

# TROPICAL FOCK-GONCHAROV COORDINATES FOR $SL_3$ -WEBS ON SURFACES II: NATURALITY

DANIEL C. DOUGLAS AND ZHE SUN

ABSTRACT. In a companion paper [DS20] we constructed non-negative integer coordinates  $\Phi_{\mathcal{T}}$  for a distinguished collection  $\mathcal{W}_{3,\widehat{S}}$  of  $SL_3$ -webs on a finite-type punctured surface  $\widehat{S}$ , depending on an ideal triangulation  $\mathcal{T}$  of  $\widehat{S}$ . We prove that these coordinates are natural with respect to the choice of triangulation, in the sense that if a different triangulation  $\mathcal{T}'$  is chosen then the coordinate change map relating  $\Phi_{\mathcal{T}}$  and  $\Phi_{\mathcal{T}'}$  is a prescribed tropical cluster transformation. Moreover, when  $\widehat{S} = \square$  is an ideal square, we provide a topological geometric description of the Hilbert basis (in the sense of linear programming) of the non-negative integer cone  $\Phi_{\mathcal{T}}(\mathcal{W}_{3,\square}) \subseteq \mathbb{Z}_{\geq 0}^{12}$ , and we prove that this cone canonically decomposes into 42 sectors corresponding topologically to 42 families of  $SL_3$ -webs in the square.

## 1. INTRODUCTION

For a finitely generated group  $\Gamma$  and for a suitable Lie group  $G$ , a primary object of study in higher Teichmüller theory [Wie18] is the  $G$ -character variety

$$\mathcal{R}_{G,\Gamma} = \{\rho : \Gamma \longrightarrow G\} // G$$

consisting of group homomorphisms from the group  $\Gamma$  to the Lie group  $G$ , considered up to conjugation. Here, the double bar indicates that the quotient is taken in the sense of Geometric Invariant Theory [MFK94]. In addition to generalizing classical Teichmüller theory, where the Lie group  $G = PGL_2(\mathbb{R})$ , higher Teichmüller theory intersects various mathematical fields such as Higgs bundles [Hit92], dynamics [Lab06], and representation theory [FG06].

We are interested in studying the character variety  $\mathcal{R}_{SL_n,S} := \mathcal{R}_{SL_n,\pi_1(S)}$  in the case where the group  $\Gamma = \pi_1(S)$  is the fundamental group of a finite-type surface  $S$ , and where the Lie group  $G = SL_n$  is the special linear group.

**Global aspects.** More precisely, let  $\widehat{S}$  be a decorated surface, namely a compact oriented surface together with a finite subset  $M \subseteq \partial\widehat{S}$  of preferred points, called marked points, lying on some of the boundary components of  $\widehat{S}$  and considered up to isotopy. By a puncture we mean a boundary component of  $\widehat{S}$  containing no marked points. We say the surface  $\widehat{S} = S$  is non-decorated if  $M = \emptyset$ , equivalently all of the boundary components of  $S$  are punctures. We always assume that the surface  $\widehat{S}$  admits an ideal triangulation  $\mathcal{T}$ , namely a triangulation whose vertex set is equal to the set of punctures and marked points.

---

*Date:* December 29, 2020.

This work was partially supported by the U.S. National Science Foundation grants DMS-1107452, 1107263, 1107367 “RNMS: GEometric structures And Representation varieties” (the GEAR Network). The first author was also partially supported by the U.S. National Science Foundation grants DMS-1406559 and 1711297, and the second author by the China Postdoctoral Science Foundation grant 2018T110084, the FNR AFR Bilateral grant COALAS 11802479-2, and the Huawei Young Talents Programme at IHES.

*Duality conjectures.* Fock and Goncharov [FG06] introduced a pair of mutually dual higher Teichmüller spaces  $\mathcal{X}_{\mathrm{PGL}_n, \widehat{S}}$  and  $\mathcal{A}_{\mathrm{SL}_n, \widehat{S}}$ . In the case  $\widehat{S} = S$  of non-decorated surfaces, the spaces  $\mathcal{X}_{\mathrm{PGL}_n, S}$  and  $\mathcal{A}_{\mathrm{SL}_n, S}$  are variations of the  $\mathrm{PGL}_n$ - and  $\mathrm{SL}_n$ -character varieties, generalizing the enhanced Teichmüller space [FG07a] and the decorated Teichmüller space [Pen87], respectively. Fock and Goncharov conjectured [FG06, FG09] that there is a duality mapping

$$\mathbb{I} : \mathcal{A}_{\mathrm{SL}_n, S}(\mathbb{Z}^t) \longrightarrow \mathcal{O}(\mathcal{X}_{\mathrm{PGL}_n, S}),$$

from the discrete set  $\mathcal{A}_{\mathrm{SL}_n, S}(\mathbb{Z}^t)$  of tropical integer points of the moduli space  $\mathcal{A}_{\mathrm{SL}_n, S}$  to the algebra  $\mathcal{O}(\mathcal{X}_{\mathrm{PGL}_n, S})$  of regular functions on the moduli space  $\mathcal{X}_{\mathrm{PGL}_n, S}$ , satisfying enjoyable properties; for instance, the image of  $\mathbb{I}$  should form a linear basis for the algebra of functions  $\mathcal{O}(\mathcal{X}_{\mathrm{PGL}_n, S})$ . Fock and Goncharov gave an explicit proof of their conjecture in the case  $n = 2$ , by identifying the tropical integer points with laminations on the surface, which in turn correspond to the classical trace functions on the character variety.

There are many incarnations of this so-called Fock-Goncharov Duality Conjecture. Another version is (compare [FG06, Theorem 12.3 and the following Remark] for when  $n = 2$ )

$$\mathbb{I} : \mathcal{A}_{\mathrm{PGL}_n, S}(\mathbb{Z}^t) \longrightarrow \mathcal{O}(\mathcal{X}_{\mathrm{SL}_n, S}).$$

There are also formulations of these conjectures in the setting of decorated surfaces  $\widehat{S}$ , where the moduli spaces  $\mathcal{X}_{\mathrm{PGL}_n, \widehat{S}}$  and  $\mathcal{X}_{\mathrm{SL}_n, \widehat{S}}$  are replaced [GS15, GS19] by slightly more general versions  $\mathcal{P}_{\mathrm{PGL}_n, \widehat{S}}$  and  $\mathcal{P}_{\mathrm{SL}_n, \widehat{S}}$ . These conjectures have spurred intensive development of the field, and works such as [GHKK18, GS18], by employing powerful conceptual methods, have to a large extent completely settled very general formulations of duality. On the other hand, explicit higher rank constructions, in the spirit of Fock and Goncharov's original approach in the case  $n = 2$ , have remained elusive.

Following Goncharov and Shen [GS15] (see also [FG06, Proposition 12.2]), we focus on the positive points  $\mathcal{A}_{\mathrm{PGL}_n, \widehat{S}}^+(\mathbb{Z}^t) \subseteq \mathcal{A}_{\mathrm{PGL}_n, \widehat{S}}(\mathbb{Z}^t)$  defined with respect to the tropicalized Goncharov-Shen potential function  $P^t : \mathcal{A}_{\mathrm{PGL}_n, \widehat{S}}(\mathbb{Z}^t) \rightarrow \mathbb{Z}$  by  $\mathcal{A}_{\mathrm{PGL}_n, \widehat{S}}^+(\mathbb{Z}^t) = (P^t)^{-1}(\mathbb{Z}_{\geq 0})$ . The advantage of doing so lies in yet another version of duality, closely related to the previously mentioned one,

$$(*) \quad \mathbb{I} : \mathcal{A}_{\mathrm{PGL}_n, \widehat{S}}^+(\mathbb{Z}^t) \longrightarrow \mathcal{O}(\mathcal{R}_{\mathrm{SL}_n, \widehat{S}}^{\mathrm{GS}})$$

where the space  $\mathcal{R}_{\mathrm{SL}_n, \widehat{S}}^{\mathrm{GS}}$ , introduced in [GS15, §10.2] (they denote it by  $\mathrm{Loc}_{\mathrm{SL}_n, \widehat{S}}$ ), is a generalized (twisted) version of the  $\mathrm{SL}_n$ -character variety  $\mathcal{R}_{\mathrm{SL}_n, S}$  valid for decorated surfaces  $\widehat{S}$ ; note, for non-decorated surfaces  $\widehat{S} = S$ , that  $\mathcal{R}_{\mathrm{SL}_n, S}^{\mathrm{GS}} = \mathcal{R}_{\mathrm{SL}_n, S}$  when  $n$  is odd. We will see that this last formulation  $(*)$  of duality expresses strong interactions between the geometry and topology of character varieties and the representation theory of  $\mathrm{SL}_n$ .

Because  $\mathrm{PGL}_n$  is not simply connected, the moduli space  $\mathcal{A}_{\mathrm{PGL}_n, \widehat{S}}$  does not have a cluster structure. However, the tropicalized space  $\mathcal{A}_{\mathrm{PGL}_n, \widehat{S}}^+(\mathbb{Z}^t)$  can be viewed as a subset of the space  $\mathcal{A}_{\mathrm{SL}_n, \widehat{S}}(\mathbb{R}^t)$ , which does possess a tropical cluster structure. Our goal is to find a natural topological geometric model for the space  $\mathcal{A}_{\mathrm{PGL}_n, \widehat{S}}^+(\mathbb{Z}^t)$  which also exhibits this tropical cluster structure. In particular, the present paper focuses on this problem in the case  $n = 3$ ; for the geometric theory underlying this case, see [FG07b, CTT20].

*Topological indexing of linear bases.* A key principle in studying duality is that tropical integer points should correspond naturally to topological geometric objects generalizing laminations [Thu97] on surfaces in the case  $n = 2$ . Such so-called higher laminations have been studied in [Fon12, FKK13, GMN13, Xie13, GMN14, GS15, Le16, Mar19, OT19, SWZ20] in terms of affine Grassmannians, affine buildings, spectral networks, and other related geometric objects. Our approach prefers the more topological  $n$ -webs [RT90, Kup96, MOY98, Sik05, CKM14, FP16], which are certain  $n$ -valent graphs-with-boundary embedded in the surface  $\widehat{S}$ , considered up to equivalence. Webs are motivated by the rich geometric, topological, and algebraic relationships between character varieties and skein modules [Pro76, Tur89, Wit89, BHMV95, Bul97, BFKB99, PS00, Sik01, BW11a, BW11b, GJS19].

We begin by reviewing the case  $n = 2$ . For a decorated surface  $\widehat{S}$ , define the set  $\mathcal{L}_{2,\widehat{S}}$  of 2-laminations on  $\widehat{S}$  so that  $\ell \in \mathcal{L}_{2,\widehat{S}}$  is a finite collection of mutually-non-intersecting simple loops and arcs on  $\widehat{S}$ , considered up to isotopy, such that (i) there are no contractible loops, and (ii) arcs end only on boundary components of  $\widehat{S}$  containing marked points, and there are no arcs which contract to a boundary interval containing no marked points.

In the case where the surface  $\widehat{S} = S$  is non-decorated, a lamination  $\ell \in \mathcal{L}_{2,S}$  corresponds to a trace function  $\text{Tr}_\ell$ , which is a regular function on the character variety  $\mathcal{R}_{\text{SL}_2,S}$  defined simply by taking the trace along the lamination  $\ell$ . It is well-known [Prz91, HP93, Bul97, PS00, CM12] that the trace functions  $\text{Tr}_\ell$  varying over the 2-laminations  $\ell \in \mathcal{L}_{2,S}$  form a linear basis for the algebra  $\mathcal{O}(\mathcal{R}_{\text{SL}_2,S})$  of regular functions on the character variety.

On the opposite topological extreme, consider the case where the surface  $\widehat{S} = \widehat{D}$  is a disk with  $k$  marked points  $m_i$  on its boundary, cyclically ordered. For each  $i$ , assign a positive integer  $n_i$  to the  $i$ -th boundary interval located between the marked points  $m_i$  and  $m_{i+1}$ . This determines a subset  $\mathcal{L}_{2,\widehat{D}}(n_1, \dots, n_k) \subseteq \mathcal{L}_{2,\widehat{D}}$  consisting of the 2-laminations  $\ell$  having geometric intersection number equal to  $n_i$  on the  $i$ -th boundary interval. It follows from the Clebsch-Gordan theorem (see, for instance, [Kup96, §2.2,2.3]) that the subset  $\mathcal{L}_{2,\widehat{D}}(n_1, \dots, n_k)$  of 2-laminations indexes a linear basis for the space of  $\text{SL}_2$ -invariant tensors  $(V_{n_1} \otimes \dots \otimes V_{n_k})^{\text{SL}_2}$  where  $V_{n_i}$  is the unique  $n_i$ -dimensional irreducible representation of  $\text{SL}_2$ .

In the general case of a decorated surface  $\widehat{S}$ , Goncharov-Shen's moduli space  $\mathcal{R}_{\text{SL}_2,\widehat{S}}^{\text{GS}}$  simultaneously generalizes both (a twisted version of) the character variety  $\mathcal{R}_{\text{SL}_2,S}$  for non-decorated surfaces  $\widehat{S} = S$ , as well as the space of invariant tensors  $(V_{n_1} \otimes V_{n_2} \otimes \dots \otimes V_{n_k})^{\text{SL}_2}$  for marked disks  $\widehat{S} = \widehat{D}$ . Theorem 10.14 of [GS15] says that the set of 2-laminations  $\mathcal{L}_{2,\widehat{S}}$  naturally indexes a linear basis for the algebra of functions  $\mathcal{O}(\mathcal{R}_{\text{SL}_2,\widehat{S}}^{\text{GS}})$  on the generalized character variety for the decorated surface  $\widehat{S}$ , closely related to the linear bases in the specialized cases  $\widehat{S} = S$  and  $\widehat{S} = \widehat{D}$ . Moreover, Theorem 10.15 of [GS15] identifies the set of 2-laminations  $\mathcal{L}_{2,\widehat{S}}$  with the set of positive tropical integer points  $\mathcal{A}_{\text{PGL}_2,\widehat{S}}^+(\mathbb{Z}^t)$ . Taken together, their two theorems establish a compelling form of the duality (\*).

We turn now to the case  $n = 3$ . In the setting of the disk  $\widehat{S} = \widehat{D}$  with  $k$  marked points on its boundary, the integers  $n_i$  are now replaced with highest weights  $\lambda_i$  of irreducible  $\text{SL}_3$ -representations  $V_{\lambda_i}$ , and the object of interest is the space  $(V_{\lambda_1} \otimes V_{\lambda_2} \otimes \dots \otimes V_{\lambda_k})^{\text{SL}_3}$  of  $\text{SL}_3$ -invariant tensors. Kuperberg [Kup96] realized that the correct  $\text{SL}_3$ -analogue  $\mathcal{W}_{3,\widehat{D}}(\lambda_1, \dots, \lambda_k)$  of the subset  $\mathcal{L}_{2,\widehat{D}}(n_1, \dots, n_k) \subseteq \mathcal{L}_{2,\widehat{D}}$  of 2-laminations, indexing a linear

basis for the invariant space  $(V_{\lambda_1} \otimes V_{\lambda_2} \otimes \cdots \otimes V_{\lambda_k})^{\text{SL}_3}$ , consists of the non-convex non-elliptic 3-webs  $W$  on  $\widehat{D}$  that match certain fixed boundary data corresponding to the weights  $\lambda_i$ .

On the other hand, namely for non-decorated surfaces  $\widehat{S} = S$ , Sikora [Sik01] defined, for general  $n$  and for any  $n$ -web  $W$  on  $S$ , a trace function  $\text{Tr}_W$  on the character variety  $\mathcal{R}_{\text{SL}_n, S}$  generalizing the trace functions  $\text{Tr}_\ell$  for 2-laminations  $\ell \in \mathcal{L}_{2, S}$ . When  $n = 3$ , let  $\mathcal{W}_{3, S}$  denote the subset of closed non-elliptic 3-webs on  $S$ . Sikora-Westbury [SW07] proved that the trace functions  $\text{Tr}_W$  associated to the 3-webs  $W \in \mathcal{W}_{3, S}$  form a linear basis for the algebra of functions  $\mathcal{O}(\mathcal{R}_{\text{SL}_3, S})$  on the  $\text{SL}_3$ -character variety.

For a decorated surface  $\widehat{S}$ , Frohman-Sikora's recent work [FS20] suggests that the correct definition for the 3-laminations in general is the subset  $\mathcal{W}_{3, \widehat{S}}$  of reduced non-elliptic 3-webs  $W$  on  $\widehat{S}$ , which in particular are allowed to have boundary. Indeed, this subset  $\mathcal{W}_{3, \widehat{S}}$  forms a linear basis [SW07, FS20] for the reduced  $\text{SL}_3$ -skein algebra, which can be thought of as a  $\text{SL}_3$ -version of Muller's  $\text{SL}_2$ -skein algebra [Mul16, CL19, Hig20, IY]. Generalizing the case of non-decorated surfaces, where skein algebras quantize character varieties, we suspect that Frohman-Sikora's reduced  $\text{SL}_3$ -skein algebra is a deformation quantization of Goncharov-Shen's generalized  $\text{SL}_3$ -character variety  $\mathcal{R}_{\text{SL}_3, \widehat{S}}^{\text{GS}}$ . We also expect that the set  $\mathcal{W}_{3, \widehat{S}}$  indexes a natural linear basis for  $\mathcal{O}(\mathcal{R}_{\text{SL}_3, \widehat{S}}^{\text{GS}})$  generalizing [GS15, Theorem 10.14] in the case  $n = 2$ .

*Coordinates for laminations and webs.* As in [FG06, FG07a], in the case  $n = 2$ , given a choice of ideal triangulation  $\mathcal{T}$  of the decorated surface  $\widehat{S}$  with  $N_2$  edges, assign  $N_2$  positive integer coordinates to a given 2-lamination  $\ell \in \mathcal{L}_{2, \widehat{S}}$  by taking the geometric intersection numbers of  $\ell$  with the edges of the ideal triangulation  $\mathcal{T}$ . Note that  $N_2$  is independent of the choice of  $\mathcal{T}$ . It is well-known that this determines an injective coordinate mapping

$$\Phi_{\mathcal{T}}^{(2)} : \mathcal{L}_{2, \widehat{S}} \hookrightarrow \mathbb{Z}_{\geq 0}^{N_2}$$

for the set of 2-laminations  $\mathcal{L}_{2, \widehat{S}}$ . In addition, the image of  $\Phi_{\mathcal{T}}^{(2)}$  in  $\mathbb{Z}_{\geq 0}^{N_2}$  can be characterized as the set of solutions of finitely many inequalities and parity conditions of the form

$$\frac{a + b - c}{2} \in \mathbb{Z}_{\geq 0} \quad (a, b, c \in \mathbb{Z}).$$

Moreover, these positive integer coordinates are natural, in the sense of Theorem 1.1 below, and thus provide the natural identification  $\mathcal{L}_{2, \widehat{S}} \cong \mathcal{A}_{\text{PGL}_2, \widehat{S}}^+(\mathbb{Z}^t)$  as in [GS15, Theorem 10.15].

The main theorem of this and the preceding companion paper [DS20] is a generalization of these coordinates to the setting of  $\text{SL}_3$ . More precisely, given an ideal triangulation  $\mathcal{T}$  of a decorated surface  $\widehat{S}$ , let  $N$  denote twice the number of edges (including boundary edges) plus the number of triangles of  $\mathcal{T}$ . Recall that the set  $\mathcal{W}_{3, \widehat{S}}$  consists of the reduced non-elliptic 3-webs on  $\widehat{S}$ . Define a positive integer cone to mean a sub-monoid of  $\mathbb{Z}_{\geq 0}^k$  for an integer  $k$ .

**Theorem 1.1.** *Given an ideal triangulation  $\mathcal{T}$  of  $\widehat{S}$ , there is an injection*

$$\Phi_{\mathcal{T}} : \mathcal{W}_{3, \widehat{S}} \hookrightarrow \mathbb{Z}_{\geq 0}^N$$

*satisfying the property that the image  $\Phi_{\mathcal{T}}(\mathcal{W}_{3, \widehat{S}}) \subseteq \mathbb{Z}_{\geq 0}^N$  is a positive integer cone, which can be characterized completely as the set of solutions of finitely many Knutson-Tao rhombus inequalities [KT99, Appendix 2] and modulo 3 congruence conditions of the form*

$$\frac{a + b - c - d}{3} \in \mathbb{Z}_{\geq 0} \quad \text{and} \quad \frac{a + b - c}{3} \in \mathbb{Z}_{\geq 0} \quad (a, b, c, d \in \mathbb{Z}).$$

Moreover, these coordinates are natural, in the sense that if a different ideal triangulation  $\mathcal{T}'$  is chosen, then the coordinate change map relating  $\Phi_{\mathcal{T}}$  and  $\Phi_{\mathcal{T}'}$  is given by a tropical  $\mathcal{A}$ -coordinate cluster transformation; see [FZ02] and [FG06, §10 and Equation (12.5)].

See Theorems 3.3 and 4.4. Our construction was motivated by earlier work of Xie [Xie13]. Frohman-Sikora [FS20] also recently constructed positive integer coordinates for the set  $\mathcal{W}_{3,\widehat{S}}$  of basis 3-webs, however they did not address the question of naturality.

For a  $\mathrm{SL}_n$ -version of Theorem 1.1, see [Sun].

Generalizing the  $n = 2$  case [GS15, Theorem 10.15], these positive integer coordinates provide a natural identification  $\mathcal{W}_{3,\widehat{S}} \cong \mathcal{A}_{\mathrm{PGL}_{3,\widehat{S}}}^+(\mathbb{Z}^t)$  between the set of basis 3-webs and Goncharov-Shen's positive tropical integer points. In other words, the 3-webs  $\mathcal{W}_{3,\widehat{S}}$  are a concrete topological model for  $\mathcal{A}_{\mathrm{PGL}_{3,\widehat{S}}}^+(\mathbb{Z}^t)$ . If it is also true that  $\mathcal{W}_{3,\widehat{S}}$  naturally indexes a linear basis for the algebra of functions  $\mathcal{O}(\mathcal{R}_{\mathrm{SL}_{3,\widehat{S}}}^{\mathrm{GS}})$  on Goncharov-Shen's extended  $\mathrm{SL}_3$ -character variety, thereby generalizing [GS15, Theorem 10.14], then these two results would together establish an explicit  $\mathrm{SL}_3$ -version of the duality (\*).

