exponential, and the (average) speed of convergence is captured by the absolute value of the term $\frac{2C}{b} - \frac{\sigma^2}{2}$, not by the absolute value of $\frac{2C}{b}$ (as it would be in the absence of uncertainty). Computing

$$\frac{\partial C}{\partial \sigma} = \frac{b\sigma}{3}\left[-1 - \frac{b(\sigma^2 - r)}{\sqrt{b^2(\sigma^2 - r)^2 - 12ba_2}}\right] < 0, \tag{7}$$

we observe that the speed of convergence is increasing in σ: the larger the uncertainty (that is, the larger σ), the faster the convergence to the (stochastic) steady state.

[12] Some crucial qualifications are needed though, see Konrad and Schlesinger (1997), and Treich (2010).

[13] In our setting, players' lobbying efforts are intended to change the balance of power between opposite interest groups and to push the state of the system toward their preferred direction. They do not affect the probability of "success", in contrast with Hillman and Katz (1984) for example.

[14] The general decomposition, for $i = 1, 2$, in given by

$$\frac{dx_i(\sigma, z)}{d\sigma} = \frac{\partial x_i(\sigma, z)}{\partial \sigma} + \frac{\partial x_i(\sigma, z)}{\partial z}\frac{\partial z(\sigma)}{\partial \sigma}.$$

we get in the long-term. It is critically driven by the initial condition z_0.

Now, we move to the analysis of the new class of – asymmetric – MPE (see Definition 1).

4 Asymmetric equilibria

Besides the impact of uncertainty on equilibrium lobbying strategies, we are especially interested in its (de)stabilizing power. As it will become apparent soon, asymmetric equilibria exhibit much complex dynamic behaviors than symmetric ones. The "stability" issue thus deserves much more attention than in the previous study of the S-MPE. That is why, in the coming analysis, we need to decouple the issue of existence (and uniqueness) of the equilibrium from the one of stability.

4.1 Existence of asymmetric MPE and distinctive properties

Define $\underline{\sigma}$ as: $\underline{\sigma}^2 = r + 2\sqrt{-\frac{a_2}{b}}$. The existence analysis provides the following results:

Proposition 2.

There exist two asymmetric MPE if and only if:

$$\sigma^2 \in \left(\underline{\sigma}^2,\ +\infty\right). \tag{8}$$

These MPE are characterized by the following lobbying efforts:

$$x_1^{(j)}(\cdot) = \frac{B_1^{(j)} + C_1^{(j)} z(\cdot)}{b},\ \ x_2^{(j)}(\cdot) = -\frac{B_2^{(j)} + C_2^{(j)} z(\cdot)}{b},\ \ j = 1, 2 \tag{9}$$

with

$$\begin{aligned} C_1^{(1)} &= \frac{-b(\sigma^2 - r) - \sqrt{b^2(\sigma^2 - r)^2 + 4a_2 b}}{2} \left(= C_2^{(2)}\right) < 0, \\ C_2^{(1)} &= \frac{-b(\sigma^2 - r) + \sqrt{b^2(\sigma^2 - r)^2 + 4a_2 b}}{2} \left(= C_1^{(2)}\right) < 0, \end{aligned} \tag{10}$$

and,

$$B_1^{(j)} = -\frac{a_1 b\ (C_1^{(j)} - b\sigma^2)}{b^2\sigma^4 - C_1^{(j)} C_2^{(j)}} > 0,\quad B_2^{(j)} = \frac{a_1 b\ (C_2^{(j)} - b\sigma^2)}{b^2\sigma^4 - C_1^{(j)} C_2^{(j)}} < 0,\ \ j = 1, 2. \tag{11}$$

Also, the control path based on the feedback rules in (9),

$$\left(x_1^{(j)}(\cdot), x_2^{(j)}(\cdot)\right) = \left(\frac{B_1^{(j)} + C_1^{(j)} z(\cdot)}{b}, -\frac{B_2^{(j)} + C_2^{(j)} z(\cdot)}{b}\right),$$

is catching up optimal.

Asymmetric MPE (hereafter, A-MPE) exist if and only if the level of uncertainty, as captured by the volatility parameter σ, is high enough. So compared to the deterministic case, the set of solutions is further increased provided that enough uncertainty surrounds players' interaction. It is also worth noting that these two equilibria are mirror image of each other since $C_1^{(2)} = C_2^{(1)}$, $C_2^{(2)} = C_1^{(1)}$ (and $B_1^{(1)} = -B_2^{(2)}$, $B_1^{(2)} = -B_2^{(1)}$). This means that analyzing one of them is enough to characterize the features of the A-MPE. Let us focus on the equilibrium $j = 1$ that is such that $|C_1^{(1)}| > |C_2^{(1)}|$. For the ease of presentation, hereafter we skip the superscript $j = 1$ and switch to the superscript "a", respectively "s", for the A-MPE strategies, respectively S-MPE.

The fundamental similarity between the two types of MPE is the prevalence of the so-called retaliation effect according to which players' efforts move in opposite direction following a change in z (player 1's effort being decreasing in z). That being said, we can also identify a series of differences between the S-MPE and the A-MPE. First, at the S-MPE the status quo position also represents the threshold that determines which player undertakes the largest efforts, i.e., $x_1^s(z) \geq x_2^s(z) \Leftrightarrow z \leq 0$. This is no longer the case at the A-MPE. Indeed, it is easy to check that the critical z that induces the same level of effort by the two players becomes positive: $\tilde{z} = -\frac{B_1+B_2}{C_1+C_2} > 0$. Player 1 continues to devote more effort than player 2 in order to change z, even after its realizations have turned favorable to him. Second, we have already noticed that $|C_1| > |C_2|$: player 1 is more reactive to changes in z than player 2. However, at the aggregate level – i.e., when computing the net variation of effort, $dx = [x_1'(z) - x_2'(z)]dz (= \frac{C_1+C_2}{b} dz)$, resulting from a variation in z, dz – we get that following $dz > 0$, the net effort decreases by the same magnitude as the one by which it would increase following a decrease in z of the same importance. This is also true at the S-MPE. In addition, one has:

$$\begin{aligned} \frac{\partial C_1}{\partial \sigma} &= b\sigma \left[-1 - \frac{b(\sigma^2 - r)}{\sqrt{b^2(\sigma^2-r)^2 + 4ba_2}} \right] < 0, \\ \frac{\partial C_2}{\partial \sigma} &= b\sigma \left[-1 + \frac{b(\sigma^2 - r)}{\sqrt{b^2(\sigma^2-r)^2 + 4ba_2}} \right] > 0. \end{aligned} \tag{12}$$

As uncertainty increases, player 1's reactivity gets bigger, while the reverse happens for player 2 (remember again that the two coefficients are strictly negative). So uncertainty contributes to exacerbate the differences in the marginal efforts. Overall, and as in the S-MPE, the larger σ, the larger the marginal net effort, as measured in absolute terms ($\frac{\partial(C_1+C_2)}{\partial \sigma} < 0$).

To sum-up, the discussion above allows us to address an important question raised by the present analysis: How to justify the existence of A-MPE under sufficient uncertainty? We can provide a general answer to this question. Under uncertainty, because by definition players are unsure about the evolution of the situation, there is an incentive for any given player to do more efforts (than under certainty) in order to protect himself against future bad realizations of z. This, combined with the retaliation argument that still holds, also implies that the other player must do less. As an illustration, we have taken the A-MPE $j = 1$ at which this is player 1 who endorses this role. Under large enough uncertainty, we find that asymmetric strategies may indeed be optimal. The fear of retaliation, combined with a large uncertainty about the future state, leads player 1 not only to *more often*[15] provide the highest effort but also to strengthen his reaction (to a change in z), as compared to player 2. From that point of view, we can argue that player 1 is at the same time the most active and the most reactive player.

Of course, these asymmetric responses may not be compatible with the convergence towards a point-wise stochastic steady state. In particular, there is no reason to believe that the status quo position that the system reaches at the S-MPE remains the steady state if the uncertainty is sufficiently large as stated in Proposition 2. Actually, in the next Section, we show that the A-MPE features convergence to a limit invariant distribution.

4.2 Existence and stochastic stability of invariant limit distributions

The dynamics of z at the A-MPEs are given by:

$$dz = [x_1(z) - x_2(z)]dt + \sigma z \, dW \Leftrightarrow dz = \left[\Gamma - (\sigma^2 - r)z\right] dt + \sigma z \, dW, \tag{13}$$

with $\Gamma = \frac{a_1(C_2^{(j)} - C_1^{(j)})}{b(b\sigma^4 + a_2)}$ and $C_2^{(1)} - C_1^{(1)} = \sqrt{b^2(\sigma^2 - r)^2 + 4a_2 b} > 0$. One can notice that under the parametric condition of Proposition 2, $\sigma^2 \in \left(\underline{\sigma}^2, +\infty\right)$, the denominator of Γ is always positive whatever $j = 1, 2$. Thus, $\Gamma > 0$ for the A-MPE $j = 1$, and $\Gamma < 0$ for the A-MPE $j = 2$. This further reflects the symmetry between the two A-MPE, which will be even more apparent in Proposition 3 below. Let us continue to work on the A-MPE $j = 1$ (and omit the time index).

In the deterministic case, one can see that the stochastic differential equation degenerates into $\dot{z} = \Gamma + rz$. This confirms that the A-MPE cannot bring the legislative state almost surely to the status quo, $z = 0$. Actually, no stable asymmetric equilibrium can arise in this case. If uncertainty is low enough (say $\sigma^2 < r$), then the same explosive sub-optimal behavior also prevails.

However, it is possible to show that the $z-$process admits a stationary invariant distribution with a well-identified density. Following Merton (1975), we compute this density by studying the corresponding Kolmogorov-Fokker-Planck (KFP) forward equation. Let $Q(z, t; z_0)$ be the conditional probability density for process $z(t)$ at t, given the initial value z_0. Then the KFP

[15] That is, for a larger set of realizations of z in $(-\infty, +\infty)$.

forward equation is

$$Q_t - \frac{\sigma^2}{2}\left[z^2 Q\right]_{zz} + \left[\left(\Gamma + \left(r-\sigma^2\right)z\right)Q\right]_z = 0. \tag{14}$$

It also satisfies

$$\int_{-\infty}^{\infty} Q\left(z,t;z_0\right)dz = 1. \tag{15}$$

Any candidates stationary invariant distribution for the $z-$process must solve for the steady-state KFP forward equation. Denote by $q\left(z\right)$ such a potential solution. It must satisfy

$$\frac{\sigma^2}{2}\left(z^2 q\right)'' - \left[\left(\Gamma + \left(r-\sigma^2\right)z\right)q\right]' = 0.$$

The next proposition provides a closed-form solution of such a distribution.

