THREE-DIMENSIONAL ARITHMETIC BILLIARDS

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ABSTRACT. Our billiard table is a rectangular parallelepiped with integer side lengths. A point-like ball moves with constant speed along linear segments, each making a $45°$ angle with the sides, bouncing off them upon contact. The ball can start at any point with integer distances from the sides. Building on the classical theory of two-dimensional arithmetic billiards inside rectangles, we explore the new features that occur in dimension three.

Arithmetic billiards show a nice interplay of arithmetic and geometry. In the classical case, the billiard table is a rectangle with integer side lengths, and we refer to [3, 7, 9] for an introduction to the topic. Additional teacher resources are for example [4, 6]. These two-dimensional arithmetic billiards seem to be completely understood, see [8, 5].

In this paper we consider the three-dimensional generalization, so that the billiard table is a rectangular parallelepiped with integer side lengths. A point-like ball bounces inside the billiard table, moving at a constant speed along segments and always making a $45°$ angle with each face. The ball starts at a point that has integer distance from the faces of the parallelepiped, and it only stops when it reaches a vertex of the parallelepiped.

We outline the paper. Firstly, we fully understand arithmetic billiards in the case where the side lengths of the billiard table are three pairwise coprime integers (see Section 2). Then we prove general results concerning the points in the ball path that lie on the faces of the billiard table (see Sections 3 and 4). We deduce that the billiard path can never go precisely three times through a point of the billiard table (see Theorem 7). The last part of the paper is a detailed study of the path points on the edges of the billiard table, for paths that start at a vertex of the parallelepiped (see Section 5).

The variety of billiard paths allows for further explorations. Students may collect experimental data and formulate conjectures on specific kind of paths, for example concerning the distribution of what we call the multiplicity of the points. During the preparation of this work, n-dimensional billiards were investigated (see [2] and the short note [1]), and some results hold very generally: for example, for a path starting at a vertex of the parallelepiped, all multiplicities are a power of 2 (an open question is then whether this holds for all paths: this is clear in dimension 2 and we prove it in dimension 3).

1. Background on arithmetic billiards

Basic notions. The billiard table has integer side lengths, which we call a, b, c. We may choose Cartesian coordinates (x, y, z) so that two opposite *corners* (namely, parallelepiped vertices) are the origin $(0, 0, 0)$ and the point (a, b, c).

We are interested in the ball's trajectory, its *path*. It is convenient to use a discrete notion of time, so that at every *step* each coordinate is either increased or decreased by 1: during each step, the ball makes a geometric length of $\sqrt{3}$ (this is the length of the diagonal of the unit cube).

To easily reference faces and edges of the billiard table, we call *x-faces* the faces $x = 0$ and $x = a$, and we call *x-edges* the edges parallel to the x-axis (and we define similar notions for y and z).

We call *corner path* a path that starts (and ends) in a corner and we call *closed path* a path that is not contained in a corner path (for which the trajectory is periodic).

We consider the path points that have integer coordinates and distinguish between *boundary points* (namely, the points on the faces) and the remaining *interior points*. We also call *edge points* the path points that are on the edges, excluding the corners.

We call *projection path* the orthogonal projection of the billiard path on a face, whose image is a two-dimensional billiard path.

In the figures, we open the parallelepiped faces around the basis $z = 0$ so that they all lie in the xy-plane, leaving aside the face $z = c$: we mark all boundary points, and the projection paths.

Corner paths. Suppose that the ball starts at a corner. Then the ball lands in a corner after $\operatorname{lcm}(a, b, c)$ steps (because e.g. we are on an x-face after any number of steps which is a multiple of a). We can determine the ending corner by looking at the parity of the numbers $\operatorname{lcm}(a, b, c)/a$, $\operatorname{lcm}(a, b, c)/b$, and $\operatorname{lcm}(a, b, c)/c$. Since at least one of the three ratios is odd, the starting and ending corners are different. Moreover, we have:

- the starting and ending corner are opposite if and only if all three ratios are odd;
- the starting and ending corners are on the same edge if and only if exactly one of the three ratios is odd, and it is a z-edge (respectively, x-edge and y-edge) if $\operatorname{lcm}(a, b, c)$ divided by c (respectively, by a and by b) is the odd ratio;
- the starting and ending corners are opposite vertices on the same face if and only if exactly one of the three ratios is even, and it is a z-face (respectively, x-face and y-face) if $\operatorname{lcm}(a, b, c)$ divided by c (respectively, by a and by b) is the even ratio.