As an application, Kim [Kim20] recently proved an explicit  $\mathrm{SL}_3$ -version of the original formulation of Fock-Goncharov duality, using the coordinates of Theorem 1.1 as well as our proof of naturality (see Corollary 4.5). More precisely, for non-decorated surfaces  $\widehat{S} = S$ , he used these coordinates to index a linear basis for the algebra of functions  $\mathcal{O}(\mathcal{X}_{\mathrm{PGL}_{3,S}})$  by the points of  $\mathcal{A}_{\mathrm{SL}_{3,S}}(\mathbb{Z}^t)$ . As a cautionary conventional note, when studying this original formulation of duality, one typically (as in [FG06, Xie13, Kim20]) re-scales the coordinates of Theorem 1.1 or its  $\mathrm{SL}_n$ -generalization by a factor of  $1/n$ . This is because, geometrically, what matters is then the proper subset of  $n$ -laminations having integer coordinates. For instance, in the case  $n = 2$ , this amounts to considering only those 2-laminations  $\ell$  having even geometric intersection numbers with the edges of the ideal triangulation  $\mathcal{T}$ . Lastly, Kim's approach, together with the  $\mathrm{SL}_3$ -quantum trace map [Dou20, Dou], should lead to an explicit  $\mathrm{SL}_3$ -version of quantum Fock-Goncharov duality; see [AK17] for the  $n = 2$  case.

As another future application, our work can be combined with the work of Knutson-Tao [KT99, Appendix 2] on the Clebsch-Gordan theorem to give a new proof of Kuperberg's theorem for  $\mathrm{SL}_3$  [Kup96, Theorem 6.1], in the same spirit as his direct proof of the analogous statement for  $\mathrm{SL}_2$  [Kup96, Theorem 2.4].

**Local aspects.** Theorem 1.1, minus the naturality, was proved in [DS20] in the case where the surface  $\widehat{S} = S$  is non-decorated. Essentially the same proof given there can be used to prove the more general formulation, Theorem 1.1, for decorated surfaces  $\widehat{S}$ ; see also [Kim20].

The first main result of the present work is a proof of the naturality appearing in Theorem 1.1. This is a completely local statement. Indeed, a well-known fact says that any two ideal triangulations  $\mathcal{T}$  and  $\mathcal{T}'$  are related by a sequence of diagonal flips inside of ideal squares. It thus suffices to check the desired coordinate change formula for a single square.

The construction of the coordinate map  $\Phi_{\mathcal{T}}$  from Theorem 1.1 proceeds as follows. From the ideal triangulation  $\mathcal{T}$  form the split ideal triangulation  $\widehat{\mathcal{T}}$  (Figure 3) by replacing each edge  $E$  of  $\mathcal{T}$  with two parallel edges  $E'$  and  $E''$ , in other words we fatten each edge  $E$  into a biangle. The resulting split ideal triangulation  $\widehat{\mathcal{T}}$  thus consists of biangles and triangles.

We then put a given 3-web  $W \in \mathcal{W}_{3,\widehat{S}}$  into good position with respect to the split ideal triangulation  $\widehat{\mathcal{T}}$ . The result is that most of the complexity of the 3-web  $W$  is pushed into the biangles (Figure 4), whereas over each triangle there is only a single (possibly empty)

honeycomb together with finitely many arcs lying on the corners (Figure 6). For an earlier appearance of these honeycomb webs in ideal triangles, see [Kup96, pp. 140-141], and for a similar application of split ideal triangulations  $\widehat{\mathcal{T}}$ , see [BW11b, p. 1596]. Once the web  $W$  is in good position, its coordinates  $\Phi_{\mathcal{T}}(W) \in \mathbb{Z}_{\geq 0}^N$  are easily computed in practice (Figure 7).

By a straightforward topological combinatorial count, we show that, given a 3-web  $W \in \mathcal{W}_{3,\widehat{S}}$  in good position, the restriction  $W|_{\square}$  of  $W$  to a triangulated ideal square  $(\square, \mathcal{T}|_{\square}) \subseteq (\widehat{S}, \mathcal{T})$  falls into one of 42 families  $\mathcal{F}_{\mathcal{T}|_{\square}}^k \subseteq \mathcal{W}_{3,\square}$  for  $k = 1, 2, \dots, 42$  depending on  $\mathcal{T}|_{\square}$ . Note that there are two possible ideal triangulations  $\mathcal{T}|_{\square}$  corresponding to the two diagonals of the ideal square  $\square$ . To prove the naturality in Theorem 1.1 under flipping the diagonal of  $(\square, \mathcal{T}|_{\square})$  we check the desired coordinate change formula for each of these 42 families  $\mathcal{F}_{\mathcal{T}|_{\square}}^k \subseteq \mathcal{W}_{3,\square}$  which reduces to 9 calculations up to symmetry. Key to the solution is an explicit topological description of how the 3-web restrictions  $W|_{\square}$  re-arrange themselves into good position after the flip. At first glance, our solution might seem quite complicated, because it requires a by-hand check of many cases. However, this complication arises in a natural topological way, so it is not clear at present how it can be circumvented. In summary, the topology, despite being complex, is tractable enough for us to extract a solution.

To make matters more interesting, these 42 families of 3-webs in the square appear to have a deeper geometric significance, leading to our second main result, which is of a purely local nature. Repeating ourselves, let  $\square$  be the disk with four marked points, namely the ideal square, and let  $\mathcal{T}$  be an ideal triangulation of  $\square$ , namely a choice of a diagonal of  $\square$ . Theorem 1.1 says that the collection  $\mathcal{W}_{3,\square}$  of 3-webs in  $\square$  embeds via  $\Phi_{\mathcal{T}}$  as a positive integer cone inside  $\mathbb{Z}_{\geq 0}^{12}$ . A classical theorem, commonly employed in linear programming, implies that this positive integer cone  $\Phi_{\mathcal{T}}(\mathcal{W}_{3,\square}) \subseteq \mathbb{Z}_{\geq 0}^{12}$  possesses a unique subset of minimal elements spanning the cone over  $\mathbb{Z}_{\geq 0}$ , called its Hilbert basis [Gor73, Hil90, vdC31, GP79, Sch81].

**Theorem 1.2.** *Given an ideal triangulation  $\mathcal{T}$  of the ideal square  $\square$ , namely a choice of a diagonal of  $\square$ , the Hilbert basis for the positive integer cone  $\Phi_{\mathcal{T}}(\mathcal{W}_{3,\square}) \subseteq \mathbb{Z}_{\geq 0}^{12}$  consists of 22 elements, corresponding via  $\Phi_{\mathcal{T}}$  to 22 webs  $W_{\mathcal{T}}^i \in \mathcal{W}_{3,\square}$  for  $i = 1, 2, \dots, 22$  depending on  $\mathcal{T}$ .*

*Moreover, the positive integer cone*

$$\Phi_{\mathcal{T}}(\mathcal{W}_{3,\square}) = \bigcup_{k=1}^{42} \Delta_{\mathcal{T}}^k \subseteq \mathbb{Z}_{\geq 0}^{12}$$

*can be decomposed into 42 sectors  $\Delta_{\mathcal{T}}^k$  such that (i) each sector is generated over  $\mathbb{Z}_{\geq 0}$  by 12 of the 22 Hilbert basis elements, and (ii) adjacent sectors are separated by a co-dimension 1 wall. This sector decomposition satisfies the property that the 42 sectors  $\Delta_{\mathcal{T}}^k$  are in canonical one-to-one correspondence with the 42 families  $\mathcal{F}_{\mathcal{T}}^k \subseteq \mathcal{W}_{3,\square}$  of 3-webs in the square appearing in the proof of Theorem 1.1, which we remind also depend on the ideal triangulation  $\mathcal{T}$ .*

*Lastly, each family  $\mathcal{F}_{\mathcal{T}}^k \subseteq \mathcal{W}_{3,\square}$  contains 12 distinguished 3-webs  $W_{\mathcal{T}}^{i(k)j} \in \{W_{\mathcal{T}}^i\}_i$  for  $j = 1, 2, \dots, 12$  corresponding to the 12 Hilbert basis elements generating the sector  $\Delta_{\mathcal{T}}^k$ . We refer to these 12 distinguished 3-webs  $\{W_{\mathcal{T}}^{i(k)j}\}_j$ , taken together, as the topological type of the sector  $\Delta_{\mathcal{T}}^k$ . The final statement is that two sectors  $\Delta_{\mathcal{T}}^k$  and  $\Delta_{\mathcal{T}}^{k'}$  are adjacent if and only if the topological types of those two sectors differ by exactly one distinguished 3-web.*

There is also a formulation of Theorem 1.2 using real coefficients, which is essentially the same; see Theorems 5.10 and 6.8.

For an earlier appearance of Hilbert bases, in the  $n = 2$  situation, see [AF17].

We suspect that Theorem 1.2 is related to the wall-crossing phenomenon of [KS08].

The proof of Theorem 1.2 is geometric in nature and might be of independent interest. Loosely speaking, in the theory of Fock and Goncharov, there are two dual sets of coordinates for the two dual moduli spaces of interest, respectively, the  $\mathcal{A}$ -coordinates and the  $\mathcal{X}$ -coordinates, as well as their tropical versions. For a triangulated ideal square  $(\square, \mathcal{T})$ , via the mapping  $\Phi_{\mathcal{T}}$  each 3-web  $W \in \mathcal{W}_{3,\square}$  is assigned 12 positive tropical integer  $\mathcal{A}$ -coordinates  $\Phi_{\mathcal{T}}(W) \in \mathbb{Z}_{\geq 0}^{12}$ . We show that there are also assigned to  $W$  four internal tropical integer  $\mathcal{X}$ -coordinates valued in  $\mathbb{Z}$ , two associated to the unique internal edge of  $\mathcal{T}$  and two associated to the two triangles of  $\mathcal{T}$ . It turns out that the decomposition of the cone  $\Phi_{\mathcal{T}}(\mathcal{W}_{3,\square}) \subseteq \mathbb{Z}_{\geq 0}^{12}$  into 42 sectors is mirrored by a corresponding decomposition of the lattice  $\mathbb{Z}^4$  into 42 sectors, determined by inequalities among the tropical  $\mathcal{X}$ -coordinates. We suspect that this is a concrete manifestation of Fock-Goncharov's tropicalized forgetful map

$$p^t : \mathcal{W}_{3,\square} \cong \Phi_{\mathcal{T}}(\mathcal{W}_{3,\square}) \cong \mathcal{A}_{\text{PGL}_{3,\square}}^+(\mathbb{Z}^t) \subseteq \mathcal{A}_{\text{SL}_{3,\square}}(\mathbb{R}^t) \xrightarrow{\text{FG}} \mathcal{X}_{\text{PGL}_{3,\square}}(\mathbb{R}^t).$$

The image of the mapping  $p^t$  is  $\mathcal{X}_{\text{PGL}_{3,\square}}(\mathbb{Z}^t) \cong \mathbb{Z}^4$ , and in addition  $p^t$  maps sectors of the positive integer cone  $\Phi_{\mathcal{T}}(\mathcal{W}_{3,\square}) \cong \mathcal{A}_{\text{PGL}_{3,\square}}^+(\mathbb{Z}^t)$  to sectors of the integer lattice  $\mathcal{X}_{\text{PGL}_{3,\square}}(\mathbb{Z}^t) \cong \mathbb{Z}^4$ . Moreover, the restriction of the mapping  $p^t$  to the subset  $\mathcal{W}_{3,\square}^0 \subseteq \mathcal{W}_{3,\square}$  consisting of the 3-webs in the square without corner arcs, is an injection onto the lattice  $\mathcal{X}_{\text{PGL}_{3,\square}}(\mathbb{Z}^t) \cong \mathbb{Z}^4$ .

#### ACKNOWLEDGEMENTS

We are profoundly grateful to Dylan Allegretti, Francis Bonahon, Charlie Frohman, Sasha Goncharov, Linhui Shen, and Daping Weng for many very helpful conversations and for generously offering their time as this project developed. Much of this work was completed during very enjoyable visits to Tsinghua University in Beijing, supported by a GEAR graduate internship grant, and the University of Southern California in Los Angeles. We would like to take this opportunity to extend our enormous gratitude to these institutions for their warm hospitality (and many tasty dinners).

#### 2. PRELIMINARY FOR TROPICAL POINTS

In this section, we recall some preliminaries in order to define the set  $\mathcal{A}_{\text{PGL}_{3,\hat{S}}}^+(\mathbb{Z}^t)$  of tropical points corresponding to the space of reduced webs that we study, including the Fock-Goncharov  $\mathcal{A}$  moduli space and its parametrization, the Goncharov-Shen potential.

**Definition 2.1** (Decorated flags). *Let  $E$  be a 3-dimensional vector space. A flag  $F$  in  $E$  is a maximal filtration of vector subspaces of  $E$ :*

$$\{0\} = F^{(0)} \subseteq F^{(1)} \subseteq F^{(2)} \subseteq F^{(3)} = E, \quad \dim F^{(i)} = i,$$

*denoted by  $(F^{(1)}, F^{(2)})$ .*

*A decorated flag  $(F, \varphi)$  is a pair consisting of a flag  $F$  and a collection  $\varphi$  of 2 non-zero vectors*

$$\varphi = \{\check{f}_i \in F^{(i)}/F^{(i-1)}\}_{i=1,2}.$$

*A basis for a decorated flag  $(F, \varphi)$  is a basis  $(f_1, f_2, f_3)$  for the vector space  $E$  such that*

$$f_i + F^{(i-1)} = \check{f}_i \in F^{(i)}/F^{(i-1)} \quad \text{for } i = 1, 2,$$

*and  $\Omega(f_1 \wedge f_2 \wedge f_3) = 1$  for a fixed volume form in  $\mathbb{R}^3$ .*

*The space of decorated flags is denoted by  $\mathcal{A}$ .*

**Definition 2.2** ([FG06, Definition 2.4, page 38],  $\mathcal{A}$ -moduli space  $\mathcal{A}_{\mathrm{SL}_3, \hat{S}}$ ). *The decorated surface  $\hat{S}$  is a pair  $(S, m_b)$  where  $S$  is a connected oriented topological surface of negative Euler characteristic with  $m \geq 1$  holes and  $m_b$  is the set of marked points on  $\partial S$  considered up to isotopy. Let  $m_p = \{p_1, \dots, p_n\}$  be the set of punctures, which is the set of holes without marked points. Let  $\alpha_i$  be the oriented peripheral closed curve around  $p_i$  such that  $p_i$  is on the right side of  $\alpha_i$ . A decorated  $\mathrm{SL}_3$ -local system on  $\hat{S}$  is a pair  $(\rho, \bar{\xi})$  consisting of*

- a surface group representation  $\rho \in \mathrm{Hom}(\pi_1(S), \mathrm{SL}_3)$  with unipotent boundary monodromy and
- a map  $\bar{\xi} : m_b \cup m_p \rightarrow \mathcal{A}$ , such that each  $\rho(\alpha_i)$  fixes the decorated flag  $\bar{\xi}(p_i) \in \mathcal{A}$ .

Two decorated  $\mathrm{SL}_3$ -local systems  $(\rho_1, \bar{\xi}_1), (\rho_2, \bar{\xi}_2)$  are equivalent if and only if there exists some  $g \in \mathrm{SL}_3$  such that  $\rho_2 = g\rho_1g^{-1}$  and  $\bar{\xi}_2 = g\bar{\xi}_1$ . We denote the moduli space of decorated  $\mathrm{SL}_3$ -local systems on  $\hat{S}$  by  $\mathcal{A}_{\mathrm{SL}_3, \hat{S}}$ .

**Definition 2.3** (3-triangulation). *Given an ideal triangulation  $\mathcal{T}$  of  $\hat{S}$ , we define the 3-triangulation  $\mathcal{T}_3$  of  $\mathcal{T}$  to be the triangulation of  $\hat{S}$  obtained by subdividing each triangle of  $\mathcal{T}$  into 9 triangles (as in Figure 1).*

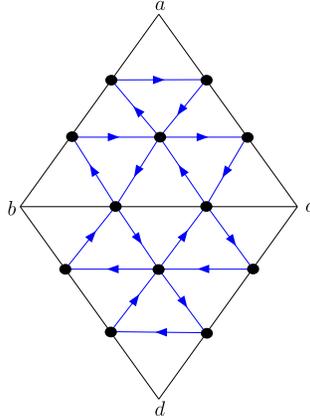


FIGURE 1. 3-triangulation.

**Notation 2.4.** *Let  $V_{\mathcal{T}}$  ( $V_{\mathcal{T}_3}$  resp.) be the set of vertices of  $\mathcal{T}$  ( $\mathcal{T}_3$  resp.).*

$$\mathcal{I}_3 := \{ V \in V_{\mathcal{T}_3} \setminus V_{\mathcal{T}} \mid V \text{ lies on an edge of } \mathcal{T} \} \text{ and } \mathcal{J}_3 := V_{\mathcal{T}_3} \setminus (V_{\mathcal{T}} \cup \mathcal{I}_3).$$

We adopt the following vertex labelling conventions:

- We denote a vertex  $V \in \mathcal{I}_3$  on an oriented ideal edge  $(a, b)$  by  $v_{a,b}^{i,3-i} = v_{b,a}^{3-i,i}$ , where  $i \geq 1$  is the least number of edges of  $\mathcal{T}_3$  from  $V$  to  $b$  (see Figure 1).
- We denote a vertex  $V \in \mathcal{I}_3 \cup \mathcal{J}_3$  on a triangle  $(a, b, c)$  by  $v_{a,b,c}^{i,j,k} = v_{b,a,c}^{j,i,k} = v_{b,c,a}^{j,k,i}$ , where three non-negative integers  $i, j, k$  sum to 3 are the least number of edges of  $\mathcal{T}_3$  from  $V$  to  $\overline{bc}$ , from  $V$  to  $\overline{ac}$  and from  $V$  to  $\overline{ab}$  respectively (see Figure 1).

**Definition 2.5** ( $\mathcal{A}$  coordinate). *Fix an ideal triangulation  $\mathcal{T}$  of  $\hat{S}$  and its 3-triangulation  $\mathcal{T}_3$ . Let  $\tilde{\mathcal{T}}$  be all the lifts of  $\mathcal{T}$  in the universal cover  $\tilde{S}$ . Given a vertex  $V \in \mathcal{I}_3 \cup \mathcal{J}_3$  contained in the anticlockwise oriented marked ideal triangle  $\Delta$ , let the marked ideal triangle  $(a, b, c)$  be a lift of  $\Delta$  in  $\tilde{\mathcal{T}}$  where the position of the lift of  $V$  in  $(a, b, c)$  is determined by the triple*

of non-negative integers  $(i, j, k)$  sum to 3. For  $(\rho, \bar{\xi}) \in \mathcal{A}_{\text{SL}_3, \hat{s}}$ , let  $\bar{\xi}_\rho$  be the map from all the lifts  $\widetilde{m_b \cup m_p}$  to  $\mathcal{A}$  obtained by deck transformations of  $\bar{\xi}$ . Choose bases

$$(a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3)$$

for the respective decorated flags  $\bar{\xi}_\rho(a)$ ,  $\bar{\xi}_\rho(b)$ ,  $\bar{\xi}_\rho(c)$ . Fix a non-zero volume form  $\Omega$  of  $E$ . By [FG06, Section 9], the (Fock–Goncharov)  $\mathcal{A}$ -coordinate at  $V$  is

$$A_V(\rho, \bar{\xi}) := A_V := A_{a,b,c}^{i,j,k} := \Omega(a^i \wedge b^j \wedge c^k).$$

Given a quiver defined as in Figure 1 with respect to the orientation of the surface, let

$$\varepsilon_{VW} = \#\{\text{arrows from } V \text{ to } W\} - \#\{\text{arrows from } W \text{ to } V\}.$$

The (Fock–Goncharov)  $\mathcal{X}$ -coordinate at  $V$  (which is not on the boundary) is

$$X_V(\rho, \bar{\xi}) := X_V := \prod_W A_W^{\varepsilon_{VW}}.$$

The higher Teichmüller space  $\mathcal{A}_{\text{SL}_3, \hat{s}}(\mathbb{R}_{>0})$  is the set of positive real points of  $\mathcal{A}_{\text{SL}_3, \hat{s}}$  with respect to all the  $\mathcal{A}$  coordinates for some ideal triangulation  $\mathcal{T}$ .

The moduli space  $\mathcal{A}_{\text{PGL}_3, \hat{s}}$  is defined in a similar way. But  $\mathcal{A}_{\text{PGL}_3, \hat{s}}(\mathbb{R})$  can not be parameterized by the determinants since the determinants are not invariant under rescaling the vectors. By [FG06, Section 10], the moduli space  $\mathcal{A}_{\text{SL}_3, \hat{s}}$  has the cluster algebraic structure [FZ02] with respect to the quiver in Figure 1. The transition maps between coordinate patches are always fractions of two polynomials with positive integer coefficients, thus send positive coordinates to positive coordinates. Hence all the  $\mathcal{A}$  coordinates of  $\mathcal{A}_{\text{SL}_3, \hat{s}}(\mathbb{R}_{>0})$  for any ideal triangulation are positive.

**Definition 2.6** (Goncharov–Shen potential). *Let*

$$\mathcal{D} := \{(2, 1, 0), (1, 2, 0), (1, 1, 1)\}.$$

Suppose  $(a, b, c)$  are anticlockwise oriented. For  $(i, j, k) \in \mathcal{D}$ , the monomial

$$(1) \quad \alpha_{a;b,c}^{i,j,k} := \frac{A_{a,b,c}^{i-1,j,k+1} \cdot A_{a,b,c}^{i+1,j-1,k}}{A_{a,b,c}^{i,j,k} \cdot A_{a,b,c}^{i,j-1,k+1}}, \quad (A_{a,b,c}^{3,0,0} := 1 \text{ by convention}),$$

introduced in [GS15, Lemma 3.1] corresponds to the rhombuses in Figure 2.

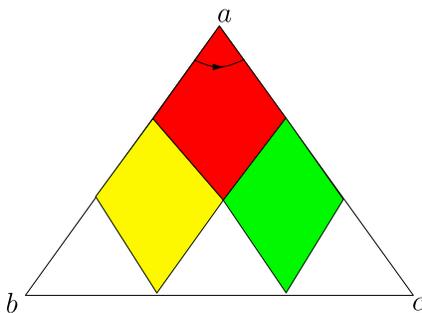


FIGURE 2. Red rhombus for  $\alpha_{a;b,c}^{2,1,0}$ , yellow rhombus for  $\alpha_{a;b,c}^{1,2,0}$ , green rhombus for  $\alpha_{a;b,c}^{1,1,1}$ .