Proposition 3. *Define the function $\mu(z)$ as*

$$\mu\left(z\right) = \left|z\right|^{4-2r/\sigma^2} e^{2\Gamma/\left(\sigma^2 z\right)}.$$

Then:

1. *If $\Gamma > 0$, then the density of the invariant distribution is given by*

$$q\left(z\right) = \begin{cases} 0 & \textit{if } z \leq 0, \\ k_1/\mu\left(z\right) & \textit{if } z > 0 \end{cases}$$

   where

$$k_1 = 1/\int_0^{\infty}\frac{dz}{\mu\left(z\right)}.$$

2. *If $\Gamma < 0$, then we have*

$$q\left(z\right) = \begin{cases} k_2/\mu\left(z\right) & \textit{if } z < 0, \\ 0 & \textit{if } z \geq 0 \end{cases}$$

   with

$$k_2 = 1/\int_{-\infty}^{0}\frac{dz}{\mu\left(z\right)}.$$

For an illustration, we can draw the representative curve of the function $q(z)$ for the A-MPE $j=1$ (see Figure 1).

This quite remarkable finding deserves further discussion. First, it is worth pointing out the quite peculiar and intriguing shape of the invariant distributions. For a given sign of Γ, a unique distribution exists: it reduces to zero as z approaches $z=0$ from one side and remains identically zero on the half-line on the opposite side of $z=0$. Moreover, the two invariant

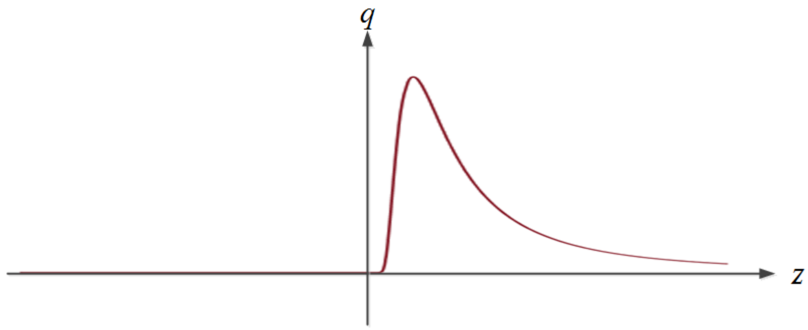

Figure 1: Probability density function $q(z)$ for the A-MPE $j = 1$.

redistributions corresponding to the A-MPE $j = 1$ ($\Gamma > 0$) and $j = 2$ ($\Gamma < 0$) respectively are symmetric with respect to the origin. Even more interesting is the observation that the invariant distribution corresponding to $j = 1$ is only nonzero when $z > 0$. This means that under stochastic convergence, if the feedback strategies corresponding to $j = 1$ are systematically applied, the probability to converge to a negative z is zero. That is by no way surprising: indeed by Proposition 2, this A-MPE satisfies C_1 and C_2 both strictly negative and $C_2 - C_1 > 0$. As player 1 is more reactive than player 2, he manages to push the legislative state towards its preferred region, i.e., the one with $z > 0$, in the long run.[16]

The comparison between Propositions 1 and 3 provides us with highly interesting and sharply contrasted results as to the asymptotic implications of S- vs A-MPE. At the S-MPE, the outcome – convergence almost surely to the status quo $z = 0$ – comes from the combination of two characteristics of equilibrium strategies. First, players' efforts move along opposite directions, see (4), i.e., one's efforts increase with z and the rival player's efforts decrease with z. Second, their efforts end up balancing each other and the legislative state converges almost surely to $z_\infty = 0$. At the A-MPE, the former characteristic is still present but not the latter, though there exists $\tilde{z} > 0$ that equalizes players' efforts. This implies that the z-process will not in general reach a single value in the long run. Instead, it will exhibit an invariant distribution on the real line. As a consequence, the cost of lobbying may be in the end much higher than with the latter class of equilibrium.

We now turn to the analysis of the much trickier issue of stochastic stability in the context where an invariant distribution exists, which has been established in Proposition 3. To determine the stability of q, define u as $u = Q - q$. Since equation (14) is linear, u satisfies

$$u_t - \frac{\sigma^2}{2}\left[z^2 u\right]_{zz} + \left[\left(\Gamma + \left(r - \sigma^2\right) z\right) u\right]_z = 0 \tag{16}$$

[16] An analogous outcome arises for the A-MPE $j = 2$.

and,

$$\int_{-\infty}^{\infty} u(z,t)\,dz = 0 \tag{17}$$

for any $t \geq 0$.

We can then prove the following stability result.

Theorem 1. *Any solution $u(z,t)$ to equation (16) subject to the constraint (17) and with the initial value $u_0 \in L^1(\mathbb{R})$ converges to zero in $L^1(\mathbb{R})$ as $t \to \infty$.*

The proof – quite long and tricky – is developed in details in the Appendix. We essentially show that the induced eigenvalue problem

$$\frac{\sigma^2}{2}\left[z^2\phi\right]'' - \left[\left(\Gamma + \left(r - \sigma^2\right)z\right)\phi\right]' = \lambda\phi \qquad \text{on } (-\infty,\infty) \tag{18}$$

subject to the constraint

$$\int_{-\infty}^{\infty} \phi(z)\,dz = 0 \tag{19}$$

has only negative eigenvalues.

We now move to a discussion of the economic implications of our results, for the lobbying problem studied so far.

5 Economic implications

The coming discussion will rely on two positional statistics of any distribution, the average and the mode. More precisely, we will first characterize the expected value of z, and its most likely position, for the invariant distribution determined above. Next, we will examine their determinants, and emphasize some economic implications. We will finally interpret our results in terms of the rent dissipation problem.

5.1 Critical asymptotic political states

The expected asymptotic political state, denoted by $\tilde{z}$, is obtained by computing

$$\tilde{z} = \int_{-\infty}^{\infty} zq(z)\,dz.$$

This expected value yields the average political state at the asymmetric MPE. Besides, the most likely asymptotic political state, denoted by $\hat{z}$, is the point at which the probability density,

$q(z)$, reaches the maximum. Related papers typically restrict their attention to the latter when discussing the features of the long run equilibrium (see for instance Jorgensen and Yeung, 1996). If they do so, this is because they deal with a unique equilibrium. In our analysis, considering these two remarkable states turns out to be extremely valuable for the comparison between the S- vs. A-MPE. The next Proposition summarizes their main formal properties.

Proposition 4. *The expected and most-likely states are respectively given by*

$$\tilde{z} = \frac{\Gamma}{\sigma^2 - r}, \qquad \widehat{z} = \frac{\Gamma}{2\sigma^2 - r}.$$

- *For $j = 1$, both $\tilde{z}$ and $\hat{z}$ are positive and satisfy $\hat{z} < \tilde{z}$. For $j = 2$, both are negative and satisfy $\tilde{z} < \hat{z}$.*
- *For $j = 1$, $x_1(z) > x_2(z)$ for $z < \tilde{z}$, $x_1(z) < x_2(z)$ for $z > \tilde{z}$, and $V_1(z) > V_2(z)$ for any $0 \leq z \leq \tilde{z}$, where $V_i(z)$ represents player i's value function. In particular these inequalities are true at $\hat{z}$. For $j = 2$, the reversed inequalities hold.*

The most likely outcome is that the state variable will be located in the neighborhood of $\widehat{z}$. As this level is positive, the A-MPE $j = 1$ turns out to be more favorable to player 1. How to explain that this particular A-MPE is more favorable to player 1? Well, we can refer again to the properties of this solution to conclude that the system will end up in a region that suits more to the player who both deploys the largest means the more often, and is more reactive to any change in the state variable than his opponent. In fact, it is quite easy to check that $x_1(\widehat{z}) > x_2(\widehat{z})$ and $V_1(\widehat{z}) > V_2(\widehat{z})$. Of course, the converse would have been true if we had considered the A-MPE $j = 2$.

A more politically-oriented interpretation of the result is that the uncertainty in the dynamics of legislation, or institutions, may induce opposing groups to adopt strategies that are not generally compatible with the convergence to the status quo $z_\infty = 0$, that arises at the S-MPE. Indeed, following the lobbying strategies of the A-MPE, anything can happen asymptotically, that is, any level of legislation can be achieved with a positive probability, and the economy can end up with very bad or very good political and economic institutions. However, in the end, depending on the A-MPE that emerges, there is one most active and reactive player who manages to bring the most likely state in her preferred region.

5.2 Symmetric vs asymmetric equilibria and the rent dissipation problem

Asymmetric equilibria converge to an invariant distribution while the symmetric equilibrium goes almost surely to $z_\infty = 0$ where players exercise the same lobbying effort, equal to $\frac{a_1}{br - C}$. This clearly indicates that the former type of equilibrium can produce (much) larger rent dissipation than the latter with positive probability. To get the full picture, we need to characterize more finely the rent dissipation differential between S- and A-MPE. To ease the exposition, we again focus on the A-MPE $j = 1$. Rent dissipation is measured by total lobbying efforts, and our aim is to compare its level at the most likely state of the asymmetric equilibrium with its level at the almost sure steady state of the symmetric equilibrium. Direct computations yield

$$x^s(0) = \frac{12a_1}{6br + b(\sigma^2 - r) + \sqrt{b^2(\sigma^2 - r)^2 - 12a_2 b}},$$

$$x^a(\tilde{z}) = \frac{2a_1\left[b(\sigma^2 - r) - 2a_2 b\right]}{b(b\sigma^4 + a_2)(\sigma^2 - r)}, \qquad x^a(\hat{z}) = \frac{a_1\left[b\sigma^2(5\sigma^2 - 3r) - 4a_2\right]}{b(b\sigma^4 + a_2)(2\sigma^2 - r)},$$

where $x^l = \sum_{i=1,2} x_i^l$ for $l = a, s$. Comparing these aggregate values, we can establish that:

Proposition 5. *Consider the average position, $\tilde{z}$, then the ranking between aggregate efforts is $x^s(0) \geq x^a(\tilde{z})$ if and only if σ satisfies condition (8). As of the most likely position, $\hat{z}$, there exists $\overline{\sigma}$, with $\overline{\sigma}^2 > \underline{\sigma}^2$, such that the following ranking holds:*

$$x^s(0) < x^a(\widehat{z}) \text{ if } \sigma^2 \in (\underline{\sigma}^2, \overline{\sigma}^2), \; x^s(0) > x^a(\widehat{z}) \text{ else.}$$

Our results in terms of rent dissipation are much more involved than in the preexisting related deterministic differential games literature. First note that when A-MPE exist (that's when condition (8) holds), the average rent dissipation is lower with these equilibria relative to the S-MPE asymptotically. This might be surprising as A-MPE converge to invariant stationary distributions, and therefore they may lead to arbitrarily large losses with positive probability. To have a more accurate picture of the rent dissipation problem in our stochastic setting, we further exploit the closed-form solution of the stationary densities obtained and perform a comparison of lobbying efforts at the most likely state of these distributions with those at the almost sure asymptotic state of the S-MPE. It turns out that rent dissipation is superior in the most likely state of the A-MPE provided uncertainty is high enough but moderate. In other words, the interplay between the retaliation and uncertainty-driven behaviors in the A-MPE leads to a larger lobbying effort than under the S-MPE only under those conditions. If uncertainty is too high, lobbying becomes anyway too risky, efforts drop in both types of equilibria. But the specific uncertainty-induced mechanism being inherently more affected, rent dissipation is lower in the A-MPE.

