CONGRUENCE THEOREMS FOR CONVEX POLYGONS INVOLVING SIDES, ANGLES, AND DIAGONALS

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Abstract. We first investigate Congruence Theorems for convex polygons involving only sidelengths and angle measures (the sides and angles that we know have a fixed relative position, as it is customary for triangles). Moreover, we prove that a convex n-gon is determined up to congruence in the following cases: for $n \geqslant 4$, if we know n sides and all but $2n-7$ diagonals; for $n \geqslant 7$, if we know all but $n-5$ diagonals; for $n \geqslant 5$, if we know all but $n-3$ sides or diagonals. Notice that the results are optimal, as they would not hold for n smaller than the given bound, or by increasing the number of exceptions.

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1. Introduction

Two subsets of the Euclidean plane are said to be *congruent* if one is the image of the other via an isometry. The general aim of *Congruence Theorems* is specifying enough information on such a set as to be able to determine it up to congruence. Congruence Theorems for triangles are an important topic in geometry and in school mathematics, however not much is known for polygons.

We prove Congruence Theorems for convex n-gons, where n can be arbitrarily large: the information are the length of various sides and diagonals, and the measure of various angles (by which we mean interior angles). All interior angles of a convex n-gon are assumed to be strictly smaller than $180°$. Following the usual convention, when we say for example that we know one side length in the triangle ABC, then we mean that there is a specific side, e.g. AB, of which we know the length.

Congruence Theorems for triangles are well-known since Euclid [7]. Congruence Theorems for convex quadrilaterals have been studied following the classical approach for triangles (see [5], where it is shown for example that at least 5 conditions are necessary) and also with a modern point of view (see [3], where the slightly weaker notion of congruent-like is considered). We refer the reader to [4] for an axiomatic introduction (didactically interesting) to the general topic of

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congruence for polygons.
Some interesting work has recently been made concerning how many measurements are needed for generic Congruence Theorems for polygons (see [1], despite the focus of the article is on general polyhedra). In particular, for some exceptional polygons, the amount of measurements needed is larger than expected.
The novelty of the results that we present in this article consists in being as precise as possible on how many sides or diagonals or angles measurements are needed, so not only counting the total amount of measurements.

We prove the following general result for convex quadrilaterals:

Theorem 1. *For a convex quadrilateral, suppose to know the length of x sides, the measure of y angles, and the length of z diagonals. Those triples (x, y, z) for which the convex quadrilateral is determined up to congruence are:*

$$(4, \geqslant 0, \geqslant 1) \quad (4, \geqslant 1, \geqslant 0) \quad (3, \geqslant 2, 0) \quad (3, \geqslant 0, 2) \quad (2, \geqslant 3, 2)\,.$$

For convex polygons, part of the following result may be found in [6].

Theorem 2. *For $n \geqslant 3$, a convex n-gon is determined up to congruence if we know at least one of the following:*

(1) *the length of n sides and the measure of $n-3$ angles;*
(2) *the length of $n-1$ sides and the measure of $n-1$ angles;*
(3) *the length of $n-1$ sides and the measure of $n-2$ angles, such that the two unknown angles are both at the unknown side;*
(4) *the length of $n-2$ sides and the measure of $n-1$ angles, such that the two unknown sides are consecutive.*

The assumptions in the above statement are optimal in the following sense:

Theorem 3. *For $n \geqslant 4$, a convex n-gon is not necessarily determined up to congruence if we know exactly one of the following:*

(1) *the length of $n-1$ sides and the measure of $n-2$ angles, provided that the two unknown angles are not both at the unknown side;*
(2) *the length of $n-2$ sides and the measure of $n-1$ angles, provided that the two unknown sides are not consecutive;*
(3) *the length of $n-t$ sides and the measure of $n-4+t$ angles for some $t \in \{0, 1, 2, 3\}$.*

Most notably, we prove three general Congruence Theorems involving sides and diagonals:

Theorem 4 (Perucca)**.** *For $n \geqslant 4$, a convex n-gon is determined up to congruence if we know the length of all sides and of all but $2n-7$ diagonals.*

Theorem 5 (Perucca)**.** *For $n \geqslant 7$, a convex n-gon is determined up to congruence if we know the length of all diagonals with at most $n-5$ exceptions.*

Theorem 6 (Perucca)**.** *For $n \geqslant 5$, a convex n-gon is determined up to congruence if we know the length of all sides and diagonals with at most $n-3$ exceptions.*

Notice that the number of exceptions in the above statements is optimal, and the results would not hold for n smaller than the given bound. Moreover, notice that for $n = 4, 5, 6$ knowing all diagonals is not sufficient to determine a convex n-gon up to congruence by Theorem 1, Example 8 and Example 9 respectively.

