THREE-DIMENSIONAL ARITHMETIC BILLIARDS

ABSTRACT. The billiard table is a parallelepiped with integer side lengths. A pointwise ball moves with constant speed along segments making a $45°$ angle with the sides and bounces on these. We allow the ball to start at any point with integer distances from the sides: either the ball lands in a corner or the trajectory is periodic. The geometry of the path depends on the arithmetic properties of the side lengths (for example if these are pairwise coprime). This generalizes a previous work by the author joint with Reguengo De Sousa and Tronto concerning the two-dimensional analogue: notice however that new interesting features occur in the three-dimensional context.

1. PRELIMINARY REMARKS

1.1. **Setting.** Consider the three-dimensional generalization of the arithmetic billiard, where the *billiard table* is a parallelepiped with integer side lengths a, b, c. The ball bounces inside the billiard table, moving at constant speed along segments making a $45°$ angle with each face. The ball only stops when it reaches a *corner*, by which we mean a vertex of the parallelepiped. We place the origin in one corner and let the opposite corner be the point (a, b, c). At every *step* each coordinate is either increased or decreased by 1. We call *x-face* of the parallelepiped the face where $x = 0$ and the face where $x = a$, and we similarly define this notion for y and z.

1.2. **Corner paths and closed paths.** If we shoot the ball from a corner, then we are on an x-face after any number of step which is a multiple of a (and we are back to the initial x-face after any number of step which is an even multiple of a), and similarly for the other coordinates. We deduce that after $\mathrm{lcm}(a, b, c)$ steps we reach a corner and hence the length of the path is $\sqrt{3}\,\mathrm{lcm}(a, b, c)$. Moreover, we can also determine the ending corner by looking at the parity of the numbers $\mathrm{lcm}(a, b, c)/a$, $\mathrm{lcm}(a, b, c)/b$, $\mathrm{lcm}(a, b, c)/c$. In particular, we have: since at least one of the three ratios is odd, then the starting and ending corners are different (which means that for at least one projection, their projections are different); starting and ending corner are opposite if and only if the three ratios are odd; starting and ending corner are on a same edge, supposing w.l.o.g. that it is a z-edge, if and only if $\mathrm{lcm}(a, b, c)/c$ is the only odd ratio; starting and ending corner are opposite vertices on a same face, supposing w.l.o.g. that it is a z-face, if and only if $\mathrm{lcm}(a, b, c)/a$ and $\mathrm{lcm}(a, b, c)/a$ are the odd ratios.

We call *corner path* a path starting in a corner: up to reversing the orientation of the path (i.e. swapping starting and ending corner) there are clearly four corner paths. We also consider paths starting at any point of the billiard table with integer coordinates. If we have choose a starting segment belonging to one of the corner paths, then the ball lands in a corner: this case is not very interesting because the path is simply a subset of a corner path.

Now start from a diagonal segment of the billiard table which does not belong to a corner path: the path then never reaches a corner so the ball keeps bouncing on the sides and the trajectory is periodic. The length of the period is $2\,\mathrm{lcm}(a,b,c)$ steps (thus the geometric length is $2\sqrt{3}\,\mathrm{lcm}(a,b,c)$) because we want to be back to the starting point and move in the same direction as at the beginning. We call such a path a *closed path*. For closed paths we restrict to one period and suppose w.l.o.g. that the starting point is on the face $z=0$.

1.3. **Symmetry for corner paths.** There are, neglecting the orientation, exactly four corner paths. If the starting and ending corner are opposite or not does not depend on the starting corner. If not, then there is exactly one coordinate which is the same for starting and ending corner, and which coordinate it is does not depend on the choice of the corner path.

Two corner paths are symmetric images of one another if and only if there is a symmetry of the parallelepiped mapping starting and ending corner of one path to starting and ending corner of the other. Because the "relation" between starting and ending corner described above does not depend on the choice of the corner path, then there is a symmetry of the parallelepiped mapping one corner to the other. This symmetry can be a plane symmetry w.r.t. the intermediate plane between two parallel faces or a composition of up to three plane symmetries.

Moreover, one corner path in itself is symmetric, where the symmetry of the billiard to be considered is the one exchanging starting and ending points. For example, if starting and ending points are opposite (this is precisely the case when abc is odd), then the requested symmetry is the point-symmetry at the center of the parallelepiped.

Up to a billiard symmetry we may then suppose that the starting corner is the origin.

1.4. **Mirrored billiards.** Consider a corner path and suppose w.l.o.g. that it starts from the origin. Build the cube having a vertex at the origin and the opposite vertex at the point $(\mathrm{lcm}(a,b,c),\mathrm{lcm}(a,b,c),\mathrm{lcm}(a,b,c))$. The cube consists of finitely many copies of the billiard tables, namely the original billiard table and further copies that we orient in such a way that a common face between two billiard tables represents the same face in both. We represent the corner path as the diagonal between the two above vertices of the cube, similarly as for two-dimensional arithmetic billiards. Now we can consider the projection of the cube and the diagonal corner path onto a cube face: the projection of the path becomes the diagonal of a square, and on the face we find the mirrored billiards construction for the corner path of a two-dimensional arithmetic billiard. The only difference is that now the length of the projection path on the face e.g. $x=0$ is $\mathrm{lcm}(a,b,c)$ and not $\mathrm{lcm}(a,b)$, so what we are doing here is moving back and forth on the corner path of the face $x=0$ (bouncing at the starting and ending corner).

We can immediately generalize the construction of the mirrored billiards to closed paths: we may suppose w.l.o.g. that the starting point is on the face $z=0$, and up to a billiard symmetry we may suppose that in the first step all coordinates are increasing. Then the closed path becomes a segment on the line from the starting point and parallel to $x = y = z$. The projection of the path onto the coordinate plane $z = 0$ becomes a segment in the plane parallel to $x = y$.

1.5. **Type of points, multiplicites.** We focus on the points of the path having integer coordinates. We call *edge points* the points as such that are on the edges and excluding the corners, we call *side points* the points as such on the sides and not on the edges, and we call *interior points* the remaining points of the path with integer coordinates. Finally we call *boundary points* the points of the path which are on the faces (they can be either corners, edge points, side points).

Notice that the ball cannot go twice through one same segment (neither in the same direction nor in the opposite direction), but the ball can go through one same point up to 4 times.

We call *multiplicity* of a point (for a point of the path with integer coordinates) the number of times that the path goes through it. For example, the multiplicity is 1 for the starting and ending corner of a corner path, or for an edge point (this is because the path cannot go twice through one same segment). The multiplicity for a side point can be either 1 or 2. We will prove that for an interior point of a corner path the multiplicity can be either 1, 2, or 4.

