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Dipl.-Math. Robert BAUMGARTH
Born in Löbau (Germany)

Scattering Theory for the Hodge Laplacian and Covariant Riesz Transforms

Dissertation defence committee

Dr Anton THALMAIER (Dissertation supervisor)
Professor, Université du Luxembourg

Dr Giovanni PECCATI (Chairman)
Professor, Université du Luxembourg

Dr Marc ARNAUDON (Vice Chairman)
Professor, Université de Bordeaux

Dr Batu GÜNEYSU (Member)
Professor, Universität Bonn

Dr Max von RENESSE (Member)
Professor, Universität Leipzig

À Martin et Martina.

Un enfant qui meurt de faim est un enfant assassiné.

Jean Ziegler

Damit hat die noematische Geltungsreflexion unter allen Weisen des Denkens überhaupt, aber auch unter allen Weisen der Reflexion eine *absolute Vorrangstellung* erworben. Sie ist das Denken, das nicht einfach bloß Theorien aufbaut, sondern als einziges Denken gleichzeitig auch der Gültigkeit dessen, was es aufbaut, absolut sicher zu sein vermag.

Hans Wagner, Philosophie und Reflexion

Many worlds might have been botched and bungled, throughout an eternity, ere this system was struck out; much labour lost, many fruitless trials made; and a slow, but continued improvement carried on during infinite ages in the art of world-making.

David Hume, Dialogues Concerning Natural Religion

Scattering Theory for the Hodge Laplacian and Covariant Riesz Transforms

Abstract

Based on gradient estimates obtained by probabilistic Bismut type formulae for the heat semigroup on the full exterior bundle of square-integrable Borel forms defined by spectral calculus, we show results for two distinct problems:

I

We prove using an integral criterion the existence and completeness of the wave operators $W_{\pm}(\Delta_h^{(k)}, \Delta_g^{(k)}, I_{g,h}^{(k)})$ corresponding to the Hodge Laplacians $\Delta_\nu^{(k)}$ acting on differential k-forms, for $\nu \in \{g, h\}$, induced by two quasi-isometric Riemannian metrics g and h on a complete open smooth manifold M. In particular, this result provides a criterion for the absolutely continuous spectra $\sigma_{\mathrm{ac}}(\Delta_g^{(k)}) = \sigma_{\mathrm{ac}}(\Delta_h^{(k)})$ of $\Delta_\nu^{(k)}$ to coincide. By these localised formulae, the integral criterion only requires local curvature bounds and some upper local control on the heat kernel acting on functions provided the Weitzenböck curvature endomorphism is in the Kato class, but no control on the injectivity radii. A consequence is a stability result of the absolutely continuous spectrum under a Ricci flow. For applications we concentrate on the important case of conformal perturbations, and specify our results under global curvature bounds and ε-close Riemannian metrics.

II

We prove a Li-Yau type heat kernel bound of $\nabla e^{-t\Delta^{(k)}}$ and an exponentially weighted L^p-bound for the heat kernel of $\nabla e^{-t\Delta^{(k)}}$, if the curvature tensor and its covariant derivative are bounded. We show that the covariant derivative of the heat semigroup acting on k-forms $\nabla e^{-t\Delta^{(k)}}$ is bounded in L^p for all $1 < p < \infty$ if the curvature tensor and its covariant derivative are bounded. We derive a second order Davies-Gaffney estimate for small times, if the Weitzenböck curvature endomorphism is bounded from below. Based on these results, a Corollary is that the covariant local Riesz transform $\nabla(\Delta^{(k)} + a)^{-1/2}$ is weak $(1, 1)$ and bounded in L^p for all $1 < p \leqslant 2$ without a volume doubling assumption. In particular, our Corollary implies the L^p-Calderón-Zygmund inequality for such p. From our results we can formulate a conjecture for all $1 < p < \infty$, and explain its implications to geometric analysis.

Key words Scattering theory, Wave operators, Bismut type derivative formulae, Hodge Laplacian, Riesz transform, Heat kernel

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Introduction

The deep connection between Brownian motion and the Laplacian being its generator seamlessly opens ways to study the local and global geometry of manifolds, or more generally vector bundles, by virtue of the paths of this stochastic process. Notably, so called Bismut type formulae, first introduced by Bismut [Bis84] in 1984, provide probabilistic derivative formulae for diffusion semigroups on possibly non-compact manifolds:

Let M be a (possibly non-compact) complete Riemannian manifold (M, g) without boundary and Δ_M be the Laplace-Beltrami operator on M. A Brownian motion on a manifold M starting from x at time 0 with lifetime $\zeta(x)$ is given as the M-valued process $X(x)$ generated by $\frac{1}{2}\Delta_M$. Let f be bounded measurable and $u(x,t) = P_t f(x)$ be the (minimal) solution to the heat equation

$$\partial_t u = \frac{1}{2}\Delta_M u, \qquad u(\cdot, 0) = f.$$

Using Itô's lemma we find the stochastic representation of the heat semigroup P_t as

$$P_t f(x) = \mathbb{E}\left(f(X_t(x))\mathbb{1}_{\{t<\zeta(x)\}}\right). \tag{1}$$

For any solution to this heat equation, we find the Bismut derivative formula [TW98; AT10]

$$\langle \nabla P_t f, v\rangle = -\mathbb{E}\left(f(X_t(x))\mathbb{1}_{\{t<\zeta(x)\}}\int_0^\tau \left\langle \mathcal{Q}_s \dot{\ell}_s, \mathrm{d}B_s\right\rangle\right) \qquad \forall v \in \mathrm{T}_x M\ \forall x \in M, \tag{2}$$

where:

- $\tau = \tau_D(x) \wedge t$, with

  $$\tau_D(x) := \inf\left\{t > 0 : X_t(x) \notin D\right\}$$

  is the first exit time of $X_t(x)$ from some (arbitrary small) relatively compact open neighbourhood D of x
- B is the associated anti-development of the Brownian motion $X(x)$, i.e. a Brownian motion in $\mathrm{T}_x M$ related to X qua $\mathrm{d}B := /\!/^{-1} \circ \mathrm{d}X(x)$, and $/\!/_t : \mathrm{T}_x M \to \mathrm{T}_{X_t} M$ is the stochastic parallel transport along X
- the process $\mathcal{Q}$ is a linear transform taking values in the group of linear automorphisms of $\mathrm{T}_x M$, defined by the pathwise covariant ordinary differential equation

  $$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{Q}_t = -\frac{1}{2}\operatorname{Ric}_{/\!/_t}(\mathcal{Q}_t), \qquad \mathcal{Q}_0 = \mathrm{id}_{\mathrm{T}_x M},$$

  with $\operatorname{Ric}_{/\!/_t} := /\!/_t^{-1} \circ \operatorname{Ric}^\sharp_{X_t} \circ /\!/_t$ is a linear transform in $\mathrm{T}_x M$ and we identify $\left\langle \operatorname{Ric}^\sharp_z v, w\right\rangle = \operatorname{Ric}_z(v, w)$ for any $v, w \in \mathrm{T}_z M$ $(z \in M)$

- ℓ is any bounded adapted process taking values in $\mathsf{T}_x M$ with absolutely continuous paths such that $\left(\int_0^\tau |\dot{\ell}_s|^2 \,\mathrm{d}s\right)^{1/2} \in \mathsf{L}^{1+\varepsilon}$ for some $\varepsilon > 0$ and

$$\ell_0 = v, \qquad \ell_\tau = 0.$$

Note the remarkable fact that in the stochastic derivative formulae (2) Ricci curvature only enters *locally around the point* x and no derivative of the heat equation appears on the righthand side of the equation. This potentially opens ways of studying gradient estimates. In addition, the process ℓ may be chosen nicely as long as it starts in $v \in \mathsf{T}_x M$ ($x \in M$) and is 0 as Brownian motion hits the boundary of D. In comparison, the stochastic representation of the semigroup (1) requires lower Ricci bounds as Brownian motion explores the whole manifold for arbitrary small $t > 0$. The idea in proving such formulae is to use integration by parts to get a suitable local martingale, say N_s, and stay on the *local* martingale level as long as possible. Using that a *true* martingale has constant expectation, one then shows that N_s is indeed a martingale and takes expectations at times $s = 0$ and $s = t \wedge \tau$.

Formulae of the form (2) have been heavily investigated in and extended to various contexts [EL94a; EL94b; Tha97; TW98; DT01; Hsu07; Hsu02; APT03; Tho19].

In this thesis, we make use of those methods to first derive local and global Bismut type formulae, and prove localised gradient estimates for the heat semigroup defined by spectral calculus on the full exterior bundle of square-integrable Borel forms in § 2. By these formulae, we obtain results for two distinct problems:

In § 3, we prove using an integral criterion the existence and completeness of the wave operators $W_\pm(\Delta_h^{(k)}, \Delta_g^{(k)}, I_{g,h}^{(k)})$ corresponding to the Hodge Laplacians $\Delta_\nu^{(k)}$ acting on differential k-forms, for $\nu \in \{g, h\}$, induced by two quasi-isometric Riemannian metrics g and h on a complete open smooth manifold M. In particular, this result provides a criterion for the absolutely continuous spectra $\sigma_{\mathrm{ac}}(\Delta_g^{(k)}) = \sigma_{\mathrm{ac}}(\Delta_h^{(k)})$ of $\Delta_\nu^{(k)}$ to coincide. The integral criterion only requires local curvature bounds and some upper local control on the heat kernel acting on functions provided the Weitzenböck curvature endomorphism is in the Kato class, but no control on the injectivity radii. A consequence is a stability result of the absolutely continuous spectrum under a Ricci flow § 3.6.1 and state the main result in the case of differential k-forms § 3.6.2. As an application we concentrate on the important case of conformal perturbations § 3.6.3, specify our results for global curvature bounds § 3.6.4 and ε-close Riemannian metrics § 3.6.5.

In § 4, we make use of the global and local covariant derivative formulae of the heat semigroup developed in § 2: We show a Li-Yau type heat kernel bound of $\nabla e^{-t\Delta^{(k)}}$ and an exponentially weighted L^p-bound for the heat kernel of $\nabla e^{-t\Delta^{(k)}}$, if the curvature tensor and its covariant derivative are bounded. We show that $\nabla e^{-t\Delta^{(k)}}$ is bounded in L^p for all $1 < p < \infty$ if the curvature tensor and its covariant derivative are bounded. In addition, we derive a second order Davies-Gaffney estimate in this case for small times, if the Weitzenböck curvature endomorphism is bounded from below. From these results, a Corollary is that the covariant local Riesz transform $\nabla(\Delta^{(k)} + a)^{-1/2}$ is weak $(1,1)$ and bounded in L^p for all $1 < p \leqslant 2$ without a volume doubling assumption. In particular, our Corollary implies the L^p-Calderón-Zygmund inequality. From our results we can formulate a conjecture for all $1 < p < \infty$, and explain its implications to geometric analysis.

Main results and outline of the thesis

In the § 1 we agree upon the notation used throughout this thesis and recall well-known notions and results from differential theory, spectral theory, geometric analysis and stochastic calculus.

The thesis presents a concise overview of the results that are contained in the following papers:

- **PI** Scattering Theory for the Hodge Laplacian, 45 p., *submitted*, arXiv:2007.06447.
- **PII** Estimates for the covariant derivative of the heat semigroup on differential forms and covariant Riesz transforms, joint work with Batu Güneysu & Baptiste Devyver, 39 p., arxiv.org:2107.00311.

Although the application may seem distant, both results are based on the same key technical tool, to wit: global and local Bismut type formulae on the (full) exterior bundle of square-integrable Borel forms developed in § 2. The thesis then splits into two parts. Part I is concerned with the scattering theory of the Hodge Laplacian. Part II gives estimates for the covariant derivative of the heat semigroup and its kernel, the covariant Riesz transform and implications to geometric analysis.

Bismut type formulae and gradient estimates

In § 2, we first elaborate on the techniques to prove Bismut type formulae in the abstract setting of vector bundles and introduce the so called Kato class of potentials, i.e. a sufficiently rich class of potentials for which we can still expect the Feynman-Kac formula to make sense pointwise (cf. § 2.1). Using those methods, we then derive Bismut type formulae on the full exterior bundle of square-integrable Borel forms: a global and a local covariant Bismut formula in § 2.2 and Bismut formulae for the exterior derivative and codifferential in § 2.3. From the localised Bismut formulae, we obtain localised gradient estimates in § 3.3 that will be the key technical tool in the proof of our main result in § 3.

Scattering Theory for the Hodge Laplacian

The Hodge Laplacian $\Delta_g^{(k)}$ acting on differential k-forms carries important geometric and topological information about M, of particular interest is the spectrum $\sigma(\Delta_g^{(k)})$ of $\Delta_g^{(k)}$. If M is non-compact, then the spectrum contains some absolutely continuous part. A natural question to ask is, to what extent can we control the absolutely continuous part of $\sigma(\Delta_g^{(k)})$ and under which assumptions on the geometry of (M, g)?

A systematic approach to control the absolutely continuous part of the spectrum $\sigma_{\mathrm{ac}}(\Delta_g^{(k)})$ is inspired by quantum mechanics, namely scattering theory: Assume that there is another Riemannian metric h on M such that h is quasi-isometric to g, i.e. there exists a constant $C \geqslant 1$ such that $(1/C)g \leqslant h \leqslant Cg$. We show that under suitable assumptions the wave operators

$$W_{\pm}(\Delta_h^{(k)}, \Delta_g^{(k)}, I_{g,h}^{(k)}) = \operatorname*{s-lim}_{t\to\pm\infty} \mathrm{e}^{it\Delta_h^{(k)}} I_{g,h}^{(k)} \mathrm{e}^{-it\Delta_g^{(k)}} \mathrm{P}_{\mathrm{ac}}(\Delta_g^{(k)})$$

exist and are complete, where the limit is taken in the strong sense, and $I_{g,h}^{(k)}$ denotes a bounded identification operator between the Hilbert spaces of equivalence classes of square-integrable Borel k-forms on M corresponding to the metric g and h respectively (cf. Theorem 3.5 and § 3.2 for details). Then as well-known, it follows in particular that $\sigma_{\mathrm{ac}}(\Delta_h^{(k)}) = \sigma_{\mathrm{ac}}(\Delta_g^{(k)})$.

Similar problems have been investigated: In [MS07; HPW14] considered Laplacians acting on functions on M. However, using analytics methods, strong assumptions are needed to control the injectivity radii. Bei, Güneysu & Müller [BGM17] generalised previous results in the case of conformally equivalent metrics on differential forms under a mild first order control on the conformal factor. Recently, [GT20] established a rather simple integral criterion induced by two quasi-isometric Riemannian metrics using stochastic methods. In particular, no injectivity radii assumptions are made. Using a similar method, very recently Boldt & Güneysu [BG20] extended the result to a non-compact spin manifold with a fixed topological spin structure and two complete Riemannian metrics with bounded sectional curvatures.

In this chapter, we address the natural question: Can we extend the results to the setting of differential k-forms?

We will show that previous results can be extended to the setting of differential k-forms, for a large class of potentials (i.e. in the Kato class) assuming an integral criterion only depending on a local upper bound on the heat kernel and certain explicitly given local curvature bounds. Our main result of this chapter, Theorem 3.32, reads as follows:

Main result. *Assume that g and h are two geodesically complete and quasi-isometric Riemannian metrics on M, denoted $g \sim h$, and assume that there exists $C < \infty$ such that $\left|\delta_{g,h}^{\nabla}\right| \leqslant C$, and that for both $\nu \in \{g, h\}$, $\left|\mathscr{R}_\nu\right|_\nu$ is in the Kato class and it holds*

$$\int \max\left\{\delta_{g,h}(x), \delta_{g,h}^{\nabla}(x) + \Xi_g(x,s), \Psi_\nu(x,s)\right\} \Phi_\nu(x,s)\,\mathrm{vol}_\nu(\mathrm{d}x) < \infty, \qquad \textit{some } s > 0,$$

where

- vol_ν *denotes the Riemannian volume measure with respect to the metric ν,*
- $\mathscr{R}_\nu \in \Gamma(\mathrm{End}\,\Omega(M,\nu))$ *denotes the Weitzenböck curvature endomorphism,*
- $\Psi_\nu(x,s) : M \to (0,\infty)$ *is a function explicitly given terms of local curvature bounds and a finite constant $c_\gamma(\underline{\mathscr{R}}^-)$,*

- *$\Xi_\nu(\cdot, s) : M \to (0, \infty)$ is a function explicitly given in terms of $\Psi_\nu(x, s)$ and an additional local bound on the derivative of curvature,*
- *$\Phi_\nu(\cdot, s) : M \to (0, \infty)$ is a local upper bound on the heat kernel acting on functions,*
- *$\delta_{g,h} : M \to (0, \infty)$ a zeroth order deviation of the metrics from each other,*
- *$\delta^{\nabla}_{g,h} : M \to [0, \infty)$ a first order deviation of the metrics.*

Then the wave operators $W_\pm(\Delta_h, \Delta_g, I_{g,h})$ exist and are complete. Moreover, $W_\pm(\Delta_h, \Delta_g, I_{g,h})$ are partial isometries with initial space $\operatorname{ran} \mathsf{P}_{\mathrm{ac}}(\Delta_g)$ *and final space* $\operatorname{ran} \mathsf{P}_{\mathrm{ac}}(\Delta_h)$. *In particular,*

$$\sigma_{\mathrm{ac}}(\Delta_g) = \sigma_{\mathrm{ac}}(\Delta_h).$$

In § 3.1 we briefly motivate the notion of wave operators (cf. Definition 3.2) and cite the abstract Belopol'skii-Birman Theorem 3.5 which is a well-known tool to prove existence and completeness of the wave operators. A direct consequence is that the absolutely continuous spectra coincide. In the following Section § 3.2, we introduce notions needed, of particular interest the quasi-isometry of two Riemannian metrics, cf. Definition 3.12. It turns out that quasi-isometry can be characterised by boundedness of a zeroth order deviation of the metrics from each other (3.14) defined in terms of a smooth vector bundle morphism $\mathscr{A}_{g,h}$ relating the two quasi-isometric metrics which is given by (3.9). In particular, we obtain estimates for the covariant derivative of $\mathscr{A}_{g,h}$ and a representation of the codifferential in the quasi-isometric metric (cf. Lemma 3.15). In § 3.3 we use the localised gradient estimates obtained in § 2 to prove gradient estimates for the covariant derivative, exterior derivative and codifferential of the heat semigroup transformed by $\mathscr{A}_{g,h}$. We then explain the main result of this chapter in § 3.4 and its proof in § 3.5. We close § 3 with applications to the Ricci flow § 3.6.1, state the main result in the case of differential k-forms 3.6.2, the particularly important cases of conformal perturbations § 3.6.3, specify our results for global curvature bounds § 3.6.4 and ε-close Riemannian metrics § 3.6.5.

Covariant derivative estimates and Riesz transforms

The Riesz transform $\nabla(\Delta^{(0)} + \lambda)^{-1/2}$ on a Riemannian manifold, considered by Strichartz [Str83], has been intensively studied and extended in various frameworks [Bak85a; Bak85b; CD99; TW04; Aus+04]. A direct application to geometric analysis is given by the L^p-Calderón-Zygmund inequalities, cf. e.g. [GP15; Pig20]. However, the study of Riesz transform normally involves assuming a volume doubling property of M.

For the following results we only assume that the Riemann curvature tensor and its covariant derivative are bounded by some constant $A < \infty$, i.e.

$$\max\left(\|\mathsf{R}\|_\infty, \|\nabla \mathsf{R}\|_\infty\right) \leqslant A, \tag{3}$$

where ∇ denotes the Levi-Civita connection on M, and $\|\mathsf{R}\|_\infty$ the $\|\cdot\|_\infty$-norm of $\mathsf{R} \in \boldsymbol{\Gamma}_{C^\infty}(\mathsf{T}^{(0,4)}M)$ read as a $(0,4)$-tensor, analogously for $\nabla \mathsf{R}$ read as $(0, 4+1)$-tensor.

We first state our main results of this chapter. In Theorem 4.4 we show a Li-Yau type heat kernel bound for $\nabla e^{-t\Delta^{(k)}}$ for all $1 \leqslant k \leqslant m$:

$$\left|\nabla_x \mathrm{e}^{-t\Delta^{(k)}}(x,y)\right| \leqslant \frac{C}{\mathrm{vol}(\mathrm{B}(x,\sqrt{t}))} t^{-1/2} \mathrm{e}^{Ct} \mathrm{e}^{-D\frac{\mathrm{d}(x,y)^2}{t}} \qquad \forall t > 0 \ \forall x, y \in M,$$

where $\mathrm{d}(x,y)$ denotes the geodesic distance and $\mathrm{B}(x,r)$ the induced open balls. The positive constants C and D only depend on A and the dimension m of M.

From this theorem we can deduce an exponentially weighted L^p-bound for the heat kernel of $\nabla e^{-t\Delta^{(k)}}$ for all $1 \leqslant k \leqslant m$:

$$\int \left|\nabla_x \mathrm{e}^{-t\Delta^{(k)}}(x,y)\right|^p \mathrm{e}^{\frac{\gamma \mathrm{d}(x,y)^2}{t}} \mathrm{vol}(\mathrm{d}x) \leqslant \frac{C\mathrm{e}^{Ct}}{t^{p/2}\,\mathrm{vol}(\mathrm{B}(y,\sqrt{t}))^{p-1}} \qquad \forall t > 0,$$

where the positive constant C only depends on A, the dimension m of M and p.

Next, Theorem 4.6 states that $\nabla e^{-t\Delta^{(k)}}$ is bounded in L^p, for all $1 < p < \infty$, i.e. we have, for all $1 \leqslant k \leqslant m$,

$$\left\|\nabla \mathrm{e}^{-t\Delta^{(k)}}\big|_{\Gamma_{\mathrm{L}^2 \cap \mathrm{L}^p}(\bigwedge^k \mathrm{T}^* M)}\right\|_{p,p} \leqslant C\mathrm{e}^{tC} t^{-1/2} \qquad \forall t > 0, \tag{4}$$

where the constant C only depends on A, the dimension m of M and p.

A direct consequence is Corollary 4.7: the operator $\nabla(\Delta^{(k)} + \lambda)^{-1/2}$ is weak $(1,1)$ and, by interpolation, the boundedness of the covariant local Riesz transform in L^p for all $1 < p \leqslant 2$,

$$\left\|\nabla(\Delta^{(k)} + \lambda)^{-1/2}\big|_{\Gamma_{\mathrm{L}^2 \cap \mathrm{L}^p}(\bigwedge^k \mathrm{T}^* M)}\right\|_{p,p} \leqslant C, \tag{5}$$

where the positive constant C only depends on A, the dimension m of M, p and λ.

In particular, Corollary 4.7 does not involve any volume assumptions and directly implies the L^p-*Calderón-Zygmund inequality*, i.e. that there is a positive constant D depending on A, the dimension m of M and p such that

$$\|\mathrm{Hess}\, u\|_p \leqslant D\left(\|\Delta u\|_p + \|u\|_p\right) \qquad \forall u \in \mathrm{C}_c^\infty(M).$$

Based on our results, we can formulate Conjecture 4.8 that Corollary 4.7 holds for all $1 < p < \infty$. Our conjecture is based on a result by [Aus+04] in the case $k = 0$, i.e. on functions, that for $p > 2$, estimate (4) implies (and is actually equivalent to) (5). We can also generalise a central tool in the proof of the scalar case, i.e. a second order Davies-Gaffney estimate for small times:

Theorem 4.11. *There are universal constants $c_1, c_2 > 0$ such that for all $1 \leqslant k \leqslant m$ with $\mathscr{R}^{(k)} \geqslant -A$ for some constant $A \geqslant 0$, all $t > 0$, all Borel subsets $E, F \subset M$ with compact closure, and all $\alpha \in \Gamma_{\mathrm{L}^2}(\bigwedge^k \mathrm{T}^* M)$ with $\mathrm{supp}\,\alpha \subset E$, we have*

$$\left\|\mathbb{1}_F \mathrm{e}^{-t\Delta^{(k)}}\alpha\right\|_2 + \left\|\mathbb{1}_F \sqrt{t}\nabla \mathrm{e}^{-t\Delta^{(k)}}\alpha\right\|_2 + \left\|\mathbb{1}_F t\Delta^{(k)} \mathrm{e}^{-t\Delta^{(k)}}\alpha\right\|_2 \leqslant c_1\left(1 + \sqrt{tA}\right) \mathrm{e}^{-\frac{c_2 \varrho(E,F)^2}{t}} \left\|\mathbb{1}_E \alpha\right\|_2.$$

It turns out, for this result, boundedness on the Weitzenböck curvature endomorphism $\mathscr{R}^{(k)}$ from below is sufficient. The Proof is given in § 4.6 and is based on analytics tools, to wit the Phragmen-Lindelöf's inequality 4.16. Following the strategy of the proof given in [Aus+04] the only part we could not adjust so far is where the local Poincaré inequality is used explicitly.

In § 4.3, we prove the heat kernel bound for $\nabla e^{-t\Delta^{(k)}}$ if the curvature tensor and its covariant derivative are bounded by making use of the global Bismut formula, Theorem 2.24, applied to Brownian bridge measure, and its Corollary 4.5 using Li-Yau estimates. In § 4.4, we prove that the covariant derivative formula of the heat semigroup on k-forms is bounded in L^p for all $1 < p < \infty$ if the curvature tensor and its covariant derivative are bounded. We therefore use the covariant Bismut formula and similar techniques to prove the gradient estimates developed on § 2 under global curvature bounds. In § 4.5 we prove Corollary 4.7, i.e. that the operator $\nabla(\Delta^{(k)} + \lambda)^{-1/2}$ is weak $(1, 1)$ and the boundedness of the covariant local Riesz transform in L^p for all $1 < p \leqslant 2$.

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Luxembourg, in December 2020 Robert Baumgarth

Index of notation

Unless otherwise stated, **functions** are maps whose codomain is $\mathbb{R}^m$ whereas the term **map(ping)** normally refers to a map between arbitrary manifolds. Binary operations between functions such as $f \pm g$, $f \cdot g$, $f \wedge g$, $f \vee g$, comparisons $f \leqslant g$, $f < g$ or limits $f_j \xrightarrow{j\to\infty} f$, $\lim_j f_j$, $\liminf_j f_j$, $\sup_j f_j$ are understood pointwise. «Positive» and «negative» means «$\geqslant 0$» and «$\leqslant 0$», respectively.

Throughout, we will use the notation

$$x \lesssim y \qquad :\Longleftrightarrow \qquad \exists C > 0 : \quad x \leqslant Cy,$$

and

$$x \simeq y \qquad :\Longleftrightarrow \qquad x \lesssim y \quad \wedge \quad y \lesssim x.$$

Einstein Summation Convention If the same index variable appears twice in a term, both as an upper and a lower index, it is assumed to be summed over all possible values of that index (usually ranging from 1 to the dimension $m =: \dim M$). For example, we write

$$a_i b^i \quad \text{instead of} \quad \sum_i a_i b^i \qquad \text{or} \qquad a^{ijkl} b_{il} c_j \quad \text{instead of} \quad \sum_{i,j,k} a^{ijkl} b_{il} c_j.$$

Analysis and measure theory

$\mathbb{N}$ [$\mathbb{N}_0$]	natural numbers [incl. 0]
$\mathbb{R}$	real numbers
$\inf \varnothing$	$\inf \varnothing = +\infty$
$a \wedge b, a \vee b$	minimum and maximum
$\mathbb{1}_A$	$\mathbb{1}_A(x) := \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$
pr_E	projection on E
$\operatorname{supp} f$	support of f, $\overline{\{f \neq 0\}}$
$\operatorname{supp} \eta$	support of η, $\overline{\{\eta \neq 0\}}$

Sets

$\hat{M}$	$\hat{M} := M \cup \{\infty\}$
E^*	dual space of a set E

Differential geometry

M	smooth manifold
$\mathsf{T}M$ [T^*M]	[co]tangent bundle
T_xM [T^*_xM]	[co]tangent space at $x \in M$
$\pi : E \to M$	fibre or vector bundle
$\Gamma(E)$	smooth sections of fibre bundle E, $\Gamma(E) \equiv \Gamma_{C^\infty}(E)$, 3
$\Gamma_{L^2}(E)$	L^2-sections of fibre bundle E

$\Omega^k(M)$	smooth differential k-forms on M, $\Omega^k(M) \equiv \Omega^k_{C^\infty}(M)$, 4
$\Omega(M)$	smooth differential forms on M, $\Omega(M) \equiv \Omega_{C^\infty}(M)$
$\Omega^k_{L^2}(M)$	L^2 k-forms on M
$\Omega_{L^2}(M)$	L^2 total forms on M
$\sharp, \flat$	musical isomorphisms, 9
∇	covariant derivative, 5
grad	gradient
tr	trace, contraction
$\mathbf{d}$	exterior derivative, 4
$\mathbf{d}^{(k)}$	——— acting on k-forms
$\boldsymbol{\delta}_g$	codifferential (w.r.t. g), 11
$\boldsymbol{\delta}_g^{(k)}$	——— acting on k-forms
$\mathbf{D}_g$	Dirac operator $\mathbf{d} + \boldsymbol{\delta}_g$
$\Delta_{\mathbb{R}^m}$	Laplace operator, $\sum_{i \leqslant n} \partial_i^2$
Δ_M	Laplace-Beltrami operator, 10
Δ_g	Hodge-de Rham operator on total forms, 12
$\Delta_g^{(k)}$	——— acting on k-forms
$\Box \equiv \nabla^*\nabla$	connection Laplacian on manifold M, 42
T	torsion tensor, 9
R	curvature tensor, 12
Ric	Ricci tensor, 12
$\mathscr{R}$	Weitzenböck curvature endomorphism, 13
$\mathscr{R}^{(k)}$	——— acting on k-forms
Q	curvature operator, 51
vol_g	volume measure (w.r.t. g)
$g = (\cdot, \cdot)_g$	Riemannian metric, 8
Metr M	smooth Riemannian metrics on M
$\lvert\cdot\rvert_g$	induced fibre norm
$\langle \cdot, \cdot \rangle_g$	inner product on $\Gamma_{L^2}(E)$
$\lVert\cdot\rVert_g$	norm on $\Gamma_{L^2}(E)$

Spaces (of sets)

$\mathscr{B}(E)$	space of Borel-measurable functions $f : E \to \mathbb{R}$
$\mathscr{B}_b(E)$	space of bounded, Borel-measurable functions $f : E \to \mathbb{R}$
$C^\infty(E)$	space of continuous functions $f : E \to \mathbb{R}$, 1
$C_c(E)$	space of continuous functions $f : E \to \mathbb{R}$ with compact support, 1
$C_c^\infty(E)$	space of smooth functions $f : E \to \mathbb{R}$ with compact support, 1
$\mathscr{S}(E)$	space of all continuous semimartingales on E
$\mathscr{M}(E)$	space of all local martingales on E
$\mathscr{A}(E)$ [$\mathscr{A}_0(E)$]	space of all continuous finite variation processes [starting at zero] on E
Hom(E, F)	space of all homomorphisms from E to F
End(E, F)	space of all endomorphisms from E to F

Operator theory & spectral calculus

$\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$	bounded linear operators $\mathscr{H}_1 \to \mathscr{H}_2$, 16
$\mathscr{J}^q$	Schatten class operator of order q, 16
$\mathscr{J}^1$	trace class operators, 16
$\mathscr{J}^2$	Hilbert-Schmidt class operators, 16
$\mathscr{J}^\infty$	compact operators, 16
dom $\mathbf{H}$	domain of an operator $\mathbf{H}$
ran $\mathbf{H}$	range of an operator $\mathbf{H}$
$\mathbf{H}^*$	adjoint of an operator $\mathbf{H}$
$\sigma(\mathbf{H})$	spectrum of an operator $\mathbf{H}$, 17

$\sigma_{\mathrm{ac}}(\mathbf{H})$	absolutely continuous spectrum of $\mathbf{H}$, 70
$e^{-s\mathbf{H}}$	semigroup generated by $\mathbf{H}$
$e^{-s\mathbf{H}}(x,y)$	kernel corresponding to $e^{-s\mathbf{H}}$
$p_t^{(k)}(x,y)$	heat kernel acting on k-forms
$p_t^{(0)}(x,y)$	heat kernel on functions
$\mathfrak{q}$	sesquilinear form corresponding to s.a. operator
$W_\pm(\mathbf{H}_2,\mathbf{H}_1,I)$	wave operators
$\lvert\cdot\rvert_{\mathrm{op}}$	operator norm
s-lim	strong limit
$\xrightarrow{\mathrm{s}}$	strong convergence

Scattering theory

$g \sim h$	g quasi-isometric to h, 78
$\mathscr{A}_{g,h}(x)$	vector bundle homomorphism, 74
$\delta_{g,h}(x)$	0^{th} order deviation of g and h, 78
$\delta^{\nabla}_{g,h}(x)$	1^{st} order deviation of g and h, 78
$I_{g,h}$	bounded identification operator, 84
$I^{(k)}_{g,h}$	––– acting on k-forms
$\mathsf{K}(M)$	Kato class, 45
$\mathsf{D}(M)$	Dynkin class, 45

Probability theory

$\sim$	distributed as
$\overset{\mathrm{m}}{=}$	modulo local martingales
a.s.	almost sure(ly)
a.e.	almost every(where)
$(\Omega,\mathscr{F},\mathbb{P})$	(underlying) probability space
$\mathbb{E}(\,\cdot\mid\mathscr{H})$	conditional expectation w.r.t. a σ-algebra $\mathscr{H}$

Stochastic processes and SDEs

$\mathscr{F}_t = \mathscr{F}_{t+}$	$\mathscr{F}_{t+} := \bigcap_{r>t}\mathscr{F}_r$, $t \geqslant 0$, right-continuous filtration
$X_t \in \mathscr{F}_t$	X_t measurable w.r.t $\mathscr{F}_t$
$\tau = \tau(x,r)$	first exit time from the open ball $\mathrm{B}(x,r)$, 49
$X_t^\tau = X_{t\wedge\tau}$	stopped process $X_{t\wedge\tau}$
ζ	lifetime, 29
$[X]$	square bracket $[X,X]$
$(P_t)_{t\geqslant 0}$	heat semigroup, $P_t f(x) := \mathbb{E}f(X_t(x))$
$B=(B_t)_{t\geqslant 0}$	Brownian motion (BM)
$\mathrm{BM}(E)$	BM on $E \subset \mathbb{R}^n$
$\mathrm{BM}(M,g)$	BM on (M,g), 28
SDE	Stochastic differential equation
$\circ$	Stratonovich circle, 29
$\mathsf{CM}(t,\xi,E)$	finite energy process in E

Chapitre 1

Éléments d'Analyse Géométrique and Calcul Stochastique

Dans ce chapitre, nous introduisons les notions préliminaires nécessaires et convenons de la notation utilisée. En particulier, nous revisiterons les définitions bien connues de la théorie de géométrie différentielle et de la théorie des probabilités.

Remarque 1.1 (Convention de sommation d'Einstein). Désormais, nous utiliserons la convention commode de sommation d'Einstein. Il s'agit d'une convention de notation importante qui est couramment utilisée dans la théorie des variétés, car nous devons souvent traiter des vecteurs et des covecteurs et de l'inévitable superflu des signes de sommation: si un index apparaît deux fois, une fois en indice et une fois en exposant, nous omettons le symbole de sommation. Par exemple nous écrivons

$$v^i \partial_i, \quad \omega_i \mathbf{d}x^i, \quad \frac{\partial x^i}{\partial x^{\bar{i}}} \frac{\partial x^j}{\partial x^{\bar{j}}} F_{ij},$$

à la place de

$$\sum_{i=1}^{n} v^i \partial_i, \quad \sum_{i=1}^{n} \omega_i \mathbf{d}x^i, \quad \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial x^i}{\partial x^{\bar{i}}} \frac{\partial x^j}{\partial x^{\bar{j}}} F_{ij}.$$

Par conséquent, donnée un espace vectoriel V, nous faisons la distinction et écrivons les vecteurs (contravariants) $e_1, \ldots, e_n \in V$ toujours avec un indice et les covecteurs du repère dual correspondant $\varepsilon^1, \ldots, \varepsilon^n \in V^*$ avec un exposant.

Pour $x \in \mathbb{R}^m$ et quelconque $r > 0$, on dénote par

$$\mathrm{B}(x,r) := \left\{ y \in \mathbb{R}^m : \|x-y\| < r \right\} \qquad \text{et} \qquad \mathrm{B}[x,r] := \left\{ y \in \mathbb{R}^m : \|x-y\| \leqslant r \right\}.$$

la boule ouverte et la boule fermée, respectivement (dans le norme habituelle sur $\mathbb{R}^m$).

Notons par $\mathrm{C}^\infty(E)$, $\mathrm{C}_c^\infty(E)$, et $\mathrm{C}_c(E)$, pour toutes les fonctions $f : E \to \mathbb{R}^m$ lisses, lisses et disparaissant à l'infini et lisses à supports compacts.

On définit encore les espaces de courbes suivant: Soit $I \subset \mathbb{R}$, on denote par $\mathrm{C}([a,b], \mathbb{R}^m)$ l'ensemble des courbes continues $\gamma : [a,b] \to \mathbb{R}^m$ et $\mathrm{L}^1([a,b], \mathbb{R}^m)$ l'ensemble des courbes intégrables. En outre, une courbe $\gamma : [a,b] \to \mathbb{R}^m$ est **absolument continue** si pour tout $\varepsilon > 0$ il existe un $\delta > 0$ tel que pour toute partition $a \leqslant s_1 < t_1 \leqslant \ldots \leqslant s_n < t_n \leqslant b$

$$\sum_{i=1}^{n} (t_i - s_i) < \delta \quad \Longrightarrow \quad \sum_{i=1}^{n} \left| \gamma(t_i) - \gamma(s_i) \right| < \varepsilon.$$

De manière équivalente une courbe $\gamma : [a,b] \to \mathbb{R}^m$ est absolument continue si γ est différentiable presque partout avec $\dot{\gamma} \in \mathrm{L}^1([a,b], \mathbb{R}^m)$ et telle que

$$\gamma(t) = \gamma(a) + \int_a^t \dot{\gamma}(x) \mathrm{d}s.$$

Finalement, on définit l'espace

$$\mathsf{L}^{1,2}([a,b],E) := \left\{ \gamma : [a,b] \to E : \gamma \text{ différentiable p.p. et } \int_a^b |\dot{\gamma}(t)|^2 \, \mathrm{d}t < \infty \right\}.$$

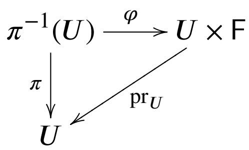
1.1 Théorie de Géométrie Différentielle

Une introduction approfondie aux variétés différentielles est donnée dans [Lee18], à la géométrie riemannienne dans [Jos17] et [Pet16]. Étant un lecteur allemand nous renvoyons le lecteur à [HT94, Chapitre 7]. Une référence classique pour la géométrie différentielle nécessaire est [KN63; KN69].

Soit M une variété topologique de dimension $\dim M =: m$, i.e. un espace topologique de Hausdorff et à base dénombrable assimilable localement à un espace euclidien. Combinée avec une structure différentielle on appelle M une **variété différentielle**.

1.1.1 Fibrés différentiels Un fibré (différentiel) est un espace qui est localement le produit de deux espaces, mais peut avoir une structure topologique différente globalement. Nous nous limitons à des fibrés entre variétés de façon à ce que l'espace total E est une variété au lieu d'un espace topologique. Ainsi, les trivialisations deviennent des difféomorphismes (pas homeomorphismes).

Definition 1.2. Soient E, M et F des variétés. Une application lisse $\pi : E \to M$ ou, plus précisément, le quadruplet (E, M, π, F) est appelé **fibré (différentiel) sur M de fibre** F, si π est une submersion surjective et pour tous $p \in M$ il existe un voisinage U de p dans M et un difféomorphisme $\varphi : \pi^{-1}(U) \xrightarrow{\sim} U \times F$ sur U tel que le diagramme



soit commutatif. On appelle E l'**espace total**, π la **projection**, M la **base** et (U,φ) une **trivialisation locale** du fibré. De plus, F est la **fibre** et $E_p := \pi^{-1}\{p\}$ la **fibre sur** $p \in M$.

Définition 1.3. Un **fibré vectoriel de rang k sur M** est un fibré $\pi : E \to M$ de fibre espace vectoriel V de dimension k, avec trivialisations locales $\varphi_U : \pi(U) \to U \times V$ telles que toute restriction $\varphi_U\big|_x : E_x \to \{x\} \times V$ est un isomorphisme d'espaces vectoriels.

Définition 1.4. Le **fibré dual** d'un fibré $\pi : E \to M$ est le fibré

$$E^* := \bigcup_{x \in M} E_x^* \to M, \qquad x \in M,$$

où $E_x^* = \mathsf{Hom}_{\mathbb{R}}(E_x, \mathbb{R})$ est l'espace vectoriel dual de E_x.

Définition 1.5. Si $\pi : E \to M$ est un fibré de fibre F, une **section** est une application différentiable $\sigma : M \to E$ telle que $\pi \circ \sigma = \mathrm{id}_M$. L'espace des sections différentiable d'un fibré E est donné par

$$\Gamma(E) := \{\sigma : M \to E \text{ différentiable} \mid \pi \circ \sigma = \mathrm{id}_M\} \\ \equiv \{\sigma : M \to E \text{ différentiable} \mid \sigma(x) \in \mathsf{T}_x M \ \forall x \in M\}.$$

Exemple 1.6. (a) Le **fibré trivial** $E = M \times \mathsf{F}$ de fibre F est la projection

$$\pi := \mathrm{pr}_M : M \times F \to M,$$

i.e. $(x, y) \mapsto x$. Si $E = M \times \mathbb{R}$, les sections de E sont les fonctions lisses à valeurs réels sur M, i.e. $\Gamma^\infty(M) = \mathsf{C}^\infty(M)$.

(b) Si $E = M \times \mathbb{R}^k$ ($k \in \mathbb{N}$), les sections de E sont les fonctions vectorielles sur M.

Exemple 1.7. (a) Le **fibré tangent** sur $\mathsf{T}M \xrightarrow{\pi} M$ est l'union disjointe des espaces tangents $\mathsf{T}M = \bigcup_{x \in M} \mathsf{T}_x M$. Les sections $X \in \Gamma(\mathsf{T}M)$ sont données par champ de vecteurs sur M. Comme d'habitude, on identifie les champs de vecteurs sur M avec les dérivations de $\mathsf{C}^\infty(M)$ à valeurs dans $\mathsf{C}^\infty(M)$, i.e.

$$\Gamma(\mathsf{T}M) := \mathrm{Der}(\mathsf{C}^\infty(M), \mathsf{C}^\infty(M)) \\ := \{A : \mathsf{C}^\infty(M) \to \mathsf{C}^\infty(M) \ \mathbb{R}\text{-linear} \mid A(fg) = fA(g) + gA(f) \ \forall f, g \in \mathsf{C}^\infty(M)\},$$

où le champ de vecteurs $A \in \Gamma(\mathsf{T}M)$ est considéré comme une $\mathbb{R}$-dérivation qua

$$A(f)(x) := (\mathbf{d}f)_x A(x) \in \mathbb{R}, \qquad \forall x \in M,$$

utilisé le différential (ou push-forward) $(f_x)_* = \mathbf{d}f_x : \mathsf{T}_x M \to \mathbb{R}$ de f dans x.

(b) Le **fibré cotangent** sur $\mathsf{T}^*M \xrightarrow{\pi} M$ est l'union disjointe des espaces cotangents $\mathsf{T}^*M = \bigcup_{x \in M} \mathsf{T}^*_x M$, où $\mathsf{T}^*_x M$ est l'espace vectoriel dual linéaire de l'espace tangent $\mathsf{T}_x M$ pour tout $x \in M$. Les sections $\eta \in \Gamma(\mathsf{T}^*M)$ sont données par les formes différentielles d'ordre un.

Exemple 1.8. (a) Le **fibré tensoriel de type** (k, l) sur un fibré $\pi : E \to M$ est le produit tensoriel

$$\mathsf{T}^{(k,l)}E := \bigcup_{x \in M} (E_x^*)^{\oplus k} \otimes E_x^{\oplus l} \to M.$$

(b) Le k-**ème produit extérieur** d'un fibré $\pi : E \to M$ est le fibré

$$\bigwedge^k E := \bigcup_{x \in M} \bigwedge^k E_x \to M,$$

où $\bigwedge^k E_x$ est le sous-espace de l'espace $\mathsf{T}^{(k,l)}E$ défini par toutes $\alpha \in \mathsf{T}^{(k,0)}E$ alternée. Si le fibré E est de rang m, le fibré $\bigwedge^k E$ est de rang $\binom{m}{k}$.

(c) Le **fibré des endomorphismes** d'un fibré $\pi : E \to M$ est le fibré

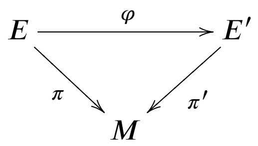
$$\mathrm{End}\, E := \bigcup_{x \in M} \mathrm{End}\, E_x \to M,$$

où $\mathrm{End}\, E_x$ est l'ensemble des applications linéaires sur chaque fibre E_x. En particulier, il y a un isomorphisme de fibrés $\mathrm{End}\, E \cong E \otimes E^*$.

Exemple 1.9. Pour tout $k \geqslant 0$, une **forme différentielle d'ordre** k sur M, ou k-forme, est une section $\eta : M \to \bigwedge^k \mathsf{T}^* M$ du k-ème produit extérieur du fibré cotangent, où $\bigwedge^0 \mathsf{T}^* M = M \times \mathbb{R}$. L'ensemble des k-formes différentielles sur M est noté par $\Omega^k(M) = \boldsymbol{\Gamma}(\bigwedge^k \mathsf{T}^* M)$.

En particulier, $\Omega^0(M) = \mathsf{C}^\infty(M)$, $\Omega^m(M)$ est de rang 1 et $\Omega^k(M) = 0$ si $k > m$.

Définition 1.10. Soient E et E' des fibrés vectoriels sur M. Une application lisse $\varphi : E \to E'$ est appelée un **homomorphisme de fibrés vectoriels** si linéaire sur chaque fibre, i.e. $\pi' \circ \varphi = \pi$, et préservant les fibres



Définition 1.11 (Section, champ de vecteurs le long d'une application). Soit $f : M \to N$ une fonction lisse entre deux variétés, E un fibré vectoriel sur N. Les éléments de

$$\boldsymbol{\Gamma}(f^* E) := \left\{ A : M \to E \mid A \text{ lisse avec } \pi \circ A = f \right\}$$

s'appellent **sections le long de** f, particulièrement dans le cas $\boldsymbol{\Gamma}(f^* \mathsf{T} N)$ il s'appelle **champ de vecteurs le long de** f. Soit $I \subset \mathbb{R}$ une intervalle et $\gamma : I \to N$ une courbe continue, on a que

$$\boldsymbol{\Gamma}(\gamma^* I) = \left\{ \sigma : I \to E \mid \sigma \text{ lisse avec } \sigma(t) \in E_{\gamma(t)} \quad \forall i \in I \right\}.$$

Le champ de vecteurs $\dot\gamma \in \boldsymbol{\Gamma}(\gamma^* \mathsf{T} N)$, $\dot\gamma_t := \dot\gamma(t)$, induit par γ est s'appelle **champ de vecteurs tangentiels le long de** γ.

Lemme 1.12 (et Définition). *Soit $f : M \to N$ une fonction lisse entre deux variétés. La* **différentielle de** *f est l'application $\mathsf{d}f : \mathsf{T}M \to \mathsf{T}N$, définit par*

$$(\mathsf{d}f)_x v = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f \circ \gamma(t),$$

où $\gamma : (-\varepsilon, \varepsilon) \to M$ telle que $\gamma(0) = x \in M$ et $\dot\gamma(0) = v \in \mathsf{T}_x M$ pour tout $\varepsilon > 0$. En particulier, pour tout $x \in M$ on a $(\mathsf{d}f)_x : \mathsf{T}_x M \to \mathsf{T}_{f(x)} M$ et $\mathsf{d}(\mathrm{id}_M)_x = \mathrm{id}_{\mathsf{T}_x M}$.

Définition 1.13. On appelle **différentielle extérieure** l'opérateur unique

$$\mathbf{d} : \Omega^k(M) \to \Omega^{k+1}(M), \qquad \eta \mapsto \mathbf{d}\eta, \qquad \forall k \geqslant 0,$$

avec les propriétés suivantes:

(i) $\mathbf{d}$ est $\mathbb{R}$-linéaire.

(ii) $\mathbf{d}$ est une dérivation graduée: Si $\eta_1 \in \Omega^k(M)$, $\eta_2 \in \Omega^q(M)$, alors

$$\mathbf{d}(\eta_1 \wedge \eta_2) = \mathbf{d}\eta_1 \wedge \eta_2 + (-1)^k \eta_1 \wedge \mathbf{d}\eta_2.$$

(iii) $\mathbf{d}$ dérivation de carré nul: $\mathbf{d}^2 = \mathbf{d} \circ \mathbf{d} = 0$.

(iv) Pour tout $f \in \Omega^0(M) \equiv \mathrm{C}^\infty(M)$, on a

$$\mathbf{d}f(X) = Xf \qquad \forall X \in \mathbf{\Gamma}(\mathrm{T}M),$$

i.e. pour toute fonction lisse f, la 1-forme $\mathbf{d}f$ est la différentielle de f.

Remarque 1.14. Si nécessaire, on indique par $\mathbf{d}^{(k)}$ le degré k de la forme différentielle sur qui la différentielle extérieure agit.

Dans le cas $k = 0$, on a

$$\mathbf{d}^{(0)} : \mathrm{C}^\infty(M) \to \Omega^1(M), \qquad \mathbf{d}^{(0)}f(X) = Xf, \qquad \forall X \in \mathbf{\Gamma}(\mathrm{T}M),$$

donc $\mathbf{d}^{(0)}f$ coïncide avec la différentielle de f, $\mathrm{d}f : M \to \mathrm{T}^*M$, $x \mapsto (\mathrm{d}f)_x$ avec

$$(\mathrm{d}f)_x : \mathrm{T}_xM \to \mathrm{T}_{f(x)}\mathbb{R} \cong \mathbb{R}, \qquad (\mathrm{d}f)_x(X_x) = X_xf := X^i(x)\partial_{x^i}f\big|_x.$$

De plus, dans le cas $p = 1$, pour tout $X, Y \in \mathbf{\Gamma}(\mathrm{T}M)$,

$$\mathbf{d}^{(1)} : \Omega^1(M) \to \Omega^2(M), \qquad \mathbf{d}^{(1)}\eta(X, Y) = X\eta(Y) - Y\eta(X) - \eta([X, Y]),$$

où $[X, Y] := XY - YX$ denote le **crochet de Lie**.

Définition 1.15. Le **support d'une fonction** f sur M est l'ensemble

$$\operatorname{supp} f := \overline{\{x \in M : f(x) \neq 0\}},$$

où $\overline{\bullet}$ indique la fermeture topologique. De même, le **support d'une forme différentielle** η sur M est l'ensemble

$$\operatorname{supp} \eta := \overline{\{x \in M : \eta(x) \neq 0\}}.$$

1.1.2 Connexions et transport parallèle Étant donné un fibré vectoriel $E \xrightarrow{\pi} M$ sur M, p. ex. $E = \mathrm{T}M$, chaque fibre est un espace tangent dans un point $x \in M$. Soit $\gamma : [0, 1] \to M$ une courbe lisse avec $\gamma(0) = p$ et $\gamma(1) = y$. Un transport parallèle (ou de manière équivalente une connexion) est un moyen naturel de transporter des éléments d'une fibre à une autre avec leur géométrie locale de $v \in E_x$ le long de γ à E_y. L'idée clé des connexions est de généraliser la dérivée directionnelle d'un champ vectoriel d'une manière invariante par changement de coordonnées.

Définition 1.16 (Connexion sur un fibré vectoriel). Une **connexion sur** E est une application $\mathbb{R}$-linéaire

$$\begin{aligned} &\nabla : \mathbf{\Gamma}(E) \to \mathbf{\Gamma}(\mathrm{T}^*M \otimes E), \\ &\quad \nabla(fX) = \mathbf{d}f \otimes X + f\nabla X, \qquad \forall X \in \mathbf{\Gamma}(E)\ \forall f \in \mathrm{C}^\infty(M). \end{aligned}$$

Une section $X \in \mathbf{\Gamma}(E)$ est dite **parallèle** si $\nabla X = 0$. Parce que, l'on a

$$\mathbf{\Gamma}(\mathrm{T}^*M \otimes E) \cong \mathrm{Hom}_{C^\infty(M)}(\mathbf{\Gamma}(\mathrm{T}M), \mathbf{\Gamma}(E))$$

on peut également voir ∇ comme une application $\mathbb{R}$-bilinéaire sur E

$$\begin{aligned} \Gamma(\mathrm{T}M) \times \Gamma(E) &\to \Gamma(E) \\ (X, Y) &\mapsto \nabla_X Y := (\nabla Y)X. \end{aligned}$$

Donc

(i) $X \mapsto \nabla_X Y$ est $\mathrm{C}^\infty(M)$-**linéaire**, c'est-à-dire,

$$\nabla_{fX_1+gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y, \qquad \forall f, g \in \mathrm{C}^\infty(M)$$

(ii) et $\nabla_X Y$ satisfait une règle du produit de type dérivation,

$$\nabla_X (fY) = f\nabla_X Y + (Xf)Y, \qquad \forall f \in \mathrm{C}^\infty(M).$$

∇ est dit **del** et $\nabla_X Y$ la **dérivée covariante de Y dans la direction** X.

Proposition 1.17. *La différence entre deux connexions est une application*

$$A : \Gamma_{\mathrm{C}^\infty}(\mathrm{T}M) \to \operatorname{End} \Gamma(E)$$

à valeurs dans $\operatorname{End}_{\mathrm{C}^\infty(M)} \Gamma(E) \cong \Gamma(\operatorname{End} E)$*, i.e. une* 1*-forme différentielle à valeurs de* $\operatorname{End} \mathrm{T}M$

$$A \in \operatorname{Hom}_{\mathrm{C}^\infty(M)} \left(\Gamma_{\mathrm{C}^\infty(M)}(\mathrm{T}M), \Gamma(\operatorname{End} E)\right) \cong \Omega^1(M) \otimes_{\mathrm{C}^\infty(M)} \Gamma(\operatorname{End} E) = \Omega^1(M; \operatorname{End} E).$$

Par conséquent, l'ensemble des connexions sur E est un espace affine qui est isomorphe à $\Omega^1(M; \operatorname{End} E)$.

Proof. Soient ∇ et $\widetilde{\nabla}$ deux connexions sur E et $A = \nabla - \widetilde{\nabla}$. Alors pour tout $X \in \Gamma(\mathrm{T}M)$ et pour toute $f \in \mathrm{C}^\infty(M)$, $Y \in \Gamma(E)$, on a

$$\begin{aligned} A_X(fY) = \left(\nabla - \widetilde{\nabla}\right)(fY) &= Xf + f\nabla_X Y - Xf - f\widetilde{\nabla}_X Y \\ &= f\left(\nabla_X - \widetilde{\nabla}_X\right)(Y). \end{aligned}$$

Donc, pour tout $X \in \Gamma(\mathrm{T}M)$, l'application $A_X : \Gamma(E) \to \Gamma(E)$ est le push-forward d'un morphisme $E \to E$, autrement dit $A : \Gamma(\mathrm{T}M) \to \Gamma(\operatorname{End} E)$ provient d'un morphisme de fibrés

$$\begin{aligned} \mathrm{T}M \otimes E &\iff E \to \mathrm{T}^*M \otimes E \\ &\iff \mathrm{T}M \to \operatorname{End} E \iff \mathrm{T}^*M \otimes \operatorname{End} E. \end{aligned}$$

■

Définition 1.18. Soit $\pi : E \to M$ est un fibré vectoriel sur une variété M. Une **connexion sur** E est une dérivée covariante d'E (ou équivalent un transport parallèle dans E, cf. Définition 1.22 ci-dessous).

Une connexion sur $\mathrm{T}M$ est dite souvent **connexion sur** M, c'est-à-dire, une application

$$\nabla : \Gamma(\mathrm{T}M) \otimes \Gamma(\mathrm{T}M) \to \Gamma(\mathrm{T}M), \qquad (X, Y) \mapsto \nabla_X Y,$$

qui est $\mathrm{C}^\infty(M)$-linéaire dans la variable X et une $\mathrm{C}^\infty(M)$-dérivation dans la variable Y. Une connexion peut être également vue comme une dérivée covariante

$$\nabla : \Gamma(\mathrm{T}M) \to \Omega^1(M) \cong \Omega^1(M) \otimes_{\mathrm{C}^\infty(M)} \Gamma(\mathrm{T}M).$$

Les **symboles de Christoffel** d'une telle connexion sont des fonctions locales Γ_{ij}^k telles que $\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k \partial_k$, pour tout $i, j = 1, \ldots, m$.

Soit $\gamma : I \to M$ une courbe lisse, où $I \subset \mathbb{R}$ est une intervalle. Un champ de vecteurs le long de γ est une application lisse $\gamma : I \to \mathsf{T}M$ avec $v(t) \in \mathsf{T}_{\gamma(t)}M$ pour tout $t \in I$. En utilisant la connexion $\nabla^{\mathsf{T}M}$ sur $\mathsf{T}M$ on peut donner un sens à la dérivée directionnelle d'un champ vectoriel le long d'une courbe.

Pour tous $X \in \Gamma(E)$ et $v \in \mathsf{T}_x M$ $(x \in M)$ la **dérivée covariante** $\nabla_v X$ **de** X **dans la direction** v

$$\nabla_v X := \nabla_{\dot{\gamma}}(X \circ \gamma)(0) \in E_p, \tag{1.1}$$

est bien définie, où $\gamma : (-\varepsilon, \varepsilon) \to M$ est une courbe lisse avec $\gamma(0) = 0$ et $\dot{\gamma}(p) = v$. Soit $X \in \Gamma(E)$. Pour tout $v \in \mathsf{T}_x M$ $(x \in M)$ il existe $A \in \Gamma(\mathsf{T}M)$ avec $A_x = v$. Alors $\nabla_v X := (\nabla_A X)_x \in E_x$, la **dérivée covariante de** X **dans la direction de** v, est bien définie. Pour une section $X \in \Gamma(\gamma^* E)$ le long de γ on appelée $\nabla_D X \in \Gamma(\gamma^* E)$ **parallèle le long de** γ si $D = \frac{\mathrm{d}}{\mathrm{d}t}$ est le champ du vecteur canonique on I.

Pour $X \in \Gamma(\gamma^* E)$ le long de γ on appele $\nabla_{\dot{\gamma}} X \in \Gamma(\gamma^* E)$ la **dérivée covariante de** X **le long de** γ. Donc $\mathsf{D}_t = \frac{\mathrm{d}}{\mathrm{d}t}$ est le champ des vecteur canonique sur I. Une section $X \in \Gamma(\gamma^* E)$ le long de γ est **parallèle le long de** γ **(par rapport à** ∇**)**, si $\nabla_{\dot{\gamma}} X = 0$. On note par $\Gamma_{\mathrm{par}}(\gamma^* E)$ le sous-espace de $\Gamma(\gamma^* E)$, la famille de toutes sections parallèles le long de γ.

Définition 1.19 (Géodésique). Soit M une variété et ∇ la dérivée covariante sur M. Une courbe lisse $\gamma : I \to M$ sur M s'appelle **géodésique** si son vecteur tangent $\dot{\gamma} \in \Gamma(\gamma^* \mathsf{T}M)$ est parallèle le long de γ (par rapport à ∇), i.e. si $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$.

Définition 1.20 (Variété complète). Une variété M est dite **(géodésiquement) complète** si toutes les géodésiques sont définies sur $\mathbb{R}$.

Theorem 1.21. *Soit ∇ la dérivée covariante sur un fibré vectoriel E sur M et $\gamma : I \to M$ une courbe lisse, $t_0 \in I$ et $e \in E_{\gamma(t_0)}$. Alors il existe une unique section parallèle $X \in \Gamma_{\mathrm{par}}(\gamma^* E)$ le long de γ avec $X_{t_0} = e$.*

Définition 1.22. Soit ∇ une dérivée covariante sur fibré vectoriel E sur M et $\gamma : I \to M$ une courbe lisse. On définit un isomorphisme, pour tout $s, t \in I$,

$$\begin{aligned} /\!/_{s,t} : E_{\gamma(s)} &\to E_{\gamma(t)} \\ /\!/_{s,t}(e) &:= X_t, \end{aligned}$$

où $X \in \Gamma_{\mathrm{par}}(\gamma^* E)$ avec $X_s = e$, le **transport parallèle** $/\!/_{s,t}$ **de** $E_{\gamma(s)}$ **à** $E_{\gamma(t)}$ **le long de** γ.

Évidemment, on a $/\!/_{s,t}^{-1} = /\!/_{t,s}$ et $/\!/_{t,t} = \mathrm{id}_{E_{\gamma(t)}}$. Donc, nous noterons $/\!/_t := /\!/_{0,t}$ pour faire court.

Lemme 1.23. *Le transport parallèle $/\!/_{s,t}$ constitue la dérivée covariante sous-jacent: Soit*

$X \in \Gamma(E)$, $v \in \mathsf{T}_x M$ et $\gamma : I \to M$ telle que $\dot\gamma(0) = v$. Alors on a

$$\nabla_v X = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(//_t^{-1} X_{\gamma(t)}\right) \in E_p. \tag{1.2}$$

Donc, le transport parallèle dans E et la dérivée covariante sur E définissent la même structure.

1.1.3 Géométrie riemannienne Une métrique riemannienne est une structure supplémentaire sur la variété M en équipant M avec un produit intérieur défini positif g sur chaque espace tangent $\mathsf{T}_x M$ en chaque point $x \in M$. Cette métrique permet de définir la longueur d'un chemin entre deux points, le volume vol_g sur M, le plus important la connexion de Levi-Civita qui donne naissance à une notion de courbure sur M.

Définition 1.24. Une **metrique riemannienne** $g = (\cdot,\cdot)_g$ sur M est une section globale lisse en tout point définie positive du fibré vectoriel des formes bilinéaires symétriques de M. Une **variété riemannienne** est la paire (M, g), où M est variété différentielle de la dimension $m := \dim M$ et $g(\cdot,\cdot) = (\cdot,\cdot)_g$ est une métrique riemannienne sur M.

Définition 1.25. Par $\mathsf{Metr}M$ on note l'ensemble de toutes les métriques riemanniennes différentielles sur M.

Localement, la métrique s'écrit, pour tout $X = X^i\partial_i$ et $Y = Y^j\partial_j$

$$\left(X_x, Y_x\right)_{g,x} = X^i(x)Y^j(x)\left(\partial_i, \partial_j\right)_{g,x} = g_{ij}X^i(x)Y^j(x),$$

où g_{ij} est la matrice $g_{ij}(x) := g(\partial_i, \partial_j)(x)$, i.e.

$$g = g_{ij}\mathbf{d}x^i \otimes \mathbf{d}x^j.$$

Exemple 1.26. L'exemple le plus simple d'une variété riemannienne est $M = \mathbb{R}^m$ avec sa **métrique euclidienne** $\overline{g}$ définie comme le produit scalaire sur chaque espace tangent $\mathsf{T}_x\mathbb{R}^m \cong \mathbb{R}^m$ pour tout $x \in M$. En coordonnées cartésiennes, on a

$$\overline{g} = \delta_{ij}\mathbf{d}x^i\mathbf{d}x^j = \sum_i \mathbf{d}x^i\mathbf{d}x^i = \sum_i \left(\mathbf{d}x^i\right)^2. \tag{1.3}$$

Alors, la matrice dans ces coordonnées est juste $\overline{g} = \delta_{ij}$. Appliquer aux vecteurs $v, w \in \mathsf{T}_x M$, cela donne

$$\overline{g}_x(v, w) = \delta_{ij}v^i w^j = \sum_{i=1}^n v^i w^i = v \cdot w.$$

Par consequent, $\overline{g}$ est le 2-champ tensoriel dont la valeur en chaque point est le produit scalaire euclidien.

Définition 1.27 (Les isomorphismes musicaux). Une métrique riemannienne détermine le produit scalaire sur chaque espace tangent $\mathsf{T}_x M$, qui est généralement dénoté par $(X, Y)_g := g(X, Y)$ pour tout $X, Y \in \mathsf{T}_x M$.

Par le théorème de représentation de Riesz A.2, g fournit un isomorphisme naturel entre l'espace tangent et cotangent donné par $X \mapsto (X, \cdot)_g$,

$$\mathrm{T}M \underset{\sharp}{\overset{\flat}{\rightleftarrows}} \mathrm{T}^*M.$$

Plus précisément, on définit l'**operateur bémol (flat operator)**

$$\flat^g : \mathrm{T}M \to \mathrm{T}^*M, \qquad X^{\flat^g}(Y) := g(X, Y).$$

En coordonnées, on a

$$X^{\flat^g} = \left(X^i\partial_i, \cdot\right)_g = g(X^i\partial_i, \cdot) = g_{ij}X^i\mathbf{d}x^j = X_j\mathbf{d}x^j, \quad \text{où } X_j := g_{ij}X^i.$$

Puisque g est inversible, on définit l'**operateur dièse (sharp operator)** analoguement par

$$\sharp^g : \mathrm{T}^*M \to \mathrm{T}M, \qquad \omega(\eta) = g(\omega^{\sharp^g}, \eta),$$

en coordonnées locales l'on obtient

$$\omega^{\sharp^g} = \sharp^g(\omega_i\mathbf{d}x^i) = \left(\omega_i\mathbf{d}x^i, \cdot\right)_g = g(\omega_i\mathbf{d}x^i, \cdot) = g^{ij}\omega_j\partial_j = \omega^i\partial_i, \quad \text{où } \omega^i := g^{ij}\omega_j,$$

et g^{ij} sont les composants de l'inverse du tenseur métrique, pour lesquels $g_{ij}g^{jk} = \delta_i^k$.

La métrique euclidienne induit la **mesure de volume** vol_g sur M (par rapport à la métrique g), i.e. la mesure borélienne lisse vol_g uniquement déterminée, telle que pour chaque carte locale lisse $((x^1, \dots, x^m), U)$ et tout ensemble borélien $N \subset U$, on a

$$\mathrm{vol}_g(N) = \int \sqrt{\det g(x)}\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m,$$

où $\det g(x)$ est le déterminant de la matrice $g_{ij}(x) := g(\partial_i, \partial_j)(x)$.