For the marked triangle  $(a, b, c)$  as a lift of the marked ideal triangle  $T$ , we define

$$P(T) := P(a; b, c) := \alpha_{a;b,c}^{2,1,0} + \alpha_{a;b,c}^{1,2,0} + \alpha_{a;b,c}^{1,1,1}.$$

Let  $\Theta$  be the collection of anticlockwise oriented marked ideal triangles of  $\mathcal{T}$ , the Goncharov–Shen potential is

$$P = \sum_{T \in \Theta} P(T).$$

**Remark 2.7.** In [HS19, Section 4], the Goncharov–Shen potentials are understood as generalized horocycle lengths. In [GS15] and [GHKK18], the Goncharov–Shen potentials are understood as the mirror Landau–Ginzburg potentials.

The tropicalization  $f^t$  of a positive Laurent polynomial  $f$  on  $\mathcal{A}_{\mathrm{SL}_3, \hat{S}}(\mathbb{R}_{>0})$  is

$$f^t(x_1, \dots, x_k) = \lim_{C \rightarrow +\infty} \frac{\log f(e^{Cx_1}, \dots, e^{Cx_k})}{C}.$$

Then the tropical semifield  $\mathbb{R}^t$  is to take the usual multiplication  $x \cdot y$  into addition  $x + y$  and the usual addition  $u + v$  into  $\max\{u, v\}$ . The isomorphism  $x \rightarrow -x$  sends  $(\mathbb{R}^t, +, \max)$  to  $(\mathbb{R}^t, +, \min)$ . We use  $(\mathbb{R}^t, +, \min)$  for the tropical points of  $\mathcal{A}_{\mathrm{SL}_3, \hat{S}}$ . By Equation (1)

$$(2) \quad \left(\alpha_{a;b,c}^{i,j,k}\right)^t = \left(A_{a,b,c}^{i-1,j,k+1}\right)^t + \left(A_{a,b,c}^{i+1,j-1,k}\right)^t - \left(A_{a,b,c}^{i,j,k}\right)^t - \left(A_{a,b,c}^{i,j-1,k+1}\right)^t,$$

and

$$P^t = \min \left\{ \left(\alpha_{a;b,c}^{i,j,k}\right)^t \right\}_{\text{any } \alpha_{a;b,c}^{i,j,k} \text{ of } P}.$$

**Definition 2.8.** The space of positive real tropical points of  $\mathcal{A}_{\mathrm{SL}_3, \hat{S}}$  is

$$\mathcal{A}_{\mathrm{SL}_3, \hat{S}}^+(\mathbb{R}^t) := \left\{ x \in \mathcal{A}_{\mathrm{SL}_3, \hat{S}}(\mathbb{R}^t) \mid P^t(x) \geq 0 \right\}.$$

We define  $\mathcal{A}_{\mathrm{SL}_3, \hat{S}}^+(\mathbb{Z}^t)$  and  $\mathcal{A}_{\mathrm{SL}_3, \hat{S}}^+(\frac{1}{3}\mathbb{Z}^t)$  similarly.

Let  $\mathbb{Z}_+$  be the set of non-negative integers. The ratios  $\{\alpha_{a;b,c}^{i,j,k}\}$  of the determinants can be viewed as functions of  $\mathcal{A}_{\mathrm{PGL}_3, \hat{S}}$  which determine  $\mathcal{A}_{\mathrm{PGL}_3, \hat{S}}$ . Since  $\mathcal{A}_{\mathrm{PGL}_3, \hat{S}}^+(\mathbb{R}^t) = \mathcal{A}_{\mathrm{SL}_3, \hat{S}}^+(\mathbb{R}^t)$ , we have the set of web-tropical points

$$\mathcal{A}_{\mathrm{PGL}_3, \hat{S}}^+(\mathbb{Z}^t) = \left\{ x \in \mathcal{A}_{\mathrm{SL}_3, \hat{S}}(\mathbb{R}^t) \mid \text{for any } \alpha_{a;b,c}^{i,j,k} \text{ of } P^t : \left(\alpha_{a;b,c}^{i,j,k}\right)^t(x) \in \mathbb{Z}_+ \right\},$$

which satisfies

$$\mathcal{A}_{\mathrm{SL}_3, \hat{S}}^+(\mathbb{Z}^t) \subseteq \mathcal{A}_{\mathrm{PGL}_3, \hat{S}}^+(\mathbb{Z}^t) \subseteq \mathcal{A}_{\mathrm{SL}_3, \hat{S}}^+\left(\frac{1}{3}\mathbb{Z}^t\right).$$

Moreover the space of lamination-tropical points is

$$\mathcal{A}_{\mathrm{PGL}_3, \hat{S}}(\mathbb{Z}^t) := \left\{ x \in \mathcal{A}_{\mathrm{SL}_3, \hat{S}}(\mathbb{R}^t) \mid \text{for any } \alpha_{a;b,c}^{i,j,k} \text{ of } P^t : \left(\alpha_{a;b,c}^{i,j,k}\right)^t(x) \in \mathbb{Z} \right\},$$

which satisfies

$$\mathcal{A}_{\mathrm{SL}_3, \hat{S}}(\mathbb{Z}^t) \subseteq \mathcal{A}_{\mathrm{PGL}_3, \hat{S}}(\mathbb{Z}^t) \subseteq \mathcal{A}_{\mathrm{SL}_3, \hat{S}}\left(\frac{1}{3}\mathbb{Z}^t\right).$$

From now on, whenever we talk about the tropical  $\mathcal{A}$  coordinates of  $\mathcal{A}_{\text{PGL}_3, \hat{S}}^+(\mathbb{Z}^t)$ , we view  $\mathcal{A}_{\text{PGL}_3, \hat{S}}^+(\mathbb{Z}^t)$  as contained in  $\mathcal{A}_{\text{SL}_3, \hat{S}}(\mathbb{R}^t)$ .

**Remark 2.9.** *The space  $\mathcal{A}_{\text{PGL}_3, \hat{S}}(\mathbb{Z}^t)$  of lamination-tropical points is the space of balanced points studied by [Kim20].*

By [GS15, Section 3], when  $\hat{S}$  is a disk with three marked points on the boundary, the set  $\mathcal{A}_{\text{SL}_3, \hat{S}}^+(\mathbb{Z}^t)$  is canonically identified with the Knutson–Tao hive cone [KT99].

### 3. FROM WEBS TO TROPICAL POINTS

Let  $\mathcal{W}_{\hat{S}}$  be the space of reduced webs up to equivalence; note that  $\mathcal{W}_{\hat{S}}$  is what we called  $\mathcal{W}_{3, \hat{S}}$  in §1. Given an ideal triangulation  $\mathcal{T}$  of  $\hat{S}$ , let  $\mathcal{A}_{\mathcal{T}}$  be the tropical  $\mathcal{A}$  coordinate chart of  $\mathcal{A}_{\text{PGL}_3, \hat{S}}^+(\mathbb{Z}^t)$  with respect to  $\mathcal{T}$ . In order to send all the coordinates to non-negative integers, let us consider the isomorphic coordinate chart

$$(3) \quad \mathcal{C}_{\mathcal{T}} := \{x \mid -x/3 \in \mathcal{A}_{\mathcal{T}}\}.$$

In [DS20], we introduce the map  $\Phi_{\mathcal{T}} : \mathcal{W}_{\hat{S}} \rightarrow \mathcal{C}_{\mathcal{T}}$  using the split ideal triangulation introduced by Bonahon–Wong [BW11b]. The *split ideal triangulation* of  $\mathcal{T}$  is defined by splitting each ideal edge of the ideal triangulation  $\mathcal{T}$  into two disjoint isotopic ideal edges, which is still denoted by  $\mathcal{T}$  without loss of ambiguity. Then  $\hat{S}$  is cut into ideal triangles and bigons by the split ideal triangulation  $\mathcal{T}$  as in Figure 3.

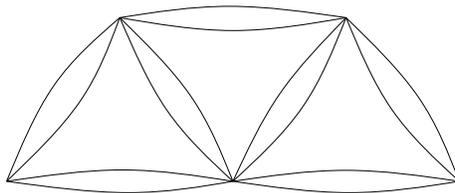


FIGURE 3

By [DS20] (also in [FS20]), we can put any reduced web  $W$  in good position such that

- (1)  $W$  intersects minimally with the split ideal triangulation  $\mathcal{T}$ ;
- (2) the restriction of the web  $W$  in good position to any bigon is a ladder (e.g. Figure 4(1)).
- (3) the restriction of the web  $W$  to any ideal triangle is the combination of oriented honeycomb in the middle and oriented arcs in the corners (e.g. Figure 6(1)).

Figure 4(2) is the *schematic diagram for the ladder* where each “H” is replaced by a crossing.

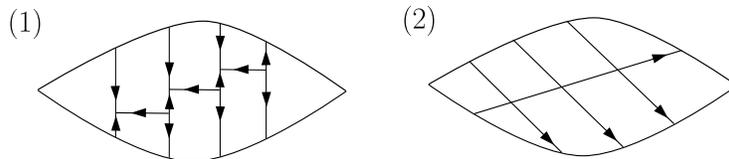


FIGURE 4. Web in bigon.

**Definition 3.1.** *Given the split ideal triangulation as in Figure 5, suppose that two oriented arcs intersect with each other in the bigon along the ideal edge  $\overline{bc}$ . The intersection is called*

- (1) non-admissible crossing if they go towards a common ideal triangle (Figure 5 (1));  
 (2) admissible crossing if they go towards two different ideal triangles (Figure 5 (2)).

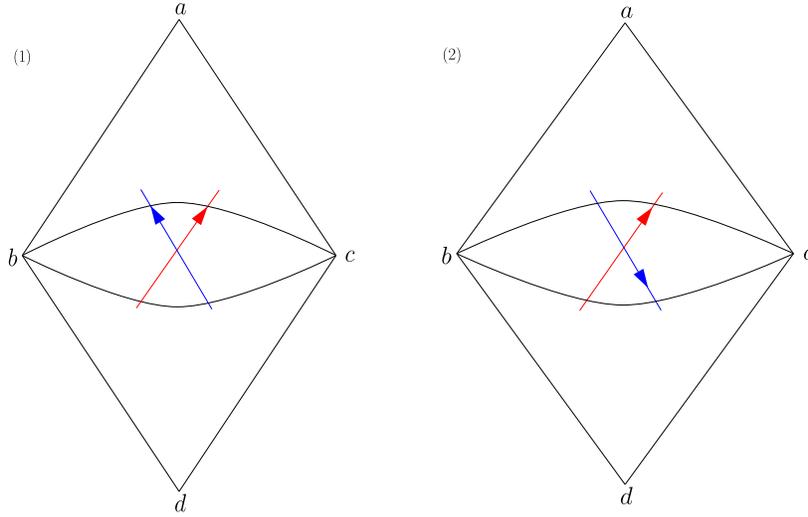


FIGURE 5. (1) Non-admissible crossing; (2) Admissible crossing.

**Lemma 3.2** ([DS20]). *For any reduced web  $W$  in good position with respect to the split ideal triangulation  $\mathcal{T}$ , the schematic diagram of the restriction of  $W$  to any bigon is a ladder, thus with only admissible crossings.*

The *schematic diagram* for the restriction of the reduced web  $W$  to any ideal triangle is shown in Figure 6(2) assigned by the number  $x$  of the oriented arcs intersecting one side of the ideal triangle. The *schematic diagram for the oriented arcs* is the oriented arcs assigned by the non-negative integer numbers  $(y, z, t, u, v, w)$  of the oriented arcs with the same isotopy classes.

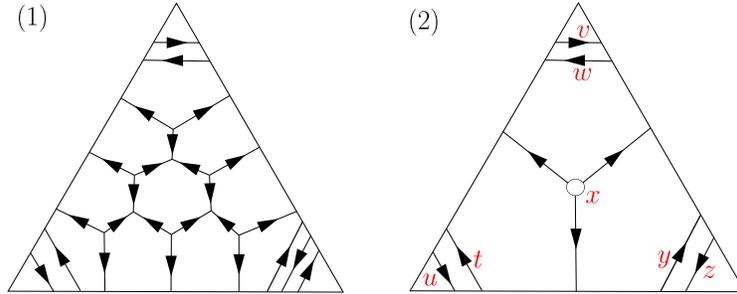


FIGURE 6. Web in triangle. Here  $x = 3$ ,  $y = 2$ ,  $z = t = u = v = w = 1$ .

Let the *ideal triangle*  $\Delta$  be a disk with three marked points  $(a, b, c)$  on the boundary. We firstly define the map  $\Phi$  for each ideal triangle. The image of the  $SL_3$ -webs  $R_a, L_a, R_b, L_b, R_c, L_c, T_{in}, T_{out}$  under  $\Phi$  are shown in Figure 7. Then the image of any reduced web in Figure 6(2) under  $\Phi$  is the linear combination of these webs ( $T_{out}, R_a, L_a, R_b, L_b, R_c, L_c$ ) (or  $(T_{in}, R_a, L_a, R_b, L_b, R_c, L_c)$  depending on the orientation of the oriented honeycomb) with coefficients  $(x, w, v, u, t, y, z) \in \mathbb{Z}_+^7$ . Then the map  $\Phi_{\mathcal{T}}$  for the surface is defined by gluing the map  $\Phi$  for all the ideal triangles together.

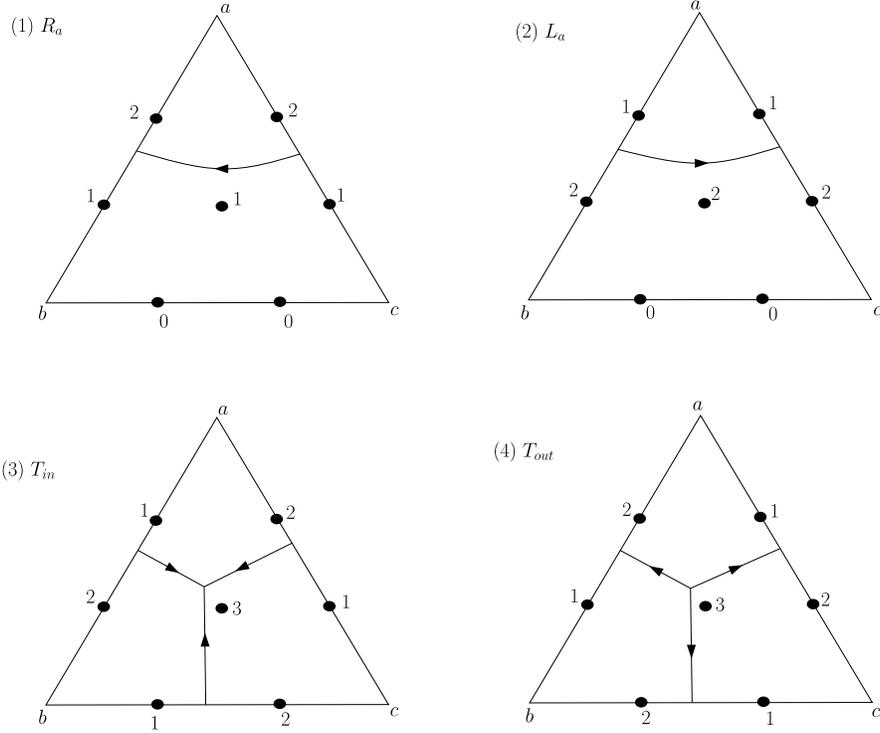


FIGURE 7. Webs and coordinates in  $\mathcal{C}_{\mathcal{T}}$  for  $\Delta$ .

**Theorem 3.3** ([DS20]). *Given an ideal triangulation  $\mathcal{T}$ , the map*

$$(4) \quad \Phi_{\mathcal{T}} : \mathcal{W}_{\hat{S}} \rightarrow \mathcal{C}_{\mathcal{T}}$$

*is a bijection.*

**Remark 3.4.** *This is a slightly more general formulation of the theorem in [DS20], but the proof is essentially the same.*

When we glue the ideal triangles together, the coordinates along same edges defined by different ideal triangles are identical, since they are given by the number of the oriented arcs in one direction plus twice of the number of the oriented arcs in other direction. Thus the map  $\Phi_{\mathcal{T}}$  is well defined. Moreover, by definition

**Lemma 3.5** ([DS20]). *For any two disjoint  $W, W' \in \mathcal{W}_{\hat{S}}$ , we have*

$$\Phi_{\mathcal{T}}(W) + \Phi_{\mathcal{T}}(W') = \Phi_{\mathcal{T}}(W \cup W').$$

#### 4. MAPPING CLASS GROUP EQUIVARIANCE

Let the square  $\square$  be a disk with four marked points on the boundary. In this section, we classify all the possible reduced webs in the square. In each case, we represent the reduced web before and after the flip. Then we compute explicitly the change of tropical  $\mathcal{A}$  coordinates under the flip. We prove that the bijection  $\Phi_{\mathcal{T}}$  from the webs to the tropical points is equivariant under the flip along the diagonal of the square. As a consequence, the map  $\Phi_{\mathcal{T}}$  for the decorated surface  $\hat{S}$  with proper ideal triangulations is equivariant under the extended mapping class group action.

Given a split ideal triangulation  $\mathcal{T}$  of  $\square$ , let  $W$  be a reduced web in good position. Then restricted to each ideal triangle of  $\mathcal{T}$ , the reduced web  $W$  looks like Figure 6(2). By Lemma 3.2, the schematic diagram of  $W$  in the bigon has only admissible crossings. By running through all the possibilities of the gluing, we obtain

**Proposition 4.1.** *Given a split ideal triangulation  $\mathcal{T}$  of the square  $\square$ , any reduced web  $W$  in good position, modulo the 8 oriented corner arcs with weights, is one of the following 9 cases in Figure 8 up to symmetry and orientation reversing. There are 42 cases in total.*

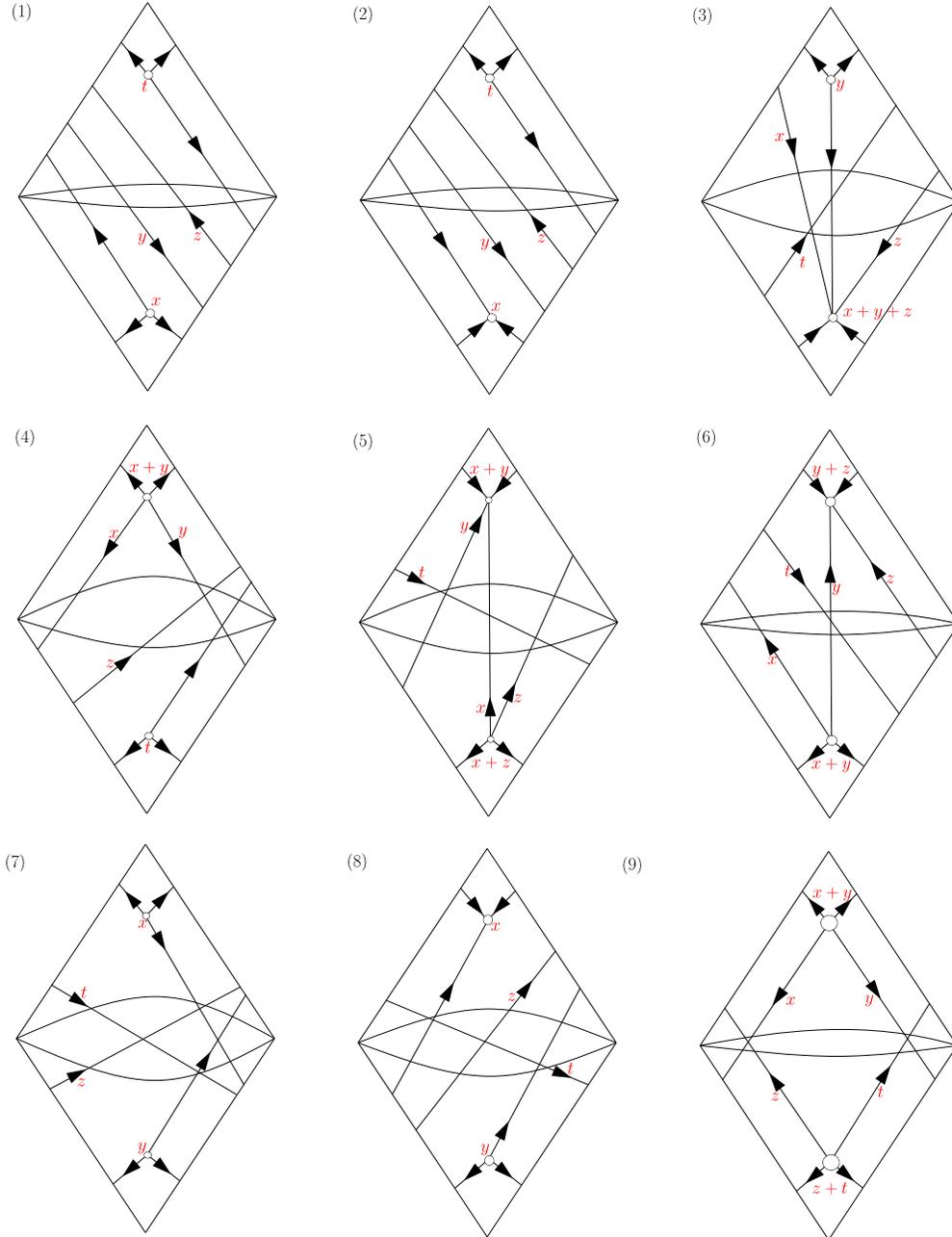


FIGURE 8. 9 cases up to symmetry and orientation reversing.