1.6. **Projection paths.** We call *projection path* the orthogonal projection of the path on a face, whose image is a two-dimensional billiard path: there we distinguish the *boundary points* on the face sides and the *self-intersection points* which are the points (that necessarily have integer coordinates and are not on the face sides) where the projection path crosses itself. The projection paths on two parallel faces are clearly the same, so we may concentrate on the three projections onto the coordinate planes.

A point of the path clearly has all three projections which are on the corresponding projection path.

The projection of a point can be of *cross-type* or else of *line-type*: with the former we mean that the projection is a self-intersection point of the projection path, and with the latter that it is a point on the projection path which is not a self-intersection point. If the three projections of a path point are of line type, then clearly the multiplicity of the point must be 1. Since the segments of the path form $45°$ angle with all billiard faces, then it cannot be that exactly one projection is of cross type. We can have two projections of cross-type and one projection of line-type: we will prove that for corner paths the multiplicity of such a path point must be 2 (in particular, if a side point is a self-intersection point of the projection path, then the multiplicity is 2). Finally (only for the interior points) we can have all three projections of cross-type: we will prove that for corner paths the multiplicity of such a path point must be 4.

1.7. **The closest face.** Consider the six distances to the faces from a point of the billiard table with integer coordinates. Given a path point P as such, we can fix one face F minimizing the distance. Then consider the four lines through P which make $45°$

angles with all billiard faces. These lines intersect F into four distinct points. Then the multiplicity of P is simply the number (from 1 to 4) telling how many among these four points on F are on the path (then they are side, edge, or corner points).

2. The special case a, b, c pairwise coprime

Consider w.l.o.g. the corner path starting at the origin. The path points with integer coordinates are such that all coordinates have the same parity (this is the case for the starting point, and at every step each coordinate changes parity).

With a counting argument we show the following:

Theorem 1. *The path points with integer coordinates are exactly the points in the billiard table whose coordinates are integers with the same parity. Moreover, their multiplicity is maximal (1 for corner and edge points, 2 for side points, 4 for interior points). There are precisely*

$$\begin{array}{ll} (a-1)+(b-1)+(c-1) & \textit{edge points} \\ ((a-1)(b-1)+(a-1)(c-1)+(b-1)(c-1))/2 & \textit{side points} \\ (a-1)(b-1)(c-1)/4 & \textit{interior points}. \end{array}$$

Finally, the path is the intersection of the billiard table with the grid of octaeders having edges making $45°$ angles with all coordinate planes, having edges of length $\sqrt{3}$ and such that the grid goes through the origin (the vertices of the octaeders are exactly the points in the billiard table whose coordinates all have the same parity).

Proof. For the proof we may suppose w.l.o.g. that abc is odd or ab is odd and c is even.

There are $a+b+c-2$ points on the edges such that all coordinates have the same parity (if abc is odd, then we have all coordinates even only on the three edges through the origin and we have all coordinates odd only on the three edges through (a, b, c); if c is even, then we have all coordinates even only on the five edges through the origin or $(0, 0, c)$ and all coordinates odd only on the edge from $(a, b, 0)$ to (a, b, c)).

On the face $z = 0$ there are $(a-1)(b-1)/4$ points not on the edges such that all coordinates are even. On the face $z = c$ we have the same amount of points not on the edges with all coordinates odd (respectively, even) if c is odd (respectively, even). The analogous count for the other faces shows that on all faces (excluding the edges) there are exactly $((a-1)(b-1)+(a-1)(c-1)+(b-1)(c-1))/2$ points whose coordinates have the same parity. Finally in the parallelepiped (excluding the faces) there are $(a-1)(b-1)(c-1)/4$ points whose coordinates have the same parity. Taking all these points with the maximal multiplicities we get a total of $abc-1$, which is the correct amount because the path consists of abc steps. An aside remark: When the reader is obtaining the above quantities for the points having coordinates of the same parity, they will most likely obtain $c-1$ as $(c-1))/2+(c-1))/2$ if c is odd and as $(c/2-1)+c/2$ if c is even. The last assertion is clear because we described all interior points as the points inside the billiards having coordinates with the same parity and because all these points have multiplicity 4. $\square$

If a, b, c are pairwise coprime, then we can deduce that each point of the billiard table belongs to exactly one of the four corner paths (because its multiplicity in a corner path is maximal). Moreover, supposing w.l.o.g. that ab is odd, then the partition given by the four corner paths is into four sets determined by the following two conditions: the coordinates x, z have (respectively, do not have) the same parity; the coordinates y, z have (respectively, do not have) the same parity. Since there is no closed path for these values of a, b, c we have completely understood arithmetic billiards in the case where a, b, c are pairwise coprime.

3. Boundary points

3.1. **A point on each face.** To describe the boundary points on a given face we need to fix one of them. We write below how to determine such a point, distinguishing between corner paths and closed paths. The choice of the point does not matter, we only need to have a boundary point (path corner, edge point, or side point) on the given face.

Consider w.l.o.g. the corner path starting at the origin, and fix some face F. We now choose a boundary point on F. If F is $x = 0$ or $y = 0$ or $z = 0$, then we choose the origin. If F is $x = a$ or $y = b$ or $z = c$ and it contains a path corner, then it contains precisely the ending corner and we choose this point (recall that it is immediate to compute the ending corner by looking at the parity of the numbers $\operatorname{lcm}(a, b, c)/w$ for $w = a, b, c$). Now suppose that F is $x = a$ or $y = b$ or $z = c$ and it does not contains a path corner: then we choose the path point at times a, b, c respectively. We now determine the path point at time c, the calculations for the times a and b are analogous. The requested point is:

$$\begin{array}{ll} ((c \bmod a), (c \bmod b), c) & \text{if } \lfloor c/a \rfloor \text{ and } \lfloor c/b \rfloor \text{ are even} \\ (a - (c \bmod a), (c \bmod b), c) & \text{if } \lfloor c/a \rfloor \text{ is odd and } \lfloor c/b \rfloor \text{ is even} \\ ((c \bmod a), b - (c \bmod b), c) & \text{if } \lfloor c/a \rfloor \text{ is even and } \lfloor c/b \rfloor \text{ is odd} \\ (a - (c \bmod a), b - (c \bmod b), c) & \text{if } \lfloor c/a \rfloor \text{ and } \lfloor c/b \rfloor \text{ are odd.} \end{array}$$

In all cases we are selecting either the ending corner or the path point on the face which is found at the smallest possible time ≥ 0. Notice that all faces contain a corner if and only if $v_2(a) = v_2(b) = v_2(c)$. Also notice that if the given face contains no path corners but at least edge points, then it would make sense to select our boundary point among the edge points.

