

Gaussian approximation for sums of region-stabilizing scores*

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Abstract

We consider the Gaussian approximation for functionals of a Poisson process that are expressible as sums of region-stabilizing (determined by the points of the process within some specified regions) score functions and provide a bound on the rate of convergence in the Wasserstein and the Kolmogorov distances. While such results have previously been shown in Lachièze-Rey, Schulte and Yukich (2019), we extend the applicability by relaxing some conditions assumed there and provide further insight into the results. This is achieved by working with stabilization regions that may differ from balls of random radii commonly used in the literature concerning stabilizing functionals. We also allow for non-diffuse intensity measures and unbounded scores, which are useful in some applications. As our main application, we consider the Gaussian approximation of number of minimal points in a homogeneous Poisson process in $[0, 1]^d$ with $d \geq 2$, and provide a presumably optimal rate of convergence.

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1 Introduction

Let $(\mathbb{X}, \mathcal{F})$ be a Borel space and let \mathbb{Q} be a σ -finite measure on $(\mathbb{X}, \mathcal{F})$. For $s \geq 1$, let \mathcal{P}_s denote a Poisson process with intensity measure $s\mathbb{Q}$. Our main object of study is the sum of score functions $(\xi_s)_{s \geq 1}$ given by

$$H_s = H_s(\mathcal{P}_s) := \sum_{x \in \mathcal{P}_s} \xi_s(x, \mathcal{P}_s), \quad s \geq 1, \tag{1.1}$$

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when the sum converges. While H_s is a functional of the whole point process, this representation implicitly assumes that the functional can be decomposed as a sum of local contributions at each point $x \in \mathcal{P}_s$. Indeed, in the vast literature on limit theorems for sums of score functions over points in a Poisson process (see, e.g., [14, 15, 16]), it is usually assumed that the score function at a point x depends on the whole point process only through the set of its points within some small (random) distance to x , prohibiting any long-range interactions. Conditions like exponential decay of the tail distribution of this distance, so-called ‘radius of stabilization’, and bounds on certain moments of the score functions are crucial to derive a quantitative central limit theorem. The idea of using *stabilization* for studying limit theorems started with the works [13, 14]. Subsequently, important further works advanced such quantitative results for the Gaussian approximation of stabilizing functionals, see, e.g., [2, 15, 17]. But all these results provided bounds that had an extraneous logarithmic factor multiplied to the inverse of the square root of the variance. The results in this area culminated in [10], where, using Malliavin-Stein approach, this logarithmic factor was removed, and further in [9], providing presumably optimal rates and ready-to-use conditions illustrated with numerous applications.

The comparative simplicity of the bounds provided in [9] comes at the cost of assuming a few conditions on the underlying space and the score functions. Even though these conditions are satisfied in many important examples as demonstrated therein, they are not applicable in some cases, especially, in examples exhibiting long-range interactions. A notable example is the number of minimal (or Pareto optimal) points in \mathcal{P}_s restricted to the unit cube $[0, 1]^d$, $d \geq 2$. This example violates all existing stabilization conditions usually assumed in the context of quantitative limit theorems. In particular, the appearance of stabilization regions that can be arbitrarily thin and long makes the radius of stabilization too large to obtain a meaningful bound using results from [9]. As a result, [9] could only manage to handle (in the problem of counting maximal points, which is equidistributed as the number of minimal points) a modified setting, by replacing the cube with a domain of the form $\{x \in [0, \infty)^d : F(x) \leq 1\}$, where $F : [0, \infty)^d \rightarrow [0, \infty)$ is strictly increasing in each coordinate with $F(0) < 1$, is continuously differentiable, and has continuous partial derivatives that are bounded away from zero and infinity. Even though one can define a function F to obtain a domain that is arbitrarily close to the cube, the behavior of the number of maximal points is very sensitive to small changes in the shape of the domain: while the variance of H_s is of the order of $s^{(d-1)/d}$ in the setting of [9], its order becomes $\log^{d-1} s$ in the case of the cube, see [1].

The main aim of this paper is to develop a more versatile notion of stabilization that enables us to handle various examples with long-range interactions, most notably the example of minimal points in the cube. We achieve this by generalizing the concept of stabilization radius to allow for regions of arbitrary shape, that is, by replacing balls of random radii with general sets, called stabilization regions. It is unlikely to achieve this by amending the metric on the carrier space, since the shape of these stabilization regions may be random and depend heavily on the reference point, and also since the stabilization region may be empty. The only additional condition we assume is that the stabilization region is monotonically decreasing in the point configuration, which is a natural condition satisfied by all common examples.

In addition, we also extend the results to non-diffuse intensity measures and to score functions with non-uniform bounds on their moments. The extension to non-diffuse intensity measures results from getting rid of some regularity assumption on \mathbb{Q} imposed in [9]. This makes it possible to handle examples with multiple points at deterministic locations, like Poisson processes on lattices. The extension to scores with unbounded moments is crucial in examples where the score functions are not simple indicators but

rather involve unbounded weight functions, or when the intensity measure is infinite. Such an extension is a byproduct of our generalization of [10, Theorem 6.1], which involves non-uniform bounds on the $(4 + p)$ -th moment of the first order difference operator for some $p > 0$, see Theorem 5.1. We present two examples concerning isolated points in the two-dimensional integer lattice and a random geometric graph in \mathbb{R}^d , $d \geq 2$, to demonstrate further applications of our general bounds. Apart from the fact that our approach is more versatile than that of [9], to the best of our knowledge, working with general monotonically decreasing stabilization sets is new in the relevant literature and thus our work opens a new direction of investigation. It should be noted that the very comprehensive setting in [9] also covers the cases of Poisson processes with marks, as well as the setting of binomial processes. Our results can be extended to these settings by adapting the scheme elaborated in [9] to our approach relying on stabilization regions. Indeed, Theorem 4.2 in [9] providing a bound on Gaussian approximation for functionals of a binomial process can be modified to the setting with a non-uniformly bounded $(4 + p)$ -th moment of the difference operator in the same way we modify Theorem 6.1 in [10] in our Theorem 5.1. Once this key step is achieved, one can follow our line of argument to obtain a result paralleling our Theorem 2.1 for binomial processes.

Let us now explicitly describe our setup. For a Borel space $(\mathbb{X}, \mathcal{F})$, denote by \mathbf{N} the family of σ -finite counting measures μ on \mathbb{X} equipped with the smallest σ -algebra \mathcal{N} such that the maps $\mu \mapsto \mu(A)$ are measurable for all $A \in \mathcal{F}$. We write $x \in \mu$ if $\mu(\{x\}) \geq 1$. Denote by 0 the zero counting measure. Further, μ_A denotes the restriction of μ onto the set $A \in \mathcal{F}$, and δ_x is the Dirac measure at $x \in \mathbb{X}$. For $\mu_1, \mu_2 \in \mathbf{N}$, we write $\mu_1 \leq \mu_2$ if the difference $\mu_2 - \mu_1$ is non-negative.

For each $s \geq 1$, a *score function* ξ_s associates to each pair (x, μ) with $x \in \mathbb{X}$ and $\mu \in \mathbf{N}$, a real number $\xi_s(x, \mu)$. Throughout, we assume that the function $\xi_s : \mathbb{X} \times \mathbf{N} \rightarrow \mathbb{R}$ is measurable with respect to the product σ -algebra $\mathcal{F} \otimes \mathcal{N}$ for all $s \geq 1$.

With H_s as in (1.1), our aim is to find an upper bound on the distance between the distributions of the normalized sum of scores $(H_s - \mathbf{E}H_s)/\sqrt{\text{Var} H_s}$ and a standard normal random variable N in an appropriate distance. We consider two very commonly used distances, namely, the Wasserstein and the Kolmogorov distances. The Wasserstein distance between (the distributions of) real-valued random variables X and Y is given by

$$d_W(X, Y) := \sup_{h \in \text{Lip}_1} |\mathbf{E} h(X) - \mathbf{E} h(Y)|,$$

where Lip_1 denotes the class of all Lipschitz functions $h : \mathbb{R} \rightarrow \mathbb{R}$ with Lipschitz constant at most one. The Kolmogorov distance between X and Y is defined by taking the test functions to be indicators of half-lines, and is given by

$$d_K(X, Y) := \sup_{t \in \mathbb{R}} |\mathbf{P}\{X \leq t\} - \mathbf{P}\{Y \leq t\}|.$$

Following [9], a score function stabilizes if $\xi_s(x, \mu)$ remains unaffected when the configuration μ is altered outside a ball of radius $r_x = r_x(\mu)$ (the radius of stabilization) centered at x . For this, it is assumed that \mathbb{X} is a semimetric space and \mathbb{Q} satisfies a technical condition concerning the \mathbb{Q} -content of an annulus in the space \mathbb{X} , which in particular implies that \mathbb{Q} is diffuse. In [9], under an exponential decay condition on the tail distribution of the stabilization radius r_x as $s \rightarrow \infty$ and assuming that the $(4 + p)$ -th moment of the score function at x is uniformly bounded by a constant for all $s \geq 1$ and $x \in \mathbb{X}$ for some $p \in (0, 1]$, a universal bound on the Wasserstein and Kolmogorov distances between the normalized sum of scores and N was derived.

The setting of stabilization regions as balls centered at $x \in \mathcal{P}_s$ with radius r_x can be thought of as a special case of a more general concept of stabilization regions which are

sets depending on x and the Poisson process. Indeed, in some examples, it is not optimal to assume that the stabilization region is a ball. The region can be made substantially smaller if it is allowed to be of a general shape. Adjusting the theory to deal with such stabilization regions is the main contribution of our work. Our general setting of non-spherical stabilization regions also eliminates the need of extra technical assumptions on the intensity measure imposed in [9]. As an illustration, we show how to handle the example of minimal points in the unit cube, which does not fit into the framework of [9]. We also allow for multiple points and for a non-uniform bound on the $(4 + p)$ -th moment of the score functions, which is particularly important in examples involving infinite intensity measures, like stationary Poisson processes. Apart from examples presented in the current paper, further applications of our method has been elaborated in [5], where a quantitative central limit theorem is obtained for functionals of growth processes that result in generalized Johnson-Mehl tessellations, and in [4], where such a result is obtained in the context of minimal directed spanning trees in dimensions three and higher, respectively.

2 Notation and main results

Throughout the paper, for $s \geq 1$, we consider a $\mathcal{F} \otimes \mathcal{N}$ -measurable score function $\xi_s(x, \mu)$. Assume that if $\xi_s(x, \mu_1) = \xi_s(x, \mu_2)$ for some $\mu_1, \mu_2 \in \mathbf{N}$ with $0 \neq \mu_1 \leq \mu_2$, then

$$\xi_s(x, \mu_1) = \xi_s(x, \mu') \quad \text{for all } \mu' \in \mathbf{N} \text{ with } \mu_1 \leq \mu' \leq \mu_2. \quad (2.1)$$

This is a natural condition to expect for any reasonably well-behaved score function. We will need a few more assumptions on the score functions. The first assumption is a generalization of the concept of stabilization radius.

(A1) *Stabilization region:* For all $s \geq 1$, there exists a map R_s from $\{(x, \mu) \in \mathbb{X} \times \mathbf{N} : x \in \mu\}$ to \mathcal{F} such that

(A1.1) the set

$$\{(x, y_1, y_2, \mu) : \{y_1, y_2\} \subseteq R_s(x, \mu + \delta_x)\}$$

is measurable with respect to the product σ -algebra on $\mathbb{X}^3 \times \mathbf{N}$,

(A1.2) the map R_s is monotonically decreasing in the second argument, i.e.

$$R_s(x, \mu_1) \supseteq R_s(x, \mu_2), \quad \mu_1 \leq \mu_2, \quad x \in \mu_1,$$

(A1.3) for all $\mu \in \mathbf{N}$ and $x \in \mu$, $\mu_{R_s(x, \mu)} \neq 0$ implies $(\mu + \delta_y)_{R_s(x, \mu + \delta_y)} \neq 0$ for all $y \notin R_s(x, \mu)$,

(A1.4) for all $\mu \in \mathbf{N}$ and $x \in \mu$,

$$\xi_s(x, \mu) = \xi_s(x, \mu_{R_s(x, \mu)}).$$

By taking the intersection of the set from (A1.1) with the set $\{(x, y, y, \mu) : \mu \in \mathbf{N}\} \subseteq \mathbb{X}^3 \times \mathbf{N}$ (which is also measurable) and then applying the bijective projection on \mathbf{N} we see that

$$\{\mu \in \mathbf{N} : y \in R_s(x, \mu + \delta_x)\} \in \mathcal{N} \quad (2.2)$$

for all $(x, y) \in \mathbb{X}^2$. Furthermore, Fubini's theorem implies that

$$\mathbf{P} \{y \in R_s(x, \mathcal{P}_s + \delta_x)\} \quad \text{and} \quad \mathbf{P} \{\{y_1, y_2\} \subseteq R_s(x, \mathcal{P}_s + \delta_x)\} \quad (2.3)$$

are Lebesgue measurable functions of $(x, y) \in \mathbb{X}^2$ and $(x, y_1, y_2) \in \mathbb{X}^3$, respectively. Even though, assumption (A1.1) is sufficient for our result, it is indeed enough to assume (2.2)

and (2.3). Thus, when simpler, we will verify the conditions (2.2) and (2.3) instead of (A1.1).

Note that (A1) holds trivially if one takes R_s to be identically equal to the whole space \mathbb{X} . If (A1) holds with a non-trivial R_s , then the score function is called *region-stabilizing*. Also note that a condition like [9, Eq. (2.3)], requiring stabilization with 7 additional points, trivially holds in our set up due to the monotonicity assumption (A1.2) and (2.1).

We also assume the standard $(4 + p)$ -th moment condition, stated here in terms of the norm for notational simplicity. In the following, $\|\cdot\|_{4+p}$ denotes the L^{4+p} -norm.

(A2) L^{4+p} -norm: There exists a $p \in (0, 1]$ such that, for all $\mu \in \mathbb{N}$ with $\mu(\mathbb{X}) \leq 7$,

$$\left\| \xi_s(x, \mathcal{P}_s + \delta_x + \mu) \right\|_{4+p} \leq M_{s,p}(x), \quad s \geq 1, \quad x \in \mathbb{X},$$

where $M_{s,p} : \mathbb{X} \rightarrow \mathbb{R}$, $s \geq 1$, are measurable functions.

If the score function is an indicator random variable, Condition (A2) is trivially satisfied with $M_{s,p} \equiv 1$ for any $p \in (0, 1]$ and $s \geq 1$. For notational convenience, in the sequel we will write M_s instead of $M_{s,p}$, and generally drop p from all subscripts.