**Proposition 4.2.** *Recall the bijection  $\Phi_{\mathcal{T}} : \mathcal{W}_{\square} \rightarrow \mathcal{C}_{\mathcal{T}}$  in Equation (4). Let  $\mu$  be the compositions of mutations from  $\mathcal{C}_{\mathcal{T}}$  of  $\mathcal{A}_{\text{PGL}_{3,\square}^+(\mathbb{Z}^t)}$  to  $\mathcal{C}_{\mathcal{T}'}$  induced by the unique flip along the diagonal from  $\mathcal{T}$  to  $\mathcal{T}'$ . Then*

$$\Phi_{\mathcal{T}'} = \mu \circ \Phi_{\mathcal{T}}.$$

*Proof.* For any web  $W \in \mathcal{W}_{\square}$ , suppose that its image under  $\Phi_{\mathcal{T}}$  is shown in Figure 9(1) and its image under  $\Phi_{\mathcal{T}'}$  is shown in Figure 9(2). By Equation (3),  $\min\{u, v\}$  in  $\mathcal{A}_{\mathcal{T}}$  is replaced by  $\max\{u, v\}$  in  $\mathcal{C}_{\mathcal{T}}$ . By the tropical mutation formulas, the theorem is equivalent to verify that

$$(5) \quad \max\{x_2 + y_3, y_1 + x_3\} - y_2 = z_2,$$

$$(6) \quad \max\{y_1 + x_6, x_7 + y_3\} - y_4 = z_4,$$

$$(7) \quad \max\{x_1 + z_4, x_8 + z_2\} - y_1 = z_1,$$

$$(8) \quad \max\{z_2 + x_5, z_4 + x_4\} - y_3 = z_3.$$

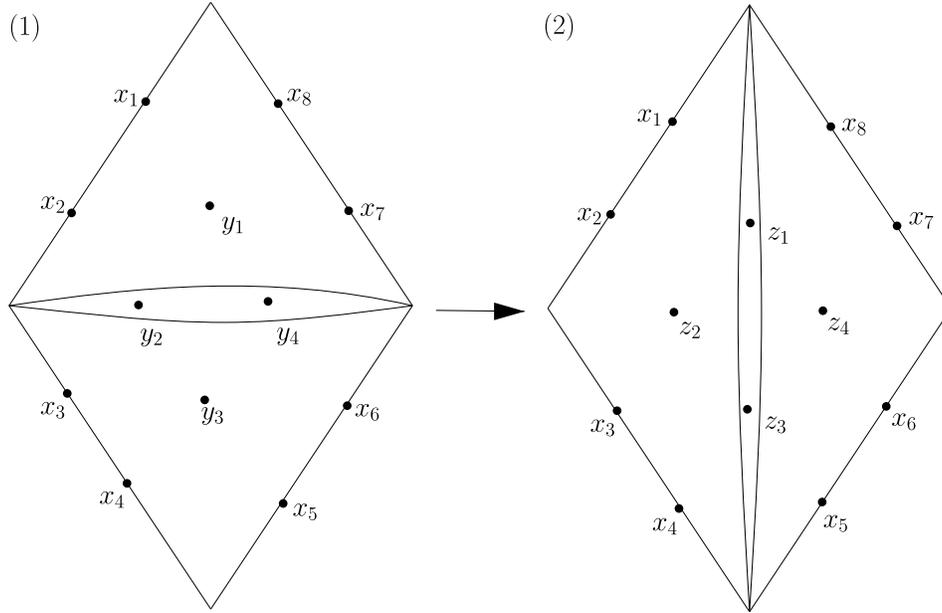


FIGURE 9.  $\mathcal{C}_{\mathcal{T}}$  coordinates before and after the flip.

The difficulty to prove this proposition is that for any reduced web  $W$  in good position with respect to the ideal triangulation  $\mathcal{T}$ , after the flip, we have to put the reduced web  $W$  in good position with respect to  $\mathcal{T}'$ . It is not trivial to do so one by one in the first place since there are infinitely many reduced webs.

Firstly, by direct computation (Figure 26(1)-(8)), we obtain that the set  $R = \{[R_a], [L_a], [R_b], [L_b], [R_c], [L_c], [R_d], [L_d]\}$  of oriented corner arcs do verify Equations (5)(6)(7)(8). Let  $\mathcal{R}$  be the set of combinations of corner arcs with integer coefficients.

**Lemma 4.3.** *For any mutually disjoint  $W \in \mathcal{R}$  and  $W' \in \mathcal{W}_{\square}$ , we have*

$$\mu(\Phi_{\mathcal{T}}(W)) + \mu(\Phi_{\mathcal{T}}(W')) = \mu(\Phi_{\mathcal{T}}(W \cup W')).$$

*Proof.* By Lemma 3.5, since  $W$  and  $W'$  are mutually disjoint, we get

$$\Phi_{\mathcal{T}}(W) + \Phi_{\mathcal{T}}(W') = \Phi_{\mathcal{T}}(W \cup W').$$

For any corner arc in Figure 26(1)-(8), thus for any  $W \in \mathcal{R}$ , the left sides of Equations (5)(6)(7)(8) are always of the form  $\max\{u, u\} - v$ . Since

$$(\max\{u, u\} - v) + (\max\{x, y\} - z) = \max\{u + x, u + y\} - (v + z),$$

we obtain

$$\mu(\Phi_{\mathcal{T}}(W)) + \mu(\Phi_{\mathcal{T}}(W')) = \mu(\Phi_{\mathcal{T}}(W \cup W')).$$

□

By Lemma 4.3 and Proposition 4.1, up to symmetry and orientation reversing, we only need to check the 9 cases in Figure 8. For any reduced web  $W$  in each case, we put  $W$  in good position after the flip in a uniform way as follows.

1. For the web in Figure 10(1) with the assignments  $(x, y, z, t) \in \mathbb{Z}_+$ , the web after flip in good position is draw in Figure 10(2). We check that they do verify Equations (5)(6)(7)(8):

$$\max\{x + 2y + z + t + 3x + 2y + z + 2t, 2x + y + 2z + 3t + 2x\} - (x + 2y + z + 2t) = 3x + 2y + z + t;$$

$$\max\{2x + y + 2z + 3t + x + y + 2z + t, 2t + 3x + 2y + z + 2t\} - (2x + y + 2z + t) = x + y + 2z + 3t;$$

$$\max\{2x + y + 2z + 2t + x + y + 2z + 3t, t + 3x + 2y + z + t\} - (2x + y + 2z + 3t) = x + y + 2z + 2t;$$

$$\max\{3x + 2y + z + t + 2x + 2y + z + 2t, x + y + 2z + 3t + x\} - (3x + 2y + z + 2t) = 2x + 2y + z + t.$$

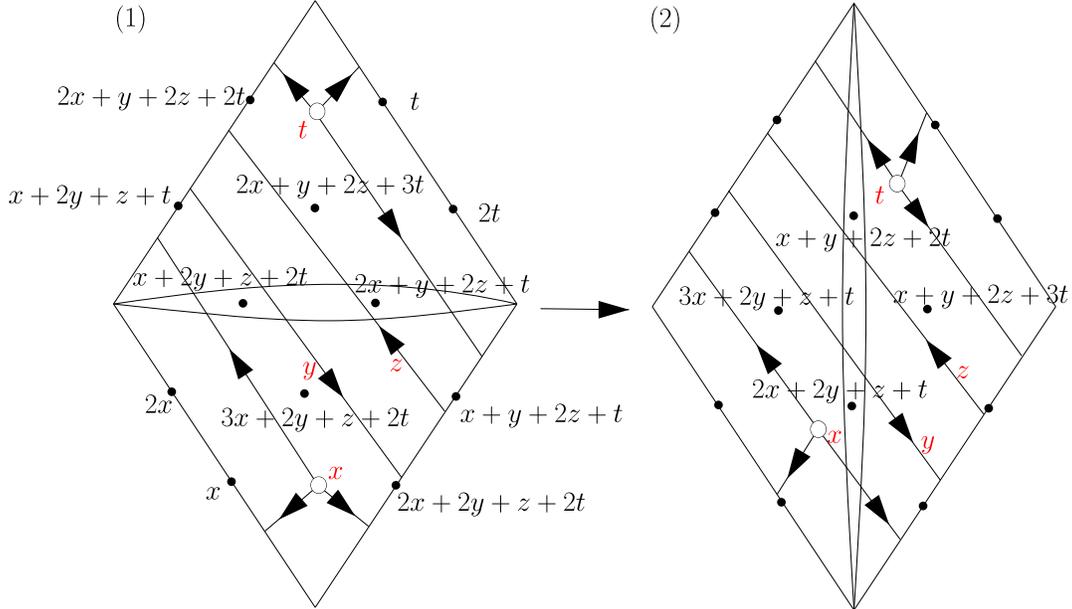


FIGURE 10. Case 1.

2. We verify Equations (5)(6)(7)(8) for the web in Figure 11 using the assigned tropical coordinates.

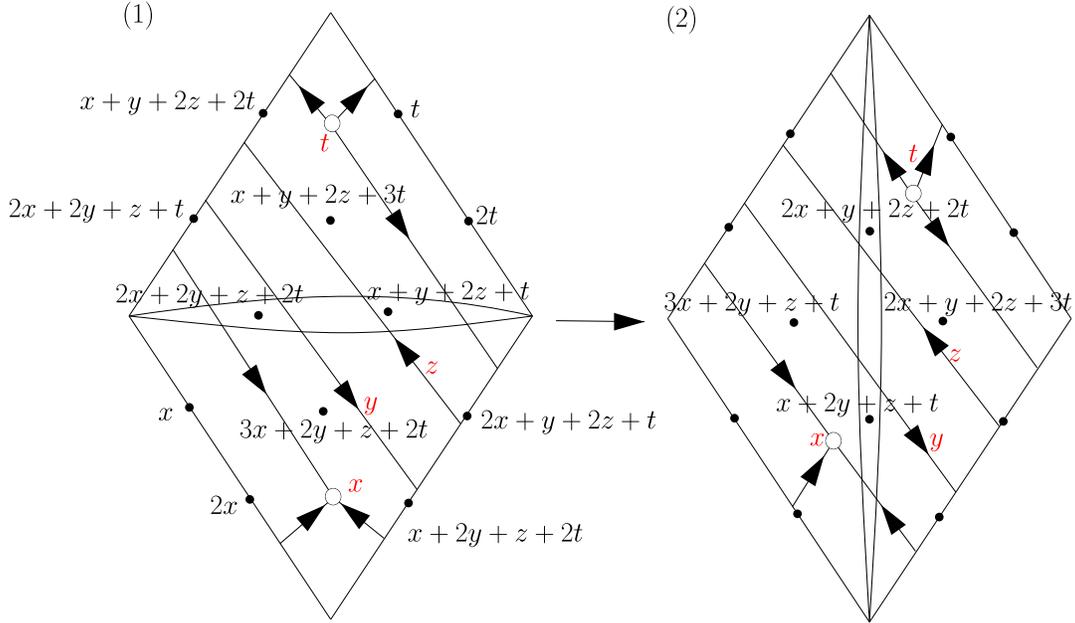


FIGURE 11. Case 2.

3. We verify Equations (5)(6)(7)(8) for the web in Figure 12.

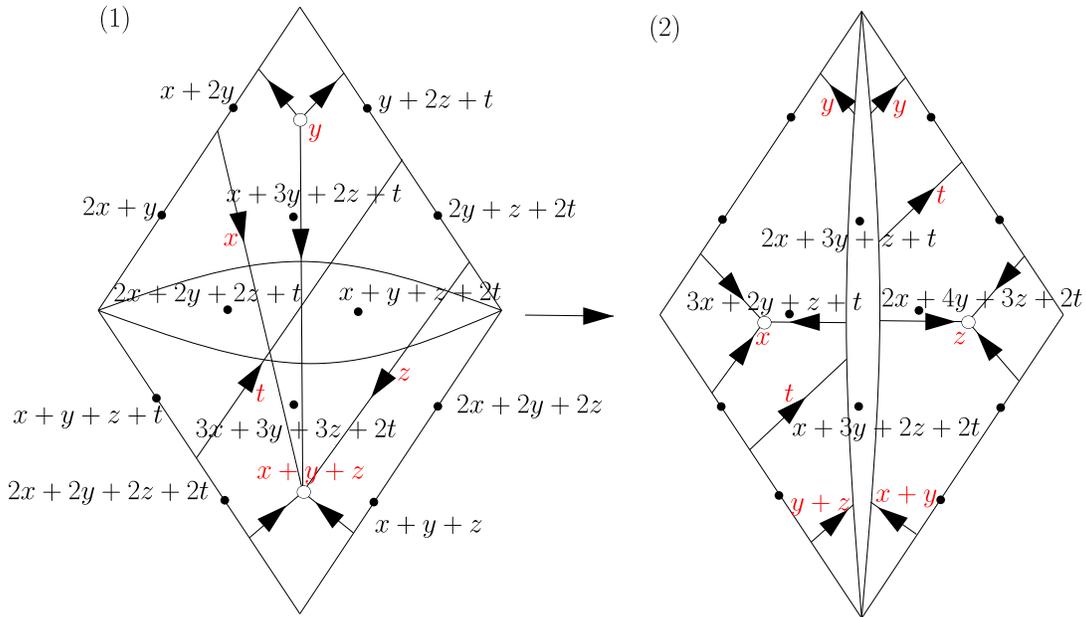


FIGURE 12. Case 3.

The schematic diagram of the web restricted to the bigon in Figure 12(2) is the ladder shown in Figure 13(1). An example where  $x = y = z = t = 1$  is shown in Figure 13(2).

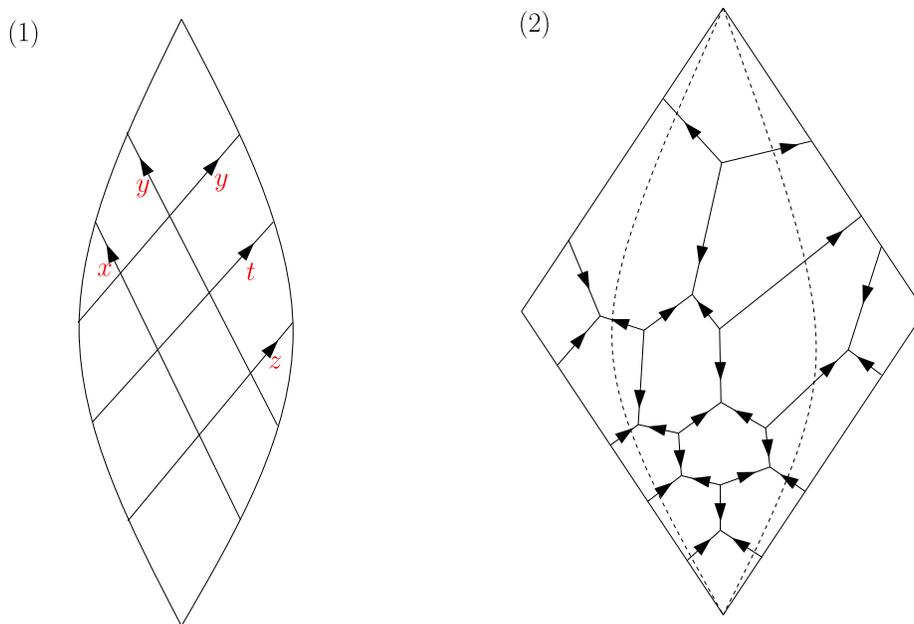


FIGURE 13. (1) Ladder part after the flip. (2) An example.

4. We verify Equations (5)(6)(7)(8) for the web in Figure 14.

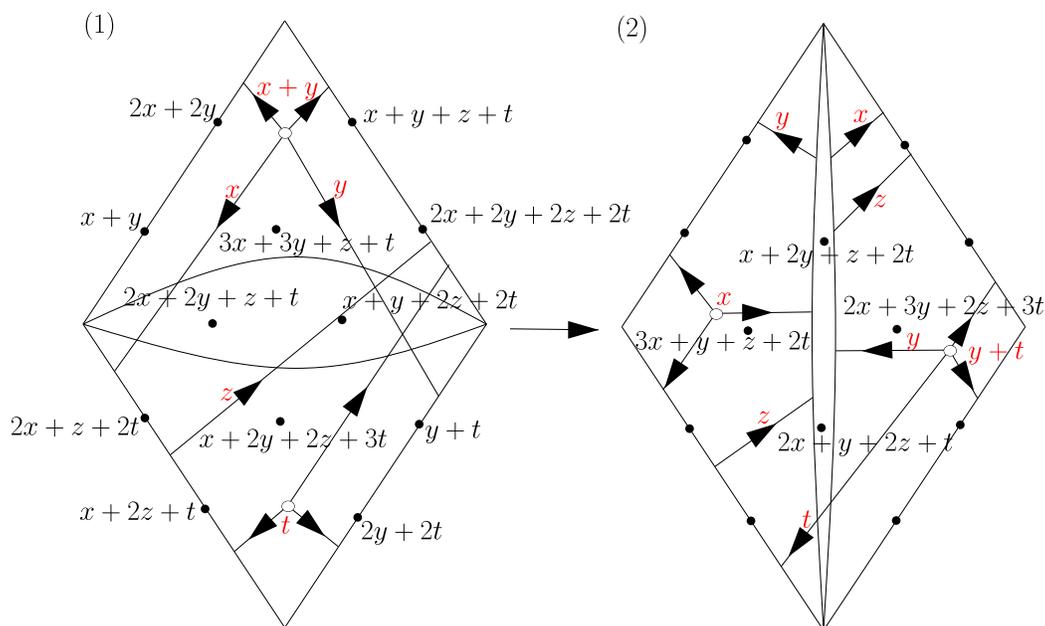


FIGURE 14. Case 4.

The schematic diagram of the web restricted to the bigon in Figure 14(2) is the ladder shown in Figure 15(1). An example where  $x = y = z = t = 1$  is shown in Figure 15(2).

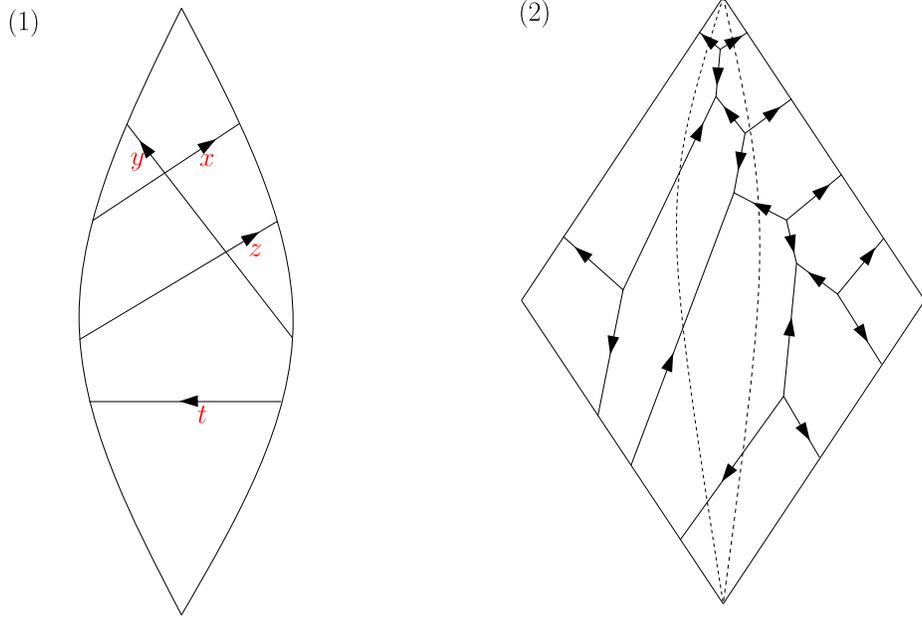


FIGURE 15. (1) Ladder part after the flip. (2) An example.

5. We verify Equations (5)(6)(7)(8) for the web in Figure 16.

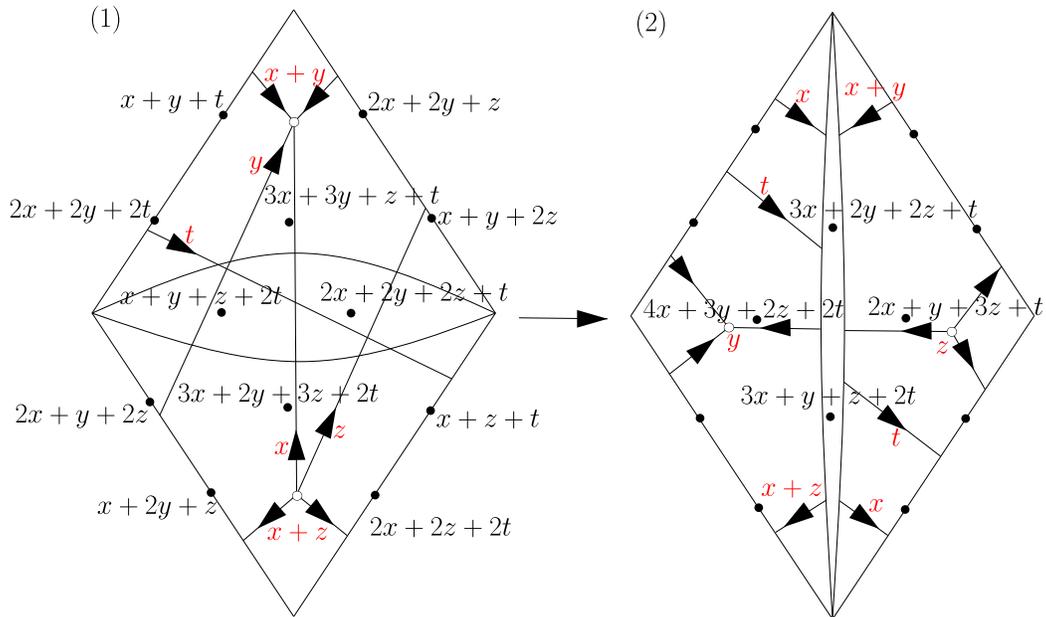


FIGURE 16. Case 5.

The schematic diagram of the web restricted to the bigon in Figure 16(2) is the ladder shown in Figure 17(1). An example where  $x = y = z = t = 1$  is shown in Figure 17(2).

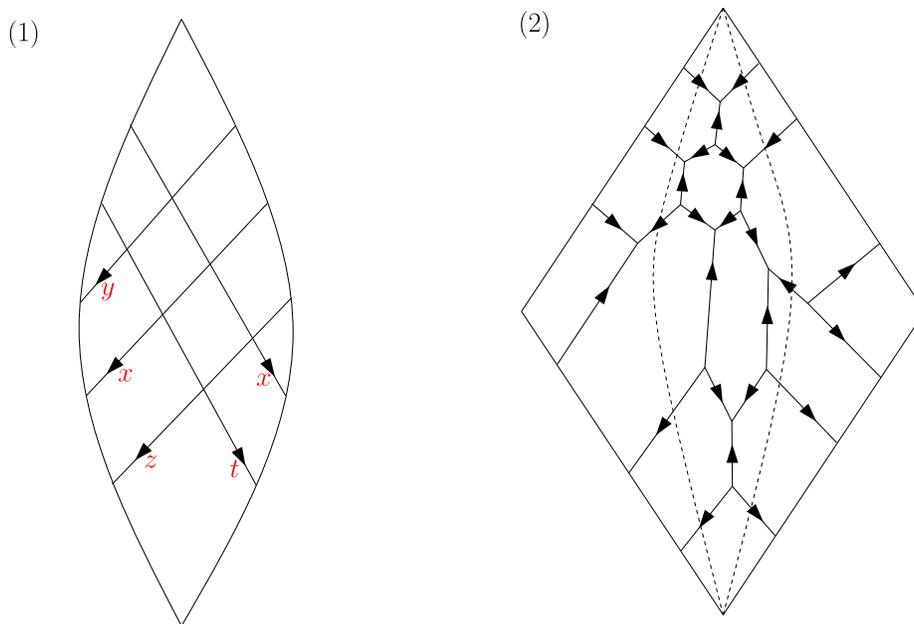


FIGURE 17. (1) Ladder part after the flip. (2) An example.

