
Original Article

Optimal mix of funded and unfunded pension systems: The case of Luxembourg

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ABSTRACT Financing of the Luxembourg pension system is based on a pay-as-you-go (PAYG) system and hence on an intergenerational contract. As is the case for most other European countries, this system will be exposed to the effects of demographic ageing over the coming decades. The aim of this article is to develop a model that allows evaluating the efficiency of a diversified pension system financed partly by a PAYG scheme and partly by capitalization. The efficiency is measured by the long-term sustainability of the system. We compare the sustainability of our model with the one of a pure PAYG system.

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INTRODUCTION

Financing of the Luxembourg pension system is based on a pay-as-you-go (PAYG) system and hence on an intergenerational contract. As is the case for most other European countries, this system will be exposed to the effects of demographic ageing over the coming decades.

The aim of this article is to develop a model that allows evaluating the efficiency of a diversified pension system financed partly by a PAYG scheme and partly by capitalization. The efficiency is measured by the long-term sustainability of the system. We compare the sustainability of our model with the one of a pure PAYG system.

We will proceed in three steps:

In a first step, we highlight the evolution of salaries in Luxembourg. To this end, we use the recent statistical group-based trajectory model

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of D. Nagin.¹ We estimate model parameters from a single database, provided by the general social security inspection office (IGSS), containing annual salaries of all wage earners in the Luxembourg private sector. As a result, we divide the population into nine groups, each with its own mean salary trajectory in time and its relative weight in society.

In a second step, we evaluate the pension system by means of a new criterion. This is the sustainability coefficient, which we define as the average amount (in euros) that the labour force has to earn to fund one euro of pension payments, based on current legislation. Our estimations show the high sensitivity of the coefficient to demographic variables and highlight the risks threatening the Luxembourg PAYG system.

To this aim, we develop a theoretical model consistent with our statistical analysis. This model allows for the determination of both the initial level and the evolution of the pensions for each of the groups identified in the first step of our work. Knowing the weight of each of the groups within the Luxembourg population, we are then able to evaluate the sustainability coefficient of the system. This corresponds to comparing the sum of all incomes of the labour force with that of all the pensions paid to pensioners at a given date while taking into account the growth rate of the population in time.

Finally, we interpret our results in the light of other sustainability criteria, such as the rate of contributions, which ensures long-term stability of the pension level under the current legislation or the pension level that current contributions are able to finance in the long run.

In the third and final step, we analyse the impact of lump sum payments into the pension regime. Specifically, we consider a retirement savings plan financed through a constant annuity and invested at a random interest rate. We then examine the problem of identifying optimal annuities for the nine groups previously identified, and deduce the corresponding ratios between annuity and lump sum in each case. As before, our evaluation criterion is based on the amount of euros that have to be earned in

order to guarantee financing of a single euro of retirement payments, as well as the variance of that amount. However, this time, we take into account not only the collective financing effort (through the PAYG regime), but also individual efforts (through savings towards lump sum payments) involving the savings plan. The new goal is then to manage the variation of the total effort under this new regime while making sure the sustainability coefficient is greater than it was within the previous system. Various results in terms of the ratio between PAYG and lump sum payments on the one hand and annuities on the other are obtained as a function of different levels of the sustainability coefficient chosen *a priori*.

THE PENSION SYSTEM IN LUXEMBOURG

The statutory pension system in Luxembourg is composed of a general scheme for private sector employees and the non-salaried and of a special scheme for civil servants. The system includes old-age, invalidity and widowers' pensions. In what follows, only the general scheme will be considered and special interest will be paid to old-age pensions.

Funding for the general scheme is based on a PAYG system, which gets reviewed every 7 years with the mandatory accumulation of a minimum reserve equal to 1.5 times the annual amount of benefits. The contribution rate is fixed at the beginning of each period of coverage to ensure funding of the scheme throughout the period. The rate is 24 per cent of taxable payroll, split evenly between workers, employers and the state. A new period of coverage was initiated in 2006. The compensation reserve amounts to EUR 8046.4 million as of 31 December 2007 and represents 3.42 times the amount of annual benefits. In what follows, we will ignore this reserve as it is a pure compensation reserve and must not be understood as a capital coverage guarantee of future benefits.

The net replacement rate after a full career in this regime is one of the highest in Europe: it reaches 96 per cent of average pre-retirement income for a worker on average earnings who has contributed for 40 years.²

ANALYSIS OF THE SALARY TRAJECTORIES

In order to evaluate the Luxembourg pension system, we need to determine the average trajectories of salaries in this country. To this end, we apply a recent method based on trajectories of development. We use an SAS procedure called Proc Traj to estimate model parameters from a database containing the annual salaries of all employees of the Luxembourg private sector. We thus obtain a regrouping of salary trajectories around common groupings.

A statistical method based on clustering

Longitudinal data form the empirical basis of research on various subjects in the social sciences and in medicine. The common statistical aim of these various application fields is the modelling of the evolution of an age- or time-based phenomenon.³ Models and techniques such as the generalized mixed model assuming a normal distribution of unobserved heterogeneity,⁴ latent growth curve modelling^{5,6} and the nonparametric mixture model, based on a discrete distribution of heterogeneity,⁷ emerged in the 1990s. We have retained the present variation of the generalized mixed model because of the growing interest in this approach in the context of questions about atypical subpopulations.⁸

The nonparametric mixed model is also called a semiparametric mixture model⁹ or latent class analysis for growth curves.¹⁰ It is designed specifically to detect the presence of distinct subgroups among a set of trajectories and represents an interesting compromise between analyses centred on a single mean trajectory and case studies.¹¹

Similar to many time-varying economic data, incomes and pensions can be collected in longitudinal research designs if they are measured at constant time intervals. They can be part of hierarchical models where individuals belong to sub-groups themselves included in one or several groups. There are hence two aspects to pensions. On the one hand, the pension of an individual depends on their income trajectory (intra-group variability), but on the other it also depends on

their profession, which links them to a part of society (inter-group variability).