Finally, note that the analysis above provides some information about the ranking between the S vs. A-MPE. Actually, when there exist multiple equilibria of different types, it is quite natural to search for a criterion to discriminate between them. The related literature often uses Pareto dominance to determine what is the “best” solution (see, for instance, Amir et al., 2008). In the current framework, and for the specific issue under scrutiny, looking at the rent dissipation measure (that is, the social cost of lobbying) appears to be a meaningful alternative. Then we get that from the social point of view, it is worth playing the A-MPE when the level of uncertainty is high.

6 Conclusion

In this paper, we have designed a stochastic dynamic game in order to study legislative lobbying. The problem has been motivated by a series of examples taken from environmental and health policies lobbying in Western countries. In both sets of examples, uncertainty and dynamics are essential ingredients which cannot be dismissed. In this context, we prove that uncertainty produces a new mechanism that may break the symmetry of equilibria, as it is generally observed in the deterministic counterpart, adding to the more traditional retaliation-based mechanism outlined in the (deterministic) literature. We also show that these asymmetric equilibria emerge when players are exposed to high enough uncertainty. Moreover, in such a case, the associated dynamics do converge to a well-defined invariant distribution.

More interestingly, we highlight that along these new asymmetric equilibria, rent dissipation may be superior compared to the symmetric equilibrium, and we characterize the conditions under which this holds. We have therefore produced in the context of lobbying games, a new mechanism for equilibrium symmetry-breaking and highlighted its implications in terms of stochastic stability on one hand, and for relevant economic problems related to lobbying (notably the rent dissipation problem) on the other.

A Appendix

A.1 Proof of Propositions 1 and 2

The proof of Propositions 1 and 2 can be done altogether, since they both come from the solution of the same stochastic dynamic game. Also, the proof for the catching up optimality is long and technical. Therefore we save it to the last part of the Subsection.

We first present the general calculation.

Denote

$$F_i(x_i, z) = \omega_i(z) - \beta(x_i).$$

To make it clear, we restate the the dynamic game as following: objective of player 1 is

$$\max_{x_1} \mathbb{E} \int_0^{+\infty} F_1(x_1, z) e^{-rt} dt = \mathbb{E} \int_0^{+\infty} e^{-rt} \left[a_0 + a_1 z + \frac{a_2}{2} z^2 - \frac{b}{2} x_1^2 \right] dt$$

and the objective of player 2 is

$$\max_{x_2} \mathbb{E} \int_0^{+\infty} F_2(x_2, z) e^{-rt} dt = \mathbb{E} \int_0^{+\infty} e^{-rt} \left[a_0 - a_1 z + \frac{a_2}{2} z^2 - \frac{b}{2} x_1^2 \right] dt,$$

with constants a_0, a_1, b positive and a_2 negative. The common state constraint is

$$dz = (x_1 - x_2) dt + \sigma z dW.$$

It is easy to see this is a standard linear-quadratic stochastic differential game. To obtain the stationary MPE, we define the value function of player i as

$$V_i(z) = A_i + B_i z + \frac{C_i}{2} z^2, \quad i = 1, 2,$$

with A_i, B_i, C_i undetermined coefficients. C_i should be negative to ensure strict concavity of the value function as the objective functions are strictly concave in (x_i, z) and the state function is linear. Thus these value functions must check the following Hamilton-Jacob-Bellman equations:

$$rV_i(z) = \max_{x_i} \left[F_i(x_i, z) + \frac{dV_i}{dz} (x_1 - x_2) + \frac{\sigma^2 z^2}{2} \frac{d^2 V_i}{dz^2} \right], \quad i = 1, 2. \tag{20}$$

The standard first order (necessary and sufficient) conditions on the right hand sides of (20) yield the optimal choice of player 1 and 2:

$$x_1^* = \frac{1}{b} \frac{dV_1}{dz} = \frac{B_1 + C_1 z}{b} \text{ and } x_2^* = -\frac{1}{b} \frac{dV_2}{dz} = -\frac{B_2 + C_2 z}{b}. \tag{21}$$

Substituting these optimal choices into the right hand sides of equation (20) and comparing the coefficients of term z on both left and right hand sides of (20), we obtain the following equation system for coefficients:

$$\begin{cases} rA_1 = a_0 + \frac{B_1^2}{2b} + \frac{B_1 B_2}{b}, \\ rB_1 = a_1 + \frac{B_1 C_1}{b} + \frac{B_1 C_2 + B_2 C_1}{b}, \\ rC_1 = a_2 + \frac{C_1^2}{b} + \frac{2 C_1 C_2}{b} + \sigma^2 C_1 \end{cases} \tag{22}$$

and

$$\begin{cases} rA_2 = a_0 + \frac{B_2^2}{2b} + \frac{B_1 B_2}{b}, \\ rB_2 = -a_1 + \frac{B_2 C_2}{b} + \frac{B_1 C_2 + B_2 C_1}{b}, \\ rC_2 = a_2 + \frac{C_2^2}{b} + \frac{2C_1 C_2}{b} + \sigma^2 C_2. \end{cases} \tag{23}$$

Combining the last equation of (22) and (23) together and rearranging terms, it yields

$$\left(r - \sigma^2\right)\ (C_1 - C_2) = \frac{(C_1 - C_2)(C_1 + C_2)}{b}.$$

Thus, two groups of solutions are possible:

$$C_1 = C_2$$

and

$$C_1 \neq C_2, \ \text{ then } \ C_2 = b(r - \sigma^2) - C_1.$$

A.1.1 Proof of Proposition 1, Items (i) and (ii)

Substituting $C_1 = C_2 = C$ into the last equation of (22) (or (23)), it yields that

$$C_1 = C_2 = \frac{-b(\sigma^2 - r) \pm \sqrt{b^2(\sigma^2 - r)^2 - 12ba_2}}{6},$$

which is always real, given $a_2 < 0$. For shortening the notation, we denote the above $C_1 = C_2$ as $C^{(l)}$, $l = 1, 2$, with $C^{(1)}$ taking negative in front of the square root term, while $C^{(2)}$ taking the positive one.

Furthermore, substituting $C_1 = C_2$ into the second equations of (22) and (23), it yields

$$B_1 + B_2 = 0, \ \text{ or } \ B_1 = -B_2.$$

Thus, by the second equation of (22) again, we have

$$B_1 = \frac{ba_1}{br - C} = -B_2.$$

Substituting B_i, C_i $(i = 1, 2)$ into the first equations of (22) and (23), we can obtain A_1 and A_2.

We next notice that for $j = 2$, the value functions are, for $i = 1, 2$,

$$V_i^{(2)} = A_i^{(2)} + B_i^{(2)} z + \frac{C_i^{(2)}}{2} z^2$$

with $C_1^{(2)} = C_2^{(2)} = \frac{-b(\sigma^2-r)+\sqrt{b^2(\sigma^2-r)^2-12ba_2}}{6} > 0$, given $a_2 < 0$. Thus, the value functions, $V_i^{(2)}$, are strictly convex. Following the arguments in Section 9.2 and 9.4 of Stokey et al (1989), $j = 2$ cannot be an optimal choice to the linear-quadratic games. Therefore, the only optimal symmetric choice is $j = 1$ which is given by Propositions 1.

Substituting now the above two equilibrium strategies into the stochastic state equation, we have

$$dz = [x_1 - x_2]dt + \sigma z\, dW = \frac{2C}{b} z\, dt + \sigma z\, dW,$$

which is a linear homogenous stochastic differential equation with $z = 0$ as one long-run solution. From the $AK-$type model of Boucekkine et al. (2018), it is easy to check that $z = 0$ is almost surely stochastically stable if and only if

$$\frac{2C}{b} - \sigma^2 < 0.$$

That completes the proof of Items (i) and (ii) in Propositions 1.

A.1.2 Proof of Proposition 2, first part

Substituting $C_2 = b(r - \sigma^2) - C_1$ into the last equation of (22) and rearranging terms, it follows:

$$C_1^{(j)} = \frac{-b(\sigma^2 - r) \mp \sqrt{b^2(\sigma^2 - r)^2 + 4ba_2}}{2}, \quad j = 1, 2$$

with $C_1^{(1)}$ taking negative sign of the square root term and $C_1^{(2)}$ taking positive one. Thus,

$$C_2^{(j)} = b(r - \sigma^2) - C_1^{(j)} = \frac{-b(\sigma^2 - r) \pm \sqrt{b^2(\sigma^2 - r)^2 + 4ba_2}}{2}, \quad j = 1, 2.$$

Remark. To guarantee that the square root term is real, some conditions on the parameters are needed. Here, we impose that

$$\sigma^2 \geq r + 2\sqrt{\frac{-a_2}{b}}.$$

Given $a_2 < 0$, we have $C_i^{(j)} < 0$, $i = 1, 2$ and $j = 1, 2$.

Combining the above expression $C_i^{(j)}$ into the second equations of (22) and (23), we obtain the $B_i^{(j)}$, $i = 1, 2$ and $j = 1, 2$.

That finishes the proof the first part of Proposition 2.

A.1.3 Proof of catching up optimality

We prove that the control paths in Propositions 1 and 2 are catching up optimal. We first extend a result in Dockner *et al* (2000) as a preparation.

Let $V(z,t)$ be a function that satisfies the HJB equation

$$\begin{aligned} rV(z,t) - V_t(z,t) = \max\nolimits_{x\in U(z,t)} \{ & F(z,x,t) + V_z(z,t) f(z,x,t) \\ & + \tfrac{1}{2} tr\left[V_{zz}(z,t)\,\sigma(z,x,t)\,\sigma(z,x,t)'\right] \}, \end{aligned} \tag{24}$$

where $U(z,t)$ is the set of feasible controls at (z,t), and let $\Phi(z,t)$ denote the set of $x \in U(z,t)$ maximizing the right-hand side of (24). One sufficient condition for a feasible control path $x(\cdot) \in \Phi(z(\cdot),\cdot)$ and a corresponding state trajectory $z(\cdot)$, is catching up optimal if $V(z,t)$ is bounded below and

$$\limsup_{t\to\infty} e^{-rt}\mathbb{E}_{x(\cdot)} V(z(t),t) \le 0. \tag{25}$$

(Cf. Theorem 8.4 in Dockner *et al.* (2000).) In the case where $V(x,t)$ is not bounded below, one may use a finite-horizon approximation. For any $T > 0$ we consider a function $V(z,t;T)$ that satisfies

$$\begin{aligned} rV(z,t;T) - V_t(z,t;T) = \max\nolimits_{x\in U(z,t)} \{ & F(z,x,t) + V_z(z,t;T) f(z,x,t) \\ & + \tfrac{1}{2} tr\left[V_{zz}(z,t;T)\,\sigma(z,x,t)\,\sigma(z,x,t)'\right] \} \end{aligned} \tag{26}$$

in for $t \in (0,T)$ and the terminal condition

$$V(z,T;T) = 0. \tag{27}$$

The following is an extension of Theorem 8.4 in Dockner *et al* (2000).