Let $r_s : \mathbb{X} \times \mathbb{X} \rightarrow [0, \infty]$ be a measurable function such that

$$\mathbf{P} \{y \in R_s(x, \mathcal{P}_s + \delta_x)\} \leq e^{-r_s(x,y)}, \quad x, y \in \mathbb{X}. \quad (2.4)$$

For the following it is essential that r_s does not vanish, and then (2.4) becomes an analog of the usual exponential stabilization condition from [9]. Note that we allow r_s to be infinite and the probability in (2.4) is well defined due to assumption (A1.1).

For $x_1, x_2 \in \mathbb{X}$, denote

$$q_s(x_1, x_2) := s \int_{\mathbb{X}} \mathbf{P} \{ \{x_1, x_2\} \subseteq R_s(z, \mathcal{P}_s + \delta_z) \} \mathbf{Q}(dz), \quad (2.5)$$

noticing that the probability in the integral is well defined and q_s is measurable due to Fubini's theorem and (2.3).

For $p \in (0, 1]$ as in (A2) and $\zeta := p/(40 + 10p)$, let

$$g_s(y) := s \int_{\mathbb{X}} e^{-\zeta r_s(x,y)} \mathbf{Q}(dx), \quad h_s(y) := s \int_{\mathbb{X}} M_s(x)^{4+p/2} e^{-\zeta r_s(x,y)} \mathbf{Q}(dx), \quad (2.6)$$

$$G_s(y) := \widetilde{M}_s(y) + \tilde{h}_s(y)(1 + g_s(y)^4), \quad y \in \mathbb{X}, \quad (2.7)$$

where for $y \in \mathbb{X}$,

$$\widetilde{M}_s(y) := \max\{M_s(y)^2, M_s(y)^4\} \quad \text{and} \quad \tilde{h}_s(y) := \max\{h_s(y)^{2/(4+p/2)}, h_s(y)^{4/(4+p/2)}\}.$$

For $\alpha > 0$, let

$$f_\alpha(y) := f_\alpha^{(1)}(y) + f_\alpha^{(2)}(y) + f_\alpha^{(3)}(y), \quad y \in \mathbb{X}, \quad (2.8)$$

where

$$\begin{aligned} f_\alpha^{(1)}(y) &:= s \int_{\mathbb{X}} G_s(x) e^{-\alpha r_s(x,y)} \mathbf{Q}(dx), \\ f_\alpha^{(2)}(y) &:= s \int_{\mathbb{X}} G_s(x) e^{-\alpha r_s(y,x)} \mathbf{Q}(dx), \\ f_\alpha^{(3)}(y) &:= s \int_{\mathbb{X}} G_s(x) q_s(x, y)^\alpha \mathbf{Q}(dx). \end{aligned} \quad (2.9)$$

Finally, define the function

$$\kappa_s(x) := \mathbf{P} \{ \xi_s(x, \mathcal{P}_s + \delta_x) \neq 0 \}, \quad x \in \mathbb{X}. \quad (2.10)$$

Our main result is the following abstract theorem, which generalizes Theorem 2.1(a) in [9]. For an integrable function $f : \mathbb{X} \rightarrow \mathbb{R}$, denote $\mathbf{Q}f := \int_{\mathbb{X}} f(x) \mathbf{Q}(dx)$.

Theorem 2.1. Assume that $(\xi_s)_{s \geq 1}$ satisfy conditions (A1), (A2) and let H_s be as in (1.1). Then, for p as in (A2) and $\beta := p/(32 + 4p)$,

$$d_W \left(\frac{H_s - \mathbf{E}H_s}{\sqrt{\text{Var } H_s}}, N \right) \leq C \left[\frac{\sqrt{s\mathbb{Q}f_\beta^2}}{\text{Var } H_s} + \frac{s\mathbb{Q}((\kappa_s + g_s)^{2\beta}G_s)}{(\text{Var } H_s)^{3/2}} \right],$$

and

$$d_K \left(\frac{H_s - \mathbf{E}H_s}{\sqrt{\text{Var } H_s}}, N \right) \leq C \left[\frac{\sqrt{s\mathbb{Q}f_\beta^2} + \sqrt{s\mathbb{Q}f_{2\beta}}}{\text{Var } H_s} + \frac{\sqrt{s\mathbb{Q}((\kappa_s + g_s)^{2\beta}G_s)}}{\text{Var } H_s} + \frac{s\mathbb{Q}((\kappa_s + g_s)^{2\beta}G_s)}{(\text{Var } H_s)^{3/2}} \right. \\ \left. + \frac{(s\mathbb{Q}((\kappa_s + g_s)^{2\beta}G_s))^{5/4} + (s\mathbb{Q}((\kappa_s + g_s)^{2\beta}G_s))^{3/2}}{(\text{Var } H_s)^2} \right]$$

for all $s \geq 1$, where N is a standard normal random variable and $C \in (0, \infty)$ is a constant depending only on p .

In order to obtain a useful bound, it is necessary that $\mathbb{Q}(\widetilde{M}_s \kappa_s)$ is finite. This is surely the case if \mathbb{Q} is finite and \widetilde{M}_s is bounded.

As an application of our abstract result, we consider an example regarding *minimal points* in a Poisson process. Let \mathbb{Q} be the Lebesgue measure on $\mathbb{X} := [0, 1]^d$, $d \geq 2$, and let \mathcal{P}_s be a Poisson process with intensity $s\mathbb{Q}$ for $s \geq 1$. A point $x \in \mathbb{R}^d$ is said to dominate a point $y \in \mathbb{R}^d$ if $x - y \in \mathbb{R}_+^d \setminus \{0\}$. We write $x \succ y$, or equivalently, $y \prec x$ if x dominates y . Points in \mathcal{P}_s that do not dominate any other point in \mathcal{P}_s are called minimal (or Pareto optimal) points of \mathcal{P}_s . The interest in studying dominance and number of minima and maxima is due to its numerous applications related to multivariate records, e.g., in the analysis of linear programming and in maxima-finding algorithms, see the references in [1] and [7]. In the following result, we derive non-asymptotic bounds on the Wasserstein and Kolmogorov distances between the normalized number of minimal points in \mathcal{P}_s , and a standard Gaussian random variable.

Theorem 2.2. Let \mathcal{P}_s be a Poisson process on $[0, 1]^d$ with intensity measure $s\mathbb{Q}$ and $s \geq 1$, where \mathbb{Q} is the Lebesgue measure, and let

$$F_s := \sum_{x \in \mathcal{P}_s} \mathbb{1}_{x \text{ is a minimal point in } \mathcal{P}_s}. \tag{2.11}$$

If $d \geq 2$, then

$$\max \left\{ d_W \left(\frac{F_s - \mathbf{E}F_s}{\sqrt{\text{Var } F_s}}, N \right), d_K \left(\frac{F_s - \mathbf{E}F_s}{\sqrt{\text{Var } F_s}}, N \right) \right\} \leq \frac{C}{\log^{(d-1)/2} s}, \quad s \geq 1,$$

for a constant $C > 0$ depending only on the dimension d . In addition, the bound on the Kolmogorov distance is of optimal order, i.e., there exists a constant $0 < C' \leq C$ depending only on d such that $d_K \left(\frac{F_s - \mathbf{E}F_s}{\sqrt{\text{Var } F_s}}, N \right) \geq C' / \log^{(d-1)/2} s$.

In the setting of binomial point process with $n \in \mathbb{N}$ i.i.d. points in the unit cube, [1] showed that the Wasserstein distance between the normalized number of minimal points and the standard normal random variable is of the order $(\log n)^{-(d-1)/2} (\log \log n)^{2d}$ using a log-transformation trick first suggested in [3], and, as a consequence, derived the order $(\log n)^{-(d-1)/4} (\log \log n)^d$ for the Kolmogorov distance. It is useful to note here that the variance of the number of minimal points in the binomial case is of the order $\log^{d-1} n$, see, e.g., [1], where the corresponding computations in the Poisson case are also available. Hence, the Wasserstein distance is of the order of the square root of the variance multiplied by an extraneous logarithmic factor, which, as mentioned before,

has commonly appeared in such contexts. Furthermore, the bound on the Kolmogorov distance is vastly suboptimal. Our result in the Poisson setting substantially improves these rates to the square root of the variance of F_s , which is optimal for the Kolmogorov distance and presumably optimal for the Wasserstein distance.

It should be noted that, in the example of Pareto optimal points, we are working with a simple Poisson process and a finite intensity measure \mathbb{Q} . Further examples confirm that our abstract bound applies also for Poisson processes with a non-diffuse or infinite intensity measure \mathbb{Q} . Note that for measures with infinite intensity, [9] requires that the score function decays exponentially with respect to the distance to some set K , and the bound in Eq. (2.10) therein becomes trivial if this set K is the whole space and \mathbb{Q} is infinite.

The rest of the paper is organized as follows. In Section 3 we prove Theorem 2.2. Section 4 provides two examples in settings, where either the intensity measure is infinite and non-diffuse or the $(4 + p)$ -th moments of the score functions are unbounded over the space \mathbb{X} , and provide bounds on the rate of convergences in the Wasserstein and the Kolmogorov distances for Gaussian approximation of certain statistics related to isolated points in these models. Finally, in Section 5 we prove Theorem 2.1 which relies on a modified version of Theorem 6.1 in [10], see Theorem 5.1. The proof of the latter is presented in the Appendix.

3 Number of minimal points in the hypercube

In this section, we apply Theorem 2.1 to prove Theorem 2.2 providing a quantitative limit theorem for the number of minimal points in a Poisson process on the hypercube. Throughout this section, \mathbb{Q} is taken to be the Lebesgue measure on $\mathbb{X} := [0, 1]^d$ with $d \in \mathbb{N}$, and \mathcal{P}_s is a Poisson process on \mathbb{X} with intensity measure $s\mathbb{Q}$ for $s \geq 1$. We omit \mathbb{Q} in integrals and write dx instead of $\mathbb{Q}(dx)$. The functional F_s from (2.11) can be expressed as in (1.1) with the score functions

$$\xi_s(x, \mu) := \mathbb{1}_{x \text{ is a minimal point in } \mu}, \quad x \in \mu, \mu \in \mathbf{N}. \tag{3.1}$$

As a convention, we let $\xi_s(x, 0) = 0$. It is straightforward to see that $(\xi_s)_{s \geq 1}$ satisfies (2.1). We will show that conditions (A1) and (A2) also hold, so that Theorem 2.1 is applicable.

For $x := (x^{(1)}, \dots, x^{(d)}) \in \mathbb{X}$, let $[0, x] := [0, x^{(1)}] \times \dots \times [0, x^{(d)}]$, and denote the volume of $[0, x]$ by

$$|x| := x^{(1)} \dots x^{(d)}.$$

Given a counting measure $\mu \in \mathbf{N}$ and $x \in \mu$, define the stabilization region as

$$R_s(x, \mu) := \begin{cases} [0, x] & \text{if } \mu([0, x] \setminus \{x\}) = 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

To begin with, we note here that the region R_s can be the empty set in our case, which rules out any possibility of it being represented as a ball in some metric on the space \mathbb{X} . Since for any $x \in \mathbb{X}$, the mapping $\mathbf{N} \ni \mu \mapsto \mu([0, x] \setminus \{x\})$ is measurable, the condition in (2.2) follows. Next, it is easy to see that for $x, y \in \mathbb{X}$,

$$\mathbf{P} \{y \in R_s(x, \mathcal{P}_s + \delta_x)\} = \mathbb{1}_{x \succ y} e^{-s|x|}, \tag{3.2}$$

which is clearly measurable. Denote by $x_1 \vee \dots \vee x_n$ the coordinatewise maximum of $x_1, \dots, x_n \in \mathbb{X}$, while $x_1 \wedge \dots \wedge x_n$ denotes their coordinatewise minimum. For $x_1, x_2 \in \mathbb{X}$, notice that $\{x_1, x_2\} \subseteq R_s(z, \mathcal{P}_s + \delta_z)$ if and only if $z \succ (x_1 \vee x_2)$ and $[0, z] \setminus \{z\}$ has no points of \mathcal{P}_s . Thus

$$\mathbf{P} \{\{x_1, x_2\} \subseteq R_s(z, \mathcal{P}_s + \delta_z)\} = \mathbb{1}_{z \succ x_1 \vee x_2} e^{-s|z|}, \tag{3.3}$$

which is also a measurable function of $(z, x_1, x_2) \in \mathbb{X}^3$, confirming (2.3). Clearly, R_s is monotonically decreasing in its second argument. It is straightforward to check (A1.3). Finally, with ξ_s as defined at (3.1), it is easy to see that (A1.4) is satisfied. Furthermore, condition (A2) holds trivially with $M_s \equiv 1$ for all $p \in (0, 1]$ and $s \geq 1$, since ξ_s is an indicator function. For definiteness, take $p = 1$.

For ξ_s as in (3.1), by (3.2) the inequality (2.4) turns into an equality with $r_s(x, y) := s|x|$ if $y \prec x$ and $r_s(x, y) := \infty$ if y is not dominated by x .

Throughout the section, for a function $f : [1, \infty) \rightarrow \mathbb{R}_+$, we will write $f(s) = \mathcal{O}(\log^{d-1} s)$ to mean that $f(s)/\log^{d-1} s$ is uniformly bounded for all $s \geq 1$. It is well known (see, e.g., [1]) that for all $\alpha > 0$,

$$s \int_{\mathbb{X}} e^{-\alpha s|x|} dx = \mathcal{O}(\log^{d-1} s). \tag{3.4}$$

In particular, by the Mecke formula, $\mathbf{E}F_s = s \int_{\mathbb{X}} e^{-s|x|} dx = \mathcal{O}(\log^{d-1} s)$. Further, by the multivariate Mecke formula (see, e.g., [11, Th. 4.4]),

$$\begin{aligned} \text{Var}(F_s) &= \mathbf{E}F_s - (\mathbf{E}F_s)^2 \\ &+ s^2 \iint_D \mathbf{P}\{x \text{ and } y \text{ are both minimal points in } \mathcal{P}_s + \delta_x + \delta_y\} dx dy, \end{aligned}$$

where D is the set of $(x, y) \in \mathbb{X}^2$ such that x and y are incomparable, i.e., $x \not\prec y$ and $y \not\prec x$. Hence, following the proof of Theorem 1 in [1], there exist finite positive constants C_1 and C_2 such that

$$C_1 \log^{d-1} s \leq \text{Var}(F_s) \leq C_2 \log^{d-1} s, \quad s \geq 1. \tag{3.5}$$

For $\alpha > 0$, $s > 0$, and $d \in \mathbb{N}$, define the function $c_{\alpha,s} : \mathbb{X} \rightarrow \mathbb{R}_+$ as

$$c_{\alpha,s}(y) := s \int_{\mathbb{X}} \mathbb{1}_{x \succ y} e^{-\alpha s|x|} dx. \tag{3.6}$$

In view of the Mecke formula and the Poisson empty space formula, $c_{1,s}(y)$ is the expected number of minimal points in \mathcal{P}_s that dominate $y \in \mathbb{X}$. Also note that $g_s(y)$ and $h_s(y)$ from (2.6) is equal to $c_{\zeta,s}(y)$ with $\zeta = p/(40 + 10p) = 1/50$, so that $G_s(y) \leq 3 + 2c_{\zeta,s}(y)^5$.