Compared with subjective classification methods, the nonparametric mixed model has the advantage of providing a formal framework to test for the existence of distinct groups of trajectories. This method does not assume *a priori* that there is necessarily more than one group in the population. Rather, an adjustment index is used to determine the optimal number of subgroups. This is a significant advance over other categorical methods that determine the number of groups only subjectively.¹¹ Moreover, this method allows to evaluate the accuracy of the assignment of individuals to different sub-groups and to consider the variation of this accuracy in subsequent analyses.¹²

The SAS procedure Proc Traj, programmed by Daniel Nagin and Bobby Jones, is able to estimate the parameters of a semiparametric mixture model for longitudinal data that follow a normal (censored), a Poisson or a Bernoulli distribution. The subgroup trajectories can be modelled by polynomials of degree up to four. The procedure can calculate the posterior probability of group membership in terms of risk factors that are stable in time. Moreover, time-dependent covariates can influence the trajectories and cause different effects in different subgroups.

Nagin's nonparametric mixed model starts from a set of individual trajectories and tries to divide the population into a number of homogeneous sub-populations and to estimate a mean trajectory for each of these sub-populations.

Consider a statistical variable Y defined on a population of size N . Let $Y_i = \gamma_{i1}, \gamma_{i2}, \dots, \gamma_{iT}$ denote a longitudinal sequence of measurements on individual i over T periods.

Let $P(Y_i)$ denote the probability of Y_i . If Y_i is a counter variable, $P(Y_i)$ is specified as the Poisson distribution, for censored data it is specified as the censored normal distribution, and for binary data it is specified as the binary logit distribution. In our application, we assume that it follows a censored normal distribution.

The purpose of the analysis is to find r trajectories of a given type, in general polynomials of degree 4, $P(t) = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3 + \beta_4 t^4$.

Let $P^j(Y_i)$ denote the probability of obtaining the observed data for individual i given membership in group j and π_j the probability of an individual chosen at random to belong to the group number j . Hence, π_j is the size of group j .

We try to estimate a set of parameters $\Omega = \{\beta_0^j, \beta_1^j, \beta_2^j, \beta_3^j, \beta_4^j, \pi_j; j = 1, \dots, r\}$, which maximizes the probability of Y_i . The optimal number of groups r is also an outcome of the analysis. For a given group, conditional independence is assumed for the sequential realizations of the elements of Y_i , y_{it} , over the T periods of measurement. The likelihood L of the sample is then given by

$$L = \frac{1}{\sigma} \prod_{i=1}^N \sum_{j=1}^r \pi_j \prod_{t=1}^T \phi \left(\frac{y_{it} - \beta^j x_{it}}{\sigma} \right)$$

where ϕ denotes the density function of the standard normal distribution. These equations are too complicated to hope to obtain an algebraic solution. Bobby L. Jones (Carnegie Mellon University) has programmed an SAS procedure based on a quasi-Newtonian maximum search procedure.¹³ The estimated standard deviations are obtained by inverting the observed information matrix.

One problem is that numerical estimates of probabilities of group membership π_j must be contained between 0 and 1. This is a difficult constraint when using iterative approximation procedures. Hence, the problem is circumvented by setting

$$\pi_j = \frac{e^{\theta_j}}{\sum_{j=1}^r e^{\theta_j}}$$

where the θ_j are real parameters. As the sum of all the π_j is equal to 1, it suffices to estimate $r-1$ parameters θ_j . By convention, we set $\theta_1 = 0$.

The final form of the likelihood is hence equal to

$$L = \frac{1}{\sigma} \prod_{i=1}^N \sum_{j=1}^r \frac{e^{\theta_j}}{\sum_{j=1}^r e^{\theta_j}} \prod_{t=1}^T \phi \left(\frac{y_{it} - \beta^j x_{it}}{\sigma} \right)$$

A hard part of the problem is selecting the right model, in particular the optimal number of

trajectory groups. One possible strategy is to use the likelihood ratio test. Unfortunately, the null hypothesis of this test is on the boundary of the parameter space, which invalidates the asymptotic results.¹⁴ Because of this, the mathematical criterion generally used is the Bayesian Information Criterion BIC defined by

$$BIC = \log(L) - 0,5k \log(N)$$

where k denotes the number of parameters in the model. The difference of the BIC between two models can then be used as an approximation of the logarithm of the Bayes factor.¹⁵ Keribin¹⁶ shows that, under certain conditions, this approximation is valid to test for the number of components in a mixture model. Finally, we choose the model with the highest BIC, meaning the one with the BIC closest to zero.

Nagin's model also allows determining which group a given individual belongs to. The posterior probability $P(j/Y_i)$ for an individual i to belong to group number j is indeed given by the Bayes theorem:

$$P(j/Y_i) = \frac{P(Y_i/j) \hat{\pi}_j}{\sum_{j=1}^r P(Y_i/j) \hat{\pi}_j}$$

A large posterior probability estimate for a small group requires that Y_i be so strongly consistent with the small group that $P(Y_i/j)$ for that group is very large in comparison to its companion probabilities for the big groups (Nagin, 2005).

The IGSS database

The analysis is based on a file containing the salaries of all employees of the Luxembourg private sector. The data cover the period from 1940 to 2006. As the file documents the careers of people who joined the workforce from the beginning of the 1940s onwards, it is not complete during the first years, but becomes so gradually. In particular, it includes all the employees of the Luxembourg private sector from the beginning of the 1980s until 2006.

This file originates from the IGSS. The main variables are the net annual taxable salary,

measured in constant (2006 equivalent) euros, gender, age at first employment, residence and nationality (Luxembourg national living in Luxembourg, foreigner living in Luxembourg or people working in Luxembourg but ordinarily resident in a neighbouring country) and the type of employment contract (blue- or white-collar worker).