For a closed path we can reason analogously because again such a path has points on every face. W.l.o.g. we fix the starting point to be a point on the face $z = 0$. Depending on the starting point and on the initial direction (for a side points there are two possibilites) we can determine the points on each face. Suppose that the starting point is $(x_0, y_0, 0)$. Then for the face $z = 0$ we choose the starting point, and for the face $z = c$ we choose the point at time c; for the faces $x = 0$ and $x = a$ we choose the points at time $\pm x_0$ and $\pm(a - x_0)$, the two signs depending on the initial direction of the path;

for the faces $y = 0$ and $y = b$ we similarly choose the points at time $\pm y_0$ or $\pm(b - y_0)$. Since only a finite case distinction is involved it is possible to write closed formulas for the coordinates of the points that we select on each face.

3.2. **Result.** Fix a path (corner path or closed path), without supposing anything on the numbers a, b, c or the starting point. Let us investigate the boundary points on some face F. We may suppose w.l.o.g. that F is a z-face. Clearly the z-projection of a boundary point lies on the z-projection path. Moreover, the path touches F regularly every $2c$ steps. A boundary point on F has multiplicity 1 or 2 and it can have multiplicity 2 only if its z-projection is a self-intersection point of the z-projection path (in particular it must be a side point, but it is clear that path corners and edge points have multiplicity 1). More precisely, we have:

Theorem 2. *Let F be a z-face, and fix a boundary point P_0 on F with z-projection P_0'. Finding the boundary points P on F (with a prescribed direction of the path at P) is the same as finding their z-projections $P' = (x_P, y_P)$ (with a prescribed slope of the z-projection path at P'). Consider the z-projection path to be a periodic path with length $2\operatorname{lcm}(a,b)$. Then the points P' as above are those points on the z-projection path which differ from P_0' by τ steps, where $0 \leq \tau < 2\operatorname{lcm}(a,b)$ is any multiple of $2\operatorname{lcm}(g_a, g_b)$. The multiplicity of P is 2 if and only if P' is a self-intersection point of the z-projection path and we have $\operatorname{lcm}(g_b, g_c) \mid x_P$ or $\operatorname{lcm}(g_a, g_c) \mid y_P$.*

Proof. Two distinct boundary points on F cannot have the same z-projection. A corner path, or a closed path during one period, can go at most twice through one same boundary point on F and if they go twice, then they arrive at the point from two distinct directions (their z-projections are perpendicular).

Proof of the characterization of the boundary points: The number of steps between boundary points on F is a multiple of $2c$, so the time difference between their images in the z-projection path is a multiple of $\gcd(2c, 2\operatorname{lcm}(a,b)) = 2\operatorname{lcm}(g_a, g_b)$.

By considering the numbers τ as in the statement we get at most

$$2\operatorname{lcm}(a,b)/2\operatorname{lcm}(g_a,g_b) = 2\operatorname{lcm}(a,b,c)/2c$$

boundary points counted with multiplicity hence for a closed path we have found all boundary points on F with the correct multiplicity.

For a corner path, let its z-projection start at the origin. The time at which we arrive at P_0' is then a multiple of $\gcd(c, 2\operatorname{lcm}(a,b))$ and hence of $\operatorname{lcm}(g_a, g_b)$. The number of boundary points on the two z-faces counted with multiplicity is $\operatorname{lcm}(a,b,c)/c + 1$. The number of boundary points on F counted with multiplicity is (recalling that F contains precisely one path corner if and only if $\operatorname{lcm}(a,b,c)/c$ is odd):

$\operatorname{lcm}(a,b,c)/2c + 1/2$	*if F contains precisely one path corner;*
$\operatorname{lcm}(a,b,c)/2c + 1$	*if F contains two path corners;*
$\operatorname{lcm}(a,b,c)/2c$	*if F contains no path corners.*

So by varying τ as in the statement we find twice the same points with a given slope (with the exception of the corners) because $\operatorname{lcm}(a,b)$ is a multiple of $\operatorname{lcm}(g_a, g_b)$. We thus find at most the number of boundary points on F counted with multiplicity and we similarly conclude.

Proof of the assertion on the multiplicity for closed paths: The assertion for corner paths will be a consequence of the next result, so consider a closed path. Let P be a boundary point such that P' is a self-intersection point of the z-projection path (else the multiplicity of P is 1). Suppose first that the z-projection path is a closed path. Then this goes twice through P' during its period, and we may suppose w.l.o.g. that the y-coordinate both increase (respectively, both decrease) at both passages. We claim that in this case the multiplicity of P' is 2 only if $\mathrm{lcm}(g_b, g_c) \mid x_P$ (the other case is analogous with the condition $\mathrm{lcm}(g_a, g_c) \mid y_P$). The two times τ (which sum up to $2\,\mathrm{lcm}(a, b)$) between the two occurrences of P' in the z-projection path satisfy

$$\tau \equiv 0 (\mathrm{mod} 2b) \qquad \tau \equiv \pm 2x_P (\mathrm{mod} 2a)\,.$$

Call $L = \mathrm{lcm}(g_a, g_b)$. Since P is a boundary point, then P is a second time on the path if and only if $\tau = k2L$ for some integer k. So we need to solve (for each sign choice) the congruence system

$$kL \equiv 0 (\mathrm{mod} b) \qquad kL \equiv \pm x_P (\mathrm{mod} a)\,.$$

Since $g_b \mid a$ and $g_b \mid L$, we must have $g_b \mid x_P$. Moreover, if the two congruences are compatible, then we must have $g_c \mid x_P$. Now suppose that $\mathrm{lcm}(g_b, g_c) \mid x_P$. Write $\mathrm{lcm}(a, b) = GAB$ such that the prime divisors of G, A, B are the primes p such that $\alpha = \beta$, $\alpha > \beta$, $\alpha < \beta$, where p^α (respectively, p^β) is the highest power of p dividing a (respectively, b). Then the two congruences are equivalent to

$$kL \equiv \pm x_P GB (GB \bmod A)^{-1} (\mathrm{mod}\, \mathrm{lcm}(a, b))$$

and we find a solution k because $g_b \mid x_P$ and $g_a \mid g_b GB$.