Up to reversing the orientation of the path, there are four corner paths. We may suppose without loss of generality (up to a symmetry of the billiard table) that the corner path starts at the origin. To study corner paths we are allowed to do a rescaling and suppose that $\gcd(a, b, c) = 1$.

Closed paths. If we start from a diagonal segment of the billiard table which does not belong to a corner path, the path never reaches a corner and the trajectory is periodic. If

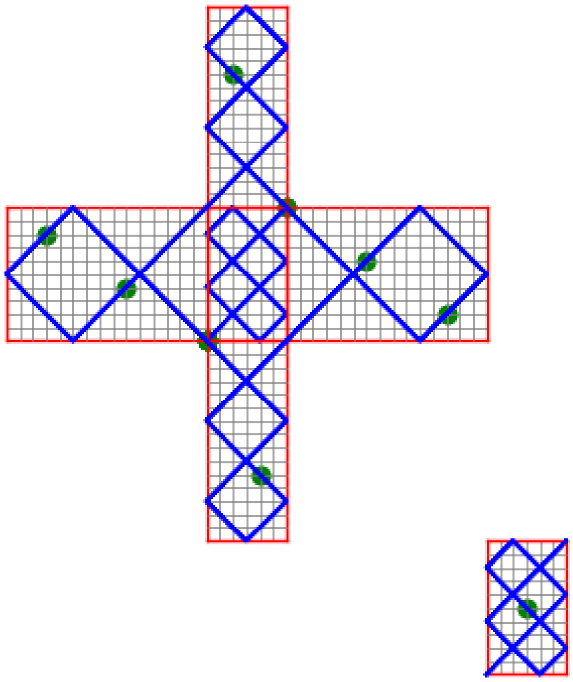

FIGURE 1. The boundary points and the projection paths for the corner path at the origin of the billiard table $(6, 10, 15)$.

the diagonal segment connecting two points of the billiard table with integer coordinates that are one step away from each other is not part of any corner path, it is part of a path that never reaches a corner and whose trajectory is periodic. The period is $2\,\mathrm{lcm}(a, b, c)$ steps long, because the period ends when the path is back at the starting point and moves in the same direction as that at the beginning. We may restrict ourselves to one period and suppose without loss of generality that the starting point is on the face $z = 0$. Notice that there are points which belong to both a corner path and a closed path (e.g. in the billiard table $(a, b, c) = (15, 9, 7)$).

An example of a closed path where it is not possible to suppose with a rescaling that $\gcd(a, b, c) = 1$ is the path starting at $(1, 0, 0)$, where the billiard table is the cube of side 2 and the trajectory is a square.

Multiplicity. The ball cannot go twice through one same segment (neither in the same direction nor in the opposite direction). We call *multiplicity* of a point (belonging to the path and with integer coordinates) the number of times that the path goes through it.

The multiplicity is clearly 1 for the starting and ending corners of a corner path, and for an edge point, as the path cannot go twice through the same segment. Similarly, the multiplicity for a boundary point can be either 1 or 2. For an interior point we will prove the multiplicity can never be 3: it can be 1 or 2 or 4.

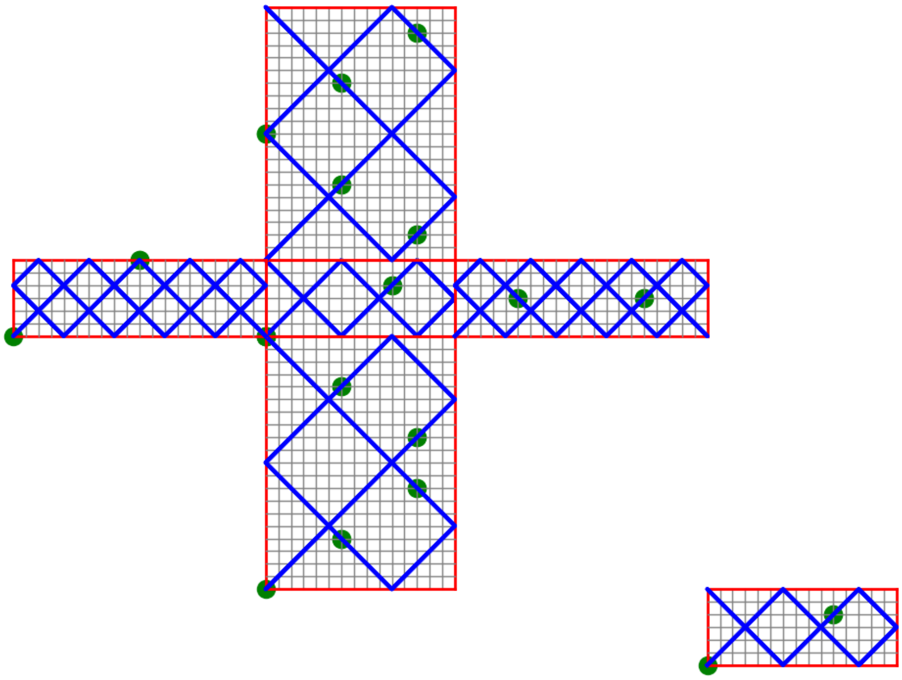

FIGURE 2. The boundary points and the projection paths for the corner path at the origin of the billiard table $(15, 6, 20)$.