Next, we specify the function q_s from (2.5). By (3.3), we have

$$q_s(x_1, x_2) = s \int_{\mathbb{X}} \mathbb{1}_{z \succ (x_1 \vee x_2)} e^{-s|z|} dz = c_{1,s}(x_1 \vee x_2).$$

Studying the function $c_{\alpha,s}$ is essential to understand the behaviour of minimal points. Note that $c_{\alpha,s}$ satisfies the scaling property

$$c_{\alpha,s}(y) = \alpha^{-1} c_{1,\alpha s}(y), \quad \alpha > 0, s > 0. \tag{3.7}$$

This will often enable us to take $\alpha = 1$ without loss of generality. The following lemma demonstrates the asymptotic behaviour of the function $c_{\alpha,s}$ for large s . Before we state the result, notice that for $i \in \mathbb{N} \cup \{0\}$ and $\alpha > 0$,

$$\int_0^\infty |\log w|^i e^{-\alpha w} dw \leq \int_0^1 |\log w|^i dw + \int_1^\infty w^i e^{-\alpha w} dw \leq \int_0^1 |\log w|^i dw + \frac{\Gamma(i+1)}{\alpha^{i+1}}.$$

Since any positive integer power of logarithm is integrable near zero, for all $i \in \mathbb{N} \cup \{0\}$ and $\alpha > 0$,

$$\int_0^\infty |\log w|^i e^{-\alpha w} dw < \infty. \tag{3.8}$$

Lemma 3.1. For all $\alpha > 0$ and $s > 0$,

$$c_{\alpha,s}(y) \leq \frac{D}{\alpha} e^{-\alpha s|y|/2} \left[1 + |\log(\alpha s|y|)|^{d-1} \right], \quad y \in \mathbb{X}$$

for a constant D that depends only on the dimension $d \in \mathbb{N}$.

Proof. The result is trivial when $d = 1$, so we assume $d \geq 2$. By (3.7), we can also assume that $\alpha = 1$. The following derivation is motivated by those used to calculate the mean of the number of minimal points in [1, Sec. 2]. Changing variables $u = s^{1/d}x$ in the definition of $c_{1,s}$ to obtain the first equality, and letting $z^{(i)} = -\log u^{(i)}$, $i = 1, \dots, d$, in the second, for $y \in \mathbb{X}$, we obtain

$$\begin{aligned} c_{1,s}(y) &= \int_{\times_{i=1}^d [s^{1/d}y^{(i)}, s^{1/d}]} e^{-|u|} du \\ &= \int_{\times_{i=1}^d [-d^{-1} \log s, -d^{-1} \log s - \log y^{(i)}]} \exp \left\{ -e^{-\sum_{j=1}^d z^{(j)}} - \sum_{j=1}^d z^{(j)} \right\} dz. \end{aligned}$$

Next, we change variables by letting $v = (v^{(1)}, \dots, v^{(d)})$ with $v^{(i)} := z^{(i)} + \dots + z^{(d)}$, $i = 1, \dots, d$. Note that the integrand is only a function of $v^{(1)}$. Taking into account the integration bounds on $z^{(i)}$, we have

$$v^{(1)} - \left(-\frac{i-1}{d} \log s - \sum_{j=1}^{i-1} \log y^{(j)} \right) \leq v^{(i)} \leq -\frac{d-i+1}{d} \log s - \sum_{j=i}^d \log y^{(j)}, \quad 2 \leq i \leq d.$$

Thus, for each $2 \leq i \leq d$, the integration variable $v^{(i)}$ belongs to an interval of length at most $(-\log(s|y|) - v^{(1)})$. Using the substitution $w = e^{-v^{(1)}}$ in the second step and Jensen's inequality in the last one, we obtain

$$\begin{aligned} c_{1,s}(y) &\leq \int_{-\log s}^{-\log(s|y|)} \left(-\log(s|y|) - v^{(1)} \right)^{d-1} \exp \left\{ -e^{-v^{(1)}} - v^{(1)} \right\} dv^{(1)} \\ &= \int_{s|y|}^s \left(\log w - \log(s|y|) \right)^{d-1} e^{-w} dw \\ &\leq 2^{d-2} e^{-s|y|/2} \left[|\log(s|y|)|^{d-1} + \int_{s|y|}^s |\log w|^{d-1} e^{-w/2} dw \right]. \end{aligned}$$

The result now follows by (3.8). □

Before proceeding to estimate the bound in Theorem 2.1, we need some estimates of integrals involving $c_{\alpha,s}$ and $|x|$. We will often use the following representation: for $\alpha > 0$, $s \geq 1$ and $i \in \mathbb{N}$,

$$\begin{aligned} s \int_{\mathbb{X}} c_{\alpha,s}(x)^i dx &= s \int_{\mathbb{X}} \prod_{j=1}^i \left(s \int_{\mathbb{X}} \mathbb{1}_{z_j \succ x} e^{-\alpha s \sum_{j=1}^i |z_j|} dz_j \right) dx \\ &= s^{i+1} \int_{\mathbb{X}^i} |z_1 \wedge \dots \wedge z_i| e^{-\alpha s \sum_{j=1}^i |z_j|} d(z_1, \dots, z_i). \end{aligned} \tag{3.9}$$

Lemma 3.2. For all $i \in \mathbb{N}$ and $\alpha > 0$,

$$s \int_{\mathbb{X}} c_{\alpha,s}(y)^i dy = \mathcal{O}(\log^{d-1} s), \tag{3.10}$$

$$s \int_{\mathbb{X}} \left(s \int_{\mathbb{X}} e^{-\alpha s|x \vee y|} dx \right)^i dy = \mathcal{O}(\log^{d-1} s), \tag{3.11}$$

$$s \int_{\mathbb{X}} \left(s \int_{\mathbb{X}} c_{\alpha,s}(x \vee y) dx \right)^i dy = \mathcal{O}(\log^{d-1} s), \tag{3.12}$$

where the constants in the bounds on the right-hand sides may depend on i .

Proof. As in Lemma 3.1, without loss of generality let $\alpha = 1$ and $s \geq 1$. We first prove (3.10). For $i \in \mathbb{N}$, by Lemma 3.1 and Jensen's inequality, we have

$$s \int_{\mathbb{X}} c_{1,s}(y)^i dy \leq 2^{i-1} D^i \left[s \int_{\mathbb{X}} e^{-is|y|/2} dy + s \int_{\mathbb{X}} e^{-is|y|/2} |\log(s|y|)|^{i(d-1)} dy \right], \quad (3.13)$$

with D as in Lemma 3.1. The first summand is of the order of $\log^{d-1} s$ by (3.4). For the second summand, we employ a similar substitution as in Lemma 3.1 and [1]:

$$\begin{aligned} & s \int_{\mathbb{X}} e^{-is|y|/2} |\log(s|y|)|^{i(d-1)} dy \leq \int_{[0, s^{1/d}]^d} e^{-|u|/2} |\log |u||^{i(d-1)} du \quad (u = s^{1/d}x) \\ & = \int_{[-d^{-1} \log s, \infty)^d} \exp \left\{ -e^{-\frac{1}{2} \sum_{j=1}^d z^{(j)}} - \sum_{j=1}^d z^{(j)} \right\} \left| \sum_{j=1}^d z^{(j)} \right|^{i(d-1)} dz \quad (z^{(j)} = -\log u^{(j)}) \\ & \leq \int_{-\log s}^{\infty} (\log s + v^{(1)})^{d-1} \exp \left\{ -e^{-v^{(1)}/2} - v^{(1)} \right\} |v^{(1)}|^{i(d-1)} dv^{(1)} \quad (v^{(i)} = \sum_{j=i}^d z^{(j)}) \\ & = \int_0^s (\log s - \log w)^{d-1} e^{-\sqrt{w}} |\log w|^{i(d-1)} dw \quad (w = e^{-v^{(1)}}) \\ & \leq 2^{d-2} \left[\log^{d-1} s \int_0^{\infty} e^{-\sqrt{w}} |\log w|^{i(d-1)} dw + \int_0^{\infty} e^{-\sqrt{w}} |\log w|^{(i+1)(d-1)} dw \right], \end{aligned}$$

where the last step is due to Jensen's inequality. Finally, by substituting $t = \sqrt{w}$ and using that $te^{-t/2} \leq 2$ for $t \geq 0$, we have

$$\int_0^{\infty} e^{-\sqrt{w}} |\log w|^j dw = \int_0^{\infty} 2^{1+j} t e^{-t} |\log t|^j dt \leq 2^{2+j} \int_0^{\infty} e^{-t/2} |\log t|^j dt, \quad j \in \mathbb{N}.$$

The result now follows by (3.8).

Next, we move on to proving (3.11). For $x \in \mathbb{X}$ and $I \subseteq \{1, \dots, d\}$, we write x^I for the subvector $(x^{(i)})_{i \in I}$. Assume that $x \vee y = (x^I, y^J)$ with $J := I^c$. Note that by Jensen's inequality, we have

$$s \int_{\mathbb{X}} \left(s \int_{\mathbb{X}} e^{-s|x \vee y|} dx \right)^i dy \leq 2^{(i-1)d} \sum_{I \subseteq \{1, \dots, d\}} s \int_{\mathbb{X}} \left(s \int_{\mathbb{X}} \mathbb{1}_{x^I \succ y^I, x^J \prec y^J} e^{-s|x^I||y^J|} dx \right)^i dy. \quad (3.14)$$

First, if $I = \emptyset$, splitting the exponential into the product of two exponentials with the power halved, using $t^i e^{-t} \leq i!$ for $t \geq 0$, and referring to (3.4) yield that

$$s \int_{\mathbb{X}} \left(s \int_{\mathbb{X}} \mathbb{1}_{x \prec y} e^{-s|y|} dx \right)^i dy = s \int_{\mathbb{X}} (s|y|)^i e^{-is|y|} dy = \mathcal{O}(\log^{d-1} s).$$

Next, assume that I is nonempty and of cardinality m , with $1 \leq m \leq d$. As a convention, let $|y^\emptyset| := 1$ for all $y \in \mathbb{X}$. Using Lemma 3.1 with $\alpha = 1$ and Jensen's inequality in the second step, we obtain

$$\begin{aligned} s \int_{\mathbb{X}} \left(s \int_{\mathbb{X}} \mathbb{1}_{x^I \succ y^I, x^J \prec y^J} e^{-s|x^I||y^J|} dx \right)^i dy &= s \int_{\mathbb{X}} \left(s|y^J| \int_{[0,1]^m} \mathbb{1}_{x^I \succ y^I} e^{-s|x^I||y^J|} dx^I \right)^i dy \\ &\leq D^i 2^{i-1} s \int_{\mathbb{X}} e^{-is|y|/2} [1 + |\log(s|y|)|^{i(m-1)}] dy, \end{aligned}$$

with D as in Lemma 3.1. The two summands can be bounded in the same manner as it was done for (3.13), providing a bound of the order of $\log^{d-1} s$. The bound in (3.11) now follows from (3.14).

Finally, we confirm (3.12). Using that $te^{-t} \leq 1$ for $t \geq 0$ in the first inequality, we have

$$\begin{aligned} & s \int_{\mathbb{X}} \left(s \int_{\mathbb{X}} c_{1,s}(x \vee y) dx \right)^i dy \\ &= s^{2i+1} \int_{\mathbb{X}} \int_{\mathbb{X}^i} \left[\prod_{j=1}^i \int_{\mathbb{X}} \mathbb{1}_{z_j \succ x_j \vee y} e^{-s \sum_{j=1}^i |z_j|} dz_j \right] d(x_1, \dots, x_i) dy \\ &= s^{i+1} \int_{\mathbb{X}^i} \left(s^i \prod_{j=1}^i |z_j| e^{-s \sum_{j=1}^i |z_j|/2} \right) |z_1 \wedge \dots \wedge z_i| e^{-s \sum_{j=1}^i |z_j|/2} d(z_1, \dots, z_i) \\ &\leq 2^i s^{i+1} \int_{\mathbb{X}^i} |z_1 \wedge \dots \wedge z_i| e^{-s \sum_{j=1}^i |z_j|/2} d(z_1, \dots, z_i) \\ &\leq 2^i s \int_{\mathbb{X}} c_{1/2,s}(x)^i dx = \mathcal{O}(\log^{d-1} s), \end{aligned}$$

where we have also used (3.9) in the penultimate step and (3.10) for the final step. \square

Now we are ready to derive the bound in Theorem 2.1. Recall from Section 2 the constants $\beta = p/(32 + 4p)$ and $\zeta = p/(40 + 10p)$, which, in particular, satisfy that $\zeta < 2\beta$. For our example, it suffices to let $p = 1$. Nonetheless, the following bounds are derived for any β and ζ , satisfying the above condition.