Now consider a closed path such that the z-projection is a corner path, so that in particular g_c divides x_P and y_P. The z-projection path, starting at a corner, passes though P' at four times $\tau_1 \le \tau_2 \le \tau_3 \le \tau_4$. Suppose w.l.o.g. that the path goes through P at time τ_1, the other cases being analogous. Then $\tau_4 - \tau_1$ cannot be a multiple of $2L$ (the closed path within one period cannot go twice through a same segment, not even with the opposite orientation). Similary, at most one between $\tau_3 - \tau_1$ and $\tau_2 - \tau_1$ is a multiple of $2L$. Requesting that one of these two times is a multiple of $2L$ gives in one case the equivalent condition $\mathrm{lcm}(g_b, g_c) \mid x_P$ and in the other case $\mathrm{lcm}(g_a, g_c) \mid y_P$. These two conditions cannot hold simultaneously because otherwise (checking this involves a case distinction but it is easy considering the next result and the canonical choice of a boundary point on F) the point P lies on one of the two corner paths with the same z-projection path: since the multiplicity of P in this corner path is 2 (by the next result), no closed path can go through P. □

Consider a closed path and a side point such that its projection on that face is a point of self-intersection of the projection path: the multiplicity of the side point can be 1 and it can be 2.

Example 3. Choosing $(a, b, c) = (10, 15, 6)$ and starting at $(1, 0, 0)$, the boundary point $P = (5, 6, 0)$ is such that $(5, 6)$ is a self-intersection point of the z-projection path: we have $\mathrm{lcm}(g_b, g_c) = 10$ and $\mathrm{lcm}(g_a, g_c) = 15$, so the multiplicity of P is 1.
Choosing $(a, b, c) = (4, 6, 9)$ and starting at $(1, 0, 0)$, the boundary point $P = (2, 3, 9)$ is such that $(2, 3)$ is a self-intersection point of the z-projection path: we have $\mathrm{lcm}(g_b, g_c) = 2$ and $\mathrm{lcm}(g_a, g_c) = 3$, so the multiplicity of P is 2.

Theorem 4. *Consider a corner path. Let F be a z-face let P_0 be a boundary point on F, with z-projection $P_0' = (x_0, y_0)$. Then a point P on F is a boundary point if and only if its z-projection $P' = (x_P, y_P)$ lies on the z-projection path and $2g_b \mid (x_P - x_0)$ and $2g_a \mid (y_P - y_0)$. If P on F is a boundary point, then we have: $g_b \mid x_P$ and $g_a \mid y_P$; the multiplicity of P is 2 if and only if P' is a self-intersection point of the z-projection path.*

Proof. We make use of the characterization of the boundary points from the previous result, and we again consider the z-projection path to be periodic of period $2\,\mathrm{lcm}(a, b)$. Clearly if P is a boundary point on F, then $g_b \mid x_P$ and $g_a \mid y_P$. We deduce that $2g_b \mid (x_P - x_0)$ and $2g_a \mid (y_P - y_0)$ because the number of steps in the projection path from P_0' to P' is a multiple of $2\,\mathrm{lcm}(g_a, g_b)$. Now suppose that P is such that P' is on the z-projection path and $2g_b \mid (x_P - x_0)$ and $2g_a \mid (y_P - y_0)$. Then the number of steps in the z-projection path between P_0' and P' is a multiple of $2\,\mathrm{lcm}(g_a, g_b)$ (because $2g_b \mid (x_P - x_0)$ and $g_b \mid a$, and analogously for the y-coordinates).

For the multiplicity result, suppose that P is a boundary point on F such that P' is a self-intersection point of the z-projection path. Then we have $\mathrm{lcm}(g_b, g_c) \mid x_P$ and $\mathrm{lcm}(g_a, g_c) \mid y_P$. By reasoning as in the proof of the previous result we can show that the z-projection path goes four times through P' in each period and hence the multiplicity of P is two. Indeed, considering the times τ_1 to τ_4 (and supposing again w.l.o.g. that the path goes through P at time τ_1) then we similarly find that $\tau_2 - \tau_1$ and $\tau_3 - \tau_1$ are multiples of $2\,\mathrm{lcm}(g_a, g_b)$, and the same holds for $\tau_4 - \tau_1 = 2\,\mathrm{lcm}(a, b) - 2\tau_1$ because we start at a corner hence $\gcd(c, 2\,\mathrm{lcm}(a, b))$ divides τ_1. $\square$

Proposition 5. *The multiplicity of an interior point cannot be 3.*

Proof. We suppose that the multiplicity of an interior point P is at least 3 and prove it that it is 4. Consider the closest face F to P and suppose w.l.o.g. that it is the face $z = 0$. Moving towards P in the four possible directions we find four points $F_i = (x_i, y_i, 0)$ on F, where $i = 1, 2, 3, 4$. Their z-projections F_i' are clearly on the z-projection path because the assumption on the multiplicity implies that the z-projection P' of P is a self-intersection point of the z-projection path. Because of the assumption on the multiplicity we may suppose w.l.o.g. that F_1 to F_3 are boundary points and that F_1 and F_3 are opposite with respect to P. We conclude by proving that F_4 is also a boundary point.

If we have a corner path, by the characterization of the boundary points it suffices to show that $2g_b \mid (x_4 - x_2)$ and $2g_a \mid (y_4 - y_2)$. This holds because F_1 and F_3 are boundary points on F and $x_4 - x_2 = x_3 - x_1$ and $y_4 - y_2 = y_3 - y_1$.

If we have a closed path, then the time difference along the projection path from going to F_1' to F_3' or conversely is a multiple of $2\,\mathrm{lcm}(g_a, g_b)$ hence the same holds for going from F_2' to F_4'. Since F_2 is a boundary point, then F_4 is a boundary point as well. $\square$

4. Edge points

The following results are a consequence of our characterization of the boundary points.

Theorem 6. *On a z-face there are points on both parallel edges $x = 0$ and $x = a$ if and only if $v_2(a) > v_2(c)$, provided that there are points on one of these edges.*

Proof. Let $L = \operatorname{lcm}(g_a, g_b)$. We find points on both edges if and only if there is some integer k such that $2kL \equiv a(\bmod 2a)$. This means that $2kL$ is an odd multiple of a, which is possible if and only if $v_2(2L) \le v_2(a)$. Since $v_2(L) = \min(\max(v_2(a), v_2(b)), v_2(c))$, this condition is equivalent to $v_2(a) > v_2(c)$. □

Theorem 7. *Let F be a z-face and let $P_0 = (x_0, y_0, z_0)$ be a boundary point on F ($z_0 \in \{0, c\}$). There is a boundary point on the edge $x = a$ of F if and only if:*

$$\begin{array}{lll} v_2(a) > v_2(c) & \textit{and} & 2g_b \mid x_0 \\ v_2(a) \le v_2(c) & \textit{and} & g_b \mid x_0 \ \textit{ and } \ v_2(x_0) = v_2(a) \end{array}$$

Moreover, to study the edge $x = 0$ we can apply the above criterion replacing x_0 by $a - x_0$.