We conclude by giving some directions of future research on the topic of Congruence Theorems. As an exercise, the reader could make precise the relative position of the known objects in Theorem 1 (as done e.g. in Theorem 2 (3)), or generalize that result to convex pentagons. In general, one can try to determine the triples x, y, z such that a convex n-gon is determined up to congruence if we know x sides, y angles, and z diagonals. One could also consider for example the length of medians, or the angle between two diagonals. Notice that we did not address Similarity Theorems, but these would also be interesting. Finally, one could remove the assumption of convexity, or even investigate further shapes beyond polygons, possibly working in a higher-dimensional Euclidean space (very few results are known in this general higher dimensional setting, see [1]).

Part of the above-mentioned problems (for convex quadrilaterals or pentagons) can be assigned as open-ended exercises for a Math Circle, while the investigation of more general results is an accessible research project for undergraduates. Moreover, the known results can be a source of school exercises and of problems for mathematical competitions. Last but not least, school pupils can understand the statements and the research questions presented in this article, and with them they can gain a new perspective on the Congruence Theorems for triangles that they are most likely to learn in school.

Acknowlegdements. We are indebted to Louis-Hadrien Robert for his counterexample (see Example 9) which solved the tricky question whether a convex hexagon is determined up to congruence if the length of all its diagonals are known. We also sincerely thank Serena Dipierro, Enrico Valdinoci, Hugo Parlier, Lassina Dembélé, Carolina Oliveira Costa, Jean-Marc Schlenker, Bryan Advocaat, Flavio Perissinotto for inspiring discussions and feedback.

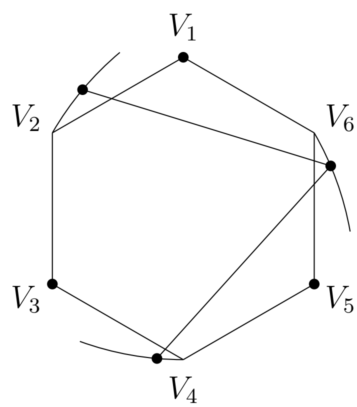
2. Convex quadrilaterals, pentagons, and hexagons

We say that a convex n-gon with some given properties has a *minimal deformation* if for every real number $\varepsilon > 0$ there is a convex n-gon with the same properties which is not congruent to it and which is obtained by moving each vertex at a distance at most ε. Provided that ε is sufficiently small, moving the vertices at a distance at most ε gives again in a convex n-gon (no three vertices become aligned and the angles are still less than π).

Lemma 7. *A convex quadrilateral of whom we know the length of all sides (respectively, the length of one side and the measure of all angles) has a minimal deformation.*

Proof. Let $ABCD$ be the vertices of the quadrilateral in cyclic order. Suppose first that we know all sides. We slightly rotate BC around B: if the rotation is sufficiently small, then we may get back the original length of CD by slightly rotating DA around A. In this way we do not change the length of the sides but, provided that the rotations are small, we alter the measure of the angles at A and B. For the second assertion we slightly move C and D on the lines containing BC and DA without changing the direction of CD: we preserve all angles and AB but we alter BC and DA. □

Example 8. We construct a minimal deformation for the regular hexagon preserving the length of all diagonals but one. By removing one vertex, this also gives a minimal deformation for some convex pentagon preserving the length of all diagonals.