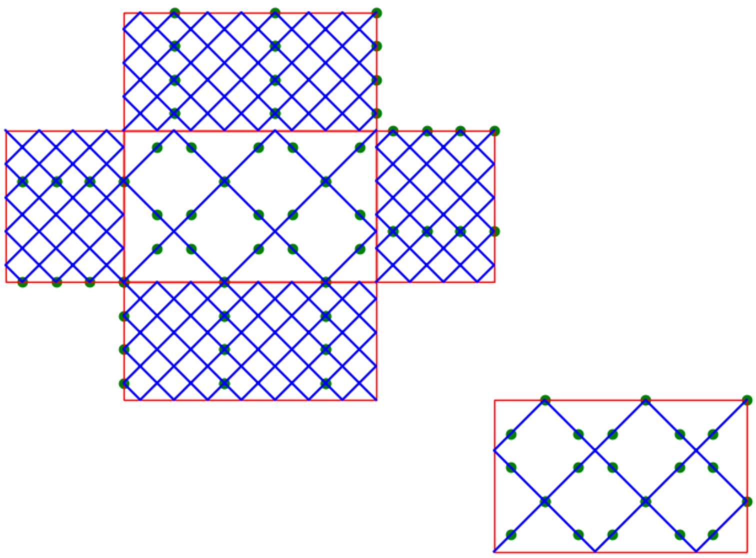

FIGURE 3. The boundary points and the projection paths for the corner path at the origin of the billiard table $(15, 9, 7)$.

2. THE CASE a, b, c PAIRWISE COPRIME

In this section we suppose that a, b, c are pairwise coprime. Without loss of generality we consider the corner path starting at the origin. The path points with integer coordinates are such that all coordinates have the same parity, since this is the case for the starting point and at every step each coordinate changes parity. With a counting argument we show the following:

Theorem 1. *Suppose that a, b, c are pairwise coprime and consider the corner path starting at the origin. The path points with integer coordinates are the points in the billiard table whose coordinates are integers with the same parity. The multiplicity is maximal: 1 for corner and edge points; 2 for the remaining boundary points; 4 for interior points. There are precisely*

$(a-1)+(b-1)+(c-1)$	*edge points*
$((a-1)(b-1)+(a-1)(c-1)+(b-1)(c-1))/2$	*boundary points not on edges*
$(a-1)(b-1)(c-1)/4$	*interior points.*

The path is the intersection of the billiard table with the grid of octahedra having edges of length $\sqrt{3}$ and making $45°$ angles with all coordinate planes, and such that the grid goes through the origin.

Proof. Suppose without loss of generality that abc is odd or that ab is odd and c is even. To count the edge points:

- if abc is odd, then all coordinates are even only on the three edges through the origin and all coordinates are odd only on the three edges through (a, b, c), and for each of the two x-edges (respectively, y-edges and z-edges) there are $(a-1)/2$ (respectively, $(b-1)/2$ and $(c-1)/2$) points with coordinates of the same parity, excluding corners;
- if ab is odd and c is even, then all coordinates are even on the five edges through the origin or $(0, 0, c)$ and all coordinates are odd only on the edge from $(a, b, 0)$ to (a, b, c). For each of the two x-edges (respectively, y-edges) there are $(a-1)/2$ (respectively, $(b-1)/2$) points whose coordinates are all even, excluding corners. For the z-edges, one contains $c/2-1$ points whose coordinates are all even (excluding corners), and the other contains $c/2$ points whose coordinates are all odd.

To count the face points not on the edges:

- if abc is odd, on the face $z=0$ there are $(a-1)(b-1)/4$ points whose coordinates are all even, and on the face $z=c$ we have the same amount of points with all coordinates odd: the analogous count can be done for the other faces;
- if ab is odd and c is even, then on each of the z-faces there are $(a-1)(b-1)/4$ points whose coordinates are all even. On the $x=0$ face (respectively, $y=0$ face) there are $(b-1)(c-2)/4$ (respectively, $(a-1)(c-2)/4$) points whose coordinates are all even, and on the $x=a$ face (respectively, $y=b$ face) there are $(b-1)c/4$ (respectively, $(a-1)c/4$) points whose coordinates are all odd.