Lemma 3.3. *For all $\beta \in (0, 1/2)$, $\zeta \in (0, 2\beta)$ and $f_{2\beta}$ defined at (2.8),*

$$s \int_{\mathbb{X}} f_{2\beta}(x_1) dx_1 = \mathcal{O}(\log^{d-1} s).$$

Proof. We first bound the integral of $f_{2\beta}^{(1)}$ defined at (2.9). By (3.10),

$$s \int_{\mathbb{X}} s \int_{\mathbb{X}} e^{-2\beta r_s(x_2, x_1)} dx_2 dx_1 = s \int_{\mathbb{X}} s \int_{\mathbb{X}} \mathbb{1}_{x_2 \succ x_1} e^{-2\beta s|x_2|} dx_2 dx_1 = \mathcal{O}(\log^{d-1} s).$$

If $x_2 \succ x_1$, then $c_{\zeta,s}(x_2) \leq c_{\zeta,s}(x_1)$. Since $\zeta < 2\beta$, by (3.10),

$$\begin{aligned} s \int_{\mathbb{X}} s \int_{\mathbb{X}} c_{\zeta,s}(x_2)^5 e^{-2\beta r_s(x_2, x_1)} dx_2 dx_1 &\leq s \int_{\mathbb{X}} c_{\zeta,s}(x_1)^5 s \int_{\mathbb{X}} \mathbb{1}_{x_2 \succ x_1} e^{-2\beta s|x_2|} dx_2 dx_1 \\ &\leq s \int_{\mathbb{X}} c_{\zeta,s}(x_1)^6 dx_1 = \mathcal{O}(\log^{d-1} s). \end{aligned} \tag{3.15}$$

Since $G_s(y) \leq 3 + 2c_{\zeta,s}(y)^5$, combining the above two bounds, we obtain

$$s \int_{\mathbb{X}} f_{2\beta}^{(1)}(x_1) dx_1 = \mathcal{O}(\log^{d-1} s).$$

We move on to $f_{2\beta}^{(2)}$. Using again that $te^{-t} \leq 1$ for $t \geq 0$ and (3.4), we have

$$\begin{aligned} s \int_{\mathbb{X}} s \int_{\mathbb{X}} e^{-2\beta r_s(x_1, x_2)} dx_2 dx_1 &= s \int_{\mathbb{X}} s \int_{\mathbb{X}} \mathbb{1}_{x_2 \prec x_1} e^{-2\beta s|x_1|} dx_2 dx_1 \\ &= s \int_{\mathbb{X}} s|x_1| e^{-2\beta s|x_1|} dx_1 \leq s\beta^{-1} \int_{\mathbb{X}} e^{-\beta s|x_1|} dx_1 = \mathcal{O}(\log^{d-1} s). \end{aligned}$$

Also, $\zeta < 2\beta$ and (3.10) yield that

$$\begin{aligned} & s \int_{\mathbb{X}} s \int_{\mathbb{X}} c_{\zeta,s}(x_2)^5 e^{-2\beta r_s(x_1, x_2)} dx_2 dx_1 \\ &\leq s \int_{\mathbb{X}} c_{\zeta,s}(x_2)^5 \left(s \int_{\mathbb{X}} \mathbb{1}_{x_1 \succ x_2} e^{-\zeta s|x_1|} dx_1 \right) dx_2 = s \int_{\mathbb{X}} c_{\zeta,s}(x_2)^6 dx_2 = \mathcal{O}(\log^{d-1} s). \end{aligned}$$

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Thus,

$$s \int_{\mathbb{X}} f_{2\beta}^{(2)}(x_1) dx_1 = \mathcal{O}(\log^{d-1} s).$$

It remains to bound the integral of $f_{2\beta}^{(3)}$. For $\alpha < 1$ and $x \in \mathbb{X}$, we have

$$\begin{aligned} c_{1,s}(x)^\alpha &= e^{-\alpha s|x|} \left(s \int_{\mathbb{X}} \mathbb{1}_{z \succ x} e^{-s(|z|-|x|)} dz \right)^\alpha \leq e^{-\alpha s|x|} \left[1 + s \int_{\mathbb{X}} \mathbb{1}_{z \succ x} e^{-s(|z|-|x|)} dz \right] \\ &\leq e^{-\alpha s|x|} \left[1 + s \int_{\mathbb{X}} \mathbb{1}_{z \succ x} e^{-\alpha s(|z|-|x|)} dz \right] = e^{-\alpha s|x|} + c_{\alpha,s}(x). \end{aligned} \quad (3.16)$$

Thus, noticing that $2\beta < 1$ and using Lemma 3.2,

$$\begin{aligned} s \int_{\mathbb{X}} s \int_{\mathbb{X}} q_s(x_1, x_2)^{2\beta} dx_2 dx_1 &= s \int_{\mathbb{X}} s \int_{\mathbb{X}} c_{1,s}(x_1 \vee x_2)^{2\beta} dx_2 dx_1 \\ &\leq s^2 \int_{\mathbb{X}^2} e^{-2\beta s|x_1 \vee x_2|} d(x_1, x_2) + s^2 \int_{\mathbb{X}^2} c_{2\beta,s}(x_1 \vee x_2) d(x_1, x_2) = \mathcal{O}(\log^{d-1} s). \end{aligned}$$

Finally, using (3.16) and that $\zeta < 2\beta$ for the inequality, write

$$\begin{aligned} s \int_{\mathbb{X}} s \int_{\mathbb{X}} c_{\zeta,s}(x_2)^5 q_s(x_1, x_2)^{2\beta} dx_2 dx_1 &= s \int_{\mathbb{X}} s \int_{\mathbb{X}} c_{\zeta,s}(x_2)^5 c_{1,s}(x_1 \vee x_2)^{2\beta} dx_2 dx_1 \\ &\leq s^8 \int_{\mathbb{X}} \int_{\mathbb{X}} \int_{\mathbb{X}^5} \mathbb{1}_{z_1, \dots, z_5 \succ x_2} \int_{\mathbb{X}} \mathbb{1}_{z_6 \succ x_1 \vee x_2} \exp \left\{ -\zeta s \sum_{i=1}^6 |z_i| \right\} dz_6 d(z_1, \dots, z_5) dx_2 dx_1 \\ &\quad + s \int_{\mathbb{X}} s \int_{\mathbb{X}} c_{\zeta,s}(x_2)^5 e^{-2\beta s|x_1 \vee x_2|} dx_2 dx_1 := A_1 + A_2. \end{aligned}$$

By (3.9) and (3.10),

$$\begin{aligned} A_1 &= s^8 \int_{\mathbb{X}^6} |z_6| |z_1 \wedge \dots \wedge z_6| \exp \left\{ -\zeta s \sum_{i=1}^6 |z_i| \right\} d(z_1, \dots, z_6) \\ &\leq 2s^7 \int_{\mathbb{X}^6} |z_1 \wedge \dots \wedge z_6| \exp \left\{ -\zeta s \sum_{i=1}^6 |z_i|/2 \right\} d(z_1, \dots, z_6) \\ &= 2s \int_{\mathbb{X}} c_{\zeta/2,s}(x)^6 dx = \mathcal{O}(\log^{d-1} s). \end{aligned}$$

Furthermore, by the Cauchy-Schwarz inequality and Lemma 3.2,

$$A_2 \leq \left(s \int_{\mathbb{X}} c_{\zeta,s}(x_2)^{10} dx_2 \right)^{1/2} \left(s \int_{\mathbb{X}} \left(s \int_{\mathbb{X}} e^{-2\beta s|x_1 \vee x_2|} dx_1 \right)^2 dx_2 \right)^{1/2} = \mathcal{O}(\log^{d-1} s).$$

Therefore,

$$s \int_{\mathbb{X}} f_{2\beta}^{(3)}(x_1) dx_1 = \mathcal{O}(\log^{d-1} s),$$

concluding the proof. □

Lemma 3.4. For $\alpha_1, \alpha_2 > 0$,

$$s \int_{\mathbb{X}} \left(s \int_{\mathbb{X}} c_{\alpha_1,s}(x)^5 e^{-\alpha_2 s|x \vee y|} dx \right)^2 dy = \mathcal{O}(\log^{d-1} s).$$

Proof. Since $c_{\alpha,s}$ is decreasing in α and in view of (3.7), it suffices to prove the result with both α_1 and α_2 replaced by 1. We split the inner integral into integration domains corresponding to the cases when $x \vee y = (x^I, y^J)$ with $J = I^c$ for $I \subseteq \{1, \dots, d\}$. First, if $I = \{1, \dots, d\}$, then using monotonicity of $c_{1,s}$ and (3.10), we have

$$\begin{aligned} & s \int_{\mathbb{X}} \left(s \int_{\mathbb{X}} \mathbb{1}_{x \succ y} c_{1,s}(x)^5 e^{-s|x \vee y|} dx \right)^2 dy \\ & \leq s \int_{\mathbb{X}} c_{1,s}(y)^{10} \left(s \int_{\mathbb{X}} \mathbb{1}_{x \succ y} e^{-s|x|} dx \right)^2 dy \leq s \int_{\mathbb{X}} c_{1,s}(y)^{12} dy = \mathcal{O}(\log^{d-1} s). \end{aligned}$$

By writing the function $|\cdot|$ as the product of coordinates and passing to the one-dimensional case, it is easy to see that for $a, b, y \in \mathbb{X}$,

$$|a \wedge y| |b \wedge y| \leq |a \wedge b \wedge y| |y|. \tag{3.17}$$

Hence, when $I = \emptyset$,

$$\begin{aligned} & s \int_{\mathbb{X}} \left(s \int_{\mathbb{X}} \mathbb{1}_{x \prec y} c_{1,s}(x)^5 e^{-s|x \vee y|} dx \right)^2 dy = s \int_{\mathbb{X}} e^{-2s|y|} \left(s \int_{\mathbb{X}} \mathbb{1}_{x \prec y} c_{1,s}(x)^5 dx \right)^2 dy \\ & \leq s^{13} \int_{\mathbb{X}} \int_{\mathbb{X}^2} \mathbb{1}_{x_1, x_2 \prec y} \int_{\mathbb{X}^{10}} \mathbb{1}_{z_1, \dots, z_5 \succ x_1} \mathbb{1}_{z_6, \dots, z_{10} \succ x_2} e^{-s|y| - s \sum_{i=1}^{10} |z_i|} d(z_1, \dots, z_{10}) d(x_1, x_2) dy \\ & = s^{13} \int_{\mathbb{X}} \int_{\mathbb{X}^{10}} |z_1 \wedge \dots \wedge z_5 \wedge y| |z_6 \wedge \dots \wedge z_{10} \wedge y| e^{-s|y| - s \sum_{i=1}^{10} |z_i|} d(z_1, \dots, z_{10}) dy \\ & \leq s^{13} \int_{\mathbb{X}} \int_{\mathbb{X}^{10}} |z_1 \wedge \dots \wedge z_{10} \wedge y| |y| e^{-s|y| - s \sum_{i=1}^{10} |z_i|} d(z_1, \dots, z_{10}) dy, \end{aligned}$$

where in the final step, we have used (3.17) with $a := z_1 \wedge \dots \wedge z_5$ and $b := z_6 \wedge \dots \wedge z_{10}$. Splitting the exponential into product of two exponentials with powers halved, and using the fact that

$$s|y| e^{-s|y|/2 - s \sum_{i=1}^{10} |z_i|/2} \leq 2,$$

we obtain by (3.10) that the last integral is bounded by

$$\begin{aligned} & 2s^{12} \int_{\mathbb{X}} \int_{\mathbb{X}^{10}} |z_1 \wedge \dots \wedge z_{10} \wedge y| e^{-s|y|/2 - s \sum_{i=1}^{10} |z_i|/2} d(z_1, \dots, z_{10}) dy \\ & = 2s \int_{\mathbb{X}} \left(s \int_{\mathbb{X}} \mathbb{1}_{y \succ x} e^{-s|y|/2} dy \right) \prod_{i=1}^{10} \left(s \int_{\mathbb{X}} \mathbb{1}_{z_i \succ x} e^{-s|z_i|/2} dz_i \right) dx \\ & = 2s \int_{\mathbb{X}} c_{1/2,s}(x)^{11} dx = \mathcal{O}(\log^{d-1} s). \end{aligned}$$

Next, assume that $d \geq 2$ and I is nonempty of cardinality m with $1 \leq m \leq d - 1$. Using monotonicity of $c_{1,s}$ in the first step and Lemma 3.1 in the last step upon identifying the integral as the function given by (3.6) in the space of dimension m , we have

$$\begin{aligned} & s \int_{\mathbb{X}} \left(s \int_{\mathbb{X}} \mathbb{1}_{x^I \succ y^I, x^J \prec y^J} c_{1,s}(x)^5 e^{-s|x^I| |y^J|} dx \right)^2 dy \\ & \leq s \int_{\mathbb{X}} \left(s \int_{\mathbb{X}} \mathbb{1}_{x^I \succ y^I, x^J \prec y^J} c_{1,s}(x^J, y^I)^5 e^{-s|x^I| |y^J|} dx \right)^2 dy \\ & = s \int_{\mathbb{X}} \left(\int_{[0,1]^m} \mathbb{1}_{x^I \succ y^I} e^{-s|x^I| |y^J|} dx^I \right)^2 \left(s \int_{[0,1]^{d-m}} \mathbb{1}_{x^J \prec y^J} c_{1,s}(x^J, y^I)^5 dx^J \right)^2 dy \\ & \leq D^2 s \int_{\mathbb{X}} \frac{e^{-s|y|}}{s^2 |y^J|^2} \left(1 + |\log(s|y|)|^{2(m-1)} \right) \left(s \int_{[0,1]^{d-m}} \mathbb{1}_{x^J \prec y^J} c_{1,s}(x^J, y^I)^5 dx^J \right)^2 dy, \end{aligned} \tag{3.18}$$

with D as in Lemma 3.1. We will now estimate the integral inside (3.18). Using Lemma 3.1 and Jensen’s inequality in the first step, substituting $u = (s|y^I|)^{1/(d-m)}x^J$ in the second step, letting $z^{(i)} = \log u^{(i)}$, $i = 1, \dots, d - m$, in the third one, $v^{(1)} = \sum_{i=1}^{d-m} z^{(i)}$ in the fourth, $w = e^{-v^{(1)}}$ in the fifth, and, finally, Jensen’s inequality in the penultimate step, we obtain that

$$\begin{aligned} & s|y^I| \int_{[0,1]^{d-m}} \mathbb{1}_{x^J \prec_{y^J} c_{1,s}}(x^J, y^I)^5 dx^J \\ & \leq 16D^5 s|y^I| \int_{[0,1]^{d-m}} \mathbb{1}_{x^J \prec_{y^J} e^{-5s|x^J||y^I|/2}} \left(1 + |\log(s|x^J||y^I|)|^{5(d-1)}\right) dx^J \\ & = 16D^5 \int_{[0,(s|y^I|)^{\frac{1}{d-m}}]^{d-m}} \mathbb{1}_{u \prec (s|y^I|)^{\frac{1}{d-m}} y^J} e^{-\frac{5}{2}|u|} \left(1 + |\log(|u|)|^{5(d-1)}\right) du \\ & = 16D^5 \int_{\times_{j \in J} [-\frac{\log(s|y^I|)}{d-m} - \log y^{(j)}, \infty)} \exp \left\{ -e^{-\frac{5}{2} \sum_{i=1}^{d-m} z^{(i)}} - \sum_{i=1}^{d-m} z^{(i)} \right\} \\ & \quad \times \left(1 + \left| \sum_{i=1}^{d-m} z^{(i)} \right|^{5(d-1)}\right) dz \\ & \leq 16D^5 \int_{-\log s|y|}^{\infty} \left(v^{(1)} + \log(s|y|)\right)^{d-m-1} \exp \left\{ -e^{-\frac{5}{2}v^{(1)}} - v^{(1)} \right\} \left(1 + |v^{(1)}|^{5(d-1)}\right) dv^{(1)} \\ & = 16D^5 \int_0^{s|y|} e^{-w^{5/2}} \left(\log(s|y|) - \log w\right)^{d-m-1} \left(1 + |\log w|^{5(d-1)}\right) dw \\ & \leq 16D^5 2^{d-m-2} \left[|\log(s|y|)|^{d-m-1} \int_0^{s|y|} \left(1 + |\log w|^{5(d-1)}\right) dw \right. \\ & \quad \left. + \int_0^{s|y|} |\log w|^{d-m-1} \left(1 + |\log w|^{5(d-1)}\right) dw \right] \\ & \leq D' s|y| \left[1 + \sum_{i=1}^{6(d-1)-m} |\log(s|y|)|^i \right] \end{aligned}$$

for a constant D' depending only on d and m , so that the bound on the last integral in (3.18) is obtained by dividing by $|y^I|$ on both sides. The last step relies on an elementary inequality, saying that, for $l \in \mathbb{N} \cup \{0\}$ and $a > 0$, there exists a constant $b_l > 0$ depending only on l such that

$$\int_0^a |\log w|^l dw \leq b_l a \left[1 + \sum_{i=1}^l |\log a|^i \right].$$

Plugging this in (3.18) and using Jensen’s inequality, we obtain

$$\begin{aligned} & s \int_{\mathbb{X}} \left(s \int_{\mathbb{X}} \mathbb{1}_{x^I \succ_{y^I}, x^J \prec_{y^J} c_{1,s}}(x) e^{-s|x^I||y^J|} dx \right)^2 dy \\ & \leq D'' s \int_{\mathbb{X}} \frac{e^{-s|y|}}{s^2|y^J|^2} \left(1 + |\log(s|y|)|^{2(m-1)}\right) s^2|y^J|^2 \left(1 + |\log(s|y|)|^{12(d-1)-2m}\right) dy \\ & = \mathcal{O}(\log^{d-1} s) \end{aligned}$$

for some constant D'' depending on d and m , where the last step is argued similarly as for (3.13). Summing over all possible $I \subseteq \{1, \dots, d\}$ yields the desired conclusion. \square