Initially, the file consisted of about 7 000 000 lines showing the salaries of some 718 054 workers. Each line gave the annualized salary of a worker. We converted the initial file into a file with 718 054 lines with the wages from 1940 to 2006 as variables. Most lines contained many zeros as no individuals worked continuously from 1940 to 2006. In addition, many careers are incomplete for many reasons. Moreover, for immigrant workers, the data only document the part of their careers spent in Luxembourg and there is no information about work done in their country of origin. Finally, the percentage of employees who quit prematurely, whether on a disability pension or through pre-retirement or even for family reasons (for example, women stopping work or interrupting their work to look after their children) is around 50 per cent.

Domestic employment (which includes foreign residents working in Luxembourg) has experienced strong growth since the mid-1980s, with an average increase of 3.5 per cent annually and an increase of more than 110 000 jobs between 1986 and 2001 (compared with 20 000 jobs in the period 1975-1985) (*Source*: STATEC). The development of the financial marketplace and the growing needs of the public sector have been key drivers of this evolution. Today, the services sector represents more than three quarters of total employment. These changes are not without consequences in terms of professional status, such that typical career progressions up to the 1980s are necessarily significantly different from those of the last 25 years.

We have therefore decided to focus on careers of individuals who began working in Luxembourg between 1982 and 1986 and who have worked for at least 20 years. Using macros programmed in Mathematica, we selected persons who meet

these criteria. In our file were 487 052 persons who started to work in Luxembourg from 1982 onwards. The final database used for our analysis includes data covering 22 203 private sector employees and workers: this is the full population of individuals who started work after 1982 and who have worked in the private sector for at least 20 years. We note that, in Luxembourg, the maximum contribution ceiling on pension insurance is five times the minimum wage, or € 7577 (2006 equivalent euros) per month. Wages in our data are thus also capped at that number.

The mean salary trajectories in Luxembourg

We used the SAS procedure Proc Traj, programmed by Daniel Nagin and Bobby Jones, to determine the mean salary trajectories of 22 203 people who began working in the Luxembourg private sector between 1982 and 1987 and who kept working for at least 20 years.

We established the trajectories for models with between 4 and 20 groups. As the salary trajectories form more or less a continuum in the continuous functions with domain [1000, 4000] and range [1200, 7577], the BIC adjustment criterion for determining the optimal number of groups is not well suited. Indeed, BIC increases with the number of groups. This is quite normal, as it just shows that if one assumes more groups one can necessarily represent reality with more details. On the other hand, a larger number of groups implies the resulting groups are smaller and the explanatory model more complicated to use. After discussion with the IGSS, we decided to retain a nine-group solution, as it gives a good representation of career development in Luxembourg. Solutions with more groups add essentially parallel paths to those present in our model.

To test the stability of trajectories in time, we also established trajectories for the first 15 years of different working lives (careers starting between 1985 and 1992) and for full careers of 40 years (careers starting between 1960 and 1967). The trajectories of the first 15 years are very close to the first 15 years of trajectories for 20-year careers. The sizes of the groups vary

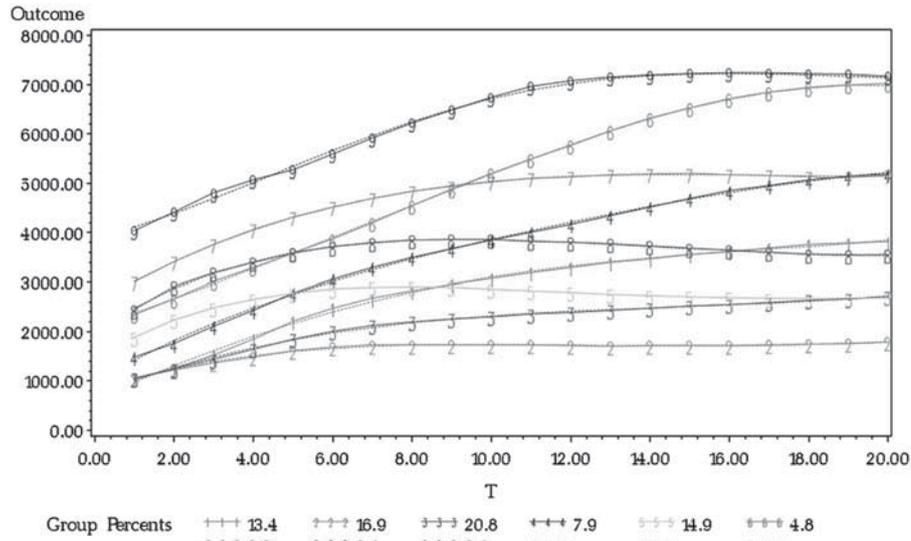


Figure 1: Average salary curves across the nine groups.

between 2 and 6 per cent compared with the ones we found and the changes are due to gains or losses to groups with similar salaries. As moreover the macroeconomic situation has not changed dramatically during the last 20 years, we are fairly confident that the trajectories remain valid, except in cases of severe economic shocks that could certainly change the situation completely. The only thing that might change over the coming years is the percentage of foreign resident workers in the different groups. As the total number of employees increases faster than the population does, this percentage will continue to grow in all groups. The trajectories for full careers are also quite similar to ours, except that they show a clear wage decline during the steel crisis for the trajectories representing high wages.

Figure 1 shows the average salary trajectories across our nine groups.

THE PENSION MODEL

In a general context where both the numbers of contributors and pensioners vary significantly, it is important to develop a model to estimate the urgency and effectiveness of reforms of the Luxembourg pension system.

This model must allow for the identification of both roles and dependencies of all system

parameters: duration and rates of contribution, life expectancy, demographic changes, replacement ratios, saving rates and so on. At the same time, it must also allow for a faithful representation of these variables and of the links between them. Finally, it must be as simple and realistic as possible.

To meet these requirements, we have calibrated wage trajectories using the Proc Traj procedure in SAS. We then formulate assumptions concerning the evolution of relevant variables of the model.

Using the salary trajectories

By highlighting nine salary paths and being able to assign to each a weight that reflects the proportion of society that it represents, we obtain a tool that is both stable in time (as it is based on 20 years of wage data) and independent of any individual approach, as the trajectories are collective. We can then get mean results, obtain a measure of dispersion or dynamically analyse situations while retaining the possibility of incorporating new parameters.