Lemma 1. *Let $V(z,t)$ satisfy (24) for all $t > 0$ and let $x(\cdot) \in \Phi(z(\cdot),\cdot)$, corresponding to the state trajectory $z(\cdot)$, that satisfies (25). In addition, suppose for all sufficiently large $T > 0$ there exists a continuously differentiable function $V(\cdot,\cdot;T) : Z \times [0,T] \to \mathbb{R}$ which solves the terminal value problem (26) and (27) and satisfies*

$$\lim_{T\to\infty} V(z,t;T) = V(z,t) \tag{28}$$

pointwise in $Z \times [0,\infty)$. Then, the control path $x(\cdot)$ is catching up optimal.

Proof. For any feasible control path $x(\cdot)$ we denote

$$J_T(x(\cdot)) = \mathbb{E}_{x(\cdot)} \int_0^T e^{-rt} F(z(t),x(t),t)\,dt.$$

By (24) and Itô's lemma (Lemma 8.2 in Dockner *et al* (2000)),

$$\begin{aligned} J_T(x(\cdot)) &= \mathbb{E}_{x(\cdot)} \int_0^T e^{-rt} \left[rV(z(t),t) - V_t(z(t),t) - V_z(z(t),t) f(z(t),x(t),t) \right. \\ &\quad \left. -\frac{1}{2} tr V_{zz}(z(t),t) \sigma(z(t),x(t),t) \sigma(z(t),x(t),t)' \right] dt \\ &= \mathbb{E}_{x(\cdot)} \int_0^T d\left[-e^{-rt} V(z(t),t) \right] = V(z_0,0) - \mathbb{E}_{x(\cdot)} e^{-rT} V(z(T),T). \end{aligned}$$

Consider a feasible path $\tilde{x}(\cdot)$ with corresponding state trajectory $\tilde{z}(\cdot)$. Using Itô's lemma to $G(z,t) = e^{-rt} V(z,t;T)$ and $g(t) = G(\tilde{z}(t),t)$ yields

$$\begin{aligned} -dg(t) &= e^{-rt} \left\{ rV(\tilde{z}(t),t;T) - V_t(\tilde{z}(t),t;T) - V_z(\tilde{z}(t),t;T) f(\tilde{z}(t),\tilde{x}(t),t) \right. \\ &\quad \left. -\frac{1}{2} tr \left[V_{zz}(\tilde{z}(t),t;T) \sigma(\tilde{z}(t),\tilde{x}(t),t) \sigma(\tilde{z}(t),\tilde{x}(t),t)' \right] \right\} dt \\ &\quad -e^{-rt} V_z(\tilde{z}(t),t;T) \sigma(\tilde{z}(t),\tilde{x}(t),t) dW \\ &\geq e^{-rt} F(\tilde{z}(t),\tilde{x}(t),t) dt - e^{-rt} V_z(\tilde{z}(t),t;T) \sigma(\tilde{z}(t),\tilde{x}(t),t) dW. \end{aligned}$$

Integrating over $[0,T]$, we obtain

$$V(z_0,0;T) \geq \int_0^T e^{-rt} F(\tilde{z}(t),\tilde{x}(t),t) dt - \int_0^T e^{-rt} V_z(\tilde{z}(t),t;T) \sigma(\tilde{z}(t),\tilde{x}(t),t) dW.$$

Applying the expectation operator $\mathbb{E}_{\tilde{u}(\cdot)}$, it follows that $V(z_0,0;T) \geq J_T(\tilde{x}(\cdot))$. As a result,

$$J_T(x(\cdot)) - J_T(\tilde{x}(\cdot)) \geq V(z_0,0) - V(z_0,0;T) - e^{-rT} \mathbb{E}_{x(\cdot)} V(z(T),T).$$

Hence, from (28) we obtain

$$\liminf_{T\to\infty} \left[J_T(x(\cdot)) - J_T(\tilde{x}(\cdot)) \right] \geq 0.$$

This completes the proof.

The above result can be easily extended to differential games. In particular, for a two-player model, suppose that $V_1(z,t)$ and $V_2(z,t)$ satisfy the equations

$$\begin{aligned} rV_i(x,t) - \partial_t V_i(x,t) &= \max_{x_i \in U_i(z,t)} \left\{ F_i(z,x_1,x_2) + \partial_z V_i(z,t) f(z,x_1,x_2,t) \right. \\ &\quad \left. + \frac{1}{2} tr \left[\partial_z^2 V_i(z,t) \sigma(z,x_1,x_2,t) \sigma(z,x_1,x_2,t)' \right] \right\} \end{aligned}$$

for $i = 1,2$ and that the control path $(x_1(\cdot), x_2(\cdot))$ maximize the right-hand side of the above equation. The finite-horizon approximations $V_1(z,t;T)$ and $V_2(z,t;T)$ satisfy

$$\begin{aligned} rV_1(z,t;T) - \partial_t V_1(z,t;T) &= \max\nolimits_{\tilde{x}_1 \in U_1(z,t)} \left\{ F_1(z,\tilde{x}_1,x_2(t),t) + \partial_z V_1(z,t;T) f(z,\tilde{x}_1,x_2(t),t) \right. \\ &\quad \left. + \tfrac{1}{2} tr \left[\partial_z^2 V_1(z,t;T) \sigma(z,\tilde{x}_1,x_2(t),t) \sigma(z,\tilde{x}_1,x_2(t),t)' \right] \right\} \\ rV_2(z,t;T) - \partial_t V_2(z,t;T) &= \max\nolimits_{\tilde{x}_2 \in U_2(z,t)} \left\{ F_2(z,x_1(t),\tilde{x}_2,t) + \partial_z V_2(z,t;T) f(z,x_1(t),\tilde{x}_2,t) \right. \\ &\quad \left. + \tfrac{1}{2} tr \left[\partial_z^2 V_2(z,t;T) \sigma(z,x_1(t),\tilde{x}_2,t) \sigma(z,x_1(t),\tilde{x}_2,t)' \right] \right\} \end{aligned} \tag{29}$$

for $t < T$ and

$$V_1(z,T;T) = V_2(z,T;T) = 0 \quad \text{for } z \in Z.$$

If this terminal value problem has a solution $(V_1(z,t;T), V_2(z,t;T))$ that satisfies

$$\lim_{T\to\infty} V_i(z,t;T) = V_i(z,t) \quad \text{for } i = 1,2$$

and the condition

$$\limsup_{t\to\infty} e^{-rt}\mathbb{E}_{(x_1(\cdot),x_2(\cdot))}V_i(z(t),t) = 0 \quad \text{for } i = 1,2 \tag{30}$$

then the control path $(x_1(\cdot), x_2(\cdot))$ is catching up optimal.

For the $(x_1(\cdot), x_2(\cdot))$ in Proposition 1 that satisfies (4), where $z(\cdot)$ satisfies (6), since z almost surely converges to $z_\infty = 0$, it follows that $\mathbb{E}_{(x_1(\cdot),x_2(\cdot))}z(t) \to 0$ as $t \to \infty$. As a result

$$\lim_{t\to\infty} \mathbb{E}_{(x_1(\cdot),x_2(\cdot))}V_i(z(t)) = 0 \quad \text{for } i = 1,2.$$

Hence (30) holds in this case. Similarly, for the value functions $V_i^{(j)}(z)$ in Proposition 2, the control path $\left(x_1^{(j)}(\cdot), x_2^{(j)}(\cdot)\right)$ satisfy (9) for $j = 1,2$, where $z^{(j)}(\cdot)$ satisfies (13). It follows that the expected value $\mathcal{Z}^{(j)}(t) = \mathbb{E}_{(x_1(\cdot),x_2(\cdot))}z^{(j)}(t)$ satisfies the equation

$$d\mathcal{Z}^{(j)} = \left[\Gamma - \left(\sigma^2 - r\right)\mathcal{Z}^{(j)}\right]dt.$$

Since $\sigma^2 > r$, $\mathcal{Z}(t)$ converges to the limit $\Gamma/\left(\sigma^2 - r\right)$. Hence $\mathbb{E}_{\left(x_1^{(j)}(\cdot),x_2^{(j)}(\cdot)\right)}V_i^{(j)}\left(z^{(j)}(t)\right) = V_i^{(j)}(\mathcal{Z}(t))$ converges to a finite limit. As a result,

$$\lim_{t\to\infty} e^{-rt}\mathbb{E}_{\left(x_1^{(j)}(\cdot),x_2^{(j)}(\cdot)\right)}V_i^{(j)}\left(z^{(j)}(t)\right) = 0.$$

This proves (30) for A-MPE.

It remains to show that the finite-horizon approximation, $V_1(z,t;T)$ and $V_2(z,t;T)$ satisfy

$$\lim_{T\to\infty} V_i(z,t;T) = V_i(z) \qquad \text{for } i = 1,2. \tag{31}$$

Eq. (29) takes the form

$$\begin{aligned} rV_1(z,t;T) - \partial_t V_1(z,t;T) &= \max\nolimits_{\tilde{x}_1}\{\omega_1(z) - \beta(\tilde{x}_1) \\ &\quad + \partial_z V_1(z,t;T)[\tilde{x}_1 - x_2(z)] + \tfrac{\sigma^2}{2}\partial_z^2 V_1(z,t;T)\Big\} \\ rV_2(z,t;T) - \partial_t V_2(z,t;T) &= \max\nolimits_{\tilde{x}_2}\{\omega_2(z) - \beta(\tilde{x}_2) + \\ &\quad + \partial_z V_2(z,t;T)[x_1(z) - \tilde{x}_2] + \tfrac{\sigma^2}{2}\partial_z^2 V_2(z,t;T)\Big\}, \end{aligned} \tag{32}$$

where

$$x_1(z) = \frac{1}{b}(B + Cz), \quad x_2(z) = \frac{1}{b}(B - Cz)$$

for S-MPE, and

$$x_1(z) \equiv x_1^{(j)}(z) = \frac{1}{b}\left(B_1^{(j)} + C_1^{(j)} z\right), \qquad x_2(z) \equiv x_2^{(j)}(z) = -\frac{1}{b}\left(B_2^{(j)} + C_2^{(j)} z\right)$$