Lemma 3.5. For $\beta \in (0, 1/2)$, $\zeta \in (0, \beta)$ and f_β defined at (2.8),

$$s \int_{\mathbb{X}} f_\beta(x_1)^2 dx_1 = \mathcal{O}(\log^{d-1} s).$$

Proof. As in Lemma 3.3, we consider integrals of squares of $f_\beta^{(i)}$ for $i = 1, 2, 3$ separately. By (3.10),

$$s \int_{\mathbb{X}} \left(s \int_{\mathbb{X}} e^{-\beta r_s(x_2, x_1)} dx_2 \right)^2 dx_1 = s \int_{\mathbb{X}} \left(s \int_{\mathbb{X}} \mathbb{1}_{x_2 \succ x_1} e^{-\beta s|x_2|} dx_2 \right)^2 dx_1 = \mathcal{O}(\log^{d-1} s).$$

Arguing as in (3.15), using monotonicity of $c_{\zeta, s}$, $\zeta < \beta$, and (3.10), we have

$$\begin{aligned} s \int_{\mathbb{X}} \left(s \int_{\mathbb{X}} c_{\zeta, s}(x_2)^5 e^{-\beta r_s(x_2, x_1)} dx_2 \right)^2 dx_1 &\leq s \int_{\mathbb{X}} c_{\zeta, s}(x_1)^{10} \left(s \int_{\mathbb{X}} \mathbb{1}_{x_2 \succ x_1} e^{-\beta s|x_2|} dx_2 \right)^2 dx_1 \\ &\leq s \int_{\mathbb{X}} c_{\zeta, s}(x_1)^{12} dx_1 = \mathcal{O}(\log^{d-1} s). \end{aligned}$$

Recalling that $G_s(y) \leq 3 + 2c_{\zeta, s}(y)^5$, combining the above bounds and using Jensen's inequality yield

$$s \int_{\mathbb{X}} f_\beta^{(1)}(x_1)^2 dx_1 = \mathcal{O}(\log^{d-1} s).$$

Next, we integrate the square of $f_\beta^{(3)}$. Using (3.16) and Lemma 3.2,

$$\begin{aligned} s \int_{\mathbb{X}} \left(s \int_{\mathbb{X}} q_s(x_1, x_2)^\beta dx_2 \right)^2 dx_1 &= s \int_{\mathbb{X}} \left(s \int_{\mathbb{X}} c_{1, s}(x_1 \vee x_2)^\beta dx_2 \right)^2 dx_1 \\ &\leq 2s \int_{\mathbb{X}} \left(s \int_{\mathbb{X}} e^{-\beta s|x_1 \vee x_2|} dx_2 \right)^2 dx_1 + 2s \int_{\mathbb{X}} \left(s \int_{\mathbb{X}} c_{\beta, s}(x_1 \vee x_2) dx_2 \right)^2 dx_1 = \mathcal{O}(\log^{d-1} s). \end{aligned} \tag{3.19}$$

Again using (3.16),

$$\begin{aligned} s \int_{\mathbb{X}} \left(s \int_{\mathbb{X}} c_{\zeta, s}(x_2)^5 q_s(x_1, x_2)^\beta dx_2 \right)^2 dx_1 &= s \int_{\mathbb{X}} \left(s \int_{\mathbb{X}} c_{\zeta, s}(x_2)^5 c_{1, s}(x_1 \vee x_2)^\beta dx_2 \right)^2 dx_1 \\ &\leq 2s \int_{\mathbb{X}} \left(s \int_{\mathbb{X}} c_{\zeta, s}(x_2)^5 e^{-\beta s|x_1 \vee x_2|} dx_2 \right)^2 dx_1 + 2s \int_{\mathbb{X}} \left(s \int_{\mathbb{X}} c_{\zeta, s}(x_2)^5 c_{\beta, s}(x_1 \vee x_2) dx_2 \right)^2 dx_1 \\ &:= 2(A_1 + A_2). \end{aligned}$$

By Lemma 3.4, we have $A_1 = \mathcal{O}(\log^{d-1} s)$. For $x_1 \in \mathbb{X}$ and $(x_{21}, x_{22}) \in \mathbb{X}^2$, denote

$$\begin{aligned} A(x_1, x_{21}, x_{22}) &:= \left\{ (z_1, \dots, z_{12}) \in \mathbb{X}^{12} : \right. \\ &\quad \left. z_1, \dots, z_5 \succ x_{21}, z_6, \dots, z_{10} \succ x_{22}, z_{11} \succ x_1 \vee x_{21}, z_{12} \succ x_1 \vee x_{22} \right\}. \end{aligned}$$

By applying (3.17) twice we have

$$|a \wedge x| |b \wedge y| |x \wedge y| \leq |a \wedge b \wedge x \wedge y| |x| |y| \leq |a \wedge b \wedge x \wedge y| (|x| + |y|)^2, \quad a, b, x, y \in \mathbb{X}.$$

Using this with $a := z_1 \wedge \dots \wedge z_5$, $b := z_6 \wedge \dots \wedge z_{10}$, $x := z_{11}$, $y := z_{12}$ in the third step, (3.9) in the penultimate step, and (3.10) in the last one, we obtain

$$\begin{aligned} A_2 &\leq s^{15} \int_{\mathbb{X}} \int_{\mathbb{X}^2} \int_{A(x_1, x_{21}, x_{22})} e^{-\zeta s \sum_{i=1}^{12} |z_i|} d(z_1, \dots, z_{12}) d(x_{21}, x_{22}) dx_1 \\ &= s^{15} \int_{\mathbb{X}^{12}} e^{-\zeta s \sum_{i=1}^{12} |z_i|} |z_1 \wedge \dots \wedge z_5 \wedge z_{11}| |z_6 \wedge \dots \wedge z_{10} \wedge z_{12}| |z_{11} \wedge z_{12}| d(z_1, \dots, z_{12}) \\ &\leq s^{15} \int_{\mathbb{X}^{12}} e^{-\zeta s \sum_{i=1}^{12} |z_i|} |z_1 \wedge \dots \wedge z_{12}| (|z_{11}| + |z_{12}|)^2 d(z_1, \dots, z_{12}) \\ &\leq (8/\zeta^2) s^{13} \int_{\mathbb{X}^{12}} e^{-\zeta s \sum_{i=1}^{12} |z_i|/2} |z_1 \wedge \dots \wedge z_{12}| d(z_1, \dots, z_{12}) \\ &= (8/\zeta^2) s \int_{\mathbb{X}} c_{\zeta/2, s}(x)^{12} dx = \mathcal{O}(\log^{d-1} s), \end{aligned}$$

where for the last inequality we have used that

$$s^2(|z_{11}| + |z_{12}|)^2 e^{-\zeta s \sum_{i=1}^{12} |z_i|/2} \leq 8/\zeta^2.$$

Combining the bounds on A_1 and A_2 with (3.19) yields that

$$s \int_{\mathbb{X}} f_{\beta}^{(3)}(x_1)^2 dx_1 = \mathcal{O}(\log^{d-1} s).$$

For the integral of the square of $f_{\beta}^{(2)}$, arguing as in Lemma 3.3 and using the inequality $t^2 e^{-t} \leq 2$ for $t \geq 0$, we have

$$\begin{aligned} s \int_{\mathbb{X}} \left(s \int_{\mathbb{X}} e^{-\beta r_s(x_1, x_2)} dx_2 \right)^2 dx_1 &= s \int_{\mathbb{X}} \left(s \int_{\mathbb{X}} \mathbf{1}_{x_2 \prec x_1} e^{-\beta s|x_1|} dx_2 \right)^2 dx_1 \\ &= s/\beta^2 \int_{\mathbb{X}} (\beta s|x_1|)^2 e^{-2\beta s|x_1|} dx_1 \leq 2s/\beta^2 \int_{\mathbb{X}} e^{-\beta s|x_1|} dx_1 = \mathcal{O}(\log^{d-1} s). \end{aligned}$$

Changing order of integration in the second step, using the Cauchy–Schwarz inequality in the third one, and referring to (3.10) in the last step yield that

$$\begin{aligned} s \int_{\mathbb{X}} \left(s \int_{\mathbb{X}} c_{\zeta, s}(x_2)^5 e^{-\beta r_s(x_1, x_2)} dx_2 \right)^2 dx_1 &= s \int_{\mathbb{X}} \left(s \int_{\mathbb{X}} \mathbf{1}_{x_2 \prec x_1} c_{\zeta, s}(x_2)^5 e^{-\beta s|x_1|} dx_2 \right)^2 dx_1 \\ &= s^2 \int_{\mathbb{X}^2} c_{\zeta, s}(x)^5 c_{\zeta, s}(y)^5 c_{\beta, s}(x \vee y) d(x, y) \leq \left(s \int_{\mathbb{X}} c_{\zeta, s}(x)^{10} dx \right)^{1/2} A_2^{1/2} = \mathcal{O}(\log^{d-1} s), \end{aligned}$$

where A_2 is defined above. Thus,

$$s \int_{\mathbb{X}} f_{\beta}^{(2)}(x_1)^2 dx_1 = \mathcal{O}(\log^{d-1} s).$$

Combining, we obtain the desired result. □

Since $2\beta = 2p/(32 + 4p) < 1$, to compute the bound, it suffices to provide a bound on the integral of $(\kappa_s + g_s)^\beta G_s$ for any $\beta \in (0, 1)$.

Lemma 3.6. *For $\beta, \zeta \in (0, 1)$, let G_s and κ_s be as in (2.7) and (2.10) respectively. Then*

$$s \int_{\mathbb{X}} G_s(x) (\kappa_s(x) + g_s(x))^\beta dx = \mathcal{O}(\log^{d-1} s).$$

Proof. First note that

$$\kappa_s(x) = \mathbf{P} \{ \xi_s(x, \mathcal{P}_s + \delta_x) \neq 0 \} = e^{-s|x|}, \quad x \in \mathbb{X}.$$

Using the Cauchy–Schwarz inequality in the second step, by (3.4) and (3.10),

$$\begin{aligned} s \int_{\mathbb{X}} G_s(x) \kappa_s(x)^\beta dx &\leq 3s \int_{\mathbb{X}} (1 + c_{\zeta, s}(x)^5) e^{-\beta s|x|} dx \\ &\leq 3s \int_{\mathbb{X}} e^{-\beta s|x|} dx + 3 \left(s \int_{\mathbb{X}} c_{\zeta, s}(x)^{10} dx \right)^{1/2} \left(s \int_{\mathbb{X}} e^{-2\beta s|x|} dx \right)^{1/2} = \mathcal{O}(\log^{d-1} s). \end{aligned}$$

Since $\beta \in (0, 1)$, arguing as in (3.16),

$$c_{\zeta, s}(x)^\beta \leq e^{-\beta \zeta |x|} + c_{\beta \zeta, s}(x).$$

An application of (3.4) and (3.10) now yields

$$\begin{aligned} s \int_{\mathbb{X}} G_s(x) g_s(x)^\beta dx &\leq 3s \int_{\mathbb{X}} (1 + c_{\zeta, s}(x)^5) c_{\zeta, s}(x)^\beta dx \\ &\leq 3s \int_{\mathbb{X}} e^{-\beta \zeta |x|} dx + 3s \int_{\mathbb{X}} c_{\beta \zeta, s}(x) dx + 3s \int_{\mathbb{X}} c_{\zeta, s}(x)^{5+\beta} dx = \mathcal{O}(\log^{d-1} s). \end{aligned}$$

Combining the above bounds, we obtain the desired conclusion. □

Proof of Theorem 2.2. By (3.5), $\text{Var}(F_s) \geq C_1 \log^{d-1} s$ for all $s \geq 1$. An application of Theorem 2.1 with Lemmas 3.3, 3.5 and 3.6 now yields the desired upper bound.

The proof of the optimality of the bound on the Kolmogorov distance follows by a general argument employed in the proof of [6, Theorem 1.1, Eq. (1.6)], which shows that the Kolmogorov distance between any integer-valued random variable, suitably normalized, and a standard normal random variable is always lower bounded by a constant times the inverse of the standard deviation, see Section 6 therein for further details. The variance upper bound in (3.5) now yields the result. \square

4 Non-diffuse intensity measures and unbounded scores

As discussed in the introduction, in addition to working with general stabilization regions, our approach generalizes results in [9] in two more ways. First, we allow for non-diffuse intensity measures and, second, we can consider score functions that do not have uniformly bounded moments over $x \in \mathbb{X}$. In this section, we demonstrate this with two examples. In Example 4.1, we consider a Poisson process on the two dimensional integer lattice with the counting measure as the intensity, which is non-diffuse. We derive a quantitative central limit theorem for the number of isolated points in this setup.