The sizes of the different groups as a percentage of the total population are shown in Table 1.

The nine curves highlight nine average rates of wage growth λ_i , $i = 1, \dots, 9$. Each λ_i is determined from the trajectory $\gamma_i = P_i(t)$. If β_i is the slope of

Table 1: Group sizes

Group 1 $p_1=13.4\%$	Group 2 $p_2=16.9\%$	Group 3 $p_3=20.8\%$	Group 4 $p_4=7.9\%$	Group 5 $p_5=14.9\%$
Group 6 $p_6=4.8\%$	Group 7 $p_7=6.6\%$	Group 8 $p_8=8.4\%$	Group 9 $p_9=6.4\%$	

Table 2: Wage growths in the nine groups

Curve 1 $\lambda_1=3.07\%$	Curve 2 $\lambda_2=0.96\%$	Curve 3 $\lambda_3=1.45\%$	Curve 4 $\lambda_4=2.82\%$	Curve 5 $\lambda_5=0.19\%$
Curve 6 $\lambda_6=2.58\%$	Curve 7 $\lambda_7=1.28\%$	Curve 8 $\lambda_8=0.48\%$	Curve 9 $\lambda_9=1.09\%$	

the least squares line through the cloud of points $(t, \ln P_i(t))$, then $\lambda_i = \exp(\beta_i) - 1$ (Table 2).

Model assumptions

Each salary curve models the evolution of a proportion p_i of the workforce. This evolution is characterized by a growth rate λ_i and leads to a pension trajectory for a proportion p'_i of retirees. Assume that $p_i = p'_i$ and that all pension curves grow by r per cent every year (in reality, the government decides on the adjustment every 2 years).

The population size (active and retired) changes each year to keep pace with the demographic intergenerational rate d (for details on this rate, see Box 1). It consists of individuals who, after working for T years, have a life expectancy of S years at retirement.

We will assume that $T = 40$, $S = 20$ and d follows a uniform law on the interval [0 per cent, 4 per cent].

At any time t , there are n_i individuals who started working i years before t . This leads to a cloud of points $(i, \ln(n_i))$, which is fitted by the regression line $\ln(n_i) = a + bi$.

Under these assumptions, the number of pensioners per 100 contributors, called the load factor, depends explicitly on d as shown in Figure 2.

If d is very close to 0, there are about two times more active workers than retirees; if d is of the order of 4 per cent, there are seven times more. Note, however, that the variations of d can be only gradual. Indeed, each estimate of d

Box 1: Evolution of the demographic intergenerational rate

Writing

$$N_0 = \exp(a)$$

$$d = \exp(-b) - 1$$

gives the best approximation of the real population (n_0, \dots, n_{T+S}) by the model population (N_0, \dots, N_{T+S}) . We define the number d by

$$N_i = \frac{N_0}{(1+d)^i}$$

d is then directly linked to the active structure of the country and seems to be strongly correlated to economic indicators such as the GDP.

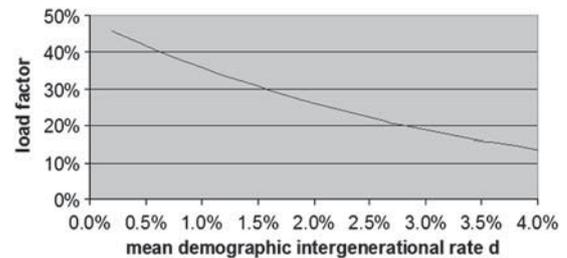


Figure 2: Load factor for old-age pension recipients.

is a snapshot of a society in which 60 years separate individuals newly entering the workforce from retirees dying of old age.

Each year, the number of people joining the workforce changes but the population structure over the remaining 59 years remains the same: the previous year's starters are now one year into their careers, the people they replace now have 2 years of work experience and so on. After 20 years, two-thirds of the workforce are still individuals present at the start unless one accepts the idea of significant variability between years (through major variations in mortality over time, foreign employees returning to their native country and so on) (Figure 3).

Studying the method of financing a pension plan is tantamount to link within a given equilibrium the inflows of contributions and the outflows of benefits. Historically, Luxembourg has followed a PAYG policy. This is not a pure PAYG policy as a reserve has been accumulated

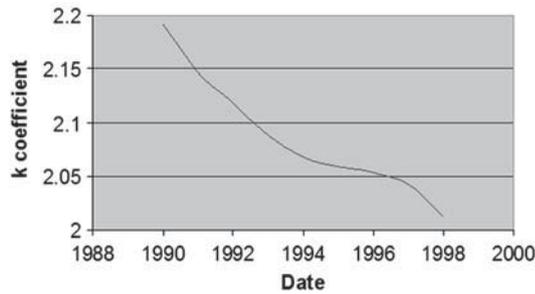


Figure 3: Variation of the k coefficient between 1990 and 1998.

in recent years. However, we do not take this into account in our reasoning, since all existing studies (by the International Labor Office in Geneva, the Luxembourg Central Bank and the General Inspectorate of Social Security) claim that the money invested will be insufficient to ensure the sustainability of the system. They predict, under conventional economic growth assumptions, that the system will have used up all reserves within the coming 30 years.

Analysis of the PAYG system

As part of our work, we have developed a new indicator of sustainability for PAYG pension regimes: the sustainability coefficient τ_1 . We have additionally evaluated τ_1 for the part of the Luxembourg system that is based on PAYG policy, taking into account a mean intergenerational population growth between 0 and 4 per cent.

Definition of the sustainability coefficient

The principle of distribution currently prevailing in the Grand Duchy is a principle of solidarity between generations. The pensions paid to retirees are directly financed by contributions levied on earnings. With this type of approach, it is essential to assess the regime's ability to meet the payments to future pensioners. To this end, cash flows generated by the contributors must at all times be compared with those paid to retirees. The sustainability coefficient is a criterion that supports this comparison.