Exemple 1.28. Par Exemple 1.26, on a $\overline{g} = 1$. Donc sur $M = \mathbb{R}^m$, on appelle **forme volume standard sur** $\mathbb{R}^m$ la forme définie en coordonnées cartesiennes $(x^1, ..., x^m)$ par

$$\mathrm{vol} = \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m =: \mathrm{d}x^1 \ldots \mathrm{d}x^m,$$

où $\mathrm{d}x \equiv \mathrm{d}x^1 \ldots \mathrm{d}x^m$ est la mesure de Lebesgue sur $\mathbb{R}^m$.

Définition 1.29. (a) On appelle **tenseur de torsion** T de la connexion ∇ l'application

$$T : \Gamma(\mathrm{T}M) \times \Gamma(\mathrm{T}M) \to \Gamma(\mathrm{T}M)$$
$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

qui est $\mathrm{C}^\infty(M)$-linéaire dans les deux variables et antisymmétrique en X et Y. La torsion T peut donc être vue comme un $(1, 2)$-tenseur $\mathrm{C}^\infty(M)$-linéaire

$$T : \Gamma(\mathrm{T}M) \to \Omega^1(M)$$

à valeurs dans les 1-forme, ou bien aussi comme une 2-forme $T \in \Omega^2(M)$. La connexion est dit **symétrique** ou **à torsion nulle** si sa torsion disparaît, i.e. si

$$\nabla_X Y - \nabla_Y X \equiv [X, Y]. \tag{1.4}$$

(b) Soit (M, g) est une variété riemannienne. La métrique g est parallèle $\nabla g \equiv 0$, i.e. tous champs du vecteurs $X, Y, Z \in \Gamma(TM)$ satisfont la règle du produit suivant

$$X\,(Y, Z)_g = \left(\nabla_X Y, Z\right)_g + \left(Y, \nabla_X Z\right)_g. \tag{1.5}$$

Un exemple facile d'une connexion sur $\mathbb{R}^m$ est la connexion euclidienne, donnée par

$$\overline{\nabla}_X(Y^j \partial_j) := (XY^j)\partial_j, \tag{1.6}$$

i.e. $\overline{\nabla}_X Y$ est juste un champ du vecteur dont les composantes sont les dérivées directionnelles ordinaires des composantes de Y dans la direction X. De plus, $\overline{\nabla}$ a la propriété agréable (1.5), qui peut être facilement vérifiée en calculant en termes de base standard. Le théorème suivant montre que sur toute variété riemannienne, il existe naturellement une connexion unique satisfaisant (1.5) et (1.4). Cela motive la définition de parallélisme de g.

Théorème 1.30 (Levi-Civita). *Soit (M, g) une variété riemannienne. Alors il existe une unique connexion ∇ sur M, appelée* **connexion de Levi-Civita**, *telle que g est parallèle et symétrique.*

Définition 1.31. Soit M une variété equipée avec une connexion lineáire ∇. Pour tout $\eta \in \Omega^1(M)$ et $A \in \Gamma(TM)$ on définit $\nabla_A \eta \in \Omega^1(M)$ d'une 1-forme par

$$\nabla\eta(A, B) \equiv \nabla_A\eta(B) := A(\eta B) - \eta(\nabla_A B) \qquad \forall B \in \Gamma(TM). \tag{1.7}$$

En particulier, si $\eta = \mathbf{d}^{(0)} f$ avec $f \in C^\infty(M)$, on appelle

$$\nabla \mathbf{d}^{(0)} f \in \Gamma(T^*M \otimes T^*M)$$
$$\nabla_A \mathbf{d}^{(0)} f(B) \equiv \nabla \mathbf{d}^{(0)} f(A, B) = ABf - (\nabla_A B)f$$

le **tenseur hessien** de f.

Définition 1.32. Soit (M, g) une variété riemannienne et ∇ la connexion de Levi-Civita sur M. Pour $f \in C^\infty(M)$ l'opérateur de **Laplace-Beltrami** Δ_M est défini par

$$\Delta_M f := \operatorname{tr} \nabla \mathbf{d}^{(0)} f \in C^\infty(M).$$

Plus précisément, ça veut dire pour tous repères orthonormals $e_1, \dots, e_m$ de $T_x M$, on a

$$\Delta_M f(x) = \sum_{i=1}^{m} \nabla \mathbf{d}^{(0)} f(e_i, e_i).$$

Example 1.33. Soit $M = \mathbb{R}^m$ avec sa métrique euclidienne $\overline{g}$ (cf. Exemple 1.26 et ∇ la connexion de Levi-Civita. En utilisant coordonnées locales il est bien connu que

$$\nabla_{\partial_i}\partial_j = 0, \tag{1.8}$$

d'où

$$\nabla_v w \overset{\text{def}}{=} v\; w^i \partial_i = v\; w^i e_i = \left(v(w^1), \dots, v(w^n)\right). \tag{1.9}$$

De plus,

$$\nabla \mathbf{d}^{(0)} f(\partial_i, \partial_j) \overset{\text{def}}{=} \partial_i \partial_j f - \nabla_{\partial_i} \partial_j f \overset{(1.8)}{=} \partial_i \partial_j f.$$

Donc, on retrouve l'opérateur laplacien euclidien

$$\Delta_M f(x) \overset{\text{def}}{=} \operatorname{tr} \nabla \mathbf{d}^{(0)} f = \sum_{i=1}^n \nabla \mathbf{d}^{(0)} f(\partial_i, \partial_i) = \sum_{i=1}^n \partial_i \partial_i f = \sum_{i=1}^n \partial_i^2 f.$$

1.1.4 Espace L^2 de formes différentielles, la codifférentielle $\boldsymbol{\delta}$ et le laplacien de Hodge-de Rham, représentations métriques de $\mathbf{d}$ et $\boldsymbol{\delta}$ On pose $\Omega^1_{\mathsf{L}^2}(M) := \mathbf{\Gamma}_{\mathsf{L}^2}(\bigwedge^1 \mathsf{T}^* M)$ pour l'espace de Hilbert séparable de classes d'équivalence α de 1-formes différentielles boréliennes de carré sommable sur M, i.e. la complétion de $\mathsf{C}^\infty_c(\mathsf{T}^* M)$ par rapport au produit scalaire

$$\langle \omega_1, \omega_2 \rangle_g^{(1)} := \langle \omega_1, \omega_2 \rangle_{\Omega^1_{\mathsf{L}^2}(M,g)} := \int_M \big(\omega_1(x), \omega_2(x)\big)_g \operatorname{vol}_g(\mathrm{d}x) \qquad \forall \omega_1, \omega_2 \in \mathbf{\Gamma}_{\mathsf{C}^\infty_c}(\mathsf{T}^* M).$$

De façon similaire, le produit scalaire g induit un produit scalaire g sur chaque produit tenseur $\mathsf{T}M \otimes \ldots \otimes \mathsf{T}M$ et donc sur chaque k-ème produit extérieur $\bigwedge^k \mathsf{T}^* M$, et donc un produit scalaire global

$$\langle \eta_1, \eta_2 \rangle_g^{(k)} = \langle \eta_1, \eta_2 \rangle_{\Omega^k_{\mathsf{L}^2}(M,g)} := \int_M \big(\eta_1(x), \eta_2(x)\big)_g \operatorname{vol}_g(\mathrm{d}x) \qquad \forall \eta_1, \eta_2 \in \mathbf{\Gamma}_{\mathsf{C}^\infty_c}(\textstyle\bigwedge^k \mathsf{T}^* M).$$

La complétion est notée par $\Omega^k_{\mathsf{L}^2}(M) := \mathbf{\Gamma}_{\mathsf{L}^2}(\bigwedge^k \mathsf{T}^* M)$.

Définition 1.34. On appelle **codifférentielle** l'opérateur unique

$$\boldsymbol{\delta}_g^{(k)} : \Omega^k(M,g) \to \Omega^{k-1}(M,g), \qquad \eta \mapsto \boldsymbol{\delta}_g^{(k)} \eta, \qquad \forall k \geqslant 0,$$

défini comme l'opérateur adjoint de le différentielle extérieure, i.e.

$$\left\langle \eta_1, \boldsymbol{\delta}_g^{(k+1)} \eta_2 \right\rangle_g = \left\langle \mathbf{d}^{(k)} \eta_1, \eta_2 \right\rangle_g, \qquad \forall \eta_1 \in \Omega^k(M,g)\ \forall \eta_2 \in \Omega^{k+1}(M,g).$$

Définition 1.35. Le **produit intérieur** correspond à la contraction de $\alpha \in \Omega^k(M)$ avec un champ de vecteurs $X \in \mathbf{\Gamma}(\mathsf{T}M)$, i.e.

$$X \lrcorner_g \alpha(X_1, \ldots, X_{k-1}) := \alpha(X, X_1, \ldots, X_{k-1}), \qquad \forall X_1, \ldots, X_{k-1} \in \mathbf{\Gamma}(\mathsf{T}M),$$

et est une anti-dérivation:

$$X \lrcorner_g (\alpha \wedge \beta) = (X \lrcorner \alpha) \wedge \beta + (-1)^k \alpha \wedge (X \lrcorner \beta) \qquad \forall \alpha \in \Omega^k(M)\ \forall \beta \in \Omega(M).$$

Pour la preuve de notre résultat principal dans le chapitre 3, on profitera de la représentation métrique suivante de différentielle exterieure et codifférentielle. Deux preuves distinctes peuvent être trouvées dans [Ros97, Lemma 2.39] ou [Jos17, Lemma 4.3.4].

Proposition 1.36. *Soit (e_i) est un repère local orthonormal et (ε^i) son repère dual, i.e. $\varepsilon^j(e_i) = \delta_i^j$. Soit ∇ la connexion de Levi-Civita. Alors*

$$\mathbf{d}^{(k)} = \varepsilon^i \wedge \nabla_{e_i} \qquad \textit{et} \qquad \boldsymbol{\delta}_g^{(k)} = -e_i \lrcorner \nabla_{e_i}. \tag{1.10}$$

Finalement, on définit l'opérateur laplacien sur les sections de fibré de formes différentielles sur une variété riemannienne.

Définition 1.37. On défini l'opérateur **laplacien de Hodge-de Rham** par

$$\Delta_g^{(k)} := \boldsymbol{\delta}_g^{(k+1)} \mathbf{d}^{(k)} + \mathbf{d}^{(k-1)} \boldsymbol{\delta}_g^{(k)} : \Omega^k(M) \to \Omega^k(M).$$

On note que $\Delta_g^{(k)}$ est symmétrique et non-négatif sur $\Omega_{C_c^\infty}(M)$.

Example 1.38. Utilisant Proposition 1.36, on trouve que

$$\Delta_g^{(0)} f = \boldsymbol{\delta}_g^{(1)} \mathbf{d}^{(0)} f = -e_i \lrcorner \nabla_{e_i} \mathbf{d}^{(0)} f = -\operatorname{tr} \nabla^2 f \qquad \forall f \in C^\infty(M).$$

1.1.5 Courbure sur une variété Le grand motif de géométrie différentielle est de pouvoir étudier des espaces non-plats. Ainsi on introduit la courbure d'une connexion R en examinant les pièces antisymétriques de «hessien» d'un champ de vecteur. La courbure donne une mesure de détecter sur la façon dont notre espace est non-plat. Une variété riemannienne est appelée plate si elle est localement isométrique à l'espace euclidien avec le produit intérieur euclidien habituel. Il s'avère qu'un espace est plat si et seulement si la courbure est zéro.

Définition 1.39. La **courbure d'une connexion** ∇ sur M est une application $C^\infty(M)$-linéaire

$$\begin{aligned} R : \Gamma(TM) \otimes_C \Gamma(TM) \otimes_C \Gamma(TM) \to \Gamma(TM), \\ (X,Y,Z) \mapsto R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \end{aligned}$$

Étant anti-symétrique en (X,Y), la courbure est une 2-forme $R \in \Omega^2(M, \operatorname{End} TM)$.

Définition 1.40. On appelle **courbure de Ricci de M** l'application de trace

$$\begin{aligned} \operatorname{Ric} : \Gamma(TM) \otimes_C \Gamma(TM) \to C^\infty(M), \\ (X,Y) \mapsto \operatorname{Ric}(X,Y) := \operatorname{tr}(Z \mapsto R(Z,Y)X). \end{aligned}$$

c'est-à-dire, localement, Ric s'exprime comme

$$\operatorname{Ric}(\partial_i, \partial_j) = \operatorname{tr}\left(\partial_k \mapsto R(\partial_k, \partial_j)\partial_i\right) = \sum_k R^k_{kj,i} =: R_{ij}.$$

Donc, la courbure de Ricci est une mesure sur la façon dont le volume d'un petit morceau d'une boule géodésique diffère de son homologue euclidien.

Comme on le verra au § 1.3.3 s'il est indispensable que la courbure de Ricci est bornée par le bas pour assurer la complétude stochastique de variété. Il est également bien connu que les bornes inférieures (locales) sur la courbure de Ricci sont un outil essentiel pour les estimations de gradient, cf. [Wan14a] pour une étude détaillée.

1.1.6 La technique de Bochner ou comment associer les opérateurs laplaciens? Soit η une k-forme différentielle, i.e. un tenseur d'ordre $(0,k)$. Alors, $\nabla\eta$ est un tenseur d'ordre

$(0, k+1)$ et $\nabla^2\eta$ un tenseur d'ordre $(0, k+2)$. La trace est définie par la contraction tensorielle

$$\operatorname{tr} \nabla^2\eta(\bullet) := \sum \nabla^2\eta(\bullet, e_i, e_i),$$

qui est indépendant du choix de repère orthonormal (e_i).

Lemme 1.41. *Pour tous repères orthonormals $(e_i)_{i\in\mathbb{N}}$, on peut decomposer*

$$\operatorname{tr} \nabla^2\eta = \sum_i \left(\nabla_{e_i}\nabla_{e_i}\eta - \nabla_{\nabla_{e_i}e_i}\eta\right) \qquad \forall \eta \in \Omega^k(M).$$

La formule de Weitzenböck associe l'opérateur laplacien de Hodge-de Rham à la dérivée covariante sur (M, g). Le **laplacien de Bochner** est défini via la connexion

$$\square := \nabla^*\nabla : \Gamma(E) \xrightarrow{\nabla^E} \Gamma(\mathrm{T}^*M \otimes E) \xrightarrow{\nabla^{\mathrm{T}^*M\otimes E}} \Gamma(\mathrm{T}^*M \otimes \mathrm{T}^*M \otimes E) \xrightarrow{\operatorname{tr}} \Gamma(E),$$

i.e. un opérateur de second ordre agissant sur les sections du fibré E. Le laplacien de Bochner ne diffère de l'opérateur Laplace-Beltrami que par un signe:

$$\nabla^*\nabla = -\operatorname{tr} \nabla^2.$$

Le Théorème 1.42 suivant montre que les deux diffèrent par un opérateur linéaire d'ordre zéro n'impliquant que la courbure.

Théorème 1.42 (Formule de Weitzenböck). *Pour toutes k-formes différentielles $\eta \in \Omega(M)$, on a*

$$\Delta_g\eta = -\operatorname{tr} \nabla^2\eta - \mathscr{R}, \tag{1.11}$$

*où l'***endomorphisme de courbure de Weitzenböck** *est donné par*

$$\mathscr{R} = -\sum_{k,l=1}^{m} e_l^\flat \wedge \left(e_k \lrcorner \mathrm{R}(e_l, e_k)\right), \tag{1.12}$$

pour tout repère orthonormal $(e_i)_{i\leqslant m}$.

La proposition suivant une conséquence directe.

Proposition 1.43 (Formule de Bochner). *Pour toute k-forme différentielle η, on a*

$$-\frac{1}{2}\Delta_g |\eta|^2 = -\left(\Delta_g\eta, \eta\right)_g + |\nabla\eta|^2 + \left(\varepsilon^i \wedge (e_j \lrcorner \mathrm{R}(e_i, e_j)\eta, \eta\right)_g,$$

pour tout repère orthonormal $(e_i)_{i\leqslant m}$ et repère dual $(\varepsilon^i)_{i\leqslant m}$.

Corollaire 1.44. *Pour toutes 1-formes différentielles $\eta \in \Omega^1(M)$, on a*

$$-\frac{1}{2}\Delta_g |\eta|^2 = -\left(\Delta_g\eta, \eta\right)_g + |\nabla\eta|^2 + \operatorname{Ric}(\eta^\sharp, \eta^\sharp).$$

1.1.7 Longueur, application exponentielle, boule géodésique et Théorème de Hopf-Rinow

Étant donné une courbe $\gamma : [a,b] \to M$, la **longueur** d'une courbe C^1 par morceaux, est définie par

$$\mathsf{L}(\gamma,[a,b]) := \int_a^b |\dot{\gamma}(t)|_{g(\gamma(t))}\,\mathrm{d}t.$$

La longueur est invariante par reparamétrisation régulier et induit aussi une distance sur M par

$$d(x,y) := \inf\left\{\mathsf{L}(\gamma,[a,b]) : \gamma : [a,b] \to M, \mathrm{C}^1 \text{ par morceaux } \gamma(a) = x,\ \gamma(b) = y\right\}$$

Donc, avec cette distance, l'espace (M,d) devient un espace métrique.

Pour $v \in \mathrm{T}_x M$, il existe une unique géodésique γ_v telle que $\gamma_v(0) = x$ et de vecteur tangent initial $\gamma_v'(0) = v$. On définit un sous-ensemble $\mathscr{E} \subset \mathrm{T}M$, **le domaine de l'application exponentielle**, par

$$\mathscr{E} := \left\{v \in \mathrm{T}M : \gamma_V \text{ définit sur une intervalle contenant } [0,1]\right\}.$$

et encore une application différentiable, **l'application exponentielle** $\exp : \mathscr{E} \to M$, par

$$\exp(v) = \gamma_v(1).$$

Pour tout $x \in M$, la restriction de l'application exponentielle $\exp_x$ est la restriction d'exp à l'ensemble $\mathscr{E}_x := \mathscr{E} \cap \mathrm{T}_x M$. Alors $\exp_x(t) = \gamma_v(t)$ pour tout $t \in \mathbb{R}$. Soit $\varepsilon > 0$. Si

$$\exp_x : \mathsf{B}(0,\varepsilon) \to \exp(\mathsf{B}(0,\varepsilon))$$

est un difféomorphisme, alors

$$\mathsf{B}(x,\varepsilon) := \exp_x(B(0,\varepsilon)) = \{y \in M : \mathrm{d}(x,y) < \varepsilon\}$$

est appelé la **boule géodésique** dans M avec centre $x \in M$ et de rayon $r > 0$.

Si M est (géodésiquement) complète, alors pour tout point $x \in M$ l'application exponentielle $\exp_x$ d'origine x est définie sur $\mathrm{T}_x M$. Par le théorème de Hopf-Rinow [Jos17, Theorem 1.7.1.] pour toutes M variétés riemanniennes connexes (sans bord) les propriétés suivantes sont
équivalentes :

(i) M est un espace métrique complet (i.e. toutes suites de Cauchy convergent)

(ii) M est (géodésiquement) complète

(iii) les parties fermées et bornées sont compactes

1.2 Operator Theory and Spectral Calculus

Next, we recall well-known notions and results for self-adjoint operators and their relation to sesquilinear forms in Hilbert spaces. The main tool to define the heat semigroup on the Hilbert space of square-integrable Borel forms on M will be the Spectral Theorem 1.50. A particularly important example is given by the Friedrichs realisation 1.62 giving a canonical self-adjoint extension of a non-negative densely defined symmetric operator. We close this section by introducing and collecting necessary facts about smooth heat kernels on metric vector bundles. For proofs, we refer the reader to Simon & Reed [RS72; RS75; RS79] and Weidmann [Wei80; Wei00; Wei03]. We greatly benefited from Güneysu [Gün17a] and Kato's lovely monograph [Kat95], in particular, [Kat95, Chapter Six] gives a detailed discussion about sesquilinear forms in Hilbert spaces and associated operators.

Let $\mathscr{H}$ be a complex separable Hilbert space. The underlying scalar product (antilinear in its first slot) will be denoted by $\langle\cdot,\cdot\rangle$ and the induced norm (as well as the induced operator norm) is denoted by $\|\cdot\|$. The convergence in $\mathscr{H}$ is understood to be norm convergence if not stated otherwise.

Given a linear operator T in $\mathscr{H}$, we denote by $\operatorname{dom}\mathsf{T}\subset\mathscr{H}$ its domain, by $\operatorname{ran}\mathsf{T}\subset\mathscr{H}$ its range, and by $\ker\mathsf{T}\subset\mathscr{H}$ its kernel.

An operator $(\mathsf{T},\operatorname{dom}\mathsf{T})$ is called an **extension** of $(\mathsf{S},\operatorname{dom}\mathsf{S})$, denoted $\mathsf{S}\subset\mathsf{T}$, if

$$\operatorname{dom}\mathsf{S}\subset\operatorname{dom}\mathsf{T} \qquad \text{and} \qquad \mathsf{T}\big|_{\operatorname{dom}\mathsf{S}}=\mathsf{S}.$$

An operator $(\mathsf{T},\operatorname{dom}\mathsf{T})$ is **closed** if the set

$$\Gamma_{\mathsf{T}} := \left\{(u,\mathsf{T}u)\,:\,u\in\operatorname{dom}\mathsf{T}\subset\mathscr{B}\times\mathscr{B} \text{ is closed in } (\mathscr{B}\times\mathscr{B},\|\cdot\|)\right\}.$$

An operator $(\mathsf{T},\operatorname{dom}\mathsf{T})$ is called **closable** if there exists a closed extension $\left(\widetilde{\mathsf{T}},\operatorname{dom}\widetilde{\mathsf{T}}\right)$ and $\widetilde{\mathsf{T}}\subset\mathsf{T}$. The **closure** $(\overline{\mathsf{T}},\operatorname{dom}\overline{\mathsf{T}})$ of T is the minimal closed extension of T.

1.2.1 Normal, adjoint, symmetric operators and operators semibounded from below

For any densely defined operator T, the **adjoint** $(\mathsf{T}^*,\operatorname{dom}\mathsf{T}^*)$ of T is defined as

$$\operatorname{dom}(\mathsf{T}^*) := \left\{\forall u\in\mathscr{H}\ \exists u^*\in\mathscr{H}\ :\quad \langle u^*,v\rangle\ =\langle u,\mathsf{T}v\rangle\quad \forall v\in\operatorname{dom}\mathsf{T}\right\}$$

and then $\mathsf{T}^*u:=u^*$.

A densely defined T is called **symmetric** if $\mathsf{T}\subset\mathsf{T}^*$ (i.e. T^* is an extension of T), **self-adjoint** if $\mathsf{T}=\mathsf{T}^*$ and **normal** if

$$\operatorname{dom}\mathsf{T}=\operatorname{dom}\mathsf{T}^* \qquad \text{and} \qquad \|\mathsf{T}u\|=\left\|\mathsf{T}^*u\right\| \qquad \forall u\in\operatorname{dom}\mathsf{T}.$$

Hence, typical examples of normal operators include symmetric and self-adjoint operators.

An operator T is called **semibounded (from below)** if there is a constant $C\geqslant 0$ such that $\mathsf{T}\geqslant -C$, i.e.

$$\langle\mathsf{T}u,u\rangle\geqslant -C\,\|u\|^2 \qquad \forall u\in\operatorname{dom}\mathsf{T}.$$

Using the polarisation equality, on *complex* Hilbert spaces, semibounded operators are automatically symmetric.

If T is symmetric, then T is called **essentially self-adjoint** if $\overline{\mathsf{T}}$ is self-adjoint.

Proposition 1.45. *Let T be a bounded self-adjoint (normal) operator in a real (complex) Hilbert space. Then*

$$\|\mathsf{T}\| = \max_{z\in\sigma(\mathsf{T})}|z|\,.$$

Let $(\mathscr{H}_1, \langle\cdot,\cdot\rangle_{\mathscr{H}_1})$ and $(\mathscr{H}_2, \langle\cdot,\cdot\rangle_{\mathscr{H}_2})$ be two Hilbert space (inner product space is sufficient). An **isometry** $A : \mathscr{H}_1 \to \mathscr{H}_2$ is an isomorphism such that

$$\langle Av, Aw\rangle_{\mathscr{H}_2} = \langle v, w\rangle_{\mathscr{H}_1} \qquad \forall v, w \in \mathscr{H}_1.$$

A linear operator $\mathsf{U} : \mathscr{H} \to \mathscr{H}$ is called a **partial isometry** if the restriction U into the orthogonal complement of $\ker \mathsf{U}$ is an isometry, i.e.

$$\|\mathsf{U}x\| = \|x\| \qquad \forall x \in (\ker \mathsf{U})^{\perp}.$$

The space $(\ker \mathsf{U})^{\perp}$ is called **initial space** and the image $\operatorname{ran} \mathsf{U}$ is called **final space**. In other words, a partial isometry is an isometry between its initial space and final space.

Example 1.46. (a) Every isometry is partial isometries such that $\ker \mathsf{U} = \{0\}$. In particular, every unitary operator is a partial isometry.

(b) Any orthogonal projection $P : \mathscr{H} \to \mathscr{H}$ is one with common initial and final subspace.

Partial isometries play in important rôle in the *polar decomposition* of linear operators which is reflected in the following Theorem. We will make use of this idea to define suitable bounded operators in the proof of the decomposition theorem, Lemma 3.36, needed to prove our Main Result in § 3.

Lemma 1.47. *(a) Every positive self-adjoint operator* T *has exactly one positive self-adjoint square root.*

(b) Every densely defined closed operator $\mathsf{T} : \mathscr{H}_1 \to \mathscr{H}_2$ *can be written in its* **polar decomposition** $\mathsf{T} = U\,\|\mathsf{T}\|$ *with* $\|\mathsf{T}\| = (\mathsf{T}^*\mathsf{T})^{1/2}$ *and some partial isometry* U *with initial space* $\overline{\operatorname{ran}\|\mathsf{T}\|}$ *and final space* $\overline{\operatorname{ran}\mathsf{T}}$*. Moreover,* $\operatorname{dom}\|\mathsf{T}\| = \operatorname{dom}\mathsf{T}$ *and* $\||\mathsf{T}|\,x\| = \|\mathsf{T}\|$ *for all* $x \in \operatorname{dom}\mathsf{T}$*, i.e.* T *and* $\|\mathsf{T}\|$ *are metrically equivalent.*

1.2.2 Schatten class of operators, trace class operators and Hilbert-Schmidt operators

Let $\mathscr{H}_1, \mathscr{H}_2$ be complex separable Hilbert spaces.

The linear space of bounded linear operators $\mathscr{H}_1 \to \mathscr{H}_2$ is denoted by $\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$. If $\mathscr{H} := \mathscr{H}_1 \equiv \mathscr{H}_2$, we simply write $\mathscr{L}(\mathscr{H}) := \mathscr{L}(\mathscr{H}, \mathscr{H})$.

For any orthonormal basis (e_i) of $\mathscr{H}$, we say that $\mathsf{K} \in \mathscr{L}(\mathscr{H})$ is a **Hilbert-Schmidt operator** if the **Hilbert-Schmidt norm** $\|\mathsf{K}\|_{\mathrm{HS}}$ is finite, i.e. $\|\mathsf{K}\|_{\mathrm{HS}}^2 := \sum_i \|\mathsf{K}e_i\|^2 < \infty$.

For any $q \in [1, \infty)$, an operator $\mathsf{K} \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ is a **Schatten operator of class** q, denoted $\mathscr{J}^q(\mathscr{H}_1, \mathscr{H}_2)$, if $\operatorname{tr}|\mathsf{K}^*\mathsf{K}|^q < \infty$, i.e. if for arbitrary orthonormal sequences (φ_i) in $\mathscr{H}_1$ and (ψ_i) in $\mathscr{H}_2$ we have $\sum_i |\langle \mathsf{K}\varphi_i, \psi_i\rangle|^q < \infty$. An operator $\mathsf{K} \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ is **compact**, denoted $\mathsf{K} \in \mathscr{J}^{\infty}(\mathscr{H}_1, \mathscr{H}_2)$, if for every orthonormal sequence (φ_i) in $\mathscr{H}_1$ and (ψ_i) in $\mathscr{H}_2$ we have $\langle \mathsf{K}\varphi_i, \psi_i\rangle \xrightarrow{n\to\infty} 0$. We write $\mathscr{J}^{\bullet}(\mathscr{H}_1) := \mathscr{J}^{\bullet}(\mathscr{H}_1, \mathscr{H}_1)$ for short.

Example 1.48. (i) The operator class $\mathscr{J}^1$ is called the **trace class**.

(ii) The operator class $\mathscr{J}^2$ is called the **Hilbert-Schmidt class** and every Hilbert-Schmidt operator is compact.

Given two metric vector bundles E, $\widetilde{E}$ over a manifold M and a bounded operator

$$\mathsf{K} \in \mathscr{L}(\Gamma_{L^2}(E), \Gamma_{L^2}(\widetilde{E}))$$

such that there exists a corresponding jointly smooth integral kernel $k(x, y)$ of K

$$M \times M \ni (x, y) \mapsto k(x, y) \in \mathrm{Hom}\left(E_y, \widetilde{E}_x\right),$$

the uniquely determined map such that we have

$$\mathsf{K}a(x) = \int_M k(x, y)a(y)\,\mathrm{vol}_\bullet(\mathrm{d}y).$$

Then K is Hilbert-Schmidt, if

$$\iint_{M\times M} |k(x, y)|^2\,\mathrm{vol}_\bullet(\mathrm{d}x)\,\mathrm{vol}_\bullet(\mathrm{d}y) < \infty.$$

It is well-known [RS79] that the product of two Hilbert-Schmidt operators is trace class, and the product of a bounded operator and a trace class operator (resp. Hilbert-Schmidt operator) is again trace class (resp. Hilbert-Schmidt).

1.2.3 Spectrum, spectral theorem and spectral calculus

The *spectral calculus* or *Borel functional calculus* is a rigorous way to apply an arbitrary Borel function to a normal (self-adjoint) operator. By the Spectral Theorem 1.50, for any normal (self-adjoint) operator on a Hilbert space $\mathscr{H}$ we find a representation of a linear operator in form of an integral with respect to a certain measure, the spectral measure (the spectral resolution).

The **resolvent set** $\varrho(\mathsf{T})$ of T is the set of all regular values of T, i.e. of all $z \in \mathbb{C}$ such that $\mathsf{T} - z$ is invertible as a linear map $\mathrm{dom}\,\mathsf{T} \to \mathscr{H}$ and bounded as a linear operator from $\mathscr{H} \to \mathscr{H}$. If T is closed and $(\mathsf{T} - z)^{-1}$ is invertible, then $(\mathsf{T} - z)^{-1}$ is bounded by the closed graph theorem. The complement of the resolvent set

$$\sigma(\mathsf{T}) := \mathbb{C} \setminus \varrho(\mathsf{T})$$

is the **spectrum** of T. As the resolvent set of closed operators are open, the spectrum of a closed operator is always closed.

Given a set Ω, σ-algebra $\mathscr{F}$ and Hilbert space $\mathscr{H}$, we define the **spectral measure** (or **projection-valued measure (PVM)**) as the σ-additive mapping $\mathsf{E} : \mathscr{F} \to \mathscr{H}$ such that for all $F \in \mathscr{F}$, $\mathsf{E}(F)$ is idempotent and self-adjoint and such that $\mathsf{E}(\varnothing) = 0$ and $\mathsf{E}(\Omega) = I$. In particular, E is monotone, i.e. for $F_1, F_2 \in \mathscr{F}$ with $F_1 \subset F_2$ it follows $\mathsf{E}(F_1)\mathscr{H} \subset \mathsf{E}(F_2)\mathscr{H}$, and for two disjoint sets the corresponding spectral measures are orthogonal.

Example 1.49. Let E be a spectral measure on $(\Omega, \mathscr{F}, \mathscr{H})$. Let $x, y \in \mathscr{H}$ be fixed. Then $\mathsf{E}_{x,y}$ with

$$\mathsf{E}_{x,y} := \langle x, \mathsf{E}(F)y\rangle, \qquad \forall F \in \mathscr{F}, \tag{1.13}$$

is a σ-additive, complex measure on $(\Omega, \mathscr{F})$ and $\left\|\mathsf{E}_{x,y}\right\| \leqslant \|x\|\ \|y\|$. In particular, for $x = y$ we get that $\mathsf{E}_{x,y}$ is a classical measure.

Theorem 1.50 (Spectral theorem). *Let $\mathsf{T} \in \mathscr{L}(\mathscr{H})$ be a normal operator. Then there exists a uniquely determined spectral measure on the Borel-σ-algebra $\mathscr{B}$ of $\sigma(\mathsf{T})$, such that*

(i) T *can be written as* $\mathsf{T} = \int_{\sigma(\mathsf{T})} \lambda \, \mathsf{E}(\mathrm{d}\lambda)$*, and moreover*

$$f(\mathsf{T}) = \int_{\sigma(\mathsf{T})} f(\lambda)\, \mathsf{E}(\mathrm{d}\lambda) \qquad \forall f \in \mathscr{B}(\sigma(\mathsf{T})). \tag{1.14}$$

(ii) If $S \in \sigma(\mathsf{T})$ *is open and non-empty, then* $\mathsf{E}(S) \neq 0$*, i.e.* $\operatorname{supp} E = \sigma(\mathsf{T})$.

The mapping

$$\Phi_{\mathsf{T}} : \begin{array}{ccc} \mathscr{B}(\Omega) & \to & \mathscr{L}(\mathscr{H}) \\ f & \mapsto & f(\mathsf{T}) = \int_{\sigma(\mathsf{T})} f(\lambda)\mathsf{E}(\mathrm{d}\lambda) \end{array}$$

is called **Borel functional calculus**.

Analogously to the theory of probability measures and their distribution functions, we may define the **spectral family** via $\mathsf{E}_\lambda : \mathbb{R} \to \mathscr{L}(\mathscr{H})$, $\mathsf{E}_\lambda := \mathsf{E}(-\infty, \lambda]$ $(\lambda \in \mathbb{R})$. Then E_λ is an orthogonal projection, monotone, right-continuous, with $E_\lambda \xrightarrow{s} 0$ and $E_\lambda \xrightarrow{s} I$, and

$$\operatorname{supp}\left\{\mathsf{E}_\lambda : \lambda \in \mathbb{R}\right\} = \overline{\left\{\lambda \in \mathbb{R} : \mathsf{E}_\lambda \neq 0, \mathsf{E}_\lambda \neq I\right\}}.$$

Moreover, if the support of $(\mathsf{E}_\lambda)_{\lambda\in\mathbb{R}}$ is compact, we may also define an integral uniquely determined by the relation

$$\left\langle \left(\int_{-\infty}^{\infty} f(\lambda) \mathrm{d}\mathsf{E}_\lambda \right) x, y \right\rangle = \int_{-\infty}^{\infty} f(\lambda)\, \mathrm{d}\left\langle \mathsf{E}_\lambda x, y\right\rangle, \qquad \forall x, y \in \mathscr{H},$$

where the integral on the righthand side is defined in the Riemann-Stieltjes sense by common approximation arguments. Thus

$$\int_{-\infty}^{\infty} f(\lambda)\mathrm{d}\mathsf{E}_\lambda = \int_{\mathbb{R}} f(\lambda)\mathsf{E}(\mathrm{d}\lambda).$$

Example 1.51. A **spectral resolution** P on $\mathscr{H}$ is a map $\mathsf{P} : \mathbb{R} \to \mathscr{L}(\mathscr{H})$ such that

(i) $\mathsf{P}(\lambda)$ is an orthogonal projection, i.e. $\mathsf{P}(\lambda) = \mathsf{P}(\lambda)^*$ and $\mathsf{P}(\lambda)^2 = \mathsf{P}(\lambda)$ $(\lambda \in \mathbb{R})$

(ii) P is monotone, i.e. for $\lambda_1 \leqslant \lambda_2 \implies \operatorname{ran}\mathsf{P}(\lambda_1) \subset \operatorname{ran}\mathsf{P}(\lambda_2)$

(iii) P is right-continuous (in the strong topology of $\mathscr{L}(\mathscr{H})$)

By definition, for every $x \in \mathscr{H}$, the function

$$\lambda \mapsto \langle \mathsf{P}(\lambda)x, x\rangle = \|\mathsf{P}(\lambda)x\|^2 =: \rho_x(\lambda). \tag{1.15}$$

is right-continuous and increasing. Constructing a Riemann-Stieljes integral this induces a Borel measure on $\mathbb{R}$, denoted $\langle \mathsf{P}(\mathrm{d}\lambda)f, f\rangle$ with total mass

$$\langle \mathsf{P}(\mathbb{R})x, x\rangle = \|x\|^2 .$$

Given P and further a Borel function $f : \mathbb{R} \to \mathbb{C}$, the set

$$\mathsf{D}_{\mathsf{P},x} := \left\{ x \in \mathscr{H} : \int_{\mathbb{R}} |f(\lambda)|^2 \langle \mathsf{P}(\mathrm{d}\lambda)x, x\rangle < \infty \right\}$$

is a dense linear subspace of $\mathscr{H}$, and defines a linear operator $f(\mathsf{P})$ by $\operatorname{dom} f(\mathsf{P}) := \mathsf{D}_{\mathsf{P},x}$ in $\mathscr{H}$, where $\langle f(\mathsf{P})x, y\rangle = \int f(\lambda)\rho_{x,y}(\mathrm{d}\lambda)$ and $\rho_{x,y}$ is defined by complex polarisation of the identity (1.15).

In particular, such an operator has the following properties: Let $f, g : \mathbb{R} \to \mathbb{C}$ be Borel functions, then

(i) the operator $f(\mathsf{P})$ is normal with $f(\mathsf{P})^* = \overline{f}(\mathsf{P})$. In particular,

$$f(\mathsf{P}) \text{ is self-adjoint} \qquad \Longleftrightarrow \qquad f \text{ is real-valued}$$

(ii) we get $\|f(\mathsf{P})\| \leqslant \sup_{\mathbb{R}} |f(\mathsf{P})| \in [0, \infty]$

(iii) if $f \geqslant -K$ for some constant $K \geqslant 0$, then $f(\mathsf{P})$ is semibounded below with bound $-K$, i.e. $f(\mathsf{P}) \geqslant -K$

(iv) $f(\mathsf{P}) + g(\mathsf{P}) \subset (f+g)(\mathsf{P})$ with $\operatorname{dom}(f(\mathsf{P}) + g(\mathsf{P})) = \operatorname{dom}((|f| + |g|)(\mathsf{P}))$

(v) $f(\mathsf{P})g(\mathsf{P}) \subset (f+g)(\mathsf{P})$ with $\operatorname{dom}(f(\mathsf{P})g(\mathsf{P})) = \operatorname{dom}((fg)(\mathsf{P})) \cap \operatorname{dom} g(\mathsf{P})$

(vi) if g is bounded, we get $f(\mathsf{P}) + g(\mathsf{P}) = (f+g)(\mathsf{P})$ and $f(\mathsf{P})g(\mathsf{P}) = (fg)(\mathsf{P})$

(vii) for every $x \in \operatorname{dom} f(\mathsf{P})$, we have

$$\|f(\mathsf{P})x\|^2 = \int_{\mathbb{R}} |f(\lambda)|^2 \langle \mathsf{P}(\mathrm{d}\lambda)x, x\rangle$$

By the Spectral Theorem 1.50, for every self-adjoint operator $\mathbf{H}$ in $\mathscr{H}$ there is exactly one spectral resolution $\mathsf{P}_{\mathbf{H}}$, called **spectral resolution of H**, such that $\mathbf{H} = \mathrm{id}_{\mathbb{R}}(\mathsf{P}_{\mathbf{H}})$. By (1.14), $\mathsf{P}_{\mathbf{H}}$ is concentrated on the spectrum, i.e. $f(\mathsf{P}_{\mathbf{H}}) = (\mathbb{1}_{\sigma(\mathbf{H})}f)(\mathsf{P}_{\mathbf{H}})$ and we get $(fg)(\mathsf{P}_{\mathbf{H}}) = f(\mathsf{P}_{g(\mathsf{P}_{\mathbf{H}})})$. If f is also continuous, then $\sigma(f(\mathsf{P}_{\mathbf{H}})) = \overline{f(\sigma(\mathbf{H}))}$.

A prominent example is the strongly continuous (C_0) unitary semigroup $\left(\mathrm{e}^{-it\mathbf{T}}\right)_{t\in\mathbb{R}}$ of bounded operators defined in the following Theorem.

Theorem 1.52 (Stone's theorem). *Let $\mathbf{T}$ be a self-adjoint operator in a complex Hilbert space $\mathscr{H}$. Then*

$$\left(\mathbf{U}_t\right)_{t\in\mathbb{R}} := \left(\mathrm{e}^{-it\mathbf{T}}\right)_{t\in\mathbb{R}}$$

defines a (C_0) unitary, i.e. $\mathbf{U}\mathbf{U}^* = \mathbf{U}^*\mathbf{U} = I$, semigroup. Its infinitesimal generator is given by $-i\mathbf{T}$. For every $x \in \operatorname{dom}\mathbf{T}$, $\psi(t) = \exp(-it\mathbf{T})x$ is the uniquely determined solution to the initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = -i\mathbf{T}\psi(t), \quad \psi(0) = x \qquad (t \in \mathbb{R}).$$

1.2.4 Sesquilinear forms in Hilbert spaces We introduce some facts about (possibly unbounded) sesquilinear forms in Hilbert spaces. On a finite-dimensional complex inner product space, the notion of sesquilinear form and that of a linear operator coincide and symmetric forms correspond to symmetric operators. The theory can be extended to infinite-dimensional Hilbert spaces, although we restrict ourselves to bounded forms and bounded operators. A further generalisation is given for non-symmetric forms and operators (under certain restrictions) by using the closed theory connecting semibounded symmetric forms and semibounded self-adjoint operators. For a detailed review of what follows, we refer the reader to Kato's monograph [Kat95, Chapter Six] and [Gün17a].

Let $\mathscr{H}$ be a complex separable Hilbert space. A **sesquilinear form q** on $\mathscr{H}$ is a map

$$\mathsf{q} : \operatorname{dom}\mathsf{q} \times \operatorname{dom}\mathsf{q} \to \mathbb{C},$$

where $\operatorname{dom}\mathsf{q} \subset \mathscr{H}$ is a linear subspace, the **domain of definition of q**, such that q is anti-linear in the first and linear in the second slot.

The quadratic form q associated to the sesquilinear form is the map

$$\mathsf{q} : \operatorname{dom}\mathsf{q} \ni a \mapsto \mathsf{q}(a,a).$$

There is a one-to-one mapping between a sesquilinear form and a quadratic form. Moreover from a quadratic form $\mathsf{q}(\cdot)$ we recover the underlying sesquilinear form by polarisation:

$$\mathsf{q}(a,b) = \frac{1}{4}\left(\mathsf{q}(a+b) - \mathsf{q}(a-b) + \frac{1}{i}\left(\mathsf{q}(a+ib) - \mathsf{q}(a-ib)\right)\right).$$

In what follows, let q and q' be sesquilinear forms on $\mathscr{H}$.

The sum $\mathsf{q}+\mathsf{q}'$ of q and q' is the sesquilinear form with its domain of definition given by $\operatorname{dom}(\mathsf{q}+\mathsf{q}') = \operatorname{dom}\mathsf{q} \cap \operatorname{dom}\mathsf{q}'$.

A form q' is called **extension of q**, denoted $\mathsf{q} \subset \mathsf{q}'$, if $\operatorname{dom}\mathsf{q} \subset \operatorname{dom}\mathsf{q}'$ and both forms coincide on $\operatorname{dom}\mathsf{q}$. A form q is called **symmetric**, if $\mathsf{q}(a,a) = \mathsf{q}(a,a)^*$, and **semibounded (from below)**, denoted $\mathsf{q} \geqslant -C$, if there is a constant $C \geqslant 0$ such that

$$\mathsf{q}(a,a) \geqslant -C\,\|a\|^2 \qquad \forall a \in \operatorname{dom}\mathsf{q}.$$

Every semibounded form is symmetric by polarisation.

Given a sequence $(a_n) \subset \operatorname{dom}\mathsf{q}$ and $a \in \operatorname{dom}\mathsf{q}$, then we write $a \xrightarrow[n\uparrow\infty]{\mathsf{q}} 0$, if $a_n \to a$ in $\mathscr{H}$ and

$$\mathsf{q}(a_n - a_m, a_n - a_m) \longrightarrow 0 \qquad \text{as } n,m \to \infty.$$

A form q is **closed**, if

$$a_n \xrightarrow{\mathsf{q}} a \qquad \Longrightarrow \qquad a \in \operatorname{dom}\mathsf{q}.$$

A semibounded form q is closed, if and only if, for some (hence every) $C \geqslant 0$ with $\mathsf{q} \geqslant -C$ the induced scalar product on $\operatorname{dom}\mathsf{q}$ given by

$$\langle a,b\rangle_{\mathsf{q},C} := (1+C)\,\langle a,b\rangle + \mathsf{q}(a,b)$$

turns $\operatorname{dom}\mathsf{q}$ into a Hilbert space. The form q is called **closable** if it has a closed extension. If q is closed, then a linear subspace $D \subset \operatorname{dom}\mathsf{q}$ is called **core of q** if $\overline{\mathsf{q}|_D} = \mathsf{q}$.

Example 1.53. Let $\mathscr{H}$ be a Hilbert space and $\mathsf{T} : \operatorname{dom}\mathsf{T} \to \mathscr{H}$ a symmetric operator such that there is a $K > 0$ such that $\langle \mathsf{T}a, a\rangle \geqslant -K\,\|a\|^2$ for all $u \in \mathscr{H}$. Then the form

$$\mathsf{q} : \operatorname{dom}\mathsf{T} \times \operatorname{dom}\mathsf{T} \to \mathbb{C}, \quad \mathsf{q}(a,b) = \langle \mathsf{T}a, b\rangle$$

is closable. Its closure is denoted by q_{T}.

Definition 1.54. Let $\mathbf{q}$ be symmetric. If $\operatorname{dom}\mathbf{q} \subset \operatorname{dom}\mathbf{q}'$, then $\mathbf{q}'$ is called **q-bounded with bound** < 1, if there are constants $\delta \in [0, 1)$, $L \in [0, \infty)$ such that

$$\left|\mathbf{q}'(a, a)\right| \leqslant L \|a\|^2 + \delta \mathbf{q}(a, a) \qquad \forall a \in \operatorname{dom}\mathbf{q}. \tag{1.16}$$

The form $\mathbf{q}'$ is called **infinitesimally q-bounded**, if for every $\gamma \in [0, \infty)$ there exists a constant $L = L(\gamma) \in [0, \infty)$ such that (1.16) holds.

Proposition 1.55. *Let $\mathbf{q}$ be a closed form on $\mathscr{H}$. Then*

$$\begin{aligned} \operatorname{dom}\mathbf{T} &= \{a \in \operatorname{dom}\mathbf{q} \,:\, \exists c \in \mathscr{H}\ \forall b \in \operatorname{dom}\mathbf{q} \,:\, \mathbf{q}(a, b) = \langle a, c\rangle\} \\ \mathbf{T}a &= c \end{aligned}$$

defines a lower semi-bounded, self-adjoint operator on $\mathscr{H}$ and $\mathbf{q} = \mathbf{q}_{\mathbf{T}}$. The operator $\mathbf{T}$ is called the generator of $\mathbf{q}_{\mathbf{T}}$.

The KLMN Theorem, named after Kato, Lions, Lax, Milgram and Nelson, is a famous result from perturbation theory. It may be seen as the «quadratic form version» of the Kato-Rellich theorem.

Theorem 1.56 (KLMN). *Let $\mathbf{q}$ be semibounded and closed, and let $\mathbf{q}'$ be symmetric and $\mathbf{q}$-bounded with bound < 1. Then $\mathbf{q} + \mathbf{q}'$ is semibounded and closed on its natural domain $\operatorname{dom}\mathbf{q} \cap \operatorname{dom}\mathbf{q}' = \operatorname{dom}\mathbf{q}$. Moreover, every form core of $\mathbf{q}$ is also one of $\mathbf{q} + \mathbf{q}'$, and for every constant $K \geqslant 0$ with $\mathbf{q} \geqslant -K$ and every L, δ as in (1.16) we get the explicit lower bound*

$$\mathbf{q} + \mathbf{q}' \geqslant -(1 - \delta)K - L.$$

The next definition reflects the connection between forms and self-adjoint operators.

Definition 1.57. Given a self-adjoint operator $\mathbf{T}$ in $\mathscr{H}$, the densely defined and symmetric sesquilinear form $\mathbf{q}_{\mathbf{T}}$ in $\mathscr{H}$ given by

$$\begin{aligned} \operatorname{dom}\mathbf{q}_{\mathbf{T}} &:= \operatorname{dom}\sqrt{\mathbf{T}} \\ \mathbf{q}_{\mathbf{T}}(a, b) &:= \left\langle \sqrt{\mathbf{T}}a, \sqrt{\mathbf{T}}b \right\rangle \end{aligned}$$

is called the **form associated to T**.

The fundamental relation between densely defined, semibounded, closed forms and semibounded self-adjoint operators is reflected in the following two Theorems.

Theorem 1.58. *For every self-adjoint semibounded operator $\mathbf{T}$ in $\mathscr{H}$, the form $\mathbf{q}_{\mathbf{T}}$ is densely defined, semibounded and closed. Conversely, for every densely defined, closed and semibounded sesquilinear form $\mathbf{q}$ in $\mathscr{H}$, there is precisely one self-adjoint semibounded operator $\mathbf{T}_{\mathbf{q}}$ in $\mathscr{H}$ such that $\mathbf{q} = \mathbf{q}_{\mathbf{T}_{\mathbf{q}}}$. The operator $\mathbf{T}_{\mathbf{q}}$ is called the* **operator associated with q**.

Theorem 1.59. *Let $\mathbf{q}$ be densely defined, closed and semibounded. Then* *(i)* $\mathsf{T}_{\mathbf{q}}$ *is the uniquely determined self-adjoint and semibounded operator in* $\mathscr{H}$ *such that* $\operatorname{dom}\mathsf{T}_{\mathbf{q}} \subset \operatorname{dom}\ \mathbf{q}$ *and*

$$\left\langle \mathsf{T}_{\mathbf{q}} a, a \right\rangle = \mathbf{q}(a, b) \qquad \forall a \in \operatorname{dom}\mathbf{q}_1\ \forall b \in \operatorname{dom}\mathbf{q}_2.$$

(ii) $\operatorname{dom}\mathsf{T}_{\mathbf{q}}$ *is a core of* $\mathbf{q}$ *and some* $a \in \operatorname{dom}\mathbf{q}$ *is in* $\operatorname{dom}\mathsf{T}_{\mathbf{q}}$*, if and only if, there is* $b \in \mathscr{H}$ *and a core* D *of* $\mathbf{q}$ *with*

$$\mathbf{q}(a, c) = \langle b, c\rangle \qquad \forall c \in D,$$

and then $\mathsf{T}_{\mathbf{q}} a = b$.

(iii) We have

$$\operatorname{dom}\mathbf{q} = \left\{ a \in \mathscr{H} \,:\, \lim_{t\to 0^+} \left\langle \frac{a - e^{-t\mathsf{T}_{\mathbf{q}}} a}{t}, a \right\rangle < \infty \right\}$$
$$\mathbf{q}(a, a) = \lim_{t\to 0^+} \left\langle \frac{a - e^{-t\mathsf{T}_{\mathbf{q}}} a}{t}, a \right\rangle,$$

(iv) and

$$\begin{aligned}\min \sigma(\mathsf{T}_{\mathbf{q}}) &= \inf\{\mathbf{q}(a, a) \,:\, a \in \operatorname{dom}\mathbf{q},\ \|f\| = 1\}\\ &= \inf\left\{\left\langle \mathsf{T}_{\mathbf{q}} a, a\right\rangle \,:\, a \in \operatorname{dom}\mathbf{q},\ \|a\| = 1\right\}.\end{aligned}$$

Example 1.60. The quadratic form $\mathbf{q}_{\mathsf{T}}$ associated to a self-adjoint operator T. The form domain is given by

$$\operatorname{dom}\mathbf{q}_{\mathsf{T}} := \operatorname{dom}\sqrt{|\mathsf{T}|} = \left\{ \varphi \in \mathscr{H} \,:\, \langle \varphi, |\mathsf{T}|\,\varphi\rangle = \int |x|\, \mathsf{E}_\varphi(\mathrm{d}x) < \infty \right\},$$

where E_φ denotes the spectral measure associated to T and φ. The quadratic form is given by

$$\mathbf{q}_{\mathsf{T}} = \langle \varphi, \mathsf{T}\varphi\rangle \ = \int \lambda\, \mathsf{E}_\varphi(\mathrm{d}\lambda).$$

Notation 1.61. If $\mathbf{q}, \mathbf{q}'$ are symmetric, we write $\mathbf{q} \geqslant \mathbf{q}'$, if and only if, $\operatorname{dom}\ \mathbf{q} \subset \operatorname{dom}\mathbf{q}'$ and $(a, a) \geqslant \mathbf{q}'(a, a)$ for all $a \in \operatorname{dom}\ \mathbf{q}$.

Example 1.62 (Friedrich realisation). Let T be a positive symmetric operator and $\mathbf{q}(a, b) = \langle a, \mathsf{T}b\rangle$ for $a, b \in \operatorname{dom}\mathsf{T}$. Then $\mathbf{q}$ is a closable quadratic form and its closure $\overline{\mathbf{q}}$ is the quadratic form of a unique self-adjoint operator $\overline{\mathsf{T}}$. $\overline{\mathsf{T}}$ is a positive extension of T and the lower bound of its spectrum is the lower bound of $\mathbf{q}$. Further, $\overline{\mathsf{T}}$ is the only self-adjoint extension of T whose domain is contained in the form domain of $\overline{\mathbf{q}}$.

For a Riemmanian manifold M and $\pi : E \to M$ a vector bundle over M which is endowed with a Riemannian connection, let $\Gamma_{\mathrm{L}^2}(E)$ be the Hilbert space of square-integrable sections of E with inner product

$$\langle a, b\rangle_{\Gamma_{\mathrm{L}^2}(E)} := \int_M (a, b)_{E_x}\, \mathrm{vol}(\mathrm{d}x),$$

where $(\cdot,\cdot)_{E_x}$ is the corresponding fibre norm ($x \in M$). The operator $\Box = \operatorname{tr} \nabla^2$ is non-positive and formally self-adjoint, i.e., for all compactly supported $a,\ b \in \Gamma_{C_c^\infty}(E)$,

$$\langle \Box a, b\rangle_{\Gamma_{L^2}(E)} = -\langle \nabla a, \nabla b\rangle_{\Gamma_{L^2}(TM\otimes E)}\,.$$

By the Weitzenböck formula 1.42, we want to construct the canonical self-adjoint extension of $\Box - \mathscr{R}$, where $\mathscr{R} \in \Gamma(TM)$ is assumed to be symmetric: We suppose that $(\Box - \mathscr{R})\big|_{\Gamma_c(E)}$ is bounded from above, i.e.

$$\lambda_0(\mathscr{R}) = \sup_{0\neq a\in\Gamma_c(E)} \frac{\langle(\Box - \mathscr{R})a, a\rangle_{\Gamma_{L^2}(E)}}{\langle a, a\rangle_{\Gamma_{L^2}(E)}} < \infty.$$

We define $\mathscr{E}(a,b) := -\langle \nabla a, \nabla b\rangle_{\Gamma_{L^2}(E)} - \langle \mathscr{R}a, b\rangle_{\Gamma_{L^2}(E)}$ for all $a, b \in \operatorname{dom}\mathscr{E} := \Gamma_c(E)$. Then for any $c > \lambda_0(\mathscr{R})$,

$$\mathsf{q}(a,b) := -\mathscr{E}(a,b) + c\,\langle a,b\rangle_{\Gamma_{L^2}(E)}$$

is a positive quadratic form on $\operatorname{dom}\mathscr{E}$. Completing $\operatorname{dom}\overline{\mathscr{E}} := \overline{\operatorname{dom}\mathscr{E}}^{\mathsf{q}}$ and extending $\mathscr{E}$ by continuity to a closed form $\bar{\mathsf{q}}$ on $\operatorname{dom}\overline{\mathscr{E}}$, we get

$$\overline{\mathscr{E}}(a,b) = \left\langle \overline{(\Box - \mathscr{R})}a, b\right\rangle_{\Gamma_{L^2}(E)}$$

for some self-adjoint operator $\overline{(\Box - \mathscr{R})}$ with form domain $\operatorname{dom}\overline{\mathscr{E}} \subset \Gamma_{L^2}(E)$. This operator is called Friedrichs extension of $(\Box - \mathscr{R})\big|_{\Gamma_c(E)}$.

1.2.5 Smooth heat kernels on vector bundles In this subsection, let $E \to M$ be a smooth metric vector bundle over M and vol the volume measure of M. Moreover, let H be an elliptic, formally self-adjoint and semibounded operator and $\overline{\mathsf{H}}$ its Friedrich realisation. The heat semigroup

$$\left(\mathrm{e}^{-s\overline{\mathsf{H}}}\right) \subset \mathscr{L}(\Gamma_{L^2}(E))$$

is defined by the Spectral Theorem 1.50. A heat semigroup of this kind is always induced by a jointly smooth heat kernel on M reflected in the following Theorem. For a much more general setting cf. e.g. [Gün17a, Chapter II].

Theorem 1.63. *Let H be elliptic and formally self-adjoint and $\overline{\mathsf{H}}$ its Friedrich realisation in $\mathscr{L}(\Gamma_{L^2}(E))$. Then:*

(i) There is a unique smooth map

$$(0,\infty)\times M\times M \ni (s,x,y) \mapsto \mathrm{e}^{-s\overline{\mathsf{H}}}(x,y) \in \operatorname{Hom}(E_y, E_x)$$

the **heat kernel** *of $\overline{\mathsf{H}}$, such that for all $s > 0$, $a \in \Gamma_{L^2}(E)$ and a.e. $x \in M$*

$$\mathrm{e}^{-s\overline{\mathsf{H}}}a(x) = \int_M \mathrm{e}^{-s\overline{\mathsf{H}}}(x,y)a(y)\,\mathrm{vol}(\mathrm{d}y).$$

(ii) For any $a \in \Gamma_{L^2}(E)$, the section

$$(0,\infty)\times M \ni (s,x) \mapsto a(s,x) := \int_M \mathrm{e}^{-s\overline{\mathsf{H}}}(x,y)a(y)\,\mathrm{vol}(\mathrm{d}y) \in E_x$$

is smooth and we have

$$\partial_s a(s,x) = -\overline{\mathsf{H}}a(s,x), \qquad \forall s>0\ \forall x\in M.$$

(iii) For all $s > 0$ and $x \in M$, we get

$$\int_M \left| e^{-s\overline{\mathsf{H}}}(x,z) \right| \operatorname{vol}(dz) < \infty.$$

(iv) For all $s > 0$ and $x, y \in M$, we get adjoints of finite-dimensional operators

$$e^{-s\overline{\mathsf{H}}}(y,x) = e^{-s\overline{\mathsf{H}}}(x,y)^*$$

(v) For all $s, t > 0$, $x, y \in M$, we get the Chapman-Kolmogorov equations

$$e^{-(t+s)\overline{\mathsf{H}}}(x,y) = \int_M e^{-t\overline{\mathsf{H}}}(x,z) e^{-s\overline{\mathsf{H}}}(z,y) \operatorname{vol}(dz).$$

Example 1.64 (continued from Example 1.62). Using the Spectral Theorem 1.50 we define the semigroup on $\Gamma_{L^2}(E)$ by

$$P_s a = e^{\frac{s}{2}\overline{(\square - \mathscr{R})}} a \qquad \forall a \in \Gamma_{C^\infty}(E).$$

By Theorem 1.63, for all $a \in \Gamma_{L^2}(E)$,

(a) If $a \in \overline{(\square - \mathscr{R})}$, then $a \in \Gamma_{C^\infty}(E)$.

(b) The map $(s,x) \mapsto P_s a(x)$ is smooth on $(0,\infty) \times M$, for $a \in \Gamma_{C^\infty}(E)$ on $[0,\infty)$. In addition, there is a kernel $(s,x,y) \mapsto p_s(x,y) \in \operatorname{Hom}(E_y, E_x)$ which is smooth on $(0,\infty) \times M \times M$ such that

$$P_s a(x) = \int_M p_s(x,y) a(y) \operatorname{vol}(dy)$$

for the C^∞-version of $P_s a$.

Remark 1.65. If the manifold M is complete, then $(\square - \mathscr{R})\big|_{\Gamma_c(E)}$ is essentially self-adjoint.

1.3 Stochastic Processes and Brownian Motion on Manifolds

Finally, we recall notions from the theory of probability theory, stochastic processes and stochastic calculus on manifolds. We will see that Brownian motion is an M-valued process that is naturally associated to Laplace-Beltrami operator $\frac{1}{2}\Delta_M$ on M, as solution to the martingale problem and as a stochastic flow of that operator. Thus, Brownian motion will be a local object by definition. However, its stochastic behaviour determines global aspects of the topology and geometry of the manifold.

A brief and concise overview to basic notations of stochastic differential geometry are given in [Tha16]. Moreover we refer the reader to the original work by Émery [Éme89] and Elworthy: a lecture given at St. Flour [Elw82] and the monograph [Elw88]. [HT94, Chapter 7] provides a systematic treatment of the modern differential geometry necessary to understand the notion of stochastic analysis on manifolds. [Hsu02] treats the subject with less generality and requires a less extensive background in differential geometry.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. For a sub-σ-algebra $\mathscr{G} \subset \mathscr{F}$, denote by $\sigma(\mathscr{G})$ the smallest σ-algebra containing $\mathscr{G}$. A **filtration** $(\mathscr{F}_t)_{t \geqslant 0}$ is a family of sub-σ-algebras $\mathscr{F}$ such that $\mathscr{F}_s \subset \mathscr{F}_t$ for all $s \leqslant t$. We set

$$\mathscr{F}_{t+} := \bigcap_{s>t} \mathscr{F}_s \qquad \text{and} \qquad \mathscr{F}_\infty := \sigma\Big(\bigcup_{t\geqslant 0} \mathscr{F}_t\Big).$$

and say the filtration is **right-continuous** if $\mathscr{F}_{t+} = \mathscr{F}_t$ for all $t \geqslant 0$. We call $(\mathscr{F}_t)_{t\geqslant 0}$ **complete** if $\mathscr{F}_0$ contains all subsets of $\mathbb{P}$-null sets i.e.

$$\{M \subset \Omega;\ \exists N \subset \mathscr{F} : M \subset N,\ \mathbb{P}(N) = 0\} \subset \mathscr{F}_0.$$

We always suppose that $\mathscr{F}_t$ satisfies the **usual conditions**, i.e. $\mathscr{F}_t$ is right-continuous and complete. A random variable $\tau : \Omega \to [0, \infty]$ is called **stopping time** if $\{\tau \leqslant t\} \in \mathscr{F}_t$ for all $t \geqslant 0$.

An E-**valued stochastic process** $(X_t)_{t\geqslant 0}$ is a family of random variables $X_t : \Omega \to E$, for all $t \geqslant 0$. The canonical filtration of a process $(X_t)_{t\geqslant 0}$ is given by $\mathscr{F}_t^X := \sigma(X_s, s \leqslant t)$. A stochastic process is **adapted to the filtration** $(\mathscr{F}_t)_{t\geqslant 0}$ if $X_t \in \mathscr{F}_t$, i.e. X_t is $\mathscr{F}_t$-measurable for all $t \geqslant 0$ – which is equivalent to saying that $\mathscr{F}_t^X \subset \mathscr{F}_t$ for all $t \geqslant 0$. If not stated otherwise, we always consider the canonical filtration, i.e. $\mathscr{F}_t = \mathscr{F}_t^X$.

Remark 1.66. Note that we use the different but common notations for the sample space Ω and the differential k-forms $\Omega^k(M)$ (cf. Definition 1.9).

A stochastic process $(X_t)_{t\geqslant 0}$ is called a **martingale** if it is adapted to the filtration $(\mathscr{F}_t)_{t\geqslant 0}$ and $X_t \in \mathrm{L}^1(\mathbb{P})$ for all $t \geqslant 0$ such that[1]

$$\mathbb{E}(X_t \mid \mathscr{F}_s) = X_s \qquad \forall s \leqslant t.$$

A **continuous local martingale** is an adapted continuous process X for which there exists a sequence of stopping times $(\sigma_n)_{n\geqslant 0}$ such that $\sigma_n \uparrow \infty$ a.s. and for every $n \geqslant 0$, the process $X_t^{\sigma_n} \mathbb{1}_{\{\sigma_n>0\}}$ is a (true) martingale. Equality modulo continuous local martingales will be denoted by $\overset{\mathrm{m}}{=}$, i.e. $X \overset{\mathrm{m}}{=} Y$ if and only if $X - Y$ is a continuous local martingale.

A process X has **finite variation** if it is adapted and each **path** $\omega \to X_t(\omega)$ is of bounded variation over every finite time interval a.s.

1.3.1 Semimartingales on a manifold M

Let $\mathcal{S}(E)$ be the family of all continuous semimartingales on a set E, i.e.

$$\mathcal{S} = \mathcal{M} \oplus \mathcal{V}_0, \tag{1.17}$$

where $\mathcal{M}$ is the family of all continuous local martingales and $\mathcal{V}_0$ the family of all continuous finite variation processes starting at zero (a.s.). Such a unique (canonical) decomposition always exists, cf. [Pro05, p. 131] or [RW00, p. 358]. We sometimes suppress the adjective *continuous*.

[1] A short recap of the conditional expectation $\mathbb{E}(X_t \mid \mathscr{F}_s)$ of X_t given $\mathscr{F}_s$ and martingales can be found in Appendix A.1.

Definition 1.67. Let X be a continuous adapted process taking values in a manifold M. Then X is a **semimartingale on** M, denoted $X \in \mathcal{S}(M)$, if the composition

$$f(X) = \big(f(X_t)\big)_{t \geqslant 0} \qquad \forall f \in \mathrm{C}^\infty(M)$$

is a real-valued semimartingale.