Finally in the parallelepiped (excluding the faces) there are $(a-1)(b-1)(c-1)/4$ points whose coordinates have the same parity. Taking all these points with the maximal multiplicities we get a total of $abc-1$, which is expectedly the correct amount since the path consists of abc steps.

The last assertion is because the vertices in the grid of octahedra (that are inside the billiard table) are precisely the points having coordinates with the same parity and because the multiplicity of the path points is maximal. □

We deduce that each point of the billiard table belongs to exactly one of the four corner paths (which are disjoint because the multiplicity of their path points is maximal). Supposing without loss of generality that ab is odd, the partition given by the four corner paths is determined by the following two conditions:

- the coordinates x, z have (respectively, do not have) the same parity;
- the coordinates y, z have (respectively, do not have) the same parity.

Since there is no closed path for these values of a, b, c we have completely understood arithmetic billiards in the case where a, b, c are pairwise coprime.

3. Temporal characterization of boundary points

To describe the boundary points on a given face we may need to fix one of them: if applicable, we may choose the starting or ending corner, else we may choose the first point on the given face (there are closed formulas with a finite case distinction that hold for all a, b, c and for all starting points and starting directions).

Without loss of generality we investigate the boundary points on a z-face F. A boundary point on F can have multiplicity 2 only if its z-projection is a self-intersection point of the z-projection path. Considering that distinct boundary points on F have distinct z-projections, finding the boundary points P on F (with a prescribed direction of the path at P) is the same as finding their z-projections P' (with a prescribed slope of the z-projection path at P').

Remark 2. *Consider the corner path starting at the origin. The number of boundary points on the z-faces counted with multiplicity is $\operatorname{lcm}(a,b,c)/c+1$. The number of boundary points on F counted with multiplicity is then*

$\operatorname{lcm}(a,b,c)/2c + 1/2$	*if F contains precisely one corner of the path;*
$\operatorname{lcm}(a,b,c)/2c + 1$	*if F contains both corners of the path;*
$\operatorname{lcm}(a,b,c)/2c$	*if F contains no corners of the path.*

For a closed path, the number of boundary points on F is $2\operatorname{lcm}(a,b,c)/2c$.

It may be convenient to see a corner path as a periodic path by considering bounces also at corners. To ease notation, we define

$$L := \operatorname{lcm}(\gcd(a,c), \gcd(b,c)) .$$

Theorem 3. *The z-projection of the boundary points on F appear regularly in the z-projection path (considered as a periodic path) with intervals of $2L$ steps.*

Proof. The z-projection path has period $2\operatorname{lcm}(a,b)$. The number of steps between boundary points on F is a multiple of $2c$, so the step difference between their images in the z-projection path is a multiple of

$$\gcd(2c, 2\operatorname{lcm}(a,b)) = 2L$$

(the last equality can be shown e.g. by focusing on the powers of any fixed prime number that intervenes). Then we get at most

$$2\operatorname{lcm}(a,b)/2L = 2\operatorname{lcm}(a,b,c)/2c$$

boundary points counted with multiplicity. Hence for a closed path we have found all boundary points on F with the correct multiplicity. Consider now a corner path, without loss of generality the one starting at the origin. The z-projection of a boundary point on F appear on the projection path after a multiple of $\gcd(c, 2\operatorname{lcm}(a,b))$, and hence of L, steps. By considering time intervals that are multiples of $2L$ we find twice the same projection points with a given slope (with the exception of the corners) because $\operatorname{lcm}(a,b)$ is a multiple of L. So we find at most $\lfloor \operatorname{lcm}(a,b)/2L \rfloor + 1$ points if F is $z=0$, and $\lfloor (\operatorname{lcm}(a,b) - L)/2L \rfloor + 1$ points if F is $z=c$. We then find the correct number of boundary points with multiplicity, which is $\lfloor \operatorname{lcm}(a,b,c)/2c \rfloor + 1$ for the face $z=0$ and $\lfloor (\operatorname{lcm}(a,b,c) - c)/2c \rfloor + 1$ for the face $z=c$. □

4. Multiplicities of points

If n is a positive integer and p is a prime number, we denote by $v_p(n)$ the exponent of the largest power of p which divides n.