($j = 1, 2$) for A-MPE. It is easy to see that the right-hand sides of the first two equations in (32) are maximized at

$$\tilde{x}_1 = \frac{1}{b}\partial_z V_1(z, t; T) \text{ and } \tilde{x}_2 = -\frac{1}{b}\partial_z V_2(z, t; T),$$

respectively. Hence,

$$\begin{aligned} rV_1(z,t;T) - \partial_t V_1(z,t;T) &= a_0 + a_1 z + \frac{a_2}{2}z^2 + \frac{1}{2b}\left[\partial_z V_1(z;t;T)\right]^2 \\ &\quad + \frac{1}{b}(B_2 + C_2 z)\partial_z V_1(z,t;T) + \frac{\sigma^2}{2}\partial_z^2 V_1(z,t;T), \\ rV_2(z,t;T) - \partial_t V_2(z,t;T) &= a_0 - a_1 z + \frac{a_2}{2}z^2 + \frac{1}{2b}\left[\partial_z V_2(z;t;T)\right]^2 \\ &\quad + \frac{1}{b}(B_1 + C_1 z)\partial_z V_2(z,t;T) + \frac{\sigma^2}{2}\partial_z^2 V_2(z,t;T). \end{aligned}$$

We seek solutions in the form

$$V_i(z,t;T) = D_i(T-t) + E_i(T-t)z + \frac{F_i(T-t)}{2}z^2 \qquad \text{for } i = 1, 2$$

where D_i, E_i, and F_i are functions defined on $[0, \infty)$. using a change of variable $\tau = T - t$, these functions satisfy the differential equations

$$\begin{aligned} rD_1 + D_1' &= a_0 + \frac{1}{2b}E_1^2 + \frac{1}{b}B_2 E_1, \\ rE_1 + E_1' &= a_1 + \frac{1}{b}E_1 F_1 + \frac{1}{b}(C_2 E_1 + B_2 F_1), \\ \frac{r}{2}F_1 + F_1' &= \frac{a_2}{2} + \frac{1}{2b}F_1^2 + \frac{1}{b}C_2^{(j)} F_1 + \frac{\sigma^2}{2}F_1, \end{aligned}$$

and

$$\begin{aligned} rD_2 + D_2' &= a_0 + \frac{1}{2b}E_2^2 + \frac{1}{b}B_1 E_2, \\ rE_2 + E_2' &= -a_1 + \frac{1}{b}E_2 F_2 + \frac{1}{b}(C_1 E_2 + B_1 F_2), \\ \frac{r}{2}F_2 + F_2' &= \frac{a_2}{2} + \frac{1}{2b}F_2^2 + \frac{1}{b}C_1 F_2 + \frac{\sigma^2}{2}F_2, \end{aligned}$$

for $\tau > 0$, where "$'$" stands for "$d/d\tau$," and the initial conditions

$$D_i(0) = E_i(0) = F_i(0) = 0.$$

We first consider S-MPE. In this case $C_1 = C_2 = C$ that satisfy (5). The equations for F_1 and F_2 can be written as

$$F_i' = \frac{1}{2}\left[\frac{1}{b}F_i^2 + \left(\sigma^2 - r + \frac{2}{b}C\right)F_i + a_2\right] \qquad \text{for } i = 1, 2. \tag{33}$$

The quadratic function on the right-hand side has two roots,

$$\frac{b}{2}\left[-\left(\sigma^2 - r + \frac{2}{b}C\right) \pm \sqrt{\left(\sigma^2 - r + \frac{2}{b}C\right)^2 - \frac{4a_2}{b}}\right].$$

Since $a_2 < 0$, the two roots are real and have opposite signs. Since $F_i(0) = 0$ and $F_i' < 0$ for F_i between the two roots, it follows that $F_i(\tau)$ converges to the negative root. Note that C is a root of the quadratic polynomial on the right-hand side of (33) and it is negative. Therefore

$$\lim_{\tau\to\infty} F_i(\tau) = C \qquad \text{for } i = 1, 2.$$

We then solve the linear equations for E_1 and E_2 to get

$$\begin{aligned} E_1(\tau) &= \frac{1}{m(\tau)}\int_0^\tau m(s)\left[a_1 + \frac{1}{b}B_2F_1(s)\right]ds, \\ E_2(\tau) &= \frac{1}{m(\tau)}\int_0^\tau m(s)\left[-a_1 + \frac{1}{b}B_1F_2(s)\right]ds \end{aligned}$$

where

$$m(\tau) = e^{\int_0^\tau\left[r - \frac{1}{b}(F_1(s)+C)ds\right]}$$

(Notice that $F_1 = F_2$.) Using l'Hôpital's Rule we find

$$\lim_{\tau\to\infty} E_1(\tau) = \lim \frac{a_1 + \frac{1}{b}B_2F_1(\tau)}{r - \frac{1}{b}(F_1(\tau)+C)} = \frac{a_1 + \frac{1}{b}B_2C}{r - \frac{2}{b}C}.$$

A similar derivation leads to

$$\lim_{\tau\to\infty} E_2(\tau) = \frac{-a_1 + \frac{1}{b}B_1C}{r - \frac{2}{b}C}.$$

Using

$$B_1 = -B_2 = \frac{ba_1}{br - C}$$

we find

$$\lim_{\tau\to\infty} E_1(\tau) = \frac{a_1b}{br - C} = B_1, \qquad \lim_{\tau\to\infty} E_2(\tau) = -\frac{a_1b}{br - C} = B_2.$$

Finally, solving the linear equations for D_1 and D_2 we find

$$\begin{aligned} D_1(\tau) &= e^{-r\tau}\int_0^\tau e^{rs}\left[a_0 + \frac{1}{2b}E_1^2(s) + \frac{1}{b}B_2E_1(s)\right]ds, \\ D_2(\tau) &= e^{-r\tau}\int_0^\tau e^{rs}\left[a_0 + \frac{1}{2b}E_2^2(s) + \frac{1}{b}B_1E_2(s)\right]ds. \end{aligned}$$

Taking limits as $\tau \to \infty$, we find

$$\begin{aligned} \lim_{\tau\to\infty} D_1(\tau) &= \frac{1}{r}\left[a_0 + \frac{1}{2b}B_1^2 + \frac{1}{b}B_1B_2\right], \\ \lim_{\tau\to\infty} D_2(\tau) &= \frac{1}{r}\left[a_0 + \frac{1}{2b}B_2^2 + \frac{1}{b}B_1B_2\right]. \end{aligned}$$

Comparing with Eqs. (22) and (23), we find $\lim_{\tau\to\infty} D_1(\tau) = A_1$ and $\lim_{\tau\to\infty} D_2(\tau) = A_2$.

As a result, it follows that

$$\begin{aligned} \lim_{T\to\infty} V_i(z,t;T) &= \lim_{\tau\to\infty}\left[D_i(\tau) + E_i(\tau)z + \frac{F_i(\tau)}{2}z^2\right] \\ &= A_i + B_iz + \frac{C_i}{2}z^2 = V_i(z). \end{aligned}$$

This proves (31) for S-MPE.

For A-MPE with $j = 1, 2$, the equations for F_1 and F_2 are

$$\begin{aligned} F_1' &= \frac{1}{2}\left[\frac{1}{b}F_1^2 + \left(\sigma^2 - r + \frac{2}{b}C_2^{(j)}\right)F_1 + a_2\right], \qquad F_1(0) = 0; \\ F_2' &= \frac{1}{2}\left[\frac{1}{b}F_2^2 + \left(\sigma^2 - r + \frac{2}{b}C_1^{(j)}\right)F_2 + a_2\right], \qquad F_2(0) = 0. \end{aligned}$$

Since $a_2 < 0$, the quadratic polynomials on the right-hand sides of the equations have two real roots of opposite signs. Therefore $F_i(\tau)$ converges to the negative root of the corresponding polynomial. Furthermore, since $C_1^{(j)}$ and $C_2^{(j)}$ are negative roots of the right-hand side of the first and second equations, respectively, it follows that

$$\lim_{\tau\to\infty} F_i(\tau) = C_i^{(j)} \qquad \text{for } i = 1, 2. \tag{34}$$

We then solve the equations for $E_1(\tau)$ and $E_2(\tau)$ to get

$$\begin{aligned} E_1(\tau) &= \frac{1}{m_1(\tau)}\int_0^\tau m_1(s)\left[a_1 + \frac{1}{b}B_2^{(j)}F_1(s)\right]ds, \\ E_2(\tau) &= \frac{1}{m_2(\tau)}\int_0^\tau m_2(s)\left[-a_1 + \frac{1}{b}B_1^{(j)}F_2(s)\right]ds, \end{aligned}$$

where

$$m_1(\tau) = e^{\int_0^\tau\left[r - \frac{1}{b}F_1(s) - \frac{1}{b}C_2^{(j)}\right]ds}, \qquad m_2(\tau) = e^{\int_0^\tau\left[r - \frac{1}{b}F_2(s) - \frac{1}{b}C_1^{(j)}\right]ds}$$

Taking the limit as $\tau \to \infty$ and using (34) we find

$$\lim_{\tau\to\infty} E_1(\tau) = \frac{a_1 + \frac{1}{b}B_2^{(j)}C_1^{(j)}}{r - \frac{1}{b}\left(C_1^{(j)} + C_2^{(j)}\right)}, \qquad \lim_{\tau\to\infty} E_2(\tau) = \frac{-a_1 + \frac{1}{b}B_1^{(j)}C_2^{(j)}}{r - \frac{1}{b}\left(C_1^{(j)} + C_2^{(j)}\right)}.$$

Comparing to the second equations in (22) and (23), we find that

$$\lim_{\tau \to \infty} E_i(\tau) = B_i^{(j)} \qquad \text{for } i = 1, 2.$$

Finally, solving the equations for $D_1(\tau)$ and $D_2(\tau)$ we find

$$\begin{aligned} D_1(\tau) &= e^{-r\tau} \int_0^\tau e^{rs} \left[a_0 + \frac{1}{2b} E_1^2(s) + \frac{1}{b} B_2^{(j)} E_1(s) \right] ds, \\ D_2(\tau) &= e^{-r\tau} \int_0^\tau e^{rs} \left[a_0 + \frac{1}{2b} E_2^2(s) + \frac{1}{b} B_1^{(j)} E_2(s) \right] ds. \end{aligned}$$

Taking the limit as $\tau \to \infty$, we find

$$\begin{aligned} \lim_{\tau \to \infty} D_1(\tau) &= \frac{1}{r} \left[a_0 + \frac{1}{2b} \left(B_1^{(j)} \right)^2 + \frac{1}{b} B_1^{(j)} B_2^{(j)} \right], \\ \lim_{\tau \to \infty} D_2(\tau) &= \frac{1}{r} \left[a_0 + \frac{1}{2b} \left(B_2^{(j)} \right)^2 + \frac{1}{b} B_1^{(j)} B_2^{(j)} \right]. \end{aligned}$$

Comparing with the first equations in (22) and (23), we see that

$$\lim_{\tau \to \infty} D_i(\tau) = A_i^{(j)} \qquad \text{for } i = 1, 2.$$

This leads to

$$\begin{aligned} \lim_{T \to \infty} V_i(z, t; T) &= \lim_{\tau \to \infty} \left[D_i(\tau) + E_i(\tau) z + \frac{F_i(\tau)}{2} z^2 \right] \\ &= A_i^{(j)} + B_i^{(j)} z + \frac{C_i^{(j)}}{2} z^2 = V_i^{(j)}(z). \end{aligned}$$

This proves (31). Hence, the control path $\left(x_1^{(j)}(\cdot), x_2^{(j)}(\cdot) \right)$ is catching up optimal for all three cases.