Definition 1.68. For $X, Y \in \mathcal{S}(M)$, let

$$X \circ \mathrm{d}Y := X\mathrm{d}Y + \frac{1}{2}[X, Y] \tag{1.18}$$

be the **Stratonovich differential**. Here $X\mathrm{d}Y$ denotes the classical Itô differential and $\mathrm{d}[X, Y] = \mathrm{d}X\mathrm{d}Y$ differential of quadratic covariation of X and Y. The integral

$$\int X \circ \mathrm{d}Y := \int X\mathrm{d}Y + \frac{1}{2}[X, Y]$$

is called the **Stratonovich integral** of X with respect to Y.

Formula (1.18) gives the relation between the Stratonovich integral and the usual Itô integral. The Stratonovich differential is associative, i.e. $X \circ (Y \circ \mathrm{d}Z) = (XY) \circ \mathrm{d}Z$ and respects the product rule, i.e.

$$\mathrm{d}(XY) = X \circ \mathrm{d}Y + Y \circ \mathrm{d}X. \tag{1.19}$$

Proof. By Itô's formula,

$$\mathrm{d}(XY) = X\mathrm{d}Y + Y\mathrm{d}X + \mathrm{d}X\mathrm{d}Y = X \circ \mathrm{d}Y + Y \circ \mathrm{d}X. \qquad \blacksquare$$

Example 1.69. Let $X \in \mathcal{S}(\mathbb{R}^m)$ and $f \in \mathrm{C}^3(\mathbb{R}^m)$. Then

$$\mathrm{d}f(X) = \mathrm{D}_i f(X) \circ \mathrm{d}X^i = (\nabla f(X), \circ \mathrm{d}X) .$$

Hence, the Stratonovich integral obeys the chain rule of classical analysis, so it is more common to work with the Stratonovich integral in the manifold setting.

1.3.2 Diffusions as (stochastic) flows to a PDO and Brownian motion as a flow to $\frac{1}{2}\Delta_M$

In the classical theory, there is a dynamical point of view to vector fields on manifolds: It associates to each vector field a dynamical system given by the flow of the vector field.

Given vector field A on M, we consider the smooth curve $t \mapsto x(t)$ in M via

$$x(0) = x_0 \qquad \text{and} \qquad \dot{x}(t) = A(x(t)).$$

For each $A \in \Gamma(\mathrm{T}M)$, the corresponding **flow curve** (or integral curve) to A at x, $t \mapsto \varphi_t(x) := x(t)$, is given by

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\varphi_t &= A\varphi_t \\ \varphi_0 &= \mathrm{id}_M . \end{aligned} \tag{1.20}$$

For any $f \in \mathrm{C}_c^\infty(M)$, it follows that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} f(\varphi_t) &= Af(\varphi_t) \\ f(\varphi_0) &= f, \end{aligned} \tag{1.21}$$

by the very definition of the exterior derivative 1.13 (d) and the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\varphi_t) = (\mathbf{d}f)_{\varphi_t} \dot{\varphi}_t \overset{(1.20)}{=} (\mathbf{d}f)_{\varphi_t} A\varphi(t) \overset{1.13\,(\mathrm{d})}{=} Af(\varphi_t). \tag{1.22}$$

By integrating the last equation, we may rewrite (1.21) as

$$f(\varphi_t(x)) - f(x) - \int_0^t Af(\varphi_s(x))\mathrm{d}s = 0 \qquad \forall t \geqslant 0 \; \forall x \in M.$$

A natural question to ask is whether there exists a flow to a second order differential operator?

Let A be a second order partial differential operator (PDO) on M, e.g. of the form

$$\mathsf{A} = \sum_{i=1}^{r} A_i^2,$$

where $A_1, ..., A_r \in \Gamma(\mathrm{T}M)$ $(r \in \mathbb{N})$ and $A_i^2 f := A_i(A_i f)$.

Example 1.70. The Laplace operator $\Delta_{\mathbb{R}^m}$ on $M = \mathbb{R}^m$, is defined as the sum of all the unmixed second partial derivatives in the Cartesian coordinates

$$\Delta_{\mathbb{R}^m} = \sum_{i=1}^{m} \partial_i^2 = \sum_{i=1}^{m} \partial_i \partial_i,$$

i.e. $A_0 := 0$ and $A_i = \partial_i$ for $i = 1, \dots, m$.

Definition 1.71 (Stochastic Flow Process). Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathbb{P})$ be a filtered probability space. A continuous adapted (stochastic) process

$$X_\bullet(x) \equiv \big(X_t(x)\big)_{t\geqslant 0}$$

taking values in M is called **flow process to A** (**A-diffusion**) starting in $X_0(x) = x$ if

$$N_t^f(x) := f(X_t(x)) - f(x) - \int_0^t (\mathsf{A}f)(X_s(x))\mathrm{d}s \qquad \forall f \in \mathrm{C}_c^\infty(M) \; \forall t \geqslant 0 \tag{1.23}$$

is a martingale, i.e.

$$\mathbb{E}^{\mathscr{F}_s} \underbrace{\left(f(X_t(x)) - f(X_s(x)) - \int_s^t (\mathsf{A}f)(X_r(x))\mathrm{d}r \right)}_{= N_t^f(x) - N_s^f(x)} = 0 \qquad \forall s \leqslant t.$$

Note that, by definition, flow processes to a second order PDO depend on an additional random parameter $\omega \in \Omega$. In contrast to the classical case, the defining equation (1.22) for flow curves only holds under conditional expectations, i.e. equation (1.22) translates to the martingale property (1.23). The theory of stochastic flows has been studied in detail by Kunita [Kun90].

As a flow to a vector field, a flow process on a manifold may only be defined up to maximal, possibly finite, **lifetime** ζ, i.e.

$$\{\zeta < \infty\} \subset \left\{ \lim_{t\uparrow\zeta} X_t = \infty \text{ in } \hat{M} := M \sqcup \{\infty\} \right\} \qquad \text{a.s.}$$

Then $f(X)$ is well-defined as a process globally in $\mathbb{R}_+$, for all $f \in C_c(\mathbb{R}^m)$ where $\hat{M}$ is the one-point compactification (with convention $f(\infty) := 0$). So, in general,

$$f(X) = \big(f(X_t)\big)_{t\geqslant 0} \qquad \forall f \in C^\infty(M)$$

is only a semimartingale with lifetime ζ.

An important example is the **A**-diffusion generated by $\frac{1}{2}\Delta_M$, namely, a Brownian motion X on M.

Definition 1.72 (Brownian motion on (M,g)). Let (M,g) be a Riemannian manifold and X an adapted M-valued process with maximal lifetime ζ. The process X is a **Brownian motion on** (M,g) if, for every $f \in C^\infty(M)$, the real process

$$f(X) - \frac{1}{2}\int_0^t \Delta_M f(X)\mathrm{d}t, \qquad t < \zeta(x),$$

is a local martingale (with lifetime ζ). The family of all Brownian motions on (M,g) will be denoted by $\mathrm{BM}(M,g)$.

Remark 1.73. Note that Δ_M depends on the Riemannian metric, so a $\mathrm{BM}(M,g)$ is locally controlled by the Riemannian metric and thus a local object by definition. However, its stochastic behaviour determines global aspects of the topology and geometry of the manifold.

1.3.3 Stochastic completeness

As we saw in the previous Subsection § 1.3.2, Brownian motion may explode in finite time. We therefore make the following definition.

Definition 1.74. We say that a Riemannian manifold (M,g) is **stochastically complete** if $\zeta(x) = \infty$ a.s. for all $x \in M$.

Equivalently M is stochastically complete if and only if the (minimal) heat kernel is conservative, i.e. characterised by the parabolic condition on the heat kernel that

$$\int_M p_t(x,y)\,\mathrm{vol}(\mathrm{d}y) = 1 \qquad \forall t > 0\ \forall x \in M.$$

Note that (geodesically) completeness is not sufficient for the stochastically completeness of a Riemannian manifold (M,g). If a Riemannian manifold (M,g) is uniformly complete, then (M,g) is stochastically complete (cf. [Gli97, Theorem 15.2]). A very important and direct consequence is

Theorem 1.75. *Any compact Riemannian manifold is stochastically complete.*

Finally, we point out two sufficient conditions to guarantee stochastic completeness.

Lemma 1.76 (Yau, [Yau78]). *A complete Riemannian manifold is stochastically complete if its Ricci curvature is bounded from below.*

Lemma 1.77 (Grigor'yan, [Gri86]). *Let (M, g) be a complete Riemannian manifold. Let* $\mathrm{B}(x_0, r)$ *be the geodesic ball centred at some point $x_0 \in M$ of metric radius r. If*

$$\int_0^\infty \frac{r\,\mathrm{d}r}{\log \mathrm{vol}(B(x_0, r))} = \infty$$

then (M, g) is stochastically complete.

1.3.4 Stochastic differential equations (SDEs) on a manifold M A flow process may only have a finite lifetime ζ. Then ζ is a predictable stopping time and X defined on $[0, \zeta)$ such that on $\{\zeta < \infty\}$ holds: $X_t \xrightarrow{\text{a.s.}} \infty$ in the one-point compactification of M at $t \nearrow \zeta$. In this case there exists a continuous extension $(X_t)_{t \geqslant 0}$ with values in $\hat{M}$ by setting $X_t(\omega) := \infty$ for $t \geqslant \zeta(\omega)$ and $f(\infty) := 0$ by definition, $f \in \mathrm{C}_c^\infty(M)$.

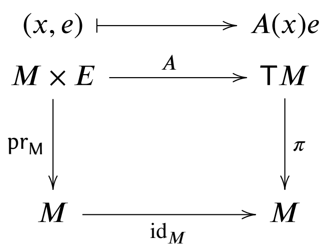
Definition 1.78. The pair (A, Z) is called a **stochastic differential equation on a manifold** M (SDE on M) if

(i) Z is a continuous semimartingale with values in a finite dimensional real vector space E,

(ii) $A : M \times E \to \mathrm{T}M$ is a vector bundle homomorphism on M.

We will denote the SDE (A, Z) as $\mathrm{d}X = A(X) \circ \mathrm{d}Z$. Herein, «$\circ$» denotes the Stratonovich circle.

More precisely, condition (ii) constitutes the following commutative diagramme



and for every $x \in M$ the map $A(x) : E \to \mathrm{T}_x M$ is linear on each fibre; in particular $A(\cdot)e \in \Gamma(\mathrm{T}M)$ for $e \in E$. The semimartingale $Z = (Z_t)_{t \geqslant 0}$ is defined on a standard filtrated probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geqslant 0}, \mathbb{P})$ and we can write $Z = Z^i e_i$, where $(e_i)_{1 \leqslant i \leqslant r}$ is any basis for E and Z^i are real semimartingales.

Definition 1.79. Let (A, Z) be an SDE and $x_0 : \Omega \to M$ an $\mathscr{F}_0$-measurable random variable. A **solution to the stochastic differential equation**

$$\mathrm{d}X = A(X) \circ \mathrm{d}Z \tag{1.24}$$

with **initial condition** $X_0 = x_0$ is a continuous adapted process $(X_t)_{t < \zeta}$ with values in M such that for every test function $f \in \mathrm{C}_c^\infty(M)$ the composed process $f(X)$ is a real semimartingale and satisfies the integral equation

$$f(X_\tau) = f(x_0) + \int_0^\tau (\mathbf{d}f)_X A(X) \circ \mathrm{d}Z, \quad \mathbb{P}-\text{a.s.}, \tag{1.25}$$

for every stopping time τ with $0 \leqslant \tau < \zeta$. A solution to (1.24) with maximal lifetime is called **maximal solution** of the SDE (1.24); then we say the SDE is **nonexplosive**. In this case (if necessary after passing over to the extension of f on $\hat{M}$ and X on $[0,\infty)\times\Omega$) up to indistinguishability

$$f(X_t) = f(X_0) + \int_0^t (\mathbf{d}f)_X A(X) \circ \mathrm{d}Z, \quad t \geqslant 0. \tag{1.26}$$

More precisely, maximal lifetime of the continuous M-valued process X means that

$$\{\zeta < \infty\} \subset \left\{ \lim_{t \nearrow \zeta} X_t = \infty \text{ in } \hat{M} \right\}, \quad \mathbb{P}-\text{a.s.} \tag{1.27}$$

A solution to (1.24) is a semimartingale on M by definition (in the sense of L. Schwartzm cf. Definition 1.67): Every adapted M-valued process X is a **semimartingale on** M if for every $f \in \mathrm{C}_c^\infty(M)$ the composition $f(X)$ is a real-valued semimartingale. Mind that for the maximal lifetime of X the semimartingale $f(X)$ is well-defined on the hole line $[0,\infty)$. Moreover, the compositions $f(X)$ with smooth functions f are real semimartingales but, in general, only defined up to the lifetime of X.

For every $x \in M$ the composition

$$E \xrightarrow{A(x)} \mathrm{T}_x M \xrightarrow{(\mathbf{d}f)_x} \mathbb{R}$$

is linear by definition. Therefore, if we write the semimartingale Z with a fixed basis $(e_i)_{1\leqslant i\leqslant r}$ for E as $Z = Z^i e_i$, we get

$$(\mathbf{d}f)_X A(X) \circ \mathrm{d}Z \equiv (\mathbf{d}f)_X A(X) e_i \circ \mathrm{d}Z^i.$$

The bundle homomorphism A is naturally determined through the vector fields $A_i := A(\cdot)e_i$ for $i = 1,\dots,r$. Thus, we can symbolically write (1.24) as

$$\mathrm{d}X = A_i(X) \circ \mathrm{d}Z^i, \tag{1.28}$$

which should be read, for every test function $f \in \mathrm{C}_c^\infty(M)$, as

$$\mathrm{d}f(X) = (\mathbf{d}f)_X A_i(X) \circ \mathrm{d}Z^i.$$

But $(\mathbf{d}f)_x A_i(x) = (A_i f)(x)$ so that the equation above is equal to

$$\mathrm{d}f(X) = (A_i f)(X) \circ \mathrm{d}Z^i, \quad f \in \mathrm{C}_c^\infty(M).$$

Conversely, for a fixed basis $(e_i)_{1\leqslant i\leqslant r}$ for E and arbitrary vector fields $A_1,\dots,A_r \in \Gamma(\mathrm{T}M)$ on M there is a unique bundle homomorphism $A \in \Gamma(\mathrm{Hom}(M\times E, \mathrm{T}M))$ with $A_i := A(\cdot)e_i$. Thus, the equations (1.24) and (1.28) are equivalent. Consequently, without loss of generality we can assume $E = \mathbb{R}^m$.

Example 1.80. Let $E = \mathbb{R}^{m+1}$ and $Z = (t, Z^1, \dots, Z^m)$, where Z^i are real semimartingales. Let $A_0, A_1, \dots, A_m \in \Gamma(\mathrm{T}M)$ be given. Then (1.28) reads as

$$\mathrm{d}X = A_0(X)\mathrm{d}t + A_i(X) \circ \mathrm{d}Z^i. \tag{1.29}$$

Thus, the composition $f(X)$ is a real semimartingale, for every $f \in \mathrm{C}_c^\infty(M)$, and

$$\mathrm{d}f(X) = (A_0 f)(X)\mathrm{d}t + (A_i f)(X)\mathrm{d}Z^i + \frac{1}{2}\mathrm{d}\left((A_i f)(X)\right)\mathrm{d}Z^i,$$

by the conversion formula (1.18) from Stratonovich to Itô differentials. Since[2]

$$\mathrm{d}(A_i f(X)) = (A_0 A_i f)(X)\mathrm{d}t + \sum_{j=1}^m (A_j A_i f)(X) \circ \mathrm{d}Z^j,$$

we get $\mathrm{d}\left((A_i f)(X)\right)\mathrm{d}Z^i = \sum_{j=1}^m (A_j A_i f)(X)\mathrm{d}Z^j\mathrm{d}Z^i$, i.e.

$$\mathrm{d}f(X) = (A_0 f)(X)\mathrm{d}t + \frac{1}{2}(A_i A_j f)(X)\mathrm{d}[Z^i, Z^j] + (A_i f)(X)\mathrm{d}Z^i.$$

In particular, if we set $Z = (t, B^1, \dots, B^m)$ where B is an m-dimensional Brownian motion we get, for every $f \in \mathrm{C}_c^\infty(M)$,

$$\mathrm{d}f(X) = (A_0 f)(X)\mathrm{d}t + \frac{1}{2}\sum_{i=1}^m (A_i^2 f)(X)\mathrm{d}t + (A_i f)(X)\mathrm{d}B^i,$$

where we used $\mathrm{d}B^i\mathrm{d}B^j = \delta_{ij}\mathrm{d}t$. But this means for $\mathsf{A} = A_0 + \frac{1}{2}\sum_1^m A_i^2$

$$\mathrm{d}f(X) - (\mathsf{A}f)(X)\mathrm{d}t = \mathrm{d}(\text{martingale}).$$

Corollary 1.81. *Every maximal solution to the SDE*

$$\mathrm{d}X = A_0(X)\mathrm{d}t + A_i(X) \circ \mathrm{d}B^i, \qquad X_0 = x \in M$$

is a flow process X staring in x with generator $\mathsf{A} = A_0 + \frac{1}{2}\sum_{i=1}^m A_i^2$.

Theorem 1.82 (Existence and Uniqueness). *Let (A, Z) be an SDE on M and x_0 an $\mathscr{F}_0$-measurable random variable. Then there exists a unique maximal solution X of (1.24) with lifetime $\zeta > 0$ $\mathbb{P}$-a.s. and initial condition $X_0 = x_0$. Uniqueness holds in the following sense: For any other solution $(Y_t)_{t<\tau}$ of (1.24) with the same initial condition, it holds $(X_t)_{t<\tau} = Y$ $\mathbb{P}$-a.s. for every $\tau \leqslant \zeta$.*

The proof of Theorem 1.82 is based on the famous Whitney Embedding Theorem (cf. e.g. [Lee13, Theorem 6.15]) that every smooth m-manifold (with or without boundary) admits a proper smooth embedding into $\mathbb{R}^{2m+1}$ considered as a closed submanifold.

The idea is simple. Taking such a (Whitney) embedding $\imath$, we can identify M with its image $M \overset{\imath}{\longrightarrow} \imath(M) \subset \mathbb{R}^{2m+1}$, so that it is a submanifold of $\mathbb{R}^{2m+1}$. Using a C^∞ partition of unity to extend A to a map on $\mathbb{R}^{2m+1} \times \mathbb{R}^{2m+1}$ and see that if X is a solution to (1.24) on M with $X_0 = x_0$, then $\overline{X} := \imath \circ X$ is a solution to the new SDE $\mathrm{d}X = \overline{A}(X) \circ \mathrm{d}Z$ on $\mathbb{R}^{2m+1}$

[2] Note that i is fixed in this line.

with $\overline{X}_0 = \iota \circ x_0$. Therefore, also uniqueness follows. The main problem is to show that $\{t < \zeta\} \subset \{X_t \in M\}$ holds for every solution $\overline{X}$. This approach is often called *extrinsic*, since it relies on embedding the manifold in the ambient Euclidean space by a proper extension.

A solution X of (1.24) on M is, by definition, an M-valued semimartingale in the sense that all compositions $f(X)$ with $f \in \mathrm{C}^\infty(M)$ are continuous real semimartingales on $[0,\zeta)$ (with ζ the lifetime of X). The converse is also true:

Theorem 1.83 (M-valued semimartingales as solutions of SDEs). *Every semimartingale on a manifold can be written as the solution to* (1.24).

1.3.5 Γ-operators and quadratic variation on a manifold M

Definition 1.84. Let $\mathbf{L} : \mathrm{C}^\infty(M) \to \mathrm{C}^\infty(M)$ be linear. The **Γ-operator associated to $\mathbf{L}$** (or **l'operator carré du champ**) is a bilinear map

$$\Gamma : \mathrm{C}^\infty(M) \times_{\mathrm{C}} \mathrm{C}^\infty(M) \to \mathrm{C}^\infty(M)$$
$$\mathbf{\Gamma}(f,g) := \frac{1}{2}\left(\mathbf{L}(fg) - f\mathbf{L}g - g\mathbf{L}f\right).$$

Example 1.85. Let $\mathbf{L}$ be a second order PDO on M without constant term (i.e. $\mathbf{L}1 = 0$). In a local chart (x,U) for L the operator $\mathbf{L}$ can be written as

$$\mathbf{L}\big|_{\mathrm{C}^\infty_U(M)} = a^{ij}\partial_i\partial_j + b^i\partial_i,$$

where $\mathrm{C}^\infty_U(M) := \{f \in \mathrm{C}^\infty(M) : \operatorname{supp} f \subset U\}$. Then

$$\mathbf{\Gamma}(f,g) = a^{ij}(\partial_i f)(\partial_j g) \qquad \forall f,g \in \mathrm{C}^\infty_U(M).$$

In the special case of $M = \mathbb{R}^m$ and $L = \Delta_{\mathbb{R}^m}$, we find $\mathbf{\Gamma}(f,f) = \|\nabla f\|^2$.

Remark 1.86. Let $\mathbf{L}$ be a second order PDO. Then

$$\mathbf{\Gamma}(f,g) = 0 \quad \forall f,g \in \mathrm{C}^\infty(M) \qquad \Longleftrightarrow \qquad \mathbf{L} \in \mathbf{\Gamma}(\mathrm{T}M), \text{ i.e. is of first order.}$$

For example,

$$\mathbf{L} = A_0 + \sum_{i=1}^r A_i^2 \qquad \Longrightarrow \qquad \mathbf{\Gamma}(f,g) = \sum_{i=1}^r (A_i f)(A_i g).$$

In particular

$$\Gamma \equiv 0 \qquad \Longleftrightarrow \qquad A_1 = A_2 = \cdots = A_r = 0.$$

Proposition 1.87. *Let* $\mathbf{L} : \mathrm{C}^\infty(M) \to \mathrm{C}^\infty(M)$ *be linear and* $X \in \mathcal{S}(M)$ *such that*

$$N_t^f := f(X_t) - f(X_0) - \int_0^t \mathbf{L}f(X_r)\,\mathrm{d}r \qquad \forall f \in \mathrm{C}^\infty(M)$$

is a continuous local martingale (of same lifetime as X*). Then, for all* $f,g \in \mathrm{C}^\infty(M)$*, the* **quadratic variation** $[f(X), g(X)]$ *of* $f(X)$ *and* $g(X)$ *is given by*

$$\mathrm{d}\,[f(X), g(X)] \equiv \mathrm{d}\left[N_t^f, N_t^g\right] = 2\mathbf{\Gamma}(f,g)(X)\mathrm{d}t.$$

In particular, $\mathbf{\Gamma}(f,f)(X) \geqslant 0$ *a.s.*

Lemma 1.88. *For an $\mathbb{R}$-linear map $\mathsf{L} : \mathrm{C}^\infty(M) \to \mathrm{C}^\infty(M)$ the following are equivalent:*

(i) L *is a second order PDO (without constant term)*

(ii) L *satisfies the second order chain rule, i.e.*

$$\mathsf{L}\varphi(f) = \mathrm{D}_i\varphi(f)(\mathsf{L}f^i) + \mathrm{D}_i\mathrm{D}_j\varphi(f)\Gamma(f^i, f^i) \qquad \forall f \in \mathrm{C}^\infty(M, \mathbb{R}^r)\ \forall \varphi \in \mathrm{C}^\infty(\mathbb{R}^r).$$

Corollary 1.89. *Let $\mathsf{L} : \mathrm{C}^\infty(M) \to \mathrm{C}^\infty(M)$ be an $\mathbb{R}$-linear mapping. Suppose that for each $x \in M$ there is an $X \in \mathcal{S}(M)$ such that $X_0 = x$ and such that*

$$f(X_t) - f(x) - \int_0^t \mathsf{L}f(X_r)\mathrm{d}r \qquad \forall f \in \mathrm{C}^\infty(M)$$

is a local martingale. Then L is a PDO of order at most 2. In addition,

$$[f(X), f(X)] = 0 \quad \forall f \in \mathrm{C}^\infty(M) \qquad \Longleftrightarrow \qquad \mathsf{L} \text{ is first order.}$$

1.3.6 Quadratic variation and integration of 1-forms

For a Brownian motion $\mathrm{BM}(M, g)$ on a manifold M, we also find Lévy's characterisation. To this end, we need a generalisation of the quadratic variation for a semimartingale on M. But on a manifold, the usual notion of multiplication does not make sense, so the idea is to replace it by a twice covariant tensor.

Proposition 1.90. *Let $X \in \mathcal{S}(M)$. Then there exists a unique linear map*

$$\Gamma(\mathrm{T}^*M \otimes \mathrm{T}^*M) \to \mathcal{A},$$

denoted by $b \mapsto \int b(\mathrm{d}X, \mathrm{d}X)$, such that, for all $f, g \in \mathrm{C}^\infty(M)$,

$$\mathbf{d}f \otimes \mathbf{d}g \mapsto [f(X), g(X)] \tag{1.30}$$

$$f \cdot b \mapsto \int f(X) b(\mathrm{d}X, \mathrm{d}X). \tag{1.31}$$

By definition, $b(\mathrm{d}X, \mathrm{d}X) := \mathrm{d}\int b(\mathrm{d}X, \mathrm{d}X)$. The quadratic variation $\int b(\mathrm{d}X, \mathrm{d}X)$ depends only on the symmetric part of b. In particular, if b is antisymmetric, then $\int b(\mathrm{d}X, \mathrm{d}X) = 0$.

The process $\int b(\mathrm{d}X, \mathrm{d}X)$ is said to be the **integral of b along X** or **b-quadratic variation of X**. Its value at time t will be denoted $\int_0^t b(\mathrm{d}X_s, \mathrm{d}X_s)$ instead of $\left(\int b(\mathrm{d}X, \mathrm{d}X)\right)_t$.

Proposition 1.91. *Let $f : M \to N$ be a smooth map between manifolds and $b \in \Gamma(\mathrm{T}^*N \otimes \mathrm{T}^*N)$. For any $X \in \mathcal{S}(M)$, we have*

$$\int f^*b(\mathrm{d}X, \mathrm{d}X) = \int b\left(\mathrm{d}f(X), \mathrm{d}f(X)\right). \tag{1.32}$$

Proposition 1.92. *Let $X \in \mathcal{S}(M)$. Then there exists a unique linear map*

$$\Omega^1(M) \to \mathcal{S}, \qquad \eta \mapsto \int \eta(\circ \mathrm{d}X) =: \int_X \eta$$

such that, for all $f \in \mathrm{C}^\infty(M)$,

$$\mathbf{d}f \mapsto f(X) - f(X_0) \tag{1.33}$$

$$f\eta \mapsto \int f(X) \circ \eta(\circ \mathrm{d}X). \tag{1.34}$$

By definition, $f(X) \circ \eta(\circ \mathrm{d}X) := f(X) \circ \mathrm{d}\left(\int_X \eta\right)$.

The process $\int_X \eta$ is called **Stratonovich integral of the (differential) form η along X.**

Example 1.93. Let X be a smooth deterministic M-valued curve, i.e. $\mathcal{S}(M) \ni X_t = x(t)$, then

$$\int_X \eta = \int \eta(\dot{x}(t))\mathrm{d}t, \quad \eta \in \Omega^1(M).$$

1.3.7 Stochastic parallel transport and stochastically moving frames The problem in former definition of Brownian motion (Definition 1.72) lies in the manifold itself: There does not exist a Hörmander-type representation of the Laplace-Beltrami operator (there is no canonical way of writing Δ_M as sums of squares) if M is not parallelisable, i.e. the tangent bundle $\mathrm{T}M \xrightarrow{\pi} M$ is not trivial. But it holds the fundamental relation (cf. Theorem 1.101 below)

$$\Delta_{\mathcal{O}(M)}\pi^* = \pi^*\Delta_M, \tag{1.35}$$

i.e. there exists a *lifted* version of the Laplace-Beltrami operator, called *horizontal Laplacian*, on the orthonormal frame bundle $\mathcal{O}(M) \to M$ over M. Each element $u \in \mathcal{O}(M)$ is an isometry $u : \mathbb{R}^m \to \mathrm{T}_{\pi(u)}M$. The set of tangent vectors of horizontal curves passing through a fixed point $u \in \mathcal{O}(M)$ is the horizontal splitting $H_u\mathcal{O}(M)$ with

$$\mathrm{T}_u\mathcal{O}(M) = H_u\mathcal{O}(M) \oplus V_uO(M),$$

and m well-defined unique horizontal vectors $L_i(u) \in H_u\mathcal{O}(M)$ whose projection is the ith unit vector ue_i of the orthonormal frame, i.e. $\pi_* L_i(u) = ue_i$, where (e_i) is the canonical basis for $\mathbb{R}^m$. Using this relation, it is due to Malliavin, Eells and Elworthy that there always exists a lifted Brownian motion as solution to the globally defined SDE

$$\mathrm{d}U_t = L_i(U_t) \circ \mathrm{d}B_t^i,$$

where B is an m-dimensional Brownian motion. A solution is a diffusion generated by $\Delta_{\mathcal{O}(M)}$. By Itô's formula for $\widetilde{f} \in \mathrm{C}^\infty(\mathcal{O}(M))$

$$\mathrm{d}\widetilde{f}(U_t) = L_i\widetilde{f}(U_t)\mathrm{d}B_t^i + \frac{1}{2}\Delta_{\mathcal{O}(M)}\widetilde{f}(U_t)\mathrm{d}t.$$

Applying this to the lift $\widetilde{f} := f \circ \pi$ we get, using (1.35),

$$\mathrm{d}f(X_t) = L_i f(X_t)\mathrm{d}B_t^i + \frac{1}{2}\Delta_M f(X_t)\mathrm{d}t,$$

where $X_t = \pi(U_t)$ is the projection of the lifted Brownian motion U_t on the manifold M. It follows that X_t is a Brownian motion on M starting from $X_0 = \pi(U_0)$. Therefore, the key idea was to solve conversely the SDE on the orthonormal frame bundle $\mathcal{O}(M)$ and project the solution back down to M by $\pi : \mathcal{O}(M) \to M$, cf. [Elw82], [Mal78].

In geometrical terms, the idea is to «roll» our manifold M by means of the (stochastic) parallel transport along the paths of an $\mathbb{R}^m$-valued Brownian motion («rolling without slipping»), known as *stochastic development*. Starting in $x \in M$, the resulting Brownian motion X on M can be thought of as footprints left behind by the paths of the Euclidean Brownian motion B in the tangent space $\mathsf{T}_x M \cong \mathbb{R}^m$ if M is rolled along the paths of B. The procedure is known as *Cartan development* in the deterministic case. We will see that it can be adopted to work with a suitable Stratonovich SDE.

For an m-dimensional manifold we denote by $P = \mathscr{F}(\mathsf{T}M)$ its frame bundle, the prototypical example of a G-principal bundle $\pi : P \to M$ whose structure group is the general linear group $G := \mathrm{GL}(m, \mathbb{R})$. For $x \in M$, the fibre P_x consists of linear isomorphisms $u : \mathbb{R}^m \to \mathsf{T}_x M$ (the **frames** of $\mathsf{T}_x M$), where $u \in P_x$ can be identified with a basis $(e_1, \dots, e_m)$ for $\mathbb{R}^m$ via

$$(u_1, \dots, u_m) := (u \lhd e_1, \dots, u \lhd e_m),$$

i.e. $\mathrm{GL}(m, \mathbb{R})$ acts on $\mathscr{F}(\mathsf{T}M)$ from the right

$$u \lhd g : \mathbb{R}^m \xrightarrow{g} \mathbb{R}^m \xrightarrow{u} \mathsf{T}_x M,$$

where $g = (g_{ij}) \in G$. Then $ug \in \mathrm{GL}(M)$ by $(ug)_j = \sum_i g_{ij} u_i$.

Restricting the structure group $\mathrm{GL}(n, \mathbb{R})$ to $\mathrm{O}(n)$, the frames $u \in P_x$ at a point $x \in M$ become isometries. We call $P = \mathscr{O}(M) \to M$ consisting of all frames $u \in P_x$ at a point $x \in M$, i.e. linear isometries $u : \mathbb{R}^m \to \mathsf{T}_x M$, the **orthonormal frame bundle over** M.

Remark 1.94. For simplicity, we restrict ourselves to the Riemannian case (M, g) with a Levi-Civita connection $\nabla = \nabla^{\mathrm{LC}}$. More generally, M may be a smooth manifold equipped torsion-free connection, then the frame bundle $P = \mathscr{F}(\mathsf{T}M)$, considered as a manifold, is also parallelisable but every $u \in P$ is read as an isomorphism.

A linear connection $\mathsf{T}M$ induces canonically a G-**connection in** P given as a G-invariant differentiable splitting h in the following exact sequence of vector bundles over P:

$$0 \longrightarrow \ker \mathrm{d}\pi \longrightarrow \mathsf{T}P \underset{\overset{\nwarrow}{h}}{\xrightarrow{\mathrm{d}\pi}} \pi^* \mathsf{T}M \longrightarrow 0$$

The splitting (1.3.7) induces a decomposition of $\mathsf{T}P$

$$\mathsf{T}P = V \oplus H := \ker \mathrm{d}\pi \oplus h(\pi^* \mathsf{T}M).$$

For each $u \in P$, the **horizontal space** H_u **at** u is constituted via the G-invariance: $H_{u \lhd g} P = (\lhd g)_* H_u P$, for the G-right action $\lhd g$ on M. The **vertical space** V_u **at** u is given by $V_u = \{ v \in \mathsf{T}_u P : (\mathrm{d}\pi) v = 0 \}$. The bundle isomorphism

$$h : \pi^* \mathsf{T}M \xrightarrow{\sim} H \hookrightarrow \mathsf{T}P$$

is called **horizontal lift** of the G-connection, i.e. fibrewise it is given as

$$h_u : \mathsf{T}_{\pi(u)} M \xrightarrow{\sim} H_u.$$

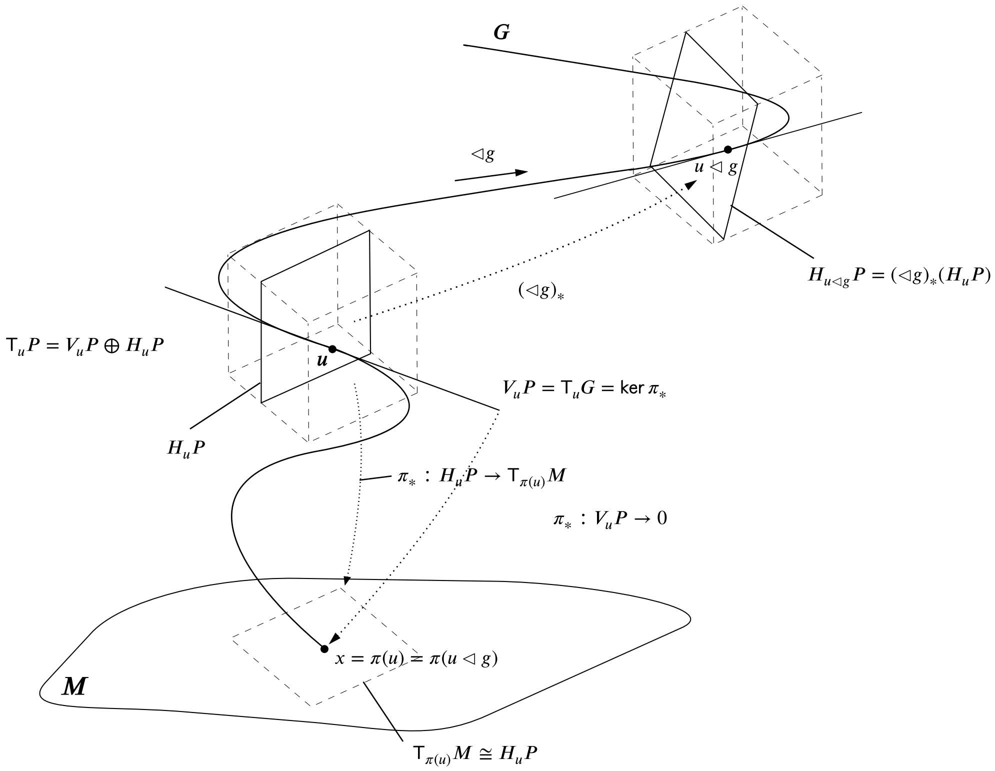


Figure 1.1: G-connection of a principal G-bundle

By means of the G-connection in P each vector field $X \in \Gamma(\mathsf{T}P)$ may be decomposed in a horizontal and a vertical part:

$$X = X^{\text{hor}} + X^{\text{vert}}.$$

Every $u \in P$ defines an embedding $I_u : G \hookrightarrow P$, $g \mapsto u \triangleleft g$. Its differential at the Einselement $e \in \mathfrak{g}$,

$$\iota_u := (\mathrm{d}I_u)_e : \mathsf{T}_eG \longrightarrow \mathsf{T}_uP, \quad A \mapsto \tilde{A}(u), \tag{1.36}$$

provides an identification $\kappa_u : \mathfrak{g} \xrightarrow{\sim} V_u$ from the Lie algebra $\mathfrak{g} := \mathsf{T}_eG$ of G with vertical fibre V_u at u. The vertical vector field $\tilde{A} \in \Gamma(\mathsf{T}P)$, defined by (1.36), is the **standard vertical vector field on P associated to $A \in \mathfrak{g}$**. By

$$\tilde{\omega}_u(X_u) := \kappa_u^{-1}(X^{\text{vert}})_u, \quad X \in \Gamma(\mathsf{T}P), \tag{1.37}$$

we define a $\mathfrak{g}$-valued 1-form $\tilde{\omega} \in \Gamma(\mathsf{T}^*P \otimes \mathfrak{g})$ on P, called **connection form** of the G-connection. The connection form is by definition **horizontal**, i.e. $\tilde{\omega}(X) = 0$ if and only if X is a horizontal vector field on P.

We get the following Theorem (cf. [Tha16, Remark 5.23] or [HT94, Satz 7.131]).

Theorem 1.95. *The orthonormal frame bundle $P = \mathcal{O}(M)$ as a Riemannian manifold is parallelisable, i.e. the tangent bundle $\mathsf{T}\mathcal{O}(M) \to \mathcal{O}(M)$ is a trivial bundle.*

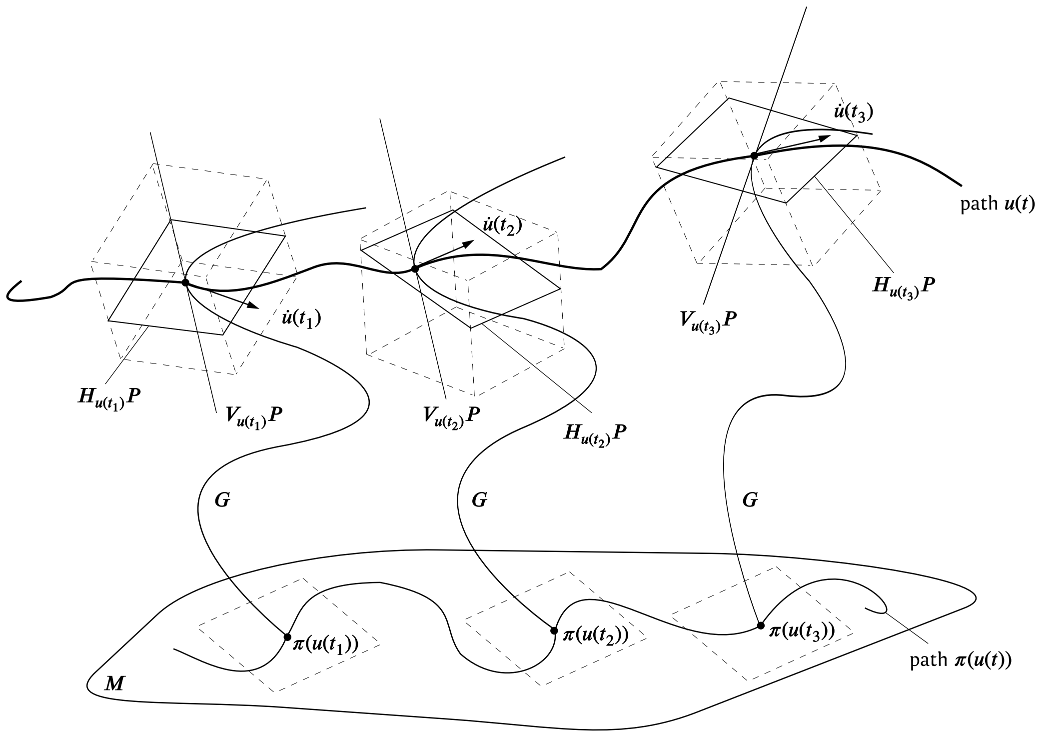


Figure 1.2: Horizontal lift $u(t)$ through principal G-bundle

Proof. Let $P = \mathcal{O}(M)$. Choose a G-connection in P with $G = \mathrm{O}(n)$ and split $\mathsf{T}P = V \oplus H$. A canonical trivialisation for $\mathsf{T}P$ is given as follows: the vertical subbundle V is trivialised by the standard vertical vector fields $\tilde{A}$ to A, where A passes through a basis for $\mathfrak{g}$. The horizontal subbundle H is trivialised by the **standard horizontal vector fields** $L_1, \dots, L_n$ in $\Gamma(\mathsf{T}P)$, defined by $L_i(u) := h_u(ue_i)$. Then, for every $u \in P$,

$$\big(\tilde{A}(u), L_i(u) \,:\, A \in \text{basis for } \mathfrak{g},\; i = 1, \dots, m\big)$$

is a basis for $\mathsf{T}_u P = V_u \oplus H_u$, since $\mathfrak{g} \xrightarrow{\sim} V_u$, $A \mapsto \tilde{A}(u)$ and $h_u \,:\, \mathsf{T}_{\pi(u)} M \xrightarrow{\sim} H_u$ are isomorphisms. ■

Recall that we restrict ourselves to the principal G-bundle $P = \mathcal{O}(M)$ over a Riemannian manifold M with $G = \mathrm{O}(n)$. The associated Lie algebra is given by the algebra of matrices $\mathfrak{g} = \big\{A \in \mathrm{M}(n \times n; \mathbb{R}) \,:\, A \text{ skew-symmetric}\big\}$. Fix a G-connection in P with

$$\vartheta \in \Gamma(\mathsf{T}^* P \otimes \mathbb{R}^m), \quad \vartheta_u(X_u) := u^{-1}(\mathbf{d}\pi X_u), \quad u \in P \text{ with } X \in \Gamma(\mathsf{T}P), \tag{1.38}$$

the so called **canonical** 1-**form** on the principal bundle $\pi : P \to M$. Note that the definition of a connection form depends on the G-connection, but not the canonical 1-form ϑ.

Theorem 1.96. *Let $\pi : P \to M$ be a principal G-bundle over M with a G-connection. Let $x : I \to M$, $t \mapsto x(t)$ be a smooth curve and $t_0 \in I$. Then there exists for every $u_0 \in P$ with $\pi(u_0) = x(t_0)$ a unique horizontal curve $u : I \to P$ with $u(t_0) = u_0$ above $t \mapsto x(t)$, i.e. $\pi \circ u(t) = x(t)$ and $\dot u(t) \in H_{u(t)}$ for every $t \in I$.*

Corollary 1.97. *Let P be the principal G-bundle over a manifold M. Every G-connection in P naturally defines a parallel displacement on P along smooth curves $t \mapsto x(t)$ in M, namely for $t_0, t_1 \in I$ as*

$$\overline{/\!/_{t_0,t_1}} : P_{x(t_0)} \xrightarrow{\sim} P_{x(t_1)}, \quad u_0 \mapsto u(t_1), \tag{1.39}$$

where $t \mapsto u(t)$ is the uniquely determined horizontal lift of $t \mapsto x(t)$ on P with $u(t_0) = u_0$.

The standard vertical, and horizontal vector fields respectively, are given by

$$\vartheta(\tilde{A}) = 0 \text{ and } \vartheta(L_i) = e_i \quad \text{resp.} \quad \tilde{\omega}(\tilde{A}) = 0 \text{ and } \tilde{\omega}(L_i) = 0.$$

Definition 1.98. The second order differential operator

$$\Delta_{\mathcal{O}(M)} := \sum_{i=1}^{n} L_i^2$$

is called **horizontal Laplacian** on $\mathcal{O}(M)$.

For every $U \in \mathcal{S}(P)$ the Stratonovich integral $\int_U \tilde{\omega}$ provides a semimartingale with values in the Lie algebra $\mathfrak{g}$ (namely component-by-component with respect for a basis $\mathfrak{g}$). We call U **horizontal** if $\int_U \tilde{\omega} = 0$ a.s. If $X \in \mathcal{S}(M)$, then we call $U \in \mathcal{S}(P)$ **horizontal lift of** X if U is horizontal and $\pi \circ U = X$ a.s.

Obviously, the concept of horizontal lifts of semimartingales generalises the concept of horizontal lifts of M-valued smooth curves (cf. Theorem 1.96) according to which a curve $t \mapsto u(t)$ above $t \mapsto x(t)$ is horizontal, i.e. $\pi \circ u = x$ and $\tilde{\omega}(\dot{u}) = 0$ (cf. Example 1.93).

Definition 1.99 (Anti-development). Let $X \in \mathcal{S}(M)$ and U its horizontal lift with values in $P = \mathcal{O}(M)$. The $\mathbb{R}^m$-valued semimartingale

$$Z = \int_U \vartheta = \int \vartheta(\circ \mathrm{d}U)$$

is the $\mathbb{R}^m$**-anti-development of** X (with initial basis U_0). In particular, with respect to the standard basis $\mathbb{R}^m$ we get $Z = (Z^1, \dots, Z^n)$ with $Z^i = \int_U \vartheta^i$.

The next, fundamental theorem shows the existence of horizontal lifts to M-valued semimartingales (cf. [Tha16, Theorem 5.30] or [HT94, Satz 7.141]).

Theorem 1.100. *Let P be a principal G-bundle over a manifold M with G-connection. Let x_0 be an M-valued random variable and u_0 a P-valued random variable above x_0, i.e. $\pi \circ u_0 = x_0$ a.s. Then, for every M-valued semimartingale X with $X_0 = x_0$ there is a unique horizontal lift U on P with $U_0 = u_0$ a.s.*

Theorem 1.101. *Let M be a Riemannian manifold equipped with Levi-Civita connection. Then*

$$\Delta_{\mathcal{O}(M)} \pi^* = \pi^* \Delta_M. \tag{1.40}$$

Proof. For $u \in \mathcal{O}(M)$, we have

$$\sum_{i=1}^{n} L_i^2(f \circ \pi)(u) = \sum_{i=1}^{n} \nabla \mathbf{d} f(ue_i, ue_i) = (\operatorname{tr} \nabla \mathbf{d} f)\pi(u) = (\Delta_M f) \circ \pi(u). \qquad \blacksquare$$

Let us briefly summarise the construction: For a semimartingale $X \in \mathcal{S}(M)$, its horizontal lift U to $P = \mathcal{O}(\mathsf{T}M)$ and anti-development $Z = \int_U \vartheta$ into $\mathbb{R}^m$ each of the three processes X, U, Z defines the other two (modulo choice of initial conditions $X_0 = x$ and $U_0 = u$) in the following way:

(a) Z determines U as solution to the SDE $\mathrm{d}U = L_i(U) \circ \mathrm{d}Z^i$ with $U_0 = u_0$,

(b) U determines X by $X = \pi \circ U$,

(c) X determines Z as $Z = \int_U \vartheta$, where U is the uniquely determined horizontal lift of X with $U_0 = u_0$.

Mind that this procedure depends only trivially on the choice of u_0 above x_0. Usually, one starts vice versa, cf. [Tha16, Theorem 5.35]: Choose a continuous $\mathbb{R}^m$-valued semimartingale Z with $Z_0 = 0$ and fix an $\mathscr{F}_0$-measurable random variable $u_0 : \Omega \to P$ as initial value. Define U on P as the maximal solution to

$$\mathrm{d}U = L_i(U) \circ \mathrm{d}Z^i, \quad U_0 = u_0,$$

and set $X := \pi \circ U$ as the projection from U on M with initial value $X_0 = \pi \circ u_0$. Thus, we obtain a horizontal process U over X with $U_0 = u_0$. In particular,

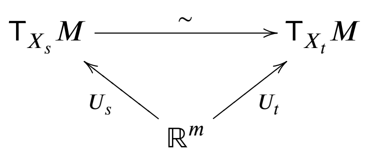
$$\mathrm{d}X = Ue_i \circ \mathrm{d}Z^i = U \circ \mathrm{d}Z.$$

Since $L_i(U) \circ \mathrm{d}Z^i = h_U(Ue_i) \circ \mathrm{d}Z^i$, also

$$\mathrm{d}U = h_U(\circ \mathrm{d}X).$$

Hence, we regain the original process up to the lifetime of U by the $\mathbb{R}^m$-anti-development $Z = \int_U \vartheta$ of X. We say X is the **stochastic development** of Z.

Definition 1.102 (Stochastic parallel transport). Let $X \in \mathcal{S}(M)$ and U a horizontal lift of X on $\mathcal{O}(M)$. For every $0 \leqslant s \leqslant t$, we define $/\!/_{s,t} := U_t \circ U_s^{-1}$ by



and $/\!/_{t,s} := /\!/_{s,t}^{-1}$. We call the isometries $/\!/_t := /\!/_{0,t} : \mathsf{T}_{X_0}M \to \mathsf{T}_{X_t}M$ **stochastic parallel transport along** X.

The stochastic parallel transport leads to an intrinsic version of Itô's formula: For $f \in \mathsf{C}^2(M)$ and for all $t \geqslant 0$, we get almost surely

$$f(X_t) = f(X_0) + \int_0^t U_s e_i f(X_s) \mathrm{d}Z_s^i + \frac{1}{2} \int_0^t U_s e_i U_s e_j f(X_s) \mathrm{d}\left[Z^i, Z^j\right]_s, \tag{1.41}$$

where $X \in \mathcal{S}(M)$ with a horizontal lift U and anti-development Z, and $(e_1, \dots, e_m)$ is an orthonormal basis for $\mathbb{R}^m$. Note that we employ the usual Einstein summation convention over repeated indices. More succinctly (1.41) can be written as

$$\mathrm{d}f(X_t) = \langle \nabla f(X_t), U_t \mathrm{d}Z_t \rangle + \frac{1}{2} \operatorname{tr} \nabla \mathbf{d}_{X_t} f(U_t, U_t) \mathrm{d}[Z]_t.$$

Example 1.103. If $M = \mathbb{R}^m$ then, we can choose $U_s = \mathrm{id}_{\mathbb{R}^m}$ and $Z = X$ and this formula reduces to

$$f(X_t) = f(X_0) + \int_0^t \partial_i f(X_s) \mathrm{d}X_s^i + \frac{1}{2} \int_0^t \partial_{ij} f(X_s) \mathrm{d}\left[X^i, X^j\right]_s.$$

for any $t \geqslant 0$ almost surely and $X \in \mathcal{S}(\mathbb{R}^m)$ is an $\mathbb{R}^m$-valued continuous semimartingale.

Chapter 2

Bismut Formulae and Gradient Estimates

In this chapter we will derive localised Bismut formulae and prove gradient estimates for the heat semigroup defined by spectral calculus on the full exterior bundle of square-integrable Borel forms $\Gamma_{L^2}(\bigwedge T^*M)$.

The so called Bismut(-type) formulae provide derivative formulae of heat (diffusion) semigroups on manifolds. First introduced by Bismut [Bis84] in 1984, they have been extended to various frameworks: Notably, by Elworthy & Li [EL94a; EL94b], Thalmaier [Tha97] and in the general setting using martingale methods for sections of vector bundles by Driver & Thalmaier [DT01].

In this chapter, let (M,g) be a complete smooth Riemannian manifold without boundary and $(\cdot,\cdot)$ its Riemannian metric. We write vol for the corresponding volume measure. On a vector bundle $E \to M$ the corresponding fibre norms are denoted by $|\cdot| := \sqrt{(\cdot,\cdot)}$ and $\Gamma(E) := \Gamma_{C^\infty}(E)$ denotes all *smooth* sections of E and $\Gamma_{L^2}(E)$ the L^2-section of E.

We write $\Omega_{L^2}(M) := \Gamma_{L^2}(\bigwedge T^*M)$ for the complex separable Hilbert space of equivalence classes of square-integrable Borel forms on M such that

$$\|\alpha\|^2 := \|\alpha\|_{\Omega_{L^2}(M)} := \int_M |\alpha(x)|^2 \operatorname{vol}(\mathrm{d}x) < \infty,$$

with inner product

$$\langle \alpha, \beta \rangle_g := \langle \alpha, \beta \rangle_{\Omega_{L^2}(M)} := \int_M (\alpha(x), \beta(x)) \operatorname{vol}(\mathrm{d}x).$$

Analogously, we write $\Omega^k_{L^2}(M)$ for the Hilbert space of Borel k-forms. In particular,

$$\Omega_{L^2}(M) = \bigoplus_{k=0}^{m} \Omega^k_{L^2}(M).$$

To relax notation, we set

$$\Omega(M) := \Omega_{C^\infty}(M) \qquad \text{and} \qquad \Omega^k(M) := \Omega^k_{C^\infty}(M).$$

for the set of all *smooth* forms, and *smooth* k-forms respectively, on (M,g).

Let $(\Omega, \mathscr{F}, (\mathscr{F}_s)_{s\geqslant 0}, \mathbb{P})$ be a filtered probability space satisfying the usual hypotheses. Let $X(x)$ be a $\mathrm{BM}(M,g)$ starting at $x \in M$ and $\zeta(x)$ its maximal lifetime. Further, let B the stochastic anti-development of X to T_xM, which is a standard Brownian motion on $T_xM \cong \mathbb{R}^m$.

Let $E, \widetilde{E}$ be two Riemannian vector bundles over M, endowed with a metric connection ∇^E and $\nabla^{\widetilde{E}}$ respectively. The corresponding parallel transport will be specified by a superscript, e.g. $/\!/^E_s : E_x \to E_{X_s}$. The covariant derivatives ∇^{TM}, ∇^E and $\nabla^{\widetilde{E}}$ induce covariant derivatives on any vector bundle $\mathscr{E}$ over M constructed via the tensor product of the bundles $\mathrm{T}M$, E and $\widetilde{E}$ and their dual bundles. To relax notation, the corresponding induced covariant derivative on this bundle will be denoted ∇ and the corresponding stochastic parallel transport by $/\!/_s : \mathscr{E}_x \to \mathscr{E}_{X_s}$.

Given multiplication map $m \in \Gamma(\mathrm{Hom}(\mathrm{T}^*M \otimes E, \widetilde{E})) \cong \Gamma(\mathrm{T}M \otimes E^* \otimes \widetilde{E})$, we consider the Dirac-type operator

$$\mathbf{D}_m := m\nabla : \Gamma(E) \to \Gamma(\widetilde{E})$$

which is understood as the composition

$$\Gamma(E) \xrightarrow{\nabla^E} \Gamma(\mathrm{T}^*M \otimes E) \xrightarrow{m} \Gamma(\widetilde{E}).$$

A multiplication map m is said to be compatible with ∇ provided $\nabla m = 0$, i.e.

$$\nabla_v^{\widetilde{E}}(m_U\alpha) = m_{\nabla_v^{\mathrm{T}M}U}\alpha + m_U(\nabla_v^E\alpha) \qquad \forall U \in \Gamma(\mathrm{T}M)\ \forall\alpha \in \Gamma(E)\ \forall v \in \mathrm{T}M,$$

where $m_v\xi := m((v,\cdot)\otimes\xi) \in \widetilde{E}_x$ for all $\xi \in E_x$.

The horizontal Laplacian $\square$ is the second order differential operator given by the following decomposition

$$\square := \nabla^*\nabla : \Gamma(E) \xrightarrow{\nabla^E} \Gamma(\mathrm{T}^*M \otimes E) \xrightarrow{\nabla^{\mathrm{T}^*M\otimes E}} \Gamma(\mathrm{T}^*M \otimes \mathrm{T}^*M \otimes E) \xrightarrow{\mathrm{tr}} \Gamma(E).$$

Driver and Thalmaier [DT01, p. 48] propose the following formalism: Let $\mathbf{L}$ and $\widetilde{\mathbf{L}}$ are given second order differential operators on $\Gamma(E)$ and $\Gamma(\widetilde{E})$ respectively that satisfy the following two conditions.

(1) The operators $\mathbf{D}_m$, $\mathbf{L}$ and $\widetilde{\mathbf{L}}$ obey the commutation rule, for some $\varrho \in \Gamma(\mathrm{Hom}(E,\widetilde{E}))$,

$$\mathbf{D}_m\mathbf{L} = \widetilde{\mathbf{L}}\mathbf{D}_m - \varrho. \tag{2.1}$$

(2) The operators $\mathscr{R} := \square - \mathbf{L} : \Gamma(E) \to \Gamma(E)$ and $\widetilde{\mathscr{R}} := \widetilde{\square} - \widetilde{\mathbf{L}} : \Gamma(\widetilde{E}) \to \Gamma(\widetilde{E})$ are zeroth order operators, i.e. $\mathscr{R}$ and $\widetilde{\mathscr{R}}$ are section in $\Gamma(\mathrm{End}\,E)$ and $\Gamma(\mathrm{End}\,\widetilde{E})$, provided m is compatible with the Levi-Civita connection.

In geometrically natural situations, we have $\varrho = 0$ or $\rho \in \Gamma(\mathrm{Hom}(E,\widetilde{E}))$ is of zeroth order. Under those assumptions, Driver and Thalmaier [DT01] can prove derivative formulae for the heat semigroup in the general setting of vector bundles using martingale methods. For a detailed discussion we refer the reader to [DT01].

Let us note two important examples.

Example 2.1. The exterior bundle of total forms $E = \bigwedge \mathrm{T}^*M \to M$ with its natural connection

$$\nabla^{\bigwedge \mathrm{T}^*M} := \bigoplus_{k=0}^{m} \nabla^{\bigwedge^k \mathrm{T}^*M}$$

and Clifford action $c : \mathrm{T}M \to \mathrm{End}(\bigwedge \mathrm{T}^*M)$, $c(\alpha)\beta := \alpha\wedge\beta - \alpha^\sharp \lrcorner \beta$. The Hodge Laplacian Δ is related to the horizontal Laplacian $\square$ by the Weitzenböck formula (cf. Theorem 1.42)

$$\Delta = \square - \mathscr{R}, \tag{2.2}$$

where Weitzenböck curvature operator $\mathscr{R} \in \Gamma(\operatorname{End}\Omega_{C^\infty}(M))$ is a symmetric field of endomorphisms. Acting on k-forms, the field of endomorphisms is specified again by an index

$$\mathscr{R}^{(k)} := \mathscr{R}|_{\Omega^k_{C^\infty}(M)}.$$

In particular, note that $\mathscr{R}^{(1),\mathrm{tr}} = \mathrm{Ric}$ and $\mathscr{R}^{(0),\mathrm{tr}} = 0$. Moreover, it can be written explicitly (cf. e.g. [DT01, Lemma A.7]), for any orthonormal basis $(e_k)_{1\leqslant k\leqslant m}$,

$$\mathscr{R}^{(k)} = -\sum_{i,j=1}^{m} \mathsf{R}(e_j, e_i)(e_j^\flat \wedge \bullet)(e_i \lrcorner \bullet),$$

where $\mathsf{R}(e_j, e_i)$ is the curvature tensor acting on k-forms (cf. [DT01, Lemma A.9]). Then

$$\mathbf{D}_g \equiv \mathbf{D}_c = \mathbf{d} + \boldsymbol{\delta}_g, \qquad \mathbf{L} = \widetilde{\mathbf{L}} = \Delta, \qquad m = c \quad \text{and} \quad \varrho = 0.$$

In particular, for $E := \bigwedge^k \mathsf{T}^*M$ and $\widetilde{E} := \bigwedge^{k+1} \mathsf{T}^*M$, then $\quad \varrho = 0 \quad$ with

$$\mathbf{D}_m = \mathbf{d}|_{\Omega^k}, \qquad \mathbf{L} = -\Delta^{(k)}, \quad \widetilde{\mathbf{L}} = -\Delta^{(k+1)},$$
$$\mathscr{R} = \mathscr{R}^{(k)}, \quad \widetilde{\mathscr{R}} = \mathscr{R}^{(k+1)}, \qquad m(\alpha \otimes \beta) = \alpha \wedge \beta.$$

If instead $\widetilde{E} := \bigwedge^{k-1} \mathsf{T}^*M$, then again $\quad \varrho = 0 \quad$ but with

$$\mathbf{D}_m = -\boldsymbol{\delta}|_{\Omega^k}, \qquad \mathbf{L} = -\Delta^{(k)}, \quad \widetilde{\mathbf{L}} = -\Delta^{(k-1)},$$
$$\mathscr{R} = \mathscr{R}^{(k)}, \quad \widetilde{\mathscr{R}} = \mathscr{R}^{(k-1)}, \qquad m(\alpha \otimes \beta) = -(\alpha^\sharp \lrcorner \beta).$$

Example 2.2 ([DT01, cf. Proposition 2.15]). Let $\widetilde{E} = \mathsf{T}^*M \otimes E$ and $m = \mathrm{id}_{\widetilde{E}}$. For $\mathbf{D}_m = \nabla$ and given $\mathscr{R} \in \operatorname{End} E$, we set

$$\widetilde{\mathscr{R}} = \mathrm{Ric}^{\mathrm{tr}} \otimes \mathbf{1}_E - 2\mathsf{R}^E \cdot + \mathbf{1}_{\mathsf{T}^*M} \otimes \mathscr{R} \in \Gamma(\operatorname{End} \widetilde{E}), \tag{2.3}$$
$$\varrho = \nabla \cdot \mathsf{R}^E + \nabla^{\operatorname{End} E} \mathscr{R} \in \Gamma(\operatorname{Hom}(E, \widetilde{E})), \tag{2.4}$$

where R^E denotes the Riemannian curvature tensor to ∇ on E and

$$\mathrm{Ric}^{\mathrm{tr}} \in \Gamma(\operatorname{End} \mathsf{T}^*M)$$

denotes the transpose of the Ricci curvature tensor $\mathrm{Ric} \in \Gamma(\operatorname{End} \mathsf{T}M)$ on M. Further for any $\eta \in \widetilde{E}_x$, $v \in \mathsf{T}_xM$, $\alpha \in E_x$, (e_i) an orthonormal frame for T_xM, we have

$$\left(\mathsf{R}^E \cdot \eta\right)(v) := \sum_{i=1}^{m} \mathsf{R}^E(v, e_i)\eta(e_i),$$
$$\left(\nabla \cdot \mathsf{R}^E \alpha\right)(v) := \sum_{i=1}^{m} \nabla_{e_i} \mathsf{R}^E(e_i, v)\alpha,$$
$$(\nabla \mathscr{R} \alpha)(v) := (\nabla_v \mathscr{R})\alpha.$$

Choosing $m = \mathrm{id}$, for $\mathbf{L} := \square - \mathscr{R}$, $\widetilde{\mathbf{L}} = \square - \widetilde{\mathscr{R}}$, it follows that $\varrho \in \Gamma(\operatorname{Hom}(E, \widetilde{E}))$.

Next, we introduce the stochastic representation of the semigroup. To this end, recall that $/\!/_s^E : E_x \to E_{X_s}$ is the parallel transport along our diffusion $X_s = X_s(x)$ started at $x \in M$. We define the linear operators on E_x and $\widetilde{E}_x$, respectively,

$$\mathscr{R}_{/\!/_s} := (/\!/_s^E)^{-1} \mathscr{R} \, /\!/_s^E \qquad \text{and} \qquad \widetilde{\mathscr{R}}_{/\!/_s} := (/\!/_s^{\widetilde{E}})^{-1} \widetilde{\mathscr{R}} \, /\!/_s^{\widetilde{E}}$$

along the paths of $X(x)$ in the following way: Via the stochastic parallel transport we get to a random point on the tangent space at X_s and apply the curvature (in case $k = 1$ just Ric and $E = \mathrm{T}M$) considered as a linear transformation. Then we parallel transport back to where the diffusion started. Thus, we get a linear mapping $E_x \to E_x$ which now depends on random, i.e.:

$$\begin{array}{ccc} E_x & \overset{\mathscr{R}_{/\!/_s^E}}{\dashrightarrow} & E_x \\ {\scriptstyle /\!/_s^E}\downarrow & & \uparrow{\scriptstyle /\!/_s^{E,-1}} \\ E_{X_s} & \xrightarrow[\mathscr{R}_{X_s}]{} & E_{X_s} \end{array}$$

Let $\mathcal{Q}_s$ be the End(E_x)-valued, and $\widetilde{\mathcal{Q}}_s$ the End$(\widetilde{E}_x)$-valued respectively, pathwise solutions to the ordinary differential equations

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{Q}_s &= -\frac{1}{2}\mathscr{R}_{/\!/_s}\mathcal{Q}_s, & \mathcal{Q}_0 &= \mathrm{id}_{E_x}, \\ \frac{\mathrm{d}}{\mathrm{d}s}\widetilde{\mathcal{Q}}_s &= -\frac{1}{2}\widetilde{\mathscr{R}}_{/\!/_s}\widetilde{\mathcal{Q}}_s, & \widetilde{\mathcal{Q}}_0 &= \mathrm{id}_{\widetilde{E}_x}. \end{aligned} \tag{2.5}$$

The composition $\mathcal{Q} \circ /\!/^{E,-1}$ maps from a random point X_s back to the starting point x and is called the (inverse) **damped parallel transport** along the paths of $X(x)$.