To ease notation we define

$$g_a := \gcd(b,c) \qquad g_b := \gcd(a,c) \qquad g_c := \gcd(a,b)\,.$$

Theorem 4. *Consider a closed path, a boundary point P on a z-face and its z-projection $P' = (x_P, y_P)$. The multiplicity of P is 2 if and only if P' is a self-intersection point of the z-projection path and* $\operatorname{lcm}(g_b, g_c) \mid x_P$ *or* $\operatorname{lcm}(g_a, g_c) \mid y_P$.

Proof. We suppose that P' is a self-intersection point of the z-projection path (else the multiplicity of P would be 1). Suppose first that the z-projection path is a closed path. Then it goes twice through P' during its period. Notice that either there is a y-coordinate increase at the next step for both passages, or there is an x-coordinate increase at both passages. We suppose the y-coordinates are the ones increasing and we claim that in this case the multiplicity of P' is 2 only if $\operatorname{lcm}(g_b, g_c) \mid x_P$ (the case in which the x-coordinates increase is analogous with the condition $\operatorname{lcm}(g_a, g_c) \mid y_P$). By Theorem 3, the boundary point P has multiplicity 2 if and only if the two times τ and τ' (which sum up to $2\operatorname{lcm}(a,b)$) between the two occurrences of P' in the z-projection path, are multiples of $2L$, say $\tau = 2Lk$ and $\tau' = 2Lk'$ for some integers k and k'. Moreover, by the assumption on the coordinates, they satisfy:

$$\begin{cases} 2Lk \equiv 0 (\bmod 2b) \\ 2Lk \equiv 2x_P (\bmod 2a) \end{cases} \qquad \begin{cases} 2Lk' \equiv 0 (\bmod 2b) \\ 2Lk' \equiv -2x_P (\bmod 2a) \end{cases} \tag{1}$$

Since $g_b \mid a$ and $g_b \mid L$, we must have $g_b \mid x_P$. Moreover, if the two congruences are compatible, then we must have $g_c \mid x_P$.

Suppose now that $\operatorname{lcm}(g_b, g_c) \mid x_P$, and we prove a solution exists for k (and the same can be done for k'). Write $\operatorname{lcm}(a, b) = AB$ such that the prime divisors of A are the primes p such that $v_p(a) > v_p(b)$ and the prime divisors of B are the primes p such that $v_p(a) \le v_p(b)$. Then the system (1) is equivalent to

$$Lk \equiv \pm x_P B(B \bmod A)^{-1} (\operatorname{mod} \operatorname{lcm}(a, b))$$

and we find a solution k because $g_b \mid x_P$ and $g_a \mid g_b B$.

Now suppose that the z-projection is a corner path, and in particular g_c divides x_P and y_P. The z-projection path (considered to be periodic with period $2 \operatorname{lcm}(a, b)$ and starting at a corner) passes though P' at four times $\tau_1 \le \tau_2 \le \tau_3 \le \tau_4$. Suppose that the path goes through P at a time that is congruent to τ_1 modulo $2 \operatorname{lcm}(a, b)$, the other cases being analogous, and apply Theorem 3. Then $\tau_4 - \tau_1$ cannot be a multiple of $2L$ (the closed path within one period cannot go twice through a same segment). Similary, at most one of $\tau_3 - \tau_1$ and $\tau_2 - \tau_1$ is a multiple of $2L$. One of these numbers is a multiple of $2L$ if and only if we have $\operatorname{lcm}(g_b, g_c) \mid x_P$ or $\operatorname{lcm}(g_a, g_c) \mid y_P$. □

Example 5. We may compute that, choosing (a, b, c) to be $(10, 15, 6)$ and starting at $(1, 0, 0)$, the boundary point $P = (5, 6, 0)$ is such that its z-projection is a self-intersection point of the z-projection path, and the multiplicity of P is 1. On the other hand, choosing (a, b, c) to be $(4, 6, 9)$ and starting at $(1, 0, 0)$, the boundary point $P = (2, 3, 9)$ is such that its z-projection is a self-intersection point of the z-projection path, and the multiplicity of P is 2.

Theorem 6. *Consider the corner path starting at the origin and a z-face F.*

1. *If P_0 is a boundary point on F, then a point P on F is a boundary point if and only if its z-projection lies on the z-projection path and we have both $2g_b \mid (x_P - x_{P_0})$ and $2g_a \mid (y_P - y_{P_0})$.*
2. *A boundary point P on F has multiplicity 2 if and only if its z-projection is a self-intersection point of the projection path. Moreover, we have $\operatorname{lcm}(g_b, g_c) \mid x_P$ and $\operatorname{lcm}(g_a, g_c) \mid y_P$.*

Proof. First we prove (1). Let P' and P'_0 be the z-projections of P and P_0. We consider the z-projection path to be periodic of period $2 \operatorname{lcm}(a, b)$.