The proof of Propositions 1 and 2 is complete.

A.2 Proof of Proposition 3

A steady-state $q(z)$ satisfies

$$\frac{\sigma^2}{2} \left(z^2 q \right)'' - \left[\left(\Gamma + \left(r - \sigma^2 \right) z \right) q \right]' = 0.$$

By integration, we find

$$\frac{\sigma^2}{2} \left(z^2 q \right)' - \left(\Gamma + \left(r - \sigma^2 \right) z \right) q = K$$

for some constant K. The first order linear equation can be written as

$$\frac{1}{2} \sigma^2 z^2 q' + \left[\left(2\sigma^2 - r \right) z - \Gamma \right] q = K.$$

Let $\mu(z)$ be the integrating factor

$$\mu(z) = |z|^{4-2r/\sigma^2} e^{2\Gamma/\left(\sigma^2 z\right)}.$$

This leads to

$$[\mu(z) q(z)]' = 2K \frac{\mu(z)}{\sigma^2 z^2}.$$

Integrating the both sides over an interval $[a, z]$, for any $a \in \mathbb{R}$, we obtain

$$q(z) = \frac{\mu(a) q(a)}{\mu(z)} + \frac{2K}{\sigma^2 \mu(z)} \int_a^z \frac{\mu(\xi)}{\xi^2} d\xi.$$

We show that $K = 0$. Note that $e^{2\Gamma/\sigma^2 z} \to 1$ as $z \to \pm\infty$. There is $M > |a|$ such that

$$1/2 \leq e^{2\Gamma/\sigma^2 z} \leq 2 \qquad \text{if } |z| \geq M.$$

Thus

$$\begin{aligned} \int_a^z \frac{\mu(\xi)}{\xi^2} d\xi &\geq \int_a^M \frac{\mu(\xi)}{\xi^2} d\xi + \frac{1}{2} \int_M^z \xi^{2-2r/\sigma^2} d\xi \\ &= \int_a^M \frac{\mu(\xi)}{\xi^2} d\xi + \frac{1}{2\left(3 - 2r/\sigma^2\right)} \left[z^{3-2r/\sigma^2} - M^{3-2r/\sigma^2}\right] \\ &= O\left(z^{3-2r/\sigma^2}\right). \end{aligned}$$

Since $\mu(z) = O\left(z^{4-2r/\sigma^2}\right)$ as $z \to \infty$, it follows that

$$\frac{1}{\mu(z)} \int_a^z \frac{\mu(\xi)}{\xi^2} d\xi = O\left(z^{-1}\right) \qquad \text{as } z \to \infty.$$

This function is not integrable on $(-\infty, \infty)$. Thus $q(z)$ cannot satisfy

$$\int_{-\infty}^{\infty} q(z)\, dz = 1 \tag{35}$$

unless $K = 0$. As a result,

$$q(z) = \mu(a) q(a) / \mu(z) \tag{36}$$

for some $a \in \mathbb{R}$.

We next show that $q(z) = 0$ for $z < 0$ if $\Gamma > 0$. If this were not true, then there is $a < 0$ such that $q(a) > 0$ and (36) holds for $z \in \mathbb{R}$. However,

$$\frac{1}{\mu(z)} = |z|^{-4+2r/\sigma^2} e^{-2\Gamma/\left(a^2 z\right)} \geq O\left(|z|^{-2}\right) \qquad \text{as } z \to 0^-. \tag{37}$$

It cannot be integrable on $(-\infty, 0)$. This again fails (35). Therefore, $q(z) = 0$ for $z < 0$. On the other hand, since $\sigma^2 > r$, it follows that

$$\frac{1}{\mu(z)} = O\left(|z|^{-2}\right) \qquad \text{as } z \to \infty$$

and

$$\lim_{z \to 0^+} \frac{1}{\mu(z)} = \lim_{z \to 0^+} |z|^{-4+2r/\sigma^2} e^{-2\Gamma/\left(a^2 z\right)} = 0.$$

Therefore $1/\mu(z)$ is integrable on $(0, \infty)$. Hence $q(z)$ has the form

$$q(z) = \begin{cases} 0 & \text{if } z \leq 0, \\ k_1/\mu(z) & \text{if } z > 0 \end{cases}$$

if $\Gamma > 0$, where

$$k_1 = 1/\int_0^\infty \frac{dz}{\mu(z)}$$

by (35).

A similar argument show that

$$q(z) = \begin{cases} k_2/\mu(z) & \text{if } z < 0, \\ 0 & \text{if } z \geq 0 \end{cases}$$

if $\Gamma < 0$, where

$$k_2 = 1/\int_{-\infty}^0 \frac{dz}{\mu(z)}.$$

A.3 Proof of Theorem 1

We first show that the eigenvalue problem

$$\frac{\sigma^2}{2}\left[z^2\phi(z)\right]'' - \left[\left(\Gamma + \left(r - \sigma^2\right)z\right)\phi(z)\right]' = \lambda\phi(z) \qquad \text{on } (-\infty, \infty) \tag{38}$$

subject to the constraint

$$\int_{-\infty}^\infty \phi(z)\, dz = 0 \tag{39}$$

has only negative eigenvalues in the space $L^1(\mathbb{R})$.

It is well-known that the principal (largest) eigenvalue possesses an eigenfunction of one sign. Suppose λ_1 is the principal eigenvalue of (38) (without the constraint (39)), and that ϕ_1 is a nonnegative eigenfunction corresponding to λ_1. Integrating the two sides of (38) over $(-\infty, \infty)$ leads to

$$\lambda \int_{-\infty}^\infty \phi(z)\, dz = \frac{\sigma^2}{2}\left[z^2\phi(z)\right]' - \left(\Gamma + \left(r - \sigma^2\right)z\right)\phi(z)\Big|_{-\infty}^\infty. \tag{40}$$

We first show that the second term on the right-hand side vanishes. Note that Eq. (38) is well-approximated by the Cauchy-Euler equation

$$\frac{\sigma^2}{2}\left[z^2\psi(z)\right]'' + \left[\left(\sigma^2 - r\right)z\psi(z)\right]' - \lambda\psi(z) = 0$$

for $|z|$ sufficiently large. A general solution of the above equation has the form $c_1|z|^{m_1} + c_2|z|^{m_2}$ or $c_1|z|^{m_1} + c_2|z|^{m_1}\ln|z|$ for some constants m_1 and m_2. Thus $\phi_1(z) = o(|z|^m)$ as $z \to \pm\infty$ for some $m \in \mathbb{R}$. Since ϕ_1 is integrable on $(-\infty, \infty)$, it follows that $m \leq -1$. Therefore, $z\phi_1(z) \to 0$ as $z \to \pm\infty$. As a result,

$$\left(\Gamma + \left(r - \sigma^2\right)z\right)\phi_1(z)\Big|_{-\infty}^{\infty} = 0.$$

Hence, by (40),

$$\lambda\int_{-\infty}^{\infty}\phi_1(z)\,dz = \frac{\sigma^2}{2}\left[z^2\phi_1(z)\right]'\Big|_{-\infty}^{\infty} \equiv \sigma^2\left[z\phi_1(z) + \frac{z^2}{2}\phi_1'(z)\right]\Big|_{-\infty}^{\infty}.$$

Since $z\phi_1(z) \to 0$ as $z \to \pm\infty$, it follows that

$$\lambda\int_{-\infty}^{\infty}\phi_1(z)\,dz = \frac{\sigma^2}{2}z^2\phi_1'(z)\Big|_{-\infty}^{\infty}.$$

If $\lambda > 0$, then the left-hand side is positive. Thus

$$\lim_{z\to\infty} z^2\phi_1'(z) > \lim_{z\to-\infty} z^2\phi_1'(z). \tag{41}$$

On the other hand, since $\phi_1(z)$ is nonnegative and $\phi_1(z) \to 0$ as $z \to \pm\infty$, it follows that

$$\liminf_{z\to\infty} z^2\phi_1'(z) \leq 0, \qquad \limsup_{z\to-\infty} z^2\phi_1'(z) \geq 0.$$

Therefore, (41) cannot hold. This proves that the principal eigenvalue of (38) cannot be positive.

It is easy to see that $q(z)$ is an eigenfunction corresponding to $\lambda = 0$. Therefore, the principal eigenvalue of (38) is zero. On the other hand, $q(z)$ does not satisfy (39), and from the derivation of $q(z)$ we can see that any eigenfunction in $L^1(\mathbb{R})$ corresponding to $\lambda = 0$ is a constant multiple of $q(z)$. Such functions do not satisfy (39). Therefore any eigenvalue corresponding to an eigenfunction that satisfies (39) must be negative.

We now show that the solution u to Problem (16) and (17) has the limit

$$\lim_{t\to\infty}\|u(\cdot,t)\|_{L^1(\mathbb{R})} = 0.$$

For any $\lambda < 0$ we let $\phi(z;\lambda)$ denote an eigenfunction of Problem (38)-(39), which is normalized so that $\|\phi(\cdot,\lambda)\|_{L^1(\mathbb{R})} = 1$. We also let $\psi(z;\lambda)$ be an eigenfunction of the adjoint eigenvalue problem

$$\frac{\sigma^2}{2}\left[z^2\psi'(z)\right]' + \left(\Gamma + \left(r - 2\sigma^2\right)z\right)\psi'(z) = \lambda\psi(z) \qquad \text{on } (-\infty, \infty)$$

in $C^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, normalized so that $\|\psi(\cdot,\lambda)\|_{L^\infty(\mathbb{R})} = 1$. Therefore, any $L^1(\mathbb{R})$ function f that satisfies

$$\int_{-\infty}^{\infty} f(z)\, dz = 0$$

can be expanded to

$$f(z) = \int_{-\infty}^{0} C(\lambda)\, \phi(z,\lambda)\, d\lambda$$

where

$$C(\lambda) = \int_{-\infty}^{\infty} f(z)\, \psi(z,\lambda)\, dz.$$

In addition, $C(\lambda)$ satisfies

$$\begin{aligned} |C(\lambda)| &\leq \int_{-\infty}^{\infty} |f(z)|\, \|\psi(\cdot,\lambda)\|_{L^\infty(\mathbb{R})}\, dz \\ &= \int_{-\infty}^{\infty} |f(z)|\, dz = \|f\|_{L^1(\mathbb{R})}. \end{aligned}$$

