ON GENERALIZED IWASAWA MAIN CONJECTURES AND p-ADIC STARK CONJECTURES FOR ARTIN MOTIVES

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ABSTRACT. We continue the study of Selmer groups associated with an Artin representation endowed with a p-stabilization which was initiated in [Mak20]. We formulate a main conjecture and an extra zeros conjecture at all unramified odd primes p, which are shown to imply the p-part of the Tamagawa number conjecture for Artin motives at $s = 0$. We also relate our new conjectures with various cyclotomic Iwasawa main conjectures and p-adic Stark conjectures that appear in the literature. In particular, they provide a natural interpretation for recent conjectures on p-adic L-functions attached to (the adjoint of a) weight one modular form. In the case of monomial representations, we prove that our conjectures are essentially equivalent to some newly introduced Iwasawa-theoretic conjectures for Rubin-Stark elements.

1. INTRODUCTION

Iwasawa theory traditionally focuses on the construction of p-adic L-functions and on their relation with arithmetic invariants of number fields. Concurrently with the first major achievements of the theory [Kat78, DR80, MW84, Wil90, Rub91], several attempts were made in order to define the conjectural p-adic L-function of a motive [CPR89, Coa91, Gre94] and its corresponding Selmer group [Gre89, Gre91]. The motive in question was assumed to admit a critical value in the sense of Deligne and to be ordinary at p, hypotheses which were circumvented in a work of Perrin-Riou [PR95] using the so-called big exponential map and in a work of Benois [Ben14] allowing the treatment of "trivial zeros" via Nekovář's theory of Selmer complexes. However, the statement of Perrin-Riou's conjecture is "maximalist" (in her own words) and is rather not appropriate in certain practical settings. In particular, one would expect that the arithmetic invariants could be expressed in terms of a Greenberg-style Selmer group for p-ordinary motives that are not necessarily critical.

While being ordinary at unramified primes, motives coming from Artin representations are seldom critical and the arithmetic of their L-values at $s = 0$ is described by "refined Stark conjectures", thanks to the work of various mathematicians beginning with Stark's and Rubin's influential papers [Sta75, Sta80, Rub96] and culminating in [Bur11]. In the particular case of monomial Artin motives, that is, when the associated Artin representation is induced from a one-dimensional character, a long-term strategy to tackle these conjectures with the aid of Iwasawa theory was presented in [BKS17]. A general main conjecture (called "higher rank Iwasawa main conjecture") and an extra zero conjecture (called "Iwasawa-theoretic Mazur-Rubin-Sano conjecture") are formulated in terms of Rubin-Stark elements, but neither a p-adic L-function nor an $\mathcal{L}$-invariant play a part in this work. Nevertheless, the authors verify that their extra zero conjecture generalizes the Gross-Stark conjecture. Not long after that, Büyükboduk and Sakamoto [BS19] proved via Coleman theory that it also implies an extra zero conjecture for Katz's p-adic L-functions.

This aim of our paper is to provide a unifying approach to the cyclotomic Iwasawa theory for general Artin motives and to the study of their L-values generalizing many aspects of [BKS17, BS19] which not only encompasses classical conjectures on Deligne-Ribet's or Katz's p-adic L-functions, but also allows a natural interpretation of very recent constructions of p-adic L-functions and of new variants of the Gross-Stark conjecture in the context of (the adjoint of a) weight one modular form [Mak20, DLR16]. More specifically, one of the central objects in this paper is the Selmer group $X_\infty(\rho,\rho^+)$ introduced and studied in [Mak20] (see also [GV20]). It depends on the choice of an ordinary p-stabilization ρ^+ of the p-adic realization ρ of the Artin motive. Although there is no canonical choice for ρ^+ when ρ is non-critical, the key idea is to think of ρ^+ as an *additional parameter*, following the viewpoint of Perrin-Riou.