Call the vertices V_1 to V_6 in cyclic order, and fix V_1, V_3, V_5 (preserving V_1V_3, V_1V_5, V_3V_5). We slightly rotate V_2 around V_5, V_4 around V_1, V_6 around V_3 (preserving V_1V_4, V_2V_5, V_3V_6). We obtain again a convex hexagon, and we conclude by showing that we may also preserve V_2V_6, V_4V_6. Rotating slightly V_2 towards V_1 has the effect of decreasing the length of V_2V_6. By rotating V_6 slightly towards V_5 we can make sure that V_2V_6 is preserved. Finally, by rotating V_4 towards V_3 we can make sure that V_4V_6 is preserved (we can do this if V_6 stays in a sufficiently small circle around its original position, and this holds provided that the rotation of V_2 was sufficiently small).

The proof of Theorem 1. Recall that having $y = 3$ is equivalent to having $y = 4$.

We exclude $(4, 0, 0)$ by considering all rhombi with a given side length. We exclude $(1, 4, 2)$ by considering an isosceles trapezoid and taking a parallel shift of the small basis (provided that the diagonals are almost perpendicular to the oblique sides at the vertices of the small basis). We exclude $(2, 4, 1)$ by considering a parallelogram of which we know the two longer sides and the short diagonal, and by taking a parallel shift of one of these sides (if the short diagonal and the oblique side are almost perpendicular). We exclude $(2, 2, 2)$ by constructing two quadrilaterals $ABCD$ and $ABC'D$ as follows: consider a half-circle with diameter AB and two non-congruent inscribed right triangles ABC and ABC'; the perpendicular bisector of CC' passes through the middlepoint of AB, and we select a point D on it outside the semicircle. We exclude $(3, 2, 1)$ by cutting the given quadrilateral with the known diagonal, and working with the triangle of which we only know two sides: if the two known angles are opposite to the known diagonal, then we can exploit a counterexample to the SSA Congruence Theorem for triangles.

To determine the convex quadrilateral up to congruence, it suffices to determine the position of all vertices and hence it suffices to know three sides and the two angles between them. This is clear for $(3, 4, 0)$, while for $(3, 0, 2)$ it can be seen by applying the SSS Congruence Theorem to both triangles made by two known sides. To prove that $(4, 0, 1)$ is suitable we apply the SSS Congruence Theorem to both triangles made by the known diagonal and two sides. Then, to prove that $(4, 1, 0)$ is suitable, it suffices to know a diagonal: we apply the SAS Congruence Theorem to the triangle made by the two sides at the known angle. Finally, we prove that $(2, 4, 2)$ is suitable: this is immediate if the known sides are consecutive, else apply the SsA Congruence Theorem to the triangle made by a diagonal and two sides such that the angle opposite to the diagonal is not acute. □

3. Congruence Theorems involving sides and angles

Convex n-gons are determined by their n vertices, each of which has two Cartesian coordinates. They live in a space with $2n$ degrees of freedom, as by slightly moving some vertices of a convex n-gon one gets again a convex n-gon. Congruence is preserved by the isometries, which are the composition of translations, rotations at the origin, and the line symmetry at the x-axis. Allowing for one line symmetry does not decrease the number of degrees of freedom (it is a discrete transformation, and it cannot be constructed by a continuous process). Translations are determined by the image of the origin and rotations around the origin are characterised by one angles. Thus convex n-gons up to congruence live in a space with $2n - 3$ degrees of freedom. Consequently, no congruence n-gon is determined up to congruence if we know less than $2n-3$ measurements (lengths or angle measures) and, prescribing more than $2n-3$ such measurements, the n-gon is over determined (most likely, it won't exist).

The proof of Theorem 2. Recall that knowing $n-1$ angles is equivalent to knowing n angles. Moreover, by the Congruence Theorems for triangles we may suppose $n \geqslant 4$. Under assumption (2) or (3), the polygonal line obtained by removing the unknown side is determined up to congruence hence so is the given n-gon. Now consider assumption (4), and let AB, BC be the unknown sides. The $(n-1)$-gon obtained by removing B is determined up to congruence by assumption (3), and we conclude by the ASA Congruence Theorem at ABC. Finally consider assumption (1) and let ABC be the vertices at which we do not know the angles. The polygonal line consisting of the sides between A and B (respectively, B and C or C and A) is determined up to congruence. In particular we know AB, BC, CA. We conclude because by the SSS Congruence Theorem at ABC (thanks to the convexity of the n-gon) the three polygonal lines fit in a prescribed way. □