6. We verify Equations (5)(6)(7)(8) for the web in Figure 18.

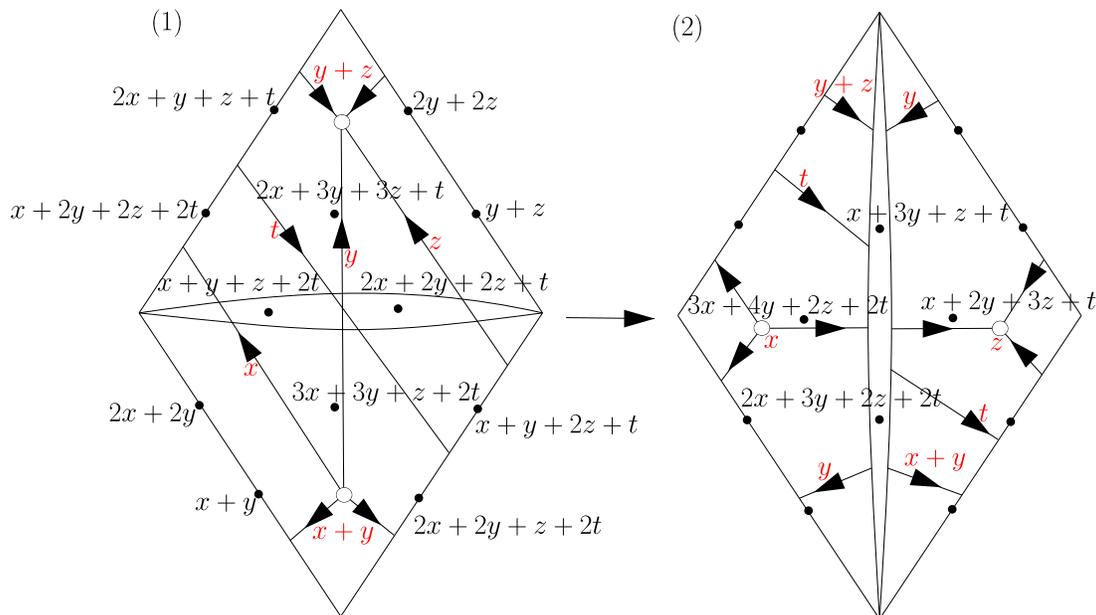


FIGURE 18. Case 6.

The schematic diagram of the web restricted to the bigon in Figure 18(2) is the ladder shown in Figure 19(1). An example where  $x = y = z = t = 1$  is shown in Figure 19(2).

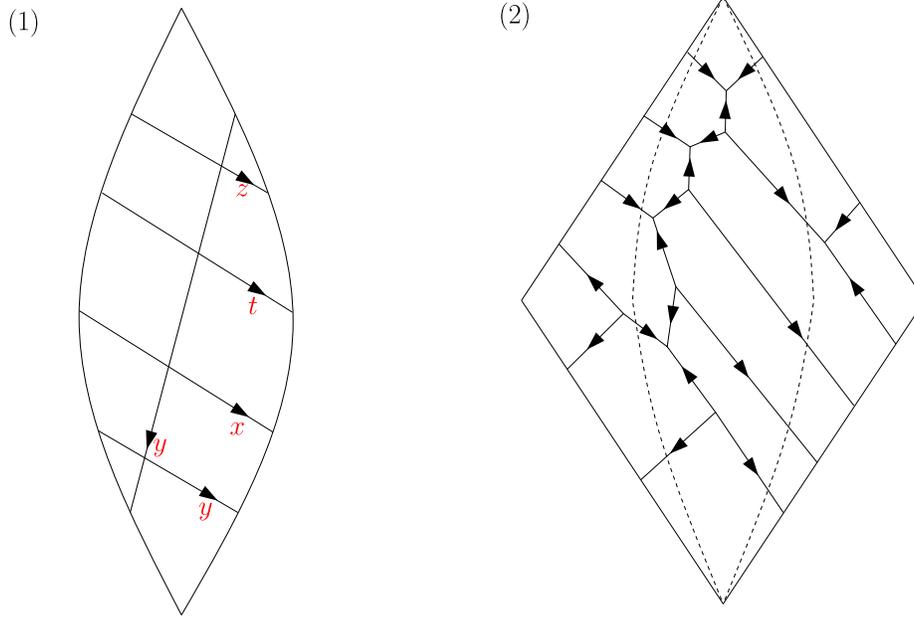


FIGURE 19. (1) Ladder part after the flip. (2) An example.

7. We verify Equations (5)(6)(7)(8) for the web in Figure 20 when  $z \geq t$  (similarly for  $z \leq t$ ).

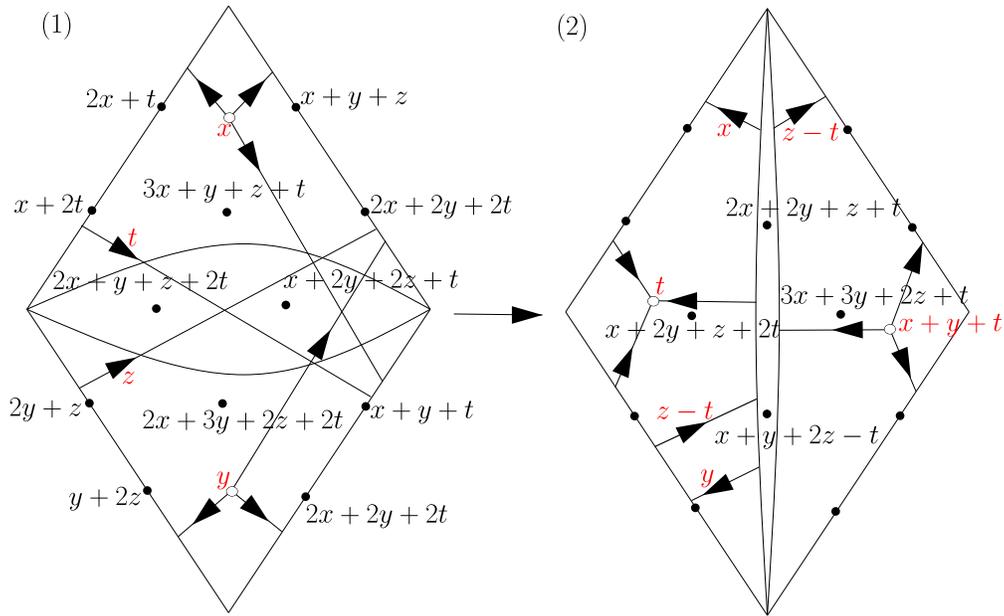


FIGURE 20. Case 7.

The schematic diagram of the web restricted to the bigon in Figure 20(2) is the ladder shown in Figure 21(1). An example where  $x = y = t = 1$  and  $z = 2$  is shown in Figure 21(2).

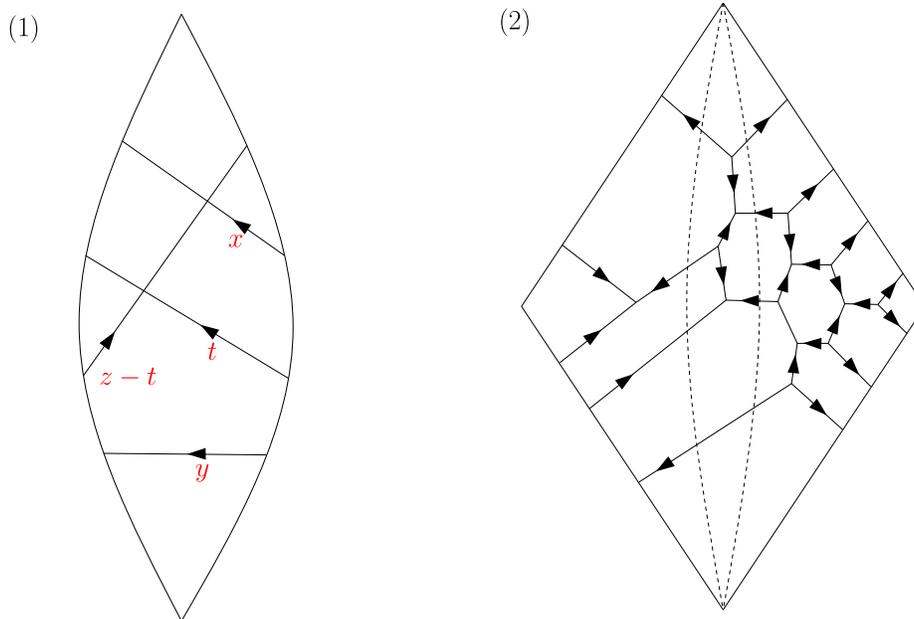


FIGURE 21. (1) Ladder part after the flip. (2) An example.

8. We verify Equations (5)(6)(7)(8) for the web in Figure 22 when  $z \geq t$  (similarly for  $z \leq t$ ).

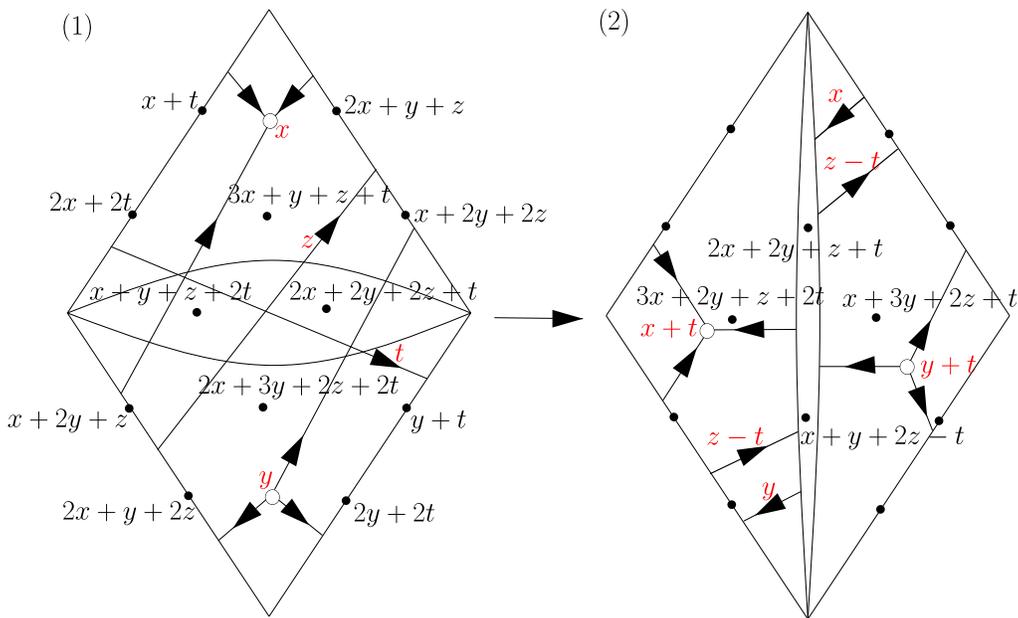


FIGURE 22. Case 8.

The schematic diagram of the web restricted to the bigon in Figure 22(2) is the ladder shown in Figure 23(1). An example where  $x = y = t = 1$  and  $z = 2$  is shown in Figure 23(2).

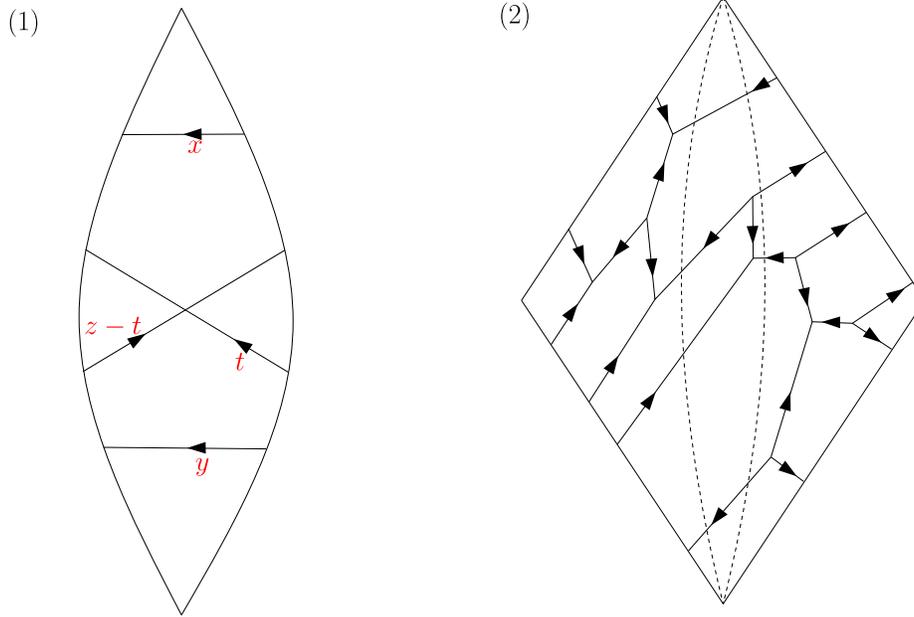


FIGURE 23. (1) Ladder part after the flip. (2) An example.

9. We verify Equations (5)(6)(7)(8) for the web in Figure 24.

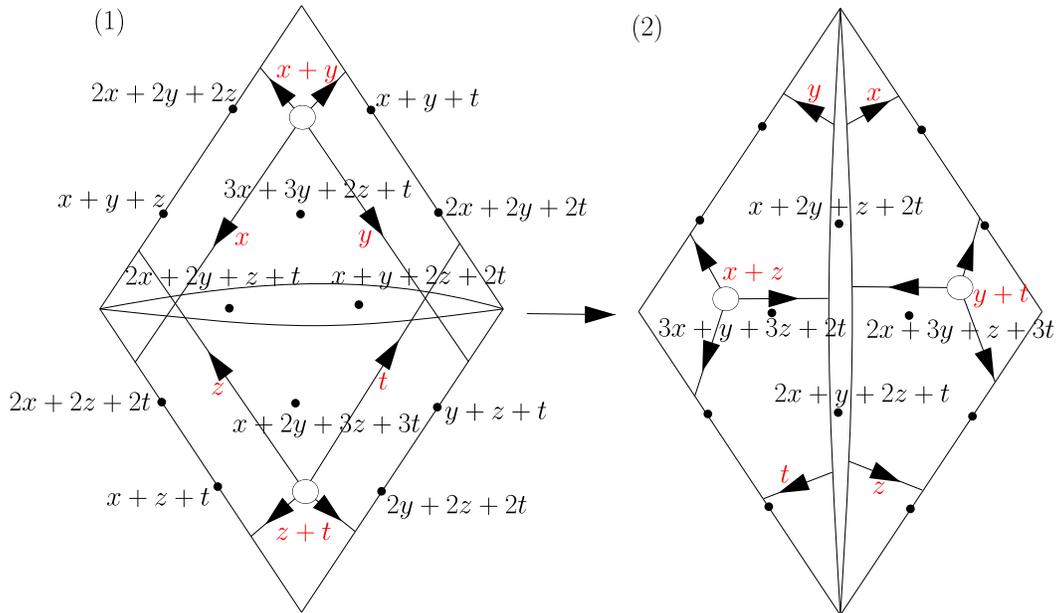


FIGURE 24. Case 9.

The schematic diagram of the web restricted to the bigon in Figure 24(2) is the ladder shown in Figure 25(1). An example where  $x = y = z = t = 1$  is shown in Figure 25(2).

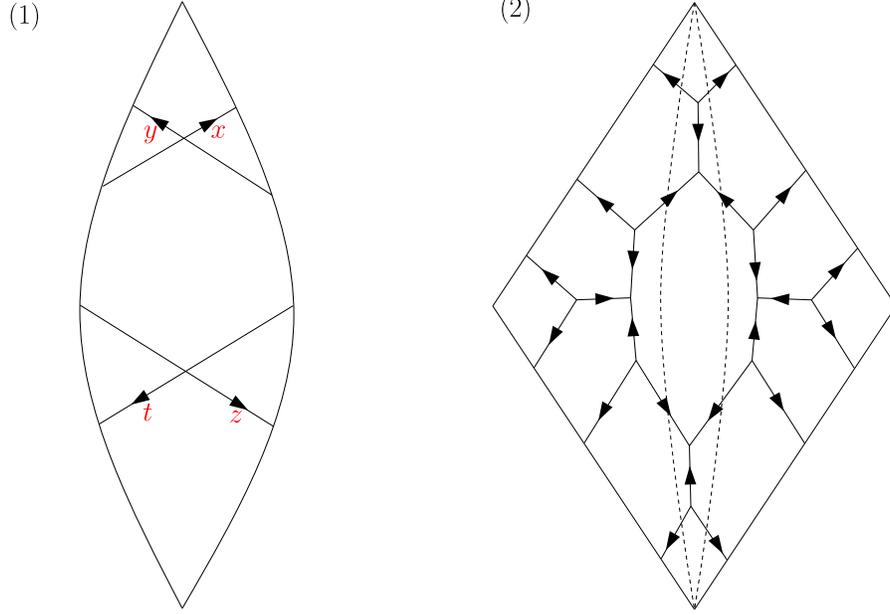


FIGURE 25. (1) Ladder part after the flip. (2) An example.

□

Since any flip of the decorated surface  $\hat{S}$  is along the diagonal of a square:

**Theorem 4.4.** *For any decorated surface  $\hat{S}$ , the bijection  $\Phi_{\mathcal{T}} : \mathcal{W}_{\hat{S}} \rightarrow \mathcal{C}_{\mathcal{T}}$  in Equation (4) is equivariant with respect to the extended mapping class group action.*

In [Kim20, Definition 1.8], Kim consider the space  $\widetilde{\mathcal{W}}_{\hat{S}}$  of  $A_2$ -laminations (he denotes this space by  $\mathcal{A}_L(S, \mathbb{Z})$ ) which extends the space  $\mathcal{W}_{\hat{S}}$  of reduced webs by allowing negative integer weights around the peripheral arcs/loops. By [Kim20, Proposition 1.12], Kim extend our bijection  $\Phi_{\mathcal{T}}$  to the bijection  $\overline{\Phi}_{\mathcal{T}} : \widetilde{\mathcal{W}}_{\hat{S}} \rightarrow \mathcal{A}_{\text{PGL}_3, \hat{S}}(\mathbb{Z}^t)$ . Since Lemma 4.3 works for the oriented corner arcs with integer coefficients, Proposition 4.2 works for the space of  $A_2$ -laminations. Thus

**Corollary 4.5.** *For any decorated surface  $\hat{S}$ , the bijection  $\overline{\Phi}_{\mathcal{T}} : \widetilde{\mathcal{W}}_{\hat{S}} \rightarrow \mathcal{A}_{\text{PGL}_3, \hat{S}}(\mathbb{Z}^t)$  is equivariant with respect to the extended mapping class group action.*

## 5. HILBERT BASIS FOR THE SQUARE CASE

In this section, we study the algebraic structure of  $\mathcal{A}_{\text{PGL}_3, \square}^+(\mathbb{Z}^t)$  (Definition 2.8) as an intersection of a lattice with a polyhedra convex cone.

**5.1. Hilbert basis in linear programming.** The terminology Hilbert basis is used in many different mathematical fields. Here we mean the following.

**Definition 5.1** (Hilbert basis). *A closed subset  $C$  of  $\mathbb{R}^k$  is a cone if for any  $x \in C$  and  $\lambda \geq 0$ , we have  $\lambda \cdot x \in C$ . A cone  $C$  is called*

- convex if for any  $x, y \in C$  and  $\alpha, \beta \geq 0$ , we have  $\alpha \cdot x + \beta \cdot y \in C$ ;
- polyhedral if there is some matrix  $A$  such that  $C = \{x \in \mathbb{R}^k \mid Ax \geq 0\}$ ;
- pointed if  $x, -x \in C$  implies  $x = 0$ .

Let  $C$  be a pointed convex polyhedral cone. Let  $L$  be a lattice in  $\mathbb{R}^k$ . Let  $U = C \cap L$ . A Hilbert basis of  $U$  is a minimal set  $H_0$  of elements in  $U$  such that for any  $x \in U$ , there exists  $x_1, \dots, x_m \in H_0$  and  $\lambda_1, \dots, \lambda_m \in \mathbb{Z}_+$  satisfying

$$x = \lambda_1 \cdot x_1 + \dots + \lambda_m \cdot x_m.$$

An element  $x \in C$  is called irreducible if  $x = y + z$  where  $y, z \in C$ , then  $y = 0$  or  $z = 0$ .

**Example 5.2.** Let the ideal triangle  $\Delta$  be a disk with three marked points on the boundary. Let  $\mathcal{C}_\Delta$  be the coordinate chart of  $\mathcal{A}_{\text{PGL}_3, \Delta}^+(\mathbb{Z}^t)$ . By [DS20], the elements in Figure 7

$$\Phi(R_a), \Phi(L_a), \Phi(R_b), \Phi(L_b), \Phi(R_c), \Phi(L_c), \Phi(T_{in}), \Phi(T_{out})$$

form the Hilbert basis of  $\mathcal{C}_\Delta$ .

**Theorem 5.3.** [GP79, Hil90] For any  $U = C \cap L$  where  $C$  is a pointed convex polyhedral cone and  $L$  is a lattice of  $\mathbb{R}^k$ , the Hilbert basis  $H_0$  of  $U$  has finitely many elements.

**Theorem 5.4.** [Sch81] For any  $U = C \cap L$  where  $C$  is a pointed convex polyhedral cone and  $L$  is a lattice of  $\mathbb{R}^k$ , the Hilbert basis  $H_0$  of  $U$  is unique. And  $H_0$  is the set of all irreducible elements of  $U$ .

**5.2. A linear isomorphism.** Given an ideal triangulation  $\mathcal{T}$  of the square  $\square$ , recall the coordinate chart  $\mathcal{C}_\mathcal{T}$  of  $\mathcal{A}_{\text{PGL}_3, \square}^+(\mathbb{Z}^t)$  in Equation (3). We want to show that  $\mathcal{C}_\mathcal{T}$  is an intersection of a lattice with a pointed convex polyhedral cone in  $\mathbb{R}^k$ , and thus admits a unique Hilbert basis formed by finitely many irreducible elements. In order to do so, we send  $\mathcal{C}_\mathcal{T}$  to another subset by a linear isomorphism using the monomials in the Goncharov–Shen potential which is easier to study. Remember these monomials are ratios of determinants, viewed as functions of  $\mathcal{A}_{\text{PGL}_3, \square}$  which determine  $\mathcal{A}_{\text{PGL}_3, \square}$ ; see §2.