The question of the torsionness of Selmer groups over the Iwasawa algebra is a recurring theme in Iwasawa theory, and it was shown in [Mak20] that delicate conjectures coming from p-adic transcendence theory arise through the study of $X_\infty(\rho,\rho^+)$. A first crucial aspect of this work is to obtain, under an unramifiedness assumption at p, finer information on the structure of $X_\infty(\rho,\rho^+)$ with the aid of modified Coleman maps and of classical freeness results on $\mathbb{Z}_p$-towers of global units.

On the analytic side, our conjectural p-adic L-function will interpolate the algebraic part of Artin L-values at $s=0$ given by a recipe of Stark [Sta75, Tat84]. "Extra zeros" in the sense of Benois abound in this setting, and we will also formulate an extra zero conjecture which is compatible with the one in [Ben14]. It involves a new $\mathcal{L}$-invariant which computes Benois' $\mathcal{L}$-invariant when the Artin motive is crystalline at p, and it also gives back various $\mathcal{L}$-invariants that appear in the literature [Gro81, RR21, RRV21, BS19].

While it might seem somewhat artificial to introduce p-stabilizations to study Artin L-values in general, they are proving to be useful even in the setting of monomial Artin representations where they do not naturally appear. As a striking example, we prove in this case that the existence of a p-stabilization with a non-vanishing $\mathcal{L}$-invariant implies what the authors in [BKS17] call the "Gross's finiteness conjecture" in their main theorem (see Theorem 3.8.6 for a more general statement which includes unconditional results). The p-stabilizations also naturally occur in the setting of the (adjoint of the) Deligne-Serre representation of a classical weight one modular form f, when we deform f via Hida theory. The techniques relying on p-adic variation turn out to be omnipresent in arithmetic geometry, and we believe that the recent results on the geometry of Eigenvarieties at weight one points [BD16, BDF20] should help advance Iwasawa theory in those settings.

In the next section we formulate our principal conjecture.

1.1. **The main conjecture.**

We fix once and for all an embedding $\iota_\infty:\overline{\mathbb{Q}}\hookrightarrow\mathbb{C}$ as well as $\iota_\ell:\overline{\mathbb{Q}}\hookrightarrow\overline{\mathbb{Q}}_\ell$ for all primes ℓ. Let

$$\rho: G_{\mathbb{Q}}\longrightarrow \mathrm{GL}_E(W)$$

be a d-dimensional Galois representation of $G_{\mathbb{Q}}=\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of finite image with coefficients in a number field $E\subseteq\overline{\mathbb{Q}}$. We will always assume that W does not contain the trivial representation, *i.e.*, one has $\mathrm{H}^0(\mathbb{Q},W)=0$. For any character $\eta: G_{\mathbb{Q}}\longrightarrow\overline{\mathbb{Q}}^\times$ of finite order, we denote by W_η the underlying space of $\rho\otimes\eta$, by $E_\eta\subseteq\overline{\mathbb{Q}}$ its coefficient field and by $H_\eta\subseteq\overline{\mathbb{Q}}$ the Galois extension cut out by $\rho\otimes\eta$. Our fixed embedding ι_∞ allows us to see $\rho\otimes\eta$ as an Artin representation, and we let $L(\rho\otimes\eta,s)$ be its Artin L-function. It is a meromorphic function over $\mathbb{C}$, and it is known that $L(\rho\otimes\eta,1)\in\mathbb{C}^\times$ provided that $\mathrm{H}^0(\mathbb{Q},W_\eta)=0$. Assuming further that η is even, the functional equation implies that $L((\rho\otimes\eta)^\vee,s)$ has a zero of order $d^+=\dim\mathrm{H}^0(\mathbb{R},W)$ at $s=0$. The non-abelian Stark's conjecture describes the transcendental part of its leading term $L^*((\rho\otimes\eta)^\vee,0)$ at $s=0$ as follows. Set $\mathrm{H}^1_{\mathrm{f}}((\rho\otimes\eta)^\vee(1)) = \mathrm{Hom}_{G_{\mathbb{Q}}}(W_\eta, E_\eta\otimes_{\mathbb{Z}}\mathcal{O}^\times_{H_\eta})$, where $\mathcal{O}^\times_{H_\eta}$ is the group of units of H_η. It is a E_η-vector space of dimension d^+ by Dirichlet's unit theorem, and it conjecturally coincides with the group of extensions of the trivial motive $E_\eta(0)$ by the arithmetic dual of $\rho\otimes\eta$, which have good reduction everywhere. There is a natural $\mathbb{C}$-linear perfect pairing