The proof of Theorem 3 (1),(2). Consider (1), and let AB be the unknown side. If the unknown angles are at two further vertices X, Y, then we can slightly decrease their two angles, keeping invariant all known sides and all angles at the vertices different from A, B, X, Y, and also preserve the direction of AB (hence the angles at A and B are preserved). If the unknown angles are at A and at some vertex $C \neq A, B$, then ABC is not determined up to congruence (the known angle is not between the two known sides). Then we may replace BC and CA by polygonal lines (for example those given by consecutive sides of a regular m-gon with m large enough to ensure convexity) on the outside of ABC and construct two non-congruent n-gons as requested. Now consider (2): for $n = 4$ consider two non-congruent rectangles with the same basis and for $n \geqslant 5$ replace the known sides of these rectangles by polygonal lines with the appropriate number of sides (we may choose the number of sides separating the two unknown ones). □

The proof of Theorem 3 (3). For $t = 0$, the vertices at the unknown angles form a convex quadrilateral where only the sides are known, and this has a minimal deformation by Lemma 7: moving accordingly the polygonal curves between two unknown angles, we obtain a minimal deformation of the n-gon.

For $t = 1$, let AB be the unknown side. If the unknown angles are at A, B, X, then we can slightly decrease the angle at X and ensure that the polygonal line consisting of the sides between B and X (respectively, X and A) gets mapped to a congruent one. If the unknown angles are at A, X, Y (with A, B, X, Y distinct and in cyclic order), then it suffices to find a minimal deformation of $ABXY$ preserving all sides except AB and the angle at B: it exists because we can fix B and X and slightly rotate Y around X and move A on the line AB. If the unknown angles are at X, Y, Z (with A, B, X, Y, Z distinct and in cyclic order), then it suffices to find a minimal deformation of $ABXYZ$ preserving all sides except AB and the angles at A, B: we can slightly increase AB preserving the angles at A, B and AZ, BX, and also slightly increase the angle at Y so that also XY , YZ are preserved.

For $t = 2$, by Theorem 3(2) we may suppose that the unknown sides are consecutive, so call them AB and BC. If the two unknown angles are among A, B, C, then ABC is not determined up to congruence and we conclude. If the unknown angles are at A, X, with $X \neq A, B, C$, then it suffices to find a minimal deformation for $ABCX$ preserving CX, XA and the angles at B, C. Suppose that $ABCX$ is a rectangle: fix the angle at C and CX, and slightly rotate XA; by adjusting the length of AB, BC we also preserve the angle at B. If the unknown angles are at B, X, with $X \neq A, B, C$, then the $(n-1)$-gon obtained by removing B has a minimal deformation where only the angles at X, A, C and AC change, thus by rescaling ABC we construct a minimal deformation for the n-gon. Finally suppose that the unknown angles are at X, Y, with A, B, C, X, Y distinct and in cyclic order. It suffices to find a minimal deformation for $ABCXY$ preserving the angles at A, B, C and CX, XY, YA: we fix XY and slightly rotate CX and YA such that the sum of the angles at X and Y is invariant, and we preserve the angles at A, C (and at B, because the sum of all angles is invariant).

For $t = 3$, if the three unknown sides are consecutive, then it suffices to apply Lemma 7 to construct a minimal deformation. If the unknown sides are AB, BC, XY (with A, B, C, X, Y distinct and in cyclic order), suppose that AC, XY are parallel: we can slightly decrease the length of AC, XY preserving CX, YA, and all angles of $ACXY$; we then replace ABC by a smaller similar triangle to complete the construction of a minimal deformation. If the unknown sides are AB, CD, EF, where A, B, C, D, E, F are in cyclic order and form a regular hexagon, then we can find a minimal deformation of $BCDE$ such that we are only altering CD, BE, and by Lemma 7 we can find a minimal deformation of $ABEF$ such that we are only altering AB, BE, EF. Provided that BE is altered in the same way, we get a minimal deformation for the n-gon. $\square$

4. The Congruence Theorem knowing all sides and some diagonals

We prove Theorem 4, where by Theorem 1 we may suppose $n \geqslant 5$. Since a convex n-gon has $n(n-3)/2$ diagonals, by assumption we know $\frac{n(n-7)}{2} + 7$ diagonals. Notice that this number of diagonals is optimal (for $n = 4$ see Theorem 1, while for $n \geqslant 5$ the known diagonals could be all in the $(n-2)$-gon obtained by removing two consecutive vertices hence we could move these by Lemma 7).