2.0.1 Probabilistic representation of the semigroup Given a potential $w : M \to \mathbb{C}$, the Feynman-Kac semigroup

$$P_s^w f(x) := \mathbb{E}\left(\mathrm{e}^{-\frac{1}{2}\int_0^s w(X_u(x))\mathrm{d}u} f(X_s(x))\mathbb{1}_{\{s<\zeta(x)\}}\right)$$

acts on (bounded) measurable functions f on M. Further, let $\underline{\mathscr{R}}^E = \sigma_{\min}(\mathscr{R}^E)$, i.e.

$$\underline{\mathscr{R}}^E(x) := \min\left\{\left(\mathscr{R}_x v, v\right), v \in E_x, |v| = 1\right\}.$$

By uniform continuity, $\underline{\mathscr{R}}^E$ is a continuous function on M. By Gronwall's inequality,

$$\left|\mathcal{Q}_s\right|_{\mathrm{op}} \leqslant \exp\left(-\frac{1}{2}\int_0^s \underline{\mathscr{R}}^E(X_u(x))\mathrm{d}u\right).$$

We have the following probabilistic representation of the semigroup, for all $s > 0$ and every $x \in M$,

$$P_s a(x) = \int_M p_s(x,y)a(y)\,\mathrm{vol}_g(\mathrm{d}y) = \mathbb{E}^x\left(\mathcal{Q}_s^{\mathrm{tr}} /\!/_s^{E,-1} a(X_s)\mathbb{1}_{\{s<\zeta\}}\right) \qquad \forall a \in \Gamma_{\mathrm{L}^2}(E),$$

provided the scalar semigroup $P_s^{\underline{\mathscr{R}}^E}\,|\alpha|\,(x) < \infty$ (cf. [DT01, Theorem B.4.]). In particular, it holds semigroup domination

$$\left|P_s a(x)\right| \leqslant P_s^{\underline{\mathscr{R}}^E}\,|a|\,(x). \tag{2.6}$$

2.0.2 Kato classes and semigroup domination The existence of the Feynman-Kac semigroup will be essential to prove the Bismut-type formulae below (cf. Theorem 2.19 and Theorem 2.25). To ensure the existence of the scalar semigroup in formula (2.6), we therefore will always assume that $\underline{\mathscr{R}}^E \in \mathsf{K}(M)$ is in the so called Kato class:

Definition 2.3 (Kato class). Let $w : M \to \mathbb{C}$ be a measurable function. Then w is said to be in the **contractive Dynkin class** $\mathsf{D}(M)$ (also **extended Kato class**), if there is a $s > 0$ with

$$\sup_{x\in M} \int_0^s \mathbb{E}^x \left(\mathbb{1}_{\{u<\zeta\}} \left| w(X_u) \right| \right) \mathrm{d}u < 1,$$

and w is in the **Kato class** $\mathsf{K}(M)$, if

$$\lim_{s\searrow 0} \sup_{x\in M} \int_0^s \mathbb{E}^x \left(\mathbb{1}_{\{u<\zeta\}} \left| w(X_u) \right| \right) \mathrm{d}u = 0.$$

The function w is said to be in the **local Dynkin class** $\mathsf{D}_{\mathrm{loc}}(M)$ or **local Kato class** $\mathsf{K}_{\mathrm{loc}}(M)$, if $\mathbb{1}_K w \in \mathsf{D}(M)$ or $\mathbb{1}_K w \in \mathsf{K}(M)$, respectively, for all compact $K \subset M$.

Remark 2.4. (i) The Kato class $\mathsf{K}(M)$ plays an important rôle in the study of Schrödinger operators and their associated semigroups, cf. [AS82; Sim82]. The contractive Dynkin class $\mathsf{D}(M)$ appears in [Voi86] to study properties of semigroups associated to Schrödinger operators. In the case of a non-compact manifold, it is well-known that there are many technical difficulties with the behaviour of the potentials at ∞. The Kato class defines a sufficiently rich class of potentials for which we can still expect the Feynman-Kac formula to make sense pointwise not only vol-a.e. x, but *for all* $x \in M$.

(ii) In particular (cf. [Gün17a, Remark VI.2.]), in the Euclidean space $\mathbb{R}^m$, we get $\mathsf{L}^q(\mathbb{R}^m) \subset \mathsf{K}(\mathbb{R}^m)$, for $m \geqslant 2$ and $q > \frac{m}{2}$. Then it is well-known, that the Coulomb potential $\frac{1}{|x|}$ is in $\mathsf{K}(\mathbb{R}^3)$.

(iii) Clearly, all four classes depend on the Riemannian structure of M and we have

$$\mathsf{K}(M) \subset \mathsf{D}(M) \quad \text{and} \quad \mathsf{K}_{\mathrm{loc}}(M) \subset \mathsf{D}_{\mathrm{loc}}(M).$$

In view of those implications and since it is more common to work with Kato classes, we note that in what follows all assumptions may be relaxed from $\mathsf{K}(M)$ to $\mathsf{D}(M)$.

Lemma 2.5 ([Gün17a, Lemma VI.8.]). *For any $w \in \mathsf{K}(M)$ and $\gamma > 1$, there is $c_\gamma = c_\gamma(w) > 0$, such that*

$$\sup_{x\in M} \mathbb{E}^x \left(\mathbb{1}_{\{s<\zeta\}} \mathrm{e}^{\int_0^s |w(X_u)| \mathrm{d}u} \right) \leqslant \gamma \mathrm{e}^{s c_\gamma} < \infty, \qquad \forall s \geqslant 0. \tag{2.7}$$

Remark 2.6. The previous Lemma 2.5 can be elaborated in the case of potentials in the Dynkin class (cf. [Gün17a, Lemma VI.8.]), namely: For any $w \in \mathsf{D}(M)$ there are $c_k = c_k(w) > 0$, for $k \in \{1,2\}$, such that

$$\sup_{x\in M} \mathbb{E}^x \left(\mathbb{1}_{\{s<\zeta\}} \mathrm{e}^{\int_0^s |w(X_u)| \mathrm{d}u} \right) \leqslant c_1 \mathrm{e}^{s c_2} < \infty, \qquad \forall s \geqslant 0.$$

2.0.3 Corresponding sesquilinear forms Let $\mathbf{H}^\nabla$ denote the Friedrichs realisation of $\frac{1}{2}\square = \frac{1}{2}\nabla^*\nabla$ and $\mathbf{q}^\nabla$ the closed densely defined symmetric sesquilinear form corresponding to $\mathbf{H}^\nabla$ given by

$$\operatorname{dom}\mathbf{q}^\nabla = \operatorname{dom}\sqrt{\mathbf{H}^\nabla}$$
$$\mathbf{q}^\nabla(a_1,a_2) := \left\langle \sqrt{\mathbf{H}^\nabla}a_1, \sqrt{\mathbf{H}^\nabla}a_2 \right\rangle = \frac{1}{2}\int_M \left(\nabla a_1, \nabla a_2\right) \,\mathrm{d}\,\mathrm{vol}\,.$$

A Borel section $V : M \to \operatorname{End} E$ in $\operatorname{End} E \to M$ is a **potential** on $E \to M$ if $V(x) = V(x)^*$ for all $x \in M$, where the adjoint is taken fibrewise with respect to the fixed metric on $E \to M$. We define

$$\operatorname{dom}\mathbf{q}_V := \left\{ a \in \Gamma_{\mathrm{L}^2}(E) : \int_M |(Va,a)| \,\mathrm{d}\,\mathrm{vol} < \infty \right\}$$
$$\mathbf{q}_V(a_1,a_2) := \int_M \left(Va_1, a_2\right) \mathrm{d}\,\mathrm{vol}\,.$$

Assume V admits a decomposition $V = V_+ + V_-$ into potentials $V_\pm$ on $E \to M$ with $V_\pm \geqslant 0$ such that $\left|V_+\right| \in \mathrm{L}^1_{\mathrm{loc}}(M)$ and $\mathbf{q}_{V_-}$ is $\mathbf{q}^\nabla$-bounded with bound < 1 (cf. Definition 1.54). Then we denote by $\mathbf{H}^\nabla_V$ the semibounded from below, self-adjoint operator in $\Gamma_{\mathrm{L}^2}(E)$ corresponding to the closed symmetric semibounded densely defined sesquilinear form

$$\mathbf{q}^\nabla_V := \mathbf{q}^\nabla + \mathbf{q}_V = \mathbf{q}^\nabla + \mathbf{q}_{V_+} + \mathbf{q}_{V_-}$$

It follows that the symmetric form $\mathbf{q}^\nabla_V$ is densely defined and by definition we have

$$\operatorname{dom}\mathbf{q}^\nabla_V := \operatorname{dom}\mathbf{q}^\nabla \cap \operatorname{dom}\mathbf{q}_V = \operatorname{dom}\mathbf{q}^\nabla \cap \operatorname{dom}\mathbf{q}_{V_+} \cap \operatorname{dom}\mathbf{q}_{V_-}$$

The following theorem will be used in the proof of the main result and can be found in [Gün17a, Theorem VII.4.].

Theorem 2.7. *Let V be a potential on $E \to M$ such that the potential $V = V_+ + V_-$ can be decomposed into potentials $V_\pm \geqslant 0$ with $\left|V_+\right| \in \mathrm{L}^1_{\mathrm{loc}}(M)$ and $\left|V_-\right| \in \mathrm{K}(M)$ (or $\left|V_-\right| \in \mathrm{D}(M)$ respectively). Then the form $\mathbf{q}_{V_-}$ is infinitesimally $\mathbf{q}^\nabla$-bounded (or $\mathbf{q}^\nabla$-bounded with bound < 1).*

2.0.4 Kato-Simon inequalities and semigroup domination By [Gün17a, Theorem VII.8.] we immediately get the following

Lemma 2.8 (Kato-Simon inequality). *Assume $\mathscr{R}^{(k)} \geqslant -K$ for some constant $K \geqslant 0$, and let $\alpha \in \Omega_{\mathrm{L}^2}(M)$. Then for all $s > 0$, we have*

$$\left|\mathrm{e}^{-s\Delta^{(k)}}\alpha\right| \leqslant \mathrm{e}^{-Ks}\mathrm{e}^{-s\Delta^{(0)}}\,|\alpha|\,. \tag{2.8}$$

2.1 Bismut type Formulae and Derivative Formulae on Vector Bundles

Next, we will outline the strategy of [DT01] to prove Bismut type formulae. Briefly, the idea is to define a suitable martingale, say N_u, and stay on the *local* martingale level as long as possible. Using that a *true* martingale has constant expectation, one then shows that N_u is indeed a martingale and takes expectations at times $u = 0$ and $u = s \wedge \tau$. Note that this method solely involves the geometry and applies *especially* in the case of non-compact manifolds.

Definition 2.9. A **finite energy process** $(\ell_u)_{0\leqslant u\leqslant s}$ **with values in** E is a bounded adapted process with sample paths in the Cameron-Martin space $\mathrm{L}^{1,2}([0,s],E)$.

We recall that a time-dependent section $(a_u)_{0\leqslant u\leqslant s} \in \Gamma(E)$ is said to be **smooth** if $(s,x) \mapsto a_s(x)$ is infinitely differentiable for $(s,x) \in (0,s)\times M$ and with derivative extending continuously to $[0,s)\times M$. The following Lemma can be found in [DT01, Proposition 3.2].

Lemma 2.10. *Let m be a multiplication map, ϱ, L and $\widetilde{\mathsf{L}}$ as in (2.1), and $\mathcal{Q}$ and $\widetilde{\mathcal{Q}}$ are defined by (2.5). Suppose further that $(a_u)_{0\leqslant u\leqslant s} \in \Gamma(E)$ is smooth time-dependent and satisfies the backwards heat equation*

$$\partial_u a_u + \frac{1}{2}\mathsf{L} a_u = 0. \tag{2.9}$$

For some u in the stochastic interval $[0, \zeta(x)\wedge s)$, let

$$N_u := \mathcal{Q}_u^{\mathrm{tr}} /\!\!/_u^{-1} a_u(X_u(x)) \qquad \text{and} \qquad \widetilde{N}_u := \widetilde{\mathcal{Q}}_u^{\mathrm{tr}} /\!\!/_u^{-1} \mathsf{D} a_u(X_u(x)).$$

Then the Itô differentials of N_u and $\widetilde{N}_u$ are given by

$$\mathrm{d}N_u = \mathcal{Q}_u^{\mathrm{tr}} /\!\!/_u^{-1} \nabla_{/\!\!/_u^{TM}\mathrm{d}B_u} a_u(X_u(x))$$

and

$$\mathrm{d}\widetilde{N}_u = \widetilde{\mathcal{Q}}_u^{\mathrm{tr}} /\!\!/_u^{-1} \nabla_{/\!\!/_u^{TM}\mathrm{d}B_u} \mathsf{D} a_u(X_u(x)) + \frac{1}{2}\widetilde{\mathcal{Q}}_u^{\mathrm{tr}} /\!\!/_u^{-1} (\varrho a_u)(X_u(x))\mathrm{d}u.$$

Proof. By Itô's lemma and its product rule for Itô differential (1.19)

$$\begin{aligned}
\mathrm{d}N &= \mathcal{Q}^{\mathrm{tr}} /\!\!/^{-1} \nabla_{/\!\!/^{TM}\mathrm{d}B} a(X(x)) \\
&\quad + \frac{1}{2}\left(-\mathcal{Q}^{\mathrm{tr}} \mathscr{R}_{/\!\!/} /\!\!/^{-1} a(X(x)) + \mathcal{Q}^{\mathrm{tr}} /\!\!/^{-1} \square a(X(x)) - \mathcal{Q}^{\mathrm{tr}} /\!\!/^{-1} \mathsf{L} a(X(x))\right) \mathrm{d}u \\
&= \mathcal{Q}^{\mathrm{tr}} /\!\!/^{-1} \nabla_{/\!\!/^{TM}\mathrm{d}B} a(X(x)),
\end{aligned}$$

where the last equality follows from

$$-\mathscr{R}_{/\!\!/} /\!\!/^{-1} + /\!\!/^{-1}\square = /\!\!/^{-1}(\square - \mathscr{R}) = /\!\!/^{-1}\mathsf{L}.$$

Similarly,

$$\begin{aligned}
\mathrm{d}\widetilde{N} &= \widetilde{\mathcal{Q}}^{\mathrm{tr}} /\!\!/^{-1} \nabla_{/\!\!/^{TM}\mathrm{d}B} \mathsf{D} a(X(x)) \\
&\quad + \frac{1}{2}\left(-\widetilde{\mathcal{Q}}^{\mathrm{tr}} \widetilde{\mathscr{R}}_{/\!\!/} /\!\!/^{-1} \mathsf{D} a(X(x)) + \widetilde{\mathcal{Q}}^{\mathrm{tr}} /\!\!/^{-1} \widetilde{\square} \mathsf{D} a(X(x)) - \widetilde{\mathcal{Q}}^{\mathrm{tr}} /\!\!/^{-1} \mathsf{D}\mathsf{L} a(X(x))\right) \mathrm{d}u \\
&= \widetilde{\mathcal{Q}}^{\mathrm{tr}} /\!\!/^{-1} \nabla_{/\!\!/\mathrm{d}B} \mathsf{D} a(X(x)) + \frac{1}{2}\widetilde{\mathcal{Q}}^{\mathrm{tr}} /\!\!/^{-1} (\varrho a)(X(x))\mathrm{d}u
\end{aligned}$$

where in contrast

$$-\widetilde{\mathscr{R}}_{/\!\!/} /\!\!/^{-1}\mathsf{D} + /\!\!/^{-1}\widetilde{\square}\mathsf{D} - /\!\!/^{-1}\mathsf{D}\mathsf{L} = /\!\!/^{-1}\left(\widetilde{\mathsf{L}}\mathsf{D} - \mathsf{D}\mathsf{L}\right) = /\!\!/^{-1}\varrho.$$

■

The next theorem shows that a suitable dual pair of N and $\widetilde{N}$ in Lemma 2.10 constitutes a local martingale.

Theorem 2.11 ([DT01, Theorem 3.7]). *Let a, N and $\widetilde{N}$ be as in Lemma 2.10, $\ell_u \in \widetilde{E}^*_x$ be a finite energy process and define the $\widetilde{E}^*_x$-valued process*

$$U^\ell_s := \int_0^s (Q^{\mathrm{tr}}_u)^{-1} /\!/_u^{-1} m^{\mathrm{tr}}_{/\!/_u \mathrm{d}B_u} /\!/_u \widetilde{Q}^{\mathrm{tr}}_u \dot{\ell}_u + \frac{1}{2} \int_0^s (Q^{\mathrm{tr}}_u)^{-1} \varrho^{\mathrm{tr}}_{/\!/_u} \widetilde{Q}^{\mathrm{tr}}_u \ell \, \mathrm{d}u, \tag{2.10}$$

where $\varrho^{\mathrm{tr}}_{/\!/_u} = /\!/_u^{-1} \varrho^{\mathrm{tr}}(X_u(x)) /\!/_u$. Then

$$Z^\ell_u := \left(\widetilde{N}_u, \ell_u \right) - \left(N_u, U^\ell_u \right) \tag{2.11}$$

is a local martingale on $[0, s \wedge \zeta(x))$ and

$$\begin{aligned} \mathrm{d}Z^\ell_u = & \left(\widetilde{Q}^{\mathrm{tr}}_u /\!/_u^{-1} \nabla_{/\!/_u \mathrm{d}B_u} \mathbf{D}a(X_u(x)), \ell_u \right) - \left(Q^{\mathrm{tr}}_u /\!/_u^{-1} \nabla_{/\!/_u \mathrm{d}B_u} a(X_u(x)), U^\ell_u \right) \\ & - \left(N_u, (Q^{\mathrm{tr}}_u)^{-1} /\!/_u^{-1} m^{\mathrm{tr}}_{/\!/_u \mathrm{d}B_u} /\!/_u \widetilde{Q}^{\mathrm{tr}}_u \dot{\ell}_u \right). \end{aligned}$$

Taking expectations in the previous Theorem 2.11 we get the following abstract derivative formula on vector bundles (cf. [DT01, Theorem 4.1]).

Theorem 2.12 (Derivative formula on vector bundles). *Let a be a solution to the backwards heat equation (2.9), and Q and $\widetilde{Q}$ are defined by (2.5). Let τ be a stopping time bounded by $s < \infty$ such that $\tau < \zeta(x)$ and $\ell \in \widetilde{E}^*_x$ be a Cameron-Martin process on $[0, \tau]$. Assume that τ and ℓ have been chosen such that*

$$\mathbb{E} \left| \left(\widetilde{Q}^{\mathrm{tr}}_\tau /\!/_\tau^{-1} \mathbf{D}a_\tau(X_\tau(x)), \ell_\tau \right) \right| < \infty \quad \textit{and} \quad \mathbb{E} \left| \left(Q^{\mathrm{tr}}_\tau /\!/_\tau^{-1} \mathbf{D}a_\tau(X_\tau(x)), U^\ell_\tau \right) \right| < \infty,$$

where U^ℓ is defined by (2.10). Finally, let Z^ℓ_s be the local martingale defined by (2.11). If we assume that $(Z^\ell)^\tau_s := Z^\ell_{s\wedge\tau}$ is a (true) martingale, then

$$\mathbb{E} \left(\mathbf{D}a_0(x), \ell_0 \right) = \mathbb{E} \left(\widetilde{Q}^{\mathrm{tr}}_\tau /\!/_\tau^{-1} \mathbf{D}a_\tau(X_\tau(x)), \ell_\tau \right) - \mathbb{E} \left(Q^{\mathrm{tr}}_\tau /\!/_\tau^{-1} a_\tau(X_\tau(x)), U^\ell_\tau \right).$$

In particular,

(a) If $\ell_\tau = 0$ and $\ell_0 = \xi \in \widetilde{E}_x$, then

$$\mathbb{E} \left(\mathbf{D}a_0(x), \xi \right) = -\mathbb{E} \left(Q^{\mathrm{tr}}_\tau /\!/_\tau^{-1} a_\tau(X_\tau(x)), U^\ell_\tau \right) \tag{2.12}$$

(b) If $\ell_0 = 0$ and $\ell_\tau = \xi \in \widetilde{E}_x$ (possibly random!), then

$$\mathbb{E} \left(\widetilde{Q}^{\mathrm{tr}}_\tau /\!/_\tau^{-1} \mathbf{D}a_\tau(X_\tau(x)), \xi \right) = \mathbb{E} \left(Q^{\mathrm{tr}}_\tau /\!/_\tau^{-1} a_\tau(X_\tau(x)), U^\ell_\tau \right) \tag{2.13}$$

Let $a_\bullet = \mathrm{e}^{(s-\bullet)/2\mathbf{L}} \alpha$ for some $\alpha \in \Gamma_{L^2}(E)$, where the semigroup $\mathrm{e}^{s/2\mathbf{L}}$ is generated by the Friedrichs extension of $\mathbf{L}$. Then the equations (2.12) and (2.13) provide stochastic representations of the derivatives $\mathbf{D}\mathrm{e}^{(s-\bullet)/2\mathbf{L}}\alpha$ and $\mathrm{e}^{(s-\bullet)/2\widetilde{\mathbf{L}}}\mathbf{D}\alpha$. The formula for $\mathbf{D}\mathrm{e}^{(s-\bullet)/2\mathbf{L}}\alpha$ relies on the fact that the local Z^ℓ given by (2.11) is indeed a *true* martingale. This can always be assured by a proper choice of the finite energy process ℓ, i.e. ℓ such that $\ell_0 = \xi$ and $\ell_u = 0$ for $u \geqslant s \wedge \tau$ where τ is the first exit time of $X(x)$ from some relatively compact neighbourhood, *irrespective of whether M is compact or complete.*

In the next two sections, we derive similar localised Bismut type formulae and prove localised gradient estimates on the full exterior bundle, i.e. in the setting of Example 2.1 for **d** and **δ** acting on the heat semigroup, and a covariant Bismut type formula in the setting of Example 2.2, following the ideas of [DT01]. Therefore:

From now on, let $E := \bigwedge \mathsf{T}^* M$ and, to shorten notation, we set $\underline{\mathscr{R}} := \underline{\mathscr{R}}^{\bigwedge \mathsf{T}^* M}$.

We denote by

$$(P_s)_{s>0} := \left(\mathrm{e}^{-\frac{s}{2}\Delta}\right)_{s>0} \subset \mathscr{L}(\Gamma_{\mathsf{L}^2}(\textstyle\bigwedge \mathsf{T}^* M))$$

the heat semigroup defined by the Spectral Theorem 1.50, where we have chosen $f : \mathbb{R} \to \mathbb{R}$, $f(\lambda) := \mathrm{e}^{-s/2\lambda}$. But recall that [DT01, Section B.2], by the spectral theorem, on a complete manifold, for any $a \in \Gamma_{\mathsf{L}^2}(E)$,

$$\mathbf{d} P_u a = P_u \mathbf{d} a \qquad \text{resp.} \qquad \boldsymbol{\delta} P_u a = P_u \boldsymbol{\delta} a.$$

This relation is no longer true dropping completeness, even if M is stochastically complete, cf. e.g. [DT01, Appendix B].

For every $r > 0$, let

$$\tau := \tau(x,r) := \inf\left\{t \geqslant 0 : X_s(x) \notin \mathsf{B}(x,r)\right\} : \Omega \to [0,\infty] \tag{2.14}$$

be the first exit time of X from the open ball $\mathsf{B}(x,r)$ with small radius, say $r = 1$, and we define

$$\overline{K}(x) := \max\left\{(\mathscr{R}(v), v) : v \in \textstyle\bigwedge \mathsf{T}_y M, |v| = 1, y \in \mathsf{B}(x,r)\right\}, \tag{2.15}$$

$$\underline{K}(x) := \min\left\{(\mathscr{R}(v), v) : v \in \textstyle\bigwedge \mathsf{T}_y M, |v| = 1, y \in \mathsf{B}(x,r)\right\}. \tag{2.16}$$

Definition 2.13. Let $r > 0$, $s > 0$, $x \in M$, and $\xi \in \mathsf{T}^*_x M \otimes E$. By $\mathsf{CM}(t,\xi,E)$ we denote the set of all finite energy processes

$$\ell : [0,s] \times \Omega \to \mathsf{T}^*_x M \otimes E$$

such that

$$|k| \leqslant 1, \quad \mathbb{E}\int_0^{s\wedge\tau(x,r)} \left|\dot{\ell}_u\right|^2 \mathrm{d}u < \infty, \quad \ell_0 = 1, \quad \ell_u = 0 \qquad \forall u \geqslant s \wedge \tau(x,r).$$

By a proper choice of the Cameron-Martin space valued process ℓ_s, according to the geometry on D, in [TW98, Proof of Corollary 5.1] Thalmaier & Wang show how to achieve explicit gradient estimates from Bismut-type derivative formulae only using the *local geometry* of the manifold (cf. also [TW11, Remark 3.2], [CTT18]). We use the construction of ℓ_s briefly summarised in the Proof of [Wan14a, Corollary 2.2.2.] to prove the following theorems 2.27 and 2.28.

Lemma 2.14. *For all $s > 0$, $r > 0$, $x \in M$, $\xi \in \mathsf{T}_x M \otimes E$ there is a process $\ell \in \mathsf{CM}(s, \xi, E)$ such that for all $1 \leqslant q < \infty$ and $\underline{K}(x)$ defined by (2.16), we find a constant $C(m, q, r, \underline{K}^-) < \infty$ satisfying*

$$|\ell| \leqslant |\xi|, \qquad \left[\mathbb{E}\left(\int_0^{s\wedge\tau(x,r)} |\dot{\ell}_u|^2 \,\mathrm{d}u\right)^q\right]^{1/q} \leqslant s^{-1/2} \mathrm{e}^{C(m,q,r,\underline{K}^-)t/2} |\xi|, \tag{2.17}$$

where

$$C(m, q, r, \underline{K}^-) := \frac{\pi}{2r}\sqrt{(m-1)\underline{K}^- + \frac{\pi^2}{4r^2}(m+q+3)}. \tag{2.18}$$

Proof. It is well-known, [Wan14a, Corollary 2.2.2.] (also [CTT18]), how to construct a bounded adapted process

$$k : [0, s] \times \Omega \longrightarrow \mathbb{R}$$

with paths in the Cameron-Martin space $\mathsf{L}^{1,2}([0, s], \mathbb{R})$, such that

$$|k| \leqslant 1, \quad \mathbb{E}\int_0^{s\wedge\tau} |\dot{k}_u|^2 \,\mathrm{d}u < \infty, \quad k_0 = 1, \quad k_u = 0 \qquad \forall u \geqslant s \wedge \tau,$$

and

$$\left[\mathbb{E}\left(\int_0^{s\wedge\tau} |\dot{k}_u|^2 \,\mathrm{d}u\right)^{q/2}\right]^{1/q} \leqslant s^{-1/2} \mathrm{e}^{C(m,q,r,\underline{K}^-)}.$$

Thus we may simply set $\ell_u := k_u \xi$. ■

Corollary 2.15. *For all $s > 0$, $r > 0$, $x \in M$, $\xi \in \mathsf{T}_x M \otimes E$ there is a process $\ell \in \mathsf{CM}(s, \xi, E)$ such that for all $1 \leqslant q < \infty$ and all constants $K \geqslant 0$ with $\mathrm{Ric} \geqslant -K$ in $\mathsf{B}(x, r)$ we find constants $C(m, q), C(m, q, K) < \infty$ satisfying*

$$|\ell| \leqslant |\xi|, \qquad \left[\mathbb{E}\left(\int_0^{s\wedge\tau} |\dot{\ell}_u|^2 \,\mathrm{d}u\right)^{q/2}\right]^{1/q} \leqslant s^{-1/2} \mathrm{e}^{\frac{sC(m,q,K)}{r} + \frac{sC(m,q)}{r^2}} |\xi|.$$

2.2 Local and Global Bismut formula for ∇

Using the strategy outlined above, we get the stochastic representation of the semigroup and a covariant Bismut formula, see also Driver & Thalmaier [DT01, Appendix B] and Thalmaier & Wang [TW04, Theorem 3.1].

Theorem 2.16 (Covariant Feynman-Kac formula). *Assume that $\underline{\mathscr{R}}^- \in \mathsf{K}(M)$. Then we have*

$$\mathrm{e}^{-\frac{s}{2}\Delta}\alpha(x) = \mathbb{E}\left(\mathcal{Q}_s /\!\!/_s^{-1} \alpha(x) \mathbb{1}_{\{s<\zeta(x)\}}\right) \qquad \forall s \geqslant 0\ \forall x \in M\ \forall \alpha \in \Gamma_{\mathsf{L}^2\cap\mathsf{L}^\infty\cap\mathsf{C}^\infty}(\textstyle\bigwedge \mathsf{T}^* M).$$

Proof. By the Weitzenböck formula 1.42, we have $\Delta = \square - \mathscr{R}$, where $\mathscr{R} \in \Gamma(\mathrm{End} \bigwedge \mathsf{T}^* M)$ is a symmetric field of endomorphisms. Suppose that $(\square - \mathscr{R})|_{\Gamma_c(\bigwedge \mathsf{T}^* M)}$ is bounded from above and let

$$P_s\alpha = \mathrm{e}^{\frac{s}{2}\overline{\square - \mathscr{R}}} a \qquad \forall \alpha \in \Gamma_{\mathsf{L}^2}(\textstyle\bigwedge \mathsf{T}^* M),$$

be the smooth version of the L^2 semigroup. Then, by [DT01, Theorem B.4],

$$P_s\alpha(x) = \mathbb{E}\left(\mathcal{Q}_s /\!/_s^{-1}\alpha(x)\mathbb{1}_{\{s<\zeta(x)\}}\right) \qquad \forall\alpha \in \Gamma_{\mathrm{L}^2}(\textstyle\bigwedge \mathrm{T}^*M)$$

holds, if $P_s^{\underline{\mathscr{R}}}|\alpha|(x) < \infty$. Since $\underline{\mathscr{R}}^- \in \mathrm{K}(M)$, the claim follows. ∎

Remark 2.17. Theorem 2.16 is true especially under global curvature bounds: Let R be the Riemannian curvature tensor. Then the **curvature operator** $Q \in \bigwedge^2 \mathrm{T}^*M$ is self-adjoint and uniquely determined by the equation

$$(Q(X \wedge Y), U \wedge V) = (\mathrm{R}(X,Y)U, V)$$

for all smooth vector fields $X, Y, U, V \in \Gamma_{\mathrm{C}^\infty}(\mathrm{T}M)$.

By the Gallot-Meyer estimate [GM75], a global bound $Q \geqslant -K$, for some constant $K > 0$, already implies that curvature endomorphism in the Weitzenböck formula 1.42 is globally bounded by

$$\mathscr{R}^{(k)} \geqslant -Kk(m-k).$$

In particular, specified to differential k-forms under global curvature bounds, we get the following

Corollary 2.18 (Covariant Feynman-Kac formula for k-forms). *Assume $\mathscr{R}^{(k)} \geqslant -K$ for some constant $K \geqslant 0$. Then, we have*

$$e^{-\frac{s}{2}\Delta^{(k)}}\alpha(x) = \mathbb{E}\left(\mathcal{Q}_s /\!/_s^{-1}\alpha(x)\right) \qquad \forall s \geqslant 0\ \forall x \in M\ \forall \alpha \in \Gamma_{\mathrm{L}^2\cap\mathrm{L}^\infty\cap\mathrm{C}^\infty}(\textstyle\bigwedge^k \mathrm{T}^*M).$$

Let us provide a short proof that does not rely on Theorem 2.16. As we assume $\mathscr{R}^{(k)} \geqslant -K$ for some $K \geqslant 0$, although M is non-compact, by Lemma 1.76 M is stochastically complete, so that the statement includes that the right-hand side coincides for *all* $x \in M$ and not only for vol-a.e. $x \in M$ with the smooth representative of $\nabla e^{-\frac{s}{2}\Delta^{(k)}}\alpha$.

Proof. By assumption $\mathscr{R}^{(k)} \geqslant -K$ for some constant $K \geqslant 0$, so M is stochastically complete. Then we have

$$\left|e^{-s\Delta^{(k)}}\alpha\right| \leqslant e^{-Ks}e^{-s\Delta^{(0)}}|\alpha| \qquad \forall s > 0, \tag{2.19}$$

by the Kato-Simon inequality (2.8).

As M is stochastically complete, we get $\zeta(x) = \infty$ $\mathbb{P}$-a.s. To prove the formula, we may assume $s > 0$. By Lemma 2.10 the process

$$N_u := \mathcal{Q}_u /\!/_u^{-1} e^{-\frac{s-u}{2}\Delta^{(k)}}\alpha(X_u(x))$$

is a continuous local martingale. By (2.19) above and using that by Gronwall's inequality $\left|\mathcal{Q}_s\right| \leqslant e^{-Ks}$ $\mathbb{P}$-a.s., we find that

$$\left|N_s\right| \leqslant e^{2|K|s}\,\|\alpha\|_\infty \int e^{-\frac{s-u}{2}\Delta}(X_u(x), y)\,\mathrm{vol}(\mathrm{d}y) \leqslant e^{2|K|s}\,\|\alpha\|_\infty,$$

so that N is a true martingale by Lemma A.11. Evaluating N_u at times $u = 0$ and $u = s$ and taking expectations proves the claim. ∎

2.2.1 Local covariant Bismut formula Next, we derive a local covariant Bismut derivative formula that will be used to obtain localised gradient estimates in § 2.4. Those estimates will play a crucial rôle in showing the main result in § 4.

Theorem 2.19 (Covariant Bismut formula). *Let $\xi \in \widetilde{E}^*_x = \mathsf{T}_x M \otimes \bigwedge \mathsf{T}_x M$. Then*

$$\left(\nabla e^{-\frac{s}{2}\Delta}\alpha(x), \xi\right) = -\mathbb{E}\left(\mathcal{Q}^{\mathrm{tr}}_{s\wedge\tau} /\!\!/^{-1}_{s\wedge\tau} e^{-\frac{(s-s\wedge\tau)}{2}\Delta}\alpha(X_{s\wedge\tau}(x)), U^{\ell}_{s\wedge\tau}\right) \qquad \forall \alpha \in \Gamma_{\mathsf{L}^2\cap\mathsf{C}^\infty}(\textstyle\bigwedge \mathsf{T}^* M), \quad (2.20)$$

where

$$U^{\ell}_s := \int_0^s \mathcal{Q}^{-1}_u (\mathrm{d}B_u \lrcorner \widetilde{\mathcal{Q}}_u \dot{\ell}_u) + \frac{1}{2}\int_0^s \mathcal{Q}^{-1}_u \varrho^{\mathrm{tr}}_{/\!\!/_u} \widetilde{\mathcal{Q}}_u \ell_u \,\mathrm{d}u, \tag{2.21}$$

$\varrho^{\mathrm{tr}}_{/\!\!/_u} := /\!\!/^{-1}_u \varrho^{\mathrm{tr}} /\!\!/_u$ with ϱ given by (2.4), $\mathcal{Q}$ and $\widetilde{\mathcal{Q}}$ are defined by (2.5) and $\widetilde{\mathcal{R}}$ by (2.3) and

- *$\tau := \tau(x,r) < \zeta(x)$ is the first exit time of X from the open ball $\mathsf{B}(x,r)$,*
- *$\mathrm{d}B := /\!\!/^{-1} \circ \mathrm{d}X(x)$ is a Brownian motion in $\mathsf{T}_x M$, i.e. the associated anti-development of the Brownian motion $X(x)$,*
- *$(\ell_u)_{0\leqslant u\leqslant s}$ is a finite energy process with values in $\mathsf{T}_x M \otimes \bigwedge \mathsf{T}_x M$ such that for some arbitrary small $\varepsilon > 0$*

$$\mathbb{E}\left(\int_0^{(s-\varepsilon)\wedge\tau(x)} \left|\dot{\ell}_u\right|^2 \mathrm{d}u\right)^{1/2} < \infty \qquad \text{and} \qquad \ell_0 = v, \quad \ell_u = 0 \qquad \forall u \geqslant (s-\varepsilon)\wedge\tau(x).$$

If, in addition, $\alpha \in \Omega_{\mathsf{L}^2}(M)$ is bounded on this neighbourhood, we may take $\varepsilon = 0$.

Before we prove Theorem 2.19, let us make the following

Definition 2.20. Let U^ℓ as in Theorem 2.19. We set

$$U^\ell_s = \ell^{(1)}_s + \frac{1}{2}\ell^{(2)}_s,$$

where we define processes

$$\ell^{(1)}_s := \int_0^s \mathcal{Q}^{-1}_u (\mathrm{d}B_u \lrcorner \widetilde{\mathcal{Q}}_u \dot{\ell}_u) \qquad \text{and} \qquad \ell^{(2)}_s := \frac{1}{2}\int_0^s \mathcal{Q}^{-1}_u \varrho^{\mathrm{tr}}_{/\!\!/_u} \widetilde{\mathcal{Q}}_u \ell_u \,\mathrm{d}u.$$

Then $\ell^{(1)}$ is a continuous local martingale and $\ell^{(2)}$ is a continuous process of finite variation.

Proof of Theorem 2.19. By Gronwall's inequality, we have

$$\left|\mathcal{Q}_s\right|_{\mathrm{op}} \leqslant \exp\left(-\frac{1}{2}\int_0^s \underline{\mathcal{R}}(X_u(x))\mathrm{d}u\right) \qquad \forall s \geqslant 0.$$

and hence

$$\left|\mathcal{Q}_s\right|_{\mathrm{op}} \leqslant e^{\underline{K}(x)s/2}, \quad \left|\widetilde{\mathcal{Q}}_s\right|_{\mathrm{op}} \leqslant e^{\underline{K}(x)s/2} \qquad \mathbb{P}\text{-a.s. on } \{s \leqslant \tau(x,r)\}. \tag{2.22}$$

As $\mathcal{Q}$ and $\widetilde{\mathcal{Q}}$ are invertible with

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{Q}^{-1}_s = \frac{1}{2}\mathcal{R}_{/\!\!/_s}\mathcal{Q}^{-1}_s, \qquad \mathcal{Q}^{-1}_0 = \mathrm{id}_{\bigwedge \mathsf{T}_x M},$$
$$\frac{\mathrm{d}}{\mathrm{d}s}\widetilde{\mathcal{Q}}^{-1}_s = \frac{1}{2}\mathcal{R}_{/\!\!/_s}\widetilde{\mathcal{Q}}^{-1}_s, \qquad \widetilde{\mathcal{Q}}^{-1}_0 = \mathrm{id}_{\mathsf{T}_x M\otimes \bigwedge \mathsf{T}_x M},$$

we also have

$$\left|\mathscr{Q}_s^{-1}\right|_{\mathrm{op}} \leqslant \mathrm{e}^{\overline{K}(x)s/2}, \qquad \left|\widetilde{\mathscr{Q}}_s^{-1}\right|_{\mathrm{op}} \leqslant \mathrm{e}^{\overline{K}(x)s/2} \qquad \mathbb{P}\text{-a.s. on } \{s \leqslant \tau(x,r)\}\,. \tag{2.23}$$

According to Lemma 2.10 and Theorem 2.11

$$N := \left(\widetilde{\mathscr{Q}}/\!\!/^{-1}\nabla \mathrm{e}^{-\frac{s-\bullet}{2}\Delta}\alpha(X(x)), \ell\right) - \left(\mathscr{Q}/\!\!/^{-1}\mathrm{e}^{-\frac{s-\bullet}{2}\Delta}\alpha(X(x)), \ell^{(1)} + \ell^{(2)}\right)$$

is a continuous local martingale, and in view of (2.22) and (2.23) and the assumptions imposed on ℓ, a bounded local martingale, hence a true martingale by Lemma A.11. Evaluating N_u at the times $r = 0$ and $r = s \wedge \tau$ and taking expectations, we get $\mathbb{E}N_0 = \mathbb{E}N_{s\wedge\tau}$ so that

$$\left(\nabla \mathrm{e}^{-\frac{s}{2}\Delta^{(1)}}\alpha(x), \xi\right) = -\mathbb{E}\left(\mathscr{Q}_{s\wedge\tau}/\!\!/_{s\wedge\tau}^{-1}\mathrm{e}^{-\frac{(s-s\wedge\tau)}{2}\Delta}\alpha(X_{s\wedge\tau}(x)), \ell_{s\wedge\tau}^{(1)} + \ell_{s\wedge\tau}^{(2)}\right),$$

which is the covariant Bismut formula with $U^\ell = \ell^{(1)} + \ell^{(2)}$. ■

An immediate consequence is the following Theorem.

Theorem 2.21. *Suppose $\alpha \in \Omega_{\mathrm{L}^2}(M)$, $\underline{\mathscr{R}}^- \in \mathrm{K}(M)$ and $\xi \in \widetilde{E}_x^* = \mathrm{T}_x M \otimes \bigwedge \mathrm{T}_x M$. Let $(\ell_u)_{0\leqslant u \leqslant s}$ a bounded adapted process with absolutely continuous paths in $\mathrm{T}_x M \otimes \bigwedge \mathrm{T}_x M$ such that $\mathbb{E}\left(\int_0^{(s-\varepsilon)\wedge\tau(x,r)} \left|\dot{\ell}_u\right|^2 \mathrm{d}u\right)^{1/2} < \infty$ and $\ell_0 = \xi$, $\ell_u = 0$ for all $u \geqslant (s-\varepsilon)\wedge\tau(x,r)$ and some arbitrary small $\varepsilon > 0$. Then,*

$$\left(\nabla e^{-\frac{s}{2}\Delta}\alpha(x), \xi\right) = -\mathbb{E}\left(\mathscr{Q}_s^{\mathrm{tr}}/\!\!/_s^{-1}\alpha(X_s(x))\mathbb{1}_{\{s<\zeta(x)\}}, U^\ell_{s\wedge\tau(x)}\right), \tag{2.24}$$

where U^ℓ is given by (2.29), $\varrho^{\mathrm{tr}}_{/\!\!/_s} := /\!\!/_s^{-1}\varrho\,/\!\!/_s$ with ϱ given by (2.4), $\mathscr{Q}^{\mathrm{tr}}$ and $\widetilde{\mathscr{Q}}^{\mathrm{tr}}$ are defined by (2.5) and $\widetilde{\mathscr{R}}$ by (2.3).

Proof. Note that, by the strong Markov property,

$$\mathscr{Q}^{\mathrm{tr}}_{s\wedge\tau}/\!\!/_{s\wedge\tau}^{-1}\left(P_{s-s\wedge\tau}\alpha\right)(X_{s\wedge\tau}(x)) = \mathbb{E}^{\mathscr{F}_{s\wedge\tau}}\left(\mathscr{Q}_s^{\mathrm{tr}}/\!\!/_s^{-1}\alpha(X_s(x))\mathbb{1}_{\{s<\zeta(x)\}}\right),$$

which is by definition a bounded $\mathscr{F}_{s\wedge\tau}$-measurable random variable. The existence of the scalar semigroup is provided by the assumption $\underline{\mathscr{R}}^- \in \mathrm{K}(M)$. Hence, by Theorem 2.19,

$$\left(\nabla e^{-\frac{s}{2}\Delta}\alpha(x), \xi\right) = -\mathbb{E}\left(\mathscr{Q}_s^{\mathrm{tr}}/\!\!/_s^{-1}\alpha(X_s(x))\mathbb{1}_{\{s<\zeta(x)\}}, U^\ell_{s\wedge\tau}\right). \qquad ■$$

2.2.2 Global covariant Bismut formula To end this section, we derive a global version of the covariant Bismut derivative formula that will be our key tool to prove Theorem 4.4. Therefore, global assumptions on the curvature are sufficient to control the process U^ℓ in (2.29), to wit, we assume that the curvature and the derivative of the curvature is bounded.

This subsection was carried out with Batu Güneysu in our joint work with Baptiste Devyver.

Assumption 2.22. We assume that the curvature and its derivative are bounded by some constant $A < \infty$, i.e.

$$\max\left(\|\mathrm{R}\|_\infty, \|\nabla \mathrm{R}\|_\infty\right) \leqslant A. \tag{2.25}$$

First, we show an a priori L^∞ bound.

Lemma 2.23. *Assume that (2.25) holds. Then there is a constant $C = C(A, m) > 0$ such that for all $1 \leqslant k \leqslant m$, $s > 0$, $x \in M$, we have*

$$\left|\mathrm{e}^{-\frac{s}{2}\Delta^{(k)}}\alpha(x)\right| \leqslant C\mathrm{e}^{Cs}s^{-1/2}\,\|\alpha\|_\infty\,, \qquad \forall \alpha \in \Gamma_{\mathsf{L}^2\cap\mathsf{L}^\infty\cap\mathsf{C}^\infty}(M).$$

Proof. In the sequel, $C(a, \dots)$ denotes a constant that only depends on $a, \dots$, and which may differ from line to line. Let $s > 0$, $r > 0$, $x \in M$, $\xi \in \mathsf{T}^*_x M \otimes \bigwedge^k \mathsf{T}^*_x M$ be arbitrary and pick a finite energy process $\ell \in \mathsf{CM}(s, \xi, \bigwedge^k \mathsf{T}^*_x M)$. Note that by assumption (2.25) M is stochastically complete, i.e. $\zeta(x) = \infty$ $\mathbb{P}$-a.s. It follows from the covariant Feynman-Kac formula, Corollary 2.18, and the Markov property (cf. Proof of Theorem 2.21) that

$$\left(\nabla \mathrm{e}^{-\frac{s}{2}\Delta^{(k)}}\alpha(x), \xi\right) = -\mathbb{E}\left(\mathcal{Q}_s /\!\!/_s^{-1}\alpha(X_s(x)), \ell^{(1)}_s + \ell^{(2)}_s\right).$$

By Gronwall's inequality, we get

$$\left|\mathcal{Q}_s\right|_{\mathrm{op}} \leqslant \mathrm{e}^{C(m,A)s}, \quad \left|\widetilde{\mathcal{Q}}_s\right|_{\mathrm{op}} \leqslant \mathrm{e}^{C(m,A)s} \qquad \mathbb{P}\text{-a.s. on } \{s \leqslant \tau\}, \tag{2.26}$$

and as $\mathcal{Q}$ and $\widetilde{\mathcal{Q}}$ are invertible, we also get

$$\left|\mathcal{Q}_s^{-1}\right|_{\mathrm{op}} \leqslant \mathrm{e}^{C(m,A)s}, \quad \left|\widetilde{\mathcal{Q}}_s^{-1}\right|_{\mathrm{op}} \leqslant \mathrm{e}^{C(m,A)s} \qquad \mathbb{P}\text{-a.s. on } \{s \leqslant \tau\}. \tag{2.27}$$

By Corollary 2.15, a proper choice of the Cameron-Martin space valued process ℓ_s gives

$$\left[\mathbb{E}\left(\int_0^{t\wedge\tau} \left|\dot{\ell}_s\right|^2 \mathrm{d}u\right)^{q/2}\right]^{1/q} \leqslant t^{-1/2}\mathrm{e}^{\frac{tC(m,q,K)}{r}+\frac{tC(m,q)}{r^2}}\,|\xi|\,.$$

By the Burkholder-Davis-Gundy inequality A.12, we get

$$\mathbb{E}\left|\int_0^s \mathcal{Q}_u^{-1}(\mathrm{d}B_u \lrcorner \widetilde{\mathcal{Q}}_u\dot{\ell}_u)\right| \leqslant \mathbb{E}\left(\int_0^{s\wedge\tau} |\mathcal{Q}_u^{-1}|^2|\widetilde{\mathcal{Q}}_u|^2|\dot{\ell}_u|^2\mathrm{d}u\right)^{1/2}.$$

Thus, we estimate

$$\mathbb{E}|\ell^{(1)}_{s\wedge\tau}| \leqslant C\ \mathrm{e}^{C(m)s}s^{-1/2}\mathrm{e}^{\frac{C(A,m)s}{r}+\frac{C(m)s}{r^2}}\,|\xi|\,,$$

using eqs. (2.26) and (2.27), and

$$\mathbb{E}|\ell^{(2)}_{s\wedge\tau}| \leqslant \mathrm{e}^{C(m)s}C(A,m)\,|\xi|\,,$$

which follows from eqs. (2.26) and (2.27), and $|\ell| \leqslant |\xi|$, $|\rho| \leqslant C(A, m)$. Using (2.26) once more, we can now estimate as follows

$$\begin{aligned}
&\left|\left(\nabla \mathrm{e}^{-\frac{s}{2}\Delta_1}\alpha(x), \xi\right)\right| \\
&\quad\leqslant \mathbb{E}\left(|\mathcal{Q}_s||\alpha(X_s(x))||\ell^{(1)}_{s\wedge\tau}|\right) + \mathbb{E}\left(|\mathcal{Q}_s||\alpha(X_s(x))||\ell^{(2)}_{s\wedge\tau}|\right) \\
&\quad\leqslant |\xi|\,C(A,m)\mathrm{e}^{C(A,m)s}\,\|\alpha\|_\infty\left(C\mathrm{e}^{C(m)s}s^{-1/2}\mathrm{e}^{\frac{C(A,m)s}{r}+\frac{C(m)s}{r^2}} + \mathrm{e}^{C(m)s}C(A,m)\right).
\end{aligned}$$

Taking $r \to \infty$, we have managed to construct some $C(A, m) < \infty$, such that for all $x \in M$, $s > 0$, we have

$$\left|\nabla \mathrm{e}^{-\frac{s}{2}\Delta^{(k)}}\alpha(x)\right| \leqslant C(A,m)\mathrm{e}^{C(A,m)s}s^{-1/2}\,\|\alpha\|_\infty\,.$$

∎

By a proper choice of the finite energy process, namely $\ell = (s - \bullet)/s\xi$, we immediately get the following

Theorem 2.24. *Assume that (2.25) holds. For every $s > 0$, $x \in M$, $\xi \in \mathsf{T}_x^* M \otimes \bigwedge^k \mathsf{T}_x M$, we have*

$$\left(\nabla e^{-\frac{s}{2}\Delta^{(k)}}\alpha(x), \xi\right) = -\mathbb{E}\left(\mathcal{Q}_s^{\mathrm{tr}} /\!/_s^{-1} \alpha(X_s(x)), U_s^{\mathrm{global}}\right), \qquad \forall \alpha \in \Gamma_{\mathsf{L}^2 \cap \mathsf{L}^\infty \cap \mathsf{C}^\infty}(M), \tag{2.28}$$

where

$$U_s^{\mathrm{global}} := \frac{1}{s}\int_0^s \mathcal{Q}_u^{-1}(\mathrm{d}B_u \lrcorner \widetilde{\mathcal{Q}}_u)\xi + \frac{1}{2s}\int_0^s \mathcal{Q}_u^{-1} \varrho_{/\!/_u}^{\mathrm{tr}} \widetilde{\mathcal{Q}}_u (s-u)\xi \, \mathrm{d}u. \tag{2.29}$$

Proof. According to Lemma 2.10 and Theorem 2.11

$$N := \left(\widetilde{\mathcal{Q}} /\!/^{-1} \nabla \mathrm{e}^{-\frac{s-\bullet}{2}\Delta^{(k)}} \alpha(X(x)), \ell\right) - \left(\mathcal{Q} /\!/^{-1} \mathrm{e}^{-\frac{s-\bullet}{2}\Delta} \alpha(X(x)), U_s^{\mathrm{global}}\right)$$

is a continuous local martingale. By Lemma 2.23,

$$\left|N_u\right| \leqslant C(A,m)\mathrm{e}^{C(A,m)s} \left|\xi\right| \left\|\alpha\right\|_\infty \left((s-u)^{1/2} + \left|U_s^{\mathrm{global}}\right|\right),$$

As in the previous Proof of Lemma 2.23, by the Burkholder-Davis-Gundy inequality A.12, eqs. (2.26) and (2.27), N is a true martingale and the claim follows from taking expectation at the times $u = 0$ and $u = s$. ■

2.3 Local Bismut Formulae for d and δ

Theorem 2.25. *Let $\alpha \in \Gamma_{\mathsf{L}^2 \cap \mathsf{C}^\infty}(\bigwedge \mathsf{T}^* M)$ and $\underline{\mathscr{R}} \in \mathsf{K}(M)$. Then for any $v \in \bigwedge \mathsf{T}_x M$, we have the following Bismut type formulae:*

$$\left((\mathbf{d}P_s\alpha)_x, v\right) = -\mathbb{E}\left(/\!/_s^{-1}\alpha(X_s(x))\mathbb{1}_{\{s<\zeta(x)\}}, \mathcal{Q}_s \int_0^s \mathcal{Q}_u^{-1}(\mathrm{d}B_u \lrcorner \mathcal{Q}_u \dot{\ell}_u)\right), \tag{2.30}$$

$$\left((\boldsymbol{\delta}P_s\alpha)_x, v\right) = -\mathbb{E}\left(/\!/_s^{-1}\alpha(X_s(x))\mathbb{1}_{\{s<\zeta(x)\}}, \mathcal{Q}_s \int_0^s \mathcal{Q}_u^{-1}(\mathrm{d}B_u \wedge \mathcal{Q}_u \dot{\ell}_u)\right), \tag{2.31}$$

where $\mathcal{Q}$ is defined by (2.5) and

- *$\tau(x,r) < \zeta(x)$ is the first exit time of X from the open ball $\mathsf{B}(x,r)$,*
- *$\mathrm{d}B := /\!/^{-1} \circ \mathrm{d}X(x)$ is a Brownian motion in $\mathsf{T}_x M$, i.e. the associated anti-development of the Brownian motion $X(x)$,*
- *$(\ell_u)_{0 \leqslant u \leqslant s}$ is any adapted process in $\bigwedge \mathsf{T}_x M$ with absolutely continuous paths such that for some arbitrary small $\varepsilon > 0$*

$$\mathbb{E}\left(\int_0^{(s-\varepsilon)\wedge\tau(x)} \left|\dot{\ell}_u\right|^2 \mathrm{d}u\right)^{1/2} < \infty \qquad \text{and} \qquad \ell_0 = v, \quad \ell_u = 0 \qquad \forall u \geqslant (s-\varepsilon)\wedge\tau(x,r).$$

If, in addition, $\alpha \in \Omega_{\mathsf{L}^2}(M)$ is bounded on this neighbourhood, we may take $\varepsilon = 0$.

Proof. Using the same strategy as in the previous Section § 2.2, again the assumption $\mathscr{R} \in \mathsf{K}(M)$ assures that the scalar semigroup is finite. More directly, using $\mathscr{R} \in \mathsf{K}(M)$, it holds semigroup domination

$$\left|P_s a(x)\right| \leqslant P_s^{\mathscr{R}} \left|a\right|(x).$$

and the result follows immediately by [DT01, Theorem 6.1]. ■

Remark 2.26. By the same method used in § 2.2.2, we can deduce global Bismut type formulae for $\mathbf{d}P_s\alpha$ and $\boldsymbol{\delta}P_s\alpha$.

2.4 Gradient Estimates

By the Bismut formulae, derived in the last sections 2.3 and 2.2, we now prove localised gradient estimates that will be the key tool in the proof of our Main Result in § 3. As curvature only enters locally around a point x, the stochastic integral can be estimated by choosing a suitable finite energy process ℓ and the Burkholder-Davis-Gundy inequality, Lemma A.12.

From now on, set $D := \mathsf{B}(x,r)$ to be a ball with small radius, say $r=1$, and we define

$$\overline{K}(x) := \max\left\{(\mathscr{R}(v),v) \,:\, v\in \bigwedge \mathsf{T}_yM, |v|=1, y\in\mathsf{B}(x,1)\right\}, \tag{2.32}$$
$$\underline{K}(x) := \min\left\{(\mathscr{R}(v),v) \,:\, v\in \bigwedge \mathsf{T}_yM, |v|=1, y\in\mathsf{B}(x,1)\right\}. \tag{2.33}$$

Then $\tau(x,1)$ is the first exit time of X from the open ball $\mathsf{B}(x,1)$ (cf. (2.14) above).

Theorem 2.27. *Let $\alpha\in\Omega_{\mathsf{L}^2}(M)$ and $\underline{\mathscr{R}}^-\in\mathsf{K}(M)$. Then, for all $s>0$,*

$$\left|(\mathbf{d}P_s\alpha)_x\right|^2 \leqslant \Psi(x,s)\Phi(x,s)\,\|\alpha\|^2_{\Omega_{\mathsf{L}^2}(M)}, \tag{2.34}$$
$$\left|(\boldsymbol{\delta}P_s\alpha)_x\right|^2 \leqslant \Psi(x,s)\Phi(x,s)\,\|\alpha\|^2_{\Omega_{\mathsf{L}^2}(M)}, \tag{2.35}$$

where

$$\Psi(x,s) := \frac{1}{\sqrt{s}}\exp\left[D(\gamma,c_\gamma(\underline{\mathscr{R}}^-),c_q^{1/q})s + \left(\pi\sqrt{(m-1)\underline{K}(x)^-} + \pi^2(m+5) + \left(\overline{K}(x)+\underline{K}(x)\right)^-\right)\frac{s}{2}\right], \tag{2.36}$$

and the finite constant D depends on the constant $c_\gamma(\underline{\mathscr{R}}^-)$ in (2.7) and the constant c_q from the Burkholder-Davis-Gundy inequality, and $\Phi(x,s)$ is defined by (3.19).

Proof. Again by Gronwall's inequality, we get

$$\left|\mathscr{Q}_s\right|_{\mathrm{op}} \leqslant \mathrm{e}^{\underline{K}(x)s/2}, \quad \left|\mathscr{Q}_s^{-1}\right|_{\mathrm{op}} \leqslant \mathrm{e}^{\overline{K}(x)s/2} \qquad \mathbb{P}\text{-a.s. on } \{s\leqslant\tau(x,r)\}. \tag{2.37}$$

By the Burkholder-Davis-Gundy inequality A.12, we get

$$\mathbb{E}\left|\int_0^s \mathscr{Q}_r^{-1}(\mathrm{d}B_r \lrcorner\, \mathscr{Q}_r\dot{\ell}_r)\right|^{2q} \leqslant c_q\mathrm{e}^{q\left(\overline{K}(x)+\underline{K}(x)\right)^- s/2}\,\mathbb{E}\left(\int_0^s \left|\dot{\ell}_r\right|^2\mathrm{d}r\right)^q. \tag{2.38}$$

Let $q\in[2,\infty)$. By Lemma 2.14, a proper choice of the Cameron-Martin space valued process ℓ_u gives

$$|\ell|\leqslant|v|, \qquad \left[\mathbb{E}\left(\int_0^{s\wedge\tau(x,1)}\left|\dot{\ell}_r\right|^2\mathrm{d}r\right)^q\right]^{1/(2q)} \leqslant \frac{1}{\sqrt{s}}\mathrm{e}^{C(m,2q,\underline{K}^-(x))s/2}\,|v|, \tag{2.39}$$

where the constant $C(m,q,r,\underline{K}^-)$ is given by (2.18).

By Lemma 2.5, for any $\gamma>1$, there is a constant $c_\gamma = c_\gamma(\underline{\mathscr{R}})$ such that

$$\sup_{x\in M}\mathbb{E}^x\left(\mathbb{1}_{\{s<\zeta\}}\mathrm{e}^{\int_0^s|\underline{\mathscr{R}}(X_u)|\mathrm{d}u}\right) \leqslant \gamma\mathrm{e}^{sc_\gamma} < \infty. \tag{2.40}$$

Now, we can estimate as follows: Let $|v| \leqslant 1$, then using Hölder and Cauchy-Schwarz inequality,

$$\begin{aligned}
\left|(\mathbf{d}P_s\alpha)_x\right| &\leqslant \left[\mathbb{E}^x\left|\alpha(X_s)\mathbb{1}_{\{s<\zeta\}}\right|^p\right]^{1/p}\left[\mathbb{E}^x\left|\mathcal{Q}_s\mathbb{1}_{\{s<\zeta\}}\int_0^s\mathcal{Q}_u^{-1}(\mathrm{d}B_u\lrcorner\,\mathcal{Q}_u\dot{\ell}_u)\right|^q\right]^{1/q}\\
&\leqslant \left[\mathbb{E}^x\left|\alpha(X_s)\mathbb{1}_{\{s<\zeta\}}\right|^p\right]^{1/p}\left[\mathbb{E}^x\left(\left|\mathcal{Q}_s\right|^{2q}\mathbb{1}_{\{s<\zeta\}}\right)\right]^{1/(2q)}\left[\mathbb{E}^x\left(\int_0^s\mathcal{Q}_u^{-1}(\mathrm{d}B_u\lrcorner\,\mathcal{Q}_u\dot{\ell}_u)\right)^{2q}\right]^{1/(2q)}\\
&\underset{(2.38)}{\overset{(2.40)}{\leqslant}} \left[\mathbb{E}\left(|\alpha|^p(X_s(x))\mathbb{1}_{\{s<\zeta(x)\}}\right)\right]^{1/p}\gamma e^{sc_\gamma}c_q^{1/q}e^{\left(\overline{K}(x)+\underline{K}(x)\right)^-s/2}\left[\mathbb{E}\left(\int_0^s\left|\dot{\ell}_r\right|^2\mathrm{d}r\right)^q\right]^{1/(2q)}\\
&\overset{(2.39)}{\leqslant} e^{C_1(\gamma,c_\gamma,c_q^{1/q})s\,+\,\left(\overline{K}(x)+\underline{K}(x)\right)^-s/2}e^{C(m,2q,\underline{K}^-(x))s/2}s^{-1/2}\left[\int_M p_s^{g,(0)}(x,y)\,|\alpha(y)|^p\,\mathrm{vol}_g(\mathrm{d}y)\right]^{1/p}\\
&\leqslant \sqrt{\Psi(x,s)}\sqrt[p]{\Phi(x,s)}\,\|\alpha\|_{\Omega_{\mathsf{L}^p}(M)}\,,
\end{aligned}$$

where $\Psi(x,s)$ is given by (2.42). In particular, for $p=2=q$ the result follows.

By an analogous calculation, we obtain the estimate (2.35). ■

Using similar techniques as in the proof of the previous Theorem 2.27, we can show the following estimate. Note that, in comparison to Theorem 2.27, the process U^ℓ in Theorem 2.21 involves the *derivative of the curvature* which is now reflected in the local bound $\Xi(x,s)$.

Theorem 2.28. *Let $\alpha \in \Omega_{\mathsf{L}^2}(M)$ and $\underline{\mathscr{R}}^- \in \mathsf{K}(M)$. Then, for all $\xi \in \mathsf{T}_xM \otimes \bigwedge \mathsf{T}_xM$ and $s>0$,*

$$\left|\langle\nabla P_s\alpha,\xi\rangle\right|^2 \leqslant |\xi|^2\,\Xi(x,s)\Phi(x,s)\,\|\alpha\|^2_{\Omega_{\mathsf{L}^2}(M)}\,, \tag{2.41}$$

where

$$\Xi(x,s) := \Psi(x,s) + s^{-3/2}\Psi(x,s)\max_{y\in\mathsf{B}(x,1)}|\nabla\mathrm{R}(y)| \tag{2.42}$$

with $\Psi(x,s)$ defined by (2.42) and $\Phi(x,s)$ is defined by (3.19).

Proof. As in the previous Proof of Theorem 2.27, we find

$$|\ell| \leqslant |\xi|\,, \qquad \left[\mathbb{E}\left(\int_0^{s\wedge\tau(x,r)}\left|\dot{\ell}_u\right|^2\mathrm{d}u\right)^q\right]^{1/q} \leqslant s^{-1/2}e^{C(m,q,r,\underline{K}^-)s/2}\,|\xi|\,, \tag{2.43}$$

where the constant $C(m,q,r,\underline{K}^-)$ is given by (2.18).

Again using Gronwall's inequality we get (2.37), and by

$$\left|\varrho(X_s(x))\right| \leqslant \max_{y\in\mathsf{B}(x,r)}|\varrho(y)| \leqslant \max_{y\in\mathsf{B}(x,r)}|\nabla\mathrm{R}(y)| \qquad \mathbb{P}\text{-a.s. on } \{s\leqslant\tau(x,r)\}\,,$$

we have

$$\mathbb{E}\left|\int_0^s\mathcal{Q}_s^{-1}\varrho_{/\!/_s}^{\mathrm{tr}}\widetilde{\mathcal{Q}}_s\ell_u\,\mathrm{d}u\right| \leqslant e^{(\overline{K}_1+\underline{K}_2)^-s/2}\max_{y\in\mathsf{B}(x,r)}|\nabla\mathrm{R}(y)|\,s\,|\xi|\,. \tag{2.44}$$

By Lemma 2.5, for any $\gamma>1$, there is a constant $c_\gamma = c_\gamma(\underline{\mathscr{R}})$ such that

$$\sup_{x\in M}\mathbb{E}^x\left(\mathbb{1}_{\{s<\zeta\}}e^{\int_0^s|\underline{\mathscr{R}}(X_u)|\mathrm{d}u}\right) \leqslant \gamma e^{sc_\gamma} < \infty.$$

As in the proof of Theorem 2.27, a similar calculation shows, using Hölder's inequality and the elementary inequality $(a+b)^c \leqslant 2^{c-1}(a^c+b^c)$,

$$\begin{aligned}
\left|\left(\nabla P_s\alpha(x),\xi\right)\right| &= \left|\mathbb{E}\left(/\!\!/_s^{-1}\alpha(X_s(x))\mathbb{1}_{\{s<\zeta(x)\}}, \mathcal{Q}_s^{\mathrm{tr}} U^{\ell}_{s\wedge\tau}\right)\right| \\
&\leqslant 2\left|\mathbb{E}\left(/\!\!/_s^{-1}\alpha(X_s(x))\mathbb{1}_{\{s<\zeta(x)\}}, \mathcal{Q}_s^{\mathrm{tr}}\ell^{(1)}_{s\wedge\tau}\right)\right| + \left|\mathbb{E}\left(/\!\!/_s^{-1}\alpha(X_s(x))\mathbb{1}_{\{s<\zeta(x)\}}, \mathcal{Q}_s^{\mathrm{tr}}\ell^{(2)}_{s\wedge\tau}\right)\right| \\
&\leqslant \left[\mathbb{E}\left|\alpha(X_s(x))\mathbb{1}_{\{s<\zeta(x)\}}\right|^p\right]^{1/p}\left(2\left[\mathbb{E}\left(\mathcal{Q}_s\mathbb{1}_{\{s<\zeta(x)\}}\ell^{(1)}_{s\wedge\tau}\right)^q\right]^{1/q} + \left[\mathbb{E}\left(\mathcal{Q}_s\mathbb{1}_{\{s<\zeta(x)\}}\ell^{(2)}_{s\wedge\tau}\right)^q\right]^{1/q}\right) \\
&\leqslant |\xi|\left[\mathbb{E}\left(|\alpha|^p(X_s(x))\mathbb{1}_{\{s<\zeta(x)\}}\right)\right]^{1/p}\gamma \mathrm{e}^{sc_\gamma}c_q^{1/q}\mathrm{e}^{\left(\overline{K}(x)+\underline{K}(x)\right)^- s/2}\left(2\left[\mathbb{E}\left(\int_0^s\left|\dot{\ell}_r\right|^2\mathrm{d}r\right)^q\right]^{1/(2q)} + \right. \\
&\qquad\qquad \left. + \left(\max_{y\in\mathrm{B}(x,1)}|\nabla \mathrm{R}(y)|\right)s\right) \\
&\leqslant |\xi|\,\mathrm{e}^{D(\gamma,c_\gamma,c_q^{1/q})s\,+\,\left(\overline{K}(x)+\underline{K}(x)\right)^- s/2}\left[\mathrm{e}^{C(m,2q,\underline{K}^-(x))s/2}s^{-1/2} + \left(\max_{y\in\mathrm{B}(x,1)}|\nabla\mathrm{R}(y)|\right)s\right]\times \\
&\qquad\qquad \times\left[\int_M p_s^{g,(0)}(x,y)\,|\alpha(y)|^p\,\mathrm{vol}_g(\mathrm{d}y)\right]^{1/p} \\
&\leqslant |\xi|\sqrt{\Xi(x,s)}\,\sqrt[p]{\Phi(x,s)}\,\|\alpha\|_{\Omega_{\mathrm{L}^p(M)}},
\end{aligned}$$

where $\Xi(x,s)$ is given by (2.42). In particular, for $p = 2 = q$ the result follows. ■

I

Scattering theory for the Hodge Laplacian

Chapter 3

Scattering Theory for the Hodge Laplacian

Let (M,g) be a non-compact geodesically complete Riemannian manifold without boundary. The Hodge Laplacian $\Delta_g^{(k)}$ acting on differential k-forms carries important geometric and topological information about M, of particular interest is the spectrum $\sigma(\Delta_g^{(k)})$ of $\Delta_g^{(k)}$. If M is compact, then the spectrum consists of eigenvalues with finite multiplicity. If M is non-compact, then the spectrum contains some absolutely continuous part (cf. [RS79; Wei80]). A natural question to ask is to what extent can we control the absolutely continuous part of $\sigma(\Delta_g^{(k)})$ and under which assumptions on the geometry of (M,g)?