By definition, the sustainability coefficient τ_1 of a PAYG system is the sum of all salaries earned by active workers, divided by the sum

of all pensions paid to retirees at time t . For every euro of pension, τ_1 euros are earned in salary. Moreover, τ_1 is a random variable whose values depend on the demographic structure of the country at time t .

Evaluation of τ_1 as a function of d

Under our assumptions, the workforce is changing year after year of length of service at a rate of mean intergenerational population growth d between 0 and 4 per cent. The SAS procedure Proc Traj allows us to estimate the average wage S_i of an active individual with i years of career experience and the average pension P_j of a retiree who began working j years ago. We infer the sum of all wages paid at time T :

$$N_a = N_0 S_0 + \dots + \frac{N_0 S_T}{(1+d)^T}$$

The calculation of d for the year 1998, for example, leads to $d = 3.4$ per cent.

In general, the contributors that we have followed over time using the IGSS database benefit from an old-age pension. It is therefore assumed for the area of old-age pension insurance that population growth corresponds to that of the active workforce. To take account of other pension plans, we introduce a multiplier k which can be used to calculate the sum of all pensions granted on the date T :

$$N_p = \frac{kN_0}{(1+d)^{T+1}} P_{T+1} + \dots + \frac{kN_0}{(1+d)^{T+S}} P_{T+S}$$

As said above, k is a new parameter of the problem, which we assume constant for simplicity. Its estimation using data in our possession gives the average value $k = 2$.

We can then see that τ_1 is an increasing function of d . The higher the rate of population growth, the more sustainable the pension system.

$$\tau_1 = \frac{N_p}{N_a} = \frac{\frac{k}{(1+d)^{T+1}} P_{T+1} + \dots + \frac{k}{(1+d)^{T+S}} P_{T+S}}{S_0 + \dots + \frac{S_T}{(1+d)^T}}$$

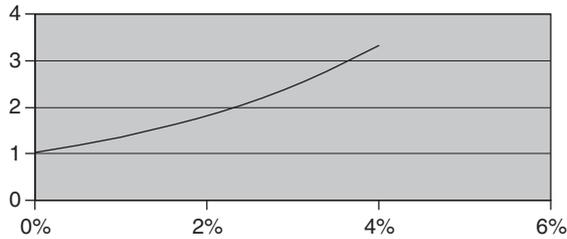


Figure 4: Sustainability coefficient as a function of the demographic rate d .

It is easy to see that, approximately, $E(\tau_1) = 2$ and $\sigma(\tau_1) = 0.8$. On average, an employee has to earn 2 € for each euro of pension paid (Figure 4).

The value of the coefficient of variation (that is, the ratio of standard deviation to mean)

$$CV(\tau_1) = \frac{0.8}{2} = 40\%$$

shows that the dispersion around this mean is important.

This is confirmed by the graph above. From a situation in which the rate of population growth is low to one where it is high, the number of euros that have to be earned by contributors to fund every dollar of retirement is multiplied by 3. This large variability in τ_1 is evidence of danger to the pension system. Whereas financing a euro pension is easy when d is large, the same task quickly becomes unmanageable when d is low. In particular, when d falls below 1 per cent, τ_1 approaches the value unity and equilibrium becomes almost impossible to maintain.

The rate of pension contributions in Luxembourg is currently set at 24 per cent of taxable earnings; this charge is divided into three equal parts of 8 per cent between the state, employees and employers.

Figure 5, drawn using the value $k=2$, shows that, for a fixed horizon to 2040 and an average intergenerational growth rate of 3 per cent (which is our baseline figure calculated in 1998), the equilibrium between wages earned and pensions requires a rate of contribution of 40 per cent! (To represent the future horizon, we extrapolate salaries earned since the 1980s

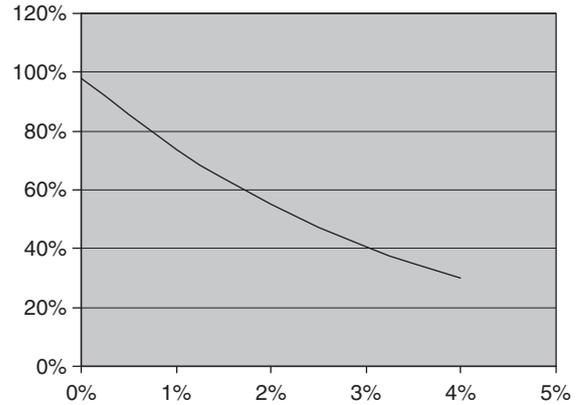


Figure 5: Rate of pension contributions as a function of d .

over $20 + 20 = 40$ years to guarantee that all individuals can be certain of being paid throughout retirement.)

Another way to see the problem is that if the contribution rate c_1 necessary for equilibrium is greater than that required by law ($C_0 = 24$ per cent), then we must reduce benefits by a factor of γ so as to obtain a ratio of pension flows to wages such that:

$$P_1 = \gamma P_0 \text{ et } \frac{P_1}{S} = c_0$$

Thus, we find:

$$\frac{\gamma P_0}{S} = c_0 \Leftrightarrow \gamma c_1 = c_0 \Leftrightarrow \gamma = \frac{c_0}{c_1}$$

With the same assumptions ($k=2$, $d=3$ per cent, $c_0=24$ per cent), the reduction γ of pensions that would recover an equilibrium of the flows is 40 per cent (Figure 6).

Thus, although the system is well managed, it contains significant risk. With an uncertain number of contributors and an influx of new rightholders, the PAYG system could at some stage become less favourable than today. Faced with this situation, what to say to people tempted to improve their pension? It may be possible to save up individually, but how can one in that case determine what proportion should be capitalized?

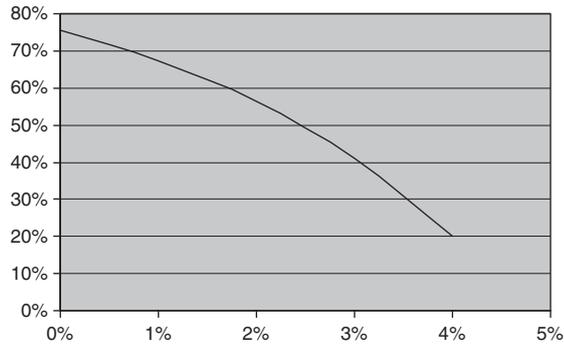


Figure 6: Fall of the replacement rate as a function of d .