If P is a boundary point on F, then $g_b \mid x_P$ and $g_a \mid y_P$ (because the times for the boundary points are multiples of c). We deduce that $2g_b \mid (x_P - x_{P_0})$ and $2g_a \mid (y_P - y_{P_0})$ because by Theorem 3 the number of steps in the z-projection path from P'_0 to P' is a multiple of $2L$.

Now suppose that a point P on F is such that P' is on the z-projection path and $2g_b \mid (x_P - x_{P_0})$ and $2g_a \mid (y_P - y_{P_0})$. Then the number of steps in the z-projection path between P'_0 and P' is a multiple of $2L$ (because $2g_b \mid (x_P - x_{P_0})$ and $g_b \mid a$, and analogously for the y-coordinates) and hence P is a boundary point by Theorem 3.

To prove (2), suppose that P is a boundary point such that P' is a self-intersection point of the z-projection path (which is a necessary condition for the multiplicity of P to be 2). Then we have $g_b \mid x_P$ and $g_a \mid y_P$. The z-projection path goes four times through P' in each period, at times τ_1 to τ_4 (and supposing without loss of generality that the path goes through P at a time that is congruent to τ_1 modulo $2\operatorname{lcm}(a,b)$). By Theorem 3, the multiplicity of P is two. Indeed, $\tau_3-\tau_1$ and $\tau_2-\tau_1$ are multiples of $2L$ because g_a divides y_P and b (respectively, g_b divides x_P and a) while $\tau_4-\tau_1 = 2\operatorname{lcm}(a,b)-2\tau_1$ is a multiple of $2L$ because we start at a corner hence $\gcd(c, 2\operatorname{lcm}(a,b))$ divides τ_1. □

Theorem 7. *The multiplicity of an interior point can never be* 3.

Proof. We suppose that the multiplicity of an interior point P is at least 3 and prove that it is 4. Consider the closest face F to P and suppose without loss of generality that it is the face $z=0$. Moving from P to F in the four admissible directions we find points $F_i=(x_i,y_i,0)$ for $i=1,2,3,4$. Their z-projections F_i' are on the z-projection path because the multiplicity of P implies that its z-projection is a self-intersection point of the z-projection path. By the multiplicity of P, we may suppose without loss of generality that F_1, F_2, and F_3 are boundary points and that F_1 and F_3 are opposite with respect to P: we conclude by proving that F_4 is also a boundary point.

If we have a corner path, by Theorem 6 it suffices to show that $2g_b \mid (x_4-x_2)$ and $2g_a \mid (y_4-y_2)$. This holds because F_1 and F_3 are boundary points on F and $x_4-x_2=x_3-x_1$ and $y_4-y_2=y_3-y_1$.

If we have a closed path, we apply Theorem 3. The time difference along the projection path from going to F_1' to F_3' (or conversely) is a multiple of $2L$, hence the same holds for going from F_2' to F_4'. As F_2 is a boundary point, F_4 is also a boundary point. □

5. Edge points on corner paths

Remark 8. *On a z-face there are points (belonging to the same path) on both parallel edges $x=0$ and $x=a$ if and only if $v_2(a)>v_2(c)$, provided that there are points on one of these edges. Indeed, these points correspond to integers k such that $2Lk\equiv a(\mathrm{mod} 2a)$. This means that $2Lk$ is an odd multiple of a, which is possible if and only if $v_2(2L)\leq v_2(a)$ (since $v_2(L)=\min(\max(v_2(a),v_2(b)),v_2(c))$, this is equivalent to $v_2(a)>v_2(c)$).*

Theorem 9. *Let F be a z-face and let $P_0=(x_0,y_0,z_0)$ be a boundary point on F. There is a boundary point on the edge $x=a$ of F if and only if:*

$$\begin{array}{lll} v_2(a)>v_2(c) & \text{and} & 2g_b \mid x_0, \text{ or} \\ v_2(a)\leq v_2(c) & \text{and} & g_b \mid x_0 \text{ and } v_2(x_0)=v_2(a) \end{array}$$

(To study the edge $x=0$ we can apply the above criterion replacing x_0 by $a-x_0$.)