A systematic approach to control the absolutely continuous part of the spectrum $\sigma_{\mathrm{ac}}(\Delta_g^{(k)})$ is inspired by quantum mechanics, namely scattering theory: Assume that there is another Riemannian metric h on M such that h is quasi-isometric to g, i.e. there exists a constant $C \geqslant 1$ such that $(1/C)g \leqslant h \leqslant Cg$. We show that under suitable assumptions the wave operators

$$W_\pm(\Delta_h^{(k)}, \Delta_g^{(k)}, I_{g,h}^{(k)}) = \operatorname*{s-lim}_{t\to\pm\infty} \mathrm{e}^{it\Delta_h^{(k)}} I_{g,h}^{(k)} \mathrm{e}^{-it\Delta_g^{(k)}} \mathrm{P}_{\mathrm{ac}}(-\Delta_g^{(k)})$$

exist and are complete, where the limit is taken in the strong sense, and

$$I_{g,h}^{(k)} : \Omega^k_{\mathsf{L}^2}(M,g) \to \Omega^k_{\mathsf{L}^2}(M,h)$$

denotes a bounded identification operator between the Hilbert spaces of equivalence classes of square-integrable Borel k-forms on M corresponding to the metric g and h respectively (cf. Theorem 3.5 and § 3.2 for details). Then as well-known, it follows in particular that

$$\sigma_{\mathrm{ac}}(\Delta_h^{(k)}) = \sigma_{\mathrm{ac}}(\Delta_g^{(k)}).$$

Considering Laplacians acting on 0-forms, i.e. functions, on M, Müller & Salomonsen [MS07] studied the existence and completeness of the wave operators corresponding to the Laplace-Beltrami operator by assuming both metrics to have a C^∞-bounded geometry and a weighted integral condition involving a second order deviation of the metrics. Hempel, Post & Weder [HPW14] improved the result of [MS07] by assuming only a zeroth order deviation of the metric g from h and a weighted integral condition involving a local lower bound of the injectivity radius and the Ricci curvatures. However, detailed control on the sectional curvature is needed to get control over the injectivity radii. In general, injectivity radii are hard to calculate.

Recently, Güneysu & Thalmaier [GT20] established a rather simple integral criterion induced by two quasi-isometric Riemannian metrics only depending on a local upper bound on the heat kernel and certain explicitly given local lower bound on the Ricci curvature using stochastic methods, namely a Bismut-type formula for the derivative of the heat semigroup [AT10].

Considering Laplacians acting on differential k-forms, Bei, Güneysu & Müller [BGM17] generalised the previous results in [MS07] for the case of *conformally equivalent metrics* under a mild first order control on the conformal factor.

Using a similar method, very recently Boldt & Güneysu [BG20] extended the result of [GT20] to a non-compact spin manifold with a fixed topological spin structure and two complete Riemannian metrics with bounded sectional curvatures. As the metrics induce Dirac operators D_g and D_h, they can show existence and completeness of the wave operators corresponding to the Dirac operators $W_\pm(\mathsf{D}_h, \mathsf{D}_g, I_{g,h})$ and their squares $W_\pm(\mathsf{D}_h^2, \mathsf{D}_g^2, I_{g,h})$.

In this chapter, we address the natural question: Can we extend the result of [GT20] to the setting of differential k-forms for two quasi-isometric Riemannian metrics?

We will show that this can be done if the Weitzenböck curvature endomorphisms is in the Kato class and assuming an integral criterion only depending on a local upper bound on the heat kernel and certain explicitly given local curvature bounds. In addition, a necessary assumption will be a bound on a weight function measuring the first order deviation of the metrics in terms of the corresponding covariant derivatives ∇^h and ∇^g which is a one-form on M with values in $\operatorname{End}\mathrm{T}M$.

Therefore, we consider the Hodge Laplacian Δ_g, also known as Laplace-de Rham operator, acting on the full exterior bundle $\Omega(M) = \Gamma(\bigwedge \mathrm{T}^*M)$, i.e. the complex separable Hilbert space of differential forms on M. The Hodge Laplacian Δ_g is related to the horizontal Laplacian $\square_g = (\nabla^g)^*\nabla^g$ by the Weitzenböck formula $\Delta_g = \square_g - \mathscr{R}_g$, where the Weitzenböck curvature operator $\mathscr{R}_g \in \Gamma(\operatorname{End}\Omega(M))$ is a symmetric field of endomorphisms. In particular, when acting on 1-forms, $\mathscr{R}_g^{\mathrm{tr}}|_{\Omega^1(M,g)} = \mathrm{Ric}_g$ and, acting on functions, $\mathscr{R}_g^{\mathrm{tr}}|_{\Omega^0(M,g)} = 0$. We assume that $\mathscr{R}_g$ is in the Kato class, i.e. that the fibrewise taken operator norm $|\mathscr{R}_g|_g$ (which is a Borel function on M) of $\mathscr{R}_g$ is in the Kato class (cf. Definition Definition 2.3). We are now in the position to state our main result, cf. Theorem 3.32 below.

Main result. *Assume that g and h are two geodesically complete and quasi-isometric Riemannian metrics on M, denoted $g \sim h$, and assume that there exists $C < \infty$ such that $\left|\delta^{\nabla}_{g,h}\right| \leqslant C$, and that for both $\nu \in \{g, h\}$, $\mathscr{R}_\nu$ is in the Kato class and it holds*

$$\int \max\left\{\delta_{g,h}(x), \delta^{\nabla}_{g,h}(x) + \Xi_g(x,s), \Psi_\nu(x,s)\right\} \Phi_\nu(x,s) \operatorname{vol}_\nu(\mathrm{d}x) < \infty, \qquad \textit{some } s > 0, \tag{3.1}$$

where

- vol_ν *denotes the Riemannian volume measure with respect to the metric ν,*
- $\Psi_\nu(x,s) : M \to (0,\infty)$ *is a function explicitly given terms of local curvature bounds (cf. (2.42) in Section 3.3) and a finite constant $c_\gamma(\underline{\mathscr{R}}^-)$ (cf. (2.7) in ∫ 2),*

- $\Xi_\nu(\cdot, s) : M \to (0,\infty)$ *is a function explicitly given in terms of* $\Psi_\nu(x,s)$ *and an additional local bound on the derivative of the curvature (cf.* (2.42) *in* ∬ *2),*
- $\Phi_\nu(\cdot, s) : M \to (0,\infty)$ *is a local upper bound on the heat kernel acting on functions on* (M,ν) *(cf.* (3.19) *in* ∬ *3.2),*
- $\delta_{g,h} : M \to (0,\infty)$ *a zeroth order deviation of the metrics from each other (cf.* (3.14) *in* ∬ *3.2),*
- $\delta^{\nabla}_{g,h} : M \to [0,\infty)$ *a first order deviation of the metrics (cf.* (3.15) *in* ∬ *3.2).*

Then the wave operators $W_\pm(\Delta_h, \Delta_g, I_{g,h})$ *exist and are complete. Moreover,* $W_\pm(\Delta_h, \Delta_g, I_{g,h})$ *are partial isometries with initial space* $\operatorname{ran} \mathsf{P}_{\mathrm{ac}}(\Delta_g)$ *and final space* $\operatorname{ran} \mathsf{P}_{\mathrm{ac}}(\Delta_h)$*. In particular,*

$$\sigma_{\mathrm{ac}}(\Delta_g) = \sigma_{\mathrm{ac}}(\Delta_h).$$

We will see that a zeroth order deviation $\delta_{g,h}$ of the metrics from each other is induced by quasi-isometry. In comparison to the case of 0-forms, i.e. functions, it turns out that working on higher degree differential forms, also a first order deviation of the metrics $\delta^{\nabla}_{g,h} = \left|\nabla^h - \nabla^g\right|_g^2$ is necessary. But note that $\nabla^h - \nabla^g$ is a one-form on M with values in $\operatorname{End} \mathrm{T}M$.

Remark 3.1. In contrast to previous results, it seems that we are the first to assume global curvature conditions in terms of the Kato class, more precisely, that the Weitzenböck curvature endomorphism is in the Kato class.

To this end, our strategy is to verify the assumptions given by a variant of the Belopol'skii-Birman theorem 3.5 which is adapted to our special case of two Hilbert spaces. The main technical difficulty is to show that the operator

$$\mathbf{T}_s^{g,h} = \Delta_h^{(k)} \mathrm{e}^{s\Delta_h^{(k)}} I_{g,h} \mathrm{e}^{s\Delta_g^{(k)}} - \mathrm{e}^{s\Delta_h^{(k)}} I_{g,h} \mathrm{e}^{s\Delta_g^{(k)}} \Delta_g^{(k)}$$

is trace class. As the product of Hilbert-Schmidt operators is trace class, our idea is to decompose the operator $\mathbf{T}_s^{g,h}$ in such a way that the terms only consist of (transformed) derivations of Hilbert-Schmidt estimates and bounded multiplication operators. In comparison to the corresponding decomposition formula in [GT20, Lemma 4.1], the analysis becomes considerably more difficult because the quadratic form associated to $\Delta_h^{(k)}$ involves not only the exterior derivative $\mathbf{d}^{(k)}$, but also the codifferential $\boldsymbol{\delta}_h^{(k)}$ which depends on the metric by definition. Moreover, we encounter quantities transformed by a smooth vector bundle morphism $\mathscr{A}_{g,h}$ induced by the quasi-isometry (cf. (3.9) below). Using the quasi-isometry of the metrics we can give a formula how to express the codifferential $\boldsymbol{\delta}_h$ with respect to the metric h in terms of the codifferential $\boldsymbol{\delta}_g$ in terms *of* g (cf. Lemma 3.15). Using the metric description for the exterior derivative and the codifferential (cf. Lemma 3.21), we can express the corresponding quantities transformed by $\mathscr{A}_{g,h}$ solely in terms of the covariant derivative ∇^g *of* g applied to the semigroup (cf. Proposition 3.25).

Our tool to obtain the Hilbert-Schmidt estimates for various derivatives of the heat semigroup will be derived probabilistic Bismut-type derivative formulae for the exterior derivative, codifferential and covariant derivative (cf. Theorem 2.27 and 2.28) following the ideas in [DT01]. The gradient estimates are then a direct consequence, and the probabilistic formulae used provide us, in particular, with *explicit* local constants.

We first look at total differential forms, then finally everything filters through the form degree to differential k-forms. The particularly important case, in which two quasi-isometric Riemannian metrics differ by a conformal metric change, is a direct consequence of our main result.

Because our result is independent of the injectivity radii we have the following application to the Ricci flow. Let R_g be the Riemannian curvature tensor with respect to the metric g and set $\dim M =: m$.

Corollary 3.37. *Let $S > 0$, $\lambda \in \mathbb{R}$ and assume that*

(a) the family $(g_s)_{0\leqslant s\leqslant S} \subset \mathsf{Metr}M$ evolves under a Ricci-type flow

$$\partial_s g_s = \lambda \operatorname{Ric}_{g_s}, \qquad \forall 0 \leqslant s \leqslant S,$$

(b) the initial metric g_0 is geodesically complete,

(c) there is some $C > 0$ such that $\left|\mathsf{R}_{g_s}\right|_{g_s}, \left|\nabla^{g_s}\mathsf{R}_{g_s}\right|_{g_s} \leqslant C \qquad \forall 0 \leqslant s \leqslant S.$

We set, for all $x \in M$,

$$\begin{aligned} M_1(x) &:= \sup\left\{\left|\operatorname{Ric}_{g_s}(v,v)\right| : 0 \leqslant s \leqslant S,\ v \in \mathsf{T}_x M,\ |v|_{g_s} \leqslant 1\right\}, \\ M_2(x) &:= \sup\Big\{\left|\nabla^{g_s}_v \operatorname{Ric}_{g_s}(u,w) + \nabla^{g_s}_u \operatorname{Ric}_{g_s}(v,w) + \nabla^{g_s}_w \operatorname{Ric}_{g_s}(u,v)\right| : 0 \leqslant s \leqslant S, \\ &\qquad\qquad u,v,w \in \mathsf{T}_x M,\ |u|_{g_s}, |v|_{g_s}, |w|_{g_s} \leqslant 1\Big\}. \end{aligned}$$

Let $\mathsf{B}_g(x,r)$ denote the open geodesic ball (with respect to g). If

$$\int \operatorname{vol}_{g_0}(\mathsf{B}_{g_0}(x,1))^{-1} \max\left\{\sinh\left(\frac{m}{4} S\,|\lambda|\, M_1(x)\right), M_2(x)\right\} \operatorname{vol}_{g_0}(\mathrm{d}x) < \infty,$$

then $\sigma_{\mathrm{ac}}(\Delta_{g_s}) = \sigma_{\mathrm{ac}}(\Delta_{g_0})$ for all $0 \leqslant s \leqslant S$.

Thereupon, we reify our main results to the case of global curvature bounds: The curvature operator (with respect to the metric g) Q_g is uniquely determined by the equation

$$\left(Q_g(X\wedge Y), U \wedge V\right)_g = \left(\mathsf{R}_g(X,Y)U, V\right)_g$$

for all smooth vector fields $X, Y, U, V \in \Gamma_{C^\infty}(\mathsf{T}M)$. By the Gallot–Meyer estimate [GM75], a global bound $Q_g \geqslant -K$, for some constant $K > 0$, already implies that the curvature endomorphism $\mathscr{R}^{(k)}_g$ in the Weitzenböck formula (1.11) is globally bounded by $\mathscr{R}^{(k)}_g \geqslant -Kk(m-k)$.

Then the function $\Xi_g(x,s)$ can be bounded from above by

$$\Theta_g(x) := \left(1 + \max_{y\in\mathsf{B}_g(x,1)} \left|\nabla^g \mathsf{R}_g(y)\right|\right)^2$$

up to constants uniform in x. In this case, our main result reads as follows.

Theorem 3.44. *Let $Q_\nu \geqslant -K$, for some constant $K > 0$ for both $\nu \in \{g, h\}$. Let $g, h \in \mathrm{Metr}M$ such that $g \sim h$ and assume that there exists $C < \infty$ such that $\left|\delta^\nabla_{g,h}\right| \leqslant C$ and that for some (then both by quasi-isometry) $\nu \in \{g, h\}$*

$$\int \max\left\{\delta_{g,h}(x), \delta^\nabla_{g,h}(x) + \Theta_g(x)\right\} \Phi_\nu(x, s)\,\mathrm{vol}_\nu(\mathrm{d}x) < \infty, \qquad \text{some } s > 0.$$

Then the wave operators $W_\pm(\Delta_h, \Delta_g, I)$ exist and are complete. Moreover, $W_\pm(\Delta_h, \Delta_g, I)$ are partial isometries with initial space $\operatorname{ran}\mathsf{P}_{\mathrm{ac}}(\Delta_g)$ and final space $\operatorname{ran}\mathsf{P}_{\mathrm{ac}}(\Delta_h)$, and we have $\sigma_{\mathrm{ac}}(\Delta_g) = \sigma_{\mathrm{ac}}(\Delta_h)$.

A direct consequence and additional application of our main result is the particularly important case of conformal perturbations under local and global curvature bounds, generalising the results in [BGM17].

A final application is provided through a result by Cheeger, Fukaya and Gromov [CFG92] known as Cheeger-Gromov's thick/thin decomposition: On any complete Riemannian m-manifold (M, g) with bounded *sectional curvature* $\left|\boldsymbol{\kappa}_g\right| \leqslant 1$, there exists a Riemannian metric g_ε on M such that g_ε is ε-quasi-isometric to g and has bounded covariant derivatives. Hence, in this case, the assumptions of our main result may be suitably relaxed, cf. Theorem 3.46.

Let us end the introduction with a short outline of this chapter. § 3.1 briefly motivates und introduces the notion and necessary definitions of the wave operators and the absolutely continuous spectrum. § 3.2 introduces the necessary notation and deviation maps. In § 3.3, we use the gradient estimates proved in § 2.4, to derive similar estimates for exterior derivative, codifferential and covariant derivative of the heat semigroup deformed by a smooth vector bundle homomorphism relating the two quasi-isometric metrics. Our main results are explained in § 3.4. After this, we prove the main result in § 3.5 by making use of a slight variant of the abstract Belopol'skii-Birman Theorem 3.5. We close in § 3.6 with applications to the Ricci flow § 3.6.1, state the main result in the case of differential k-forms 3.6.2, the particularly important cases of conformal perturbations § 3.6.3, specify our results for global curvature bounds § 3.6.4 and ε-close Riemannian metrics § 3.6.5.

3.1 Preliminaries and Motivation

3.1.1 Wave operators, existence and completeness We start this section with a brief motivation on the definition of the wave operators. A comprehensive introduction of the notions and results given can be found in e.g. [RS79, Chapter XI.3] or [Kat95, Chapter X].

Let $\mathscr{H}$ be a complex separable Hilbert space. Given a linear operator H in $\mathscr{H}$ we denote by $\operatorname{dom}\mathsf{H} \subset \mathscr{H}$ its domain, $\operatorname{ran}\mathsf{S} \subset \mathscr{H}$ its range, and $\ker\mathsf{H} \subset \mathscr{H}$ its kernel. Recall, by § 1.2, that given a projection-valued Borel (probability) measure $\mathsf{E} : \mathscr{B}(\mathbb{R}) \to \mathscr{L}(\mathscr{H})$ and a Borel function $f : \mathbb{R} \to \mathbb{R}$ by means of the spectral integral

$$f(\mathsf{P}) = \int f(\lambda)\mathsf{E}(\mathrm{d}\lambda)$$

defines a densely defined operator $f(\mathbf{H})$ in $\mathscr{H}$. Then $f(\mathbf{H})$ is bounded if and only f is bounded on the spectrum $\sigma(\mathbf{H})$ of

$$\mathbf{H} = \int \lambda \, \mathsf{E}(\mathrm{d}\lambda). \tag{3.2}$$

By the Spectral Theorem 1.50, for every self-adjoint operator $\mathbf{H}$ in $\mathscr{H}$ there is a uniquely determined projection-valued Borel (probability) measure $\mathsf{E}_{\mathbf{H}}$ on $\mathbb{R}$ satisfying (3.2). For some arbitrary Borel function f, we set $f(\mathbf{H}) := f(\mathsf{E}_{\mathbf{H}})$. If $\mathbf{H}$ is self-adjoint in $\mathscr{H}$, then for every $\psi \in \operatorname{dom} \mathscr{H}$, the *path*

$$\mathbb{R} \ni t \mapsto \psi(t) := e^{-it\mathbf{H}}\psi \in \mathscr{H}$$

is the unique (norm-)differentiable solution to the *abstract Schrödinger equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = i\mathbf{H}\psi(t), \quad \psi(0) = \psi, \qquad (t \in \mathbb{R}).$$

If $\mathbf{H}$ is also semibounded, then for every $\psi \in \mathscr{H}$

$$[0, \infty) \ni t \mapsto \psi(t) := \mathrm{e}^{-t\mathbf{H}} \in \mathscr{H}$$

is the uniquely determined continuous solution, differentiable on $(0, \infty)$, to the *abstract heat equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = -\mathbf{H}\psi(t), \quad \psi(0) = \psi, \qquad (t \in \mathbb{R}).$$

By Stone's Theorem 1.52, $\left(\mathrm{e}^{-t\mathbf{H}}\right)_{s>0}$ gives rise to a the semigroup of operators called the *heat semigroup of* $\mathbf{H}$.

Besides the explicit approach of studying the corresponding scattering matrices, another approach to scattering theory does not involve solving the Schrödinger equations. Physically we investigate the time evolution of a particle coming from a region where it interacts with a perturbation potential and leaving this region again. The particle looks *asymptotically free* for $t \to \pm\infty$ if the potential is negligible outside this region, so there is less (or almost no) interaction at large scale. For example, quarks interact weakly at high energies.

Now, let $\mathbf{H}_1$ and $\mathbf{H}_2$ be self-adjoint operators in Hilbert spaces $\mathscr{H}_1$, the *free*, and $\mathscr{H}_2$, the *interactive system*. By Stone's Theorem 1.52, they generate unitary evolution semigroups $\mathbf{U}_t^1 = \mathrm{e}^{-it\mathbf{H}_1}$ and $\mathbf{U}_t^2 = \mathrm{e}^{-it\mathbf{H}_2}$. Let $\varphi_t := \mathrm{e}^{-it\mathbf{H}_2} f$, $\varphi_0 = f$ be the solution to the Schrödinger equation. Moreover, we define an *identification operator* $I : \mathscr{H}_1 \to \mathscr{H}_2$ connecting the free and the perturbated system.

A quantum state $\varphi_t \in \mathscr{H}_2$ looks **asymptotically free** as $t \to \infty$ if we expect to have

$$\exists \mathscr{H}_1 \ni \varphi_t^{1,+} = \mathrm{e}^{-it\mathbf{H}_1} f_1^+ : \quad \left\| \varphi_t - I\varphi_t^{1,+} \right\|_{\mathscr{H}_2} \overset{t\to\infty}{\longrightarrow} 0.$$

Thus, by the definition of $\varphi_t^{1,+}$ and using that $\mathrm{e}^{-it\mathbf{H}_2}$ is unitary, we get

$$\left\| \mathrm{e}^{-it\mathbf{H}_2} \left(f - \mathrm{e}^{it\mathbf{H}_2} I \mathrm{e}^{-it\mathbf{H}_1} f_1^+ \right) \right\| \overset{t\to\infty}{\longrightarrow}_{\mathscr{H}_2} 0. \tag{3.3}$$

This gives rise to the so-called *wave operators* $W_\pm : \mathscr{H}_1 \to \mathscr{H}_2$

$$\begin{aligned} \operatorname{dom} W_\pm &:= \left\{ \varphi \in \mathscr{H}_1 \,:\, \lim_{t\to\pm\infty} \mathrm{e}^{it\mathsf{H}_2} I \mathrm{e}^{-it\mathsf{H}_1}\varphi \text{ exists in the strong sense in } \mathscr{H}_2 \right\} \\ W_\pm &:= \operatorname*{s-lim}_{t\to\pm\infty} \mathrm{e}^{it\mathsf{H}_2} I \mathrm{e}^{-it\mathsf{H}_1}\varphi \qquad \forall \varphi \in \operatorname{dom} W_\pm. \end{aligned} \tag{3.4}$$

The wave operator W_+ maps every «free state» f_1^+ to the corresponding «scattering state» f. If W_+ exists, then every state in its range eventually moves «freely» in the sense of (3.3). Although it might be physically more natural to define the wave operator as the inverse of W_+, it is a priori unclear which states are scattering states (and hence the domain of definition of such an operator). By the very definition (3.4) we can answer this question by controlling the range of W_+. Analogously the wave operator W_- reflects the behaviour in the «distant past».

Further, from a physical point of view, only the action of $W_\pm$ onto certain subspaces of the Hilbert spaces are relevant: For some $h \in \mathscr{H}$ in a Hilbert space $\mathscr{H}$, we take μ_h to be the corresponding spectral measure of $\mathscr{H}$ on the spectrum $\sigma(\mathscr{H})$. By Lebesgue's decomposition theorem (cf. e.g. [Rud87, Section 6.9, The Theorem of Lebesgue-Radon-Nikodým]) there is a unique decomposition of μ_h into three mutually parts

$$\mu_h = \mu_{\mathrm{ac}} \oplus \mu_{\mathrm{sc}} \oplus \mu_{\mathrm{pp}},$$

the absolutely continuous μ_{ac} part (with respect to the Lebesgue measure), the singular part μ_{sc} (with respect to the Lebesgue measure which is atomless), and the pure point measure μ_{pp}. By the Spectral Theorem 1.50, from this we get a decomposition

$$\mathscr{H} = \mathscr{H}_{\mathrm{ac}} \oplus \mathscr{H}_{\mathrm{sc}} \oplus \mathscr{H}_{\mathrm{pp}}, \tag{3.5}$$

where $\mathscr{H}_{\mathrm{ac}}$ consists of vectors whose spectral measures are absolutely continuous with respect to the Lebesgue measure. Analogously, we define $\mathscr{H}_{\mathrm{sc}}$ and $\mathscr{H}_{\mathrm{pp}}$, respectively. The singular subspaces $\mathscr{H}_{\mathrm{sc}}$, $\mathscr{H}_{\mathrm{pp}}$ describe physically irrelevant states whereas $\mathscr{H}_{\mathrm{ac}}$ relates to the scattering states. We therefore make the following definition.

Definition 3.2. Let H_1 and H_2 be self-adjoint operators on Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively, $I : \mathscr{H}_1 \to \mathscr{H}_2$ be a bounded operator and $\mathsf{P}_{\mathrm{ac}}(\mathsf{H}_1)$ be the projection onto the absolutely continuous subspace of H_1. The **(generalised) wave operators** $W_\pm(\mathsf{H}_2, \mathsf{H}_1, I)$ exist if the strong limits

$$W_\pm(\mathsf{H}_2, \mathsf{H}_1, I) = \operatorname*{s-lim}_{t\to\pm\infty} \mathrm{e}^{it\mathsf{H}_2} I \mathrm{e}^{-it\mathsf{H}_1} \mathsf{P}_{\mathrm{ac}}(\mathsf{H}_1)$$

exist.

Note that $W_\pm(\mathsf{H}_2, \mathsf{H}_1, I)$ may not be isometries. A further assumption is that each scattering state looks asymptotically free which is reflected in the next definition.

Definition 3.3. Suppose that $W_\pm(\mathsf{H}_2, \mathsf{H}_1, I)$ exist. We say that they are **complete** if and only if

$$\left(\ker W_\pm(\mathsf{H}_2, \mathsf{H}_1, I)\right)^\perp = \operatorname{ran} \mathsf{P}_{\mathrm{ac}}(\mathsf{H}_1), \qquad \overline{\operatorname{ran} W_\pm(\mathsf{H}_2, \mathsf{H}_1, I)} = \operatorname{ran} \mathsf{P}_{\mathrm{ac}}(\mathsf{H}_2).$$

Moreover by the decomposition (3.5), the spectrum $\sigma(\mathbf{H})$ of $\mathbf{H}$ also splits into three parts given by

$$\sigma(\mathbf{H}) = \sigma_{\mathrm{ac}}(\mathbf{H}) \cup \sigma_{\mathrm{sc}}(\mathbf{H}) \cup \sigma_{\mathrm{pp}}(\mathbf{H}),$$

where

(i) $\sigma_{\mathrm{ac}}(\mathbf{H}) := \sigma(\mathbf{H}_{\mathrm{ac}})$ is called the **absolutely continuous spectrum** of $\mathbf{H}$,

(ii) $\sigma_{\mathrm{sc}}(\mathbf{H}) := \sigma(\mathbf{H}_{\mathrm{sc}})$ is called the **singular spectrum** of $\mathbf{H}$,

(iii) $\sigma_{\mathrm{pp}}(\mathbf{H})$ is the set of eigenvalues of $\mathbf{H}$, called the **pure point spectrum** of $\mathbf{H}$.

More precisely, given a self-adjoint operator $\mathbf{H}$ in a Hilbert space $\mathscr{H}$ with its operator valued spectral measure $\mathsf{E}_{\mathbf{H}}$, we define the **$\mathbf{H}$-absolutely continuous subspace $\mathscr{H}_{\mathrm{ac}}(\mathbf{H})$ of $\mathscr{H}$** to be the space of all $f \in \mathscr{H}$ such that the Borel measure $\|\mathsf{E}_{\mathbf{H}}(\cdot)f\|^2$ on $\mathbb{R}$ is absolutely continuous with respect to the Lebesgue measure. Then $\mathscr{H}_{\mathrm{ac}}(\mathbf{H})$ becomes a closed subspace of $\mathscr{H}$ and the restriction $\mathbf{H}_{\mathrm{ac}}$ of $\mathbf{H}$ to $\mathscr{H}_{\mathrm{ac}}(\mathbf{H})$ is a well-defined self-adjoint operator. So the **absolutely continuous spectrum $\sigma_{\mathrm{ac}}(\mathbf{H})$ of $\mathbf{H}$** is defined to be the spectrum of $\mathbf{H}_{\mathrm{ac}}$.

The following, fundamental theorem provides a criteria for the existence and completeness of the wave operators that can be found e.g. in [RS79, Theorem X1.13].

Theorem 3.4 (Classical Belopol'skii-Birman Theorem). *For $j = 1, 2$, let $\mathbf{H}_j \geqslant 0$ be self-adjoint operators in a Hilbert space $\mathscr{H}_j$, $\mathbf{q}_j$ the corresponding sesquilinear form, and $\mathsf{P}_{\mathrm{ac}}(\mathbf{H}_j)$ the projection onto the absolutely continuous subspace of $\mathscr{H}_j$ corresponding to $\mathbf{H}_j$. Assume that $I \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ is a bounded operator such that the following assumptions hold:*

(1) I has a two-sided bounded inverse

(2) For any bounded interval $J \subset \mathbb{R}$:

$$\mathsf{E}_{\mathbf{H}_2}(J)\left(\mathbf{H}_2 I - I\mathbf{H}_1\right)\mathsf{E}_{\mathbf{H}_1}(J) \in \mathscr{J}^1(\mathscr{H}_1, \mathscr{H}_2)$$

(3) For any bounded interval J, the operator $(I^*I - 1)E_J(\mathbf{H}_2) \in \mathscr{J}^\infty(\mathscr{H}_1)$, i.e. is compact

(4) and either

$$I \operatorname{dom} \mathbf{H}_1 = \operatorname{dom} \mathbf{H}_2 \qquad \text{or} \qquad I \operatorname{dom} \mathbf{q}_1 = \operatorname{dom} \mathbf{q}_2$$

Then the wave operators $W_\pm(\mathbf{H}_2, \mathbf{H}_1, I)$ exist and are complete. Moreover, $W_\pm(\mathbf{H}_2, \mathbf{H}_1, I)$ are partial isometries with initial space $\operatorname{ran} \mathsf{P}_{\mathrm{ac}}(\mathbf{H}_1)$ and final space $\operatorname{ran} \mathsf{P}_{\mathrm{ac}}(\mathbf{H}_2)$, and we have

$$\sigma_{\mathrm{ac}}(\mathbf{H}_1) = \sigma_{\mathrm{ac}}(\mathbf{H}_2).$$

In the proof of Theorem 3.32, we will use a variant of the Belopol'skii-Birman Theorem 3.5, which is adapted to our special case of two Hilbert space scattering theory, originally to be found in [GT20].

Theorem 3.5 (Belopol'skii-Birman). *For $j = 1, 2$, let $\mathbf{H}_j \geqslant 0$ be self-adjoint operators in a Hilbert space $\mathscr{H}_j$ and $\mathsf{P}_{\mathrm{ac}}(\mathbf{H}_j)$ the projection onto the absolutely continuous subspace of $\mathscr{H}_j$ corresponding to $\mathbf{H}_j$. Assume that $I \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ is a bounded operator such that the following assumptions hold:*

(1) *I has a two-sided bounded inverse*

(2) *We have either* $I\,\mathrm{dom}\,\sqrt{\mathsf{H}_1} = \mathrm{dom}\,\sqrt{\mathsf{H}_2}$ *or* $I\,\mathrm{dom}\,\mathsf{H}_1 = \mathrm{dom}\,\mathsf{H}_2$

(3) *The operator* $(I^*I-1)\mathrm{e}^{-s\mathsf{H}_1} : \mathscr{H}_1 \to \mathscr{H}_1$ *is compact for some* $s > 0$

(4) *There is a trace class operator* $\mathsf{T} : \mathscr{H}_1 \to \mathscr{H}_2$ *and a number* $s > 0$ *such that for all* $\alpha_1 \in \mathrm{dom}\,\mathsf{H}_1$, $\alpha_2 \in \mathrm{dom}\,\mathsf{H}_2$ *we have*

$$\langle \alpha_2, \mathsf{T}\alpha_1 \rangle_{\mathscr{H}_2} = \left\langle \mathsf{H}_2\alpha_2, \mathrm{e}^{-s\mathsf{H}_2} I \mathrm{e}^{-s\mathsf{H}_1}\alpha_1 \right\rangle_{\mathscr{H}_2} - \left\langle \alpha_2, \mathrm{e}^{-s\mathsf{H}_2} I \mathrm{e}^{-s\mathsf{H}_1}\mathsf{H}_1\alpha_1 \right\rangle_{\mathscr{H}_2}.$$

Then the wave operators

$$W_\pm(\mathsf{H}_2, \mathsf{H}_1, I) = \operatorname*{s-lim}_{t\to\pm\infty} \mathrm{e}^{it\mathsf{H}_2} I \mathrm{e}^{-it\mathsf{H}_1} \mathsf{P}_{\mathrm{ac}}(\mathsf{H}_1)$$

exist and are complete, where completeness means that

$$\left(\ker W_\pm(\mathsf{H}_2,\mathsf{H}_1,I)\right)^\perp = \mathrm{ran}\,\mathsf{P}_{\mathrm{ac}}(\mathsf{H}_1), \qquad \overline{\mathrm{ran}\,W_\pm(\mathsf{H}_2,\mathsf{H}_1,I)} = \mathrm{ran}\,\mathsf{P}_{\mathrm{ac}}(\mathsf{H}_2).$$

Moreover, $W_\pm(\mathsf{H}_2,\mathsf{H}_1,I)$ *are partial isometries with initial space* $\mathrm{ran}\,\mathsf{P}_{\mathrm{ac}}(\mathsf{H}_1)$ *and final space* $\mathrm{ran}\,\mathsf{P}_{\mathrm{ac}}(\mathsf{H}_2)$*, and we have*

$$\sigma_{\mathrm{ac}}(\mathsf{H}_1) = \sigma_{\mathrm{ac}}(\mathsf{H}_2).$$

Proof. In view of Theorem 3.4 and the proof in [RS79, Theorem XI.13], it remains to show that for every bounded interval J the operator $(I^*I-1)\mathsf{E}_{\mathsf{H}_1}(J)$ is compact, and that there exists a trace class operator $\mathsf{T} \in \mathscr{J}^1(\mathscr{H}_1,\mathscr{H}_2)$ such that for every bounded interval J and all α_1, α_2 as above we have

$$\langle f, \mathsf{T}\alpha_1 \rangle_{\mathscr{H}_2} = \left\langle \mathsf{H}_2\alpha_2, \mathsf{E}_{\mathsf{H}_2}(J) I \mathsf{E}_{\mathsf{H}_1}(J)\alpha_1 \right\rangle_{\mathscr{H}_2} - \left\langle \alpha_2, \mathsf{E}_{\mathsf{H}_2}(J) I \mathsf{E}_{\mathsf{H}_1}(J)\mathsf{H}_1\alpha_1 \right\rangle_{\mathscr{H}_2}.$$

However, using that for all self-adjoint operators H and all Borel functions $f, g : \mathbb{R} \to \mathbb{C}$, we have

$$f(\mathsf{H})g(\mathsf{H}) \subset (fg)(\mathsf{H}), \quad \mathrm{dom}(f(\mathsf{H})g(\mathsf{H})) = \mathrm{dom}(f(\mathsf{H})g(\mathsf{H})) \cap \mathrm{dom}\,g(\mathsf{H}),$$

the required compactness becomes obvious, and furthermore

$$\mathsf{T} := \mathrm{e}^{s\mathsf{H}_2}\mathsf{E}_{\mathsf{H}_2}(J) I \mathrm{e}^{s\mathsf{H}_1}\mathsf{E}_{\mathsf{H}_1}(J)$$

has the required trace class property. ■

By a classical result of Kato (cf. [Kat95, X. Perturbation of continuous spectra and unitary equivalence] and [Kat95, p. 534, Theorem 3.5.] therein), Theorem 3.5 implies

Theorem 3.6. *Assume in the above situation that the wave operators* $W_\pm(H_2, H_1, I)$ *exist and are complete. Then the operators* $H_{1,\mathrm{ac}}$ *and* $H_{2,\mathrm{ac}}$ *are unitarily equivalent. In particular, we have*

$$\sigma_{\mathrm{ac}}(H_1) = \sigma_{\mathrm{ac}}(H_2).$$

3.2 Setting and Notation

In the remainder this chapter, we must carefully distinguish between the underlying quasi-isometric complete Riemannian metrics g and h in the notation.

Therefore, let (M, g) be a complete smooth Riemannian manifold without boundary of dimension $m := \dim M \geqslant 2$ and $(\cdot,\cdot)_g$ its Riemannian metric. We write vol_g for the corresponding volume measure (with respect to the metric g) and denote by $\mathsf{Metr}M$ the set of all smooth Riemannian metrics on M. All bundles will be understood complexified, e.g. the full exterior bundle

$$\bigwedge \mathsf{T}^* M = \bigoplus_{k=0}^{m} \bigwedge^k \mathsf{T}^* M, \qquad \text{with the usual convention } \bigwedge^0 \mathsf{T}^* M = \mathbb{C}. \tag{3.6}$$

Given smooth complex vector bundles $E_1 \to M$ and $E_2 \to M$ the complex linear space of smooth linear partial differential operators from E_1 to E_2 of order $\leqslant k \in \mathbb{N}_0$ is denoted by $\mathscr{D}^{(k)}(M; E_1, E_2)$, with shorthand notation $\mathscr{D}^{(k)}(M; E_1)$ if $E_1 \equiv E_2$. On a vector bundle $E \to M$ (e.g. $E = \bigwedge^k \mathsf{T}^* M$) the corresponding fibre norms are denoted by

$$|\varphi|_g := (\varphi, \varphi)_g^{1/2} \qquad \text{for any section } \varphi \in \Gamma(E),$$

where $\Gamma(E) := \Gamma_{\mathsf{C}^\infty}(E)$ denotes all *smooth* sections of E and $\Gamma_{\mathsf{L}^2}(E)$ the L^2-section of E.

In the case of $E = \bigwedge^k \mathsf{T}^* M$, we indicate the corresponding form degree by an index: For example, $\nabla^{g,(k)}$ or $(\cdot,\cdot)_g^{(k)}$ etc.

We denote by $\Omega_{\mathsf{L}^2}(M, g) := \Gamma_{\mathsf{L}^2}(\bigwedge \mathsf{T}^*(M, g))$ the complex separable Hilbert space of equivalence classes α of square-integrable Borel forms on M such that

$$\|\alpha\|_g^2 := \|\alpha\|^2_{\Omega_{\mathsf{L}^2}(M,g)} := \int_M |\alpha(x)|_g^2 \,\mathrm{vol}_g(\mathrm{d}x) < \infty,$$

with inner product

$$\langle \alpha, \beta \rangle_g := \langle \alpha, \beta \rangle_{\Omega_{\mathsf{L}^2}(M,g)} := \int_M (\alpha(x), \beta(x))_g \,\mathrm{vol}_g(\mathrm{d}x).$$

Analogously, we write $\Omega^k_{\mathsf{L}^2}(M, g)$ for the Hilbert space of Borel k-forms. In particular,

$$\Omega_{\mathsf{L}^2}(M, g) = \bigoplus_{k=0}^{m} \Omega^k_{\mathsf{L}^2}(M, g).$$

To relax notation, we set

$$\Omega(M, g) := \Omega_{\mathsf{C}^\infty}(M, g) \qquad \text{and} \qquad \Omega^k(M, g) := \Omega^k_{\mathsf{C}^\infty}(M, g).$$

for the set of all *smooth* forms, and *smooth* k-forms respectively, on (M, g).

Further, for some $\alpha \in \Omega^1(M)$, we denote by

$$\bullet \wedge \alpha \in \mathscr{D}^{(0)}(M; \bigwedge^k \mathsf{T}^* M, \bigwedge^{k+1} \mathsf{T}^* M)$$

the exterior product and its formal adjoint with respect to g, the interior multiplication, by

$$\bullet \lrcorner_g\, \alpha := (\bullet \wedge \alpha)^{*_g} \in \mathscr{D}^{(0)}(M; \bigwedge^k \mathsf{T}^* M, \bigwedge^{k-1} \mathsf{T}^* M).$$

The interior multiplication corresponds to the contraction of $\alpha \in \Omega^k(M)$ with a vector field $X \in \Gamma(\mathsf{T}M)$ and is an antiderivation, cf. Definition 1.35.

We denote by

$$\mathbf{d}^{(k)} \in \mathscr{D}^{(1)}(M; \bigwedge^k \mathsf{T}^*M, \bigwedge^{k+1} \mathsf{T}^*M)$$
$$\boldsymbol{\delta}_g^{(k)} \in \mathscr{D}^{(1)}(M; \bigwedge^k \mathsf{T}^*M, \bigwedge^{k-1} \mathsf{T}^*M)$$

the exterior derivative on k-forms and, respectively, the codifferential as the formal adjoint of $\mathbf{d}^{(k-1)}$. Then the Hodge Laplacian can be written as the sum

$$\Delta_g^{(k)} := -\left(\boldsymbol{\delta}_g^{(k+1)}\mathbf{d}^{(k)} + \mathbf{d}^{(k-1)}\boldsymbol{\delta}_g^{(k)}\right) \in \mathscr{D}^{(2)}(M; \bigwedge^k \mathsf{T}^*M)$$

and its Friedrichs realisation in $\Omega^k_{\mathsf{L}^2}(M,g)$ will be again denoted by $\Delta_g^{(k)} \leqslant 0$. In particular, for $k = 0$, we recover the special case of the Laplace-Beltrami operator acting on 0-forms, i.e. functions,

$$\Delta_g^{(0)} = -\boldsymbol{\delta}_g^{(1)}\mathbf{d}^{(0)} \in \mathscr{D}^{(2)}(M).$$

Furthermore, we set

$$\mathbf{d} := \bigoplus_{k=0}^{m} \mathbf{d}^{(k)} \in \mathscr{D}^{(1)}(M; \bigwedge \mathsf{T}^*M)$$
$$\boldsymbol{\delta}_g := \bigoplus_{k=0}^{m} \boldsymbol{\delta}_g^{(k)} \in \mathscr{D}^{(1)}(M; \bigwedge \mathsf{T}^*M)$$

and define the underlying Dirac-type operator $\mathbf{D}_g$, and the (total) Hodge Laplacian Δ_g

$$\mathbf{D}_g := \mathbf{d} + \boldsymbol{\delta}_g \in \mathscr{D}^{(1)}(M; \bigwedge \mathsf{T}^*M)$$
$$\Delta_g := -\mathbf{D}_g^2 \in \mathscr{D}^{(2)}(M; \bigwedge \mathsf{T}^*M)$$

where the Friedrichs realisation of Δ_g in $\Omega^k_{\mathsf{L}^2}(M,g)$ will be again denoted by $\Delta_g \leqslant 0$. In particular,

$$\Delta_g\big|_{\Omega^k(M,g)} = \Delta_g^{(k)} \in \mathscr{D}^{(2)}(M; \bigwedge^k \mathsf{T}^*M)$$

and

$$\Delta_g = \bigoplus_{k=0}^{m} \Delta_g^{(k)} \qquad \text{as self-adjoint operators.}$$

Since g is (geodesically) complete, it follows that the operators $\mathbf{D}_g, \Delta_g, \Delta_g^{(k)}$ are essentially self-adjoint on the corresponding space of smooth compactly supported forms [Str83].

Next, recall that for the k-fold exterior product of the vector space T^*M, we obtain a scalar product $(\cdot,\cdot)_g$ on $\bigwedge^k \mathsf{T}^*M$ by the bilinear extension of

$$\left(\alpha_1 \wedge \ldots \wedge \alpha_k, \beta_1 \wedge \ldots \wedge \beta_k\right)_g = \det\left(\alpha_j, \beta_l\right)_g. \tag{3.7}$$

Any $A \in \mathrm{End}(\mathsf{T}^*M)$ induces a linear map

$$\bigwedge^m A : \bigwedge^m \mathsf{T}^*M \to \bigwedge^m \mathsf{T}^*M$$
$$\alpha_1 \wedge \ldots \wedge \alpha_m \mapsto A\alpha_1 \wedge \ldots \wedge A\alpha_m.$$

As $\bigwedge^m \mathsf{T}^*M$ is one-dimensional, the map $\bigwedge^m A$ is given by multiplication with a unique number, denoted by $\det A$,

$$\bigwedge^m A(e_1 \wedge \ldots \wedge e_m) = (\det A)\, e_1 \wedge \ldots \wedge e_m,$$

where $(e_1, \ldots, e_m)$ is a basis for $\bigwedge^m \mathsf{T}_x^*M$.

A Riemannian metric $(u,v)_g = g(u,v)$ for $u, v \in \mathsf{T}_xM$ gives by definition an inner product on each tangent space T_xM $(x \in M)$. By Riesz' representation theorem, g provides a natural isomorphism between tangent and cotangent bundle given by $v \mapsto (v, \cdot)_g$,

$$\mathsf{T}M \underset{\sharp^g}{\overset{\flat^g}{\rightleftarrows}} \mathsf{T}^*M.$$

More precisely, we define the **sharp operator** $\sharp^g$ (with respect to g) by

$$\sharp^g : \mathsf{T}^*M \to \mathsf{T}M, \qquad \alpha(v) = g(\alpha^{\sharp^g}, v).$$

The Riemannian metric g defines a metric g on T^*M, the **cometric**, via

$$g(\alpha, \beta) := g(\alpha^{\sharp^g}, \beta^{\sharp^g}) \qquad \forall \alpha, \beta \in \mathsf{T}_x^*M \ \forall x \in M,$$

which extends to a metric on $\bigwedge^k \mathsf{T}^*M$ according to (3.7).

Given $g, h \in \mathrm{Metr}M$, we define a vector bundle morphism

$$A := A_{g,h} : \mathsf{T}M \xrightarrow{\sim} \mathsf{T}M, \qquad h(u,v) = g(Au, v), \qquad \forall x \in M \ \forall u, v \in \mathsf{T}_xM. \tag{3.8}$$

Note that the vector bundle morphism $A = A_{g,h}$ induces a vector bundle morphism on the cotangent bundle via

$$A : \mathsf{T}^*M \xrightarrow{\sim} \mathsf{T}^*M, \qquad \alpha \mapsto A\alpha := \alpha \circ A.$$

Lemma 3.7 (and **Definition**). *In terms of the notations above, we have*

$$h(\alpha, \beta) = g(A^{-1}\alpha, \beta) \qquad \forall \alpha, \beta \in \mathsf{T}_x^*M \ \forall x \in M.$$

Extending $A^{-1} = A_{g,h}^{-1}$ to a smooth vector bundle morphism by

$$\mathscr{A} := \mathscr{A}_{g,h}(x) := (\bigwedge A_{g,h}^{-1})_x : \bigwedge \mathsf{T}^*M \xrightarrow{\sim} \bigwedge \mathsf{T}^*M, \qquad \mathscr{A}\alpha := \alpha \circ \mathscr{A}, \tag{3.9}$$

we obtain

$$g(\mathscr{A}_{g,h}(x)\alpha, \beta) = h(\alpha, \beta) \qquad \text{for } \ x \in M, \ \alpha, \beta \in \bigwedge \mathsf{T}_x^*M.$$

In the following the induced metrics will be understood complexified (conjugate-linear in the first variable and linear in the second).

Remark 3.8. (i) By the positive-definiteness of h (or g), $\mathscr{A}_{g,h}(x)$ has only positive eigenvalues $(x \in M)$. By the symmetry of g and h the endomorphism $\mathscr{A}_{g,h}$ is fibrewise self-adjoint with respect to g and h. Therefore, the fibrewise operator norm $|\cdot|_g$ (or $|\cdot|_h$) induced by the metric g (or h) of $\mathscr{A}$ is equivalent to absolute value of the largest eigenvalue on the given fibre for both g and h. Thus to relax notation, we may suppress the metric and simply write $|\mathscr{A}|$.

(ii) By the very definition, $\mathscr{A}^{1/2}$ is a (pointwise) isometry from $(\bigwedge \mathrm{T}^*M, g)$ to $(\bigwedge \mathrm{T}^*M, h)$.

Proof of Lemma 3.7. We prove the Lemma in several steps.

1° We calculate the sharp operator in the new metric. For $x \in M$, let $v \in \mathrm{T}_xM$ and $\alpha \in \mathrm{T}^*_xM$. By duality,

$$g\left(\alpha^{\sharp g}, v\right) = \alpha(v) = h\left(\alpha^{\sharp h}, v\right) = g\left(A\alpha^{\sharp h}, v\right) \qquad \Longrightarrow \qquad \alpha^{\sharp g} = A\alpha^{\sharp h}, \tag{3.10}$$

for all $v \in \mathrm{T}_xM$.

2° Let $\alpha, \beta \in \mathrm{T}^*_xM$, then

$$\begin{aligned} h(\alpha, \beta) &= h\left(\alpha^{\sharp^h}, \beta^{\sharp^h}\right) \\ &= g\left(A\alpha^{\sharp^h}, \beta^{\sharp^h}\right) = g(\alpha^{\sharp^h}, A\beta^{\sharp^h}) = g(A^{-1}\alpha^{\sharp^g}, \beta^{\sharp^g}) = g(\alpha \circ A^{-1}, \beta), \end{aligned}$$

where we used that $A^{-1}\alpha^{\sharp g} = (\alpha \circ A^{-1})^{\sharp g}$ in the last equality.

3° For any $\alpha, \beta \in \Omega^k(M)$,

$$\begin{aligned} h(\alpha, \beta) &= h\left(\alpha_1^{\sharp h} \wedge \ldots \wedge \alpha_k^{\sharp h}, \beta_1^{\sharp h} \wedge \ldots \wedge \beta_k^{\sharp h}\right) \\ &\overset{(3.7)}{=} \det\left(\alpha_k^{\sharp h}, \beta_l^{\sharp h}\right)_h \\ &= \det\left(A\alpha_k^{\sharp h}, \beta_l^{\sharp h}\right)_g \\ &\overset{(3.10)}{=} \det\left(\alpha_k^{\sharp g}, A^{-1}\beta_l^{\sharp g}\right)_g \\ &\underset{\text{s.a.}}{\overset{A}{=}} \det\left(A^{-1}\alpha_k^{\sharp g}, \beta_l^{\sharp g}\right)_g \\ &= g\left(A^{-1}\alpha_1^{\sharp g} \wedge \ldots \wedge A^{-1}\alpha_k^{\sharp g}, \beta_1^{\sharp g} \wedge \ldots \wedge \beta_k^{\sharp g}\right) \\ &= g\left(\textstyle\bigwedge^k A^{-1}\left(\alpha_1 \wedge \ldots \wedge \alpha_k\right), \beta_1 \wedge \ldots \wedge \beta_k\right) = g\left(\mathscr{A}\alpha, \beta\right). \end{aligned}$$

■

The following estimates are necessary tools for the main proof noting that it is independent of the quasi-isometry of g and h.

Lemma 3.9. *Let $\mathscr{A} := \mathscr{A}_{g,h}$ be the smooth vector bundle morphism defined by (3.9). For any vector field $X \in \Gamma(\mathrm{T}M)$, we get*

$$\nabla^g_X \mathscr{A} = \mathscr{A}(\nabla^h_X - \nabla^g_X) + (\nabla^h_X - \nabla^g_X)^* \mathscr{A},$$

and the pointwise estimate

$$\left|\nabla^g_X \mathscr{A}\right|_g \leqslant 2\left|\mathscr{A}\right|\left|\nabla^h_X - \nabla^g_X\right|_g, \tag{3.11}$$

where $|\cdot|_g$ denotes the operator norm induced by the inner product g.

Proof. We divide the proof into two steps.

1° Differentiating the identity

$$(\alpha, \beta)_h = (\mathscr{A}\alpha, \beta)_g$$

in direction of X, on the one hand,

$$\begin{aligned} X(\alpha,\beta)_h &= \left(\nabla^h_X\alpha,\beta\right)_h + \left(\alpha,\nabla^h_X\beta\right)_h \\ &= \left(\mathscr{A}\nabla^h_X\alpha,\beta\right)_g + \left(\mathscr{A}\alpha,\nabla^h_X\beta\right)_g \end{aligned}$$

and on the other hand,

$$\begin{aligned} X(\mathscr{A}\alpha,\beta)_g &= \left(\nabla^g_X(\mathscr{A}\alpha),\beta\right)_g + \left(\mathscr{A}\alpha,\nabla^g_X\beta\right)_g \\ &= \left((\nabla^g_X\mathscr{A})\alpha,\beta\right)_g + \left(\mathscr{A}\nabla^g_X\alpha,\beta\right)_g + \left(\mathscr{A}\alpha,\nabla^g_X\beta\right)_g . \end{aligned}$$

Hence,

$$\left((\nabla^g_X\mathscr{A})\alpha,\beta\right)_g = \left(\mathscr{A}(\nabla^h_X-\nabla^g_X)\alpha,\beta\right)_g + \left(\alpha,\mathscr{A}(\nabla^h_X-\nabla^g_X)\beta\right)_g ,$$

using the self-adjointness of $\mathscr{A}$.

2^o We estimate

$$\begin{aligned} \left|\nabla^g_X\mathscr{A}\right|_g &\leqslant \sup_{|v|,|w|\leqslant 1}\left|\left((\nabla^g_X\mathscr{A})v,w\right)_g\right| \\ &\leqslant \sup_{|v|,|w|\leqslant 1}\left(\left|\left(\mathscr{A}(\nabla^h_X-\nabla^g_X)v,w\right)_g\right| + \left|\left(v,\mathscr{A}(\nabla^h_X-\nabla^g_X)w\right)_g\right|\right) \\ &\leqslant 2\,|\mathscr{A}|\,\left|\nabla^h_X-\nabla^g_X\right|_g . \end{aligned}$$

■

Lemma 3.10. *Let $\mathscr{A} := \mathscr{A}_{g,h}$ be the smooth self-adjoint vector bundle morphism defined by (3.9). Then also $\mathscr{A}^{1/2}$ and, for any $X \in \Gamma(\mathrm{T}M)$, $\nabla^g_X\mathscr{A}^{1/2}$ are self-adjoint.*

Proof. By definition, the smooth vector bundle morphism $\mathscr{A}$ is self-adjoint, so clearly is $\mathscr{A}^{1/2}$.

Let $X \in \Gamma(\mathrm{T}M)$ and $\gamma : (-\varepsilon,\varepsilon) \to M$ be a smooth curve $\gamma(0) = x$ and $\dot\gamma(0) = X$ for any $\varepsilon > 0$. By Lemma 1.23, for any $Y, Z \in \Gamma(\mathrm{T}M)$,

$$\begin{aligned} \left(\nabla_X\mathscr{A}^{1/2}(Y),Z\right)_g &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\left(/\!/_t^{-1}\mathscr{A}^{1/2}_{\gamma(t)}(Y),Z\right)_g \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\left(Y,/\!/_t^{-1}\mathscr{A}^{1/2}_{\gamma(t)}(Z)\right)_g = \left(Y,\nabla_X\mathscr{A}^{1/2}(Z)\right)_g , \end{aligned}$$

using the self-adjointness of $\mathscr{A}^{1/2}$. ■

We point out that a similar argument was recently developed in [BG20] to prove the estimates in the following

Lemma 3.11. *Let $\mathscr{A} := \mathscr{A}_{g,h}$ be the smooth vector bundle morphism defined by (3.9). For any vector field $X \in \Gamma(\mathrm{T}M)$, we get the pointwise estimates*

$$\begin{aligned} \left|\nabla^g_X\mathscr{A}^{1/2}\right|_g &\leqslant |\mathscr{A}|\,\left|\mathscr{A}^{-1/2}\right|\,\left|\nabla^h_X-\nabla^g_X\right|_g \\ \left|\nabla^g_X\mathscr{A}^{-1/2}\right|_g &\leqslant |\mathscr{A}|\,\left|\mathscr{A}^{-1/2}\right|^3\,\left|\nabla^h_X-\nabla^g_X\right|_g , \end{aligned}$$

where $|\cdot|_g$ denotes the operator norm induced by the inner product g.

Proof. We divide the proof into three steps.

1° Recall that $\mathscr{A}^{1/2}$ is an isometry, so that

$$\left(\mathscr{A}^{1/2}\alpha, \mathscr{A}^{1/2}\alpha\right)_g = (\alpha, \alpha)_h .$$

Differentiating the identity in the direction of X, on the one hand

$$X\left(\mathscr{A}^{1/2}\alpha, \mathscr{A}^{1/2}\alpha\right)_g = 2\left(\nabla^g_X \mathscr{A}^{1/2}(\alpha), \mathscr{A}^{1/2}\alpha\right)_g + 2\left(\mathscr{A}^{1/2}\nabla^g_X\alpha, \mathscr{A}^{1/2}\alpha\right)_g ,$$

whereas, on the other hand

$$X(\alpha, \alpha)_h = 2\left(\nabla^h_X\alpha, \alpha\right)_h = 2\left(\mathscr{A}\nabla^h_X\alpha, \alpha\right)_g .$$

Thus,

$$\left(\nabla^g_X\mathscr{A}^{1/2}\alpha, \mathscr{A}^{1/2}\alpha\right)_g = \left(\mathscr{A}^{1/2}\circ\left(\nabla^h_X - \nabla^g_X\right)\alpha, \mathscr{A}^{1/2}\alpha\right)_g . \tag{3.12}$$

2° By Lemma 3.10 $\mathscr{A}^{1/2}$ is self-adjoint so is $\nabla^g_X\mathscr{A}^{1/2}$. So, let λ be an eigenvalue of $\nabla^g_X\mathscr{A}^{1/2}$ with $|\lambda| = \left|\nabla^g_X\mathscr{A}^{1/2}\right|_g$ at a fixed point $x \in M$. Let $v \in \mathsf{T}_xM$ be a corresponding g-normalised eigenvector. We estimate

$$\begin{aligned}\left|\left(\mathscr{A}^{1/2}\nabla^g_X\mathscr{A}^{1/2}v, v\right)_g\right| = |\lambda|\left|\left(\mathscr{A}^{1/2}v, v\right)_g\right| &= \left|\nabla^g_X\mathscr{A}^{1/2}\right|_g\left|\left(\mathscr{A}^{1/2}v, v\right)_g\right| \\ &\geqslant \left|\nabla^g_X\mathscr{A}^{1/2}\right|_g\left|\mathscr{A}^{-1/2}\right|^{-1},\end{aligned}$$

where we used that $\mathscr{A}$, hence $\mathscr{A}^{1/2}$, has only (strictly) positive eigenvalues: More precisely, since $\mathscr{A}^{1/2}$ is self-adjoint, there are eigenvalues $\lambda_{\min}$ and $\lambda_{\max}$ of $\mathscr{A}^{1/2}$ such that

$$\lambda_{\min}(\mathscr{A}^{1/2}) \leqslant \left(\mathscr{A}^{1/2}u, u\right)_g \leqslant \lambda_{\max}(\mathscr{A}^{1/2}).$$

By definition, the smooth vector bundle morphism $\mathscr{A}$ has only strictly positive eigenvalues, so does $\mathscr{A}^{1/2}$ and we have that $\left(\mathscr{A}^{1/2}u, u\right)_g > 0$. Moreover, the eigenvalues of $\mathscr{A}^{-1/2}$ are just given by $\frac{1}{\lambda(\mathscr{A}^{1/2})}$ and thus

$$\left|\mathscr{A}^{1/2}\right| \geqslant \lambda_{\min}(\mathscr{A}^{1/2}) = \frac{1}{\lambda_{\max}(\mathscr{A}^{-1/2})} = \left|\mathscr{A}^{-1/2}\right|^{-1} .$$

Hence the first estimate follows:

$$\begin{aligned}\left|\nabla^g_X\mathscr{A}^{1/2}\right|_g &\leqslant \left|\mathscr{A}^{-1}\right|^{1/2}\sup_{|v|\leqslant 1}\left|\left(\mathscr{A}^{1/2}\nabla^g_X\mathscr{A}^{1/2}v, v\right)_g\right| \\ &\overset{(3.12)}{=} \left|\mathscr{A}^{-1}\right|^{1/2}\sup_{|v|\leqslant 1}\left|\left(\mathscr{A}\circ\left(\nabla^h_X - \nabla^g_X\right)v, v\right)_g\right| \\ &\leqslant \left|\mathscr{A}^{-1}\right|^{1/2}|\mathscr{A}|\left|\nabla^h_X - \nabla^g_X\right|_g .\end{aligned}$$

3° Covariantly differentiating the identity

$$\mathrm{id} = \mathscr{A}^{1/2}\circ\mathscr{A}^{-1/2} \quad\Longrightarrow\quad 0 = \nabla^g_X\mathscr{A}^{1/2}\circ\mathscr{A}^{-1/2} + \mathscr{A}^{1/2}\nabla^g_X\mathscr{A}^{-1/2}.$$

Thus,

$$\nabla^g_X\mathscr{A}^{-1/2} = -\mathscr{A}^{-1/2}\nabla^g_X\mathscr{A}^{1/2}\circ\mathscr{A}^{-1/2},$$

and the second estimate follows from part **2°**. ■

Definition 3.12. A smooth Riemannian metric $h \in \operatorname{Metr} M$ is called **quasi-isometric to** g, denoted $g \sim h$, if there exists a constant $C \geqslant 1$ such that (to be understood pointwise, as bilinear forms)

$$\frac{1}{C} g \leqslant h \leqslant C g.$$

Obviously, denoting by $0 < \rho_{g,h} =: \rho \in \mathrm{C}^\infty(M)$ the Radon-Nikodým density, i.e.

$$\mathrm{d}\,\mathrm{vol}_h = \rho_{g,h} \mathrm{d}\,\mathrm{vol}_g,$$

the following identities hold

$$\rho_{h,g} = \rho_{g,h}^{-1}, \quad \mathscr{A}_{h,g} = \mathscr{A}_{g,h}^{-1}, \quad \rho_{g,h} = (\det A)^{1/2}, \quad 0 < \inf \rho_{g,h} \leqslant \sup \rho_{g,h} < \infty. \tag{3.13}$$

We now define the zeroth order deviation of the two metrics (considered as multiplicative perturbations of each other) as

$$\delta_{g,h}(x) := 2 \sinh \left(\frac{m}{4} \max_{\lambda \in \sigma(A_{g,h}(x))} |\log \lambda| \right) = \max_{\lambda \in \sigma(A_{g,h}(x))} \left| \lambda^{\frac{m}{4}} - \lambda^{-\frac{m}{4}} \right| : M \to (0, \infty), \tag{3.14}$$

symmetric in g and h by quasi-isometry, i.e. $\delta_{g,h} = \delta_{h,g}$. We will make use of the fact [HPW14, Appendix A] that

$$\sup \delta_{g,h}(x) < \infty \quad \Longleftrightarrow \quad g \sim h.$$

The definition is becoming clearer in the proof of the main result in Section 4. Moreover, let

$$\delta^{\nabla}_{g,h}(x) := \left| \nabla^h - \nabla^g \right|_g^2 (x) : M \to [0, \infty) \tag{3.15}$$

be a weight function defined in terms of the corresponding covariant derivatives ∇^h and ∇^g, defined in terms of the operator norm induced by the inner product g.

Remark 3.13. Recall that by Proposition 1.17, the difference of two connections $\nabla^h - \nabla^g$ is a one-form on M with values in $\operatorname{End} \mathrm{T}M$, i.e.

$$\nabla^h - \nabla^g \in \Gamma(\mathrm{T}^* M \otimes \operatorname{End} \mathrm{T}M).$$

Example 3.14 (Conformal metric change). Let $h := g_\psi$ be a conformal perturbation of g, i.e. we set $g_\psi := \mathrm{e}^{2\psi} g$ for some smooth function $\psi : M \to \mathbb{R}$. We take $A := e^{2\psi}$, so $A^{-1} = \mathrm{e}^{-2\psi}$ and

$$\delta_{g,g_\psi}(x) = 2 \sinh \frac{m}{4} |\psi(x)| \,.$$

Hence,

$$g \sim h \quad \Longleftrightarrow \quad \psi \quad \text{bounded.}$$

By (3.36d), for any smooth vector field $X, Y \in \Gamma_{\mathrm{C}^\infty}(\mathrm{T}M)$, we have

$$\left(\nabla^{g_\psi}_X - \nabla^g_X \right) Y = \mathbf{d}\psi(X) Y + \mathbf{d}\psi(Y) X - (X, Y)_g \operatorname{grad}_g \psi. \tag{3.16}$$

By norm equivalence on finite dimensional spaces, we can work with the Hilbert-Schmidt norm for the calculation. Let $(X_i)_{i=1}^m$ a smooth local g-orthonormal frame of vector fields. Then in local coordinates,

$$
\begin{aligned}
|\mathbf{d}\psi|^2_{g\otimes \mathrm{HS}} &= \sum_{i=1}^m |X_i\psi|^2 , \\
\left|\nabla^{g_\psi} - \nabla^g\right|^2_{g\otimes \mathrm{HS}} &= \sum_{j,k=1}^m \left|\left(\nabla^{g_\psi} - \nabla^g\right)(X_j, X_k)\right|^2 \\
&= \sum_{j,k=1}^m \left|(X_j\psi)X_k + (X_k\psi)X_j - \delta_{jk}\sum_i (X_i\psi)X_i\right|^2 \\
&= \sum_{j<k} 2\left|(X_j\psi)X_k + (X_k\psi)X_j\right|^2 + \sum_{i=1}^m |X_i\psi|^2 ,
\end{aligned}
$$

so that

$$
|\mathbf{d}\psi|^2_g \lesssim \left|\nabla^{g_\psi} - \nabla^g\right|^2_g = \delta^{\nabla}_{g,g_\psi} .
$$

Next, we give a formula how to express the codifferential $\boldsymbol{\delta}_h$ with respect the metric h in terms of the codifferential $\boldsymbol{\delta}_g$ in terms of g.