A LESS RISKY APPROACH: MIXTURE OF PAY-AS-YOU-GO AND CAPITALIZATION

As for the pay-as-you-go system, we define a sustainability coefficient. This coefficient depends on the rate of return of the capital market and the annuity. Then we consider an overall sustainability coefficient, characteristic of a mixed system where x per cent of the pensions are financed by a pay-as-you-go system and $(1 - x)$ per cent by lump sum payments. This allows us to assess the gains of sustainability achieved with a mixed system compared with a pure PAYG system.

Definition of the sustainability coefficient of the funded system

In an approach combining the overall effort of sharing with a personal effort of funding, every euro of a pension is earned partly by the collective work of the labour force and partly by the individual savings of the future pensioner. Consider the time T when an individual I_j , which belongs to the j th group of salaries highlighted above, is retiring.

We have seen that if τ_1 euros earned by the active workers generate one euro of pension at that time of retirement, τ_1 is the sustainability coefficient of the PAYG system.

Symmetrically, if τ_2 euros earned by I_j generate one euro of pension, τ_2 is the sustainability coefficient of the funded system. By definition, the sustainability coefficient τ_2

of the funded system is the total sum earned by the individual during his period of activity divided by the sum of all the pensions that are paid to him thanks to the savings that he has accumulated.

Consider the case where the wages S_j of I_j in $t = 1$ is increasing by the annual rate λ_j . If he decides to save each a sum of a_j EUR invested at the market rate i , the sustainability coefficient of his funded system is given by

$$\begin{aligned} \tau_2 &= \frac{S_j(1+i)^{T-1} + S_j(1+i)^{T-2}(1+\lambda_j) + \dots + S_j(1+\lambda_j)^{T-1}}{a_j(1+i)^{T-1} + \dots + a_j} \\ &= \frac{S_j}{a_j(i-\lambda_j)} i \frac{(1+i)^T - (1+\lambda_j)^T}{(1+i)^T - 1} \end{aligned}$$

For the funded system, the amount of the savings depends of course upon the rate of return i of the investments. Figure 7 illustrates this dependence for group 1.

In all the groups, τ_2 is a decreasing function of I because if the market rate is higher less money is needed to ensure one euro of pension.

The computation of τ_2 shows that $E(\tau_2)$ and $Var(\tau_2)$ are inversely proportional to the annuity a_j and its square a_j^2 , respectively:

$$\begin{aligned} E(\tau_2) &= \frac{c}{a_j} \\ Var(\tau_2) &= \frac{K}{a_j^2} \end{aligned}$$

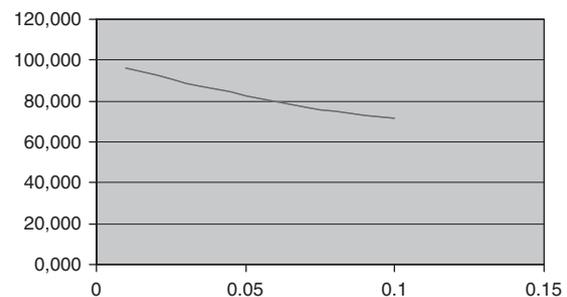


Figure 7: Sustainability coefficient of the funded system as a function of the market rate i .

Definition of the overall sustainability coefficient

Generally speaking, for every euro of pension, a proportion x may be due to the unfunded system and a proportion $(1-x)$ by the funded one. The more an individual has privately invested the smaller is x . Today, in most cases in Luxembourg $x=100$ per cent = 1, because Luxembourg has a pure PAYG system. The main advantage of this situation is a full immunization against interest rate risk, whereas its main disadvantage is its extreme dependence on the demographic risk. In contrast, a situation where we have $x=0$ per cent would present an extreme dependence on the interest rate risk and complete immunization against demographic risk.

By definition, the overall sustainability coefficient $\tau(x)$ is the number of euros needed to get a pension where x euro are derived from the PAYG system and $(1-x)$ euros by lump sum payments.

$$\tau(x) = x\tau_1 + (1-x)\tau_2, \quad x \in [0,1]$$

if we assume that τ_1 and τ_2 are independent, $\tau(x)$ is a random variable with expectation $E[\tau(x)] = xE(\tau_1) + (1-x)E(\tau_2)$ and variance $\text{Var}[\tau(x)] = x^2\text{Var}(\tau_1) + (1-x)^2\text{Var}(\tau_2)$.

The sustainability of the system

Building a mixed pension system based both on a funded and an unfunded part means diversifying the options and the risks if one adopts a portfolio management logic. Lump sum payments allow to protect itself against from demographic risk (decrease of the active workforce, increased lifespan of the retirees, and so on), whereas the PAYG system allows to protect itself against inflation and the volatility of financial markets.

Moving from a PAYG system to a mixed system forces the active workers to give up a part of their consumption as they have to pay an annuity a to a retirement saving plan. By saving a part of their income, they get a higher pension level and they have a better control of the variability of their future earnings, but they do not benefit immediately from this sum.

We hypothesize here that the effort associated with this renouncement must be as small as

possible. This means that the utility $U=U(a)$ has to be a decreasing function of a :

$$U'(a) \leq 0$$

To highlight the benefits of diversification, we fix a threshold value for the gain of sustainability

$$G(x) = \frac{\text{Var}(\tau_1) - \text{Var}[\tau(x)]}{\text{Var}(\tau_1)} \geq G^* \quad (1)$$

and seek the optimal split $x=x^*$, which maximizes the utility function $U=U(a)$.

Thus, the effort to control the risk of variability with respect to a pure PAYG situation is measured by G^* .