In Example 4.2, we consider isolated vertices in a random geometric graph built on a stationary Poisson process on \mathbb{R}^d , where two points are joined by an edge if the distance between them is at most ρ_s for some appropriate non-negative function ρ_s , $s \geq 1$. Poisson convergence for the number of such isolated vertices in different regimes has been extensively studied, see, e.g., [12, Ch. 8]. But, instead of considering the number of isolated vertices, we consider the sum of values for a general function evaluated at locations of isolated vertices, for instance, the logarithms of scaled norms. As the logarithm is unbounded near the origin, the score functions do not admit a uniform bound on their moments. We note here that in both the examples below, it should be possible to work with a binomial process as well, once a result paralleling our Theorem 5.1 is proved in this setting. As mentioned in the introduction, this can be done by following the scheme in [9] suitably adapted to incorporate general stabilization regions.

Example 4.1 (Non-diffuse intensity). Let $\mathbb{X} := \mathbb{Z}^2$ and consider a Poisson process \mathcal{P} on \mathbb{Z}^2 with the intensity measure \mathbb{Q} being the counting measure on \mathbb{Z}^2 ; so we let $s = 1$ and omit it from the subscripts. A point $x \in \mathcal{P}$ is said to be isolated in \mathcal{P} if all its nearest neighbors are unoccupied, i.e., $\mathcal{P}(x + B) = 0$, where $+$ denotes the Minkowski addition and $B := \{(0, \pm 1), (\pm 1, 0)\}$, so that $x + B$ is the set comprising the 4 nearest neighbors of $x \in \mathbb{Z}^2$. Consider a weight function $w : \mathbb{Z}^2 \rightarrow \mathbb{R}_+$, and for $i \in \mathbb{N}$ denote

$$W_i := \sum_{x \in \mathbb{Z}^2} w(x)^i.$$

Assume that $W_1 = \sum_{x \in \mathbb{Z}^2} w(x) < \infty$, which in particular implies that w is bounded. Scaling w , assume without loss of generality that w is bounded by one. Consider the statistic $H \equiv H_1(\mathcal{P}_1)$ defined at (1.1) with

$$\xi(x, \mathcal{P}) := w(x) \mathbb{1}_{\mathcal{P}(x+B)=0}, \quad x \in \mathcal{P}.$$

For $x \in \mathbb{Z}^2$, defining the stabilization region $R(x, \mathcal{P} + \delta_x) := (x + B)$ if x is isolated in $\mathcal{P} + \delta_x$ and $R(x, \mathcal{P} + \delta_x) := \emptyset$ otherwise, we see that (2.1) and (A1) are trivially satisfied. Also, (A2) holds with $p = 1$ and $M_{1,1}(x) = w(x)$, while (2.4) holds with $r(x, y) = 4$ for $x \in \mathbb{Z}^2$ and $y \in x + B$ and $r(x, y) = \infty$ otherwise. Next, notice that $\kappa(y) = e^{-4}$, $y \in \mathbb{Z}^2$, $\zeta = 1/50$,

$$g(y) = \sum_{x \in y+B, x \in \mathbb{Z}^2} e^{-4/50} = 4e^{-4/50} \quad \text{and} \quad h(y) = \sum_{x \in y+B, x \in \mathbb{Z}^2} w(x)^{9/2} e^{-4/50}, \quad y \in \mathbb{Z}^2,$$

while $q(x_1, x_2) \leq 4e^{-4}$ for $x_1, x_2 \in \mathbb{Z}^2$ with $x_2 - x_1 \in B + B$ and $q(x_1, x_2) = 0$ otherwise. Noticing that $\max\{w(x)^2, w(x)^4, w(x)^{9/2}\} = w(x)^2$, we obtain that for all $\alpha > 0$, there exists a constant C_α such that

$$f_\alpha(y) \leq C_\alpha \sum_{x-y \in (B+B) \cup (B+B+B), x, y \in \mathbb{Z}^2} w(x)^2.$$

Thus, with $\beta = 1/36$, there exists a constant $C > 0$ such that

$$\mathbb{Q}f_\beta^2 \leq CW_4, \quad \max\{\mathbb{Q}f_{2\beta}, \mathbb{Q}((\kappa + g)^{2\beta}G)\} \leq CW_2.$$

On the other hand, by the Mecke formula, we have

$$\begin{aligned} \text{Var}(H) &= \mathbf{E} \sum_{x \in \mathcal{P}} w^2(x) \mathbb{1}_{\mathcal{P}(x+B)=0} - (\mathbf{E}H)^2 \\ &\quad + \sum_{x \in \mathbb{Z}^2} \sum_{y \in (x+B)^c, y \in \mathbb{Z}^2} w(x)w(y) \mathbf{P}\{(\mathcal{P} + \delta_x + \delta_y)((x+B) \cup (y+B)) = 0\} \\ &= e^{-4}W_2 - e^{-8} \sum_{x \in \mathbb{Z}^2} \sum_{y \in (x+B)} w(x)w(y) \\ &\quad + \sum_{x \in \mathbb{Z}^2} \sum_{y \in (x+B)^c, y \in \mathbb{Z}^2} w(x)w(y) \left(\mathbf{P}\{\mathcal{P}((x+B) \cup (y+B)) = 0\} - e^{-8} \right) \\ &\geq e^{-4}W_2 + (e^{-7} - e^{-8}) \sum_{x \in \mathbb{Z}^2} \sum_{y-x \in (B+B)} w(x)w(y) - e^{-8} \sum_{x \in \mathbb{Z}^2} \sum_{y \in (x+B)} w(x)w(y). \end{aligned}$$

Finally, noticing that

$$\sum_{x \in \mathbb{Z}^2} \sum_{y \in (x+B)} w(x)w(y) \leq \sum_{x \in \mathbb{Z}^2} \sum_{y \in (x+B)} \frac{w(x)^2 + w(y)^2}{2} = 4W_2,$$

we obtain

$$\text{Var}(H) \geq (e^{-4} - 4e^{-8})W_2.$$

Hence, an application of Theorem 2.1 yields that

$$\begin{aligned} &\max \left\{ d_W \left(\frac{H - \mathbf{E}H}{\sqrt{\text{Var} H}}, N \right), d_K \left(\frac{H - \mathbf{E}H}{\sqrt{\text{Var} H}}, N \right) \right\} \\ &\leq \frac{C}{(W_2)^{1/2}} \left[1 + \sqrt{\frac{W_4}{W_2}} + \frac{1}{W_2^{1/4}} \right] \leq \frac{C}{(W_2)^{1/2}} \left[2 + \frac{1}{W_2^{1/4}} \right], \end{aligned}$$

for some constant $C > 0$, where the final step is due to the observation that $W_4 \leq W_2$. As an example, one can take $w(x) := \mathbb{1}_{x \in [-n, n]^2}$ for $n \in \mathbb{N}$ to see that the distances on the left-hand side is bounded by C/n , which is presumably optimal, since the variance is of the order n^2 . In particular, arguing as in the proof of Theorem 2.2, the bound on the Kolmogorov distance is of optimal order in this case.

Example 4.2 (Weighted sum over isolated vertices in random geometric graphs). Let $\mathbb{X} := \mathbb{R}^d$ with $d \geq 2$, and let \mathcal{P}_s be a Poisson process on \mathbb{X} with intensity measure $s\mathbb{Q}$ for $s \geq 1$ and the Lebesgue measure \mathbb{Q} . Fix $s \geq 1$. Given $\rho_s > 0$, consider a random geometric graph $G_s(\mathcal{P}_s, \rho_s)$ with the vertex set \mathcal{P}_s , where an edge joins two distinct vertices x and y if $\|x - y\| \leq \rho_s$, where $\|\cdot\|$ denotes the Euclidean norm. A vertex $x \in \mathcal{P}_s$ is called isolated if $\mathcal{P}_s(B(x, \rho_s) \setminus \{x\}) = 0$, where $B(x, \rho_s)$ denotes the closed ball of radius ρ_s centered at x . For a (possibly unbounded) weight function $w_s : \mathbb{R}^d \rightarrow \mathbb{R}_+$ with $\int_{\mathbb{R}^d} \max\{w_s(x), w_s(x)^8\} dx < \infty$, consider the statistic H_s defined at (1.1) with

$$\xi_s(x, \mathcal{P}_s) := w_s(x) \mathbb{1}_{x \text{ is isolated in } \mathcal{P}_s}, \quad x \in \mathcal{P}_s.$$

Region-stabilizing scores

For $x \in \mathbb{X}$, letting $R_s(x, \mathcal{P}_s + \delta_x) := B(x, \rho_s)$ if x is isolated in $\mathcal{P}_s + \delta_x$ and \emptyset otherwise, we see that (2.1) and (A1) are satisfied. As in Example 4.1, (A2) holds with $p = 1$ and $M_s = M_{s,1}(x) := w_s(x)$. Letting $r_s(x, y) := k_d s \rho_s^d$ for $x \in \mathbb{R}^d$ and $y \in B(x, \rho_s)$, where k_d is the volume of the unit ball in \mathbb{R}^d , and $r_s(x, y) := \infty$ otherwise, one verifies (2.4). Clearly, $\kappa_s(y) \leq e^{-k_d s \rho_s^d}$ for $y \in \mathbb{R}^d$. Also, since $\zeta = 1/50$, one has

$$g_s(y) = k_d s \rho_s^d e^{-k_d s \rho_s^d / 50} \quad \text{and} \quad h_s(y) = s e^{-k_d s \rho_s^d / 50} \int_{B(y, \rho_s)} w_s(x)^{9/2} dx, \quad y \in \mathbb{R}^d,$$

while $q_s(x_1, x_2) \leq k_d s \rho_s^d e^{-k_d s \rho_s^d}$ for $x_1, x_2 \in \mathbb{R}^d$ with $\|x_2 - x_1\| \leq 2\rho_s$ and $q_s(x_1, x_2) = 0$ otherwise. Next, we compute the variance of H_s . Denote $W_{i,s} := s \int_{\mathbb{R}^d} w_s(x)^i dx$, $i \in \mathbb{N}$. Applying the Mecke formula in the first equality, we obtain

$$\begin{aligned} \text{Var}(H_s) &= s \int_{\mathbb{R}^d} w_s(x)^2 e^{-k_d s \rho_s^d} dx - \left(s \int_{\mathbb{R}^d} w_s(x) e^{-k_d s \rho_s^d} dx \right)^2 \\ &\quad + s^2 \int_{\mathbb{R}^d} \int_{B(x, \rho_s)^c} w_s(x) w_s(y) \exp\{-\text{Vol}(B(x, \rho_s) \cup B(y, \rho_s))\} dy dx \\ &\geq e^{-k_d s \rho_s^d} W_{2,s} - s^2 e^{-2k_d s \rho_s^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d \cap B(x, \rho_s)} w_s(x) w_s(y) dy dx. \end{aligned}$$

As in the previous example,

$$s^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d \cap B(x, \rho_s)} w_s(x) w_s(y) dy dx \leq k_d s \rho_s^d W_{2,s},$$

so that

$$\text{Var}(H) \geq e^{-k_d s \rho_s^d} (1 - k_d s \rho_s^d e^{-k_d s \rho_s^d}) W_{2,s} \geq \frac{1}{2} e^{-k_d s \rho_s^d} W_{2,s},$$

where in the last step we have used that $ue^{-u} \leq 1/2$ for $u \geq 0$. Denoting $\bar{w}_s := \max\{w_s^2, w_s^4, w_s^{9/2}\}$, it is straightforward to check that

$$f_\alpha(y) \leq C s e^{-\alpha k_d s \rho_s^d} \int_{B(y, 3\rho_s)} \bar{w}_s(x) dx$$

for $\alpha > 0$ and a constant $C > 0$, so that by Jensen's inequality,

$$f_\alpha(y)^2 \leq C^2 3^d k_d s^2 \rho_s^d e^{-2\alpha k_d s \rho_s^d} \int_{B(y, 3\rho_s)} \bar{w}_s(x)^2 dx.$$

Thus, letting $\bar{W}_{i,s} := s \int_{\mathbb{R}^d} \bar{w}_s(x)^i dx$, $i \in \mathbb{N}$, and $\beta = 1/36$, and using again that $ue^{-u} \leq 1/2$ for $u \geq 0$, we have that there exists a constant C_d depending only on the dimension d such that

$$s \mathbb{Q} f_\beta^2 \leq C_d \bar{W}_{2,s}, \quad \text{and} \quad \max\{s \mathbb{Q} f_{2\beta}, s \mathbb{Q}((\kappa_s + g_s)^{2\beta} G_s)\} \leq C_d \bar{W}_{1,s}.$$

Thus, applying Theorem 2.1, we obtain for $s \geq 1$ that

$$\begin{aligned} &\max \left\{ d_W \left(\frac{H_s - \mathbf{E}H_s}{\sqrt{\text{Var} H_s}}, N \right), d_K \left(\frac{H_s - \mathbf{E}H_s}{\sqrt{\text{Var} H_s}}, N \right) \right\} \\ &\leq C'_d \left[\frac{\bar{W}_{2,s}^{1/2} + \bar{W}_{1,s}^{1/2}}{e^{-k_d s \rho_s^d} W_{2,s}} + \frac{\bar{W}_{1,s}}{(e^{-k_d s \rho_s^d} W_{2,s})^{3/2}} + \frac{\bar{W}_{1,s}^{5/4} + \bar{W}_{1,s}^{3/2}}{(e^{-k_d s \rho_s^d} W_{2,s})^2} \right] \end{aligned}$$

for some constant $C'_d > 0$ depending only on the dimension. The setting can be easily extended for functions ρ_s which depend on the position x (see [8]) and/or are random variables which, together with locations, form a Poisson process on the product space.

As an example, consider the logarithmic weight function $w_s(x) := \log \frac{s}{\|x\|} \mathbb{1}_{x \in B(0,s)}$. For $i \in \mathbb{N}$,

$$W_{i,s} = s \int_{B(0,s)} \log^i \frac{s}{\|x\|} dx = dk_d s \int_0^s r^{d-1} \log^i \frac{s}{r} dr = dk_d s^{d+1} \int_0^1 z^{d-1} \log^i \frac{1}{z} dz = \mathcal{O}(s^{d+1}),$$

so that $\overline{W}_{i,s} = \mathcal{O}(s^{d+1})$ for all $i \in \mathbb{N}$. Hence, in the regime when $s \rho_s^d - (d+1)(2k_d)^{-1} \log s \rightarrow -\infty$ as $s \rightarrow \infty$, one obtains Gaussian convergence as $s \rightarrow \infty$ with an appropriate non-asymptotic bound on the Wasserstein or Kolmogorov distances between the normalized H_s and a standard normal random variable N .

5 Modified bounds on the Wasserstein and Kolmogorov distances and proof of Theorem 2.1

In this section, we prove Theorem 2.1. The proof is primarily based on the following generalization of Theorem 6.1 in [10], incorporating a spatially inhomogeneous moment bound given by a function $c_x, x \in \mathbb{X}$. The proof, which we present for completeness in the Appendix follows closely that of [10, Theorem 6.1].