Lemma 3.15. *Let $\mathscr{A} := \mathscr{A}_{g,h}$ be the smooth vector bundle morphism defined by (3.9). Then the codifferential with respect to the metric h is given by*

$$
\boldsymbol{\delta}_h \eta = \mathscr{A}^{-1}\left(\boldsymbol{\delta}_g(\mathscr{A}\eta) - \mathbf{d}\log\rho \lrcorner_g (\mathscr{A}\eta)\right) \qquad \forall \eta \in \Omega_{C^\infty}(M).
$$

Proof. For any $\eta_1 \in \Omega_{C^\infty}(M)$, $\eta_2 \in \Omega_{C^\infty}(M)$, we calculate

$$
\begin{aligned}
\left\langle \eta_1, \mathscr{A}^{-1}\left(\boldsymbol{\delta}_g(\mathscr{A}\eta_2) - \mathbf{d}\log\rho \lrcorner_g (\mathscr{A}\eta_2)\right)\right\rangle_h &= \left\langle \rho\mathscr{A}\eta_1, \mathscr{A}^{-1}\left(\boldsymbol{\delta}_g(\mathscr{A}\eta_2) - \frac{\mathbf{d}\rho}{\rho}\lrcorner_g (\mathscr{A}\eta_2)\right)\right\rangle_g \\
&= \left\langle \eta_1, \rho\boldsymbol{\delta}_g(\mathscr{A}\eta_2) - \mathbf{d}\rho \lrcorner_g (\mathscr{A}\eta_2)\right\rangle_g \\
&= \langle \mathbf{d}(\rho\eta_1), \mathscr{A}\eta_2\rangle_g - \langle \mathbf{d}\rho\wedge\eta_1, \mathscr{A}\eta_2\rangle_g \\
&= \langle \mathbf{d}\rho\wedge\eta_1, \mathscr{A}\eta_2\rangle_g + \langle \rho\mathbf{d}\eta_1, \mathscr{A}\eta_2\rangle_g - \langle \mathbf{d}\rho\wedge\eta_1, \mathscr{A}\eta_2\rangle_g \\
&= \langle \mathbf{d}\eta_1, \eta_2\rangle_h ,
\end{aligned}
$$

where we used that $\mathscr{A}$ is fibrewise self-adjoint. ∎

In the proof of the main result, the gradient of the logarithm of the Radon-Nikodým density $\rho_{g,h}$ can be estimated in terms of smooth vector bundle morphism $\mathscr{A}_{g,h}$ and $\delta^{\nabla}_{g,h}$ which is reflected in the next Proposition 3.19. Therefore, we note two auxiliary lemmas.

Lemma 3.16. *Let A, B be two complex $m\times m$-matrices. Then,*

$$
|\mathrm{tr}(AB)| \leqslant \|A\|_{\mathrm{HS}}\,\|B\|_{\mathrm{HS}} ,
$$

where $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm.

Proof. Let A, B be two complex $m \times m$-matrices with singular values

$$\sigma_1(A) \geqslant \sigma_2(A) \geqslant \ldots \geqslant \sigma_m(A) \qquad \text{of } A \text{ and}$$
$$\sigma_1(B) \geqslant \sigma_2(B) \geqslant \ldots \geqslant \sigma_m(B) \qquad \text{of } B.$$

Then, with the Hilbert-Schmidt norm $\|\cdot\|_{\mathrm{HS}}$,

$$\|\sigma(A)\| = \sqrt{\sum_{i=1}^{m} \sigma_i^2(A)} = \sqrt{\operatorname{tr}(A^* A)} =: \|A\|_{\mathrm{HS}} \tag{3.17}$$

by norm equivalence on finite-dimensional vector spaces. By the well-known von Neumann trace formula [Mir75], we get

$$\begin{aligned} |\operatorname{tr}(AB)| &\leqslant \sum_{i=1}^{m} \sigma_i(A)\sigma_i(B) \\ &= (\sigma(A), \sigma(B))_{\mathbb{R}^m} \\ &\leqslant \|\sigma(A)\| \ \|\sigma(B)\| = \|A\|_{\mathrm{HS}} \|B\|_{\mathrm{HS}}, \end{aligned}$$

where we used Cauchy-Schwarz for the first inequality and (3.17) in the last equality. ■

Lemma 3.17 (Classical Jacobi's formula). *Let $A = A(t)$ an $m \times m$-matrix parametrised by t. If A is invertible, then*

$$\frac{\mathrm{d}}{\mathrm{d}t} \det A(t) = \det A(t) \ \operatorname{tr}\left(A(t)^{-1} \frac{\mathrm{d}}{\mathrm{d}t} A(t)\right). \tag{3.18}$$

Next, we extend the classical Jacobi formula to our setting.

Lemma 3.18 (Jacobi's formula). *Let $A := A_{g,h}$ be the smooth vector bundle morphism defined by (3.8) ($x \in M$). Then, for any $X \in \Gamma(\mathrm{T}M)$,*

$$\mathbf{d}(\det A)X = \det A \ \operatorname{tr}\left(A^{-1} \nabla_X A\right).$$

Proof. For $\varepsilon > 0$, let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a smooth curve on M such that $\gamma(0) = x$ and $\dot{\gamma}(0) = X$. Then, computing the differential using a velocity vector,

$$\begin{aligned} \mathbf{d}(\det A)_x X = \frac{\mathrm{d}}{\mathrm{d}t} \det A(\gamma(t))\Big|_{t=0} &= \det A(\gamma(t)) \ \operatorname{tr}\left(A(\gamma(t))^{-1} \frac{\mathrm{d}}{\mathrm{d}t} A(\gamma(t))\right)\Big|_{t=0} \\ &= \det A(\gamma(t)) \ \operatorname{tr}\left(A(\gamma(t))^{-1} \nabla_{\dot{\gamma}(t)} A(\gamma(t))\right)\Big|_{t=0} \\ &= \det A(x) \ \operatorname{tr}\left(A(x)^{-1} \nabla_X A(x)\right), \end{aligned}$$

using Lemma 3.17 in the second step. ■

We are now in the position to prove the aimed estimate of the Radon-Nikodým density.

Proposition 3.19. *Let $\mathscr{A} := \mathscr{A}_{g,h}$ be the smooth vector bundle morphism defined by (3.9) and $\rho = (\det A)^{1/2}$ the Radon-Nikodým density defined by (3.13). Then we can estimate as follows:*

$$|\mathbf{d} \log \rho|_g \leqslant C(m) \left|\mathscr{A}^{-1}\right| |\nabla^g \mathscr{A}|_g .$$

Proof. We first remark, that for $x \in M$,

$$\dim \bigwedge^k \mathsf{T}^*_x M = \binom{m}{k} \qquad \Longrightarrow \qquad \dim \bigwedge \mathsf{T}^*_x M = 2^m < \infty$$

is finite-dimensional. Recall that the Radon-Nikodým density ρ is given by $\rho = (\det A)^{1/2}$. By Jacobi's formula, Lemma 3.18 above,

$$\mathbf{d}(\log \rho)_x X = \frac{1}{2} \operatorname{tr}\left(A(x)^{-1} \nabla_X A(x)\right).$$

Hence, using Lemma 3.16,

$$\begin{aligned} \left| \mathbf{d}(\log \rho)_x X \right| &\leqslant \frac{1}{2} \left\|A(x)^{-1}\right\|_{\mathrm{HS}} \left\|\nabla_X A(x)\right\|_{g\otimes \mathrm{HS}} \\ &\leqslant C(m) \left|A(x)^{-1}\right| \left|\nabla_X A(x)\right|_g \\ &\leqslant C(m) \left|\mathscr{A}(x)^{-1}\right| \left|\nabla_X \mathscr{A}(x)\right|_g \end{aligned}$$

by norm equivalence on finite-dimensional spaces. ■

3.3 Gradient Estimates by Bismut formulae

For every $g \in \mathsf{Metr} M$,

$$(P^g_s)_{s>0} := \left(\mathrm{e}^{-\frac{s}{2}\Delta_g}\right)_{s>0} \subset \mathscr{L}(\Omega_{\mathsf{L}^2}(M,g))$$

is the heat semigroup defined by the spectral Theorem 1.50, choosing $f : \mathbb{R} \to \mathbb{R}$, $f(\lambda) := \mathrm{e}^{-t/2\lambda}$. Let us denote by

$$(0,\infty) \times M \times M \ni (s,x,y) \mapsto p^g_s(x,y) := \mathrm{e}^{-\frac{s}{2}\Delta_g}(x,y) \in \mathsf{Hom}\left(\bigwedge \mathsf{T}^*_y M, \bigwedge \mathsf{T}^*_x M\right)$$

the corresponding jointly smooth integral kernel of P^g_s. The smooth representative of $P_s\alpha(x)$ is given by

$$P^g_s \alpha(x) = \int_M \mathrm{e}^{-\frac{s}{2}\Delta_g}(x,y)\alpha(y) \operatorname{vol}_g(\mathrm{d}y).$$

By Theorem 1.63 (iii) we have

$$\int_M \left|p^g_s(x,y)\right|^2_g \operatorname{vol}_g(\mathrm{d}y) < \infty, \qquad \forall s > 0\ \forall x \in M.$$

The form degree k will be indicated again by round brackets $p^{g,(k)}_s$. Finally, we set

$$\Phi_g(x,s) := \sup_M p^{g,(0)}_s(x,\cdot), \tag{3.19}$$

indicating the minimal heat kernel $p^{g,(0)}_s$ acting on 0-forms, i.e. functions. Then it is well-known that, for all $(x,s) \in M \times (0,\infty)$, it follows that $\Phi_g(x,s) < \infty$. One can even show [Gün17a] that

$$\sup_K \Phi_g(\cdot,s) < \infty \qquad \forall s > 0\ \forall K \subset M \text{ compact.}$$

Remark 3.20. Note that the gradient estimates proven in §§ 2.2 and 2.3 extend naturally to the complex setting since complexifications are norm preserving.

We will now make use of gradient estimates for the covariant derivative of P_s^g, Theorem 2.28, and derive similar estimates for the exterior derivative and codifferential transformed by the smooth vector bundle morphism $\mathscr{A}_{g,h}$ defined in (3.9). The key observation will be Proposition 3.25 showing how to estimate the transformed codifferential $\boldsymbol{\delta}_g$ (with respect the metric g) applied to the semigroup in terms of covariant derivative ∇^g of g applied to the semigroup. In addition, using Lemma 3.15, a direct consequence is an analogous result (cf. Corollary 3.28) in terms of the new metric, i.e. for the transformed codifferential $\boldsymbol{\delta}_h$ (with respect the metric h) applied to the semigroup.

To this end, we will make use of the well-known metric descriptions of the exterior derivative $\mathbf{d}$ and the codifferential $\boldsymbol{\delta}_g$ (with respect the metric g) from Proposition 1.36 adapted to our setting, namely:

Lemma 3.21. *Let $e_1, \dots, e_m$ (where $m = \dim M$) be a local orthonormal frame for $\mathsf{T}_x M$ ($x \in M$) and $\varepsilon^1, \dots, \varepsilon^m$ be its dual coframe, i.e. $\varepsilon^j(e_i) = \delta_i^j$. Then*

$$\mathbf{d} = \sum_{i=1}^m \varepsilon^i \wedge \nabla_{e_i}^g \qquad \text{and} \qquad \boldsymbol{\delta}_g = -\sum_{i=1}^m e_i \lrcorner \nabla_{e_i}^g. \tag{3.20}$$

Let $\mathscr{A} := \mathscr{A}_{g,h}$ be the smooth vector bundle morphism defined by (3.9).

Proposition 3.22. *Let $\alpha \in \Omega_{\mathsf{L}^2}(M,g)$ and $\underline{\mathscr{R}}_g \in \mathsf{K}(M,g)$. Then, for any orthonormal frame $(e_i)_{i=1}^m$ for $\mathsf{T}_x M$ ($x \in M$) and dual coframe $(\varepsilon^i)_{i=1}^m$, we decompose*

$$\mathbf{d}(\mathscr{A}^{1/2} P_s\alpha) = \sum_{i=1}^m \left(\varepsilon^i \wedge \nabla_{e_i}^g P_s\alpha \circ \mathscr{A}^{1/2} + \varepsilon^i \wedge P_s\alpha \circ \left(\nabla_{e_i}^g \mathscr{A}^{1/2}\right)\right).$$

Proof. From (3.20), we get for any $\eta \in \Omega(M)$

$$\mathbf{d}(\mathscr{A}^{1/2}\eta) = \mathbf{d}(\eta \circ \mathscr{A}^{1/2}) = \sum_{i=1}^m \varepsilon^i \wedge \nabla_{e_i}^g \eta \circ \mathscr{A}^{1/2} + \sum_{i=1}^m \varepsilon^i \wedge \eta \circ \nabla_{e_i}^g \mathscr{A}^{1/2},$$

where $(e_i)_{i=1}^m$ denotes an orthonormal frame for $\mathsf{T}_x M$ ($x \in M$) and $(\varepsilon^i)_{i=1}^m$ the dual coframe. In particular, for $\eta = P_s\alpha$ the claim follows. ∎

Corollary 3.23. *Let $\alpha \in \Omega_{\mathsf{L}^2}(M,g)$ and $\underline{\mathscr{R}}_g \in \mathsf{K}(M,g)$. Then,*

$$\left|\mathscr{A}^{-1/2}\mathbf{d}(\mathscr{A}^{1/2} P_s\alpha(x))\right|_g^2 \lesssim \left(\delta_{g,h}^{\nabla}(x) + \Xi_g(x,s)\right)\Phi_g(x,s)\,\|\alpha\|_{\Omega_{\mathsf{L}^2}(M,g)}^2. \tag{3.21}$$

Proof. By Theorem 2.28, we have

$$\left|\sum_{i=1}^m \varepsilon^i \wedge \nabla_{e_i}^g P_s\alpha \circ \mathscr{A}^{1/2}\right|_g^2 \leqslant C(m)\left|\mathscr{A}^{1/2}(x)\right|^2 \Xi_g(x,s)\Phi_g(x,s)\,\|\alpha\|_{\Omega_{\mathsf{L}^2}(M,g)}^2$$

and, combined with Lemma 2.5 and Lemma 3.9,

$$\begin{aligned}\left|\sum_{i=1}^m \varepsilon^i \wedge P_s\alpha \circ \left(\nabla_{e_i}^g \mathscr{A}^{1/2}\right)\right|_g^2 &\leqslant C(m,\gamma,c_\gamma,s)\left|\nabla^g \mathscr{A}^{1/2}(x)\right|_g^2 \Phi_g(x,s)\,\|\alpha\|_{\Omega_{\mathsf{L}^2}(M,g)}^2 \\ &\lesssim \left|\mathscr{A}^{1/2}(x)\right|^2 \delta_{g,h}^{\nabla}(x)\Phi_g(x,s)\,\|\alpha\|_{\Omega_{\mathsf{L}^2}(M,g)}^2.\end{aligned}$$

Thus the claim follows:

$$\left|\mathscr{A}^{-1/2}\mathbf{d}(\mathscr{A}^{1/2}P_s\alpha(x))\right|_g^2 \lesssim \left|\mathscr{A}^{-1/2}(x)\right|^2 \left|\mathscr{A}^{1/2}(x)\right|^2 \left(\delta_{g,h}^{\nabla}(x) + \Xi_g(x,s)\right)\Phi_g(x,s)\,\|\alpha\|^2_{\Omega_{\mathsf{L}^2}(M,g)}.$$

■

Remark 3.24. By a similar calculation the estimate also holds if we interchange the rôles of $\mathscr{A}^{1/2}$ and $\mathscr{A}^{-1/2}$ in the previous Corollary 3.23.

Proposition 3.25. *Let $\alpha \in \Omega_{\mathsf{L}^2}(M,g)$ and $\underline{\mathscr{R}}_g \in \mathsf{K}(M,g)$. Then, for any orthonormal basis $(e_i)_i$ for T_xM ($x \in M$), we decompose*

$$\boldsymbol{\delta}_g(\mathscr{A}^{1/2}P_s\alpha) = -\sum_{i=1}^m e_i \lrcorner \left(\nabla^g_{e_i}P_s\alpha\right)\circ\mathscr{A}^{1/2} - \sum_{i=1}^m e_i \lrcorner P_s\alpha\circ\left(\nabla^g_{e_i}\mathscr{A}^{1/2}\right). \tag{3.22}$$

Proof. Let $\eta \in \Omega(M)$ be arbitrary and $(e_i)_{i=1}^m$ an orthonormal basis for T_xM ($x \in M$). By the metric description of the codifferential, (3.20) in Lemma 3.21 above, we get

$$\begin{aligned}\boldsymbol{\delta}_g(\mathscr{A}^{1/2}\eta) &= -\sum_{i=1}^m e_i \lrcorner \nabla^g_{e_i}(\mathscr{A}^{1/2}\eta)\\ &= -\sum_{i=1}^m \left(e_i \lrcorner (\mathscr{A}^{1/2}\nabla^g_{e_i}\eta) + e_i \lrcorner ((\nabla^g_{e_i}\mathscr{A}^{1/2})\eta)\right).\end{aligned}$$

In particular, if we set $\eta = P_s\alpha(x)$, then the following equalities hold:

$$\sum_{i=1}^m e_i \lrcorner \left(\mathscr{A}^{1/2}\nabla^g_{e_i}P_s\alpha\right) = \sum_{i=1}^m e_i \lrcorner \left(\nabla^g_{e_i}P_s\alpha\right)\circ\mathscr{A}^{1/2}$$

and

$$\sum_{i=1}^m e_i \lrcorner ((\nabla^g_{e_i}\mathscr{A}^{1/2})P_s\alpha) = \sum_{i=1}^m e_i \lrcorner P_s\alpha\circ\left(\nabla^g_{e_i}\mathscr{A}^{1/2}\right).$$

■

Corollary 3.26. *Let $\alpha \in \Omega_{\mathsf{L}^2}(M,g)$ and $\underline{\mathscr{R}}_g \in \mathsf{K}(M,g)$. Then,*

$$\left|\mathscr{A}^{-1/2}\boldsymbol{\delta}_g(\mathscr{A}^{1/2}P_s\alpha(x))\right|_g^2 \lesssim \left(\delta_{g,h}^{\nabla}(x) + \Xi_g(x,s)\right)\Phi_g(x,s)\,\|\alpha\|^2_{\Omega_{\mathsf{L}^2}(M,g)}. \tag{3.23}$$

Proof. By Theorem 2.28 we have

$$\left|\sum_{i=1}^m e_i \lrcorner \left(\nabla^g_{e_i}P_s\alpha\right)\circ\mathscr{A}^{1/2}\right|_g^2 \leqslant C(m)\left|\mathscr{A}^{1/2}(x)\right|^2 \Xi_g(x,s)\Phi_g(x,s)\,\|\alpha\|^2_{\Omega_{\mathsf{L}^2}(M,g)}$$

and, combined with Lemma 2.5 and Lemma 3.9,

$$\begin{aligned}\left|\sum_{i=1}^m e_i \lrcorner P_s\alpha\circ\left(\nabla^g_{e_i}\mathscr{A}^{1/2}\right)\right|_g^2 &\leqslant C(m,\gamma,c_\gamma,s)\left|\nabla^g\mathscr{A}^{1/2}(x)\right|_g^2 \Phi_g(x,s)\,\|\alpha\|^2_{\Omega_{\mathsf{L}^2}(M,g)}\\ &\lesssim \left|\mathscr{A}^{1/2}(x)\right|^2 \delta_{g,h}^{\nabla}(x)\Phi_g(x,s)\,\|\alpha\|^2_{\Omega_{\mathsf{L}^2}(M,g)}.\end{aligned}$$

Thus,

$$\left|\mathscr{A}^{-1/2}\boldsymbol{\delta}_g(\mathscr{A}^{1/2}P_s\alpha(x))\right|_g^2 \lesssim \left|\mathscr{A}^{-1/2}(x)\right|^2 \left|\mathscr{A}^{1/2}(x)\right|^2 \left(\delta_{g,h}^{\nabla}(x) + \Xi_g(x,s)\right)\Phi_g(x,s)\,\|\alpha\|^2_{\Omega_{\mathsf{L}^2}(M,g)},$$

so the claim follows. ■

Remark 3.27. By a similar calculation the estimate also holds if we interchange the rôles of $\mathscr{A}^{1/2}$ and $\mathscr{A}^{-1/2}$ in the previous Corollary 3.26.

Finally, we obtain a similar estimate for the transformed codifferential $\boldsymbol{\delta}_h$ (with respect the metric h) applied to the semigroup. As we can express $\boldsymbol{\delta}_h$ solely in terms of $\boldsymbol{\delta}_g$ using Lemma 3.15, we want to emphasise that the fibrewise norm and *all* the involved quantities are taken *with respect to the metric* g.

Corollary 3.28. *Let $\alpha \in \Omega_{\mathsf{L}^2}(M,g)$ and $\underline{\mathscr{R}}_g \in \mathsf{K}(M,g)$. Then,*

$$\left|\mathscr{A}^{1/2}\boldsymbol{\delta}_h(\mathscr{A}^{-1/2}P_s\alpha(x))\right|_g^2 \lesssim \left(\delta^{\nabla}_{g,h}(x) + \Xi_g(x,s)\right)\Phi_g(x,s)\,\|\alpha\|^2_{\Omega_{\mathsf{L}^2}(M,g)}\,. \tag{3.24}$$

Proof. By Lemma 3.15, recall that

$$\boldsymbol{\delta}_h(\eta) = \mathscr{A}^{-1}\left(\boldsymbol{\delta}_g(\mathscr{A}\eta) - \mathbf{d}\log\rho \lrcorner_g (\mathscr{A}\eta)\right) \qquad \forall \eta \in \Omega_{\mathsf{C}^\infty}(M),$$

so that

$$(\mathscr{A}^{1/2}\boldsymbol{\delta}_h\mathscr{A}^{-1/2})(\eta) = \mathscr{A}^{-1/2}\left(\boldsymbol{\delta}_g(\mathscr{A}^{1/2}\alpha) - \mathbf{d}\log\rho \lrcorner_g (\mathscr{A}^{1/2}\eta)\right).$$

Using Proposition 3.19 combined with Lemma 3.11, in addition

$$\begin{aligned}\mathscr{A}^{-1/2}\left|\mathbf{d}\log\rho \lrcorner (\mathscr{A}^{1/2}\eta)\right|_g &\leqslant C(m)\left|\mathscr{A}^{-1/2}\right|\left|\mathscr{A}^{-1}\right|\left|\nabla^g\mathscr{A}\right|_g\left|\mathscr{A}^{1/2}\right||\eta|_g\\ &\leqslant C(m)\left|\mathscr{A}^{-1/2}\right|\left|\mathscr{A}^{-1}\right||\mathscr{A}|\left|\nabla^h_X - \nabla^g_X\right|_g\left|\mathscr{A}^{1/2}\right||\eta|_g\\ &\leqslant C(m)\left|\nabla^h_X - \nabla^g_X\right|_g|\eta|_g\,.\end{aligned}$$

Finally, combined with Corollary 3.26,

$$\left|\mathscr{A}^{1/2}\boldsymbol{\delta}_h(\mathscr{A}^{-1/2}P_s\alpha(x))\right|_g^2 \lesssim \left(\delta^{\nabla}_{g,h}(x) + \Xi_g(x,s)\right)\Phi_g(x,s)\,\|\alpha\|^2_{\Omega_{\mathsf{L}^2}(M,g)}$$

which proves the claim. ∎

3.4 Main Results

First, we define the bounded identification operator

$$\begin{aligned}I = I_{g,h} : \Omega_{\mathsf{L}^2}(M,g) &\to \Omega_{\mathsf{L}^2}(M,h)\\ \alpha(x) &\mapsto \mathscr{A}^{-1/2}_{g,h}(x)\alpha(x),\end{aligned}$$

well-defined by $g \sim h$.

Lemma 3.29. *The adjoint I^* of the bounded identification operator I is given by*

$$\begin{aligned}I^* = I^*_{g,h} : \Omega_{\mathsf{L}^2}(M,h) &\to \Omega_{\mathsf{L}^2}(M,g)\\ \alpha(x) &\mapsto \rho_{g,h}(x)\mathscr{A}^{1/2}_{g,h}(x)\alpha(x).\end{aligned} \tag{3.25}$$

Proof. For compactly supported $\alpha \in \Omega(M,h)$ and $\beta \in \Omega(M,g)$, we get

$$\begin{aligned}\left\langle I^*_{g,h}\alpha, \beta\right\rangle_{\Omega_{\mathsf{L}^2}(M,g)} &= \left\langle \alpha, I_{g,h}\beta\right\rangle_{\Omega_{\mathsf{L}^2}(M,h)} \\ &= \int_M \left(\alpha, \mathscr{A}^{-1/2}_{g,h}\beta\right)_h \mathrm{d}\,\mathrm{vol}_h \\ &= \int_M \left(\mathscr{A}^{1/2}_{g,h}\alpha, \mathscr{A}^{1/2}_{g,h}\mathscr{A}^{-1/2}_{g,h}\beta\right)_g \rho_{g,h}\mathrm{d}\,\mathrm{vol}_g \\ &= \int_M \left(\rho_{g,h}\mathscr{A}^{1/2}_{g,h}\alpha, \beta\right)_g \mathrm{d}\,\mathrm{vol}_g = \left\langle \rho_{g,h} I^{-1}_{g,h}\alpha, \beta\right\rangle_{\Omega_{\mathsf{L}^2}(M,g)}. \end{aligned}$$ ■

Since we assume M to be geodesically complete, we can restrict ourselves to smooth compactly supported differential forms. Using the common abuse of notation, the unique realisations of the exterior derivative $\mathbf{d}$, the codifferential $\boldsymbol{\delta}_g$ and Hodge Laplacian Δ_g will be denoted by the same symbol.

In addition, we define the operators

$$\begin{aligned}(\hat{P}^g_s)_{s>0} &:= (\mathbf{d}\, P^g_s)_{s>0} \subset \mathscr{L}(\Omega_{\mathsf{L}^2}(M,g), \Omega_{\mathsf{L}^2}(M,g)),\\ (\check{P}^g_s)_{s>0} &:= (\boldsymbol{\delta}_g P^g_s)_{s>0} \subset \mathscr{L}(\Omega_{\mathsf{L}^2}(M,g), \Omega_{\mathsf{L}^2}(M,g)),\end{aligned}$$

$$\begin{aligned}(\hat{P}^{g,h}_s)_{s>0} &:= (I^{-1}_{g,h}\mathbf{d}\, I_{g,h} P^g_s)_{s>0} \subset \mathscr{L}(\Omega_{\mathsf{L}^2}(M,g), \Omega_{\mathsf{L}^2}(M,g)),\\ (\check{P}^{g,h}_s)_{s>0} &:= (I_{g,h}\boldsymbol{\delta}_g I^{-1}_{g,h} P^h_s)_{s>0} \subset \mathscr{L}(\Omega_{\mathsf{L}^2}(M,g), \Omega_{\mathsf{L}^2}(M,g)).\end{aligned}$$

Let $\hat{p}^g_s(x,y)$, $\check{p}^g_s(x,y)$, $\hat{p}^{g,h}_s(x,y)$ and $\check{p}^{g,h}_s(x,y)$ be the corresponding jointly smooth integral kernel of $\hat{P}^g_s$, $\check{P}^g_s$, $\hat{P}^{g,h}_s$ and $\check{P}^{g,h}_s$, respectively. For example, recall that this implies by Theorem 1.63 (i) and (ii) that

$$(0,\infty)\times M\times M \ni (s,x,y) \mapsto \hat{p}_s(x,y) \in \mathrm{Hom}\left(\textstyle\bigwedge \mathrm{T}^*_y M, \bigwedge \mathrm{T}^*_x M\right)$$

is the uniquely determined map such that we have

$$\hat{P}^g_s\alpha(x) = \int_M \hat{p}^g_s(x,y)\alpha(y)\,\mathrm{vol}_g(\mathrm{d}y) \qquad \forall s>0\ \forall \alpha\in\Omega_{\mathsf{L}^2}(M,g)\ \forall x\in M.$$

By Riesz' representation theorem, the next result follows from the gradient estimates, Theorem 2.27, for the exterior derivative and the codifferential.

Theorem 3.30. *For every $g \in \mathrm{Metr}M$, $(s,x)\in(0,\infty)\times M$, we have*

$$\int \left|\hat{p}^g_s(x,y)\right|^2_g \mathrm{vol}_g(\mathrm{d}y) \leqslant \Psi_g(x,s)\Phi_g(x,s), \tag{3.26}$$

$$\int \left|\check{p}^g_s(x,y)\right|^2_g \mathrm{vol}_g(\mathrm{d}y) \leqslant \Psi_g(x,s)\Phi_g(x,s). \tag{3.27}$$

By Riesz' representation theorem, the next result follows from the estimates in corollaries 3.23, 3.26 and 3.28 for the transformed exterior derivative and for the transformed codifferential with respect to g, and h, respectively.

Theorem 3.31. *For every $g \in \mathrm{Metr}M$, $(s,x) \in (0,\infty) \times M$, we have*

$$\int \left|\hat{p}_s^{g,h}(x,y)\right|_g^2 \mathrm{vol}_g(\mathrm{d}y) \lesssim \left(\delta_{g,h}^{\nabla}(x) + \Xi_g(x,s)\right) \Phi_g(x,s), \tag{3.28}$$

$$\int \left|\check{p}_s^{g,h}(x,y)\right|_g^2 \mathrm{vol}_g(\mathrm{d}y) \lesssim \left(\delta_{g,h}^{\nabla}(x) + \Xi_g(x,s)\right) \Phi_g(x,s), \tag{3.29}$$

$$\int \left|\check{p}_s^{h,g}(x,y)\right|_g^2 \mathrm{vol}_g(\mathrm{d}y) \lesssim \left(\delta_{g,h}^{\nabla}(x) + \Xi_g(x,s)\right) \Phi_g(x,s). \tag{3.30}$$

We can now state the main result on the existence and completeness of the wave operators $W_\pm(\Delta_h, \Delta_g, I)$ implying the corresponding spectra to coincide.

Theorem 3.32. *Let $g, h \in \mathrm{Metr}M$, $g \sim h$, and assume that there exists $C < \infty$ such that $\left|\delta_{g,h}^{\nabla}\right| \leqslant C$ and for both $\nu \in \{g,h\}$, we have $\left|\mathscr{R}_\nu\right|_\nu \in \mathsf{K}(M)$ and*

$$\int \max\left\{\delta_{g,h}(x), \delta_{g,h}^{\nabla}(x) + \Xi_g(x,s), \Psi_\nu(x,s)\right\} \Phi_\nu(x,s)\,\mathrm{vol}_\nu(\mathrm{d}x) < \infty, \qquad \textit{some } s > 0. \tag{3.31}$$

Then the wave operators

$$W_\pm(\Delta_h, \Delta_g, I_{g,h}) = \operatorname*{s-lim}_{t\to\pm\infty} \mathrm{e}^{it\Delta_h} I_{g,h} \mathrm{e}^{-it\Delta_g} \mathsf{P}_{\mathrm{ac}}(\Delta_g)$$

exist and are complete. Moreover, $W_\pm(\Delta_h, \Delta_g, I_{g,h})$ are partial isometries with initial space $\operatorname{ran}\mathsf{P}_{\mathrm{ac}}(\Delta_g)$ *and final space* $\operatorname{ran}\mathsf{P}_{\mathrm{ac}}(\Delta_h)$, *and we have $\sigma_{\mathrm{ac}}(\Delta_g) = \sigma_{\mathrm{ac}}(\Delta_g)$.*

The proof of Theorem 3.32 will be given in Section 3.5.

In the special case of for 0-forms, i.e. functions, and the Hodge Laplacian acting on 0-forms is the Laplace-Beltrami operator. Recall that the Weitzenböck curvature endomorphism $\mathscr{R}^{(k)}$ on 1-forms is given by the Ricci curvature, $\mathscr{R}^{(1),\mathrm{tr}} = \mathrm{Ric}$. Then we get the following result similar to the main result of [GT20, Theorem A].

Corollary 3.33. *Let $g, h \in \mathrm{Metr}M$, $g \sim h$, and assume that the function $\delta_{g,h}^{\nabla}$ is bounded, and for some $s > 0$ and both $\nu \in \{g,h\}$ satisfy (3.31). Let $\Delta_\nu \leqslant 0$ be the unique self-adjoint extensions of the Laplace-Beltrami operator for $\nu \in \{g,h\}$. Then the wave operators $W_\pm(\Delta_h, \Delta_g, I)$ exist and are complete. Moreover, $W_\pm(\Delta_h, \Delta_g, I)$ are partial isometries with initial space* $\operatorname{ran}\mathsf{P}_{\mathrm{ac}}(\Delta_g)$ *and final space* $\operatorname{ran}\mathsf{P}_{\mathrm{ac}}(\Delta_h)$, *and we have $\sigma_{\mathrm{ac}}(\Delta_h) = \sigma_{\mathrm{ac}}(\Delta_g)$.*

3.5 Proof of the Main Result

Our strategy is to show the assumptions given by a variant of the Belopol'skii-Birman Theorem 3.5 which is adapted to our special case of two Hilbert space scattering.

The next lemma shows assumption (2) in the Belopol'skii-Birman Theorem 3.5. As we will see in its proof, it is therefore necessary for the *potentials* $\mathscr{R}_\nu \in \mathsf{K}(M)$ to be in the Kato class, not only $\underline{\mathscr{R}}_\nu \in \mathsf{K}(M)$, for $\nu \in \{g,h\}$.

We denote by $\mathbf{q}_\nu$ the nonnegative closed sesquilinear form corresponding to Δ_ν, i.e. $\mathbf{q}_\nu(\alpha) = \langle \Delta_\nu \alpha, \alpha\rangle = \left\|\mathsf{D}_\nu \alpha\right\|_\nu^2$ with $\operatorname{dom}\mathbf{q}_\nu = \operatorname{dom}\sqrt{\Delta_\nu}$ for any $\nu \in \{g,h\}$.

Lemma 3.34. *Let $g, h \in \mathsf{Metr} M$, $g \sim h$, and assume that there exists $C < \infty$ such that $\left|\delta^{\nabla}_{g,h}\right| \leqslant C$ and for some $\nu \in \{g, h\}$, we have $\mathscr{R}_\nu \in \mathsf{K}(M)$. Then*

$$I_{g,h} \operatorname{dom} \mathbf{q}_g = \operatorname{dom} \mathbf{q}_h.$$

Proof. Note that, for any $\nu \in \{g, h\}$, $\operatorname{dom} \mathbf{q}_\nu$ is the closure of compactly supported forms $\Omega_{\mathsf{C}^\infty_c}(M, \nu)$ with respect to the Dirac graph norm

$$\alpha \mapsto \left(\|\alpha\|_\nu^2 + \|\mathsf{D}_\nu \alpha\|_\nu^2\right)^{1/2}.$$

Moreover let $\mathbf{q}^\nabla$ be the nonnegative closed sesquilinear form corresponding to the horizontal Laplacian $\square_\nu = (\nabla^\nu)^* \nabla^\nu$, i.e. $\mathbf{q}^\nabla_\nu(\alpha) = \langle \square_\nu \alpha, \alpha\rangle_\nu = \|\nabla^\nu \alpha\|_\nu^2$ with $\operatorname{dom}(\mathbf{q}^\nabla_\nu)$ its natural domain of definition. Recall that by the Weitzenböck formula (1.11), we thus have the relation

$$\Delta_\nu = \square_\nu - \mathscr{R}_\nu.$$

As the Weitzenböck curvature term $\mathscr{R}_\nu$ is in the Kato class, by Theorem 2.7 the corresponding form domains

$$\operatorname{dom} \mathbf{q}_\nu = \operatorname{dom} \mathbf{q}^\nabla_\nu$$

coincide. To this end, it suffices to show that

$$I \operatorname{dom} \mathbf{q}^\nabla_g = \operatorname{dom} \mathbf{q}^\nabla_h.$$

For all compactly supported $\alpha \in \Omega_{\mathsf{C}^\infty_c}(M, g)$, we write

$$\begin{aligned}\nabla^h(I\alpha) &= \nabla^h(\mathscr{A}^{-1/2}\alpha) = \nabla^h \mathscr{A}^{-1/2}(\alpha) + \mathscr{A}^{-1/2}\nabla^h \alpha \\ &= \left(\nabla^h - \nabla^g\right)\mathscr{A}^{-1/2}(\alpha) + \nabla^g \mathscr{A}^{-1/2}(\alpha) + \mathscr{A}^{-1/2}\left(\nabla^h - \nabla^g\right)\alpha + \mathscr{A}^{-1/2}\nabla^g \alpha.\end{aligned}$$

Moreover,

$$\begin{aligned}\left|\left(\nabla^h_X - \nabla^g_X\right)\mathscr{A}^{-1/2}\right|_h &= \left|\mathscr{A}^{1/2}\left(\nabla^h_X - \nabla^g_X\right)\mathscr{A}^{-1/2}\right|_g \\ &\leqslant \left|\mathscr{A}^{1/2}\right| \left|\nabla^h_X - \nabla^g_X\right|_g \left|\mathscr{A}^{-1/2}\right|\end{aligned}$$

and

$$\left|\mathscr{A}^{-1/2}\left(\nabla^h_X - \nabla^g_X\right)\right|_h = \left|\nabla^h_X - \nabla^g_X\right|_g.$$

Thus we can estimate as follows

$$\begin{aligned}\left\|\nabla^h(I\alpha)\right\|_h^2 &= \int_M \left|\nabla^h(\mathscr{A}^{-1/2}\alpha)\right|_h^2 \mathrm{d}\,\mathrm{vol}_h \\ &= \int_M \Big| \left(\nabla^h - \nabla^g\right)\mathscr{A}^{-1/2}(\alpha) + \nabla^g \mathscr{A}^{-1/2}(\alpha) + \mathscr{A}^{-1/2}\left(\nabla^h - \nabla^g\right)(\alpha) \\ &\qquad + \mathscr{A}^{-1/2}\nabla^g \alpha\Big|_h^2 \mathrm{d}\,\mathrm{vol}_h \\ &\leqslant C \int_M \Big(\left|\left(\nabla^h - \nabla^g\right)\mathscr{A}^{-1/2}(\alpha)\right|_h^2 + \left|\nabla^g \mathscr{A}^{-1/2}(\alpha)\right|_h^2 +\end{aligned}$$

$$
\begin{aligned}
&\left|\mathscr{A}^{-1/2}\left(\nabla^h-\nabla^g\right)(\alpha)\right|_h^2+\left|\mathscr{A}^{-1/2}\nabla^g\alpha\right|_h^2\Big)\rho_{g,h}\,\mathrm{d}\,\mathrm{vol}_g\\
&\leqslant C\int_M\Big(\left|\mathscr{A}^{1/2}\left(\nabla^h-\nabla^g\right)\mathscr{A}^{-1/2}(\alpha)\right|_g^2+\left|\mathscr{A}^{1/2}\nabla^g\mathscr{A}^{-1/2}(\alpha)\right|_g^2+\\
&\qquad\left|\left(\nabla^h-\nabla^g\right)(\alpha)\right|_g^2+|\nabla^g\alpha|_g^2\Big)\mathrm{d}\,\mathrm{vol}_g\\
&\leqslant C\int_M\Big(\left|\mathscr{A}^{1/2}\right|^2\left|\nabla^h-\nabla^g\right|_g^2\left|\mathscr{A}^{-1/2}\right|^2|\alpha|_g^2\\
&\qquad+\left|\mathscr{A}^{1/2}\right|^2|\mathscr{A}|^2\left|\mathscr{A}^{-1/2}\right|^6\left|\nabla_X^h-\nabla_X^g\right|_g^2|\alpha|_g^2\\
&\qquad+\left|\nabla^h-\nabla^g\right|_g^2|\alpha|_g^2+|\nabla^g\alpha|_g^2\Big)\mathrm{d}\,\mathrm{vol}_g\\
&\leqslant C\int_M\left(\left|\nabla^h-\nabla^g\right|_g^2|\alpha|_g^2+|\nabla^g\alpha|_g^2\right)\mathrm{d}\,\mathrm{vol}_g\\
&\leqslant C\int_M\left(\left\|\delta^\nabla_{g,h}\right\|_\infty|\alpha|_g^2+|\nabla^g\alpha|_g^2\right)\mathrm{d}\,\mathrm{vol}_g\\
&\leqslant C\left(\|\alpha\|_g^2+\|\nabla^g\alpha\|_g^2\right),
\end{aligned}
$$

using the elementary inequality $(a+b)^c \leqslant 2^{c-1}(a^c+b^c)$ multiple times and that δ^∇ is bounded by assumption.

Hence, we arrive at the estimate

$$\|I\alpha\|_h^2+\left\|\nabla^h I\alpha\right\|_h^2\leqslant C\left(\|\alpha\|_g^2+\|\nabla^g\alpha\|_g^2\right),$$

proving

$$I\,\operatorname{dom}\mathbf{q}_g\subset\operatorname{dom}\mathbf{q}_h.$$

Since $I^{-1}=I_{g,h}^{-1}=I_{h,g}$ and the arguments above are symmetric in g and h, this shows the claim. ■

Next, we denote by $|\cdot|:\mathbb{C}\to\mathbb{R}$ the absolute value function and by $\operatorname{sgn}:\mathbb{C}\to\mathbb{C}$ the sign-function with $\operatorname{sgn}(0)=1$. By Lemma 1.47 if P is normal operator (e.g. positive or diagonalisable), we get the (pointwise) polar decomposition $P=|P|(\operatorname{sgn}P)$, where $|P|(x)=|P(x)|\geqslant 0$ and $|\operatorname{sgn}P(x)|=1$, and where $|P|(x)$ is a non-negative endomorphism and $\operatorname{sgn}P(x)$ is unitary. More precisely, by the Spectral Theorem 1.50 choosing $f(\lambda):=|\lambda|$, we have an endomorphism

$$f\left(\hat{\mathsf{S}}_{g,h}(x)\right):\bigwedge\mathsf{T}^*M\to\bigwedge\mathsf{T}^*M$$

giving rise to a decomposition

$$\hat{\mathsf{S}}_{g,h}(x)=\left|\hat{\mathsf{S}}_{g,h}(x)\right|\operatorname{sgn}\hat{\mathsf{S}}_{g,h}(x):\bigwedge\mathsf{T}^*M\to\bigwedge\mathsf{T}^*M.$$

For the proof of Theorem 3.32, we now introduce sections

$$
\begin{aligned}
&\mathsf{S}_{g,h} : M \to \mathbb{R} \\
&\mathsf{S}_{g,h}(x) := \rho_{g,h}(x)^{1/2} - \rho_{g,h}(x)^{-1/2} = 2\sinh\frac{1}{2}\log(\rho_{g,h}(x)), \\
&\hat{\mathsf{S}}_{g,h} : M \to \mathrm{End}\left(\bigwedge \mathrm{T}^* M\right) \\
&\hat{\mathsf{S}}_{g,h}(x) := \left(\rho_{g,h}(x)\mathscr{A}_{g,h}(x)\right)^{1/2} - \left(\rho_{g,h}(x)\mathscr{A}_{g,h}(x)\right)^{-1/2} = 2\sinh\frac{1}{2}\log(\rho_{g,h}(x)\mathscr{A}_{g,h}(x)), \\
&\hat{\mathsf{S}}_{g,h;\nu} : \Omega_{\mathrm{L}^2}(M,\nu) \to \Omega_{\mathrm{L}^2}(M,\nu) \\
&\hat{\mathsf{S}}_{g,h;\nu}\alpha(x) := \left|\hat{\mathsf{S}}_{g,h}(x)\right|^{1/2}\alpha(x), \\
&\mathsf{U}_{g,h} : \Omega_{\mathrm{L}^2}(M,g) \to \Omega_{\mathrm{L}^2}(M,h) \\
&\mathsf{U}_{g,h}\alpha(x) := \mathscr{A}_{g,h}(x)^{-1/2}\alpha(x), \\
&\hat{\mathsf{U}}_{g,h} : \Omega_{\mathrm{L}^2}(M,g) \to \Omega_{\mathrm{L}^2}(M,h) \\
&\hat{\mathsf{U}}_{g,h}\alpha(x) := (\rho_{g,h}(x)\mathscr{A}_{g,h}(x))^{-1/2}\alpha(x), \\
&\widetilde{\mathsf{U}}_{g,h} : \Omega_{\mathrm{L}^2}(M,g) \to \Omega_{\mathrm{L}^2}(M,h) \\
&\widetilde{\mathsf{U}}_{g,h}\alpha(x) := (\operatorname{sgn}\hat{\mathsf{S}}_{g,h}(x))(\rho_{g,h}(x)\mathscr{A}_{g,h}(x))^{-1/2}\alpha(x).
\end{aligned}
$$

By quasi-isometry, $g \sim h$, the operators $\hat{\mathsf{S}}_{g,h;\nu}$, $\mathsf{U}_{g,h}$, $\hat{\mathsf{U}}_{g,h}$ and $\widetilde{\mathsf{U}}_{g,h}$ are bounded. Moreover, we get the pointwise estimate (see also [HPW14, Lemma 3.3], [GT20, (4.1)]).

Lemma 3.35. *We have the pointwise estimate*

$$\max\left\{\left|\mathsf{S}_{g,h}(x)\right|, \sigma_{\max}\left(\left|\hat{\mathsf{S}}_{g,h}(x)\right|\right)\right\} \leqslant \delta_{g,h}(x) \qquad \forall x \in M. \tag{3.32}$$

Proof. We write $\rho = \rho_{g,h}$ and $\mathscr{A} = \mathscr{A}_{g,h}$ for short. By definition, we have

$$\left|\hat{\mathsf{S}}_{g,h}\right| = \left|(\rho\mathscr{A})^{1/2} - (\rho\mathscr{A})^{-1/2}\right| = 2\sinh\left|\frac{1}{2}\log(\rho\mathscr{A})\right|,$$

and the i^{th} eigenvalue of $\log(\rho\mathscr{A})$ is given by

$$-\sum_{i=1}^{m}\frac{\log\alpha_i}{2} + \log\alpha_j.$$

If we choose k_0 such that $|\log\alpha_{k_0}| = \max_k\left|\log\alpha_k\right|$, then

$$\left|-\sum_{i=1}^{m}\frac{\log\alpha_i}{2} + \log\alpha_j\right| \leqslant \frac{m}{2}\left|\log\alpha_{k_0}\right|.$$

Hence,

$$\sigma_{\max}\left(\left|\hat{\mathsf{S}}_{g,h}\right|\right) \leqslant 2\sinh\frac{m}{4}\left|\log\alpha_{k_0}\right| = \delta_{g,h}(x),$$

justifying the definition of $\delta_{g,h}$. A similar calculation shows the assertion for $\mathsf{S}_{g,h}$. ■

The following lemma provides the trace class operator required in the decomposition formula in assumption (4) of the Belopol'skii-Birman Theorem 3.5.

Lemma 3.36. *Let $g, h \in \mathsf{Metr}M$, $g \sim h$. We define the unbounded operator*

$$\mathbf{T}_s^{g,h} : \Omega_{\mathsf{L}^2}(M,g) \to \Omega_{\mathsf{L}^2}(M,h)$$
$$\mathbf{T}_s^{g,h} := (\hat{P}_s^{h,g})^* \hat{\mathbf{U}}_{g,h} \hat{P}_s^{g} - (\hat{P}_s^{h})^* \mathbf{U}_{g,h} \hat{P}_s^{g,h} + (\check{P}_s^{g,h})^* \hat{\mathbf{U}}_{g,h} \check{P}_s^{g} - (\check{P}_s^{h})^* \mathbf{U}_{g,h} \check{P}_s^{h,g} - P_s^h \hat{\mathbf{S}}_{g,h;h} \widetilde{\mathbf{U}}_{g,h} \hat{\mathbf{S}}_{g,h;g} P_{s/2}^g \Delta_g P_{s/2}^g.$$

Then we get, for $\alpha_1 \in \operatorname{dom} \Delta_g$, $\alpha_2 \in \operatorname{dom} \Delta_h$ and $s > 0$,

$$\left\langle \alpha_2, \mathbf{T}_s^{g,h} \alpha_1 \right\rangle_h = \left\langle \Delta_h \alpha_2, P_s^h I P_s^g \alpha_1 \right\rangle_h - \left\langle \alpha_2, P_s^h I P_s^g \Delta_g \alpha_1 \right\rangle_h .$$

Proof. First note that

$$\mathbf{d}^2 = 0 \qquad \text{and} \qquad \boldsymbol{\delta}^2 = 0. \tag{3.33}$$

Since Δ_ν is essentially self-adjoint, for $\nu \in \{g,h\}$, we can assume $\alpha_1 \in \Omega_{\mathsf{C}_c^\infty}(M,g)$ and $\alpha_2 \in \Omega_{\mathsf{C}_c^\infty}(M,h)$ to be compactly supported. Then

$$\begin{aligned}
&\left\langle \Delta_h \alpha_2, P_s^h I P_s^g \alpha_1 \right\rangle_h - \left\langle \alpha_2, P_s^h I P_s^g \Delta_g \alpha_1 \right\rangle_h \\
&\quad = \left\langle \Delta_h P_s^h \alpha_2, I P_s^g \alpha_1 \right\rangle_h - \left\langle \Delta_g I^{-1} P_s^h \alpha_2, P_s^g \alpha_1 \right\rangle_h - \left\langle P_s^h \alpha_2, \left(I - (I^{-1})^*\right) P_s^g \Delta_g \alpha_1 \right\rangle_h \\
&\quad = - \left\langle (\mathbf{d} + \boldsymbol{\delta}_h) P_s^h \alpha_2, (\mathbf{d} + \boldsymbol{\delta}_h) I P_s^g \alpha_1 \right\rangle_h + \left\langle (\mathbf{d} + \boldsymbol{\delta}_g) I^{-1} P_s^h \alpha_2, (\mathbf{d} + \boldsymbol{\delta}_g) P_s^g \alpha_1 \right\rangle_h \\
&\qquad - \left\langle P_s^h \alpha_2, \left(I - (I^{-1})^*\right) P_s^g \Delta_g \alpha_1 \right\rangle_h \\
&\overset{(3.33)}{=} - \left\langle \mathbf{d}\, P_s^h \alpha_2, \mathbf{d}\, I P_s^g \alpha_1 \right\rangle_h - \left\langle \boldsymbol{\delta}_h P_s^h \alpha_2, \boldsymbol{\delta}_h I P_s^g \alpha_1 \right\rangle_h \\
&\qquad + \left\langle \mathbf{d}\, I^{-1} P_s^h \alpha_2, \mathbf{d}\, P_s^g \alpha_1 \right\rangle_h + \left\langle \boldsymbol{\delta}_g I^{-1} P_s^h \alpha_2, \boldsymbol{\delta}_g P_s^g \alpha_1 \right\rangle_h \\
&\qquad - \left\langle P_s^h \alpha_2, \left(I - (I^{-1})^*\right) P_s^g \Delta_g \alpha_1 \right\rangle_h .
\end{aligned} \tag{3.34}$$

Let us treat the terms separately. For the last term in (3.34),

$$\begin{aligned}
&\left\langle P_s^h \alpha_2, \left(I - (I^{-1})^*\right) P_s^g \Delta_g \alpha_1 \right\rangle_h \\
&\quad = \int_M \left\langle P_s^h \alpha_2, \left(\mathscr{A}^{-1/2} - (\rho^{-1} \mathscr{A}^{-1/2}\right) P_s^g \Delta_g \alpha_1 \right\rangle \mathrm{d\,vol}_h \\
&\quad = \int_M \left\langle P_s^h \alpha_2, \left(1 - \rho^{-1}\right) \mathscr{A}^{-1/2} P_s^g \Delta_g \alpha_1 \right\rangle \mathrm{d\,vol}_h \\
&\quad = \int_M \left\langle P_s^h \alpha_2, \hat{\mathbf{S}}_{g,h} \rho^{-1/2} \mathscr{A}^{-1/2} P_s^g \Delta_g \alpha_1 \right\rangle \mathrm{d\,vol}_h \\
&\quad = \int_M \left\langle P_s^h \alpha_2, \left|\hat{\mathbf{S}}_{g,h}\right|^{1/2} (\operatorname{sgn} \hat{\mathbf{S}}_{g,h}) (\rho \mathscr{A})^{-1/2} \left|\hat{\mathbf{S}}_{g,h}\right|^{1/2} P_s^g \Delta_g \alpha_1 \right\rangle \mathrm{d\,vol}_h \\
&\quad = \left\langle \alpha_2, P_s^h \hat{\mathbf{S}}_{g,h;h} \widetilde{\mathbf{U}}_{g,h} \hat{\mathbf{S}}_{g,h;g} P_{s/2}^g \Delta_g P_{s/2}^g \alpha_1 \right\rangle_h .
\end{aligned}$$

For the first and third term involving the exterior derivative $\mathbf{d}$, we get

$$\begin{aligned}
&\left\langle \mathbf{d}\, I^{-1} P_s^h \alpha_2, \mathbf{d}\, P_s^g \alpha_1 \right\rangle_h - \left\langle \mathbf{d}\, P_s^h \alpha_2, \mathbf{d}\, I P_s^g \alpha_1 \right\rangle_h \\
&\quad = \left\langle (I \mathbf{d}\, I^{-1}) P_s^h \alpha_2, (I^{-1})^* \mathbf{d}\, P_s^g \alpha_1 \right\rangle_h - \left\langle \mathbf{d}\, P_s^h \alpha_2, I (I^{-1} \mathbf{d}\, I) P_s^g \alpha_1 \right\rangle_h \\
&\quad = \left\langle \hat{P}_s^{h,g} \alpha_2, (I^{-1})^* \mathbf{d}\, P_s^g \alpha_1 \right\rangle_h - \left\langle \mathbf{d}\, P_s^h \alpha_2, I \hat{P}_s^{g,h} \alpha_1 \right\rangle_h
\end{aligned}$$

$$= \left\langle \alpha_2, (\hat{P}_s^{h,g})^* (I^{-1})^* \mathbf{d}\, P_s^g \alpha_1 - (\mathbf{d}\, P_s^h)^* I \hat{P}_s^{g,h} \alpha_1 \right\rangle_h$$
$$= \int \left(\alpha_2, (\hat{P}_s^{h,g})^* \rho^{-1} \mathscr{A}^{-1/2} \mathbf{d}\, P_s^g \alpha_1 - (\mathbf{d}\, P_s^h)^* \mathscr{A}^{-1/2} \hat{P}_s^{g,h} \alpha_1 \right) \mathrm{d}\,\mathrm{vol}_h$$
$$= \left\langle \alpha_2, \left((\hat{P}_s^{h,g})^* \hat{\mathbf{U}}_{g,h} \mathbf{d}\, P_s^g - (\mathbf{d}\, P_s^h)^* \mathbf{U}_{g,h} \hat{P}_s^{g,h} \right) \alpha_1 \right\rangle_h$$
$$= \left\langle \alpha_2, \left((\hat{P}_s^{h,g})^* \hat{\mathbf{U}}_{g,h} \hat{P}_s^g - (\hat{P}_s^h)^* \mathbf{U}_{g,h} \hat{P}_s^{g,h} \right) \alpha_1 \right\rangle_h .$$

Similarly, for the codifferential $\boldsymbol{\delta}$,

$$\left\langle \boldsymbol{\delta}_g I^{-1} P_s^h \alpha_2, \boldsymbol{\delta}_g P_s^g \alpha_1 \right\rangle_h - \left\langle \boldsymbol{\delta}_h P_s^h \alpha_2, \boldsymbol{\delta}_h I P_s^g \alpha_1 \right\rangle_h$$
$$= \left\langle (I \boldsymbol{\delta}_g I^{-1}) P_s^h \alpha_2, (I^{-1})^* \boldsymbol{\delta}_g P_s^g \alpha_1 \right\rangle_h - \left\langle \boldsymbol{\delta}_h P_s^h \alpha_2, I (I^{-1} \boldsymbol{\delta}_h I) P_s^g \alpha_1 \right\rangle_h$$
$$= \left\langle \check{P}_s^{g,h} \alpha_2, (I^{-1})^* \boldsymbol{\delta}_h P_s^g \alpha_1 \right\rangle_h - \left\langle \boldsymbol{\delta}_h P_s^h \alpha_2, I \check{P}_s^{h,g} \alpha_1 \right\rangle_h$$
$$= \left\langle \alpha_2, \left((\check{P}_s^{g,h})^* \hat{\mathbf{U}}_{g,h} \boldsymbol{\delta}_h P_s^g - (\boldsymbol{\delta}_h P_s^h)^* \mathbf{U}_{g,h} \check{P}_s^{h,g} \right) \alpha_1 \right\rangle_h$$
$$= \left\langle \alpha_2, \left((\check{P}_s^{g,h})^* \hat{\mathbf{U}}_{g,h} \check{P}_s^g - (\check{P}_s^h)^* \mathbf{U}_{g,h} \check{P}_s^{h,g} \right) \alpha_1 \right\rangle_h . \qquad \blacksquare$$

We are finally in the position to proof our Main Result.

Proof of Theorem 3.32. We check the assumptions of the Belopol'skii-Birman Theorem 3.5.

Since $g \sim h$, the operator $I \equiv I_{g,h}$ is well-defined and bounded and has a bounded inverse $I^{-1} \equiv I_{h,g}$, so (1) follows. By Lemma 3.34, also assumption (2) is satisfied.

Recalling that by (3.25), $I_{g,h}^* = \rho_{g,h} I_{g,h}^{-1}$, we see that the operator $(I^*I - 1)e^{-s\Delta_g}$ has the integral kernel

$$\left[(I^*I - 1)\mathrm{e}^{-s\Delta_g} \right](x,y) = \left(\rho_{g,h} - 1 \right) p_s^g(x,y)$$
$$= \rho_{g,h}^{1/2} (\operatorname{sgn} \mathbf{S}_{g,h}) \left| \mathbf{S}_{g,h} \right|^{1/2} \left| \mathbf{S}_{g,h} \right|^{1/2} p_s^g(x,y).$$

Thus by Lemma 2.5 again, for some $s > 0$,

$$\int \left| \left[(I^*I - 1)e^{-s\Delta_g} \right](x,y) \right|^2 \mathrm{vol}_g(\mathrm{d}y) \leqslant \left\| \rho_{g,h}^{1/2} \mathbf{S}_{g,h} \right\|_\infty \left| \mathbf{S}_{g,h} \right| \int p_s^g(x,y)^2 \,\mathrm{vol}_g(\mathrm{d}y)$$
$$\leqslant C(\gamma, c_\gamma, s) \left\| \rho_{g,h}^{1/2} \mathbf{S}_{g,h} \right\|_\infty \left| \mathbf{S}_{g,h} \right| \Phi_g(x,s) \int p_s^{g,(0)}(x,y) \,\mathrm{vol}_g(\mathrm{d}y),$$

and, by Lemma 3.35, we arrive at the Hilbert-Schmidt estimate

$$\iint \left| \left[(I^*I - 1)\mathrm{e}^{-s\Delta_g} \right](x,y) \right|^2 \mathrm{vol}_g(\mathrm{d}y) \,\mathrm{vol}_g(\mathrm{d}x) \lesssim \int \delta_{g,h}(x) \Xi_g(x,s) \Phi_g(x,s) \,\mathrm{vol}_g(\mathrm{d}x) < \infty.$$

So, the operator $(I^*I - 1)\mathrm{e}^{-s\Delta_g}$ is Hilbert-Schmidt, hence compact, which proves assumption (3).

Finally, we prove (4). Using Lemma 3.36 it remains to show that $\mathbf{T}_s^{g,h}$ is trace class. Since the product of Hilbert-Schmidt operators is trace class, we prove that the operators $\hat{P}_s^\nu$, $\check{P}_s^\nu$, for $\nu \in \{g,h\}$, and $\hat{P}_s^{g,h}$, $\check{P}_s^{g,h}$, $\check{P}_s^{h,g}$ are Hilbert-Schmidt. Recall that $\hat{p}_s^\nu(x,y)$, $\check{p}_s^\nu(x,y)$, $\hat{p}_s^{g,h}(x,y)$, $\check{p}_s^{g,h}(x,y)$ and $\check{p}_s^{h,g}(x,y)$ are the corresponding jointly smooth integral kernel of $\hat{P}_s^\nu$, $\check{P}_s^\nu$, $\hat{P}_s^{g,h}$, $\check{P}_s^{g,h}$ and $\check{P}_s^{h,g}$, respectively.

Then, by (3.26),

$$\iint \left|\hat{p}_s^{\nu}(x,y)\right|_{\nu}^2 \operatorname{vol}_{\nu}(\mathrm{d}y)\operatorname{vol}_{\nu}(\mathrm{d}x) \leqslant \int \Psi_{\nu}(x,s)\Phi_{\nu}(x,s)\operatorname{vol}_{\nu}(\mathrm{d}x)$$

and, by (3.27),

$$\iint \left|\check{p}_s^{\nu}(x,y)\right|_{\nu}^2 \operatorname{vol}_{\nu}(\mathrm{d}y)\operatorname{vol}_{\nu}(\mathrm{d}x) \leqslant \int \Psi_{\nu}(x,s)\Phi_{\nu}(x,s)\operatorname{vol}_{\nu}(\mathrm{d}x).$$

So similarly, we have, by (3.28),

$$\iint \left|\hat{p}_s^{g,h}(x,y)\right|_g^2 \operatorname{vol}_g(\mathrm{d}y)\operatorname{vol}_g(\mathrm{d}x) \lesssim \int \left(\delta_{g,h}^{\nabla}(x) + \Xi_g(x,s)\right)\Phi_g(x,s)\operatorname{vol}_g(\mathrm{d}x)$$

and, by (3.29),

$$\iint \left|\check{p}_s^{g,h}(x,y)\right|_g^2 \operatorname{vol}_g(\mathrm{d}y)\operatorname{vol}_g(\mathrm{d}x) \lesssim \int \left(\delta_{g,h}^{\nabla}(x) + \Xi_g(x,s)\right)\Phi_g(x,s)\operatorname{vol}_g(\mathrm{d}x)$$

Finally, by (3.30),

$$\iint \left|\check{p}_s^{h,g}(x,y)\right|_g^2 \operatorname{vol}_g(\mathrm{d}y)\operatorname{vol}_g(\mathrm{d}x) \lesssim \int \left(\delta_{g,h}^{\nabla}(x) + \Xi_g(x,s)\right)\Phi_g(x,s)\operatorname{vol}_g(\mathrm{d}x)$$

This completes the proof. ■

3.6 Applications and Examples

3.6.1 Ricci flow We first generalise a result, given in [GT20], concerning the stability of the absolutely continuous spectrum of a family of metrics evolving under a Ricci flow. Let therefore R_g be the Riemannian curvature tensor with respect to the metric g.

Corollary 3.37. *Let $S > 0$, $\lambda \in \mathbb{R}$ and assume that*

(a) the family $(g_s)_{0\leqslant s\leqslant S} \subset \operatorname{Metr} M$ evolves under a Ricci-type flow

$$\partial_s g_s = \lambda \operatorname{Ric}_{g_s}, \qquad \forall 0 \leqslant s \leqslant S,$$

(b) the initial metric g_0 is geodesically complete,

(c) there is some $C > 0$ such that $\left|\mathsf{R}_{g_s}\right|_{g_s}, \left|\nabla^{g_s}\mathsf{R}_{g_s}\right|_{g_s} \leqslant C \qquad \forall 0 \leqslant s \leqslant S.$

We set, for all $x \in M$,

$$\begin{aligned} M_1(x) &:= \sup\left\{\left|\operatorname{Ric}_{g_s}(v,v)\right| : 0 \leqslant s \leqslant S,\ v \in \mathsf{T}_x M,\ |v|_{g_s} \leqslant 1\right\}, \\ M_2(x) &:= \sup\Big\{\left|\nabla_v^{g_s}\operatorname{Ric}_{g_s}(u,w) + \nabla_u^{g_s}\operatorname{Ric}_{g_s}(v,w) + \nabla_w^{g_s}\operatorname{Ric}_{g_s}(u,v)\right| : 0 \leqslant s \leqslant S, \\ &\qquad u,v,w \in \mathsf{T}_x M,\ |u|_{g_s}, |v|_{g_s}, |w|_{g_s} \leqslant 1\Big\}. \end{aligned}$$

Let $\mathsf{B}_g(x,r)$ denote the open geodesic ball (with respect to g). If

$$\int \operatorname{vol}_{g_0}(\mathsf{B}_{g_0}(x,1))^{-1} \max\left\{\sinh\left(\frac{m}{4} S\,|\lambda|\, M_1(x)\right), M_2(x)\right\} \operatorname{vol}_{g_0}(\mathrm{d}x) < \infty,$$

then $\sigma_{\mathrm{ac}}(\Delta_{g_s}) = \sigma_{\mathrm{ac}}(\Delta_{g_0})$ for all $0 \leqslant s \leqslant S$.