We may suppose that no $n-3$ known diagonals start from a same vertex, else these partition the convex n-gon into triangles to whom we can apply the SSS Congruence Theorem. Moreover, it suffices to determine the $(n-1)$-gon obtained by removing one vertex V (apply the SSS Congruence Theorem at the triangle made by the two sides at V).

For $n=5$, calling $ABCDE$ the vertices in cyclic order, the known diagonals are w.l.o.g. AC and BE hence we determine $ABCE$ by applying the SSS Congruence Theorem to ABC and ABE.

We call J_2 *diagonal* (respectively, J_3 *diagonal*) a diagonal that cuts the convex n-gon into two parts containing 2 and $n-2$ (respectively, 3 and $n-3$) sides.

For $n=6$, at least two known diagonals start from some vertex V, and we consider the possible cases. Suppose to know the J_3 diagonal and a J_2 diagonal at V: the J_3 diagonal cuts the hexagon into two quadrilaterals, and the one containing the J_2 diagonal is determined by Theorem 1; since we know a diagonal at a vertex outside this quadrilateral, we can fix this vertex thanks to the SSS Congruence Theorem. Suppose to know the J_2 diagonals connecting three vertices: we can determine the triangle that they form, and conclude by applying the SSS Congruence Theorem to the triangles made by one of these diagonals and two sides. Suppose to know the two J_2 diagonals at V and the J_2 diagonal around V: we determine the triangle made by the sides at V and then, applying again the SSS Congruence Theorem, we fix the position of all vertices not opposite to V. Finally, suppose to know two J_2 diagonals at neighboring vertices: we determine the quadrilateral having these two diagonals; since we know a diagonal at a vertex outside this quadrilateral, we can fix this vertex thanks to the SSS Congruence Theorem.

We now prove the statement by induction for $n \geqslant 5$ (only the induction step is missing), supposing that at most $n-5$ known diagonals have a common vertex. Notice that there is a known diagonal which is a J_2 diagonal or a J_3 diagonal. If we know a J_2 diagonal around some vertex V, then we apply the result to the convex $(n-1)$-gon obtained by removing V (because there were at most $n-5$ known diagonals at V). If we know a J_3 diagonal around some consecutive vertices V,W, then we can apply the result to the convex $(n-2)$-gon obtained by removing V, W (because there were at most $2n-10$ known diagonals at V or W); since we know some diagonal at V or W, we may fix the position of an additional vertex thanks to the SSS Congruence Theorem.

Finally, we prove the statement for $n \geqslant 7$, supposing to know $n-4$ diagonals at some vertex V. Let VX be the unknown diagonal for some vertex X. If there is some vertex A such that AV and AX are sides, then the convex $(n-2)$-gon obtained by removing A and X is determined because we know all its sides and all its diagonals at V; since we know a diagonal at A or X, we may fix the position of an additional vertex thanks to the SSS Congruence Theorem. Now we may suppose that the vertices on each side of VX and distinct from X form a k-gon and a $(n-k)$-gon for some $3 \leqslant k \leqslant n-3$. Since we know all diagonals at V except VX, we can determine both polygons with the SSS Congruence Theorem. There are two known diagonals not contained in the two polygons because we have

$$\frac{k(k-3)}{2} + \frac{(n-k)(n-k-3)}{2} \leqslant \frac{n(n-7)}{2} + 3$$

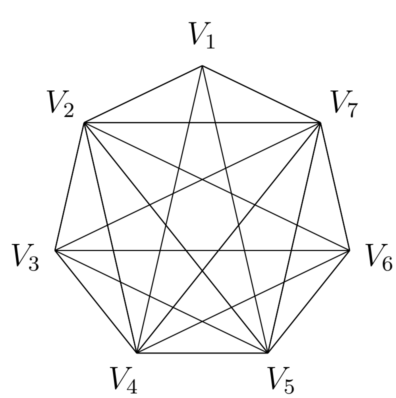
(the left-hand-side is maximal when $k(n-k)$ is minimal, hence for $k = 3, n-3$). If we know a diagonal between vertices on distinct sides of VX, then we can determine the angle of the convex n-gon at V thanks to the SSS Congruence Theorem and hence determine the $(n-1)$-gon without X. Else, we know two diagonals XA and XB for some vertices $A, B \neq V$, and we conclude by determining VX: calling Y, Z the vertices of the n-gon next to X, we know all sides and four diagonals of the hexagon $VXYZAB$ hence this is determined up to congruence.