Proof. The last assertion is evident by symmetry. Let $L = \operatorname{lcm}(g_a, g_b)$. We have an edge point on F on the edge $x = a$ if and only if there is some integer k such that $2kL \equiv a - x_0 (\mathrm{mod} 2a)$. This is equivalent to $2\gcd(L, a) \mid (a - x_0)$. If p is a prime number, then we have $v_p(\gcd(L, a)) = v_p(g_b)$. The divisibility then means: $v_p(x_0) \ge v_p(g_b)$, if p is odd; $v_2(x_0) = v_2(a) = v_2(g_b)$, if $v_2(a) \le v_2(c)$; $v_2(x_0) \ge v_2(g_b) + 1$, if $v_2(a) > v_2(c)$. □

4.1. **Examples.** Example of closed path where it is not possible to suppose w.l.o.g. that $\gcd(a, b, c) = 1$: With sides $(2d, 2d, 2d)$ and with starting point on the middle of the edge the trajectory is a square, w.l.o.g. the boundary points are $(d, 0, 0), (2d, d, d), (d, 2d, 2d), (0, d, d)$.

Consider the corner path in $15 \times 21 \times 35$. All path points have multiplicity 1 (here the situation is $a = g_b g_c$ and similarly for the other coordinates, and abc are odd and $\gcd(a, b, c) = 1$). To have w.l.o.g. a side point on the $z = 0$ face, then we need $(x, y, 0)$ where $x, y, 0$ have the same parity and x, y are multiples of g_c and x is a multiple of g_b and y is a multiple of g_a, so $x = 0, a$ and $y = 0, b$ and the point is on an edge, contradiction. There is no interior point with multiplicity 3 or 4 because with similar reasoning we cannot have all three projections which are self-intersection points of the projection path: by considering the projection onto $z = 0$ both coordinates x, y are a multiple of g_c. So we find that $a \mid x$, $b \mid y$, and $c \mid z$ (and in fact it suffices that one coordinate w.l.o.g. a equals $g_b g_c$ to arrive at this conclusion: is such a condition necessary? same question of necessity to have no side point with multiplicity 2?). There is no interior point with multiplicity 2 because supposing each coordinate to be of the form $a = g_b g_c$ then by reasoning as above we cannot get two projections which are of cross-type for an interior point. In particular this specific example does not tell us whether the above tentative result is true or false, but maybe adapting the example? (For example only one coordinate of the form $a = g_b g_c$?)

Suppose that one of the numbers a, b, c divides or it is multiple of the other two. What does that mean for the paths? Investigate these special cases with special divisibilities.

4.2. **Rescaling.** Consider corner paths. Call $g = \gcd(a, b, c)$. Up to rescaling all coordinates by g we may suppose that $g = 1$. Indeed, this does not affect the number of edge and side points, nor the multiplicity of the side points, nor the relative position of these points. Moreover, since the difference between any two coordinates has the same parity during the path, we deduce that the intersection of diagonal lines is always a point with integer coordinates (in particular the multiplicity of points of the path which do not have integer coordinates is always 1) also supposing $g = 1$. In other words, the interesting points on a corner path have all coordinates which are a multiple of g and we can find these points as points with integer coordinates in the rescaled path. So it suffices to multiply by g all coordinates of the points that we found for the rescaled path.

4.3. **Pairwise greatest common divisors.** We define $g_a = \gcd(b, c)$, $g_b = \gcd(a, c)$, $g_c = \gcd(a, b)$. These numbers are pairwise coprime because $\gcd(a, b, c) = 1$. It is not difficult to show (by comparing the largest power of a fixed prime that divides the numbers) that

$$\frac{\operatorname{lcm}(a, b)}{\operatorname{lcm}(g_a, g_b)} = \frac{\operatorname{lcm}(a, b, c)}{c}.$$

Also notice that $\gcd(c, \operatorname{lcm}(a, b)) = g_a g_b$. Similar relations hold by permuting a, b, c.

4.4. **Cases.** We distinguish cases according to the position of the ending corner (we fix the starting corner at the origin). Consider the numbers $\frac{\operatorname{lcm}(a,b,c)}{a}, \frac{\operatorname{lcm}(a,b,c)}{b}, \frac{\operatorname{lcm}(a,b,c)}{c}$. W.l.o.g. we only have three cases (and recall that $\gcd(a, b, c) = 1$):

- Case 1: Suppose that $\frac{\operatorname{lcm}(a,b,c)}{a}, \frac{\operatorname{lcm}(a,b,c)}{b}, \frac{\operatorname{lcm}(a,b,c)}{c}$ are odd. Then a, b, c are odd. The ending corner is the opposite corner (a, b, c), and there is precisely one corner on each face.
- Case 2: Suppose that $\frac{\operatorname{lcm}(a,b,c)}{a}, \frac{\operatorname{lcm}(a,b,c)}{b}$ are even and $\frac{\operatorname{lcm}(a,b,c)}{c}$ is odd. Then we must have $v_2(c) > v_2(a)$ and $v_2(c) > v_2(b)$. W.l.o.g. we may suppose that $v_2(a) = 0$. The ending corner is the corner $(0, 0, c)$: there are two faces with two corners, two faces with one corners, and two faces without corners.
- Case 3: Suppose that $\frac{\operatorname{lcm}(a,b,c)}{a}, \frac{\operatorname{lcm}(a,b,c)}{b}$ are odd and $\frac{\operatorname{lcm}(a,b,c)}{c}$ is even. Then we must have $v_2(a) = v_2(b) > v_2(c) = 0$. The ending corner is the corner $(a, b, 0)$: there is one face with two corners, there are four faces with one corner, and there is one face without corners.

4.5. **Landing in a corner.** It is not true that a path starting at a point of a corner path lands in a corner independently of the direction. For example consider $a = 15$, $b = 9$, $c = 7$: there are side points on the face $z = 0$ with that projection of line-type (so they have multiplicity 1) and which do not belong to the other corner path on that face. So the other segment through the point belongs to none of the two corner paths on the face and hence any segment in the billiard table having that other segment as projection does not belong to any corner path. We deduce that the property of landing or not in a corner not only depends on the starting point, but also on the starting direction. However, moving in a direction or the opposite direction does not change the property of landing in a corner because a segment lies on a corner path independently on which orientation the trajectory has while going through it. In the above example, the points of height 2 above the indicated side points have two projections of cross-type and one of line-type so they have multiplicity 2. By the same reasoning on the projection of line-type we deduce that the two segments through the point that are not on the given corner path do not belong to any corner path. So we also have examples of interior points of multiplicity 2 that only belong to one corner path.