To fix the ideas, consider an individual I belonging to group 1 who has already achieved to save 2000 euros. Its share x related to the unfunded system in the calculation of the sustainability coefficient $T(x)$ depends of course on the evolution of the market rate during its period of activity. The higher is the market rate, the smaller the value of x needs to be. This value is not known in advance. However, it is possible to assess the variability of $\tau(x)$ according to the different possible values of x . We show that the points $(x, \text{Var}[T(x)])$ lie on a parabola with summit (0.83, 0.56) (see below). This means that the minimum variability that I can hope for is 0.56 and that it is obtained for $x=83$ per cent. If I want to reduce this variability even more and get for instance $\text{Var}[T(x)]=0.5$, Figure 8 shows that he has to save more than his current 2000 euros!

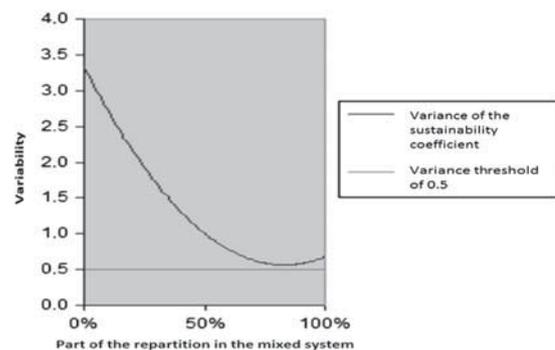


Figure 8: Variability of the sustainability coefficient as a function of x for a saving annuity of 2000€.

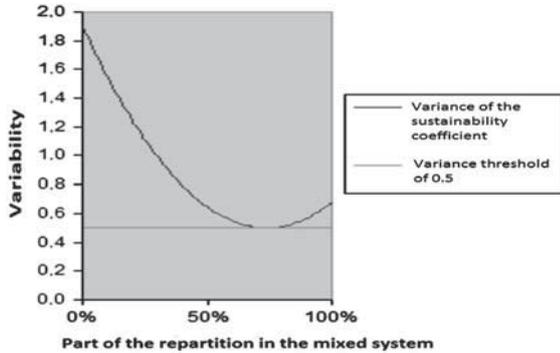


Figure 9: Variability of the sustainability coefficient as a function of x for a saving annuity of 2650€.

One can show that the minimal effort to achieve this goal is saving annuity $a=2650$ euros (Figure 9).

Let us write the formal demonstration of these results.

Assume that the individual belongs to group j . Let

$$K = \text{Var} \left[\frac{S_j}{(1-\lambda_j)} i \frac{(1+i)^T - (1+\lambda_j)^T}{(1+i)^T - 1} \right]$$

Inequality (1) implies that

$$\text{Var}[\tau(x)] \leq (1-G^*)\text{Var}(\tau_1)$$

hence

$$\begin{aligned} \text{Var}[\tau(x)] &= x^2\text{Var}(\tau_1) + (1-x)^2\text{Var}(\tau_2) \\ &= x^2\text{Var}(\tau_1) + (1-x^2)\frac{K}{a^2} \\ &\leq \text{Var}(\tau_1)(1-G^*) \end{aligned}$$

and

$$(K + a^2\text{Var}(\tau_1))x^2 - 2Kx - a^2\text{Var}(\tau_1)(1-G^*) + K \leq 0 \quad (2)$$

Inequality (2) is possible if and only if

$$\begin{aligned} \min & [(K + a^2\text{Var}(\tau_1))x^2 - a^2 \\ & - 2Kx\text{Var}(\tau_1)(1-G^*) + K] \leq 0 \end{aligned}$$

that is to say

$$-a^2\text{Var}(\tau_1) \frac{(a^2\text{Var}(\tau_1) - G^*K - G^*a^2\text{Var}(\tau_1))}{K + a^2\text{Var}(\tau_1)} \leq 0$$

as the minimum of the expression is obtained for

$$x = \frac{K}{K + a^2\text{Var}(\tau_1)}$$

Geometrically, this means that the parabola (P) of equation

$$y = (K + a^2\text{Var}(\tau_1))x^2 - a^2 - 2Kx\text{Var}(\tau_1)(1-G^*) + K$$

cuts the (Ox) axis only if its summit is lying below (Ox).

Thus,

$$a^2 \geq \frac{G^*K}{\text{Var}(\tau_1)(1-G^*)}$$

and $U(a)$ is maximum when

$$a^* = \sqrt{\frac{G^*K}{\text{Var}(\tau_1)(1-G^*)}}$$

We then have

$$\begin{aligned} \min & [(K + a^2\text{Var}(\tau_1))x^2 - 2Kx \\ & - a^2\text{Var}(\tau_1)(1-G^*) + K] = 0 \end{aligned}$$

Under these circumstances, necessarily (2) is an equality and

$$\begin{aligned} x^* &= \frac{K}{K + a^2\text{Var}(\tau_1)} \\ &= \frac{K}{K + \frac{G^*K}{\text{Var}(\tau_1)(1-G^*)}\sigma_1^2} = 1-G^* \end{aligned}$$

Conclusion: The optimal annuity

Let us summarize our results.

Assume that the utility function of an active worker $U=U(a)$ is a decreasing function of a .

The unfunded system part $x=x^*$ for which the utility U is maximal under the constraint of sustainability

$$G(x) = \frac{Var(\tau_1) - Var[\tau(x)]}{Var(\tau_1)} \geq G^*$$

is obtained for $x^* = 1 - G^*$ and requires that the individual has a retirement plan financed through a constant annuity

$$a^* = \sqrt{\frac{G^* K}{Var(\tau_1)(1 - G^*)}}$$

where

$$K = Var \left[\frac{S_j}{(1 - \lambda_j)} \right] i \frac{(1 + i)^T - (1 + \lambda_j)^T}{(1 + i)^T - 1}$$

depends on the salary trajectory.