5. The Congruence Theorem knowing some diagonals

In this section we prove Theorem 5 by induction (and present Example 9). By assumption we know $\frac{n(n-5)}{2} + 5$ diagonals, and this number is optimal (if $n-4$ diagonals are unknown, then there can be vertices V, V' such that VV' is the only known diagonal at V hence we can rotate V around V').

For the inductive step, let $n \geqslant 7$ and consider a convex $(n+1)$-gon. Removing some vertex V we obtain a convex n-gon of which we know at least $\frac{n(n-5)}{2} + 5$ diagonals, so this is determined up to congruence by the induction hypothesis. Since there were at least two known diagonals at V, we may conclude by the SSS Congruence Theorem for triangles.

Now consider a convex heptagon, naming the vertices V_1 to V_7 in cyclic order. Suppose that the unknown diagonals are V_1V_3 and V_1V_6 (in the picture we show the known diagonals).



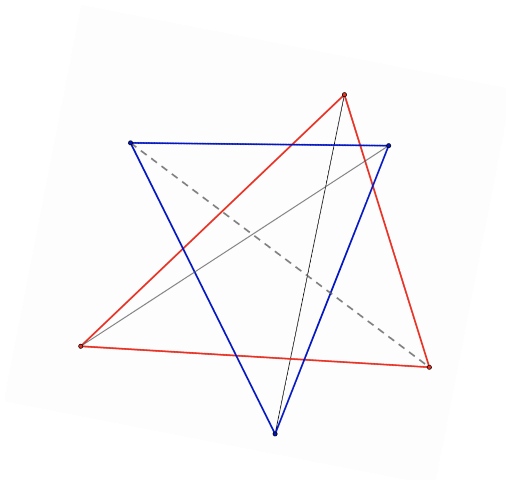
We apply several times the SSS Congruence Theorem for triangles: we determine $V_2V_5V_7$ up to congruence, so we fix the position of V_2, V_5, V_7; we determine $V_2V_4V_7$ up to congruence, so we fix the position of V_4 (thanks to the convexity of the heptagon); we determine $V_3V_5V_7$ and $V_2V_4V_6$ and $V_1V_4V_5$ up to congruence, so we fix the position of V_3, V_6, V_1.

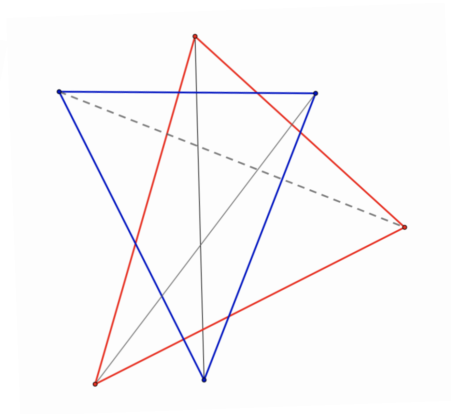
The other cases are completely analogous, so we conclude by describing the list of cases according to the two unknown diagonals. Recall that a J_2 (respectively, J_3) diagonal jumps 2 (respectively, 3) sides. If the two unknown diagonals have a common vertex, then they can be two J_2 diagonals or two J_3 diagonals, else one is a J_2 diagonal and one is a J_3 diagonal and w.l.o.g. the former is V_1V_3 and the latter is V_1V_4 or V_1V_5. Now suppose that the two unknown diagonals do not have a common vertex. They could be two J_2 diagonals (respectively, a J_2 diagonal and a J_3 diagonal), either crossing or non-crossing. Finally, they can be two crossing J_3 diagonals and w.l.o.g. the first is V_1V_4 and the second is V_2V_5 or V_2V_6.