Let \mathcal{P} be a Poisson process on a measurable space $(\mathbb{X}, \mathcal{F})$ with a σ -finite intensity measure ν . Let $F := f(\mathcal{P})$ be a measurable function of \mathcal{P} . For $x, y \in \mathbb{X}$, define the first and second order difference operators as $D_x F := f(\mathcal{P} + \delta_x) - f(\mathcal{P})$ and $D_{x,y}^2 F := D_x(D_y F)$. Also, denote by $\text{dom } D$ the collection of functions $F \in L^2_{\mathcal{P}}$ with

$$\mathbf{E} \int_{\mathbb{X}} (D_x F)^2 \nu(dx) < \infty.$$

Theorem 5.1. *Let $F \in \text{dom } D$ be such that $\text{Var } F > 0$. Assume that there exists a $q > 0$ such that, for all $\mu \in \mathbf{N}$ with $\mu(\mathbb{X}) \leq 1$,*

$$\mathbf{E} |D_x F(\mathcal{P} + \mu)|^{4+q} \leq c_x \quad \text{for } \nu\text{-a.e. } x \in \mathbb{X},$$

where c_x is a measurable function of $x \in \mathbb{X}$. Then

$$\begin{aligned} & d_W \left(\frac{F - \mathbf{E}F}{\sqrt{\text{Var } F}}, N \right) \\ & \leq \frac{12}{\text{Var } F} \left[\int_{\mathbb{X}} \left(\int_{\mathbb{X}} c_{x_1}^{2/(4+q)} \mathbf{P} \{D_{x_1, x_2}^2 F \neq 0\}^{q/(16+4q)} \nu(dx_1) \right)^2 \nu(dx_2) \right]^{1/2} + \frac{\Gamma_F}{(\text{Var } F)^{3/2}}, \end{aligned}$$

and

$$\begin{aligned} d_K \left(\frac{F - \mathbf{E}F}{\sqrt{\text{Var } F}}, N \right) & \leq \frac{12}{\text{Var } F} \left[\int_{\mathbb{X}} \left(\int_{\mathbb{X}} c_{x_1}^{2/(4+q)} \mathbf{P} \{D_{x_1, x_2}^2 F \neq 0\}^{q/(16+4q)} \nu(dx_1) \right)^2 \nu(dx_2) \right]^{1/2} \\ & \quad + \frac{\Gamma_F^{1/2}}{\text{Var } F} + \frac{2\Gamma_F}{(\text{Var } F)^{3/2}} + \frac{\Gamma_F^{5/4} + 2\Gamma_F^{3/2}}{(\text{Var } F)^2} \\ & \quad + \frac{12}{\text{Var } F} \left[\int_{\mathbb{X}^2} c_{x_1}^{4/(4+q)} \mathbf{P} \{D_{x_1, x_2}^2 F \neq 0\}^{q/(8+2q)} \nu^2(d(x_1, x_2)) \right]^{1/2}, \end{aligned}$$

with

$$\Gamma_F := \int_{\mathbb{X}} \max\{c_x^{2/(4+q)}, c_x^{4/(4+q)}\} \mathbf{P} \{D_x F \neq 0\}^{q/(8+2q)} \nu(dx).$$

For a proof of this result, see the Appendix. We derive Theorem 2.1 from Theorem 5.1 by proving a series of lemmas, following the general structure of the proof of Theorem 2.1(a) in [9]. However, our setting is more versatile, enabling us to handle new examples. The first lemma is an exact restatement of [9, Lemma 5.2], which is also contained in Remark 6.2 of [10]. Recall the definition of H_s given at (1.1).

Lemma 5.2. For $s \geq 1$, $\mu \in \mathbb{N}$ and $y_1, y_2, y_3 \in \mathbb{X}$,

$$D_y H_s(\mu) = \xi_s(y, \mu + \delta_y) + \sum_{x \in \mu} D_y \xi_s(x, \mu)$$

and

$$D_{y_1, y_2}^2 H_s(\mu) = D_{y_1} \xi_s(y_2, \mu + \delta_{y_2}) + D_{y_2} \xi_s(y_1, \mu + \delta_{y_1}) + \sum_{x \in \mu} D_{y_1, y_2}^2 \xi_s(x, \mu).$$

The next lemma shows that the difference operator D_y vanishes if y lies outside the stabilization region.

Lemma 5.3. Assume that (A1) holds and let $\mu \in \mathbb{N}$ and $x, y, y_1, y_2 \in \mathbb{X}$. Then for $s \geq 1$,

$$D_y \xi_s(x, \mu + \delta_x) = 0 \text{ if } y \notin R_s(x, \mu + \delta_x),$$

and

$$D_{y_1, y_2}^2 \xi_s(x, \mu + \delta_x) = 0 \text{ if } \{y_1, y_2\} \not\subseteq R_s(x, \mu + \delta_x).$$

Proof. By (A1.4),

$$\begin{aligned} D_y \xi_s(x, \mu + \delta_x) &= \xi_s(x, \mu + \delta_x + \delta_y) - \xi_s(x, \mu + \delta_x) \\ &= \xi_s\left(x, (\mu + \delta_x + \delta_y)_{R_s(x, \mu + \delta_x + \delta_y)}\right) - \xi_s\left(x, (\mu + \delta_x)_{R_s(x, \mu + \delta_x)}\right), \end{aligned}$$

If $(\mu + \delta_x)_{R_s(x, \mu + \delta_x)} = 0$, by the monotonicity property (A1.2), for $y \notin R_s(x, \mu + \delta_x)$ we have $(\mu + \delta_x + \delta_y)_{R_s(x, \mu + \delta_x + \delta_y)} = 0$ yielding $D_y \xi_s(x, \mu + \delta_x) = 0$. If $(\mu + \delta_x)_{R_s(x, \mu + \delta_x)} \neq 0$, then (A1.3) implies that $(\mu + \delta_x + \delta_y)_{R_s(x, \mu + \delta_x + \delta_y)} \neq 0$. Thus, for $y \notin R_s(x, \mu + \delta_x)$, by (A1.4) and (2.1) we have

$$\xi_s\left(x, (\mu + \delta_x + \delta_y)_{R_s(x, \mu + \delta_x + \delta_y)}\right) = \xi_s\left(x, (\mu + \delta_x + \delta_y)_{R_s(x, \mu + \delta_x)}\right) = \xi_s\left(x, (\mu + \delta_x)_{R_s(x, \mu + \delta_x)}\right),$$

so that $D_y \xi_s(x, \mu + \delta_x)$ vanishes.

Finally, by (A1.2), $y_1 \notin R_s(x, \mu + \delta_x)$ implies $y_1 \notin R_s(x, \mu + \delta_{y_2} + \delta_x)$. Hence, the second order difference operator vanishes, being an iteration of the first order one. If $y_2 \notin R_s(x, \mu + \delta_x)$, a similar argument applies. \square

The next lemma, which is similar to [9, Lemma 5.4(a)] provides a bound in terms of M_s on the $(4 + \varepsilon)$ -th moment of the difference operator for any $\varepsilon \in (0, p]$, where $p \in (0, 1]$ and M_s are as in (A2).

Lemma 5.4. Assume that (A2) holds. For all $\varepsilon \in (0, p]$, $s \geq 1$, $x, y \in \mathbb{X}$ and $\mu \in \mathbb{N}$ with $\mu(\mathbb{X}) \leq 6$

$$\mathbf{E} \left| D_y \xi_s(x, \mathcal{P}_s + \delta_x + \mu) \right|^{4+\varepsilon} \leq 2^{4+\varepsilon} M_s(x)^{4+\varepsilon}.$$

Proof. By Jensen's inequality, Hölder's inequality and assumption (A2),

$$\begin{aligned} &\mathbf{E} \left| D_y \xi_s(x, \mathcal{P}_s + \delta_x + \mu) \right|^{4+\varepsilon} \\ &\leq 2^{3+\varepsilon} \mathbf{E} \left(\left| \xi_s(x, \mathcal{P}_s + \delta_x + \delta_y + \mu) \right|^{4+\varepsilon} + \left| \xi_s(x, \mathcal{P}_s + \delta_x + \mu) \right|^{4+\varepsilon} \right) \leq 2^{4+\varepsilon} M_s(x)^{4+\varepsilon}. \end{aligned}$$

\square

Recall the functions g_s and h_s defined at (2.6).

Lemma 5.5. Assume that (A1) and (A2) hold. Then, there exists a constant $C_p \in [1, \infty)$ depending only on p , such that

$$\mathbf{E} \left| D_y H_s(\mathcal{P}_s + \mu) \right|^{4+p/2} \leq C_p \left[M_s^{4+p/2}(y) + h_s(y)(1 + g_s(y)^4) \right]$$

for all $y \in \mathbb{X}$, $\mu \in \mathbb{N}$ with $\mu(\mathbb{X}) \leq 1$, and $s \geq 1$.

Proof. Let $\varepsilon := p/2$. We argue as in [9]. For $\mu = 0$, using Lemma 5.2 followed by Jensen’s inequality,

$$\begin{aligned} \mathbf{E} |D_y H_s(\mathcal{P}_s)|^{4+\varepsilon} &= \mathbf{E} \left| \xi_s(y, \mathcal{P}_s + \delta_y) + \sum_{x \in \mathcal{P}_s} D_y \xi_s(x, \mathcal{P}_s) \right|^{4+\varepsilon} \\ &\leq 2^{3+\varepsilon} \mathbf{E} |\xi_s(y, \mathcal{P}_s + \delta_y)|^{4+\varepsilon} + 2^{3+\varepsilon} \mathbf{E} \left| \sum_{x \in \mathcal{P}_s} D_y \xi_s(x, \mathcal{P}_s) \right|^{4+\varepsilon}. \end{aligned}$$

By (A2), the first summand is bounded by $2^{3+\varepsilon} M_s(y)^{4+\varepsilon}$. Following the argument in [9, Lemma 5.5], the second summand can be bounded as

$$2^{3+\varepsilon} \mathbf{E} \left| \sum_{x \in \mathcal{P}_s} D_y \xi_s(x, \mathcal{P}_s) \right|^{4+\varepsilon} \leq 2^{3+\varepsilon} (I_1 + 15I_2 + 25I_3 + 10I_4 + I_5),$$

where for $i \in \{1, \dots, 5\}$,

$$I_i = \mathbf{E} \sum_{(x_1, \dots, x_i) \in \mathcal{P}_s^{i, \neq}} \mathbf{1}_{D_y \xi_s(x_j, \mathcal{P}_s) \neq 0, j=1, \dots, i} |D_y \xi_s(x_1, \mathcal{P}_s)|^{4+\varepsilon}.$$

Here $\mathcal{P}_s^{i, \neq}$ stands for the set of all i -tuples of distinct points from \mathcal{P}_s , where multiple points at the same location are considered to be different ones. Applying the multivariate Mecke formula in the first equation, Hölder’s inequality followed by Lemma 5.4 in the second step and Lemma 5.3 and (A1.2) in the third step, we obtain for $1 \leq i \leq 5$,

$$\begin{aligned} I_i &= s^i \int_{\mathbb{X}^i} \mathbf{E} \left[\mathbf{1}_{D_y \xi_s(x_j, \mathcal{P}_s + \delta_{x_1} + \dots + \delta_{x_i}) \neq 0, j=1, \dots, i} |D_y \xi_s(x_1, \mathcal{P}_s + \delta_{x_1} + \dots + \delta_{x_i})|^{4+\varepsilon} \right] \\ &\quad \mathbb{Q}^i(d(x_1, \dots, x_i)) \\ &\leq s^i \int_{\mathbb{X}^i} (2M_s(x_1))^{4+\varepsilon} \prod_{j=1}^i \mathbf{P} \{ D_y \xi_s(x_j, \mathcal{P}_s + \delta_{x_1} + \dots + \delta_{x_i}) \neq 0 \}^{\frac{p-\varepsilon}{4i+p^i}} \mathbb{Q}^i(d(x_1, \dots, x_i)) \\ &\leq 2^{4+\varepsilon} s^i \int_{\mathbb{X}^i} M_s(x_1)^{4+\varepsilon} \prod_{j=1}^i \mathbf{P} \{ y \in R_s(x_j, \mathcal{P}_s + \delta_{x_j}) \}^{\frac{p-\varepsilon}{4i+p^i}} \mathbb{Q}^i(d(x_1, \dots, x_i)). \end{aligned}$$

By (2.4),

$$\begin{aligned}
 2^{-4-\varepsilon} I_i &\leq s^i \int_{\mathbb{X}^i} M_s(x_1)^{4+\varepsilon} \prod_{j=1}^i \exp \left\{ -\frac{p-\varepsilon}{4i+pi} r_s(x_j, y) \right\} \mathbb{Q}^i(d(x_1, \dots, x_i)) \\
 &= \left(s \int_{\mathbb{X}} \exp \left\{ -\frac{p-\varepsilon}{4i+pi} r_s(x, y) \right\} \mathbb{Q}(dx) \right)^{i-1} \\
 &\quad \times \left(s \int_{\mathbb{X}} M_s(x)^{4+\varepsilon} \exp \left\{ -\frac{p-\varepsilon}{4i+pi} r_s(x, y) \right\} \mathbb{Q}(dx) \right) \\
 &\leq \left(s \int_{\mathbb{X}} \exp \left\{ -\frac{p}{40+10p} r_s(x, y) \right\} \mathbb{Q}(dx) \right)^{i-1} \\
 &\quad \times \left(s \int_{\mathbb{X}} M_s(x)^{4+\varepsilon} \exp \left\{ -\frac{p}{40+10p} r_s(x, y) \right\} \mathbb{Q}(dx) \right) \\
 &\leq g_s(y)^{i-1} h_s(y),
 \end{aligned}$$

where g_s and h_s are defined at (2.6). Since $g_s^{i-1} \leq 1 + g_s^4$ for all $i = 1, \dots, 5$, this proves the result for $\mu = 0$. If $\mu(\mathbb{X}) = 1$, the proof is similar, see the proof of [9, Lemma 5.5] for details. \square

Lemma 5.6. Assume that (A1) holds. For any $\beta > 0$, $s \geq 1$ and $x_2 \in \mathbb{X}$,

$$s \int_{\mathbb{X}} G_s(x_1) \mathbf{P} \{ D_{x_1, x_2}^2 H_s(\mathcal{P}_s) \neq 0 \}^\beta \mathbb{Q}(dx_1) \leq 3^\beta f_\beta(x_2)$$

with f_β defined at (2.8).