Proof. The Ricci flow equation together with (i) implies that $g_s \sim g_0$ for all $0 \leqslant s \leqslant S$ and all g_s are complete. Assumption (c) assures that $\Xi(x,s)$ is bounded. By the same arguments as in [GT20, Corollary B],

$$\delta_{g_s,g_0} \leqslant \sinh\left(\frac{m}{4} S\,|\lambda|\,M_1(x)\right)$$

and as in the proof of [BG20, Theorem 6.1]

$$\delta^{\nabla}_{g_s,g_0}(x) \leqslant C M_2(x)$$

and so the claim follows. ■

3.6.2 Differential k-forms

We specify our main result, Theorem 3.32, to differential k-forms. Set

$$\begin{aligned}
\overline{K}^{(k)}_g(x) &:= \max\left\{\left(\mathscr{R}^{(k)}_g v, v\right) : v \in \bigwedge\nolimits^k \mathsf{T}_y(M,g), |v|_g = 1, y \in \mathsf{B}_g(x,1)\right\},\\
\underline{K}^{(k)}_g(x) &:= \min\left\{\left(\mathscr{R}^{(k)}_g v, v\right) : v \in \bigwedge\nolimits^k \mathsf{T}_y(M,g), |v|_g = 1, y \in \mathsf{B}_g(x,1)\right\},
\end{aligned}$$

for the corresponding constants defined analogously to (2.15) and (2.16), respectively. Following the same lines as the proof of our main result, Theorem 3.32, we get the following

Corollary 3.38. *Let $g,h \in \mathrm{Metr}M$, $g \sim h$, and assume that there exists $C < \infty$ such that $\left|\delta^{\nabla}_{g,h}\right| \leqslant C$ and that for both $\nu \in \{g,h\}$, we have $\left|\mathscr{R}_\nu\right|_\nu \in \mathsf{K}(M)$ and*

$$\int \max\left\{\delta_{g,h}(x), \delta^{\nabla}_{g,h}(x) + \Xi^{(k),\pm}_g(x,s), \Psi^{(k),\pm}_\nu(x,s)\right\} \Phi_\nu(x,s)\,\mathrm{vol}_\nu(\mathrm{d}x) < \infty, \qquad \textit{some } s > 0, \tag{3.35}$$

where

$$\begin{aligned}
\Psi^{(k),\pm}_\nu(x,s) &:= \Psi^{(k),+}_\nu(x,s) \wedge \Psi^{(k),-}_\nu(x,s),\\
\Xi^{(k),\pm}_\nu(x,s) &:= \Xi^{(k),+}_\nu(x,s) \wedge \Xi^{(k),-}_\nu(x,s),
\end{aligned}$$

with

$$\Psi^{(k),\pm}_\nu(x,s) := \frac{1}{\sqrt{s}} \exp\left[D(\gamma, c_\gamma(\mathscr{R}^-_\nu), c_q^{1/q})s + \left(\pi\sqrt{(m-1)\underline{K}^{(0),-}_\nu(x)} + \pi^2(m+5) + \left(\overline{K}^{(k)}_\nu(x) + \underline{K}^{(k\pm1)}_\nu(x)\right)^-\right)\frac{s}{2}\right],$$

where the finite constant D depends on the constant $c_\gamma(\mathscr{R}^-)$ in (2.7) and the constant c_q from the Burkholder-Davis-Gundy inequality (cf. Theorem 2.27), and

$$\Xi^{(k),\pm}_\nu(x,s) := \Psi^{(k),\pm}_\nu(x,s) + s^{3/2}\Psi^{(k),\pm}_\nu(x,s) \max_{y \in \mathsf{B}_\nu(x,1)} \left|\nabla^\nu \mathsf{R}_\nu(y)\right|,$$

and $\Phi(x,s)$ is defined by (3.19).

For all $0 \leqslant k \leqslant m$, let

$$I^{(k)} := I^{(k)}_{g,h} : \Omega^k_{\mathsf{L}^2}(M,g) \to \Omega^k_{\mathsf{L}^2}(M,h), \quad \alpha \mapsto \bigwedge\nolimits^k A^{-1/2}_{g,h}(\alpha)$$

be the bounded identification operator acting on k-forms. Then, for all $0 \leqslant k \leqslant m$, the wave operators

$$W_\pm(\Delta^{(k)}_h, \Delta^{(k)}_g, I^{(k)}) = \operatorname*{s-lim}_{t\to\pm\infty} \mathrm{e}^{it\Delta^{(k)}_h} I \mathrm{e}^{-it\Delta^{(k)}_g} \mathsf{P}_{\mathrm{ac}}(\Delta^{(k)}_g)$$

exist and are complete. Moreover, $W_\pm(\Delta^{(k)}_h, \Delta^{(k)}_g, I^{(k)})$ are partial isometries with initial space $\operatorname{ran}\mathsf{P}_{\mathrm{ac}}(\Delta^{(k)}_g)$ and final space $\operatorname{ran}\mathsf{P}_{\mathrm{ac}}(\Delta^{(k)}_h)$, and we have $\sigma_{\mathrm{ac}}(\Delta^{(k)}_g) = \sigma_{\mathrm{ac}}(\Delta^{(k)}_h)$.

Proof. We omit the metric in the notation. By similar calculations as in the proofs of Theorem 2.27 and Theorem 2.28, we see that for every $\alpha \in \Omega^k_{\mathsf{L}^2}(M)$,

$$\left|(\mathbf{d}^{(k)} P_s \alpha)_x\right|^2 \leqslant \Psi^{(k),+}(x,s)\Phi(x,s)\,\|\alpha\|^2_{\Omega^k_{\mathsf{L}^2}(M)},$$
$$\left|(\boldsymbol{\delta}^{(k)} P_s \alpha)_x\right|^2 \leqslant \Psi^{(k),-}(x,s)\Phi(x,s)\,\|\alpha\|^2_{\Omega^k_{\mathsf{L}^2}(M)},$$
$$\left|\left(\nabla P_s \alpha, \xi\right)\right|^2 \leqslant \max\left\{\Xi^{(k),+}(x,s), \Xi^{(k),-}(x,s)\right\}\Phi(x,s)\,\|\alpha\|^2_{\Omega^k_{\mathsf{L}^2}(M)}.$$

Noticing that

$$\Delta_\nu = \bigoplus_{k=0}^m \Delta_\nu^{(k)} \qquad \text{and} \qquad I = \bigoplus_{k=0}^m I^{(k)}$$

the proof now follows the lines of the proof of our main result, Theorem 3.32. ■

3.6.3 Conformal perturbations We study the important case of conformally equivalent metrics: Given a smooth function $\psi : M \to \mathbb{R}$, we define peturbated metric by $g_\psi := \mathrm{e}^{2\psi} g$. Note that g and g_ψ are quasi-isometric, if and only if ψ is bounded (cf. Example 3.14 above).

The bounded identification operator is now given by

$$I := I_{g,g_\psi} : \Omega_{\mathsf{L}^2}(M,g) \to \Omega_{\mathsf{L}^2}(M,g_\psi)$$
$$I_{g,g_\psi}\eta(x) \mapsto \mathrm{e}^{-\psi(x)}\eta(x).$$

Given a smooth function ψ on M, we define

$$\tau := \bigoplus_{k=0}^m (m-2k)\mathbf{1}_{\bigwedge^k \mathsf{T}^*M} \in \mathscr{D}^{(0)}(M; \textstyle\bigwedge \mathsf{T}^*M),$$
$$\mathrm{e}^{\psi\tau} := \bigoplus_{k=0}^m \mathrm{e}^{(m-2k)\psi}\mathbf{1}_{\bigwedge^k \mathsf{T}^*M} \in \mathscr{D}^{(0)}(M; \textstyle\bigwedge \mathsf{T}^*M).$$

Next, we collect some useful transformation rules for the conformal metric g_ψ in terms of g. A standard reference for various invariants of conformal metric change in part (a) is [Bes87, 1.159 Theorem].

Proposition 3.39. *Let $\psi : M \to \mathbb{R}$ be smooth.*

(a) We have

$$(\cdot,\cdot)^{(k)}_{g_\psi} = \mathrm{e}^{-2k\psi}\,(\cdot,\cdot)^{(k)}_g \qquad \forall k \in \{0,\dots,m\} \tag{3.36a}$$

$$\mathrm{d}\,\mathrm{vol}_{g_\psi} = \mathrm{e}^{m\psi}\mathrm{d}\,\mathrm{vol}_g \tag{3.36b}$$

$$\bullet \lrcorner_{g_\psi} \alpha = \mathrm{e}^{-2\psi}\left(\bullet \lrcorner_g \alpha\right) \qquad \forall \alpha \in \Omega^1(M) \tag{3.36c}$$

$$\nabla^{g_\psi}_X Y = \nabla^g_X Y + \mathbf{d}\psi(X)Y + \mathbf{d}\psi(Y)X - (X,Y)_g\,\mathrm{grad}_g\,\psi \qquad \forall X,Y \in \Gamma_{C^\infty}(\mathsf{T}M) \tag{3.36d}$$

$$\boldsymbol{\delta}_{g_\psi}\alpha = \mathrm{e}^{-2\psi}(\boldsymbol{\delta}_g\alpha - \tau\,\mathbf{d}\psi \lrcorner_g \alpha) \qquad \forall \alpha \in \Omega^k(M) \tag{3.36e}$$

(b) If ψ is bounded, then

$$I^* = e^{m\psi} I^{-1}. \tag{3.37}$$

Remark 3.40. We note that the canonical musical isomorphisms $\sharp$ and $\flat$ between $\mathsf{T}M$ and T^*M do not agree for g and g_ψ.

Proof. (a) We show (3.36e). Using (3.36a) and (3.36b), for any $\eta_1 \in \Omega^{k-1}(M)$, $\eta_2 \in \Omega^k(M)$,

$$\begin{aligned}\left\langle \eta_1, \boldsymbol{\delta}_{g_\psi} \eta_2 \right\rangle_{g_\psi} &= \left\langle \eta_1, e^{-2\psi}\left(\boldsymbol{\delta}_g \eta_2 + (m-2p)\mathbf{d}\psi \lrcorner \eta_2\right)\right\rangle_{g_\psi} \\ &= \left\langle e^{(m-2(p-1))\psi}\eta_1, e^{-2\psi}\boldsymbol{\delta}_g \eta_2 \right\rangle_g + \left\langle e^{(m-2(p-1))\psi}\eta_1, e^{-2\psi}(m-2p)\mathbf{d}\psi \lrcorner \eta_2 \right\rangle_g \\ &= \left\langle \mathbf{d}\left(e^{(m-2p)\psi}\eta_1\right), \eta_2 \right\rangle_g + \left\langle (m-2p)e^{(m-2p)\psi}\eta_1, \mathbf{d}\psi \lrcorner \eta_2 \right\rangle_g \\ &= \left\langle e^{(m-2p)\psi}\mathbf{d}\eta_1, \eta_2 \right\rangle_g + \left\langle \mathbf{d}\left(e^{(m-2p)\psi}\right) \wedge \eta_1, \eta_2 \right\rangle_g - \left\langle (m-2p)e^{(m-2p)\psi}\mathbf{d}\psi \wedge \eta_1, \eta_2 \right\rangle_g \\ &= \langle \mathbf{d}\eta_1, \eta_2 \rangle_{g_\psi}.\end{aligned}$$

(b) Follows from (3.25). ■

Theorem 3.41. *Let $\psi : M \to \mathbb{R}$ be smooth with ψ bounded, and assume that $g, g_\psi \in \mathsf{Metr}M$ with $g_\psi = e^{2\psi} g$ such that $\left|\delta^\nabla_{g,g_\psi}\right| \leqslant C$ for some $C < \infty$ and that for both $\nu \in \{g, g_\psi\}$, we have $\left|\mathscr{R}_\nu\right|_\nu \in \mathsf{K}(M)$ and*

$$\int \max\left\{\sinh\left|\frac{m}{4}\psi(x)\right|, \delta^\nabla_{g,h}(x) + \Xi_g(x,s), \Psi_\nu(x,s)\right\} \Phi_\nu(x,s)\,\mathrm{vol}_\nu(\mathrm{d}x) < \infty, \qquad \textit{some } s > 0. \tag{3.38}$$

Then the wave operators

$$W_\pm(\Delta_{g_\psi}, \Delta_g, I) = \operatorname*{s-lim}_{t\to\pm\infty} e^{it\Delta_{g_\psi}} I e^{-it\Delta_g} \mathsf{P}_{\mathrm{ac}}(\Delta_g)$$

exist and are complete. Moreover, $W_\pm(\Delta_{g_\psi}, \Delta_g, I)$ are partial isometries with initial space $\operatorname{ran}\mathsf{P}_{\mathrm{ac}}(\Delta_g)$ *and final space* $\operatorname{ran}\mathsf{P}_{\mathrm{ac}}(\Delta_{g_\psi})$*, and we have $\sigma_{\mathrm{ac}}(\Delta_g) = \sigma_{\mathrm{ac}}(\Delta_g)$.*

Proof. Using Example 3.14, we have $\delta_{g,g_\psi} = 2\sinh\frac{m}{4}|\psi|$ and

$$g \sim g_\psi \qquad \Longleftrightarrow \qquad \psi \text{ bounded}.$$

Hence the claim follows from our Main Result, Theorem 3.32. ■

By the same argument as in the proof of Theorem 3.32, we get the following consequence for the wave operators acting on k-forms but with appropriate localised constants respecting the degree of the differential form (cf. Proof of Corollary 3.38 above).

Corollary 3.42. *Let $\psi : M \to \mathbb{R}$ be smooth with ψ bounded, and assume that $g, g_\psi \in \mathsf{Metr}M$ with $g_\psi = e^{2\psi} g$ such that $\left|\delta^\nabla_{g,g_\psi}\right| \leqslant C$ for some $C < \infty$ and that for some $\nu \in \{g, g_\psi\}$, we have $\mathscr{R}_\nu \in \mathsf{K}(M)$ and*

$$\int \max\left\{\sinh\left|\frac{m}{4}\psi(x)\right| \Xi_g(x,s), \delta^\nabla_{g,h}(x) + \Xi^{(k)}_g(x,s), \Psi^{(k)}_\nu(x,s)\right\} \Phi_\nu(x,s)\,\mathrm{vol}_\nu(\mathrm{d}x) < \infty, \quad \textit{some } s > 0,$$

where

$$\Xi_\nu^{(k)}(x,s) := \Xi_\nu^{(k),+}(x,s) \wedge \Xi_\nu^{(k),-}(x,s).$$

For all $0 \leqslant k \leqslant m$, *let*

$$I^{(k)} := I^{(k)}_{g,g_\psi} : \Omega^k_{\mathsf{L}^2}(M,g) \to \Omega^k_{\mathsf{L}^2}(M,g_\psi), \quad \alpha \mapsto \bigwedge^k A^{-1/2}(\alpha)$$

be the bounded identification operator acting on k*-forms. Then, for all* $0 \leqslant k \leqslant m$*, the wave operators*

$$W_\pm(\Delta^{(k)}_{g_\psi}, \Delta^{(k)}_g, I^{(k)}) = \operatorname*{s-lim}_{t\to\pm\infty} \mathrm{e}^{it\Delta^{(k)}_{g_\psi}} I \mathrm{e}^{-it\Delta^{(k)}_g} \mathsf{P}_{\mathrm{ac}}(\Delta^{(k)}_g)$$

exist and are complete. Moreover, $W_\pm(\Delta^{(k)}_{g_\psi}, \Delta^{(k)}_g, I^{(k)})$ *are partial isometries with initial space* $\operatorname{ran}\mathsf{P}_{\mathrm{ac}}(\Delta^{(k)}_g)$ *and final space* $\operatorname{ran}\mathsf{P}_{\mathrm{ac}}(\Delta^{(k)}_{g_\psi})$*, and we have* $\sigma_{\mathrm{ac}}(\Delta^{(k)}_g) = \sigma_{\mathrm{ac}}(\Delta^{(k)}_{g_\psi})$.

3.6.4 Global curvature bounds

Let R_ν be the Riemannian curvature tensor with respect to the metric $\nu \in \{g,h\}$. Then the curvature operator

$$Q_\nu \in \mathscr{D}^{(0)}\left(M; \bigwedge^2 \mathsf{T}^* M\right)$$

is self-adjoint and uniquely determined by the equation

$$\left(Q_\nu(X\wedge Y), U \wedge V\right)_\nu = \left(\mathsf{R}_\nu(X,Y)U, V\right)_\nu$$

for all smooth vector fields $X, Y, U, V \in \Gamma_{\mathsf{C}^\infty}(\mathsf{T}M)$.

By the Gallot–Meyer estimate [GM75], a global bound $Q_\nu \geqslant -K$, for some constant $K > 0$, already implies that curvature endomorphism in the Weitzenböck formula (1.11) is globally bounded by

$$\mathscr{R}^{(k)}_\nu \geqslant -Kk(m-k). \tag{3.39}$$

Remark 3.43. In particular, if $Q_\nu \geqslant -K$, for some constant $K > 0$, then for $k = 1$, we have

$$\operatorname{Ric}_\nu \geqslant -K(m-1).$$

We set, for $\nu \in \{g,h\}$,

$$\Theta_\nu(x) := \left(1 + \max_{y\in \mathsf{B}_\nu(x,1)} \left|\nabla^\nu \mathsf{R}_\nu(y)\right|\right)^2.$$

Then, we get the following consequential result.

Theorem 3.44. *Let* $Q_\nu \geqslant -K$*, for some constant* $K > 0$ *for both* $\nu \in \{g,h\}$*. Let* $g, h \in \operatorname{Metr} M$ *such that* $g \sim h$ *and assume that there exists* $C < \infty$ *such that* $\left|\delta^\nabla_{g,h}\right| \leqslant C$ *and that for some (then both by quasi-isometry)* $\nu \in \{g,h\}$

$$\int \max\left\{\delta_{g,h}(x), \delta^\nabla_{g,h}(x) + \Theta_g(x)\right\} \Phi_\nu(x,s)\operatorname{vol}_\nu(\mathrm{d}x) < \infty, \qquad \textit{some } s > 0.$$

Then the wave operators $W_\pm(\Delta_h, \Delta_g, I)$ *exist and are complete. Moreover,* $W_\pm(\Delta_h, \Delta_g, I)$ *are partial isometries with initial space* $\operatorname{ran}\mathsf{P}_{\mathrm{ac}}(\Delta_g)$ *and final space* $\operatorname{ran}\mathsf{P}_{\mathrm{ac}}(\Delta_h)$*, and we have* $\sigma_{\mathrm{ac}}(\Delta_g) = \sigma_{\mathrm{ac}}(\Delta_h)$.

Lemma 3.45. *Let $g, h \in \mathrm{Metr} M$, $g \sim h$ with $Q_v \geqslant -K$, for some constant $K > 0$ for both $v \in \{g, h\}$ and assume that the function $\delta^{\nabla}_{g,h}$ is bounded. Then*

$$I_{g,h} \operatorname{dom} \mathbf{q}_g = \operatorname{dom} \mathbf{q}_h.$$

Proof. Note that, for any $v \in \{g, h\}$, $\operatorname{dom} \mathbf{q}_v$ is the closure of compactly supported forms $\Omega_{C_c^\infty}(M, v)$ with respect to the Dirac graph norm

$$\alpha \mapsto \left(\|\alpha\|_v^2 + \left\|\mathsf{D}_v \alpha\right\|_v^2 \right)^{1/2}.$$

Let D be a positive constant whose value might change from line to line. For all compactly supported $\alpha \in \Omega_{C_c^\infty}(M, g)$, we get by the Weitzenböck formula $\Delta_v = \Box_v - \mathscr{R}_v$,

$$\begin{aligned} \left\|\mathsf{D}_v I\alpha\right\|_v^2 &= \left\langle \mathsf{D}_v^2 I\alpha, \alpha \right\rangle_v = \left\langle \left((\nabla^h)^* \nabla^h - \mathscr{R}_v \right) I\alpha, \alpha \right\rangle_v \\ &= \int \left|\nabla^h (I\alpha)\right|_h^2 \mathrm{d\,vol}_v - \int \left(\mathscr{R}_v I\alpha, \alpha \right)_v \mathrm{d\,vol}_v . \end{aligned}$$

By assumption the metrics are quasi-isometric and their Weitzenböck curvature term is bounded from below, so the second term is bounded by $D \|\alpha\|_v^2$.

Following the lines of the proof of Lemma 3.34, we find

$$\left\|\nabla^h (I\alpha)\right\|_h^2 \leqslant D \left(\|\alpha\|_g^2 + \|\nabla^g \alpha\|_g^2 \right).$$

Using the Weitzenböck formula once more, we get

$$\begin{aligned} \|\nabla^g \alpha\|_g^2 = \langle \nabla^g \alpha, \nabla^g \alpha \rangle_g &= \left\langle \left((\nabla^g)^* \nabla^g - \mathscr{R}_g \right) \alpha, \alpha \right\rangle_g + \left\langle \mathscr{R}_g \alpha, \alpha \right\rangle_g \\ &\leqslant \left\langle \mathsf{D}_g^2 \alpha, \alpha \right\rangle_g + D(K, k, m) \langle \alpha, \alpha \rangle_g \\ &\leqslant D(K, k, m) \left(\left\|\mathsf{D}_g \alpha\right\|_g^2 + \|\alpha\|_g^2 \right). \end{aligned}$$

Hence, we arrive at the estimate

$$\|I\alpha\|_h^2 + \left\|\mathsf{D}_h I\alpha\right\|_h^2 \leqslant D(K, k, m) \left(\|\alpha\|_g^2 + \left\|\mathsf{D}_g \alpha\right\|_g^2 \right),$$

proving that

$$I \operatorname{dom} \mathbf{q}_g \subset \operatorname{dom} \mathbf{q}_h.$$

Since $I^{-1} = I_{g,h}^{-1} = I_{h,g}$ and the arguments above are symmetric in g and h, this shows the claim. ■

Proof of Theorem 3.44. We omit the metric in the notation. By assumption $\mathscr{R}$ is bounded from below, so the tensor $\widetilde{\mathscr{R}}$, Ric and R are also bounded from below. In particular (M, g) is stochastically complete, i.e. $\zeta(x) = \infty$ $\mathbb{P}$-a.s. By Gronwall's inequality, we have $\left|\mathscr{Q}_s\right|_{\mathrm{op}}, \left|\mathscr{Q}_s^{-1}\right|_{\mathrm{op}}, \left|\widetilde{\mathscr{Q}_s}\right|_{\mathrm{op}} \leqslant \mathrm{e}^{K^- s/2}$. Following the lines of the proof of Theorem 2.28, we get by Cauchy-Schwarz

$$\begin{aligned} \left| \left(\nabla P_s \alpha(x), \xi \right) \right| &= \left| \mathbb{E} \left(\mathscr{Q}^{\mathrm{tr}}{}_s^{\mathrm{tr}} /\!\!/_s^{-1} \alpha(X_s(x)), U^{\ell}_{s \wedge \tau} \right) \right| \\ &\leqslant C(K^-, s) \left[\mathbb{E} \left| \alpha(X_s(x)) \right|^2 \right]^{1/2} \left(\left[\mathbb{E} \left(\ell^{(1)}_{s \wedge \tau} \right)^2 \right]^{1/2} + \left[\mathbb{E} \left(\ell^{(2)}_{s \wedge \tau} \right)^2 \right]^{1/2} \right). \end{aligned}$$

By (2.39) and (2.38) the first summand in the bracket is bounded by $C(K^-, s)\,|\xi|$. By (2.43) and (2.44) the second summand is bounded by $C(K^-, s)\max_{y\in B(x,1)}|\nabla R(y)|\,|\xi|$. Hence,

$$\left|\left(\nabla P_s\alpha(x), \xi\right)\right|^2 \leqslant C(K^-, s)\,|\xi|^2\,\Phi(x,s)\left(1 + \max_{y\in B(x,1)}|\nabla R(y)|\right)^2 \|\alpha\|^2_{L^2(M)},$$

and analogously

$$\left|(\mathbf{d}P_s\alpha)_x\right|^2 \leqslant C(K^-, s)\Phi(x,s)\,\|\alpha\|^2_{\Omega_{L^2}(M)},$$
$$\left|(\boldsymbol{\delta}P_s\alpha)_x\right|^2 \leqslant C(K^-, s)\Phi(x,s)\,\|\alpha\|^2_{\Omega_{L^2}(M)}.$$

Following the lines of the proof of Theorem 3.32 we see that the assumptions of Belopol'skii-Birman theorem 3.5 are satisfied except making use of Lemma 3.45 (instead of Lemma 3.34) for assumption (2). ■

3.6.5 ε-close Riemannian metrics In this section, we denote by $\boldsymbol{\kappa}_g$ the sectional curvature with respect to a smooth, complete Riemannian metric g.

In [CFG92, Theorem 1.3 & 1.7], Cheeger, Fukaya and Gromov show what is also known as Cheeger-Gromov's thick/thin decomposition:

Theorem 3.46. *For all $\varepsilon > 0$ and $n \in \mathbb{Z}_+$, there exists $\xi > 0$ and $k \in \mathbb{Z}_+$ such that if (M, g) is a complete Riemannian manifold with $\left|\boldsymbol{\kappa}_g\right| \leqslant 1$, then there is a (ξ, k)-round metric, g_ε, on M, such that*

(i) the Riemannian metric g_ε is ε-quasi-isometric to g, i.e. $C^{-\varepsilon} g_\varepsilon \leqslant g \leqslant C^{\varepsilon} g_\varepsilon$

(ii) it has bounded covariant derivatives $|\nabla^{g_\varepsilon} - \nabla^{g}| < \varepsilon$

(iii) $|\left(\nabla^{g_\varepsilon}\right)^k R_{g_\varepsilon}| < C(m, k, \varepsilon)$, where the constant C depends in addition on the order of derivative k and ε.

Assuming that the sectional curvature $\boldsymbol{\kappa}_g$ is bounded by 1, implies that the Riemannian curvature tensor R_g is bounded, and hence, the curvature operator Q_g. Following our results in § 3.6.4, we may get

Theorem 3.47. *Let $\left|\boldsymbol{\kappa}_g\right| \leqslant 1$. Then there exists a Riemannian metric g_ε as in Theorem 3.46 that is ε-quasi-isometric metric to g. If for some $\nu \in \left\{g, g_\varepsilon\right\}$*

$$\int \delta_{g,g_\varepsilon}(x)\Phi_\nu(x,s)\,\mathrm{vol}_\nu(\mathrm{d}x) < \infty, \qquad \textit{some } s > 0,$$

then the wave operators $W_\pm(\Delta_{g_\varepsilon}, \Delta_g, I)$ exist and are complete. Moreover, $W_\pm(\Delta_{g_\varepsilon}, \Delta_g, I)$ are partial isometries with initial space $\operatorname{ran} P_{ac}(\Delta_g)$ and final space $\operatorname{ran} P_{ac}(\Delta_{g_\varepsilon})$, and we have $\sigma_{ac}(\Delta_g) = \sigma_{ac}(\Delta_{g_\varepsilon})$.

Proof. By the previous Theorem 3.46 (i), the assumption $\left|\boldsymbol{\kappa}_g\right| \leqslant 1$ assures that for any $\varepsilon > 0$ there exists a Riemannian metric g_ε that is ε-quasi-isometric metric to g. Hence,

$$\sup \delta_{g,g_\varepsilon}(x) < \infty \quad \Longleftrightarrow \quad g \sim g_\varepsilon.$$

By Theorem 3.46 (ii), the covariant derivatives are bounded so that

$$\delta^{\nabla}_{g,g_\varepsilon} = |\nabla^{g_\varepsilon} - \nabla^{g}|^2_g < \varepsilon. \qquad \blacksquare$$

II

Estimates for the Covariant Derivative of the Heat Semigroup and Covariant Riesz Transforms

Chapter 4

Covariant Derivative Estimates and Riesz Transforms

The Riesz transform $\nabla(\Delta^{(0)} + \lambda)^{-1/2}$ on a Riemannian manifold, considered by Strichartz [Str83], has been intensively studied (e.g. [Bak85a; Bak85b; Bak87]). In particular, the question arises to what extend the L^p-boundedness of the Riesz transform that holds in $\mathbb{R}^n$ can be generalised to (complete) non-compact Riemannian manifolds. If the Riesz transform is bounded in L^2 by the interpolation theorem the weak $(1,1)$-property implies L^p-boundedness for $1 < p \leqslant 2$. Notably, using this idea Coulhon & Duong [CD99] showed under the doubling volume property and an optimal on-diagonal heat kernel estimate the L^p-boundedness for $1 \leqslant p \leqslant 2$. Using probabilistic methods, to wit the Bismut derivative formulae on vector bundles (cf. § 2.1), Thalmaier & Wang [TW04] prove derivative estimates for various heat semigroups on Riemannian vector bundles. As an application, the weak (1, 1) property for a class of Riesz transforms on a vector bundle is established. In [Aus+04], the results are extended to manifolds whose heat kernel satisfies Gaussian estimates from above and below: It is shown that the Riesz transform is L^p-bounded on such a manifold, for p ranging in an open interval above 2, if and only if the gradient of the heat kernel satisfies a certain L^p-estimate in the same interval of p's. A direct application to geometric analysis is given by the L^p-Calderón-Zygmund inequalities, cf. e.g. [GP15; Pig20].

However, the study of Riesz transform normally involves assuming a volume doubling property of M. For the following results we only assume that the curvature and its derivative are bounded by some constant $A < \infty$, cf. Assumption 4.3 below.

Note that, in this chapter, in alignment with the literature and for convenience we change the sign of the Laplacian to obey the *analytic sign convention*.

This chapter is based on joint work with Batu Güneysu & Baptiste Devyver. My main contribution to this chapter is reflected in Theorem 4.6. The analytic insights are due to my collaborators.

4.1 Setting and Notation

In this chapter, let M be a smooth connected (geodesically) complete Riemannian manifold of dimension $\dim M =: m$ without boundary. Recall that by $\mathrm{d}(x, y)$ we denote the geodesic distance and the induced open balls with $\mathrm{B}(x, r)$.

Given a smooth vector bundle $E \to M$ carrying a canonically given metric and a canonically given covariant derivative, we denote its fibrewise metric by $(\cdot, \cdot)$, with $|\cdot| = \sqrt{(\cdot, \cdot)}$ the fibrewise norm and its covariant derivative with

$$\nabla : \Gamma_{C^\infty}(E) \longrightarrow \Gamma_{C^\infty}(T^*M \otimes E).$$

We equip M with the Riemannian volume measure vol and sometimes use the **local volume doubling** property for the measure vol, i.e. there is a $C > 0$ such that

$$\frac{\mathrm{vol}(\mathsf{B}(z,R))}{\mathrm{vol}(\mathsf{B}(z,r))} \leqslant C\mathrm{e}^{CR}\left(\frac{R}{r}\right)^m \qquad \forall 0 < r \leqslant R \ \forall z \in M. \tag{LVD}$$

By the Bishop-Gromov comparison theorem and the well-known formula for the volume of balls in the hyperbolic space (cf. [Bis63], [Pet16, Section 7.1.2]), property (LVD) holds if $\mathrm{Ric} \geqslant -A$ for some $A \geqslant 0$ with some $C = C(A, m)$. A well-known consequence [Stu92] of (LVD) is the following volume comparison inequality: There is a constant $C > 0$ such that we have

$$\frac{\mathrm{vol}(\mathsf{B}(x_2,\sqrt{t}))}{\mathrm{vol}(\mathsf{B}(x_1,\sqrt{t}))} \leqslant C\mathrm{e}^{\frac{Ct}{\varepsilon}}\mathrm{e}^{\varepsilon\frac{\mathsf{d}(x_1,x_2)^2}{t}} \qquad \forall t,\varepsilon > 0 \ \forall x_1, x_2 \in M. \tag{VC_ε}$$

Indeed: Setting $r = \mathsf{d}(x_1, x_2)$, it follows that

$$\begin{aligned}\frac{\mathrm{vol}(\mathsf{B}(x_2,\sqrt{t}))}{\mathrm{vol}(\mathsf{B}(x_1,\sqrt{t}))} &\leqslant \frac{\mathrm{vol}(\mathsf{B}(x_1,r+\sqrt{t}))}{\mathrm{vol}(\mathsf{B}(x_1,\sqrt{t}))}\\ &\overset{\text{(LVD)}}{\leqslant} C\left(\frac{r}{\sqrt{t}}+1\right)^m \mathrm{e}^{C(r+\sqrt{t})}.\end{aligned}$$

Using the elementary inequalities

$$\mathrm{e}^{C}r \leqslant \mathrm{e}^{\frac{Ct}{8\varepsilon}}\mathrm{e}^{\frac{2\varepsilon r^2}{t}}, \qquad \mathrm{e}^{C\sqrt{t}} \leqslant C'\mathrm{e}^{Ct},$$

we get (VC_ε) (with a possibly different value for the constant C).

Given a smooth metric vector bundle $E \to M$ we get the Banach spaces $\Gamma_{\mathsf{L}^p}(E)$ given by equivalence classes of Borel sections ψ in $E \to M$ such that

$$\|\psi\|_p := \|\psi\|_{\mathsf{L}^p} := \||\psi|\|_{\mathsf{L}^p} < \infty,$$

where $\||\psi|\|_p$ denotes the norm of the function $|\psi|$ with respect to $\mathsf{L}^p(M)$. Then $\Gamma_{\mathsf{L}^2}(E)$ canonically becomes a Hilbert space with scalar product

$$\langle\psi_1,\psi_2\rangle_2 := \langle\psi_1,\psi_2\rangle_{\mathsf{L}^2} = \int (\psi_1,\psi_2)\,\mathrm{d}\,\mathrm{vol}\,.$$

Given another smooth metric bundle $F \to M$, the operator norm of a linear map

$$A : \Gamma_{\mathsf{L}^p}(E) \longrightarrow \Gamma_{\mathsf{L}^q}(F)$$

will be denoted by

$$\|A\|_{p,q} = \sup\left\{\|A\alpha\|_q : \|\alpha\|_q \leqslant 1\right\} \in [0,\infty].$$

In this chapter, we switch to the *analytic sign convention* for convenience, so that the Laplace operators acting on 0-, resp., k-forms are given by

$$\Delta^{(0)} := \boldsymbol{\delta}^{(1)}\mathbf{d}^{(0)} : \mathsf{C}^\infty(M) \longrightarrow \mathsf{C}^\infty(M),$$
$$\Delta_g^{(k)} := \boldsymbol{\delta}_g^{(k+1)}\mathbf{d}^{(k)} + \mathbf{d}^{(k-1)}\boldsymbol{\delta}_g^{(k)} : \Gamma_{\mathsf{C}^\infty}(\textstyle\bigwedge^k \mathsf{T}^*M) \longrightarrow \Gamma_{\mathsf{C}^\infty}(\textstyle\bigwedge^k \mathsf{T}^*M).$$

In particular, we have

$$\mathbf{d}^{(k-1)}\Delta^{(k-1)} = \Delta^{(k)}\mathbf{d}^{(k-1)} \qquad \text{and} \qquad \boldsymbol{\delta}^{(k-1)}\Delta^{(k-1)} = \Delta^{(k)}\boldsymbol{\delta}^{(k-1)}, \tag{4.1}$$

and, by the Weitzenböck formula 1.42,

$$\Delta^{(k)} = \nabla^*\nabla + \mathscr{R}^{(k)},$$

where $\mathscr{R} \in \Gamma(\mathrm{End} \bigwedge^k \mathrm{T}^* M)$ is fibrewise self-adjoint, of zeroth order and

$$\left|\mathscr{R}^{(k)}\right| \leqslant C(m)\,|\mathrm{R}|$$

for some $C(m) > 0$ only depending on m by the explicit representation (1.12). Recall that for $k = 1$, $\mathscr{R}^{(1),\mathrm{tr}} = \mathscr{R}|_{\Omega^1(M)} = \mathrm{Ric}$ is the Ricci curvature, and its transpose defined by duality respectively, are read as sections

$$\mathrm{Ric} \in \Gamma_{\mathrm{C}^\infty}(\mathrm{End}(\mathrm{T}M)) \qquad \text{and} \qquad \mathrm{Ric}^{\mathrm{tr}} \in \Gamma_{\mathrm{C}^\infty}(\mathrm{End}(\mathrm{T}^* M)).$$

The Riemannian curvature tensor R can be read as $(0,4)$-tensor, i.e. $\mathrm{R} \in \Gamma_{\mathrm{C}^\infty}(\mathrm{T}^{(0,4)}M)$. We then denote by $\|\mathrm{R}\|_\infty$ its $\|\cdot\|_\infty$ norm. Similarly, $\nabla\mathrm{R}$ can be read as a $(0,4+1)$-tensor and we can consider $\|\nabla\mathrm{R}\|_\infty$ in the same fashion.

As M is geodesically complete, $\Delta^{(0)}$ is essentially self-adjoint in $\mathrm{L}^2(M)$ when initially defined on $\mathrm{C}_c^\infty(M)$. Likewise, $\Delta^{(k)}$ is essentially self-adjoint in $\Gamma_{\mathrm{L}^2}(\bigwedge^k \mathrm{T}^* M)$. By a usual abuse of notation, the corresponding self-adjoint realisations will be denoted by the same symbol, i.e. $\Delta^{(0)} \geqslant 0$, and $\Delta^{(k)} \geqslant 0$ respectively. For all square-integrable k-forms $\alpha \in \Gamma_{\mathrm{L}^2}(\bigwedge^k \mathrm{T}^* M)$, the time-dependent k-form

$$(0,\infty) \times M \ni (t,x) \longmapsto \mathrm{e}^{-t\Delta^{(k)}}\alpha \in \bigwedge^k \mathrm{T}^*_x M$$

has a smooth representative, which extends smoothly to $[0,\infty) \times M$, if α is smooth. Let us again denote by $\mathrm{e}^{-t\Delta^{(k)}}(x,y)$ its corresponding jointly smooth integral kernel, i.e. the heat kernel of $\mathrm{e}^{-t\Delta^{(k)}}$.

By the classical Li-Yau heat kernel estimate [LY86], assuming $\mathrm{Ric} \geqslant -A$ for some constant $A \geqslant 0$, implies the existence of constants $C_j = C_j(A,m) > 0$ $(j = 1,2)$ and $D = D(A,m) > 0$ (where $C_2 = 0$, if $A = 0$), such that we have

$$\mathrm{e}^{-t\Delta^{(0)}}(x,y) \leqslant C_1 \,\mathrm{vol}(\mathrm{B}(x,\sqrt{t}))^{-1} \mathrm{e}^{C_2 t} \mathrm{e}^{-D\frac{\mathrm{d}(x,y)^2}{t}} \qquad \forall t > 0\ \forall x,y \in M.$$

In particular, assuming $\|\mathrm{R}\|_\infty \leqslant A$ for some $A > 0$ and using semigroup domination, we get for every $0 \leqslant k \leqslant m$,

$$\left|\mathrm{e}^{-t\Delta^{(k)}}(x,y)\right| \leqslant C \,\mathrm{vol}(\mathrm{B}(x,\sqrt{t}))^{-1} \mathrm{e}^{Ct} \mathrm{e}^{-D\frac{\mathrm{d}(x,y)^2}{t}}, \tag{UE}$$

where $C, D > 0$ only depend on A and m. Using commutation rules (4.1), there are well-known pointwise heat kernel estimates for $\mathbf{d}^{(k)}\mathrm{e}^{-t\Delta^{(k)}}$ and $\boldsymbol{\delta}^{(k)}\mathrm{e}^{-t\Delta^{(k)}}$ (cf. [Bak87], [BDG21, Appendix A]):

Lemma 4.1. *Assume that $\|\mathrm{R}\|_\infty \leqslant A$ for some a constant $A > 0$. Then there are constants $C = C(A,m) > 0$, $D = D(A,m) > 0$, such that for all $1 \leqslant k \leqslant m$, we have*

$$\left|\mathbf{d}^{(k)}\mathrm{e}^{-t\Delta^{(k)}}(x,y)\right| \leqslant C \,\mathrm{vol}(\mathrm{B}(x,\sqrt{t}))^{-1} t^{-1/2} \mathrm{e}^{Ct} \mathrm{e}^{-D\frac{\mathrm{d}(x,y)^2}{t}} \qquad \forall t > 0\ \forall x,y \in M, \tag{$\mathbf{d}$ UE}$$

and

$$\left|\boldsymbol{\delta}^{(k-1)}\mathrm{e}^{-t\Delta^{(k)}}(x,y)\right| \leqslant C\,\mathrm{vol}(\mathrm{B}(x,\sqrt{t}))^{-1}t^{-1/2}\mathrm{e}^{Ct}\mathrm{e}^{-D\frac{\mathrm{d}(x,y)^2}{t}} \qquad \forall t>0\;\forall x,y\in M. \qquad (\boldsymbol{\delta}\,\mathrm{UE})$$

The differential $\mathbf{d}^{(k)}$ is understood to act on the first variable of the heat kernel

$$\mathbf{d}^{(k)}\mathrm{e}^{-t\Delta^{(k)}}(x,y) := \mathbf{d}^{(k)}\mathrm{e}^{-t\Delta^{(k)}}(\bullet,y)(x),$$

likewise for $\boldsymbol{\delta}^{(k-1)}$.

Remark 4.2. Alternatively, Lemma 4.1 can be proved similarly to the method we will describe in § 4.3 but using the Bismut formulae (2.30) and (2.31).

Our main goal in this chapter is to establish analogous estimates for the covariant derivative of the heat kernel of the Hodge Laplacian, i.e. to show pointwise estimates of the form

$$\left|\nabla\mathrm{e}^{-t\Delta^{(k)}}(x,y)\right| \leqslant C\,\mathrm{vol}(\mathrm{B}(x,\sqrt{t}))^{-1}t^{-1/2}\mathrm{e}^{Ct}\mathrm{e}^{-D\frac{\mathrm{d}(x,y)^2}{t}}. \qquad (\nabla\mathrm{UE})$$

We note that for $k=0$, we have $\nabla=\mathbf{d}^{(0)}$, so that ($\mathbf{d}$ UE) and (∇UE) are equivalent. If M is oriented, the same holds true for $k=m$ by Hodge duality.

As we have already seen in § 2, the covariant Bismut derivative formula Theorem 2.21 not only involves the Riemannian curvature tensor, but also its derivative. Thus, for $1\leqslant k\leqslant m$, the corresponding *covariant* derivative estimates need stronger assumptions, to wit: a uniform bound on the Riemannian curvature tensor and its covariant derivative.

4.2 Main Results

We first state the main results of this chapter.

Assumption 4.3. We assume that the curvature and its derivative are bounded by some constant $A<\infty$, i.e.

$$\max\left(\|\mathrm{R}\|_\infty, \|\nabla\,\mathrm{R}\|_\infty\right) \leqslant A. \qquad (\mathrm{A})$$

Theorem 4.4. *Assume that* (A) *holds. Then there are constants* $C=C(A,m)>0$ *and* $D=D(A,m)>0$*, such that for all* $1\leqslant k\leqslant m$*, we have*

$$\left|\nabla_x\mathrm{e}^{-t\Delta^{(k)}}(x,y)\right| \leqslant \frac{C}{\mathrm{vol}(\mathrm{B}(x,\sqrt{t}))}t^{-1/2}\mathrm{e}^{Ct}\mathrm{e}^{-D\frac{\mathrm{d}(x,y)^2}{t}} \qquad \forall t>0\;\forall x,y\in M.$$

Corollary 4.5. *Assume that* (A) *holds. There is a constant* $\gamma=\gamma(A,m)>0$*, and for all* $1\leqslant p<\infty$ *a constant* $C=C(A,m,p)>0$*, such that, for all* $1\leqslant k\leqslant m$*, we have*

$$\int\left|\nabla_x\mathrm{e}^{-t\Delta^{(k)}}(x,y)\right|^p\mathrm{e}^{\frac{\gamma\mathrm{d}(x,y)^2}{t}}\,\mathrm{vol}(\mathrm{d}x) \leqslant \frac{C\mathrm{e}^{Ct}}{t^{p/2}\,\mathrm{vol}(\mathrm{B}(y,\sqrt{t}))^{p-1}} \qquad \forall t>0. \qquad (4.2)$$

Theorem 4.6. *Assume that* (A) *holds. For all* $1 < p < \infty$ *there is a* $C = C(A, m, p) > 0$*, such that for all* $1 \leqslant k \leqslant m$*, we have*

$$\left\| \nabla e^{-t\Delta^{(k)}} |_{\Gamma_{L^2 \cap L^p}(\bigwedge^k T^* M)} \right\|_{p,p} \leqslant C e^{tC} t^{-1/2} \qquad \forall t > 0.$$

The Proof of Theorem 4.4 and its Corollary 4.5 is given in § 4.3 below. The Proof of Theorem 4.4 is based on the probabilistic representation of $\nabla_x e^{-t/2\Delta^{(k)}}(x, y)$ by using the results developed in § 2, but now applied to a Brownian bridge from x to y in time t. The localisation techniques developed in § 2 are therefore hindering as they involve the first exit time of Brownian motion from an open ball $B(x, r)$ around its starting point x – as the terminal point y does not need to be in $B(x, r)$. We therefore make use of the global Bismut formula developed § 2.2.2 and the time reversal property of the Brownian bridge.

In the § 4.4, we prove the L^p-estimate given in Theorem 4.6 by means of the local covariant Bismut formula, Theorem 2.19, and a proper choice of the Cameron-Martin space valued process ℓ.

As well-known from the work of Coulhon & Duong [CD99], the weak $(1, 1)$ property implies L^p-boundedness for $1 < p \leqslant 2$. For the proof the authors assume a volume doubling assumption and establish the spatial derivative of the heat kernel to obtain results about the corresponding Riesz transform. Adapting the Proof of [CD99, Theorem 1.2] by applying the integrated heat kernel estimate (4.2) with $p = 2$, we obtain the following result for the *covariant* Riesz transform for $1 < p \leqslant 2$:

Corollary 4.7. *Assume that* (A) *holds. Then for all* $1 \leqslant k \leqslant m$*,* $\lambda > 0$*, the operator* $\nabla(\Delta^{(k)} + \lambda)^{-1/2}$ *is weak* $(1, 1)$ *type with a bound only depending on* A*,* m *and* λ*. More precisely, there is a constant* $D = D(A, m, \lambda) > 0$ *such that for all* $1 \leqslant k \leqslant m$*,* $\alpha \in \Gamma_{L^2 \cap L^1}(\bigwedge^k T^* M)$*, we have*

$$\operatorname{vol}\left\{ \left| \nabla(\Delta^{(k)} + \lambda)^{-1/2} \alpha \right| > \lambda \right\} \leqslant \frac{D}{\lambda} \|\alpha\|_1 \qquad \forall \lambda > 0.$$

In particular, for all $1 < p \leqslant 2$*, there is a constant* $C = C(A, m, p, \lambda) > 0$*, such that for all* $1 \leqslant k \leqslant m$*, we have*

$$\left\| \nabla(\Delta^{(k)} + \lambda)^{-1/2} \right\|_{p,p} \leqslant C.$$

Corollary 4.7 is proved in § 4.5, where we show that the $(1, 1)$ property implies the L^p-boundedness. This results improves a result by Thalmaier and Wang [TW04, Theorem D], in that, in [TW04, Theorem D] the same conclusion for the covariant Riesz transform is obtained, however an additional assumption on the volume growth of M is made. This volume assumption excludes, in particular, hyperbolic geometries (cf. [Pig20]), while such geometries are covered by Corollary 4.7. In light of the our main result, Theorem 4.4, and the results in [Aus+04] for the *scalar* Riesz transform, it is natural to expect that a uniform bound on R and ∇R implies that the covariant Riesz transform is bounded on L^p for all $1 < p < \infty$: Specifically, we make the following conjecture:

Conjecture 4.8. *Assume that* (A) *holds. Then for every* $1 \leqslant k \leqslant m$ *and* $1 \leqslant p \leqslant \infty$, *we have*

$$\left\| \nabla(\Delta^{(k)} + \lambda)^{-1/2} \right\|_{p,p} < \infty \qquad \forall \lambda > 0,$$

where the bound only depends on A, m, p *and* λ.

We currently do not know whether the assumption on $\nabla \mathrm{R}$ is necessary in Conjecture 4.8. However, it is known that the curvature hypotheses cannot be weakened to merely boundedness from below of the sectionnal curvature: In fact a recent result of Marini and Veronelli [MV20] shows that there are manifolds with positive sectionnal curvature, for which the covariant Riesz transform is not bounded on L^p for all $1 < p < \infty$.

It should also be noted that boundedness in L^p of the Riesz transform $\mathbf{d}^{(k)}(\Delta^{(k)} + \lambda)^{-1/2}$ instead of $\nabla(\Delta^{(k)} + \lambda)^{-1/2}$ is considerably easier, roughly, because we have the commutation rule $\mathbf{d}^{(k)}\Delta^{(k)} = \Delta^{(k+1)}\mathbf{d}^{(k)}$. In fact, a classical result by Bakry [Bak87, Theorem 5.1] states that $\mathbf{d}^{(k)}(\Delta^{(k)} + \lambda)^{-1/2}$ is bounded in L^p, if $\mathscr{R}^{(k)}$ and $\mathscr{R}^{(k+1)}$ are bounded from below by constants. More precisely, the following results holds true.

Theorem 4.9. *Assume that* $\|\mathrm{R}\|_\infty < \infty$. *Then for all* $0 \leqslant k \leqslant m$ *the operators* $\mathbf{d}^{(k)}(\Delta^{(k)} + \lambda)^{-1/2}$ *and* $\boldsymbol{\delta}^{(k)}(\Delta^{(k)} + \lambda)^{-1/2}$ *are weakly* $(1,1)$ *with a* $(1,1)$*-norm bound only depending on* A *and* m. *In particular, for every* $1 < p \leqslant 2$, *we have*

$$\left\| \mathbf{d}^{(k)}(\Delta^{(k)} + \lambda)^{-1/2} \right\|_{p,p} < \infty \qquad \text{and} \qquad \left\| \boldsymbol{\delta}^{(k)}(\Delta^{(k)} + \lambda)^{-1/2} \right\|_{p,p} < \infty,$$

with norm bounds only depending on A, m, p *and* λ.

The weak $(1,1)$ property appears to be new in this generality. The latter is established using the estimates (**d** UE) and (**δ** UE), and Coulhon-Duong theory as in the Proof of Corollary 4.7, yielding an alternative proof of the L^p-boundedness of Theorem 4.9. The details are given in § 4.5.

For applications in geometric analysis, the L^p-boundedness of $\nabla(\Delta^{(k)} + \lambda)^{-1/2}$ is more important than that of $\mathbf{d}^{(k)}(\Delta^{(k)} + \lambda)^{-1/2}$. For example, as shown in [GP15, Proof of Theorem 4.13], the former boundedness for $k = 1$ implies the L^p-Calderón-Zygmund inequality

$$\|\mathrm{Hess}\, u\|_p \leqslant D_{\mathrm{CZ}} \left(\left\|\Delta^{(0)} u\right\|_p + \|u\|_p \right) \qquad \forall u \in \mathrm{C}_c^\infty(M),$$

where D_{CZ} only depends on $\left\| \nabla(\Delta^{(1)} + \lambda)^{-1/2} \right\|_{p,p}$. Roughly, the idea is to use the spectral calculus, for all $\lambda > 0$,

$$\begin{aligned}
\left\|\nabla \mathbf{d}^{(0)} u\right\|_p &= \left\| \nabla \mathbf{d}^{(0)}(\Delta^{(0)} + \lambda)^{-1}(\Delta^{(0)} + \lambda)\, u \right\|_p \\
&= \left\| \nabla(\Delta^{(1)} + \lambda)^{-1/2} \mathbf{d}^{(0)}(\Delta^{(0)} + \lambda)^{-1/2}(\Delta^{(0)} + \lambda) u \right\|_p \\
&\leqslant \left\| \nabla(\Delta^{(1)} + \lambda)^{-1/2}|_{\Gamma_{\mathrm{L}^p \cap \mathrm{L}^2(\mathrm{T}^* M)}} \right\|_p \left\| \mathbf{d}^{(0)}(\Delta^{(0)} + \lambda)^{-1/2}|_{\mathrm{L}^p(M)\cap \mathrm{L}^2(M)} \right\|_p \left(\left\|\Delta^{(0)} u\right\|_p + \lambda \|u\|_p \right),
\end{aligned}$$

where, by the essential self-adjointness of $\Delta^{(0)}$, we used that

$$\left(\Delta^{(1)} + \lambda\right)^{-1/2} \mathbf{d}^{(0)} g = \mathbf{d}^{(0)} \left(\Delta^{(0)} + \lambda\right)^{-1/2} g$$

for all $g \in \mathrm{L}^2(M)$ with $\mathbf{d}^{(0)} g \in \Gamma_{\mathrm{L}^2}(\mathrm{T}^* M)$.

The L^p-Calderón-Zygmund inequality together with $\|\mathsf{R}\|_\infty < \infty$, in turn, implies global second order L^p-estimates (cf. [GP19, Theorem 4 b)]) for distributional solutions $v \in \mathsf{L}^p(M)$ of $\Delta^{(0)} v = f \in \mathsf{L}^p(M)$ of the form

$$\|\operatorname{Hess} v\|_p + \left\|\mathbf{d}^{(0)} v\right\|_p \leqslant C\left(\|f\|_p + \|v\|_p\right),$$

where C only depends on D_{CZ} and an upper bound for $\|\mathsf{R}\|_\infty$.

Hence Corollary 4.7 directly implies, the following

Corollary 4.10. *Assume that (A) holds. Then for all $1 < p \leqslant 2$, there is a constant $D' = D'_{A,m,p} > 0$ such that*

$$\|\operatorname{Hess}(u)\|_p \leqslant D'\left(\left\|\Delta^{(0)} u\right\|_p + \|u\|_p\right) \qquad \forall u \in \mathsf{C}_c^\infty(M), \tag{4.3}$$

and such that for every distributional solution $v \in \mathsf{L}^p(M)$ of $\Delta^{(0)} v = f \in \mathsf{L}^p(M)$ we have

$$\|\operatorname{Hess} v\|_p + \left\|\mathbf{d}^{(0)} v\right\|_p \leqslant D'(\|f\|_p + \|v\|_p). \tag{4.4}$$

In addition, the L^p-Calderón-Zygmund inequality implies precompactness results for isometric immersions, cf. [Bre15, Theorem 1.1] and [GP19]. The Calderón-Zygmund-inequality (4.3) also improves [GP15, Theorem D] as it does *not* involve a volume assumption. A recent overview article by Pigola [Pig20] contains the state-of-the art for the L^p-Calderon-Zygmund inequality for large p: it can be shown that the L^p-Calderon-Zygmund inequality holds true for all $p > \max(2, m/2)$ if only $\|\mathsf{R}\|_\infty < \infty$. In this sense, Corollary 4.10 can be considered a complementary result for small p.

However, Conjecture 4.8 for $p > 2$ is motivated on the scalar result for functions in [Aus+04, Theorem 1.6] – noting that $\nabla = \mathbf{d}^{(0)}$ on functions – which states that if M satisfies the local Poincaré inequality and has an exponential volume doubling (these conditions are satisfied under $\operatorname{Ric} \geqslant -A$ for some constant $A \geqslant 0$), then

$$\left\|\nabla \mathrm{e}^{-t\Delta^{(0)}}|_{\mathsf{L}^2 \cap \mathsf{L}^p(M)}\right\|_{p,p} \leqslant C \mathrm{e}^{tC} t^{-1/2} \qquad \forall t > 0\ \forall p > 2$$

implies (is actually equivalent to)

$$\left\|\nabla(\Delta + \lambda)^{-1/2}|_{\mathsf{L}^2 \cap \mathsf{L}^p(M)}\right\|_{p,p} \leqslant D \qquad \forall p > 2.$$

Trying to extend the results of [Aus+04], we realise that the central tools are the scalar ($k = 0$) variants of Corollary 4.5 and Theorem 4.6, as well as the scalar variant of so called **Davies-Gaffney inequality**, i.e. an L^2 off-diagonal estimates for $\mathrm{e}^{-t\Delta^{(0)}}$ and $\mathbf{d}^{(0)} \mathrm{e}^{-t\Delta^{(0)}}$. We can generalise the result for the covariant derivative of the heat kernel of the Hodge Laplacian which is proved in § 4.6:

Theorem 4.11. *There are universal constants $c_1, c_2 > 0$ such that for all $1 \leqslant k \leqslant m$ with $\mathscr{R}^{(k)} \geqslant -A$ for some constant $A \geqslant 0$, all $t > 0$, all Borel subsets $E, F \subset M$ with compact closure, and all $\alpha \in \Gamma_{\mathsf{L}^2}(\bigwedge^k \mathsf{T}^* M)$ with $\operatorname{supp} \alpha \subset E$, we have*

$$\left\|\mathbb{1}_F \mathrm{e}^{-t\Delta^{(k)}} \alpha\right\|_2 + \left\|\mathbb{1}_F \sqrt{t} \nabla \mathrm{e}^{-t\Delta^{(k)}} \alpha\right\|_2 + \left\|\mathbb{1}_F t \Delta^{(k)} \mathrm{e}^{-t\Delta^{(k)}} \alpha\right\|_2 \leqslant c_1\left(1 + \sqrt{tA}\right) \mathrm{e}^{-\frac{c_2 \varrho(E,F)^2}{t}} \|\mathbb{1}_E \alpha\|_2 .$$

Note that the bound above implies that for small $0 < t < 1$

$$\left\|\mathbb{1}_F e^{-t\Delta^{(k)}}\alpha\right\|_2 + \left\|\mathbb{1}_F \sqrt{t}\nabla e^{-t\Delta^{(k)}}\alpha\right\|_2 + \left\|\mathbb{1}_F t\Delta^{(k)} e^{-t\Delta^{(k)}}\alpha\right\|_2 \leqslant c_{1,A} e^{-\frac{c_2 d(E,F)^2}{t}} \left\|\mathbb{1}_E \alpha\right\|_2,$$

which is needed eventually for the strategy of the proof in [Aus+04].

The only place where the techniques used in [Aus+04] cannot be adjusted directly to differential forms is where the local Poincaré inequality is used explicitly (which does not make sense on differential forms). We currently do not know whether there is a different method which avoids the use of the local Poincaré, proving Conjecture 4.8.

4.3 Proof of Theorem 4.4 and Corollary 4.5

4.3.1 Brownian bridges For details on Brownian bridges, we refer the reader to e.g. [Hsu02, Section 5.4.]. An **M-valued path** x with explosion time $\zeta = \zeta(x) > 0$ may be also interpreted as a continuous map $x : [0, \infty) \to M$ such that $x_t \in M$ for $0 \leqslant t < \zeta$ and $x_t = \infty$ for all $t \geqslant \zeta$ if $\zeta < \infty$. The space of M-valued paths with explosion time is called the **path space of** M and is denoted by $W(M)$. A Brownian motion is then given as the coordinate process $X(\omega)_t = \omega_t$ on $(W(M), \mathscr{B}_t(W(M)), \mu)$, where $\mathscr{B}_t(W(M))$ is the σ-algebra generated by the coordinate maps up to time t and μ is the Wiener measure.

On the **bridge space**

$$\mathsf{L}_t^{x,y}(M) := \left\{\omega \in W(M) : \omega_0 = x,\ \omega_t = y\right\}$$

the law of the Brownian bridge from x to y in time t is a probability measure $\mathbb{P}_t^{x,y}$ on $\mathsf{L}_t^{x,y}(M)$ roughly given by

$$\mathbb{P}_t^{x,y}(B) := \mathbb{P}\big(B \mid X_t = y\big) \qquad \forall B \in \mathscr{B}_t(W(M)).$$

The measure $\mathbb{P}_t^{x,y}$ is called **Wiener measure on** $\mathsf{L}_t^{x,y}(M)$. For any $s < t$ and $F \in \mathscr{B}_s$ nonnegative function on $W(M)$ and f a nonnegative measurable function on M, it can be shown, that

$$\mathbb{E}_t^{x,y} F(X) = \frac{\mathbb{E}^x \left(F(X) p_{t-s}(X_s, y)\right)}{p_t(x,y)}, \qquad 0 \leqslant s < t, \tag{4.5}$$

by using the Markov property. In particular, it follows that the $\mathbb{P}_t^{x,y}$ as a measure on the space $W(M)$, is absolutely continuous with respect to $\mathbb{P}^x$ on $\mathscr{B}_s$ for any $s < t$ and the Radon-Nikodým derivative is given by

$$\left.\frac{d\mathbb{P}_t^{x,y}}{d\mathbb{P}^x}\right|_{\mathscr{B}_s} = \frac{p_{t-s}(X_s, y)}{p_t(x,y)} =: N_s.$$

Then $(N_s)_{0 \leqslant s < t}$ is a continuous local martingale under the probability $\mathbb{P}^x$.

The Brownian bridge admits symmetry under time reversal (**time reversal property**) of the pinned Wiener measure: The pushforward of $\mathbb{P}_t^{x,y}$ with respect to the $\mathscr{B}_s/\mathscr{B}_s$-measurable map $W(M) \to W(M)$ given by $\omega \mapsto \omega(t - \cdot)$ is $\mathbb{P}_t^{y,x}$. In particular,

$$\mathbb{E}_t^{x,y} F(\omega) = \mathbb{E}_t^{y,x} F(\omega(t - \cdot)). \tag{4.6}$$

4.3.2 Proof of Theorem 4.4 Next, we establish the estimate for the covariant derivative of the heat kernel in Theorem 4.4. Therefore, we want to make use of the covariant Bismut formula, Theorem 2.19, and disintegrate using the relation (4.5) and time reversal.

Proof of Theorem 4.4. Let $r > 0$, $\eta_1 \in \mathrm{T}_x M$, $\eta_2 \in \bigwedge^k \mathrm{T}_x^* M$. By the covariant Bismut formula, Theorem 2.19, and the disintegration formula (4.5) we get, for any $|\xi| \leqslant 1$,

$$\begin{aligned}\left|\nabla_x \mathrm{e}^{-\frac{t}{2}\Delta^{(k)}}(x,y)\right| &\leqslant \frac{1}{t}\mathrm{e}^{-\frac{t}{2}\Delta^{(0)}}(x,y)\mathbb{E}_t^{x,y}\left(|\mathcal{Q}_t|\left|\int_0^t \mathcal{Q}_s^{-1}(\mathrm{d}B_s \lrcorner \widetilde{\mathcal{Q}}_s)\right|\right)\\ &\quad + \frac{1}{2t}\mathrm{e}^{-\frac{t}{2}\Delta^{(0)}}(x,y)\frac{1}{2}\mathbb{E}_t^{x,y}\left(|\mathcal{Q}_t|\int_0^t |\mathcal{Q}_s^{-1}||\rho(X_s(x))||\dot{\ell}_s|\mathrm{d}s\right)\\ &\leqslant \frac{1}{t}\mathrm{e}^{-\frac{t}{2}\Delta^{(0)}}(x,y)\mathrm{e}^{C(A,m)t}\mathbb{E}_t^{x,y}\left(\left|\int_0^t \mathcal{Q}_s^{-1}(\mathrm{d}B_s \lrcorner \widetilde{\mathcal{Q}}_s)\right|\right)\\ &\quad + C(A,m)\mathrm{e}^{C(A,m)t}\mathrm{e}^{-\frac{t}{2}\Delta^{(0)}}(x,y).\end{aligned}$$

By the time reversal property (4.6), we get

$$\begin{aligned}&\mathrm{e}^{-\frac{t}{2}\Delta^{(0)}}(x,y)\mathbb{E}_t^{x,y}\left|\int_0^t \mathcal{Q}_s^{-1}(\mathrm{d}B_s \lrcorner \widetilde{\mathcal{Q}}_s)\right|\\ &= \mathbb{E}\left(\mathrm{e}^{-\frac{t}{2}\Delta^{(0)}}(X_{t/2}(x),y)\left|\int_0^{t/2} \mathcal{Q}_s^{-1}(\mathrm{d}B_s \lrcorner \widetilde{\mathcal{Q}}_s)\right|\right)\\ &\quad + \mathbb{E}\left(\mathrm{e}^{-\frac{t}{2}\Delta^{(0)}}(X_{t/2}(y),x)\left|\int_0^{t/2} \mathcal{Q}_s^{-1}(\mathrm{d}B_s \lrcorner \widetilde{\mathcal{Q}}_s)\right|\right).\end{aligned}$$

Next, by the Cauchy-Schwarz inequality,

$$\begin{aligned}&\mathbb{E}\left(\mathrm{e}^{-\frac{t}{2}\Delta^{(0)}}(X_{t/2}(x),y)\left|\int_0^{t/2} \mathcal{Q}_s^{-1}(\mathrm{d}B_s \lrcorner \widetilde{\mathcal{Q}}_s)\right|\right)\\ &\leqslant \left[\mathbb{E}\left(\mathrm{e}^{-\frac{t}{2}\Delta^{(0)}}(X_{t/2}(x),y)\right)^2\right]^{1/2}\left[\mathbb{E}\left|\int_0^{t/2} \mathcal{Q}_s^{-1}(\mathrm{d}B_s \lrcorner \widetilde{\mathcal{Q}}_s)\right|^2\right]^{1/2}\\ &\leqslant C(A,m)t^{-1/2}\mathrm{e}^{C(A,m)t}\left[\int \mathrm{e}^{-\frac{t}{2}\Delta^{(0)}}(x,z)\mathrm{e}^{-\frac{t}{2}\Delta^{(0)}}(z,y)\mathrm{e}^{-\frac{t}{2}\Delta^{(0)}}(z,y)\,\mathrm{vol}(\mathrm{d}z)\right]^{1/2}\\ &\leqslant C(A,m)\,\mathrm{vol}(\mathrm{B}(y,\sqrt{t/2}))^{-1/2}t^{-1/2}\mathrm{e}^{C(A,m)t}\mathrm{e}^{-t\Delta^{(0)}}(x,y)^{1/2}\\ &\leqslant C(A,m)\,\mathrm{vol}(\mathrm{B}(y,\sqrt{t/2}))^{-1}t^{-1/2}\mathrm{e}^{C(A,m)t}\mathrm{e}^{-C(A,m)\frac{\mathrm{d}(x,y)^2}{t}},\end{aligned}$$

where we have used the Li-Yau estimate

$$\mathrm{e}^{-t\Delta^{(0)}}(x_1,x_2) \leqslant \mathrm{vol}(\mathrm{B}(x_1,\sqrt{t}))^{-1}\mathrm{e}^{-C(A,m)\frac{\rho(x_1,x_2)^2}{t}}\mathrm{e}^{C(A,m)t} \qquad \forall t>0\ \forall x_1,x_2 \in M,$$

twice, and

$$\int \mathrm{e}^{-\frac{t}{2}\Delta^{(0)}}(x,z)\mathrm{e}^{-\frac{t}{2}\Delta^{(0)}}(z,y)\,\mathrm{vol}(\mathrm{d}z) = \mathrm{e}^{-t\Delta^{(0)}}(x,y).$$

Likewise, we have

$$\begin{aligned}&\mathbb{E}\left(\mathrm{e}^{-\frac{t}{2}\Delta^{(0)}}(X_{t/2}(x),y)\left|\int_0^{t/2} \mathcal{Q}_s^{-1}(\mathrm{d}B_s \lrcorner \widetilde{\mathcal{Q}}_s)\right|\right)\\ &\leqslant C(A,m)\,\mathrm{vol}(\mathrm{B}(x,\sqrt{t/2}))^{-1}t^{-1/2}\mathrm{e}^{C(A,m)t}\mathrm{e}^{-C(A,m)\frac{\mathrm{d}(x,y)^2}{t}},\end{aligned}$$

so that with local doubling (LVD) we arrive at the desired estimate. ∎

4.3.3 Proof of Corollary 4.5

Proof of Corollary 4.5. By Theorem 4.4 and (VC_ε), given $\gamma > 0$, we get

$$\begin{aligned}
&\int \left|\nabla_x e^{-t\Delta^{(k)}}(x,y)\right|^p e^{\frac{\gamma d(x,y)^2}{t}} \operatorname{vol}(dx) \\
&\quad\leqslant C(A,m,p) e^{C(A,m)t} t^{-p/2} \operatorname{vol}(B(y,\sqrt{t}))^{-p} \int e^{(\gamma - C(A,m,p))\frac{d(x,y)^2}{t}} \operatorname{vol}(dx) \\
&\quad\leqslant C(A,m,p) e^{C(A,m)t} t^{-p/2} \operatorname{vol}(B(y,\sqrt{t}))^{-p} \sum_{j=2}^{\infty} \int_{B(y,j\sqrt{t})\setminus B(y,(j-1)\sqrt{t})} e^{(\gamma - C(A,m,p))\frac{d(x,y)^2}{t}} \operatorname{vol}(dx) \\
&\qquad + C(A,m,p) e^{C(A,m)t} t^{-p/2} \operatorname{vol}(B(y,\sqrt{t}))^{-p+1},
\end{aligned}$$

where we have chosen $\gamma < C(A,m,p)$. Finally, using the local doubling (LVD) and letting $\gamma' := C(A,m,p) - \gamma > 0$, we have

$$\begin{aligned}
&\sum_{j=2}^{\infty} \int_{B(y,j\sqrt{t})\setminus B(y,(j-1)\sqrt{t})} e^{-\gamma' \frac{d(x,y)^2}{t}} \operatorname{vol}(dx) \\
&\quad\leqslant \operatorname{vol}(B(y,\sqrt{t})) \sum_{j=2}^{\infty} \frac{\operatorname{vol}(B(y,j\sqrt{t}))}{\operatorname{vol}(B(y,\sqrt{t}))} e^{-\gamma'(j-1)^2} \\
&\quad\leqslant \operatorname{vol}(B(y,\sqrt{t})) \sum_{j=2}^{\infty} j^m e^{-\gamma'(j-1)^2 + C(A,m)j} < \infty,
\end{aligned}$$

finishing the proof. ■

Remark 4.12. Note that Lemma 4.1 can be proved almost verbatim using the global Bismut formulae analogues (2.30) and (2.31).