$$(1)\qquad \mathbb{C}\otimes\mathrm{H}^0(\mathbb{R},W)\;\times\;\mathbb{C}\otimes\mathrm{H}^1_{\mathrm{f}}((\rho\otimes\eta)^\vee(1)) \longrightarrow \mathbb{C}\otimes\left(\iota_\infty^{-1}(\mathbb{R})^\times\bigcap\mathcal{O}^\times_{H_\eta}\right)\xrightarrow{1\otimes\log_\infty}\mathbb{C},$$

where $\log_\infty : \overline{\mathbb{Q}}^\times \longrightarrow \mathbb{R}$ is given by $\log_\infty(a) = -\log|\iota_\infty(a)|$ and where log is the usual real logarithm. Any choice of bases $\omega_\infty^+ \in \det_E \mathrm{H}^0(\mathbb{R},W)$ and $\omega_{\mathrm{f},\eta}\in\det_{E_\eta}\mathrm{H}^1_{\mathrm{f}}((\rho\otimes\eta)^\vee(1))$ defines a complex regulator $\mathrm{Reg}_{\omega_\infty^+}(\rho\otimes\eta)\in\mathbb{C}^\times$ and non-abelian Stark's conjecture [Sta75] implies

$$(2)\qquad \frac{L^*((\rho\otimes\eta)^\vee,0)}{\mathrm{Reg}_{\omega_\infty^+}(\rho\otimes\eta)}\overset{?}{\in}E_\eta^\times,\qquad\text{or, equivalently,}\qquad \frac{L(\rho\otimes\eta,1)}{(i\pi)^{d^-}\,\mathrm{Reg}_{\omega_\infty^+}(\rho\otimes\eta)}\overset{?}{\in}E_\eta^\times,$$

where $d^- = d - d^+$.

Fix once and for all a prime number p as well as an isomorphism $j:\mathbb{C}\simeq\overline{\mathbb{Q}}_p$ which satisfies $\iota_p = j\circ\iota_\infty$. Letting E_p be the completion of $\iota_p(E)$ inside $\overline{\mathbb{Q}}_p$, one may see ρ as a p-adic representation by putting $W_p = W\otimes_{E,\iota_p}E_p$. We will call a p-stabilization of W_p any $G_{\mathbb{Q}_p}$-stable linear subspace $W_p^+\subseteq W_p$ of dimension d^+, where $G_{\mathbb{Q}_p}$ is the local Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. Any p-stabilization (W_p^+,ρ^+) yields a p-adic analogue of the complex pairing (1) by considering

$$(3)\qquad \overline{\mathbb{Q}}_p\otimes W_p^+\;\times\;\overline{\mathbb{Q}}_p\otimes\mathrm{H}^1_{\mathrm{f}}((\rho\otimes\eta)^\vee(1))\longrightarrow\overline{\mathbb{Q}}_p\otimes\mathcal{O}^\times_{H_\eta}\xrightarrow{1\otimes\log_p}\overline{\mathbb{Q}}_p,$$