Proof. As in the proof of [9, Lemma 5.9(a)], by Lemma 5.2 and the Mecke formula, one has

$$\begin{aligned}
 &\mathbf{P} \{ D_{x_1, x_2}^2 H_s(\mathcal{P}_s) \neq 0 \} \\
 &\leq \mathbf{P} \{ D_{x_1} \xi_s(x_2, \mathcal{P}_s + \delta_{x_2}) \neq 0 \} + \mathbf{P} \{ D_{x_2} \xi_s(x_1, \mathcal{P}_s + \delta_{x_1}) \neq 0 \} + T_{x_1, x_2, s}, \quad (5.1)
 \end{aligned}$$

where

$$T_{x_1, x_2, s} := s \int_{\mathbb{X}} \mathbf{P} \{ D_{x_1, x_2}^2 \xi_s(z, \mathcal{P}_s + \delta_z) \neq 0 \} \mathbb{Q}(dz).$$

By Lemma 5.3 and (2.4), the first two summands on the right-hand side of (5.1) are bounded by $e^{-r_s(x_2, x_1)}$ and $e^{-r_s(x_1, x_2)}$, respectively. Furthermore, by Lemma 5.3 and (2.5),

$$T_{x_1, x_2, s} \leq s \int_{\mathbb{X}} \mathbf{P} \{ \{x_1, x_2\} \subseteq R_s(z, \mathcal{P}_s + \delta_z) \} \mathbb{Q}(dz) = q_s(x_1, x_2).$$

By (2.9),

$$\begin{aligned}
 &s \int_{\mathbb{X}} G_s(x_1) \mathbf{P} \{ D_{x_1, x_2}^2 H_s(\mathcal{P}_s) \neq 0 \}^\beta \mathbb{Q}(dx_1) \\
 &\leq 3^\beta \int_{\mathbb{X}} G_s(x_1) \left[e^{-\beta r_s(x_2, x_1)} + e^{-\beta r_s(x_1, x_2)} + q_s(x_1, x_2)^\beta \right] \mathbb{Q}(dx_1) = 3^\beta f_\beta(x_2). \quad \square
 \end{aligned}$$

Recall the function $\kappa_s(x)$ in (2.10).

Lemma 5.7. Assume that (A1) holds, and let $\beta > 0$. Then for all $s \geq 1$,

$$\begin{aligned}
 &s \int_{\mathbb{X}} \left(s \int_{\mathbb{X}} G_s(x_1) \mathbf{P} \{ D_{x_1, x_2}^2 H_s(\mathcal{P}_s) \neq 0 \}^\beta \mathbb{Q}(dx_1) \right)^2 \mathbb{Q}(dx_2) \leq s 3^{2\beta} \mathbb{Q} f_\beta^2, \\
 &s^2 \int_{\mathbb{X}^2} G_s(x_1) \mathbf{P} \{ D_{x_1, x_2}^2 H_s(\mathcal{P}_s) \neq 0 \}^\beta \mathbb{Q}^2(d(x_1, x_2)) \leq s 3^\beta \mathbb{Q} f_\beta, \\
 &s \int_{\mathbb{X}} G_s(x) \mathbf{P} \{ D_x H_s(\mathcal{P}_s) \neq 0 \}^\beta \mathbb{Q}(dx) \leq s \mathbb{Q}((\kappa_s + g_s)^\beta G_s).
 \end{aligned}$$

Proof. The first two assertions follow directly from Lemma 5.6. For the last one, by Lemma 5.2 and the Mecke formula, we can write

$$\begin{aligned} \mathbf{P}\{D_x H_s(\mathcal{P}_s) \neq 0\} &\leq \mathbf{P}\{\xi_s(x, \mathcal{P}_s + \delta_x) \neq 0\} + \mathbf{E} \sum_{z \in \mathcal{P}_s} \mathbf{1}_{D_x \xi_s(z, \mathcal{P}_s) \neq 0} \\ &= \kappa_s(x) + s \int_{\mathbb{X}} \mathbf{P}\{D_x \xi_s(z, \mathcal{P}_s + \delta_z) \neq 0\} \mathbf{Q}(dz) \leq \kappa_s(x) + g_s(x), \end{aligned}$$

where we used Lemma 5.3, (2.4) and (2.6) in the final step. This yields the final assertion. \square

Proof of Theorem 2.1: In view of Lemma 5.5, the condition in Theorem 5.1 is satisfied with the exponent $4 + p/2$ with $c_y := C_p [M_s(y)^{4+p/2} + h_s(y)(1 + g_s(y)^4)]$ for $y \in \mathbb{X}$. Hence,

$$\begin{aligned} &\max \left\{ c_y^{2/(4+p/2)}, c_y^{4/(4+p/2)} \right\} \\ &\leq C_p^{4/(4+p/2)} \left[\max \{ M_s(y)^2, M_s(y)^4 \} + \max \{ h_s(y)^{2/(4+p/2)}, h_s(y)^{4/(4+p/2)} \} (1 + g_s(y)^4) \right] \\ &= C_p^{4/(4+p/2)} G_s(y), \end{aligned}$$

where G_s is defined at (2.7). The result now follows from Theorem 5.1 upon using Lemma 5.7. \square

A Proof of Theorem 5.1

In this section, we prove Theorem 5.1, which is a slightly modified version of Theorem 6.1 in [10]. Recall that \mathcal{P} is a Poisson process on a measurable space $(\mathbb{X}, \mathcal{F})$ with a σ -finite intensity measure ν and $F := f(\mathcal{P})$ is a measurable function of \mathcal{P} . For $x, y \in \mathbb{X}$, recall the definitions of the first and second order difference operators $D_x F$ and $D_{x,y}^2 F$ and that of $\text{dom } D$ from Section 5.

We are generally interested in the Gaussian approximation of such a function F with zero mean and unit variance with the aim to bound the Wasserstein and the Kolmogorov distances between F and a standard normal random variable N . An important result in this direction was given in [10]. Define

$$\begin{aligned} \gamma_1 &:= 4 \left[\int_{\mathbb{X}^3} \left[\mathbf{E} (D_{x_1} F)^2 (D_{x_2} F)^2 \right]^{1/2} \left[\mathbf{E} (D_{x_1, x_3}^2 F)^2 (D_{x_2, x_3}^2 F)^2 \right]^{1/2} \nu^3(d(x_1, x_2, x_3)) \right]^{1/2}, \\ \gamma_2 &:= \left[\int_{\mathbb{X}^3} \mathbf{E} \left[(D_{x_1, x_3}^2 F)^2 (D_{x_2, x_3}^2 F)^2 \right] \nu^3(d(x_1, x_2, x_3)) \right]^{1/2}, \\ \gamma_3 &:= \int_{\mathbb{X}} \mathbf{E} |D_x F|^3 \nu(dx), \\ \gamma_4 &:= \frac{1}{2} [\mathbf{E} F^4]^{1/4} \int_{\mathbb{X}} \left[\mathbf{E} (D_x F)^4 \right]^{3/4} \nu(dx), \\ \gamma_5 &:= \left[\int_{\mathbb{X}} \mathbf{E} (D_x F)^4 \nu(dx) \right]^{1/2}, \\ \gamma_6 &:= \left[\int_{\mathbb{X}^2} \left(6 \left[\mathbf{E} (D_{x_1} F)^4 \right]^{1/2} \left[\mathbf{E} (D_{x_1, x_2}^2 F)^4 \right]^{1/2} + 3 \mathbf{E} (D_{x_1, x_2}^2 F)^4 \right) \nu^2(d(x_1, x_2)) \right]^{1/2}. \end{aligned}$$

Theorem A.1 ([10], Theorems 1.1 and 1.2). *For $F \in \text{dom } D$ having zero mean and unit variance,*

$$d_W(F, N) \leq \gamma_1 + \gamma_2 + \gamma_3,$$

and

$$d_K(F, N) \leq \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6.$$

Under additional assumptions on the difference operator, one can simplify the bound. This is done in [10, Theorem 6.1], assuming that, for some $q > 0$, the $(4 + q)$ -th moment of the difference operator $D_x F(\mathcal{P} + \mu)$ for $\mu \in \mathbf{N}$ with total mass at most one is uniformly bounded in $x \in \mathbb{X}$. However, in some applications, as is the case in the example of minimal points discussed in Section 3, such a uniform bound does not exist. In Theorem 5.1, we modify [10, Theorem 6.1] to allow for a non-uniform bound depending on x . Below, we present the proof of Theorem 5.1 for completeness, though the arguments remain largely similar to those in the proof of Theorem 6.1 in [10], with the main difference being the presence of a spatially inhomogeneous moment bound given by the function c_x .

Proof of Theorem 5.1. By our assumption, Hölder’s inequality yields that

$$\mathbf{E} (D_x F)^4 \leq \left[\mathbf{E} |D_x F|^{4+q} \right]^{4/(4+q)} \mathbf{P} \{D_x F \neq 0\}^{q/(4+q)} \leq c_x^{4/(4+q)} \mathbf{P} \{D_x F \neq 0\}^{q/(4+q)}$$

and

$$\mathbf{E} |D_x F|^3 \leq c_x^{3/(4+p)} \mathbf{P} \{D_x F \neq 0\}^{(1+q)/(4+q)} .$$

Also, using Hölder’s inequality as above and Jensen’s inequality in the second step, we have

$$\begin{aligned} \mathbf{E} (D_{x_1, x_2}^2 F)^4 &\leq \left[\mathbf{E} |D_{x_1, x_2}^2 F|^{4+q} \right]^{4/(4+q)} \mathbf{P} \{D_{x_1, x_2}^2 F \neq 0\}^{q/(4+q)} \\ &\leq 16 \min\{c_{x_1}, c_{x_2}\}^{4/(4+q)} \mathbf{P} \{D_{x_1, x_2}^2 F \neq 0\}^{q/(4+q)} . \end{aligned}$$

Thus, evaluating $(\gamma_i)_{1 \leq i \leq 6}$ for $(F - \mathbf{E}F)/\sqrt{\text{Var } F}$, we obtain

$$\begin{aligned} \gamma_1 &\leq \frac{8}{\text{Var } F} \left[\int_{\mathbb{X}^3} c_{x_1}^{2/(4+q)} c_{x_2}^{2/(4+q)} \mathbf{P} \{D_{x_1, x_3}^2 F \neq 0\}^{q/(16+4q)} \right. \\ &\quad \left. \times \mathbf{P} \{D_{x_2, x_3}^2 F \neq 0\}^{q/(16+4q)} \nu^3(d(x_1, x_2, x_3)) \right]^{1/2} \\ &= \frac{8}{\text{Var } F} \left[\int_{\mathbb{X}} \left(\int_{\mathbb{X}} c_{x_1}^{2/(4+q)} \mathbf{P} \{D_{x_1, x_2}^2 F \neq 0\}^{q/(16+4q)} \nu(dx_1) \right)^2 \nu(dx_2) \right]^{1/2} , \\ \gamma_2 &\leq \frac{4}{\text{Var } F} \left[\int_{\mathbb{X}^3} c_{x_1}^{2/(4+q)} c_{x_2}^{2/(4+q)} \mathbf{P} \{D_{x_1, x_3}^2 F \neq 0\}^{q/(8+2q)} \right. \\ &\quad \left. \times \mathbf{P} \{D_{x_2, x_3}^2 F \neq 0\}^{q/(8+2q)} \nu^3(d(x_1, x_2, x_3)) \right]^{1/2} \\ &\leq \frac{4}{\text{Var } F} \left[\int_{\mathbb{X}} \left(\int_{\mathbb{X}} c_{x_1}^{2/(4+q)} \mathbf{P} \{D_{x_1, x_2}^2 F \neq 0\}^{q/(16+4q)} \nu(dx_1) \right)^2 \nu(dx_2) \right]^{1/2} , \\ \gamma_3 &\leq \frac{1}{(\text{Var } F)^{3/2}} \int_{\mathbb{X}} c_x^{3/(4+q)} \mathbf{P} \{D_x F \neq 0\}^{(1+q)/(4+q)} \nu(dx) \leq \frac{\Gamma_F}{(\text{Var } F)^{3/2}} , \\ \gamma_4 &\leq \frac{1}{2(\text{Var } F)^2} [\mathbf{E}(F - \mathbf{E}F)^4]^{1/4} \int_{\mathbb{X}} c_x^{3/(4+q)} \mathbf{P} \{D_x F \neq 0\}^{q/(8+2q)} \nu(dx) \\ &\leq \frac{\Gamma_F}{2(\text{Var } F)^2} [\mathbf{E}(F - \mathbf{E}F)^4]^{1/4} , \\ \gamma_5 &\leq \frac{1}{\text{Var } F} \left[\int_{\mathbb{X}} c_x^{4/(4+q)} \mathbf{P} \{D_x F \neq 0\}^{q/(4+q)} \nu(dx) \right]^{1/2} \leq \frac{\Gamma_F^{1/2}}{\text{Var } F} , \end{aligned}$$

$$\begin{aligned} \gamma_6 &\leq \frac{2\sqrt{6}}{\text{Var } F} \left[\int_{\mathbb{X}^2} c_{x_1}^{4/(4+q)} \mathbf{P} \{D_{x_1, x_2}^2 F \neq 0\}^{q/(8+2q)} \nu^2(d(x_1, x_2)) \right]^{1/2} \\ &\quad + \frac{4\sqrt{3}}{\text{Var } F} \left[\int_{\mathbb{X}^2} c_{x_1}^{4/(4+q)} \mathbf{P} \{D_{x_1, x_2}^2 F \neq 0\}^{q/(4+q)} \nu^2(d(x_1, x_2)) \right]^{1/2} \\ &\leq \frac{2\sqrt{6} + 4\sqrt{3}}{\text{Var } F} \left[\int_{\mathbb{X}^2} c_{x_1}^{4/(4+q)} \mathbf{P} \{D_{x_1, x_2}^2 F \neq 0\}^{q/(8+2q)} \nu^2(d(x_1, x_2)) \right]^{1/2}. \end{aligned}$$

Finally, by [10, Lemma 4.3],

$$\begin{aligned} &\frac{\mathbf{E}(F - \mathbf{E}F)^4}{(\text{Var } F)^2} \\ &\leq \max \left\{ \frac{256}{(\text{Var } F)^2} \left[\int_{\mathbb{X}} [\mathbf{E}(D_x F)^4]^{1/2} \nu(dx) \right]^2, \frac{4}{(\text{Var } F)^2} \int_{\mathbb{X}} \mathbf{E}(D_x F)^4 \nu(dx) + 2 \right\} \\ &\leq \max \{ 256\Gamma_F^2 / (\text{Var } F)^2, 4\Gamma_F / (\text{Var } F)^2 + 2 \}, \end{aligned}$$

so that

$$\gamma_4 \leq \frac{1}{(\text{Var } F)^{3/2}} \Gamma_F + \frac{1}{(\text{Var } F)^2} \Gamma_F^{5/4} + \frac{2}{(\text{Var } F)^2} \Gamma_F^{3/2}.$$

An application of Theorem A.1 yields the results. □

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