Proof. We have a boundary point on the given edge if and only if there is some integer k such that $2Lk\equiv a-x_0(\mathrm{mod} 2a)$. This is equivalent to $\gcd(2L,2a)\mid(a-x_0)$. We may easily conclude because $\gcd(L,a)=g_b\mid a$. □

We now consider a corner path. Without loss of generality the path starts from the origin, we have $\gcd(a,b,c)=1$ and we consider a z-edge E among

$$(0,0,z),(a,0,z),(a,b,z)\,.$$

The number of edge points on E is an integer $e\geq 0$ that is clearly less than c. To ease notation, we set

$$s:=2\operatorname{lcm}(a,b) \qquad \text{and} \qquad L_c:=\frac{\operatorname{lcm}(a,b,c)}{\operatorname{lcm}(a,b)}$$

noticing that $L_c \mid c$. If we suppose that E contains a boundary point reached at time t, then the edge points on E are precisely the points at the times T such that $0<T<\operatorname{lcm}(a,b,c)$ and $T\equiv t \pmod{s}$ (as the ball returns to E every number of steps that is a multiple of both $2a$ and $2b$).

Lemma 10. *The edge $(0,0,z)$ contains edge points if and only if $c\nmid s$. The edge $(a,0,z)$ (respectively, (a,b,z)) contains edge points if and only if $v_2(a)>v_2(b)$ and $c\nmid s$ (respectively, $v_2(a)=v_2(b)$ and $c\nmid\operatorname{lcm}(a,b)$).*

Proof. The first edge point on $(0,0,z)$ occurs at time s (under the given assumption, s is less than $\operatorname{lcm}(a,b,c)$). The first edge point on $(a,0,z)$ occurs at time $\operatorname{lcm}(a,2b)$ provided that $\operatorname{lcm}(a,2b)/a$ is odd: this is possible if and only if $v_2(a)>v_2(b)$ (this implies $\operatorname{lcm}(a,2b)=s$ which, under the given assumption, is less than $\operatorname{lcm}(a,b,c)$). The first edge point on (a,b,z) occurs at time $\operatorname{lcm}(a,b)$ provided that $\operatorname{lcm}(a,b)/a$ and $\operatorname{lcm}(a,b)/b$ are odd: this is possible if and only if $v_2(a)=v_2(b)$ and, under the given assumption, we have $\operatorname{lcm}(a,b)<\operatorname{lcm}(a,b,c)$. □

From now on we suppose that E contains at least one edge point. We call P_* the first point on the edge reached by the path, we call z_* its z-coordinate, and we call t_* its time.

Proposition 11. *Consider the edges $(0,0,z)$, $(a,0,z)$, and (a,b,z). Then t_* is s, $\operatorname{lcm}(a,2b)$, and $s/2$ respectively while z_* is congruent modulo c to $\pm s$, $\pm\operatorname{lcm}(a,2b)$, and $\pm s/2$ respectively, where the plus/minus sign depends on whether $\lfloor t_*/c\rfloor$ is even/odd.*

Proof. The assertion on t_* holds by the proof of Lemma 10. The assertion on z_* easily follows because t_* and z_* are congruent modulo $2c$ and, moreover, $\pm t_*$ and z_* are congruent modulo c (the parity of $\lfloor t_*/c\rfloor$ determines whether the ball is traveling upwards or downwards at P_*). □

Theorem 12. *We have $e=\lfloor L_c/2\rfloor$, unless E is the edge $(0,0,z)$ and the ending corner is $(0,0,c)$, in which case $e=\lfloor L_c/2\rfloor-1$.*

Proof. For the edge $(0,0,z)$, the times for edge points on E are the multiples of s strictly between 0 and $\operatorname{lcm}(a,b,c)$. We easily conclude, recalling that $L_c/2=\operatorname{lcm}(a,b,c)/s$ and that $\operatorname{lcm}(a,b,c)$ is a multiple of s if and only if $(0,0,c)$ is the ending corner.

Now consider the edge $(a,0,z)$ (respectively, (a,b,z)). When E contains the ending corner, then the times t for the edge points on E are those such that $t\equiv\operatorname{lcm}(a,b,c)\pmod{s}$, and there are $\lfloor L_c/2\rfloor$ such times. Now suppose that E does not contain the ending corner, thus $\operatorname{lcm}(a,b,c)/a$ is even or $\operatorname{lcm}(a,b,c)/b$ is odd (respectively, $\operatorname{lcm}(a,b,c)/a$ or $\operatorname{lcm}(a,b,c)/b$ is even). Combining this with Lemma 10 we deduce $v_2(c) > v_2(a) > v_2(b)$ (respectively, $v_2(c) > v_2(a) \geq v_2(b)$) and in particular L_c is even. Recall from Proposition 11 that $t_* = \operatorname{lcm}(a, 2b)$ (respectively, $t_* = s/2$). So the edge points on E occur at the times $t \equiv t_* \pmod{s}$, so we have

$$e = \lfloor (\operatorname{lcm}(a,b,c) - t_*)/s \rfloor + 1 = \lfloor (L_c - 1)/2 \rfloor + 1 = L_c/2\,.$$

□

Notice that for an integer $0 \leq m < \operatorname{lcm}(a,b,c)/c$, in the time interval from mc to $(m+1)c$ the z-coordinate of a boundary point on E is strictly increasing if m is even and strictly decreasing if m is odd.