Find a counterexample for the missing case, namely an interior point of multeplicity 1: can it belong to exactly 4/2/1 corner paths (supposing abc odd)?

5. Edge Points in Corner Paths

The aim is to study the distribution of edge points for a corner path in a 3D Billiard. Because of the symmetry of the cuboid we may w.l.o.g. restrict ourselves to a z-edge and consider only the following three edges: $(0,0,z),(a,0,z),(a,b,z)$. Let E be such an edge. Recall that the multiplicity of an edge point is precisely 1 and we do not consider corners as edge points. Recall that a corner path ends at time $\operatorname{lcm}(a,b,c)$.

The number of edge points on E is a non-negative integer, and it can be zero (consider for example a=b=c). Moreover, the number of edge points on E is strictly less than c because there are $c+1$ points with integer coordinates on E including its 2 endpoints.

For notation, we call $s := 2\operatorname{lcm}(a,b)$. Also, $L_c := \frac{\operatorname{lcm}(a,b,c)}{\operatorname{lcm}(a,b)}$, which is an integer dividing c (notice that we have $L_c = c$ if and only if c is coprime to ab). Notice that $sL_c = 2\operatorname{lcm}(a,b,c)$. We also define a <u>*step*</u> to be the period of time it takes for the ball to undergo a unit translation (i.e. for each of its coordinates to change by 1).

If we suppose that E contains an edge point or the start corner or the end corner that is obtained at time t in the path, then the edge points on E are precisely the points at the times T such that $0 < T < \operatorname{lcm}(a,b,c)$ and $T \equiv t \pmod{s}$ (as the ball returns to E every number of steps that is a multiple of both $2a$ and $2b$).

Notice that for an integer $0 \leq m < \frac{\operatorname{lcm}(a,b,c)}{c}$, in the time interval from mc to $(m+1)c$ the z-coordinate is strictly increasing if m is even and strictly decreasing if m is odd.

We partition the edge points in the largest subsets such that points obtained at later times always have a larger (respectively, smaller) z-coordinate: we call such a set an <u>up-family</u> (respectively, <u>down-family</u>). Moreover, we call a set of edge points on E a <u>k-progression</u> if their z-coordinates form an arithmetic sequence with difference k and it is a maximal set with this property. Notice that an up- or down-family is specifically an s-progression, and that the converse holds true as well.

6. First Edge Point

Recall that $s := 2\operatorname{lcm}(a,b)$ and therefore $\operatorname{lcm}(a,2b)/s$ equals $1/2$ or 1 according to whether $v_2(a) > v_2(b)$ or not.

Remark 8. *An edge point p at time t is in an up-family (respectively, down-family) if and only if $\lfloor t/c \rfloor$ is even (respectively, odd).*

Lemma 9. *The edge $E = (a,0,z)$ (respectively, $E = (a,b,z)$) has edge points if and only if $v_2(a) > v_2(b)$ (respectively, $v_2(a) = v_2(b)$).*

Proof. Notice the first edge point on $E = (a,0,z)$ occurs at time $t = \operatorname{lcm}(a,2b)$ provided $\frac{\operatorname{lcm}(a,2b)}{a}$ is odd (the smallest number of steps that is a multiple of $2b$ and an odd multiple of a). This implies

$$\max\{v_2(a), v_2(b)+1\} - v_2(a) = 0 \iff v_2(a) > v_2(b).$$

Similarly, the first edge point on $E = (a,b,z)$ occurs at time $t = \operatorname{lcm}(a,b)$ provided $\frac{\operatorname{lcm}(a,b)}{a}$ and $\frac{\operatorname{lcm}(a,b)}{b}$ are both odd (the smallest number of steps that is an odd multiple of a and b). We conclude by inferring that $\max\{v_2(a), v_2(b)\} = v_2(a)$ and $\max\{v_2(a), v_2(b)\} = v_2(b)$. □

Suppose from now on that E contains at least one edge point. Define the chronologically first edge point to be E_*, call its z-coordinate z_*, and call its time t_*.

Remark 10. *It follows from Lemma 16 that for edges* $E = (0,0,z), (a,0,z)$, *and* (a,b,z), $t_* = s, \operatorname{lcm}(a,2b), s/2$ *respectively. Further, the condition* $t_* < \operatorname{lcm}(a,b,c)$ *is required for* E_* *to exist, recalling that the total time in the corner path is* $\operatorname{lcm}(a,b,c)$.

Theorem 11. *Let* $E = (0,0,z), (a,0,z)$, *or* (a,b,z). *Then* $z_* \pmod c$ *is respectively congruent to* $\pm s, \pm\operatorname{lcm}(a,2b), \pm s/2$ *where the plus/minus sign depends on whether* E_* *belongs to an up/down-family, namely whether* $\lfloor t_*/c \rfloor$ *is even/odd, accordingly.*

Proof. If we let $A := \{0,1,\dots,c\}$ and $B := \{c, c-1, \dots, 1\}$, then the ball constantly cycles through the z-coordinates $A \cup B$. Notice that if at time t the ball is travelling in the positive (respectively, negative) z-direction, then its z-coordinate is given by the element in A (respectively, B) at index t, which is synonymous with $t \pmod c$ (respectively, $-t \pmod c$). We conclude by recalling that E_* occurs at time t_*, and that travelling in the positive (respectively, negative) z-direction is synonymous with an edge point in an up-family (respectively, down-family). □

7. Number of Edge Points

Let us denote the number of edge points on an edge E by n.

Theorem 12. *The number of edge points on* $E = (0,0,z)$ *is* $n = \lfloor L_c/2 \rfloor - 1$ *or* $n = \lfloor L_c/2 \rfloor$, *the former case holding when* E *contains the end corner, namely when* $\frac{\operatorname{lcm}(a,b,c)}{a}$ *and* $\frac{\operatorname{lcm}(a,b,c)}{b}$ *are both even. For* $E = (a,0,z), (a,b,z)$, $n = \lfloor L_c/2 \rfloor$.

Proof. For $E = (0,0,z)$, as E contains the start corner that is obtained at $t = 0$, we know the precise times for edge points on E, namely the multiples of s strictly between 0 and $\operatorname{lcm}(a,b,c)$. We easily conclude, recalling that $L_c/2 = \operatorname{lcm}(a,b,c)$ and $\operatorname{lcm}(a,b,c)$ is a multiple of s if and only if $(0,0,c)$ is the end corner.