Using such expansions on $u(z,t)$ and its initial value, $u_0(z)$, we can write

$$u(z,t) = \int_{-\infty}^{0} C(t,\lambda)\, \phi(z,\lambda)\, d\lambda, \qquad u_0(z) = \int_{-\infty}^{0} C_0(\lambda)\, \phi(z,\lambda)\, d\lambda.$$

Substituting the first identity into Eq. (16) leads to the initial value problems

$$C_t(t,\lambda) = \lambda C(t,\lambda), \qquad C(0,\lambda) = C_0(\lambda).$$

The solutions are

$$C(t,\lambda) = C_0(\lambda)\, e^{\lambda t} \qquad \text{for } \lambda < 0.$$

This leads to

$$u(z,t) = \int_{-\infty}^{0} C_0(\lambda)\, e^{\lambda t} \phi(z,\lambda)\, d\lambda.$$

Taking the $L^1(\mathbb{R})$ norm and using the relations

$$\|\phi(\cdot,\lambda)\|_{L^1(\mathbb{R})} = 1, \qquad |C_0(\lambda)| \leq \|u_0\|_{L^1(\mathbb{R})},$$

we obtain

$$\begin{aligned} \|u(\cdot,t)\|_{L^1(\mathbb{R})} &\leq \int_{-\infty}^{0} e^{\lambda t}\, |C_0(\lambda)|\, \|\phi(\cdot,\lambda)\|_{L^1(\mathbb{R})}\, d\lambda \\ &\leq \int_{-\infty}^{0} e^{\lambda t}\, \|u_0\|_{L^1(\mathbb{R})}\, d\lambda = \frac{1}{t}\, \|u_0\|_{L^1(\mathbb{R})} \to 0 \end{aligned}$$

as $t \to \infty$. This proves the convergence.

A.4 Proof of Proposition 4

We find $\tilde{z}$ by evaluating the integral

$$\tilde{z} = \int_{-\infty}^{\infty} zq(z)\,dz.$$

Since for $j = 1$, $q(z) = 0$ for $z < 0$, the integral is over the interval $(0, \infty)$. Using integration by parts we find

$$\int_0^{\infty} zq(z)\,dz = \frac{z^2}{2}q(z)\Big|_0^{\infty} - \int_0^{\infty} \frac{z^2}{2}q'(z)\,dz.$$

Using the expression

$$q(z) = k_1 z^{2r/\sigma^2-4}e^{-2\Gamma/(\sigma^2 z)} \quad \text{for } z > 0$$

we see that the first term on the right-hand side vanishes. Furthermore, by differentiation,

$$q'(z) = k_1\left[\left(\frac{2r}{\sigma^2} - 4\right)z^{2r/\sigma^2-5}e^{-2\Gamma/(\sigma^2 z)} + \frac{2\Gamma}{\sigma^2}z^{2r/\sigma^2-6}e^{-2\Gamma/(\sigma^2 z)}\right].$$

It follows that

$$\begin{aligned}\frac{z^2}{2}q'(z) &= \left(\frac{r}{\sigma^2} - 2\right)k_1 z^{2r/\sigma^2-3}e^{-2\Gamma/\sigma^2 z} + \frac{\Gamma}{\sigma^2}z^{2r/\sigma^2-4}e^{-2\Gamma/(\sigma^2 z)}\\ &= \left(\frac{r}{\sigma^2} - 2\right)zq(z) + \frac{\Gamma}{\sigma^2}q(z).\end{aligned}$$

As a result,

$$\int_0^{\infty} zq(z)\,dz = -\left(\frac{r}{\sigma^2} - 2\right)\int_0^{\infty} zq(z)\,dz - \frac{\Gamma}{\sigma^2}\int_0^{\infty} q(z)\,dz.$$

Note that the second integral on the right is 1. Solving from the above equation we find

$$\int_0^{\infty} zq(z)\,dz = -\frac{\Gamma/\sigma^2}{r/\sigma^2 - 1} = \frac{\Gamma}{\sigma^2 - r}.$$

This proves the assertion.

To prove the expression of $\hat{z}$, we use Proposition 3. In the case where $\Gamma > 0$, $q(z) = 0$ for $z < 0$ and

$$q(z) = k_1 z^{2r/\sigma^2-4}e^{-2\Gamma/(\sigma^2 z)} \quad \text{for } z > 0.$$

By differentiation,

$$\begin{aligned}q'(z) &= k_1\left(\frac{2r}{\sigma^2} - 4\right)z^{2r/\sigma^2-5}e^{-2\Gamma/(\sigma^2 z)} + k_1 z^{2r/\sigma^2-4}e^{-2\Gamma/(\sigma^2 z)}\frac{2\Gamma}{\sigma^2 z^2}\\ &= k_1 z^{2r/\sigma^2-6}e^{-2\Gamma/(\sigma^2 z)}\left[\left(\frac{2r}{\sigma^2} - 4\right)z + \frac{2\Gamma}{\sigma^2}\right].\end{aligned}$$

There is one critical point at

$$\hat{z} = \frac{2\Gamma}{4\sigma^2 - 2r} = \frac{\Gamma}{2\sigma^2 - r}.$$

It is easy to see that $q'(z) > 0$ if $0 < z < \hat{z}$ and $q'(z) < 0$ if $z > \hat{z}$.

Similarly, in the case where $\Gamma < 0$, $q(z) = 0$ fir $z < 0$ and

$$q(z) = k_1 |z|^{2r/\sigma^2 - 4} e^{-2\Gamma/\left(\sigma^2 z\right)} \quad \text{for } z < 0.$$

By differentiation,

$$\begin{aligned} q'(z) &= -k_1 \left(\frac{2r}{\sigma^2} - 4 \right) |z|^{2r/\sigma^2 - 5} e^{-2\Gamma/\left(\sigma^2 z\right)} + k_1 |z|^{2r/\sigma^2 - 4} e^{-2\Gamma/\left(\sigma^2 z\right)} \frac{2\Gamma}{\sigma^2 z^2} \\ &= k_1 |z|^{2r/\sigma^2 - 6} e^{-2\Gamma/\left(\sigma^2 z\right)} \left[-\left(\frac{2r}{\sigma^2} - 4 \right) |z| + \frac{2\Gamma}{\sigma^2} \right]. \end{aligned}$$

There is again one critical point at

$$\hat{z} = -|\hat{z}| = \frac{\Gamma}{2\sigma^2 - r}.$$

It is clear that $q'(z) > 0$ for $z < \hat{z}$ and $q'(z) < 0$ if $\hat{z} < z < 0$.

From their expressions we find

$$\hat{z} = \frac{\Gamma}{2\sigma^2 - r} < \frac{\Gamma}{\sigma^2 - r} = \tilde{z}.$$

This proves the first statement of Proposition 4.

To prove the second statement, we first show that $x_1(\tilde{z}) = x_2(\tilde{z})$. From (9) we can derive that the value of z at which the two players exert the equal efforts is

$$-\frac{B_1^{(1)} + B_2^{(1)}}{C_1^{(1)} + C_2^{(1)}} = -\frac{a_1 b \left(C_1^{(1)} - C_2^{(1)} \right)}{\left(b^2 \sigma^4 - C_1^{(1)} C_2^{(1)} \right) \left(C_1^{(1)} + C_1^{(1)} \right)}.$$

By (10),

$$C_1^{(1)} + C_1^{(1)} = -b\left(\sigma^2 - r\right), \quad C_1^{(1)} C_2^{(1)} = -ba_2.$$

Hence,

$$-\frac{B_1^{(1)} + B_2^{(1)}}{C_1^{(1)} + C_2^{(1)}} = -\frac{a_1 \left(C_1^{(1)} - C_2^{(1)} \right)}{\left(b^2 \sigma^4 + ba_2\right)\left(\sigma^2 - r\right)}.$$

Recall that

$$\Gamma^{(1)} = \frac{a_1 \left(C_1^{(1)} - C_2^{(1)} \right)}{b\left(b\sigma^4 + a_2\right)}.$$

It follows that

$$-\frac{B_1^{(1)} + B_2^{(1)}}{C_1^{(1)} + C_2^{(1)}} = \frac{\Gamma}{\sigma^2 - r} = \tilde{z}.$$

To see that $x_1(z) > x_2(z)$ for $z < \tilde{z}$ and $x_1(z) < x_2(z)$ for $z > \tilde{z}$ if $j = 1$, we observe from Proposition 2 that $x_1(z)$ and $x_2(z)$ are linear functions of z, with the slopes

$$\begin{aligned} \frac{C_1^{(1)}}{b} &= \frac{-1}{2b}\left[b\left(\sigma^2 - r\right) + \sqrt{b^2\left(\sigma^2 - r\right)^2 + 4a_2 b}\right] \qquad \text{and} \\ -\frac{C_2^{(1)}}{b} &= \frac{1}{2b}\left[b\left(\sigma^2 - r\right) - \sqrt{b^2\left(\sigma^2 - r\right)^2 + 4a_2 b}\right], \end{aligned}$$

respectively. It is easy to see that the former is less than the latter. Since $x_1(\tilde{z}) = x_2(\tilde{z})$, the assertion with $j = 1$ follows. The proof for the case where $j = 2$ is similar.

It remains to show that $V_1(z) > V_2(z)$ for $0 \le z < \tilde{z}$ if $j = 1$, and the reversed inequality holds if $j = 2$. Note that

$$V_1(z) - V_2(z) = A_1^{(1)} - A_2^{(1)} + \left(B_1^{(1)} - B_2^{(1)}\right) z + \frac{C_1^{(1)} - C_2^{(1)}}{2} z^2$$

is a quadratic polynomial. The coefficients are

$$\begin{aligned} C_1^{(1)} - C_2^{(1)} &= -\sqrt{b^2\left(\sigma^2 - r\right)^2 + 4a_2 b} < 0, \\ B_1^{(1)} - B_2^{(1)} &= -\frac{a_1 b}{b^2\sigma^4 + ba_2}\left[C_1 - C_2 - 2b\sigma^2\right] > 0, \\ A_1^{(1)} - A_2^{(1)} &= \frac{1}{2rb}\left(\left(B_1^{(1)}\right)^2 - \left(B_2^{(1)}\right)^2\right) = \frac{1}{2rb}\left(B_1^{(1)} - B_2^{(1)}\right)\left(B_1^{(1)} + B_2^{(1)}\right). \end{aligned}$$

Since

$$B_1^{(1)} + B_2^{(1)} = -\frac{a_1 b}{b^2\sigma^4 + ba_2}\left[C_1^{(1)} + C_2^{(1)}\right] = \frac{a_1 b^2\left(\sigma^2 - r\right)}{b^2\sigma^4 + ba_2} > 0,$$

it follows that $A_1^{(1)} - A_2^{(1)} > 0$. Therefore $V_1(0) - V_2(0) = A_1^{(1)} - A_2^{(1)} > 0$ and

$$V_1'(z) - V_2'(z) = B_1^{(1)} - B_2^{(1)} + \left(C_1^{(1)} - C_2^{(1)}\right) z \ge 0$$

for

$$0 \le z \le -\frac{B_1^{(1)} - B_2^{(1)}}{C_1^{(1)} - C_2^{(1)}}.$$

For such values of z, we have

$$V_1(z) - V_2(z) > V_1(0) - V_2(0) > 0.$$

Note that $B_1^{(1)} - B_2^{(1)} > 0$, $C_2^{(1)} - C_1^{(1)} > 0$ and $C_2^{(1)} < 0 < -C_2^{(1)}$. It follows that

$$-\frac{B_1^{(1)} - B_2^{(1)}}{C_1^{(1)} - C_2^{(1)}} = \frac{B_1^{(1)} - B_2^{(1)}}{-C_1^{(1)} + C_2^{(1)}} > \frac{B_1^{(1)} - B_2^{(1)}}{-C_1^{(1)} - C_2^{(1)}}.$$

Also, since $B_2^{(1)} < 0$ and $-C_1^{(1)} - C_2^{(1)} > 0$, we find

$$\frac{B_1^{(1)} - B_2^{(1)}}{-C_1^{(1)} - C_2^{(1)}} > \frac{B_1^{(1)} + B_2^{(1)}}{-C_1^{(1)} - C_2^{(1)}} = \tilde{z}.$$

Hence, since $\tilde{z} > 0$ it follows that $V_1(z) > V_2(z)$ for any z that satisfies $0 \leq z \leq \tilde{z}$.