Example 9 (Robert)**.** We show that knowing all 9 diagonals is not sufficient to determine a convex hexagon up to congruence (note that 9 is the number of degrees of freedom for the space of convex hexagons up to congruences). Given a convex hexagon, we label its vertices V_1 to V_6 in cyclic order. If we know the lengths of all diagonals, then in particular we know the triangles $V_1V_3V_5$ and $V_2V_4V_6$ up to congruence. We can imagine fixing the triangle $V_1V_3V_5$ and rotating $V_2V_4V_6$ in such a way that the length of the segments V_1V_4 and V_2V_5 does not change (to achieve this, we can rotate V_4 around V_1 and V_2 around V_5 coherently so that the length V_2V_4 is preserved; finally, one places V_6 so that the triangle $V_2V_4V_6$ has changed to a congruent triangle with the same orientation). Doing this provides a one-parameter family of convex hexagons for which the length of V_3V_6 varies. At least in the case depicted in the figure below, the length of V_3V_6 achieves a local maximum and hence, moving slightly away from it in both directions, we find two convex hexagons such that all corresponding diagonals have the same length. Apart from some exceptional symmetric cases, these two hexagons are not congruent.

6. The Congruence Theorem knowing some sides or diagonals

In this section we prove Theorem 6 by induction. By assumption we know $\frac{n(n-3)}{2}+3$ sides or diagonals, and this number is optimal (if $n-2$ sides or diagonals are unknown, then there could be a vertex at which we know one side but no diagonal, and then we can construct a minimal deformation by slightly rotating the known side at this vertex). For $n = 4$, the result would not hold, as knowing 4 sides is not sufficient by Theorem 1.





The base case for the induction is $n = 5$, and there are various cases: two unknown sides (either consecutive or not); two unknown diagonals (either having a common endpoint or not); one unknown side and one unknown diagonal (either having a common endpoint or not, and in the former case we must distinguish whether the other endpoint of the diagonal is the vertex opposite to the given side or not). The idea is to show that in every case it is possible to find a triangulation of the convex pentagon and apply Congruence Theorems for triangles. We leave this easy check as an exercise for the reader.

For the inductive step, let $n \geqslant 5$ and consider a convex $(n+1)$-gon of which we know $\frac{(n+1)(n-2)}{2} + 3$ sides or diagonals. We claim that removing some suitable vertex V we obtain a convex n-gon of which we know at least $\frac{n(n-3)}{2} + 5$ diagonals, so this is determined up to congruence by the induction hypothesis. Since there were at least two known sides or diagonals at V, we may conclude by the SSS Congruence Theorem for triangles.

We conclude by proving the claim. For a vertex of the n-gon, we have a total of n sides or diagonals touching it. Supposing to pick a vertex at which there is at least one unknown side or diagonal we have removed at most $n-1$ known objects and we conclude because $\frac{(n+1)(n-2)}{2} + 3 - (n-1) = \frac{n(n-3)}{2} + 3$.

References

[1] A. Borisov, M. Dickinson and S. Hastings, *A Congruence Problem for Polyhedra*, The American Mathematical Monthly, vol. 117 (March 2010) No. 3, 232–249.

[2] M. Harvey, *Geometry illuminated. An illustrated introduction to Euclidean and hyperbolic plane geometry*, MAA Textbooks, Mathematical Association of America, Washington DC, 2015.

[3] F. Laudano and G. Vicenzi, *Congruence Theorems for Quadrilaterals*, Journal for Geometry and Graphics, vol. 21 (2017) No. 1, 45–59.

[4] J.M. Lee, *Axiomatic geometry.* Pure and Applied Undergraduate Texts 21, American Mathematical Society, Providence RI, 2013.
[5] I.E. Vance, *Minimum Conditions for Congruence of Quadrilaterals*, School Science and Mathematics vol. 82 (1982) No. 5, 403–415.
[6] J. Page, *Math Open Reference: Testing Polygons for congruence*, `https://www.mathopenref.com/congruentpolygonstests.html`, 2011.
[7] The English Wikipedia pages *Congruence (geometry)* and *Solution of triangles*, `en.wikipedia.org`. Retrieved on May 12th, 2022.

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