For $E = (a,0,z)$ (respectively, $E = (a,b,z)$), when E contains the end corner at either $(a,0,0)$ or $(a,0,c)$ (respectively, $(a,b,0)$ or (a,b,c)), obtained at $t = \operatorname{lcm}(a,b,c)$, then we know the times for edge points on E obey $T \equiv \operatorname{lcm}(a,b,c) \pmod s$, of which there are exactly $\lfloor L_c/2 \rfloor$ times.

Otherwise E does not contain the end corner so either $\frac{\operatorname{lcm}(a,b,c)}{a}$ is even or $\frac{\operatorname{lcm}(a,b,c)}{b}$ is odd (respectively, $\frac{\operatorname{lcm}(a,b,c)}{a}$ is even or $\frac{\operatorname{lcm}(a,b,c)}{b}$ is even). Then $v_2(c) > v_2(a)$ or $v_2(b) = \max\{v_2(a), v_2(c)\}$ where the latter is impossible by Lemma 16 (respectively, $v_2(c) > v_2(a) = v_2(b)$ according to Lemma 16). Recall that the first edge point occurs at $t_* = \operatorname{lcm}(a,2b)$ (respectively, $t_* = s/2$) so we know the edge points on E are at times $T \equiv t_* \pmod s$, of which there are

$$n = \lfloor (\operatorname{lcm}(a,b,c) - \operatorname{lcm}(a,2b))/s \rfloor + 1 = \lfloor (L_c - 1)/2 \rfloor + 1$$

(respectively, $n = \lfloor (\operatorname{lcm}(a, b, c) - s/2)/s \rfloor + 1 = \lfloor (L_c - 1)/2 \rfloor + 1$).

Yet we notice L_c must be even because $v_2(L_c) = \max\{v_2(a), v_2(c)\} - v_2(a) > 0$ since $v_2(c) > v_2(a)$. Thus $n = L_c/2 - 1 + 1 = L_c/2$. □

It follows that $E = (0, 0, z)$ contains edge points if and only if L_c is at least 4 or 2 according to whether $(0, 0, c)$ is the end corner or not, and that $E = (a, 0, z), (a, b, z)$ has edge points if and only if $L_c \geq 2$.

8. Distribution of Edge Points

If $s|c$ then the edge points on $E = (0, 0, z)$ are precisely the points with z-coordinate ms with $1 \leq m < c/s$, and evidently form an s-progression. From now on we thus suppose that $s \nmid c$.

Theorem 13. *The edge points on an edge E form a $2\gcd(s, c)$-progression or $gcd(s, c)$-progression, the former case holding when $v_2(s) > v_2(c)$. It follows that the z-coordinates of edge points on E are all $0 < x < c$ such that $x \equiv z_* \pmod{2\gcd(s, c)}$ or $x \equiv z_* \pmod{\gcd(s, c)}$, the former case holding when $v_2(s) > v_2(c)$ (where z_* can be replaced with any other edge point's z-coordinate).*

Proof. We proceed by cases: $v_2(s) > v_2(c)$ and otherwise.

Case 1. When $v_2(s) > v_2(c)$ we rescale s and c by a factor of $1/\gcd(s, c)$ such that s is now $s/gcd(s, c)$ and still even meaning all edge points on E have z-coordinates of the same parity. Likewise, c becomes $c/\gcd(s, c)$. Since now $\gcd(s, c) = 1$, then $2 \nmid c$ and $\gcd(\operatorname{lcm}(a, b), c) = 1$. The former implies there are precisely $\frac{c-1}{2}$ points along E with even and odd z-coordinates each (excluding endpoints).

If $E = (0, 0, z)$, the former also implies that the end corner is not $(0, 0, c)$. Therefore $n = \lfloor L_c/2 \rfloor = \lfloor c/2 \rfloor = \frac{c-1}{2}$. If $E = (a, 0, z)$ or $E = (a, b, z)$ then, too, $n = \lfloor L_c/2 \rfloor = \frac{c-1}{2}$. As there are indeed $\frac{c-1}{2}$ points on E with z-coordinates of the same parity, this implies the edge points on E form a 2-progression. We may rescale s and c back up by a factor of $\gcd(s, c)$ without altering the relative order or consistent spacing of the edge points, yielding a $2\gcd(s, c)$-progression.

Case 2. Otherwise, $v_2(s) \leq v_2(c)$ so we instead rescale s and c by a factor of $2/\gcd(s, c)$ such that $\gcd(s, c) = 2$, implying $2|c$ and $\gcd(\operatorname{lcm}(a, b), c) = 1$. By the former, there are $c/2 - 1$ and $c/2$ points on E with even and odd z-coordinates respectively.

Notice that after the rescaling, if $E = (a, 0, z)$ then E_*, with z_* being $\operatorname{lcm}(a, 2b) - mc$ or $mc - \operatorname{lcm}(a, 2b)$ (Theorem 18), will map to

$$p_1 = \frac{2(\operatorname{lcm}(a, 2b) - mc)}{\gcd(s, c)}$$

or

$$p_2 = \frac{2(mc - \operatorname{lcm}(a, 2b))}{\gcd(s, c)}$$

respectively (where $m \in \mathbb{Z}^+$). Yet $v_2(p_1) = v_2(p_2) = 0$ given $v_2\left(\frac{2\operatorname{lcm}(a,2b)}{\gcd(s,c)}\right) = 1 + v_2(a) - v_2(s) = 0$ so the first edge point, and thus all subsequent ones, will have an odd z-coordinate (recalling that the rescaled s is even).

When $E = (a, b, z)$, E_* with z_* equal to $s/2 - mc$ or $mc - s/2$ (Theorem 18) will map to either

$$p_1 = \frac{2(\operatorname{lcm}(a,b) - mc)}{\gcd(s,c)} = \frac{s - 2mc}{\gcd(s,c)}$$

or

$$p_2 = \frac{2(mc - \operatorname{lcm}(a,b))}{\gcd(s,c)} = \frac{2mc - s}{\gcd(s,c)}$$

respectively. Once again, we see $v_2(p_1) = v_2(p_2) = 0$ given $v_2(s/\gcd(s,c)) = 0$, such that all edge points on E have an odd z-coordinate after rescaling.

Hence for $E = (a, 0, z)$ and $E = (a, b, z)$, as we have $n = \lfloor L_c/2 \rfloor = c/2$, by the counting argument the edge points on E form a 2-progression.