For these values a^* and x^* , the gains in sustainability with respect to two situations of a pure unfunded, respectively, a pure funded system are

$$G^* = \frac{Var(\tau_1) - Var[\tau(x)]}{Var(\tau_1)}$$

and

$$1 - G^* = \frac{Var(\tau_2) - Var[\tau(x)]}{Var(\tau_2)}$$

A person wishing to divide by two the variability of the PAYG sustainability coefficient has to pay annuities at least as high as the ones given in Table 3:

Table 3: Minimal annuity by group allowing to divide by two the variability of the sustainability coefficient

Group	G1	G2	G3	G4	G5	G6	G7	G8	G9
Annuity	€4466	€713	€1448	€5231	€220	€6364	€2809	€743	€3140

Table 4: Sustainability index as a function of the population growth

d	Sustainability index
0	1.02
0.01	1.36
0.02	1.82
0.03	2.45
0.04	3.33
Mean	2.00
Variance	0.67
Standard deviation	0.82

Table 5: Variability of the sustainability index for the different groups

Group	G1	G2	G3	G4	G5	G6	G7	G8	G9
Variation (%)	9.7	2.9	4.4	8.9	0.6	8.1	3.9	1.4	3.3

Consider again the example of an individual belonging to group 1 who has already saved EUR 2000. His overall sustainability coefficient is already less volatile than his PAYG sustainability coefficient would have been: $Var[\tau(x)] \leq Var(\tau_1) = 0.68$. However, if he wants to further reduce the volatility and achieve a variance $Var[\tau(x)] = 0.34$ to divide by two his initial coefficient, Table 3 shows that he cannot maintain his current annuity of 2000 euros but should at least save EUR 4466 every year.

ANALYSIS OF THE RESULTS

Our results allow a simple description of the pure unfunded or funded systems. In the first case (pure PAYG), the concept of group does not matter because the effort is collective, each generation of active workers in each group agreeing to finance the pensions of their elders, whatever their affiliation. The sustainability of the system only depends on the demographic rate d . It is shown that if d varies between 0 and 4 per cent, the workers in Luxembourg have to earn on average two euros for every euro of pension paid (Table 4).

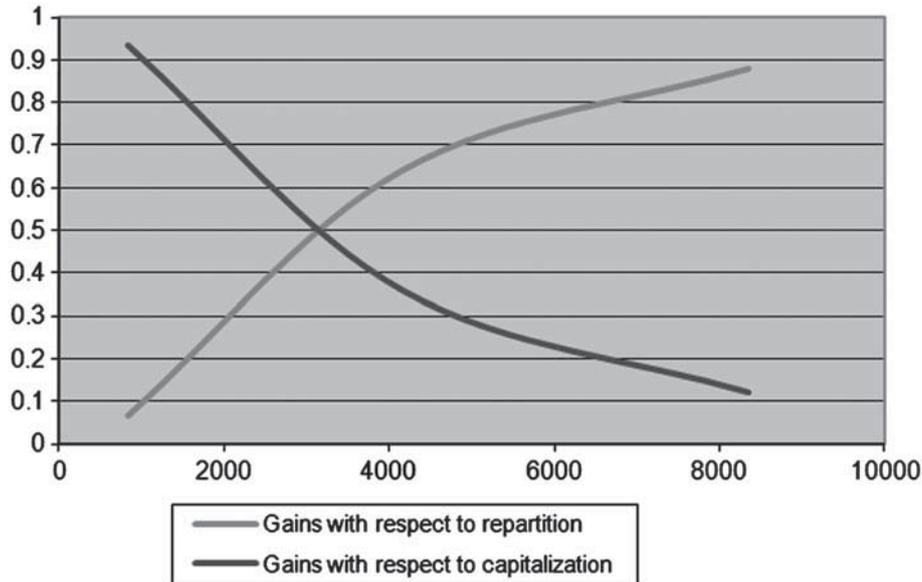
In a pure funded system, however, group membership is important. The results show that the exposure to the sustainability risk is not the same for every group. In particular, groups with a very flat salary trajectory (especially groups 5 and 8) are less exposed to changes (Table 5).

Table 6: Gain in sustainability for the different groups

Group	G1	G2	G3	G4	G5	G6	G7	G8	G9
Gain in Sustainability (%)	51.7	90.2	80.7	55.3	99.6	59.2	84.2	97.3	87.7

Table 7: Optimal repartition percentage for the different groups

Group	G1	G2	G3	G4	G5	G6	G7	G8	G9
Repartition (%)	48.3	9.8	19.3	44.7	0.4	40.8	15.8	2.7	12.3


Figure 10: Gains in sustainability as a function of the saving annuity for group 9.

In contrast, for any annuity, lump sum payment always accentuates the volatility risk for dynamic progressions. Thus, individuals with a high wage growth are less protected (groups 1, 4 and 6).

During the transition from a pure PAYG to a mixed system, one can hence expect that gains in sustainability are particularly visible for the less dynamic salary curves. However, these are not the only groups to profit from a system change. Table 6 shows that if the annuity of an individual is at least 10 per cent of its average salary, he gains at least 50 per cent in his sustainability coefficient, regardless of the group he belongs to.

An analysis of the situation highlights the role played by the number

$$K = Var \left[\frac{S_j}{(1 - \lambda_j)} i \frac{(1 + i)^T - (1 + \lambda_j)^T}{(1 + i)^T - 1} \right]$$

The higher this number, the larger the annuity must to achieve a gain of sustainability G^* of 50 per cent. More precisely, we need

$$a^* \geq \sqrt{\frac{K}{Var(\tau_1)}}$$

Under these conditions and only under these conditions, the variability of the funded system is smaller than the one of the PAYG system.

The value of G^* also determines the value of the part x^* , which goes into a funded system in the case of a mixed system. It is given in Table 7 for a gain G^* of 50 per cent.

As predicted by the theory $x^* = 1 - G^*$ and therefore the part of the PAYG system decreases gradually with the increasing of the funded part.

In general, the choice of G^* allows to compute the optimal annuity for each group. Obviously,

for every group a higher annuity implies a higher sustainability.

There is a gain both for the funded and the unfunded system but never conjointly. If one system is gaining, the other is of course losing (Figure 10 shows the situation for group 9).

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