The proof for $j = 2$ is similar. This completes the proof of Proposition 4.

A.5 Proof of Proposition 5

We first prove the statement regarding $x^a(\tilde{z})$. Simplifying the expressions of $x^s(0)$ and $x^a(\tilde{z})$ by introducing

$$s = b\left(\sigma^2 - r\right), \qquad \alpha = \sqrt{-a_2 b}, \qquad \beta = br,$$

the relation $\sigma^2 > r + 2\sqrt{-a_2/b}$ becomes $s > 2\alpha$. With these notations we have

$$x^s(0) = \frac{12a_1}{6\beta + s + \sqrt{s^2 + 12\alpha^2}}, \qquad x^a(\tilde{z}) = \frac{12a_1\left[s(s+\beta) + 2\alpha^2\right]}{6s\left[(s+\beta)^2 - \alpha^2\right]}.$$

To compare these two quantities, it suffice to compare

$$f(s) = 6\beta + s + \sqrt{s^2 + 12\alpha^2}, \qquad g(s) = \frac{6s\left[(s+\beta)^2 - \alpha^2\right]}{s(s+\beta) + 2\alpha^2}.$$

It is easy to verify that

$$f(2\alpha) = 6(\beta + \alpha) = g(2\alpha). \tag{42}$$

By differentiation,

$$f'(s) = 1 + \frac{s}{\sqrt{s^2 + 12\alpha^2}} < 2,$$

and

$$g'(s) = 6\left[1 + \frac{\alpha^2\left(3s^2 + 4s\beta + 2\beta^2 - 6\alpha^2\right)}{\left(s^2 + \beta s + 2\alpha^2\right)^2}\right].$$

Note that

$$3s^2 + 4s\beta + 2\beta^2 - 6\alpha^2 \geq 6\alpha^2 + 8\alpha\beta + 2\beta^2 > 0 \qquad \text{for } s > 2\alpha.$$

Therefore

$$g'(s) \geq 6 > f'(s) \qquad \text{for } s > 2\alpha.$$

In view of (42)

$$f(s) < g(s) \qquad \text{for } s > 2\alpha.$$

As a result

$$x^s(0) = \frac{12a_1}{f(s)} > \frac{12a_2}{g(s)} = x^a(\tilde{z}) \qquad \text{for } \sigma^2 > r + 2\sqrt{-a_2/b}.$$

This proves the statement regarding $x^a(\tilde{z})$.

We next prove the statement regarding $x^a(\hat{z})$. Substituting C defined in Proposition 1 into $x^s(0)$ and rearranging terms in $x^s(0)$ and $x^a(\hat{z})$, we have

$$x^s(0) = \frac{a_1}{\frac{b(\sigma^2+5r)+\sqrt{b^2(\sigma^2-r)^2-12ba_2}}{12}}$$

and

$$x^a(\hat{z}) = \frac{a_1}{\frac{b(2\sigma^2-r)(b\sigma^4+a_2)}{b\sigma^2(5\sigma^2-3r)-4a_2}}.$$

Thus, we only need to compare the two denominators. Denote

$$h(\sigma^2) = \frac{b(\sigma^2+5r)+\sqrt{b^2(\sigma^2-r)^2-12ba_2}}{12}$$

and

$$g(\sigma^2) = \frac{b(2\sigma^2-r)(b\sigma^4+a_2)}{b\sigma^2(5\sigma^2-3r)-4a_2}.$$

Obviously, $x^s(0) > x^a(\hat{z})$ if and only if $h(\sigma^2) < g(\sigma^2)$. Thus, in the following we only need to clarify the relationship between $h(\sigma^2)$ and $g(\sigma^2)$. Direct calculation with simplification yields

$$h(\sigma^2) - g(\sigma^2) = \frac{1}{\Phi}\left[-19b^2\sigma^6 + 5b\sigma^4\sqrt{\Delta} + 33b^2r\sigma^4 - 3br\sigma^2\sqrt{\Delta} - (15b^2r^2 + 12ba_2)\sigma^2 + 12a_2br\right] \tag{43}$$

with $\Phi = 12[b\sigma^2(5\sigma^2-3r) - 4a_2]$ and $\Delta = b^2(\sigma^2-r)^2 - 12ba_2$.

Given $\Phi > 0$, we only need to check conditions such that the numerator is positive or negative. To do so, noticing from Proposition 2 that the asymmetric equilibrium exists if

$$\sigma^2 > r + 2\sqrt{-\frac{a_2}{b}},$$

which is equivalent to

$$-4a_2 < b(\sigma^2-r)^2.$$

Thus,

$$b^2(\sigma^2-r)^2 - 12a_2b < b^2(\sigma^2-r)^2 + 3b^2(\sigma^2-r)^2 = 4b^2(\sigma^2-r)^2.$$

On the other hand, obviously, given $a_2 < 0$

$$b^2(\sigma^2-r)^2 - 12a_2b > b^2(\sigma^2-r)^2.$$

Therefore, we have

$$b(\sigma^2 - r) < \sqrt{\Delta} < 2b(\sigma^2 - r). \qquad (44)$$

We consider both the upper and lower bounds, and the general case will have similar properties:

Upper-bound

Substituting the above inequality (44) into (43), it follows

$$h(\sigma^2) - g(\sigma^2) \leq \frac{1}{\Phi}\left[-9b^2\sigma^6 + 20b^2r\sigma^4 - (12b^2r^2 + 12ba_2)\sigma^2 + 12bra_2\right]. \qquad (45)$$

If we denote

$$Y = \sigma^2(> r > 0)$$

and

$$F(Y) = -9b^2Y^3 + 20b^2rY^2 - (12b^2r^2 + 12ba_2)Y + 12bra_2,$$

then $h(\sigma^2) \gtreqless g(\sigma^2)$ is equivalent to $F(Y) \gtreqless 0$. Given $F(Y)$ is a 3rd degree polynomial, we can easily check its maximum values, roots and the fact that $\lim_{Y\to+\infty} F(Y) = -\infty < 0$.

The first order condition

$$F'(Y) = -27b^2Y^2 + 40b^2rY - (12b^2r^2 + 12ba_2) = 0,$$

yields the two real positive roots:

$$Y_{1,2} = \frac{20br \pm 2\sqrt{19b^2r^2 - 81ba_2}}{27b} > 0$$

with $Y_1 < Y_2$.

Thus, for $\sigma^2 \in (Y_1, Y_2)$, we have $F(Y) > 0$ and for $Y \in (0, Y_1)\bigcup(Y_2, +\infty)$, we have $F(Y) < 0$.

Lower-bound case

Similarly, substituting the lower bound of inequality (44) into (43), we have

$$h(\sigma^2) - g(\sigma^2) \geq \frac{1}{\Phi}\left[-14b^2\sigma^6 + 25b^2r\sigma^4 - (12b^2r^2 + 12ba_2)\sigma^2 + 12bra_2\right]. \qquad (46)$$

Denote again

$$Y = \sigma^2(> r > 0)$$

and

$$G(Y) = -14b^2Y^3 + 25b^2rY^2 - (12b^2r^2 + 12ba_2)Y + 12bra_2.$$

First order condition

$$G'(Y) = -42b^2Y^2 + 50b^{"}rY - (12b^2r^2 + 12ba_2) = 0$$

yields two positive real roots:

$$Y_{3,4} = \frac{25r \pm 2\sqrt{121r^2 - 126a_2/b}}{42}(> 0),$$

with $Y_3 < Y_4$

Thus, for $Y \in (Y_3, Y_4)$, we have $G(Y) > 0$ and for any $Y \in (0, Y_3) \bigcup (Y_4, +\infty)$, we have $G(Y) < 0$ with $\lim_{Y \to +\infty} G(Y) = -\infty < 0$.

General case

In the rest we show that $\underline{\sigma}^2 = r + 2\sqrt{-\frac{a_2}{b}}$ checks

$$\min\{Y_1, Y_3\} < \underline{\sigma}^2 < \max\{Y_2, Y_4\}.$$

Then for any $\sigma^2 > \underline{\sigma}^2$, $h(\sigma^2) - g(\sigma^2)$ is located between the upper-bound $F(\sigma^2)$ and lower-bound $G(\sigma^2)$ and therefore the above properties reserved.

It is straightforward to show that

$$Y_2 - \underline{\sigma}^2 = -\frac{7r}{27} + \frac{2\sqrt{19r^2 - \frac{81a_2}{b}}}{27} - 2\sqrt{-\frac{a_2}{b}} > 0$$

if and only if

$$\frac{2\sqrt{19r^2 - \frac{81a_2}{b}}}{27} > \frac{7r}{27} + 2\sqrt{-\frac{a_2}{b}}.$$

Taking squares on both sides and rearranging terms, we have that

$$Y_2 - \underline{\sigma}^2 > 0$$

if and only if

$$\frac{r^2}{27}\left(-\frac{a_2}{b}\right)\left[40 - \frac{28^2}{27}\right] > 0,$$

which is true for any $a_2 < 0$.

Thus, we always have

$$Y_2 > \underline{\sigma}^2$$

and similarly, we can show that

$$Y_4 > \underline{\sigma}^2.$$

On the other hand, it is easy to see

$$Y_3 - \underline{\sigma}^2 = -\frac{17r}{42} - \frac{1}{21}\sqrt{121r^2 - 126a_2/b} - 2\sqrt{-\frac{a_2}{b}} < 0$$

and similarly

$$Y_1 < \underline{\sigma}^2.$$

That completes the proof.

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