If $E = (0, 0, z)$ then notice $\max\{v_2(a), v_2(b)\} = v_2(s) - 1$ so

$$v_2\left(\frac{\operatorname{lcm}(a,b,c)}{a}\right), v_2\left(\frac{\operatorname{lcm}(a,b,c)}{b}\right) \geq v_2(c) - v_2(s) + 1 \geq 1$$

meaning the end corner is $(0, 0, c)$. Therefore $n = \lfloor L_c/2 \rfloor - 1 = \lfloor c/2 \rfloor - 1 = c/2 - 1$. Recognize again that all edge points on E have an even z-coordinate, so by the analogous counting argument they also form a 2-progression.

Rescaling s and c back up by a factor of $\gcd(s,c)/2$ validates that the edge points on E form a $\gcd(s,c)$-progression.

□

9. Features of Families

Let $0 < z_0 \leq s$ be the smallest z-coordinate among that of all edge points belonging to a particular family, and let $0 < Z_0 \leq s$ be the smallest z-coordinate among that of all edge points on E, or the minimum of all z_0.

Theorem 14. *The total number of families (up or down) on E is n if $s \geq c$, and otherwise*

$$\left\lfloor \frac{s - Z_0}{2\gcd(s,c)} \right\rfloor + 1 \text{ or } \left\lfloor \frac{s - Z_0}{\gcd(s,c)} \right\rfloor + 1$$

according to whether $v_2(s) > v_2(c)$ or not.

Proof. Notice that each family has exactly one edge point with z-coordinate $0 < x \leq s$. If $s \geq c$ then we are simply counting the total number of edge points on E, or n.

Otherwise $s < c$ so let us assume $v_2(s) > v_2(c)$. We recall that the z-coordinates of the edge points on E are given by $0 < x \leq c$ such that $x \equiv Z_0 \pmod{2\gcd(s,c)}$.

It follows that the edge point with the greatest z-coordinate less than or equal to s has coordinate $Z_0 + m(2\gcd(s,c))$ for some non-negative integer m. Then

$$Z_0 + m(2\gcd(s,c)) \leq s \iff m+1 = \left\lfloor \frac{s - Z_0}{2\gcd(s,c)} \right\rfloor + 1$$

where $m+1$ is indeed the number of edge points with z-coordinates in the interval $(0, s]$. When $v_2(s) \leq v_2(c)$, the analogous argument holds. $\square$

Notice that if $s > c$ then each family has precisely one edge point. Otherwise, notice that a particular family has

$$\left\lfloor \frac{c - z_0}{s} \right\rfloor + 1 \text{ or } \left\lfloor \frac{c - z_0}{s} \right\rfloor$$

edge points according to whether $s | c - z_0$ or not. Further, notice there is only one critical value of z_0, say z_c, such that $s | c - z_c$, and that if c and s are kept constant then for families with $0 < z_0 < z_c$ the number of edge points in each those families is the same; this is also the case for the rest of the families, namely those with $z_c \leq z_0 \leq s$. Hence the number of edge points belonging to each family on an edge E has only two possible values.

Notice that if $s < c$ then chronological families always alternate between up- and down-families. Therefore, the number of up-families and the number of down-families will either be equal or there will be one more up- than down-family. This follows by recalling that the first family will be an up-family, and that if the number of times the length of the edge is traversed, $\operatorname{lcm}(a,b,c)/c$, is even (respectively, odd) then the last family will be a down-family (respectively, up-family).

When $s > c$, there isn't as much regularity. For instance, $(a,b,c) = (4,29,21)$ yields an alternating pattern of up- and down-families, whereas $(a,b,c) = (5,11,14)$ produces exclusively down-families. Similarly, $(a,b,c) = (4,7,27)$ only has up-families. It is also possible to have a different pattern, such as when $(a,b,c) = (4,14,17)$ which results in a $\{-,+,-,-,+,-,-,+\}$ sequence where a plus (respectively, minus) denotes an up-family (respectively, down-family).

ACKNOWLEDGEMENTS

Computing many examples was made possible by the computer program written by Steve Schmol and the graphic visualization made by Sebastiano Tronto. Many thanks also to Jack Guilfoyle for interesting discussions which led to the results on the edge points.

REFERENCES

[1] Martin Gardner, *Sixth Book of Mathematical Diversions from "Scientific American"*, University of Chicago Press, Paperback 1984. ISBN: 0226282503. (See pages 211–215.)

[2] Hema Gopalakrishnan and Stephanie Furman, *Pool Table Geometry*, Math Circle Poster and Activity Session, JMM2013 San Diego, `http://sigmaa.maa.org/mcst/PosterActivitySessions/documents/PoolTableActivity.pdf`.

[3] Glenda Lappan, James T. Fey, William M. Fitzgerald, Susan N. Friel, and Elizabeth Difanis Phillips, *Comparing and Scaling: Ratio, Proportion, and Percent* in Connected Mathematics Project, Upper Saddle River NJ, Pearson Prentice Hall, 2004. ISBN: 0131656352. (See pages 64–66.)

[4] *Paper Pool Game*, NCTM (National Council of Teachers of Mathematics) Illuminations, `https://illuminations.nctm.org/Unit.aspx?id=6527`.

[5] Antonella Perucca, *Arithmetic billiards*, Plus Magazine, 24 April 2018, University of Cambridge, `https://plus.maths.org/content/arithmetic-billiards-0`.

[6] Hugo Steinhaus, *Mathematical Snapshots*, Dover Recreational Math. Series, Courier Corporation, 1999. ISBN:0486409147. (See page 63.)

[7] James Tanton, *Mathematics Galore!: The First Five Years of the St. Mark's Institute of Mathematics (Classroom Resource Materials)*, The Mathematical Association of America, 2012. ISBN: 0883857766. (See pages 145–156.)

[8] Blake Thornton, *Greatest common divisor*, The Washington University Math Teachers' Circle, 2008, `http://www.dehn.wustl.edu/~blake/circles/teacher_circle/talks/2008-Apr24-Blake_Thornton-gcd.pdf`. (See pages 3–4.)

[9] Wikipedia contributors, *Arithmetic billiards*, Wikipedia, The Free Encyclopedia, `https://en.wikipedia.org/w/index.php?title=Arithmetic_billiards&oldid=889358328` (accessed April 19, 2021).

[10] Joshua Zucker, *The Bouncing Ball Problem*, San Jose Math Circle, 2007, `http://sanjosemathcircle.org/handouts/2007-2008/20071003.pdf`. (See page 1.)