4.4 Proof of Theorem 4.6

Now we can give the

Proof of Theorem 4.6. We start by noting that it suffices to prove

$$\left\|\nabla e^{-t\Delta^{(k)}}\alpha\right\|_p \leqslant C(A,m,p) t^{-1/2} e^{C(A,m,p)t} \|\alpha\|_p \qquad \forall \alpha \in \Gamma_{C_c^\infty}\left(\bigwedge^k T^* M\right). \tag{4.7}$$

Indeed, to deduce the general case, we can pick a sequence $(\alpha_n) \subset \Gamma_{C_c^\infty}(\bigwedge^k T^*M)$ such that $\alpha_n \to \alpha$ in $L^p(M)$. Then $\nabla e^{-t\Delta^{(k)}}\alpha_n$ is convergent in $\Gamma_{L^p}(\bigwedge^k T^*M)$ by (4.7), and we have

$$\left\|e^{-t\Delta^{(k)}}(\alpha - \alpha_n)\right\|_p \leqslant e^{At} \left\|e^{-t\Delta^{(0)}} |\alpha - \alpha_n|\right\|_p$$

by the Kato-Simon inequality (2.8)

$$\left|e^{-t\Delta^{(k)}}(\alpha - \alpha_n)\right| \leqslant e^{-At} e^{-t\Delta^{(0)}} |\alpha - \alpha_n|,$$

where $A \geqslant 0$ is some lower bound on $\operatorname{Ric} \geqslant -A$, so that $e^{-t\Delta^{(k)}}\alpha_n \to e^{-t\Delta^{(k)}}\alpha$ in $\Gamma_{L^p}(\bigwedge^k T^*M)$, as $e^{-t\Delta^{(0)}}$ is a contraction in $L^p(M)$. Thus, as ∇ (acting a priori on distributions) induces a closed operator from $\Gamma_{L^p}(\bigwedge^k T^*M)$ to $\Gamma_{L^p}(T^*M \otimes \bigwedge^k T^*M)$, it follows that $\nabla e^{-t\Delta^{(k)}}\alpha_n \to \nabla e^{-t\Delta^{(k)}}\alpha$ in $\Gamma_{L^p}(T^*M \otimes \bigwedge^k T^*M)$, yielding the claimed inequality for the general case. Thus, we assume that $\alpha \in \Gamma_{C_c^\infty}(\bigwedge^k T^*M)$ for the rest of the proof.

In the sequel, $C(A,\dots)$ denotes a constant that only depends on $a,\dots$, and which may differ from line to line. Let $t>0$, $r>0$, $x \in M$, $\xi \in T_xM$ be arbitrary and pick some finite energy process $\ell \in CM(t,\xi,\bigwedge^k T^*M)$ as in Corollary 2.15. It follows from the covariant Bismut formula 2.19 that the right hand side of (2.20) can be rewritten using the Markov property of Brownian motion and using that $\zeta = \infty$ $\mathbb{P}$-a.s., since M is stochastically complete, as

$$\left\langle \nabla e^{-\frac{t}{2}\Delta^{(k)}}\alpha(x), \xi \right\rangle = -\mathbb{E}\left\langle \mathcal{Q}_t /\!/_t^{-1}\alpha(X_t(x)), \ell^{(k)}_{t\wedge\tau} + \ell^{(2)}_{t\wedge\tau} \right\rangle.$$

As in the proof of Theorem 2.19, by Gronwall's inequality, we have

$$\left|\mathcal{Q}_t\right|_{\mathrm{op}} \leqslant \exp\left(-\frac{1}{2}\int_0^t \underline{\mathcal{R}}(X_s(x))\mathrm{d}s\right) \qquad \forall t \geqslant 0,$$

and hence

$$\left|\mathcal{Q}_t\right|_{\mathrm{op}} \leqslant e^{C(m,A)t}, \quad \left|\widetilde{\mathcal{Q}}_t\right|_{\mathrm{op}} \leqslant e^{C(m,A)t} \qquad \mathbb{P}\text{-a.s. on } \{s \leqslant \tau\}. \tag{4.8}$$

As $\mathcal{Q}$ and $\widetilde{\mathcal{Q}}$ are invertible, we also have

$$\left|\mathcal{Q}_t^{-1}\right|_{\mathrm{op}} \leqslant e^{C(m,A)t}, \quad \left|\widetilde{\mathcal{Q}}_t^{-1}\right|_{\mathrm{op}} \leqslant e^{C(m,A)t} \qquad \mathbb{P}\text{-a.s. on } \{s \leqslant \tau\}. \tag{4.9}$$

Then for q with $1/q + 1/p = 1$ we have

$$\begin{aligned}\left[\mathbb{E}|\ell^{(1)}_{t\wedge\tau}|^q\right]^{1/q} &\leqslant C(q)\left[\mathbb{E}\left(\int_0^{t\wedge\tau}\left|\mathcal{Q}_s^{-1}\right|^2\left|\widetilde{\mathcal{Q}}_s\right|^2\left|\dot{\ell}_s\right|^2\mathrm{d}s\right)^{q/2}\right]^{1/q}\\ &\leqslant C(q)e^{C(m)t}t^{-1/2}e^{\frac{tC(A,q,m)}{r}+\frac{tC(q,m)}{r^2}}|\xi|,\end{aligned}$$

where we used the Burkholder-Davis-Gundy inequality A.12 and eqs. (4.8) and (4.9). Moreover,

$$\left[\mathbb{E}|\ell^{(2)}_{t\wedge\tau}|^q\right]^{1/q} \leqslant e^{C(m)t}C(m)At\,|\xi| \leqslant e^{C(m)t}C(m,A)\,|\xi|$$

which follows from eqs. (4.8) and (4.9), $|\ell| \leqslant |\xi|$, $|\rho| \leqslant C(m,A)$. We now estimate as follows

$$\begin{aligned}&\left\langle \nabla e^{-\frac{t}{2}\Delta^{(k)}}\alpha(x), \xi\right\rangle\\ &\quad\leqslant \mathbb{E}\left(|\mathcal{Q}_t||\alpha(X_t(x))||\ell^{(k)}_{t\wedge\tau}|\right) + \mathbb{E}\left(|\mathcal{Q}_t||\alpha(X_t(x))||\ell^{(2)}_{t\wedge\tau}|\right)\\ &\quad\leqslant e^{C(m)At}\mathbb{E}\left(|\alpha(X_t(x))||\ell^{(k)}_{t\wedge\tau}|\right) + e^{C(m,A)t}\mathbb{E}\left(|\alpha(X_t(x))||\ell^{(2)}_{t\wedge\tau}|\right)\\ &\quad\leqslant e^{C(m,A)t}\left[\mathbb{E}\left|\alpha(X_t(x))\right|^p\right]^{1/p}\left[\mathbb{E}|\ell^{(k)}_{t\wedge\tau}|^q\right]^{1/q} + e^{C(m,A)t}\left[\mathbb{E}\left|\alpha(X_t(x))\right|^p\right]^{1/p}\left[\mathbb{E}|\ell^{(2)}_{t\wedge\tau}|^q\right]^{1/q}\\ &\quad\leqslant e^{C(m,A)t}\left[\mathbb{E}\left|\alpha(X_t(x))\right|^p\right]^{1/p}C(q)e^{C(m)t}t^{-1/2}e^{\frac{tC(A,q,m)}{r}+\frac{tC(q,m)}{r^2}}|\xi|\\ &\qquad\quad + e^{C(m,A)t}\left[\mathbb{E}\left|\alpha(X_t(x))\right|^p\right]^{1/p}e^{C(m)t}C(m,A)\,|\xi|\\ &\quad= |\xi|\,e^{C(m,A)t}\left[\mathbb{E}\left|\alpha(X_t(x))\right|^p\right]^{1/p}\left(C(q)e^{C(m)t}t^{-1/2}e^{\frac{tC(A,q,m)}{r}+\frac{tC(q,m)}{r^2}} + e^{C(m)t}C(m,A)\right).\end{aligned}$$

Taking $r \to \infty$, we constructed $C(A, m, p) < \infty$ such that we have

$$\left|\nabla e^{-\frac{t}{2}\Delta^{(k)}} \alpha(x)\right|^p \leqslant C(A, m, p) e^{tC_2(A,m,p)} t^{-p/2} \mathbb{E} \left|\alpha(X_t(x))\right|^p \qquad \forall x \in M \ \forall t > 0,$$

and

$$\begin{aligned}\int \left|\nabla e^{-\frac{t}{2}\Delta^{(k)}} \alpha(x)\right|^p \operatorname{vol}(dx) &\leqslant C(A, m, p) e^{tC(A,m,p)} t^{-p/2} \int \mathbb{E} \left|\alpha(X_t(x))\right|^p \operatorname{vol}(dx) \\ &= C(A, m, p) e^{tC(A,m,p)} t^{-p/2} \int\int e^{-\frac{t}{2}\Delta^{(0)}}(x, y) \left|\alpha\right|^p(y) \operatorname{vol}(dy) \operatorname{vol}(dx) \\ &\leqslant C(A, m, p) e^{tC(A,m,p)} t^{-p/2} \int \left|\alpha\right|^p(y) \operatorname{vol}(dy),\end{aligned}$$

where used Fubini and

$$\int e^{-\frac{t}{2}\Delta^{(0)}}(x, y) \operatorname{vol}(dx) = \int e^{-\frac{t}{2}\Delta^{(0)}}(y, x) \operatorname{vol}(dx) \leqslant 1,$$

and so

$$\left\|\nabla e^{-\frac{t}{2}\Delta^{(k)}} \alpha\right\|_p \leqslant C(A, m, p) e^{tC(A,m,p)} t^{-1/2} \left\|\alpha\right\|_p,$$

completing the proof. ■

4.5 Proof of Corollary 4.7 and Theorem 4.9

This part of the work was carried out by Baptiste Devyver.

In this section, we explain how we can use the heat kernel estimates (UE), (∇UE), (**d** UE) and (**δ** UE) in order to get the estimates for the Riesz transforms $\nabla(\Delta^{(k)} + \lambda)^{-1/2}$ and $(\mathbf{d} + \boldsymbol{\delta})(\Delta^{(k)} + \lambda)^{-1/2}$, i.e. Corollary 4.7 and Theorem 4.9 respectively. The idea is to closely follow the Proof of [CD99, Theorem 1.2] for the localised *scalar* Riesz transform $\mathbf{d}(\Delta+\lambda)^{-1/2}$. This proof is based on the Calderón-Zygmund decomposition and kernel estimates, which we will see to follow from the assumed heat kernel estimates (UE), (∇UE), (**d** UE) and (**δ** UE). However we feel that in the Proof of [CD99, Theorem 1.2] the issue of localisation may been partly overlooked: there, it is wrongly asserted that (LVD) implies that every open ball of radius 1 in M is a doubling space, with a doubling constant that can be chosen independently of the ball. But this property depends on the geometry of balls, and not only on the (LVD) to be valid in the whole of M, and we do not see why it should hold in the context of [CD99, Theorem 1.2].

In order to clarify the matter, we decided to give full proofs for the localisation procedure that we use. Note that we stick to the original notation used in [CD99]. The first ingredient is a localised Calderón-Zygmund decomposition $f = g + b$ for a smooth section $f \in \Gamma(\bigwedge^{(k)} T^*M)$ which has support inside a ball $B = B(x, 1)$. This decomposition holds due to the local doubling assumption (LVD). The version of the Calderón-Zygmund decomposition we need is the following (cf. [Ste93] and [DR20, Appendix B]).

Lemma 4.13 ([BDG21, Lemma 5.1]). *Let $E \to M$ be a Riemannian vector bundle where M is locally doubling. Then there is a constant $C > 0$, which only depends on the local doubling* *constant and has the following property: For every ball* $\mathsf{B} = \mathsf{B}(x, 1)$*, every* $u \in \Gamma_{\mathsf{C}^\infty}(E)$ *supported in* B*, and every* $0 < \lambda < \frac{C}{\mathrm{vol}(B)} \int_B |u|$*, there is a countable collection of balls* $(B_i)_{i \in I}$*, of integrable sections* $(b_i)_{i \in I}$ *in* $\Gamma_{\mathsf{L}^1}(E)$ *and a section* $g \in \Gamma_{\mathsf{L}^\infty}(E)$ *such that:*

(1) $u = g + \sum_{i \in I} b_i$ *a.e.*

(2) the balls $(B_i)_{i \in I}$ *have the finite intersection property: There is* $N \in \mathbb{N}$ *such that for every* $i \in \mathbb{N}$*:*

$$\mathrm{Card}\{j \in \mathbb{N} : B_i \cap B_j \neq \varnothing\} \leqslant N.$$

(3) $\sum_{i \in I} \mathrm{vol}(B_i) \leqslant \frac{C}{\lambda} \int_B |u|$*.*

(4) $|g| \leqslant \lambda$ *a.e.*

(5) For all $i \in I$*,* b_i *has support inside* B_i*, and* $\int_{B_i} |b_i| \leqslant C\lambda\, \mathrm{vol}(B_i)$*.*

Furthermore, as a consequence of (2), (3) and (5), it holds for some constant C that

$$\|g\|_1 \leqslant C\, \|u\|_1 \,.$$

The proof of this version of the Calderón-Zygmund decomposition closely follows the classical one, with three differences: First, since we have only local doubling but not doubling, we have to use a modified maximal function $\mathscr{M}$, defined as follows:

$$\mathscr{M}u(x) := \sup_{B \ni x : r(B) \leqslant 8} \frac{1}{\mathrm{vol}(B)} \int_B |u| \,,$$

where $r(B)$ denotes the radius of the ball B. The particular value 8 in the definition of $\mathscr{M}$ is chosen for technical purposes later on (see the Proof of eq. (4.22)). Note that local doubling implies that $\mathscr{M}$ is weak type $(1, 1)$ and bounded on L^p for all $1 < p \leqslant \infty$, as follows from a careful inspection of the Proof of [Ste93, p. 13, Theorem 1] and the fact that the definition of $\mathscr{M}$ involves only balls with bounded radii. Secondly, in the Calderón-Zygmund decomposition localised in the ball B, the balls B_i do not have to be included inside the ball B, only inside $2B$. Lastly, the fact that we deal with sections of a vector bundle instead of functions: this does not create any real difficulty and the standard arguments apply *mutatis mutandis* if we put norms instead of absolute values everywhere if necessary.

The main steps of the Proof of Corollary 4.7 and Theorem 4.9 closely follow the approach of [CD99, Theorem 1.2]. Let $\mathbf{T}$ be either $\nabla(\Delta^{(k)} + \lambda)^{-1/2}$, or $(\mathbf{d}^{(k)} + \boldsymbol{\delta}^{(k-1)})(\Delta^{(k)} + \lambda)^{-1/2}$.

From now on, we set again $E := \bigwedge^{(k)} \mathsf{T}^* M$. We start with boundedness of $\mathbf{T}$ in L^2:

Lemma 4.14. *For all* $\lambda > 0$ *the operator* $(\mathbf{d}^{(k)} + \boldsymbol{\delta}^{(k-1)}) \left(\Delta^{(k)} + \lambda\right)^{-1/2}$*, originally defined on* $\Gamma_{\mathsf{C}^\infty_c}(\bigwedge^{(k)} \mathsf{T}^* M)$*, extends to a bounded operator on* $\Gamma_{\mathsf{L}^2}(\bigwedge^{(k)} \mathsf{T}^* M)$ *with*

$$\left\| (\mathbf{d}^{(k)} + \boldsymbol{\delta}^{(k-1)}) \left(\Delta^{(k)} + \lambda\right)^{-1/2} \right\|_{2,2} \leqslant 1.$$

If $\|\mathsf{R}\|_\infty \leqslant A < +\infty$*, then for all* $\lambda > 0$ *the operator* $\nabla \left(\Delta^{(k)} + \lambda\right)^{-1/2}$*, originally defined on* $\Gamma_{\mathsf{C}^\infty_c}(\bigwedge^{(k)} \mathsf{T}^* M)$*, extends to a bounded operator on* $\Gamma_{\mathsf{L}^2}(\bigwedge^{(k)} \mathsf{T}^* M)$ *with*

$$\left\| \nabla \left(\Delta^{(k)} + \lambda\right)^{-1/2} \right\|_{2,2} \leqslant C,$$

where the constant C only depends on A, m and λ.

Proof. Since $\mathbf{\Gamma}_{C_c^\infty}(\bigwedge^{(k)} \mathsf{T}^* M)$ is dense in $\mathbf{\Gamma}_{L^2}(\bigwedge^{(k)} \mathsf{T}^* M)$, it is enough to show that for any $f \in \mathbf{\Gamma}_{C_c^\infty}(\bigwedge^{(k)} \mathsf{T}^* M)$,

$$\left\| (\mathbf{d}^{(k)} + \boldsymbol{\delta}^{(k-1)}) \left(\Delta^{(k)} + \lambda\right)^{-1/2} f \right\|_2 \leqslant \|f\|_2 \tag{4.10}$$

and

$$\left\| \nabla \left(\Delta^{(k)} + \lambda\right)^{-1/2} f \right\|_2 \leqslant C \, \|f\|_2 . \tag{4.11}$$

The first estimate is a simple consequence of the functional calculus: since the Dirac operator $\mathbf{D} := \mathbf{d} + \boldsymbol{\delta}$ acting on smooth, compactly supported differential forms, is essentially self-adjoint on M, it follows that for all g in the domain $\mathbf{D}^2 = \Delta$ (which is included in the domain of $\mathbf{D}$), and

$$\begin{aligned} \|(\mathbf{d} + \boldsymbol{\delta}) g\|_2^2 = \langle \mathbf{D} g, \mathbf{D} g \rangle &= \langle \mathbf{D}^2 g, g \rangle \\ &\leqslant \langle (\mathbf{D}^2 + \lambda) g, g \rangle \\ &\leqslant \left\| (\mathbf{D}^2 + \lambda)^{1/2} g \right\|_2^2 . \end{aligned}$$

Applying the above inequality to $g = (\mathbf{D}^2 + \lambda)^{-1/2} f$, which is the domain of $\mathbf{D}^2$ by functional calculus, we obtain (4.10) with $C = 1$.

Next, we prove (4.11). Recall that, since M is complete, the operator $\nabla^* \nabla$ acting on smooth compactly supported differential forms is essentially self-adjoint, associated with the quadratic form $(u, v) \mapsto \langle \nabla u, \nabla v \rangle$. In particular, if $g \in \mathbf{\Gamma}_{L^2}(\bigwedge^k \mathsf{T}^* M)$ is in the domain of $\nabla^* \nabla$, then

$$\langle \nabla g, \nabla g \rangle = \langle \nabla^* \nabla g, g \rangle.$$

Hence, for such a g, using that $\left\| \mathscr{R}^{(k)} \right\|_\infty \leqslant A'$, where $A' = A'(A, m) < \infty$, we have

$$\begin{aligned} \|\nabla g\|_2^2 = \langle \nabla g, \nabla g \rangle &= \langle \nabla^* \nabla g, g \rangle \\ &\leqslant \langle (\nabla^* \nabla + \mathscr{R}^{(k)} + \lambda) g, g \rangle + A' \, \|g\|_2 \\ &\leqslant \left\| \left(\Delta^{(k)} + \lambda\right)^{1/2} g \right\|_2^2 + A' \, \|g\|_2^2 . \end{aligned}$$

We take $g = \left(\Delta^{(k)} + \lambda\right)^{-1/2} f$. Then g is in the domain of $\Delta^{(k)}$: Indeed, if we write

$$f = \left(\Delta^{(k)} + 1\right)^{-1} \left(\Delta^{(k)} + 1\right) f,$$

we have

$$g = \left(\Delta^{(k)} + 1\right)^{-1} \left(\Delta^{(k)} + 1\right)^{-1/2} \left(\left(\Delta^{(k)} + 1\right) f\right).$$

But the operator $\Delta^{(k)} \left(\Delta^{(k)} + 1\right)^{-1/2}$ is bounded in L^2 by the functional calculus, while $\left(\Delta^{(k)} + 1\right) f$ is smooth with compact support and hence in L^2. Thus, $g = \left(\Delta^{(k)} + 1\right)^{-1} h$ with h in L^2, hence g is in the domain of $\Delta^{(k)}$. It follows that

$$\begin{aligned} \left\| \nabla \left(\Delta^{(k)} + \lambda\right)^{-1/2} f \right\|_2^2 &\leqslant \|f\|_2^2 + A' \left\| \left(\Delta^{(k)} + \lambda\right)^{-1/2} f \right\|_2^2 \\ &\leqslant \left(\frac{A'}{\lambda} + 1 \right) \|f\|_2^2 , \end{aligned}$$

where we have used that $\left\|\left(\Delta^{(k)}+\lambda\right)^{-1/2}\right\|_{2,2} \leqslant \lambda^{-1/2}$ by functional calculus. This proves (4.11). ■

Finally, we prove Corollary 4.7 and Theorem 4.9. By Lemma 4.14 and using interpolation, we see that it is sufficient to prove that $\mathbf{T}$ is bounded from $\Gamma_{L^1}(\bigwedge^k T^*M)$ into the space of weakly integrable sections $\Gamma_{L^1_w}(\bigwedge^k T^*M)$, i.e., we find a constant $C > 0$ such that for all $f \in \Gamma_{L^1}(\bigwedge^{(k)} T^*M)$ and all $\lambda > 0$, we have

$$\mathrm{vol}\left(\{x \in M \,:\, |\mathbf{T}f|\,(x) > \lambda\}\right) \leqslant \frac{C}{\lambda}\, \|f\|_1\,. \tag{4.12}$$

By a density argument, it is sufficient to prove it for smooth f with compact support: So, take such an f, and fix $\lambda > 0$. Take $(x_j)_{j\in\mathbb{N}}$ a maximal 1-separated subset, hence the balls $\mathrm{B}(x_j, 1)$ cover M, while the balls $\mathrm{B}(x_j, \frac{1}{2})$ are disjoint. Local doubling then implies that the balls $\mathrm{B}(x_j, 1)$ have the finite intersection property. Let $(\varphi_j)_{j\in\mathbb{N}}$ be a smooth partition of unity associated to the covering of M by the balls $\mathrm{B}(x_j, 1)$, and write $f_j := \varphi_j f$. The fact that the covering has the finite intersection property implies that for some constant $C > 0$,

$$C^{-1}\,\|f\|_1 \leqslant \sum_{j\in\mathbb{N}} \left\|f_j\right\|_1 \leqslant C\,\|f\|_1\,.$$

Hence, it is enough to prove (4.12) for f_j (with a constant independent of j and f). In what follows, we therefore assume that $j \in \mathbb{N}$ is fixed, and we denote by $u = f_j$ and $\mathrm{B} = \mathrm{B}(x_j, 1)$. We have two cases, according to whether

$$\lambda \leqslant \frac{C}{\mathrm{vol}(B)} \int_B |u|$$

or not – with C the constant in Lemma 4.13. We first treat the case where $\lambda \leqslant \frac{C}{\mathrm{vol}(B)} \int_B |u|$, for which there are two steps: first, show that

$$\mathrm{vol}\left(\{x \in 2B \,:\, |\mathbf{T}f|\,(x) > \lambda\}\right) \leqslant \frac{C}{\lambda}\, \|u\|_1\,, \tag{4.13}$$

and then show that

$$\mathrm{vol}\left(\{x \in M \setminus 2B \,:\, |\mathbf{T}f|\,(x) > \lambda\}\right) \leqslant \frac{C}{\lambda}\, \|u\|_1\,, \tag{4.14}$$

For (4.13), notice that $\{x \in 2B \,:\, |\mathbf{T}f|(x) > \lambda\} \subset 2B$, therefore

$$\mathrm{vol}\left(\{x \in 2B \,:\, |\mathbf{T}f|\,(x) > \lambda\}\right) \leqslant \mathrm{vol}(2B) \leqslant C\,\mathrm{vol}(B) \leqslant \frac{C}{\lambda}\, \|u\|_1\,,$$

where we have used successively (LVD) and the assumption on λ. This proves (4.13).

Next, we show (4.14). By the Markov inequality, we see that (4.14) follows from the L^1-estimate

$$\int_{M\setminus 2B} |\mathbf{T}u|(x)\ \mathrm{vol}(\mathrm{d}x) \leqslant C\,\|u\|_1\,. \tag{4.15}$$

On the other hand, (4.15) can be proved as in [CD99, p. 1163], using the heat kernel estimates (∇UE), ($\mathbf{d}$ UE) and ($\boldsymbol{\delta}$ UE) respectively.

Now, we deal with the case $\lambda > \frac{C}{\operatorname{vol}(B)} \int_B |u|$. In this case, we may use the Calderón-Zygmund decomposition $u = g + \sum_{i\in I} b_i$ from Lemma 4.13. Denote by r_i the radius of B_i, and let $t_i = r_i^2$. Then, we write

$$\mathbf{T}u = \mathbf{T}g + \sum_{i\in I} \mathbf{T}\mathbb{1}_{3B} \mathrm{e}^{-t_i \Delta^{(k)}} b_i + \sum_{i\in I} \mathbf{T}(1 - \mathrm{e}^{-t_i \Delta^{(k)}}) b_i + \sum_{i\in I} \mathbf{T}\mathbb{1}_{M\setminus 3B} \mathrm{e}^{-t_i\Delta^{(k)}} b_i .$$

The weak L^1-estimate (4.12) is a consequence of the following four estimates:

$$\operatorname{vol}\left(\left\{ x \in M \,:\, |\mathbf{T}g| > \frac{\lambda}{4} \right\}\right) \leqslant \frac{C}{\lambda} \|f\|_1 , \tag{4.16}$$

$$\operatorname{vol}\left(\left\{ x \in M \,:\, \left|\sum_{i\in I} \mathbf{T}\mathbb{1}_{3B} \mathrm{e}^{-t_i\Delta^{(k)}} b_i\right| > \frac{\lambda}{4} \right\}\right) \leqslant \frac{C}{\lambda} \|f\|_1 , \tag{4.17}$$

$$\operatorname{vol}\left(\left\{ x \in M \,:\, \left|\sum_{i\in I} \mathbf{T}(1 - \mathrm{e}^{-t_i\Delta^{(k)}}) b_i\right| > \frac{\lambda}{4} \right\}\right) \leqslant \frac{C}{\lambda} \|f\|_1 , \tag{4.18}$$

$$\operatorname{vol}\left(\left\{ x \in M \,:\, \left|\sum_{i\in I} \mathbf{T}\mathbb{1}_{M\setminus 3B} \mathrm{e}^{-t_i\Delta^{(k)}} b_i\right| > \frac{\lambda}{4} \right\}\right) \leqslant \frac{C}{\lambda} \|f\|_1 . \tag{4.19}$$

First, the fact that $|g| \leqslant C\lambda$ a.e. and that $\mathbf{T}$ is bounded on L^2 leads to

$$\operatorname{vol}\left(\left\{ x \in M \,:\, |\mathbf{T}g| > \frac{\lambda}{4} \right\}\right) \leqslant \frac{16}{\lambda^2} \|g\|_2^2 \leqslant \frac{16}{\lambda^2} \|g\|_\infty \|g\|_1 \leqslant \frac{C}{\lambda} \|u\|_1 ,$$

which shows (4.16). Concerning (4.17), the same argument using the L^2-boundedness of $\mathbf{T}$ shows that (4.17) follows from the L^2-estimate

$$\left\| \sum_{i\in I} \mathbb{1}_{3B} \mathrm{e}^{-t_i\Delta^{(k)}} b_i \right\|_2^2 \leqslant C\lambda \|u\|_1 . \tag{4.20}$$

Denote by $B_i = \mathrm{B}(y_i, r_i)$. By Lemma 4.13 (5) in the Calderón-Zygmund decomposition together with the heat kernel estimate (UE) and the fact that $t_i \leqslant 2$ (since $B_i \subset 2B$) imply that

$$\begin{aligned}
|\mathrm{e}^{-t_i\Delta^{(k)}} b_i|(x) &\leqslant C\lambda \operatorname{vol}(B_i) \frac{\mathrm{e}^{-\frac{\mathrm{d}^2(x,y_i)}{Ct_i}}}{\operatorname{vol}(\mathrm{B}(x,\sqrt{t_i}))} \\
&\leqslant C\lambda \int_M \frac{\mathrm{e}^{-\frac{\mathrm{d}^2(x,y)}{Ct_i}}}{\operatorname{vol}(\mathrm{B}(x,\sqrt{t_i}))} \mathbb{1}_{B_i}(y) \operatorname{vol}(\mathrm{d}y) \\
&\leqslant C\lambda \int_M \left(1 + \frac{\mathrm{d}(x,y)}{\sqrt{t_i}}\right)^m \frac{\mathrm{e}^{-\frac{\mathrm{d}^2(x,y)}{Ct_i}}}{\operatorname{vol}(\mathrm{B}(y,\sqrt{t_i}))} \mathbb{1}_{B_i}(y) \operatorname{vol}(\mathrm{d}y) \\
&\leqslant C\lambda \int_M \frac{\mathrm{e}^{-\frac{\mathrm{d}^2(x,y)}{Ct_i}}}{\operatorname{vol}(\mathrm{B}(y,\sqrt{t_i}))} \mathbb{1}_{B_i}(y) \operatorname{vol}(\mathrm{d}y),
\end{aligned}$$

where we have used local doubling (LVD) with $t_i \leqslant 2$ in the second last line. In order to prove (4.20), it is then sufficient to prove that

$$\left\| \mathbb{1}_{3B} \sum_{i \in I} \int_M \frac{\mathrm{e}^{-\frac{\mathrm{d}^2(\cdot,y)}{Ct_i}}}{\mathrm{vol}(\mathsf{B}(y,\sqrt{t_i}))} \mathbb{1}_{B_i}(y)\ \mathrm{vol}(\mathrm{d}y) \right\|_2^2 \leqslant \frac{C}{\lambda} \|u\|_1 . \tag{4.21}$$

To estimate the L^2-norm above, we dualise against some $v \in \Gamma_{\mathrm{L}^2}(E)$ supported in $3B$: By Fubini, we have

$$\begin{aligned}
&\int_{M\times M} \sum_{i\in I} \frac{\mathrm{e}^{-\frac{\mathrm{d}^2(x,y)}{Ct_i}}}{\mathrm{vol}(\mathsf{B}(y,\sqrt{t_i}))} \mathbb{1}_{B_i}(y) v(x)\ \mathrm{vol}(\mathrm{d}x)\ \mathrm{vol}(\mathrm{d}y)\\
&\qquad = \sum_{i\in I} \int_{B_i} \frac{1}{\mathrm{vol}(\mathsf{B}(y,\sqrt{t_i}))} \left(\int_{3B} \mathrm{e}^{-\frac{\mathrm{d}^2(x,y)}{Ct_i}} v(x)\ \mathrm{vol}(\mathrm{d}x) \right) \mathrm{vol}(\mathrm{d}y).
\end{aligned}$$

Next, we prove that, for every $i \in I$ and $y \in B_i$, we have

$$\frac{1}{\mathrm{vol}(\mathsf{B}(y,\sqrt{t_i}))} \left(\int_{3B} \mathrm{e}^{-\frac{\mathrm{d}^2(x,y)}{Ct_i}} v(x)\ \mathrm{vol}(\mathrm{d}x) \right) \leqslant C\mathscr{M}u(y). \tag{4.22}$$

Indeed: For $j \in \mathbb{N}$, denote by $A_0 = B_i$ and

$$A_j := \left\{ x \in 3B : 2^j\sqrt{t_i} \leqslant \mathsf{d}(x,y) \leqslant 2^{j+1}\sqrt{t_i} \right\}, \quad j \geqslant 1.$$

Let $N \in \mathbb{N}$ be the smallest integer so that $2^{N+1}\sqrt{t_i} \geqslant 4$. Then,

$$\begin{aligned}
&\frac{1}{\mathrm{vol}(\mathsf{B}(y,\sqrt{t_i}))} \int_{3B} \mathrm{e}^{-\frac{\mathrm{d}^2(x,y)}{Ct_i}} v(x)\ \mathrm{vol}(\mathrm{d}x)\\
&\quad = \sum_{j=0}^{\infty} \frac{1}{\mathrm{vol}(\mathsf{B}(y,\sqrt{t_i}))} \int_{A_j} \mathrm{e}^{-\frac{\mathrm{d}^2(x,y)}{Ct_i}} v(x)\ \mathrm{vol}(\mathrm{d}x)\\
&\quad \leqslant \sum_{j=0}^{N} \frac{\mathrm{vol}(\mathsf{B}(y,2^{j+1}\sqrt{t_i}))}{\mathrm{vol}(\mathsf{B}(y,\sqrt{t_i}))} \mathrm{e}^{-c2^j} \frac{1}{\mathrm{vol}(\mathsf{B}(y,2^{j+1}\sqrt{t_i}))} \int_{\mathsf{B}(y,2^{j+1}\sqrt{t_i})} |v| .
\end{aligned}$$

By definition of N, we have for every $j \leqslant N$, $2^{j+1}\sqrt{t_i} \leqslant 8$, and therefore by local doubling,

$$\frac{\mathrm{vol}(\mathsf{B}(y,2^{j+1}\sqrt{t_i}))}{\mathrm{vol}(\mathsf{B}(y,\sqrt{t_i}))} \leqslant C2^{jm},$$

and it follows by definition of $\mathscr{M}$ that

$$\begin{aligned}
\frac{1}{\mathrm{vol}(\mathsf{B}(y,\sqrt{t_i}))} \left(\int_{3B} \mathrm{e}^{-\frac{\mathrm{d}^2(x,y)}{Ct_i}} v(x)\ \mathrm{vol}(\mathrm{d}x) \right) &\leqslant \sum_{j=0}^{N} 2^{jm}\mathrm{e}^{-c2^j} \mathscr{M}v(y)\\
&\leqslant \sum_{j=0}^{\infty} 2^{jm}\mathrm{e}^{-c2^j} \mathscr{M}v(y)\\
&\leqslant C\mathscr{M}v(y),
\end{aligned}$$

which proves (4.22).

According to the remark made immediately after the definition of $\mathscr{M}$, (LVD) implies that the operator $\mathscr{M}$ is bounded in L^2, so using Hölder, (3) in the Calderón-Zygmund decomposition and (4.22), we get that

$$\left\| \mathbb{1}_{3B} \sum_{i\in I} \int_M \frac{\mathrm{e}^{-\frac{\mathrm{d}^2(\cdot,y)}{Ct_i}}}{\mathrm{vol}(\mathrm{B}(y,\sqrt{t_i}))} \mathbb{1}_{B_i}(y)\,\mathrm{vol}(\mathrm{d}y) \right\|_2^2 \leqslant C\,\|\mathscr{M}v\|_2^2 \sum_{i\in I} \mathrm{vol}(B_i)$$
$$\leqslant \frac{C}{\lambda^2}\,\|v\|_2^2\,\|u\|_1\,.$$

Dividing by $\|v\|_2^2$ and taking the sup over all non-zero v, we obtain

$$\left\| \mathbb{1}_{3B} \sum_{i\in I} \int_M \frac{\mathrm{e}^{-\frac{\mathrm{d}^2(\cdot,y)}{Ct_i}}}{\mathrm{vol}(\mathrm{B}(y,\sqrt{t_i}))} \mathbb{1}_{B_i}(y)\,\mathrm{vol}(\mathrm{d}y) \right\|_2^2 \leqslant \frac{C}{\lambda}\,\|u\|_1\,,$$

which proves (4.21) and hence (4.17).

It thus remains to show (4.18) and (4.19). Both equations rely on the following

Lemma 4.15. *Assume that $\|\mathrm{R}\|_\infty \leqslant A$ for some constant $A < \infty$. Then there is a constant $C = C(A,m) > 0$, such that we have*

$$\int_{\{\mathrm{d}(\cdot,y)\geqslant\sqrt{t}\}} \left|(\mathbf{d}^{(k)} + \boldsymbol{\delta}^{(k-1)}\right| \mathrm{e}^{-s\Delta^{(k)}}(x,y)|\;\mathrm{vol}(\mathrm{d}x) \leqslant Cs^{-1/2}\mathrm{e}^{-\frac{t}{Cs}}\mathrm{e}^{Cs} \qquad \forall t,s>0\ \forall y\in M.$$

Assume that (A) holds. Then there is a constant $C = C(A,m) > 0$, such that we have

$$\int_{\left\{\mathrm{d}(\cdot,y)\geqslant\sqrt{t}\right\}} \left|\nabla\mathrm{e}^{-s\Delta^{(k)}}(x,y)\right|\;\mathrm{vol}(\mathrm{d}x) \leqslant Cs^{-1/2}\mathrm{e}^{-\frac{t}{Cs}}\mathrm{e}^{Cs} \qquad \forall t,s>0\ \forall y\in M.$$

Proof. For the second integral involving $\nabla\mathrm{e}^{-s\Delta^{(k)}}(x,y)$, the estimate is an immediate consequence of Corollary 4.5 choosing $p = 1$. The proof for the first integral follows along the lines, using (**d** UE) and (**δ** UE) instead of (∇UE) for the proof of the weighted estimate analogous to Corollary 4.5. ■

The estimates (4.18) and (4.19) follow from Lemma 4.15, in the fashion as the Proof of [CD99, Theorem 1.2]. Finally, this proves all four estimates (4.16), (4.17), (4.18) and (4.19), and concludes the Proof of Theorem 4.9 and Corollary 4.7.

4.6 Proof of Theorem 4.11

This part of the work was carried out by Batu Güneysu.

We prepare the proof with the following estimate from complex analysis that can be found in e.g. [CS08]:

Lemma 4.16 (Phragmen-Lindelöf's inequality). *Let*

$$f : \{\mathrm{Re} > 0\} \longrightarrow \mathbb{C}$$

be holomorphic, and assume that there are constants $A, B, \gamma > 0$, $b \geqslant 0$, such that

$$|f(z)| \leqslant B \qquad \forall z \in \{\mathrm{Re} > 0\},$$
$$|f(t)| \leqslant A\mathrm{e}^{bt-\frac{\gamma}{t}} \qquad \forall t > 0.$$

Then we have

$$|f(z)| \leqslant B\mathrm{e}^{-\mathrm{Re}\frac{\gamma}{z}} \qquad \forall z \in \{\mathrm{Re} > 0\}.$$

Proof of Theorem 4.11. We split the Proof into three steps.

1° We have

$$\left\|\mathbb{1}_F \mathrm{e}^{-t\Delta^{(k)}}\alpha\right\|_2 \leqslant \mathrm{e}^{-\frac{\mathrm{d}(E,F)^2}{4t}} \left\|\mathbb{1}_E\alpha\right\|_2. \tag{4.23}$$

Indeed: The inequality

$$\left|\left\langle \mathrm{e}^{-t\Delta^{(k)}}\alpha_1, \alpha_2\right\rangle\right|_2 \leqslant \mathrm{e}^{C(A)t}\mathrm{e}^{-\frac{\mathrm{d}(E,F)^2}{4t}} \left\|\alpha_1\right\|_2 \left\|\alpha_2\right\|_2$$

valid for all α_1 with support in E and α_2 with support in F has been proved in [Gün17a, Lemma XII.3.]. If we apply Phragmen-Lindelöf's inequality 4.16 with

$$f(z) = \left\langle \mathrm{e}^{-z\Delta^{(k)}}\alpha_1, \alpha_2\right\rangle, \quad b = |a|, \quad A = B = \left\|\alpha_1\right\|_2 \left\|\alpha_2\right\|_2, \quad \gamma = \mathrm{d}(E,F)^2/4,$$

noting that we may pick $A = \left\|\alpha_1\right\|_2 \left\|\alpha_2\right\|_2$ because of $\Delta^{(k)} \geqslant 0$ so that $\mathrm{e}^{-z\Delta^{(k)}}$ is a contraction, we get the bound

$$\left\|\left\langle \mathrm{e}^{-t\Delta^{(k)}}\alpha_1, \alpha_2\right\rangle\right\|_2 \leqslant \mathrm{e}^{\frac{-\mathrm{d}(E,F)^2}{4t}} \left\|\alpha_1\right\|_2 \left\|\alpha_2\right\|_2. \tag{4.24}$$

The latter inequality is equivalent to the statement of (4.23). □

2° We have

$$\left\|\mathbb{1}_F t\Delta^{(k)}\mathrm{e}^{-t\Delta^{(k)}}\alpha\right\|_2 \leqslant C\mathrm{e}^{-\frac{\mathrm{d}(E,F)^2}{6t}} \left\|\mathbb{1}_E\alpha\right\|_2, \tag{4.25}$$

where $C < \infty$ is a universal constant.

Indeed: The asserted estimate is equivalent to

$$\left|\left\langle t\Delta^{(k)}\mathrm{e}^{-t\Delta^{(k)}}\alpha_1, \alpha_2\right\rangle\right|_2 \leqslant C\mathrm{e}^{-\frac{\mathrm{d}(U_1,U_2)^2}{6t}} \left\|\alpha_1\right\|_2 \left\|\alpha_2\right\|_2, \tag{4.26}$$

where $\alpha_1 \in \Gamma_{\mathrm{L}^2}(\mathrm{T}^*M)$ is supported in E and $\alpha_2 \in \Gamma_{\mathrm{L}^2}(\mathrm{T}^*M)$ is supported in F. To see (4.25), we first note that by applying the Phragmen-Lindelöf estimate 4.16 to the estimate (4.24) we get the bound

$$\left\|\left\langle \mathrm{e}^{-z\Delta^{(k)}}\alpha_1, \alpha_2\right\rangle\right\|_2 \leqslant \mathrm{e}^{-\mathrm{d}(U_1,U_2)^2\mathrm{Re}\frac{1}{4z}} \left\|\alpha_1\right\|_2 \left\|\alpha_2\right\|_2, \tag{4.27}$$

valid for all z with $\mathrm{Re} z > 0$. By Cauchy's integral formula we have

$$\left\langle \Delta^{(k)}\mathrm{e}^{-t\Delta^{(k)}}\alpha_1, \alpha_2\right\rangle = -\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \mathrm{e}^{-t\Delta^{(k)}}\alpha_1, \alpha_2\right\rangle = -\frac{1}{2\pi i}\int_{z:|z-t|=t/2} \frac{\left\langle \mathrm{e}^{-z\Delta^{(k)}}\alpha_1, \alpha_2\right\rangle \mathrm{d}z}{(z-t)^2}.$$

Since by (4.27) we have

$$\left| \int_{z:|z-t|=t/2} \frac{\left\langle e^{-z\Delta^{(k)}}\alpha_1, \alpha_2 \right\rangle dz}{(z-t)^2} \right| \leqslant (2\pi)^{-1}\pi t \sup_{z:|z-t|=t/2} \left| \frac{\left\langle e^{-z\Delta^{(k)}}\alpha_1, \alpha_2 \right\rangle}{(z-t)^2} \right|$$

$$\leqslant \frac{1}{2} t \left\|\alpha_1\right\|_2 \left\|\alpha_2\right\|_2 e^{-\frac{\mathsf{d}(U_1,U_2)^2}{4(t+t/2)}} \left(\frac{t}{2}\right)^{-2},$$

this proves (4.25). □

3° We have

$$\left\| \mathbb{1}_F \sqrt{t}\nabla e^{-t\Delta^{(k)}}\alpha \right\|_2 \leqslant C_1(A) e^{-\frac{C_2(A)\mathsf{d}(E,F)^2}{t}} \left\|\mathbb{1}_E \alpha\right\|_2 .$$

Indeed: We pick some $\varphi \in C_c^\infty(M)$. Then we have

$$\begin{aligned}
\left\| \sqrt{t}\varphi\nabla e^{-t\Delta^{(k)}}\alpha \right\|_2^2 &= \left\langle t\nabla^*(\varphi^2\nabla e^{-t\Delta^{(k)}}\alpha), e^{-t\Delta^{(k)}}\alpha \right\rangle \\
&= 2\left\langle t\varphi\nabla_{\mathsf{d}\varphi} e^{-t\Delta^{(k)}}\alpha, e^{-t\Delta^{(k)}}\alpha \right\rangle + \left\langle t\varphi^2\nabla^*\nabla e^{-t\Delta^{(k)}}\alpha, e^{-t\Delta^{(k)}}\alpha \right\rangle \\
&= 2\left\langle t\varphi\nabla e^{-t\Delta^{(k)}}\alpha, \mathsf{d}\varphi \otimes e^{-t\Delta^{(k)}}\alpha \right\rangle + \left\langle t\varphi^2\Delta^{(k)} e^{-t\Delta^{(k)}}\alpha, e^{-t\Delta^{(k)}}\alpha \right\rangle \\
&\quad - \left\langle t\varphi^2 \operatorname{Ric}^{\mathrm{tr}} e^{-t\Delta^{(k)}}\alpha, e^{-t\Delta^{(k)}}\alpha \right\rangle \\
&\leqslant 2\left\| \sqrt{t}\varphi\nabla e^{-t\Delta^{(k)}}\alpha \right\|_2 \sqrt{t} \left\| \mathsf{d}\varphi \otimes e^{-t\Delta^{(k)}}\alpha \right\|_2 \\
&\quad + \left\| t\varphi\Delta^{(k)} e^{-t\Delta^{(k)}}\alpha \right\|_2 \left\| \varphi e^{-t\Delta^{(k)}}\alpha \right\|_2 + A^2 t \left\| \varphi e^{-t\Delta^{(k)}}\alpha \right\|_2^2 \\
&\leqslant \frac{1}{2}\left\| \sqrt{t}\varphi\nabla e^{-t\Delta^{(k)}}\alpha \right\|_2^2 + 4t\left\| \mathsf{d}\varphi \otimes e^{-t\Delta^{(k)}}\alpha \right\|_2^2 + \\
&\quad \left\| \varphi t\Delta^{(k)} e^{-t\Delta^{(k)}}\alpha \right\|_2 \left\| \varphi e^{-t\Delta^{(k)}}\alpha \right\|_2 + A^2 t \left\| \varphi e^{-t\Delta^{(k)}}\alpha \right\|_2^2,
\end{aligned}$$

and so

$$\begin{aligned}
\left\| \sqrt{t}\varphi\nabla e^{-t\Delta^{(k)}}\alpha \right\|_2^2 \leqslant ct\left\| \mathsf{d}\varphi \otimes e^{-t\Delta^{(k)}}\alpha \right\|_2^2 &+ c\left\| t\varphi\Delta^{(k)} e^{-t\Delta^{(k)}}\alpha \right\|_2 \left\| \varphi e^{-t\Delta^{(k)}}\alpha \right\|_2 \\
&+ cA^2 t \left\| t\varphi e^{-t\Delta^{(k)}}\alpha \right\|_2^2 .
\end{aligned}$$

Assume now that

$$0 \leqslant \varphi \leqslant 1, \quad \varphi|_F = 1, \quad \|\mathsf{d}\varphi\|_\infty \leqslant 1, \quad \operatorname{supp}\varphi \subset F' := \left\{ x : \mathsf{d}(x,F) \leqslant \frac{\mathsf{d}(E,F)}{3} \right\}.$$

Then we have

$$\begin{aligned}
&\left\| \mathbb{1}_F \sqrt{t}\nabla e^{-t\Delta^{(k)}}\alpha \right\|_2^2 \\
&\leqslant \left\| \sqrt{t}\varphi\nabla e^{-t\Delta^{(k)}}\alpha \right\|_2^2 \\
&\leqslant ct\left\| \mathbb{1}_{F'} e^{-t\Delta^{(k)}}\alpha \right\|_2^2 + c\left\| \mathbb{1}_{F'} t\Delta^{(k)} e^{-t\Delta^{(k)}}\alpha \right\|_2 \left\| \mathbb{1}_{F'} e^{-t\Delta^{(k)}}\alpha \right\|_2 + ctA^2 \left\| \mathbb{1}_{F'} e^{-t\Delta^{(k)}}\alpha \right\|_2^2 .
\end{aligned}$$

Using (4.23), (4.25) and

$$\mathsf{d}(E,F') \geqslant \frac{2}{3}\mathsf{d}(E,F)$$

we get

$$\left\| \mathbb{1}_F \sqrt{t}\nabla e^{-t\Delta^{(k)}}\alpha \right\|_2 \leqslant c_1(1+\sqrt{t}A) e^{-\frac{c_2\mathsf{d}(E,F)^2}{t}} \left\|\mathbb{1}_E \alpha\right\|_2 .$$ □

Altogether, the claim follows. ■

Appendix A

Appendix

A.1 Conditional expectation

For a comprehensive introduction for this subsection, we refer the reader to [Sch17] § 20-22 and § 17 treating the more general case of a σ-finite measure space and [Wil91]. First, let us recall a central theorem in the study of the geometry of Hilbert spaces. A **Hilbert space** $(\mathscr{H}, \langle\cdot,\cdot\rangle_{\mathscr{H}})$ is a complete inner product space, i.e. an inner product space where every Cauchy sequence converges. Let $\|\cdot\| = \langle\cdot,\cdot\rangle_{\mathscr{H}}^{1/2}$ the norm corresponding to the inner product $\langle\cdot,\cdot\rangle_{\mathscr{H}}$ and $\mathbb{K} := \{\mathbb{R}, \mathbb{C}\}$.

Theorem A.1 (Projection theorem). *Let $C \neq \emptyset$ be a closed convex subset of the Hilbert space $\mathscr{H}$. For every $h \in \mathscr{H}$ there is a unique minimiser $u \in C$ such that*

$$\|h - u\| = \inf_{w \in C} \|h - w\| \, .$$

This element $u = P_C h$ is called **(orthogonal) projection** *of h onto C.*

A **continuous linear functional** on $\mathscr{H}$ is a map $\Lambda : \mathscr{H} \to \mathbb{K}$, $h \mapsto \Lambda(h)$ which is linear,

$$\Lambda(\alpha g + \beta h) = \alpha\Lambda(g) + \beta\Lambda(h) \qquad \forall \alpha, \beta \in \mathscr{K}\ \forall g, h \in \mathscr{H}$$

and satisfies

$$|\Lambda(g - h)| \leqslant c(\Lambda)\,\|g - h\| \qquad \forall g, h \in \mathscr{H}$$

with a constant $c(\Lambda) \geqslant 0$ independent of $g, h \in \mathscr{H}$. In fact, all linear functional on $\mathscr{H}$ arise in this way:

Theorem A.2 (Riesz representation theorem). *For each continuous linear functional φ on the Hilbert space $\mathscr{H}$ there exists a unique $g \in \mathscr{H}$ such that*

$$\Lambda_g(h) := \langle h, g\rangle \qquad \forall h \in \mathscr{H}.$$

and $\|\Lambda_g\|_{\mathscr{H}} = \|g\|_{\mathscr{H}}$. Conversely, given $h \in \mathscr{H}$, then $h \mapsto \langle h, g\rangle$ is a continuous linear functional with operator norm $\|g\|_{\mathscr{H}}$.

From now on let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. A prototypical example of a Hilbert space is $\mathscr{H} = \mathrm{L}^2(\mathscr{F})$, i.e. the space of all functions whose (absolute) 2^{nd} moment is integrable with inner product, resp. norm

$$\langle u, v\rangle_2 := \int uv \,\mathrm{d}\mathbb{P} \qquad \text{resp.} \qquad \|u\|_2 := \left(\int |u|^2 \,\mathrm{d}\mathbb{P}\right)^{1/2}.$$

Given a sub-σ-algebra $\mathscr{G} \subset \mathscr{F}$, the idea of a conditional expectation is to make a random variable $u \in \mathsf{L}^2(\mathscr{F})$ also measurable with respect to the coarser σ-algebra, i.e. $u \in \mathscr{G}$. Formally, construct an object $\mathbb{E}^{\mathscr{G}} \in \mathscr{G}$ that is $\mathscr{G}$-measurable by definition.

Definition A.3. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and $\mathscr{G} \subset \mathscr{F}$ be a sub-σ-algebra. The **conditional expectation of** $u \in \mathsf{L}^2(\mathscr{F})$ relative to $\mathscr{G}$ is the orthogonal projection onto the closed subspace $\mathsf{L}^2(\mathscr{G})$

$$\mathbb{E}^{\mathscr{G}} : \mathsf{L}^2(\mathscr{F}) \to \mathsf{L}^2(\mathscr{G}), \quad u \mapsto \mathbb{E}^{\mathscr{G}} u.$$

It is common to write $\mathbb{E}(u \mid \mathscr{G})$ instead of $\mathbb{E}^{\mathscr{G}} u$.

Remark A.4 (Properties of $\mathbb{E}^{\mathscr{G}}$)**.** Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and $\mathscr{G} \subset \mathscr{F}$ be a sub-σ-algebra. The conditional expectation $\mathbb{E}^{\mathscr{G}}$ has the following properties, for all $u, v \in \mathsf{L}^2(\mathscr{F})$, almost surely:

(i) $\mathbb{E}^{\mathscr{G}} \in \mathsf{L}^2(\mathscr{G})$

(ii) $\left\|\mathbb{E}^{\mathscr{G}} u\right\|_{\mathsf{L}^2(\mathscr{G})} \leqslant \|u\|_{\mathsf{L}^2(\mathscr{F})}$ (Contraction)

(iii) $\left\langle \mathbb{E}^{\mathscr{G}} u, w \right\rangle = \left\langle u, \mathbb{E}^{\mathscr{G}} w \right\rangle = \left\langle \mathbb{E}^{\mathscr{G}} u, \mathbb{E}^{\mathscr{G}} w \right\rangle$ (Symmetry)

(iv) $\mathbb{E}^{\mathscr{G}} u$ is the unique minimiser in $\mathsf{L}^2(\mathscr{G})$ such that

$$\left\|u - \mathbb{E}^{\mathscr{G}} u\right\|_{\mathsf{L}^2(\mathscr{F})} = \min_{g \in \mathsf{L}^2(\mathscr{G})} \|u - g\|_{\mathsf{L}^2(\mathscr{F})}$$

(v) $u = w \implies \mathbb{E}^{\mathscr{G}} u = \mathbb{E}^{\mathscr{G}} w$

(vi) $\mathbb{E}^{\mathscr{G}}(\alpha u + \beta w) = \alpha \mathbb{E}^{\mathscr{G}} u + \beta \mathbb{E}^{\mathscr{G}} w \qquad \forall \alpha, \beta \in \mathbb{R}$ (Linearity)

(vii) If $\mathscr{G}_0 \subset \mathscr{G}$ is a another sub-σ-algebra, then $\mathbb{E}^{\mathscr{A}} \mathbb{E}^{\mathscr{G}} u = \mathbb{E}^{\mathscr{G}_0} u$ (Tower property)

(viii) $\mathbb{E}^{\mathscr{G}}(gu) = g \mathbb{E}^{\mathscr{G}} u \qquad \forall g \in \mathsf{L}^\infty(\mathscr{G})$ (Pull out)

(ix) $\mathbb{E}^{\mathscr{G}} g = g \qquad \forall g \in \mathsf{L}^2(\mathscr{G})$

(x) $0 \leqslant u \leqslant 1 \implies 0 \leqslant \mathbb{E}^{\mathscr{G}} u \leqslant 1$ (Markov property)

(xi) $u \leqslant w \implies \mathbb{E}^{\mathscr{G}} u \leqslant \mathbb{E}^{\mathscr{G}} w$ (Monotony)

(xii) $\left|\mathbb{E}^{\mathscr{G}} u\right| \leqslant \mathbb{E}^{\mathscr{G}} |u|$ (Δ-inequality)

(xiii) $\mathbb{E}^{\{\emptyset, \Omega\}} u = \mathbb{E} u$

(xiv) $\mathbb{E}\mathbb{E}^{\mathscr{F}} u = \mathbb{E} u$ (Tower property)

(xv) $0 \leqslant u_n \uparrow u \implies \mathbb{E}^{\mathscr{G}} u_n \uparrow \mathbb{E}^{\mathscr{G}} u$ (conditional Beppo Levi)

(xvi) $u_n \geqslant 0 \implies \mathbb{E}^{\mathscr{G}} \left(\liminf u_n\right) \leqslant \liminf \mathbb{E}^{\mathscr{G}} u_n$ (conditional Fatou)

(xvii) For all $n \in \mathbb{N}$, $\left|u_n\right| \leqslant w$, $\mathbb{E} w < \infty$ and

$$u_n \xrightarrow{\text{a.s.}} u \implies \mathbb{E}^{\mathscr{G}} u_n \to \mathbb{E}^{\mathscr{G}} u \qquad \text{(conditional dominated convergence)}$$

(xviii) $c : \mathbb{R} \to \mathbb{R}$ convex and $\mathbb{E}\,|c(u)| < \infty \implies \mathbb{E}^{\mathscr{G}} c(u) \geqslant c\left(\mathbb{E}^{\mathscr{G}}\right)$ (conditional Jensen)

Moreover,

$$\left\|\mathbb{E}^{\mathscr{G}} u\right\|_{\mathsf{L}^1} \leqslant \|u\|_{\mathsf{L}^1} \qquad \forall u \in \mathsf{L}^1(\mathscr{G}),$$

and $\mathbb{E}^{\mathscr{G}}$ can be extended to $\mathsf{L}^1(\mathscr{G})$ by continuity and Properties (v)-(xiv) carry over to $u \in \mathsf{L}^1(\mathscr{G})$. In abuse of notation this extension will be denoted by the same symbol.

Theorem A.5 (Classical definition of $\mathbb{E}^{\mathscr{G}}$). *Let $\mathscr{G} \subset \mathscr{F}$ be a sub-σ-algebra. For $X \in \mathsf{L}^1(\mathscr{F})$ and $Y \in \mathsf{L}^1(\mathscr{F})$ it is then equivalent:*

(i) $Y = \mathbb{E}(X \mid \mathscr{G})$ *a.s.* – Y **is a version of the conditional expectation**

(ii) $\displaystyle\int_G Y \mathrm{d}\mathbb{P} = \int_G X \mathrm{d}\mathbb{P} \qquad \forall G \in \mathscr{G},$

where (ii) holds on any $\cap$-stable generator for $\mathscr{F}$.

The proof of the following lemma follows from usual approximations arguments for Lebesgue integrals, cf. e.g. [SP14, Appendix A.2]

Lemma A.6. *Assume that $\mathscr{X}, \mathscr{Y} \subset \mathscr{F}$ are σ-algebras and let $X : (\Omega, \mathscr{F}) \to (C, \mathscr{C}) \in \mathscr{X}/\mathscr{C}$ and $Y : (\Omega, \mathscr{F}) \to (D, \mathscr{D}) \in \mathscr{Y}/\mathscr{D}$ be two random variables such that $\mathscr{X} \perp\!\!\!\perp \mathscr{Y}$. Then*

$$\mathbb{E}(\Phi(X, Y) \mid \mathscr{X}) = \mathbb{E}\Phi(x, Y)\big|_{x=X} = \mathbb{E}(\Phi(X, Y) \mid X)$$

holds for all bounded $\mathscr{C} \times \mathscr{D}/\mathscr{B}(\mathbb{R})$-measurable function $\Phi : C \times D \to \mathbb{R}$. If $\Psi : E \times \Omega \to \mathbb{R}$ is bounded and $\mathscr{C} \times \mathscr{Y}/\mathscr{B}(\mathbb{R})$-measurable, then

$$\mathbb{E}(\Psi(X(\cdot), \cdot) \mid \mathscr{X}) = \mathbb{E}\Psi(x, \cdot)\big|_{x=X} = \mathbb{E}(\Psi(X(\cdot), \cdot) \mid X)\,.$$

Corollary A.7. *Assume that $\mathscr{X}, \mathscr{Y} \subset \mathscr{F}$ are σ-algebras and let $X : (\Omega, \mathscr{F}) \to (C, \mathscr{C}) \in \mathscr{X}/\mathscr{C}$ and $Y : (\Omega, \mathscr{F}) \to (D, \mathscr{D}) \in \mathscr{Y}/\mathscr{D}$ be two random variables such that $\mathscr{X} \perp\!\!\!\perp \mathscr{Y}$. Then*

$$\mathbb{E}\Phi(X, Y) = \int \mathbb{E}\Phi(x, Y)\mathbb{P}(X \in \mathrm{d}x) = \mathbb{E}\int \Phi(x, Y)\mathbb{P}(X \in \mathrm{d}x)$$

holds for all bounded $\mathscr{C} \times \mathscr{D}/\mathscr{B}(\mathbb{R})$-measurable function $\Phi : C \times D \to \mathbb{R}$. If $\Psi : E \times \Omega \to \mathbb{R}$ is bounded and $\mathscr{C} \times \mathscr{Y}/\mathscr{B}(\mathbb{R})$-measurable, then

$$\mathbb{E}\Psi(X(\cdot), \cdot) = \int \mathbb{E}\Psi(x, \cdot)\mathbb{P}(X \in \mathrm{d}x) = \mathbb{E}\int \Psi(x, \cdot)\mathbb{P}(X \in \mathrm{d}x)\,.$$

A.2 Martingales

A **filtration** $(\mathscr{F}_t)_{t \geqslant 0}$ is a family of sub-σ-algebras $\mathscr{F}$ such that $\mathscr{F}_s \subset \mathscr{F}_t$ for all $s \leqslant t$. Now, let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geqslant 0}, \mathbb{P})$ be a **filtered probability space** satisfying the **usual hypotheses**, i.e. a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ equipped with a filtration that is right-continuous

$$\mathscr{F}_t = \mathscr{F}_{t+} := \bigcap_{s > t} \mathscr{F}_s \qquad \text{for all } t \geqslant 0,$$

and complete, i.e. $\mathscr{F}_0$ contains all subsets of $\mathbb{P}$-null sets.

Definition A.8. A stochastic process $(X_t)_{t\geqslant 0}$ is called a **martingale** if it is adapted to the filtration $(\mathscr{F}_t)_{t\geqslant 0}$ and $X_t \in \mathrm{L}^1(\mathbb{P})$ for all $t \geqslant 0$ such that

$$\mathbb{E}(X_t \mid \mathscr{F}_s) = X_s \qquad \forall s \leqslant t. \tag{A.1}$$

Definition A.9. A **continuous local martingale** is an adapted continuous process X for which there exists a sequence of stopping times $(\sigma_n)_{n\geqslant 0}$ such that $\sigma_n \uparrow \infty$ a.s. and for every $n \geqslant 0$, $X_t^{\sigma_n} \mathbb{1}_{\{\sigma_n > 0\}}$ is a (true) martingale.

Remark A.10. By the martingale property (A.1), we immediately see that a martingale has constant expectation, i.e. $\mathbb{E}N_s = \mathbb{E}N_0$ for all s.

Lemma A.11. *Let τ be an a.s. finite stopping and X be a continuous local martingale taking values in a finite-dimensional Hilbert space $\mathscr{H}$. Then*

$$\mathbb{E} \sup_{0\leqslant s\leqslant \tau} |Y_s| < \infty \qquad \Longrightarrow \qquad Y \text{ is a (true) martingale.}$$

In order to proof that a local martingale is a true martingale the common tool is given by the following inequality, originally to be found in the joint work of Burkholder, Davis & Gundy 1972 [BDG72].

Lemma A.12 (Burkholder, Davis, Gundy). *For any $p \in (0, \infty)$ there are positive constants c_p, C_p such that, for all local martingales X with $X_0 = 0$ and stopping times τ, the following inequality holds:*

$$c_p \mathbb{E}[X]_\tau^{p/2} \leqslant \mathbb{E}\left(\sup_{s\leqslant\wedge\tau} |X_s|\right)^p \leqslant C_p \mathbb{E}[X]_\tau^{p/2}.$$

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