By Equation (3), for any  $x \in \mathcal{A}_{\text{PGL}_3, \square}^+(\mathbb{Z}^t)$ , the coordinates in  $\mathcal{C}_\mathcal{T}$  is related to the tropical  $\mathcal{A}$  coordinates by

$$B_{a,b,c}^{i,j,k}(x) := -3 \left( A_{a,b,c}^{i,j,k} \right)^t(x).$$

By Equation (2), let

$$(9) \quad \beta_{a;b,c}^{i,j,k} := \frac{1}{3} \left( B_{a,b,c}^{i,j,k} + B_{a,b,c}^{i,j-1,k+1} - B_{a,b,c}^{i-1,j,k+1} - B_{a,b,c}^{i+1,j-1,k} \right) = \left( \alpha_{a;b,c}^{i,j,k} \right)^t.$$

Recall Definition 2.8, then

$$\mathcal{C}_\mathcal{T} = \left\{ \left( B_{a,b,c}^{i,j,k} \right) \in \mathbb{Z}^{12} \mid \beta_{a;b,c}^{i,j,k} \in \mathbb{Z}_+ \text{ for any } \alpha_{a;b,c}^{i,j,k} \text{ of } P \right\},$$

where

$$\begin{aligned} \left( B_{a,b,c}^{i,j,k} \right) = & \left( B_{a,b,c}^{2,1,0}, B_{a,b,c}^{2,0,1}, B_{a,b,c}^{1,2,0}, B_{a,b,c}^{1,1,1}, B_{a,b,c}^{1,0,2}, B_{a,b,c}^{0,2,1}, B_{a,b,c}^{0,1,2}, \right. \\ & \left. B_{d,b,c}^{1,2,0}, B_{d,b,c}^{1,1,1}, B_{d,b,c}^{1,0,2}, B_{d,b,c}^{2,1,0}, B_{d,b,c}^{2,0,1} \right). \end{aligned}$$

Let us consider the vector space

$$\mathcal{C}_\mathcal{T}(\mathbb{R}) := \left\{ \left( B_{a,b,c}^{i,j,k} \right) \in \mathbb{R}^{12} \right\} = \mathbb{R}^{12}$$

which is viewed as  $\mathcal{A}_{\text{SL}_3, \square}(\mathbb{R}^t) (= \mathcal{A}_{\text{PGL}_3, \square}(\mathbb{R}^t))$ . Then

$$\mathcal{C}_\mathcal{T}^+(\mathbb{R}) := \left\{ \left( B_{a,b,c}^{i,j,k} \right) \in \mathbb{R}^{12} \mid \beta_{a;b,c}^{i,j,k} \geq 0 \text{ for any } \alpha_{a;b,c}^{i,j,k} \text{ of } P \right\}$$

is a pointed convex polyhedral cone in  $\mathcal{C}_{\mathcal{T}}(\mathbb{R})$ , which is viewed as  $\mathcal{A}_{\text{SL}_3, \square}^+(\mathbb{R}^t) (= \mathcal{A}_{\text{PGL}_3, \square}^+(\mathbb{R}^t))$ .

There are six linear independent relations among  $\{\beta_{a;b,c}^{i,j,k}\}$  which are given by the tropical  $\mathcal{X}$ -coordinates (Definition 2.5):

$$(10) \quad \beta_{a;b,c}^{1,2,0} - \beta_{a;b,c}^{1,1,1} = \beta_{b;c,a}^{1,2,0} - \beta_{b;c,a}^{1,1,1} = \beta_{c;a,b}^{1,2,0} - \beta_{c;a,b}^{1,1,1} := X_1,$$

$$(11) \quad \beta_{b;c,a}^{2,1,0} - \beta_{b;d,c}^{2,1,0} = \beta_{c;b,d}^{1,2,0} - \beta_{c;a,b}^{1,1,1} := X_2,$$

$$(12) \quad \beta_{d;c,b}^{1,2,0} - \beta_{d;c,b}^{1,1,1} = \beta_{b;d,c}^{1,2,0} - \beta_{b;d,c}^{1,1,1} = \beta_{c;b,d}^{1,2,0} - \beta_{c;b,d}^{1,1,1} := X_3,$$

$$(13) \quad \beta_{c;b,d}^{2,1,0} - \beta_{c;a,b}^{2,1,0} = \beta_{b;c,a}^{2,1,0} - \beta_{b;d,c}^{1,1,1} := X_4.$$

Using the above linear relations, let us consider

$$\mathcal{B}_{\mathcal{T}} := \left\{ \left( \beta_{a;b,c}^{i,j,k} \right) \in \mathbb{Z}_+^{18} \mid \text{Equations (10), (11), (12), (13)} \right\},$$

where

$$\begin{aligned} \left( \beta_{a;b,c}^{i,j,k} \right) := & \left( \beta_{a;b,c}^{2,1,0}, \beta_{a;b,c}^{1,2,0}, \beta_{a;b,c}^{(1,1,1)}, \beta_{b;c,a}^{2,1,0}, \beta_{b;c,a}^{(1,2,0)}, \beta_{b;c,a}^{1,1,1}, \beta_{c;a,b}^{2,1,0}, \beta_{c;a,b}^{1,2,0}, \right. \\ & \left. \beta_{c;a,b}^{1,1,1}, \beta_{d;c,b}^{2,1,0}, \beta_{d;c,b}^{1,2,0}, \beta_{d;c,b}^{1,1,1}, \beta_{b;d,c}^{2,1,0}, \beta_{b;d,c}^{1,2,0}, \beta_{b;d,c}^{1,1,1}, \beta_{c;b,d}^{2,1,0}, \beta_{c;b,d}^{1,2,0}, \beta_{c;b,d}^{1,1,1} \right), \end{aligned}$$

which also determine the set  $\mathcal{A}_{\text{PGL}_3, \square}^+(\mathbb{Z}^t)$ . The vector space

$$\mathcal{B}_{\mathcal{T}}(\mathbb{R}) := \left\{ \left( \beta_{a;b,c}^{i,j,k} \right) \in \mathbb{R}^{18} \mid \text{Equations (10), (11), (12), (13)} \right\} = \mathbb{R}^{12}$$

which is viewed as  $\mathcal{A}_{\text{PGL}_3, \square}(\mathbb{R}^t) (= \mathcal{A}_{\text{SL}_3, \square}(\mathbb{R}^t))$ . Then

$$\mathcal{B}_{\mathcal{T}}^+(\mathbb{R}) := \left\{ \left( \beta_{a;b,c}^{i,j,k} \right) \in \mathcal{B}_{\mathcal{T}}(\mathbb{R}) \mid \beta_{a;b,c}^{i,j,k} \geq 0 \text{ for any } \alpha_{a;b,c}^{i,j,k} \text{ of P} \right\}$$

is a pointed convex polyhedral cone in  $\mathcal{B}_{\mathcal{T}}(\mathbb{R})$ , which is viewed as  $\mathcal{A}_{\text{PGL}_3, \square}^+(\mathbb{R}^t) (= \mathcal{A}_{\text{SL}_3, \square}^+(\mathbb{R}^t))$ .

**Lemma 5.5.** *The natural linear map  $\theta_{\mathcal{T}} : \mathcal{C}_{\mathcal{T}}(\mathbb{R}) \rightarrow \mathcal{B}_{\mathcal{T}}(\mathbb{R})$  given by Equation (9)*

$$\theta_{\mathcal{T}} \left( \left( B_{a;b,c}^{i,j,k} \right) \right) = \left( \beta_{a;b,c}^{i,j,k} \right)$$

*is a linear isomorphism.*

*Proof.* Since both sides are isomorphic to  $\mathbb{R}^{12}$ , we verify that  $\theta_{\mathcal{T}}$  has rank 12 by Equation (9). Thus  $\theta_{\mathcal{T}}$  is a linear isomorphism.  $\square$

By definition, we obtain

**Lemma 5.6.** *The restriction of  $\theta_{\mathcal{T}}$  to  $\mathcal{C}_{\mathcal{T}}(\mathbb{R})$  ( $\mathcal{C}_{\mathcal{T}}^+(\mathbb{R})$  resp.) provides a bijection between  $\mathcal{C}_{\mathcal{T}}$  and  $\mathcal{B}_{\mathcal{T}}$  ( $\mathcal{C}_{\mathcal{T}}^+(\mathbb{R})$  and  $\mathcal{B}_{\mathcal{T}}^+(\mathbb{R})$  resp.).*

**Corollary 5.7.** *The set  $\mathcal{C}_{\mathcal{T}}$  is the intersection of a lattice with a pointed convex polyhedral cone. Thus  $\mathcal{C}_{\mathcal{T}}$  has a Hilbert basis.*

*Proof.* By Lemma 5.6, the linear isomorphism  $\theta_{\mathcal{T}}$  sends  $\mathcal{C}_{\mathcal{T}}$  to  $\mathcal{B}_{\mathcal{T}}$ . Thus  $\mathcal{C}_{\mathcal{T}}$  is the intersection of a lattice with a pointed convex polyhedral cone if and only if  $\mathcal{B}_{\mathcal{T}}$  is. The set  $\mathcal{B}_{\mathcal{T}}^+(\mathbb{R})$  is a pointed convex polyhedral cone. And  $\mathbb{Z}^{18} \cap \mathcal{B}_{\mathcal{T}}(\mathbb{R})$  is a lattice in  $\mathcal{B}_{\mathcal{T}}(\mathbb{R})$ . Observe that

$$\mathcal{B}_{\mathcal{T}} = (\mathbb{Z}^{18} \cap \mathcal{B}_{\mathcal{T}}(\mathbb{R})) \cap \mathcal{B}_{\mathcal{T}}^+(\mathbb{R}).$$

Hence  $\mathcal{C}_{\mathcal{T}}$  is the intersection of a lattice with a pointed convex polyhedral cone.  $\square$

**5.3. Hilbert basis for  $\mathcal{A}_{\text{PGL}_{3,\square}}^+(\mathbb{Z}^t)$ .** Let us find explicitly the Hilbert basis elements of  $\mathcal{C}_{\mathcal{T}}$ .

**Proposition 5.8.** *Given an ideal triangulation  $\mathcal{T}$  of  $\square$ , recall the bijection  $\Phi_{\mathcal{T}}$  in Equation (4). For a reduced webs  $W$ , suppose the image  $\Phi_{\mathcal{T}}(W)$  is a Hilbert basis element of  $\mathcal{C}_{\mathcal{T}}$ , then the restriction of  $\Phi_{\mathcal{T}}(W)$  to any triangle of  $\mathcal{T}$  is a Hilbert basis element.*

*Proof.* Let  $\Delta, \Delta'$  be the two ideal triangles of  $\mathcal{T}$ . If  $\Phi_{\mathcal{T}}(W)$  is a Hilbert basis element, suppose that  $\Phi(W|_{\Delta}) = \Phi_{\mathcal{T}}(W)|_{\Delta}$  is not a Hilbert basis element. Then there are non-empty reduced webs  $A_{\Delta}$  and  $B_{\Delta}$  in  $\Delta$  such that  $\Phi(W|_{\Delta}) = \Phi(A_{\Delta}) + \Phi(B_{\Delta})$  where  $\Phi(A_{\Delta})$  and  $\Phi(B_{\Delta})$  are both non-zero. The element  $\Phi(W|_{\Delta'})$  is a  $\mathbb{Z}_+$ -combination of the Hilbert basis element in Figure 7. Then with respect to the gluing condition along the diagonal, there exist  $x \in \mathcal{C}_{\mathcal{T}}$  such that  $\Phi_{\mathcal{T}}(W) - x \in \mathbb{Z}_+^{12}$  and  $\Phi(A_{\Delta}) = x|_{\Delta}$ . Moreover, the functions  $\{\beta_{a;b,c}^{i,j,k}\}$  evaluating at  $\Phi_{\mathcal{T}}(W) - x$  are still integers under subtraction. Thus  $\Phi_{\mathcal{T}}(W) - x \in \mathcal{C}_{\mathcal{T}}$ . Then  $B = \Phi_{\mathcal{T}}^{-1}(\Phi_{\mathcal{T}}(W) - x)$  is a non-empty reduced web such that  $B|_{\Delta} = B_{\Delta}$ . Then  $\Phi_{\mathcal{T}}(W) = x + \Phi_{\mathcal{T}}(B)$ . Contradict with the fact that  $\Phi_{\mathcal{T}}(W)$  is a Hilbert basis element.  $\square$

**Notation 5.9.** Let  $H_{\mathcal{T}}^W$  be the set of 22 reduced webs in Figure 26. Let  $H_{\mathcal{T}}$  be the subset  $\Phi_{\mathcal{T}}(H_{\mathcal{T}}^W) \subseteq \mathcal{C}_{\mathcal{T}}$ .

By Proposition 5.8, the Hilbert basis elements of  $\mathcal{C}_{\mathcal{T}}$  are contained in the set  $H_{\mathcal{T}}$ . Except the corner arcs labelled by

$$[R_a], [L_a], [L_d], [R_d], [R_b], [L_b], [R_c], [L_c],$$

we labelled the rest webs in  $H_{\mathcal{T}}^W$  by  $[\cdot, \cdot]$  where the first entry represents the web restricted to the marked triangle  $(a, b, c)$  and the second entry represents the web restricted to the marked triangle  $(d, b, c)$  for the square  $\square = (a, b, d, c)$ .

**Theorem 5.10.** *The Hilbert basis of  $\mathcal{C}_{\mathcal{T}}$  is formed by the 22 elements in  $H_{\mathcal{T}}$ .*

*Proof.* By Theorem 5.4, the Hilbert basis elements are given by the irreducible elements (Definition 5.1) in  $\mathcal{C}_{\mathcal{T}}$ . By Lemma 5.6, the element  $x \in \mathcal{C}_{\mathcal{T}}$  is irreducible if and only if  $\theta_{\mathcal{T}}(x)$  is irreducible in  $\mathcal{B}_{\mathcal{T}}$ . For  $x \in H_{\mathcal{T}}$ , we compute  $\theta_{\mathcal{T}}(x)$  explicitly as follows:

$$\begin{aligned} \theta_{\mathcal{T}}(\Phi_{\mathcal{T}}([R_a])) &= (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), \\ \theta_{\mathcal{T}}(\Phi_{\mathcal{T}}([L_a])) &= (0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), \\ \theta_{\mathcal{T}}(\Phi_{\mathcal{T}}([L_d])) &= (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0), \\ \theta_{\mathcal{T}}(\Phi_{\mathcal{T}}([R_d])) &= (0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0), \\ \theta_{\mathcal{T}}(\Phi_{\mathcal{T}}([R_b])) &= (0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0), \\ \theta_{\mathcal{T}}(\Phi_{\mathcal{T}}([L_b])) &= (0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0), \\ \theta_{\mathcal{T}}(\Phi_{\mathcal{T}}([R_c])) &= (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0), \\ \theta_{\mathcal{T}}(\Phi_{\mathcal{T}}([L_c])) &= (0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1), \\ \theta_{\mathcal{T}}(\Phi_{\mathcal{T}}([T_{out}, R_b])) &= (0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0), \end{aligned}$$

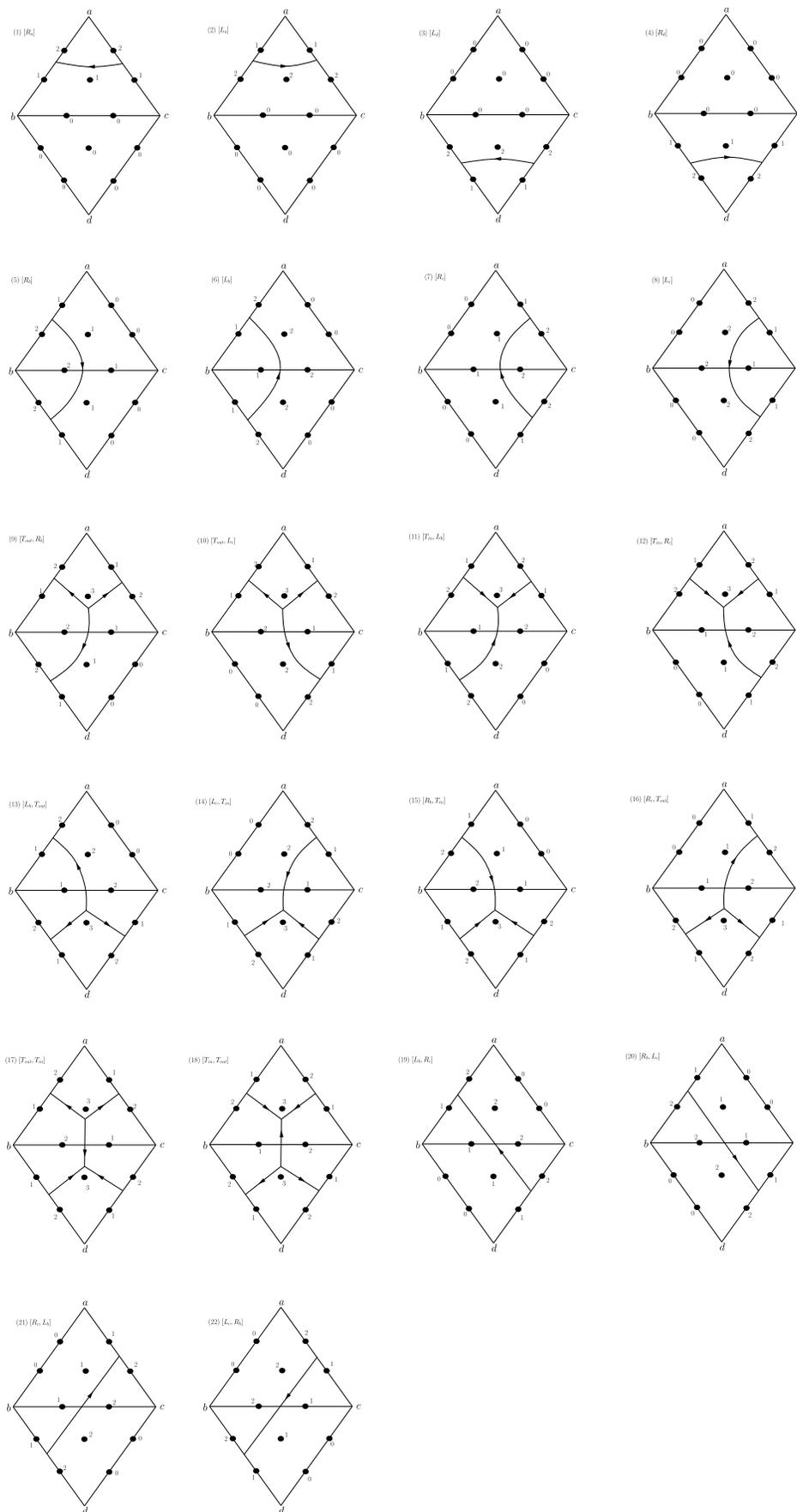


FIGURE 26. Webs and coordinates in  $\mathcal{C}_T$  for  $\square$ .

$$\begin{aligned}
 \theta_{\mathcal{T}}(\Phi_{\mathcal{T}}([T_{out}, L_c])) &= (0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1), \\
 \theta_{\mathcal{T}}(\Phi_{\mathcal{T}}([T_{in}, L_b])) &= (0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0), \\
 \theta_{\mathcal{T}}(\Phi_{\mathcal{T}}([T_{in}, R_c])) &= (0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0), \\
 \theta_{\mathcal{T}}(\Phi_{\mathcal{T}}([L_b, T_{out}])) &= (0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1), \\
 \theta_{\mathcal{T}}(\Phi_{\mathcal{T}}([L_c, T_{in}])) &= (0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0), \\
 \theta_{\mathcal{T}}(\Phi_{\mathcal{T}}([R_b, T_{in}])) &= (0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0), \\
 \theta_{\mathcal{T}}(\Phi_{\mathcal{T}}([R_c, T_{out}])) &= (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1), \\
 \theta_{\mathcal{T}}(\Phi_{\mathcal{T}}([T_{out}, T_{in}])) &= (0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0), \\
 \theta_{\mathcal{T}}(\Phi_{\mathcal{T}}([T_{in}, T_{out}])) &= (0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1), \\
 \theta_{\mathcal{T}}(\Phi_{\mathcal{T}}([L_b, R_c])) &= (0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0), \\
 \theta_{\mathcal{T}}(\Phi_{\mathcal{T}}([R_b, L_c])) &= (0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1), \\
 \theta_{\mathcal{T}}(\Phi_{\mathcal{T}}([R_c, L_b])) &= (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0), \\
 \theta_{\mathcal{T}}(\Phi_{\mathcal{T}}([L_c, R_b])) &= (0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0).
 \end{aligned}$$

For any  $x \in H_{\mathcal{T}}$ , suppose  $\theta_{\mathcal{T}}(x)$  is not irreducible in  $\mathcal{B}_{\mathcal{T}}$ . By Proposition 5.8, the Hilbert basis elements are contained in  $H_{\mathcal{T}}$ . Thus  $\theta_{\mathcal{T}}(x) = a_1\theta_{\mathcal{T}}(x_1) + \cdots + a_r\theta_{\mathcal{T}}(x_r)$  where  $x_i \neq 0 \in H_{\mathcal{T}}$ ,  $a_i \in \mathbb{Z}_{>0}$  and  $r \geq 2$ . By the above formulas, we observe that the entries of  $\theta_{\mathcal{T}}(x)$  are 0 or 1. Thus  $a_1 = \cdots = a_r = 1$ . For any mutually distinct  $y, z \in H_{\mathcal{T}}$ , by the above 22 formulas, the difference  $\theta_{\mathcal{T}}(y) - \theta_{\mathcal{T}}(z)$  is not non-negative. Thus  $\theta_{\mathcal{T}}(y) - \theta_{\mathcal{T}}(z) \notin \mathcal{B}_{\mathcal{T}}$ . Hence it is not possible to have  $\theta_{\mathcal{T}}(x) = \theta_{\mathcal{T}}(x_1) + \cdots + \theta_{\mathcal{T}}(x_r)$  for  $r \geq 2$ . Thus  $\theta_{\mathcal{T}}(x)$  is irreducible, which implies that any element  $x \in H_{\mathcal{T}}$  is irreducible. We conclude that  $H_{\mathcal{T}}$  form the Hilbert basis of  $\mathcal{C}_{\mathcal{T}}$ .  $\square$

**Proposition 5.11.** *The following 10 linear relations are independent among the 22 elements in  $H_{\mathcal{T}}$ :*

$$\begin{aligned}
 \Phi_{\mathcal{T}}([T_{in}, L_b]) + \Phi_{\mathcal{T}}([T_{out}, L_c]) &= \Phi_{\mathcal{T}}([L_a]) + \Phi_{\mathcal{T}}([L_b]) + \Phi_{\mathcal{T}}([L_c]), \\
 \Phi_{\mathcal{T}}([T_{out}, L_c]) + \Phi_{\mathcal{T}}([T_{in}, R_c]) &= \Phi_{\mathcal{T}}([L_a]) + \Phi_{\mathcal{T}}([L_c]) + \Phi_{\mathcal{T}}([L_b, R_c]), \\
 \Phi_{\mathcal{T}}([L_c, T_{in}]) + \Phi_{\mathcal{T}}([L_b, T_{out}]) &= \Phi_{\mathcal{T}}([L_b]) + \Phi_{\mathcal{T}}([L_c]) + \Phi_{\mathcal{T}}([L_d]), \\
 \Phi_{\mathcal{T}}([L_b, T_{out}]) + \Phi_{\mathcal{T}}([R_b, T_{in}]) + \Phi_{\mathcal{T}}([T_{out}, R_b]) \\
 &= \Phi_{\mathcal{T}}([L_b]) + \Phi_{\mathcal{T}}([R_b]) + \Phi_{\mathcal{T}}([L_d]) + \Phi_{\mathcal{T}}([T_{out}, L_c]), \\
 \Phi_{\mathcal{T}}([R_c, T_{out}]) + \Phi_{\mathcal{T}}([L_b, R_c]) &= \Phi_{\mathcal{T}}([L_b, T_{out}]) + \Phi_{\mathcal{T}}([R_c]), \\
 \Phi_{\mathcal{T}}([T_{out}, T_{in}]) + \Phi_{\mathcal{T}}([L_b, T_{out}]) &= \Phi_{\mathcal{T}}([L_b]) + \Phi_{\mathcal{T}}([L_d]) + \Phi_{\mathcal{T}}([T_{out}, L_c]), \\
 \Phi_{\mathcal{T}}([T_{in}, T_{out}]) + \Phi_{\mathcal{T}}([T_{out}, L_c]) &= \Phi_{\mathcal{T}}([L_a]) + \Phi_{\mathcal{T}}([L_c]) + \Phi_{\mathcal{T}}([L_b, T_{out}]), \\
 \Phi_{\mathcal{T}}([T_{out}, R_b]) + \Phi_{\mathcal{T}}([R_b, L_c]) &= \Phi_{\mathcal{T}}([R_b]) + \Phi_{\mathcal{T}}([T_{out}, L_c]), \\
 \Phi_{\mathcal{T}}([L_b, R_c]) + \Phi_{\mathcal{T}}([R_c, L_b]) &= \Phi_{\mathcal{T}}([L_b]) + \Phi_{\mathcal{T}}([R_c]), \\
 \Phi_{\mathcal{T}}([T_{out}, L_c]) + \Phi_{\mathcal{T}}([L_c, R_b]) &= \Phi_{\mathcal{T}}([T_{out}, R_b]) + \Phi_{\mathcal{T}}([L_c]).
 \end{aligned}$$

Any other linear relation is a linear combination of the above 10 linear relations.