We partition the edge points in the largest subsets such that points obtained at later times always have a larger (respectively, smaller) z-coordinate: we call such a set an *up-family* (respectively, *down-family*). By what above, an edge point at time t is in an up- (respectively, down-) family according to whether $\lfloor t/c \rfloor$ is even (respectively, odd). If $s < c$, the families chronologically alternate between up- and down-families.

We call a set of points on E (excluding the endpoints) a *k-progression* if their z-coordinates form an arithmetic sequence with difference k and it is a maximal set with this property. Notice that, if $s < c$, then an up- or down-family is precisely an s-progression of edge points on E.

If $s|c$ then the edge points on $E = (0,0,z)$ are precisely the points with z-coordinate ms with $1 \leq m < c/s$, and evidently form an s-progression. We similarly have for $(a,0,z)$ and (a,b,z) that the edge points have z-coordinates $m \operatorname{lcm}(a,2b)$ and $m \operatorname{lcm}(a,b)$ respectively, when $\operatorname{lcm}(a,2b) \mid c$ and $\operatorname{lcm}(a,b) \mid c$ respectively.

Theorem 13. *Suppose that $s \nmid c$. The edge points on E form a $2\gcd(s,c)$-progression or a $\gcd(s,c)$-progression, the former case holding when $v_2(s) > v_2(c)$.*

Proof. Suppose first that $v_2(s) > v_2(c)$. We rescale s and c by $1/\gcd(s,c)$. Then s is still even and hence all boundary points on E have z-coordinates of the same parity; c is odd. There are precisely $(c-1)/2$ points on E with integer coordinates (excluding the endpoints). For the edge $(0,0,z)$ the parity condition implies that E does not contain the ending corner hence $e = \lfloor L_c/2 \rfloor$ by Theorem 12. Since $\lfloor L_c/2 \rfloor = \lfloor c/2 \rfloor = (c-1)/2$ we deduce that the edge points on E form a 2-progression (that is what we needed to prove after rescaling).

Now suppose that $v_2(s) \leq v_2(c)$ and rescale s and c by $2/\gcd(s,c)$, so that s and c are even and $\gcd(\operatorname{lcm}(a,b),c) = 1$. We now have $\lfloor L_c/2 \rfloor = \lfloor c/2 \rfloor = c/2$. The edge $(0,0,z)$ contains the ending corner $(0,0,c)$ so for this edge $e = c/2 - 1$, while for the other edges $e = c/2$ by Theorem 12.

There are $c/2 - 1$ and $c/2$ points on E (excluding the endpoints) with even and odd z-coordinates respectively and, since s is even, all edge points have the same parity.

For the edges $(a,0,z)$ and (a,b,z) we can easily check with Proposition 11 that the rescaled value of z_* is odd, so we conclude with a counting argument that the edge points form a 2-progression (that is what we needed to prove after rescaling). For the edge $(0,0,z)$ we may reason analogously because the rescaled value of z_* is even. □

Proposition 14. *Let $0 < Z_0 \leq s$ be the smallest z-coordinate among that of all edge points on E. The total number of families on E is e if $s \geq c$, and otherwise it is*

$$\left\lfloor \frac{s - Z_0}{2\gcd(s,c)} \right\rfloor + 1 \textit{ or } \left\lfloor \frac{s - Z_0}{\gcd(s,c)} \right\rfloor + 1$$

according to whether $v_2(s) > v_2(c)$ or not.

Proof. Each family has exactly one edge point with z-coordinate $0 < x \leq s$. In particular, if $s \geq c$ we are counting the total number of edge points on E. We now suppose that $s < c$. We only present the proof for the case $v_2(s) > v_2(c)$, the other case being analogous: it suffices to recall from Theorem 13 that the z-coordinates of the edge points on E are given by $0 < x \leq c$ such that $x \equiv Z_0 \pmod{2\gcd(s,c)}$, and count those $x \leq s$. □

If $s < c$, then it is easy to see that any given family on E (calling z_0 the least coordinate for a point in the family) has

$$\left\lfloor \frac{c - z_0}{s} \right\rfloor + 1 \text{ or } \left\lfloor \frac{c - z_0}{s} \right\rfloor$$

edge points according to whether $s|(c - z_0)$ or not (and the former case occurs precisely for one value of z_0).

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