The above relations could be viewed as tropicalized skein relations.

*Proof.* Let  $E$  be a real vector space of dimension 22 with 22 variables representing the 22 elements in  $H_{\mathcal{T}}$ . By computation, we obtain that the nullity of the linear system generated by the above 10 linear equations is 10. Thus the above 10 linear relations are independent. We can not have more than 10 linearly independent relations because the elements in  $H_{\mathcal{T}}$  spans a real dimension 12 space  $\mathcal{C}_{\mathcal{T}}(\mathbb{R})$ . Hence any other linear relation is a linear combination of the above 10 linear relations.  $\square$

## 6. SECTOR DECOMPOSITION FOR THE SQUARE CASE

**Definition 6.1.** A sector  $X$  is a closed subset of  $\mathbb{R}^k$  such that there exists a basis  $(x_1, \dots, x_k)$  of  $\mathbb{R}^k$  and

$$X = \{x \in \mathbb{R}^k \mid x = a_1x_1 + \dots + a_kx_k, a_1, \dots, a_k \in \mathbb{R}_+\}.$$

By Theorem 5.10, the Hilbert basis elements in  $H_{\mathcal{T}}$  span the entire  $\mathcal{C}_{\mathcal{T}}$  for  $\square$  with coefficients in  $\mathbb{Z}_+$ . Any set of linear independent 12 Hilbert basis elements of  $\mathcal{C}_{\mathcal{T}}$  provides us a sector. Thus the union of all the possible sectors equals the pointed convex polyhedral cone  $\mathcal{C}_{\mathcal{T}}^+(\mathbb{R})$ . But these sectors are too many so that they have interior overlap. We would like to pick up some specific sectors in order to find a nice decomposition of  $\mathcal{C}_{\mathcal{T}}^+(\mathbb{R})$  into the sectors such that

- the interiors of the sectors are disjoint;
- the sectors are separated by codimension one walls;
- there is a nice description about going through the walls.

**Definition 6.2** (Sector decomposition). Let  $C$  be a pointed convex polyhedral cone in  $\mathbb{R}^k$ . Let  $H_0$  be the Hilbert basis of  $C \cap L$  for some lattice  $L$ . A sector decomposition of  $C$  is a decomposition  $C = C_1 \cup \dots \cup C_m$  such that

- for any  $i = 1, \dots, m$ , there exists  $x_1, \dots, x_k \in H_0$  such that the sector

$$C_i = \{x \in C \mid x = a_1x_1 + \dots + a_kx_k, a_1, \dots, a_k \in \mathbb{R}_+\};$$

- for any distinct  $i, j = 1, \dots, m$ ,  $C_i$  and  $C_j$  have empty interior overlap, and the intersection  $C_i \cap C_j$  is a face of both  $C_i$  and  $C_j$  of codimension at least one.

**6.1. A linear isomorphism by tropical  $\mathcal{X}$  coordinates.** Let  $\mathcal{T}$  be an ideal triangulation of  $\square = (a, b, d, c)$  with the ideal triangles  $(a, b, c)$  and  $(d, b, c)$ . In order to obtain the sector decomposition of  $\mathcal{C}_{\mathcal{T}}^+(\mathbb{R})$  with the above mentioned properties, we would like to study the sector decomposition of a linear isomorphic pointed convex polyhedral cone which is easier to study. This could be achieved by using the tropical  $\mathcal{X}$ -coordinates given by Equations (10), (11), (12), (13). Let

$$\mathcal{D}_{\mathcal{T}}(\mathbb{R}) = \left\{ \begin{array}{l} (\beta_{a;b,c}^{2,1,0}, \beta_{a;b,c}^{1,2,0}, \beta_{b;c,a}^{2,1,0}, \beta_{b;c,a}^{1,2,0}, \beta_{c;a,b}^{2,1,0}, \beta_{c;a,b}^{1,2,0}, \\ \beta_{d;c,b}^{2,1,0}, \beta_{d;c,b}^{1,2,0}, X_1, X_2, X_3, X_4) \in \mathbb{R}^{12} \quad \left| \quad \begin{array}{l} (\beta_{a,b,c}^{i,j,k}) \in \mathcal{B}_{\mathcal{T}}(\mathbb{R}), \\ \text{Equations (10), (11), (12), (13)} \end{array} \right. \end{array} \right\}.$$

**Lemma 6.3.** The natural linear map  $\phi_{\mathcal{T}} : \mathcal{B}_{\mathcal{T}}(\mathbb{R}) \rightarrow \mathcal{D}_{\mathcal{T}}(\mathbb{R})$  defined by Equations (10), (11), (12), (13)

$$\phi_{\mathcal{T}} \left( (\beta_{a,b,c}^{i,j,k}) \right) = \left( \beta_{a;b,c}^{2,1,0}, \beta_{a;b,c}^{1,2,0}, \beta_{b;c,a}^{2,1,0}, \beta_{b;c,a}^{1,2,0}, \beta_{c;a,b}^{2,1,0}, \beta_{c;a,b}^{1,2,0}, \right. \\ \left. \beta_{d;c,b}^{2,1,0}, \beta_{d;c,b}^{1,2,0}, X_1, X_2, X_3, X_4 \right)$$

is an isomorphism.

*Proof.* Since both sides are isomorphic to  $\mathbb{R}^{12}$ , we verify that the linear map  $\phi_{\mathcal{T}}$  is injective in order to show that  $\phi_{\mathcal{T}}$  is an isomorphism. If  $\phi_{\mathcal{T}}\left(\left(\beta_{a,b,c}^{i,j,k}\right)\right) = 0$ , then  $\beta_{a,b,c}^{2,1,0} = \beta_{a,b,c}^{1,2,0} = \beta_{b,c,a}^{2,1,0} = \beta_{b,c,a}^{1,2,0} = \beta_{c;a,b}^{2,1,0} = \beta_{c;a,b}^{1,2,0} = \beta_{d;c,b}^{2,1,0} = \beta_{d;c,b}^{1,2,0} = 0$ . The other ten entries are forced to be zero by Equations (10), (11), (12), (13). Thus  $\phi_{\mathcal{T}}$  is a linear isomorphism.  $\square$

**Remark 6.4.** Let  $\pi_4 : \mathbb{R}^{12} \rightarrow \mathbb{R}^4$  be the projection into the last four entries of  $\mathbb{R}^{12}$ . Then  $\pi_4 \circ \phi_{\mathcal{T}} \circ \theta_{\mathcal{T}}$  is the tropicalized Fock–Goncharov’s forgetful map

$$p^t : \mathcal{A}_{\text{SL}_3, \square}(\mathbb{R}^t) \rightarrow \mathcal{X}_{\text{PGL}_3, \square}(\mathbb{R}^t).$$

The linear isomorphism  $\phi_{\mathcal{T}} \circ \theta_{\mathcal{T}}$  could be viewed as a map  $\bar{p}^t : \mathcal{A}_{\text{SL}_3, \square}(\mathbb{R}^t) \rightarrow \mathcal{P}_{\text{PGL}_3, \square}(\mathbb{R}^t)$ .

Let  $\mathbb{R}_+$  ( $\mathbb{Z}_+$  resp.) be the set of non-negative real numbers (integers resp.). Let

$$\mathcal{D}_{\mathcal{T}}^+(\mathbb{R}) = \left\{ \begin{array}{l} (\beta_{a;b,c}^{2,1,0}, \beta_{a;b,c}^{1,2,0}, \beta_{b;c;a}^{2,1,0}, \beta_{b;c;a}^{1,2,0}, \beta_{c;a,b}^{2,1,0}, \\ \beta_{c;a,b}^{1,2,0}, \beta_{d;c,b}^{2,1,0}, \beta_{d;c,b}^{1,2,0}, X_1, X_2, X_3, X_4) \mid \left( \beta_{a,b,c}^{i,j,k} \right) \in \mathcal{B}_{\mathcal{T}}^+(\mathbb{R}), \end{array} \right\} = \mathbb{R}_+^8 \times \mathbb{R}^4,$$

which could be viewed as  $\bar{p}^t(\mathcal{A}_{\text{PGL}_3, \square}^+(\mathbb{R}^t)) = \mathcal{P}_{\text{PGL}_3, \square}^+(\mathbb{R}^t)$ . And we have  $p^t(\mathcal{A}_{\text{PGL}_3, \square}^+(\mathbb{R}^t)) = \mathcal{X}_{\text{PGL}_3, \square}(\mathbb{R}^t)$ .

Let

$$\mathcal{D}_{\mathcal{T}} = \left\{ \begin{array}{l} (\beta_{a;b,c}^{2,1,0}, \beta_{a;b,c}^{1,2,0}, \beta_{b;c;a}^{2,1,0}, \beta_{b;c;a}^{1,2,0}, \beta_{c;a,b}^{2,1,0}, \\ \beta_{c;a,b}^{1,2,0}, \beta_{d;c,b}^{2,1,0}, \beta_{d;c,b}^{1,2,0}, X_1, X_2, X_3, X_4) \mid \left( \beta_{a,b,c}^{i,j,k} \right) \in \mathcal{B}_{\mathcal{T}}, \end{array} \right\} = \mathbb{Z}_+^8 \times \mathbb{Z}^4$$

as an intersection of a lattice with the pointed convex polyhedral cone  $\mathcal{D}_{\mathcal{T}}^+(\mathbb{R})$ , which could be viewed as  $\bar{p}^t(\mathcal{A}_{\text{PGL}_3, \square}^+(\mathbb{Z}^t)) = \mathcal{P}_{\text{PGL}_3, \square}^+(\mathbb{Z}^t)$ . Moreover, by allowing the first 8 entries of  $\mathcal{D}_{\mathcal{T}}$  to be integer numbers, we have  $\bar{p}^t(\mathcal{A}_{\text{PGL}_3, \square}(\mathbb{Z}^t)) = \mathcal{P}_{\text{PGL}_3, \square}(\mathbb{Z}^t) = \mathbb{Z}^{12}$  corresponding to the space  $\widetilde{\mathcal{W}}_{\square}$  of  $A_2$ -laminations; recall the discussion after Theorem 4.4. And we have  $p^t(\mathcal{A}_{\text{PGL}_3, \square}^+(\mathbb{Z}^t)) = p^t(\mathcal{A}_{\text{PGL}_3, \square}(\mathbb{Z}^t)) = \mathcal{X}_{\text{PGL}_3, \square}(\mathbb{Z}^t)$ .

By definition

**Lemma 6.5.** *The restriction of  $\phi_{\mathcal{T}}$  to  $\mathcal{B}_{\mathcal{T}}$  ( $\mathcal{B}_{\mathcal{T}}^+(\mathbb{R})$  resp.) provides a bijection between  $\mathcal{B}_{\mathcal{T}}$  and  $\mathcal{D}_{\mathcal{T}}$  ( $\mathcal{B}_{\mathcal{T}}^+(\mathbb{R})$  and  $\mathcal{D}_{\mathcal{T}}^+(\mathbb{R})$  resp.); see §5.2.*

By Lemma 5.6 and Lemma 6.5

**Corollary 6.6.** *The restriction of  $\phi_{\mathcal{T}} \circ \theta_{\mathcal{T}}$  to  $\mathcal{C}_{\mathcal{T}}$  ( $\mathcal{C}_{\mathcal{T}}^+(\mathbb{R})$  resp.) provides a bijection between  $\mathcal{C}_{\mathcal{T}}$  and  $\mathcal{D}_{\mathcal{T}}$  ( $\mathcal{C}_{\mathcal{T}}^+(\mathbb{R})$  and  $\mathcal{D}_{\mathcal{T}}^+(\mathbb{R})$  resp.).*

Now, it is enough for us to study the sector decomposition of  $\mathcal{D}_{\mathcal{T}}^+(\mathbb{R}) = \mathbb{R}_+^8 \times \mathbb{R}^4$  for getting the sector decomposition of  $\mathcal{C}_{\mathcal{T}}^+(\mathbb{R})$  through the linear isomorphism  $\phi_{\mathcal{T}} \circ \theta_{\mathcal{T}}$ .

**6.2. Sector decomposition.** We specify a particular sector decomposition

$$\mathcal{D}_{\mathcal{T}}^+(\mathbb{R}) = C_1 \cup \cdots \cup C_{42}$$

with 42 sectors by Proposition 4.1. Each sector  $C_i$  is a combination of 12 linearly independent elements corresponding to 12 reduced webs in  $H_{\mathcal{T}}^W$  (Figure 26) with non-negative real coefficients where

- the set of these 12 elements contains the image of the set of corner arcs  $R = \{[R_a], [L_a], [R_b], [L_b], [R_c], [L_c], [R_d], [L_d]\}$  under  $\phi_{\mathcal{T}} \circ \theta_{\mathcal{T}} \circ \Phi_{\mathcal{T}}$ , and
- the other 4 elements are in the image of the other 4 reduced webs (denoted by  $Q_i$ ) of  $H_{\mathcal{T}}^W$  under  $\phi_{\mathcal{T}} \circ \theta_{\mathcal{T}} \circ \Phi_{\mathcal{T}}$ .

The set  $Q_i$  of the 4 reduced webs determines the sector  $C_i$ . By Proposition 4.1, we obtain

**Definition 6.7.** *The 42 sectors  $\{C_i\}_{i=1}^{42}$  are defined by*

- (1)  $Q_1 = \{[T_{in}, L_b], [R_b, L_c], [R_c, T_{out}], [R_c, L_b]\};$
- (2)  $Q_2 = \{[T_{out}, L_c], [R_b, L_c], [R_c, T_{out}], [R_c, L_b]\};$
- (3)  $Q_3 = \{[T_{in}, L_b], [R_b, L_c], [R_b, T_{in}], [R_c, L_b]\};$
- (4)  $Q_4 = \{[T_{out}, L_c], [R_b, L_c], [R_b, T_{in}], [R_c, L_b]\};$
- (5)  $Q_5 = \{[T_{in}, L_b], [R_b, L_c], [T_{in}, T_{out}], [R_c, T_{out}]\};$
- (6)  $Q_6 = \{[T_{in}, T_{out}], [R_b, L_c], [L_b, T_{out}], [T_{in}, R_c]\};$
- (7)  $Q_7 = \{[T_{in}, L_b], [L_c, R_b], [T_{in}, T_{out}], [R_c, T_{out}]\};$
- (8)  $Q_8 = \{[T_{in}, T_{out}], [L_c, R_b], [L_b, T_{out}], [T_{in}, R_c]\};$
- (9)  $Q_9 = \{[T_{out}, R_b], [L_c, R_b], [L_c, T_{in}], [L_b, R_c]\};$
- (10)  $Q_{10} = \{[T_{in}, R_c], [L_c, R_b], [L_c, T_{in}], [L_b, R_c]\};$
- (11)  $Q_{11} = \{[T_{out}, R_b], [L_c, R_b], [L_b, T_{out}], [L_b, R_c]\};$
- (12)  $Q_{12} = \{[T_{in}, R_c], [L_c, R_b], [L_b, T_{out}], [L_b, R_c]\};$
- (13)  $Q_{13} = \{[T_{out}, T_{in}], [T_{out}, R_b], [L_c, T_{in}], [L_b, R_c]\};$
- (14)  $Q_{14} = \{[T_{out}, L_c], [R_b, T_{in}], [T_{out}, T_{in}], [L_b, R_c]\};$
- (15)  $Q_{15} = \{[T_{out}, T_{in}], [T_{out}, R_b], [L_c, T_{in}], [R_c, L_b]\};$
- (16)  $Q_{16} = \{[T_{out}, L_c], [R_b, T_{in}], [T_{out}, T_{in}], [R_c, L_b]\};$
- (17)  $Q_{17} = \{[T_{out}, L_c], [T_{out}, R_b], [R_c, T_{out}], [R_c, L_b]\};$
- (18)  $Q_{18} = \{[T_{in}, L_b], [R_b, T_{in}], [L_c, T_{in}], [R_c, L_b]\};$
- (19)  $Q_{19} = \{[T_{in}, L_b], [L_c, R_b], [L_c, T_{in}], [T_{in}, R_c]\};$
- (20)  $Q_{20} = \{[T_{out}, R_b], [L_c, R_b], [L_b, T_{out}], [R_c, T_{out}]\};$
- (21)  $Q_{21} = \{[T_{out}, R_b], [L_c, R_b], [R_c, T_{out}], [R_c, L_b]\};$
- (22)  $Q_{22} = \{[T_{out}, R_b], [L_c, R_b], [L_c, T_{in}], [R_c, L_b]\};$
- (23)  $Q_{23} = \{[T_{in}, L_b], [L_c, R_b], [R_c, T_{out}], [R_c, L_b]\};$
- (24)  $Q_{24} = \{[T_{in}, L_b], [L_c, R_b], [L_c, T_{in}], [R_c, L_b]\};$
- (25)  $Q_{25} = \{[T_{out}, L_c], [T_{out}, R_b], [L_b, T_{out}], [L_b, R_c]\};$
- (26)  $Q_{26} = \{[T_{in}, R_c], [R_b, T_{in}], [L_c, T_{in}], [L_b, R_c]\};$
- (27)  $Q_{27} = \{[T_{in}, L_b], [R_b, L_c], [R_b, T_{in}], [T_{in}, R_c]\};$
- (28)  $Q_{28} = \{[T_{out}, L_c], [R_b, L_c], [L_b, T_{out}], [R_c, T_{out}]\};$
- (29)  $Q_{29} = \{[T_{out}, L_c], [R_b, L_c], [L_b, T_{out}], [L_b, R_c]\};$
- (30)  $Q_{30} = \{[T_{out}, L_c], [R_b, L_c], [R_b, T_{in}], [L_b, R_c]\};$
- (31)  $Q_{31} = \{[T_{in}, R_c], [R_b, L_c], [L_b, T_{out}], [L_b, R_c]\};$
- (32)  $Q_{32} = \{[T_{in}, R_c], [R_b, L_c], [R_b, T_{in}], [L_b, R_c]\};$
- (33)  $Q_{33} = \{[T_{out}, L_c], [T_{out}, R_b], [L_b, T_{out}], [R_c, T_{out}]\};$
- (34)  $Q_{34} = \{[T_{in}, L_b], [R_b, T_{in}], [L_c, T_{in}], [T_{in}, R_c]\};$
- (35)  $Q_{35} = \{[T_{in}, T_{out}], [T_{in}, L_b], [T_{in}, R_c], [R_b, L_c]\};$
- (36)  $Q_{36} = \{[T_{in}, T_{out}], [R_b, L_c], [L_b, T_{out}], [R_c, T_{out}]\};$
- (37)  $Q_{37} = \{[T_{in}, L_b], [L_c, R_b], [T_{in}, T_{out}], [T_{in}, R_c]\};$
- (38)  $Q_{38} = \{[T_{in}, T_{out}], [L_c, R_b], [L_b, T_{out}], [R_c, T_{out}]\};$
- (39)  $Q_{39} = \{[T_{out}, L_c], [T_{out}, R_b], [T_{out}, T_{in}], [L_b, R_c]\};$
- (40)  $Q_{40} = \{[T_{out}, T_{in}], [R_b, T_{in}], [L_c, T_{in}], [L_b, R_c]\};$
- (41)  $Q_{41} = \{[T_{out}, L_c], [T_{out}, R_b], [T_{out}, T_{in}], [R_c, L_b]\};$
- (42)  $Q_{42} = \{[T_{out}, T_{in}], [R_b, T_{in}], [L_c, T_{in}], [R_c, L_b]\}.$

The schematic diagram of any reduced web

$$W \in \mathcal{W}_i = \Phi_{\mathcal{T}}^{-1}(\mathbb{Z}_+ \{\Phi_{\mathcal{T}}(Q_i)\}),$$

is a combination of reduced webs in  $Q_i$  with non-negative integer coefficients  $(x, y, z, t)$ , which could be described as follows:

- (1) Figure 8(1) describes the web in  $\mathcal{W}_{29}$  ( $\mathcal{W}_{21}, \mathcal{W}_{24}, \mathcal{W}_{32}$  up to symmetry and orientation reversing).
- (2) Figure 8(2) describes the web in  $\mathcal{W}_{22}$  ( $\mathcal{W}_{23}, \mathcal{W}_{30}, \mathcal{W}_{31}$  up to symmetry and orientation reversing).
- (3) Figure 8(3) describes the web in  $\mathcal{W}_{35}$  ( $\mathcal{W}_{36}, \mathcal{W}_{37}, \mathcal{W}_{38}, \mathcal{W}_{39}, \mathcal{W}_{40}, \mathcal{W}_{41}, \mathcal{W}_{42}$  up to symmetry and orientation reversing).
- (4) Figure 8(4) describes the web in  $\mathcal{W}_{17}$  ( $\mathcal{W}_{18}, \mathcal{W}_{19}, \mathcal{W}_{20}, \mathcal{W}_{25}, \mathcal{W}_{26}, \mathcal{W}_{27}, \mathcal{W}_{28}$  up to symmetry and orientation reversing).
- (5) Figure 8(5) describes the web in  $\mathcal{W}_5$  ( $\mathcal{W}_8, \mathcal{W}_{13}, \mathcal{W}_{16}$  up to symmetry and orientation reversing).
- (6) Figure 8(6) describes the web in  $\mathcal{W}_6$  ( $\mathcal{W}_7, \mathcal{W}_{14}, \mathcal{W}_{15}$  up to symmetry and orientation reversing).
- (7) Figure 8(7) describes the web in  $\mathcal{W}_2$  ( $\mathcal{W}_3, \mathcal{W}_{10}, \mathcal{W}_{11}$  up to symmetry and orientation reversing).
- (8) Figure 8(8) describes the web in  $\mathcal{W}_1$  ( $\mathcal{W}_4, \mathcal{W}_9, \mathcal{W}_{12}$  up to symmetry and orientation reversing).
- (9) Figure 8(9) describes the web in  $\mathcal{W}_{33}$  ( $\mathcal{W}_{34}$  up to symmetry and orientation reversing).

The set of the 12 reduced webs in  $H_{\mathcal{T}}^W$  for  $C_i$  is called the *topological type* of  $C_i$ .

**Theorem 6.8.** *Given an ideal triangulation  $\mathcal{T}$ , the decomposition*

$$\mathcal{D}_{\mathcal{T}}^+(\mathbb{R}) = C_1 \cup \dots \cup C_{42}$$

*is a sector decomposition such that:*

- (1) *the sectors have empty interior overlap;*
- (2) *each pair of adjacent sectors is separated by a codimension one wall;*
- (3) *and,  $C_i$  and  $C_j$  are adjacent sectors if and only if the topological types of  $C_i$  and  $C_j$  differ by exactly one reduced web element.*

*Proof.* Recall  $R = \{[R_a], [L_a], [R_b], [L_b], [R_c], [L_c], [R_d], [L_d]\}$  is the set of 8 corner arcs. By direct computation, the set  $\phi_{\mathcal{T}} \circ \theta_{\mathcal{T}} \circ \Phi_{\mathcal{T}}(R)$  contains the first 8 elements of the standard basis of  $\mathcal{D}_{\mathcal{T}}(\mathbb{R})$ . Recall  $\pi_4 : \mathcal{D}_{\mathcal{T}}^+(\mathbb{R}) \rightarrow \mathbb{R}^4$  is the projection into the last four entries  $(X_1, X_2, X_3, X_4) \in \mathbb{R}^4$ . Let  $D_i = \pi_4(C_i)$  for  $i = 1, \dots, 42$ . Since  $\mathcal{D}_{\mathcal{T}}^+(\mathbb{R}) = \mathbb{R}_+^8 \times \mathbb{R}^4$ , we only need to show that  $\cup_{i=1}^{42} D_i$  is a sector decomposition of  $\mathbb{R}^4$ .

For  $Q_1$ , let us write the four vectors  $\pi_4(\phi_{\mathcal{T}} \circ \theta_{\mathcal{T}} \circ \Phi_{\mathcal{T}}(Q_1))$  in columns to form a  $(4 \times 4)$

matrix  $M_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix}$ . Then for  $x, y, z, t \geq 0$ , we get

$$M_1 \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} -x \\ y \\ z \\ -z - t \end{pmatrix}.$$



- (4)  $\mathbb{R}_- \times \mathbb{R}_+ \times \mathbb{R}_- \times \mathbb{R}_- = D_3 \cup D_{18}$ ;
- (5)  $\mathbb{R}_- \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ = D_6 \cup D_{31} \cup D_{35}$ ;
- (6)  $\mathbb{R}_- \times \mathbb{R}_- \times \mathbb{R}_+ \times \mathbb{R}_- = D_7 \cup D_{23} \cup D_{38}$ ;
- (7)  $\mathbb{R}_- \times \mathbb{R}_- \times \mathbb{R}_- \times \mathbb{R}_+ = D_{10} \cup D_{19}$ ;
- (8)  $\mathbb{R}_+ \times \mathbb{R}_- \times \mathbb{R}_+ \times \mathbb{R}_+ = D_{11} \cup D_{25}$ ;
- (9)  $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_- \times \mathbb{R}_+ = D_{14} \cup D_{30} \cup D_{40}$ ;
- (10)  $\mathbb{R}_+ \times \mathbb{R}_- \times \mathbb{R}_- \times \mathbb{R}_- = D_{15} \cup D_{22} \cup D_{41}$ ;
- (11)  $\mathbb{R}_- \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_- = D_1 \cup D_5 \cup D_{36}$ ;
- (12)  $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_- \times \mathbb{R}_- = D_4 \cup D_{16} \cup D_{42}$ ;
- (13)  $\mathbb{R}_- \times \mathbb{R}_- \times \mathbb{R}_+ \times \mathbb{R}_+ = D_8 \cup D_{12} \cup D_{37}$ ;
- (14)  $\mathbb{R}_+ \times \mathbb{R}_- \times \mathbb{R}_- \times \mathbb{R}_+ = D_9 \cup D_{13} \cup D_{39}$ ;
- (15)  $\mathbb{R}_+ \times \mathbb{R}_- \times \mathbb{R}_+ \times \mathbb{R}_- = D_{17} \cup D_{20} \cup D_{21} \cup D_{33}$ ;
- (16)  $\mathbb{R}_- \times \mathbb{R}_+ \times \mathbb{R}_- \times \mathbb{R}_+ = D_{26} \cup D_{27} \cup D_{32} \cup D_{34}$ .

Then  $\mathbb{R}^4 = \cup_{i=1}^{42} D_i$  is a decomposition such that the interiors of sectors are mutually disjoint and the sectors are separated by codimension one walls, thus a sector decomposition. Hence  $\mathcal{D}_7^+(\mathbb{R}) = \cup_{i=1}^{42} C_i$  is a sector decomposition.

We draw a graph as in Figure 27 with vertices  $\{D_i\}_{i=1}^{42}$  such that two vertices are connected by an edge if and only if they share a codimensional one wall.

By checking any pair  $D_i$  and  $D_j$  in Figure 27, we obtain that the topological types of  $C_i$  and  $C_j$  differ by only one reduced web element if and only if  $C_i$  and  $C_j$  share a codimension one wall.

□

**Remark 6.9.** (1) *In the above proof, we actually also decompose  $\mathcal{X}_{\text{PGL}_3, \square}(\mathbb{R}^t) \cong \mathbb{R}^4$ .*  
 (2) *We believe that Theorem 6.8 is related to the wall-crossing phenomenon [KS08].*

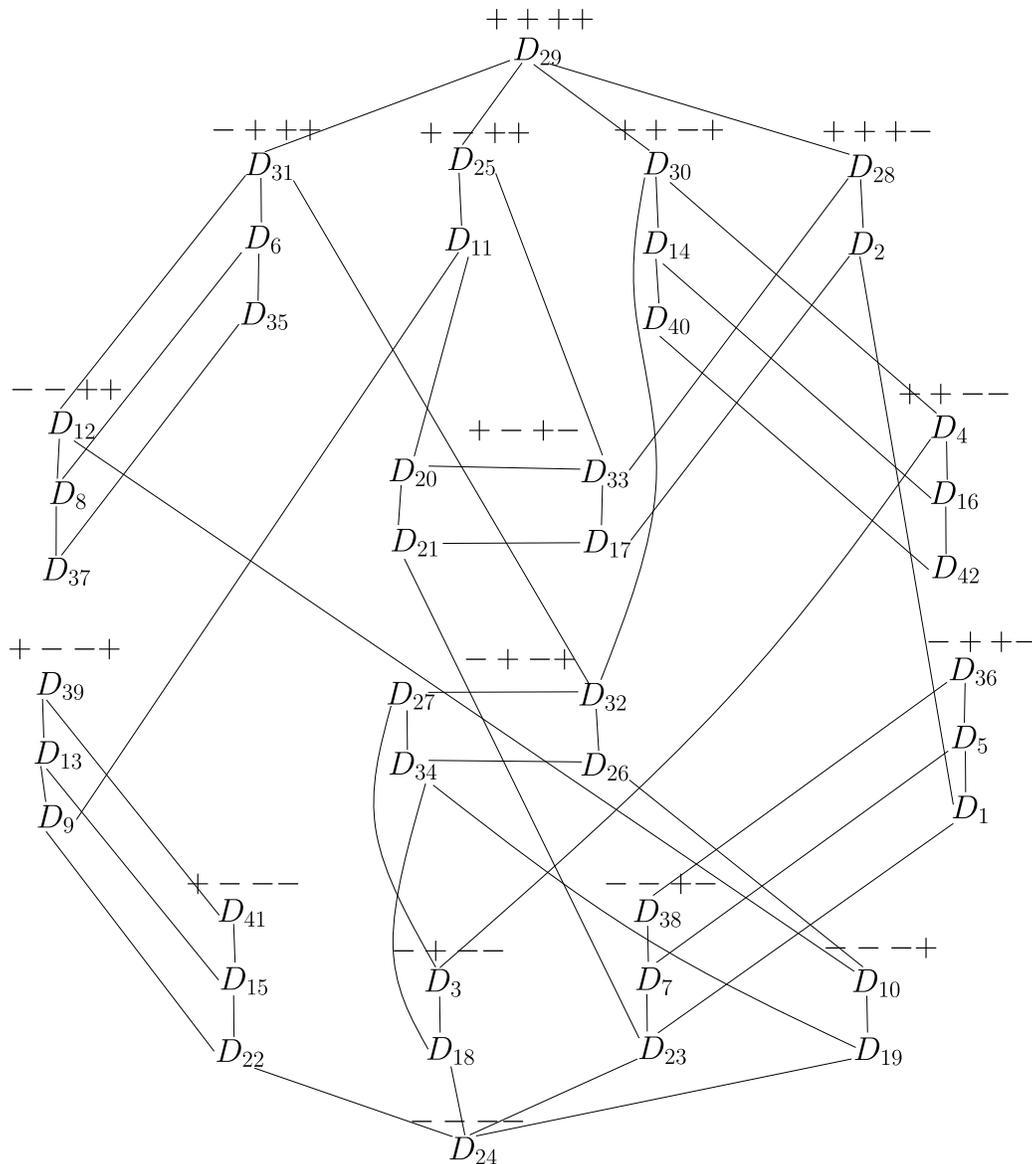


FIGURE 27. Walls.

## REFERENCES

- [AF17] N. Abdiel and C. Frohman. The localized skein algebra is Frobenius. *Algebr. Geom. Topol.*, 17:3341–3373, 2017.
- [AK17] D. G. L. Allegretti and H. K. Kim. A duality map for quantum cluster varieties from surfaces. *Adv. Math.*, 306:1164–1208, 2017.
- [BFKB99] D. Bullock, C. Frohman, and J. Kania-Bartoszyńska. Understanding the Kauffman bracket skein module. *J. Knot Theory Ramifications*, 8:265–277, 1999.
- [BHMV95] C. Blanchet, N. Habegger, G. Masbaum, and P. Vogel. Topological quantum field theories derived from the Kauffman bracket. *Topology*, 34:883–927, 1995.
- [Bul97] D. Bullock. Rings of  $SL_2(\mathbf{C})$ -characters and the Kauffman bracket skein module. *Comment. Math. Helv.*, 72:521–542, 1997.
- [BW11a] F. Bonahon and H. Wong. Kauffman brackets, character varieties and triangulations of surfaces. *Contemp. Math.*, 560:179–194, 2011.

- [BW11b] F. Bonahon and H. Wong. Quantum traces for representations of surface groups in  $SL_2(\mathbb{C})$ . *Geom. Topol.*, 15:1569–1615, 2011.
- [CKM14] S. Cautis, J. Kamnitzer, and S. Morrison. Webs and quantum skew Howe duality. *Math. Ann.*, 360:351–390, 2014.
- [CL19] F. Costantino and T. T. Q. Lê. Stated skein algebras of surfaces. <https://arxiv.org/abs/1907.11400>, 2019.
- [CM12] L. Charles and J. Marché. Multicurves and regular functions on the representation variety of a surface in  $SU(2)$ . *Comment. Math. Helv.*, 87:409–431, 2012.
- [CTT20] A. Casella, D. Tate, and S. Tillmann. Moduli spaces of real projective structures on surfaces. In *MSJ Memoirs*, volume 38. Mathematical Society of Japan, Tokyo, 2020.
- [Dou] D. C. Douglas. Quantum traces for  $SL_n(\mathbb{C})$ : the case  $n = 3$ . In preparation.
- [Dou20] D. C. Douglas. *Classical and quantum traces coming from  $SL_n(\mathbb{C})$  and  $U_q(\mathfrak{sl}_n)$* . PhD thesis, University of Southern California, 2020.
- [DS20] D. C. Douglas and Z. Sun. Tropical Fock-Goncharov coordinates for  $SL_3$ -webs on surfaces I: construction. <https://arxiv.org/abs/2011.01768>, 2020.
- [FG06] V. V. Fock and A. B. Goncharov. Moduli spaces of local systems and higher Teichmüller theory. *Publ. Math. Inst. Hautes Études Sci.*, 103:1–211, 2006.
- [FG07a] V. V. Fock and A. B. Goncharov. Dual Teichmüller and lamination spaces. *Handbook of Teichmüller theory*, 1:647–684, 2007.
- [FG07b] V. V. Fock and A. B. Goncharov. Moduli spaces of convex projective structures on surfaces. *Adv. Math.*, 208:249–273, 2007.
- [FG09] V. V. Fock and A. B. Goncharov. Cluster ensembles, quantization and the dilogarithm. *Ann. Sci. Éc. Norm. Supér.*, 42:865–930, 2009.
- [FKK13] B. Fontaine, J. Kamnitzer, and G. Kuperberg. Buildings, spiders, and geometric Satake. *Compos. Math.*, 149:1871–1912, 2013.
- [Fon12] B. Fontaine. Generating basis webs for  $SL_n$ . *Adv. Math.*, 229:2792–2817, 2012.
- [FP16] S. Fomin and P. Pylyavskyy. Tensor diagrams and cluster algebras. *Adv. Math.*, 300:717–787, 2016.
- [FS20] C. Frohman and A. S. Sikora.  $SU(3)$ -skein algebras and webs on surfaces. <https://arxiv.org/abs/2002.08151>, 2020.
- [FZ02] S. Fomin and A. Zelevinsky. Cluster algebras. I. Foundations. *J. Amer. Math. Soc.*, 15:497–529, 2002.
- [GHKK18] M. Gross, P. Hacking, S. Keel, and M. Kontsevich. Canonical bases for cluster algebras. *J. Amer. Math. Soc.*, 31:497–608, 2018.
- [GJS19] S. Gunningham, D. Jordan, and P. Safronov. The finiteness conjecture for skein modules. <https://arxiv.org/abs/1908.05233v1>, 2019.
- [GMN13] D. Gaiotto, G. W. Moore, and A. Neitzke. Spectral networks. *Ann. Henri Poincaré*, 14:1643–1731, 2013.
- [GMN14] D. Gaiotto, G. W. Moore, and A. Neitzke. Spectral networks and snakes. *Ann. Henri Poincaré*, 15:61–141, 2014.
- [Gor73] P. Gordan. Ueber die Auflösung linearer Gleichungen mit reellen Coefficienten. *Math. Ann.*, 6:23–28, 1873.
- [GP79] F. R. Giles and W. R. Pulleyblank. Total dual integrality and integer polyhedra. *Linear Algebra Appl.*, 25:191–196, 1979.
- [GS15] A. B. Goncharov and L. Shen. Geometry of canonical bases and mirror symmetry. *Invent. Math.*, 202:487–633, 2015.
- [GS18] A. B. Goncharov and L. Shen. Donaldson-Thomas transformations of moduli spaces of  $G$ -local systems. *Adv. Math.*, 327:225–348, 2018.
- [GS19] A. B. Goncharov and L. Shen. Quantum geometry of moduli spaces of local systems and representation theory. <https://arxiv.org/abs/1904.10491v1>, 2019.
- [Hig20] V. Higgins. Triangular decomposition of  $SL_3$  skein algebras. <https://arxiv.org/abs/2008.09419>, 2020.
- [Hil90] D. Hilbert. Ueber die Theorie der algebraischen Formen. *Math. Ann.*, 36:473–534, 1890.
- [Hit92] N. J. Hitchin. Lie groups and Teichmüller space. *Topology*, 31:449–473, 1992.

- [HP93] J. Hoste and J. H. Przytycki. The  $(2, \infty)$ -skein module of lens spaces; a generalization of the Jones polynomial. *J. Knot Theory Ramifications*, 2:321–333, 1993.
- [HS19] Y. Huang and Z. Sun. McShane identities for higher Teichmüller theory and the Goncharov-Shen potential, to appear in Mem. Amer. Math. Soc. <https://arxiv.org/abs/1901.02032>, 2019.
- [IY] T. Ishibashi and W. Yuasa. Skein and cluster algebras of marked surfaces without punctures for  $\mathfrak{sl}_3$ . In preparation.
- [Kim20] H. K. Kim.  $A_2$ -laminations as basis for  $PGL_3$  cluster variety for surface. <https://arxiv.org/abs/2011.14765>, 2020.
- [KS08] M. Kontsevich and Y. Soibelman. Stability structures, motivic Donaldson-Thomas invariants and cluster transformations. <https://arxiv.org/abs/0811.2435>, 2008.
- [KT99] A. Knutson and T. Tao. The honeycomb model of  $GL_n(\mathbb{C})$  tensor products. I. Proof of the saturation conjecture. *J. Amer. Math. Soc.*, 12:1055–1090, 1999.
- [Kup96] G. Kuperberg. Spiders for rank 2 Lie algebras. *Comm. Math. Phys.*, 180:109–151, 1996.
- [Lab06] F. Labourie. Anosov flows, surface groups and curves in projective space. *Invent. Math.*, 165:51–114, 2006.
- [Le16] I. Le. Higher laminations and affine buildings. *Geom. Topol.*, 20:1673–1735, 2016.
- [Mar19] G. Martone. Positive configurations of flags in a building and limits of positive representations. *Math. Z.*, 293:1337–1368, 2019.
- [MFK94] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory. Third edition.* Springer-Verlag, Berlin, 1994.
- [MOY98] H. Murakami, T. Ohtsuki, and S. Yamada. Homfly polynomial via an invariant of colored plane graphs. *Enseign. Math.*, 44:325–360, 1998.
- [Mul16] G. Muller. Skein and cluster algebras of marked surfaces. *Quantum Topol.*, 7:435–503, 2016.
- [OT19] C. Ouyang and A. Tamburelli. Limits of Blaschke metrics, to appear in Duke Math. J. <https://arxiv.org/abs/1911.02119v1>, 2019.
- [Pen87] R. C. Penner. The decorated Teichmüller space of punctured surfaces. *Comm. Math. Phys.*, 113:299–339, 1987.
- [Pro76] C. Procesi. The invariant theory of  $n \times n$  matrices. *Adv. Math.*, 19:306–381, 1976.
- [Prz91] J. H. Przytycki. Skein modules of 3-manifolds. *Bull. Polish Acad. Sci. Math.*, 39:91–100, 1991.
- [PS00] J. H. Przytycki and A. S. Sikora. On skein algebras and  $SL_2(\mathbb{C})$ -character varieties. *Topology*, 39:115–148, 2000.
- [RT90] N. Y. Reshetikhin and V. G. Turaev. Ribbon graphs and their invariants derived from quantum groups. *Comm. Math. Phys.*, 127:1–26, 1990.
- [Sch81] A. Schrijver. On total dual integrality. *Linear Algebra Appl.*, 38:27–32, 1981.
- [Sik01] A. S. Sikora.  $SL_n$ -character varieties as spaces of graphs. *Trans. Amer. Math. Soc.*, 353:2773–2804, 2001.
- [Sik05] A. S. Sikora. Skein theory for  $SU(n)$ -quantum invariants. *Algebr. Geom. Topol.*, 5:865–897, 2005.
- [Sun] Z. Sun. Tropical coordinates for  $SL_n$ -webs on surfaces. In preparation.
- [SW07] A. S. Sikora and B. W. Westbury. Confluence theory for graphs. *Algebr. Geom. Topol.*, 7:439–478, 2007.
- [SWZ20] Z. Sun, A. Wienhard, and T. Zhang. Flows on the  $PGL(V)$ -Hitchin component. *Geom. Funct. Anal.*, 30:588–692, 2020.
- [Thu97] W. P. Thurston. *Three-dimensional geometry and topology. Vol. 1.* Princeton University Press, Princeton, NJ, 1997.
- [Tur89] V. G. Turaev. Algebras of loops on surfaces, algebras of knots, and quantization. In *Braid group, knot theory and statistical mechanics*, pages 59–95. World Sci. Publ., Teaneck, NJ, 1989.
- [vdC31] J. G. van der Corput. Ueber Systeme von linear-homogenen Gleichungen und Ungleichungen. In *Proceedings Koninklijke Akademie van Wetenschappen te Amsterdam*, volume 34, pages 368–371, 1931.
- [Wie18] A. Wienhard. An invitation to higher Teichmüller theory. <https://arxiv.org/abs/1803.06870>, 2018.
- [Wit89] E. Witten. Quantum field theory and the Jones polynomial. *Comm. Math. Phys.*, 121:351–399, 1989.

- [Xie13] D. Xie. Higher laminations, webs and  $\mathcal{N} = 2$  line operators. <https://arxiv.org/abs/1304.2390>, 2013.

DEPARTMENT OF MATHEMATICS, YALE UNIVERSITY, NEW HAVEN CT 06511, U.S.A.  
*Email address:* `daniel.douglas@yale.edu`

IHES, 35 ROUTE DE CHARTRES, 91440 BURES-SUR-YVETTE, FRANCE  
*Email address:* `sun.zhe@ihes.fr`