where $\log_p:\overline{\mathbb{Q}}^\times\longrightarrow\overline{\mathbb{Q}}_p$ is the composition of Iwasawa's p-adic logarithm with the embedding ι_p. A p-stabilization W_p^+ is said to be *η-admissible* if the $\overline{\mathbb{Q}}_p$-linear pairing (3) is perfect, and it is simply called *admissible* when η is the trivial character $\mathbb{1}$. A p-stabilization might *a priori* be η-admissible or not, but in the special case where W_p^+ is *motivic*, *i.e.*, when it admits a E-rational structure, its η-admissibility follows from standard conjectures in p-adic transcendence theory (and from a theorem of Brumer when $d^+=1$, see Section 3.1). Given a basis $\omega_p^+\in\det_{E_p}W_p^+$ of any p-stabilization W_p^+, we may as well define a p-adic regulator $\mathrm{Reg}_{\omega_p^+}(\rho\otimes\eta)\in\overline{\mathbb{Q}}_p$ which vanishes precisely when its η-admissibility fails. Furthermore, the quantity

$$(4)\qquad \frac{\mathrm{Reg}_{\omega_p^+}(\rho\otimes\eta)}{j\left(\mathrm{Reg}_{\omega_\infty^+}(\rho\otimes\eta)\right)}\in\overline{\mathbb{Q}}_p$$

turns out to be independent of the choice of $\omega_{\mathrm{f},\eta}$, and it is well-defined up to multiplication by the same non-zero element of E_p for all the quotients (for varying even characters η). The ambiguity can even be reduced to a unit of the ring of integers $\mathcal{O}_p$ of E_p (and of $\mathcal{O}_p\bigcap E$ when W_p^+ is motivic) once we fix a Galois-stable $\mathcal{O}_p$-lattice T_p of W_p, if we ask ω_∞^+ and ω_p^+ to be *T_p-optimal*. That is, they should be respective $\mathcal{O}_p$-bases of $\mathrm{H}^0(\mathbb{R},T_p)$ and $T_p^+ = W_p^+\bigcap T_p$.

Let Γ be the Galois group of the $\mathbb{Z}_p$-cyclotomic extension $\mathbb{Q}_\infty = \cup_n\mathbb{Q}_n$ of $\mathbb{Q}$ and let $\widehat{\Gamma}\subseteq\mathrm{Hom}(\Gamma,\overline{\mathbb{Q}}^\times)$ be the set of $\overline{\mathbb{Q}}$-valued characters of Γ of finite order. Via ι_∞, the elements of $\widehat{\Gamma}$ correspond to (necessarily even) Dirichlet characters of p-power order and conductor, and via ι_p they become p-adic characters and may be seen as homomorphisms of $\mathcal{O}_p$-algebras

$\Lambda \longrightarrow \overline{\mathbb{Q}}_p$, where $\Lambda = \mathcal{O}_p[[\Gamma]]$ is the Iwasawa algebra. We conjecture that the special values $L^*((\rho\otimes\eta)^\vee, 0)$ for $\eta \in \widehat{\Gamma}$, suitably corrected by the quotient of regulators (4), can be p-adically interpolated by a p-adic measure which generates the characteristic ideal over Λ of the Pontryagin dual $X_\infty(\rho,\rho^+)$ of the Greenberg-style Selmer group

$$\ker\left[\mathrm{H}^1(\mathbb{Q}_\infty, D_p) \longrightarrow \mathrm{H}^1(\mathbb{Q}^{\mathrm{ur}}_{p,\infty}, D_p^-) \times \prod_{\ell\neq p} \mathrm{H}^1(\mathbb{Q}^{\mathrm{ur}}_{\ell,\infty}, D_p)\right],$$

where D_p (resp. D_p^-) is the divisible $\mathcal{O}_p$-module W_p/T_p (resp. $D_p/(\operatorname{im} W_p^+ \to D_p)$) and where $\mathbb{Q}^{\mathrm{ur}}_{\ell,\infty} \subseteq \overline{\mathbb{Q}}_\ell$ is the maximal unramified extension of the completion of $\mathbb{Q}_\infty$ along ι_ℓ. We expect $X_\infty(\rho,\rho^+)$ to be of Λ-torsion and we will denote by $\operatorname{char}_\Lambda X_\infty(\rho,\rho^+)$ its characteristic ideal. By convention, we let $\operatorname{char}_\Lambda X_\infty(\rho,\rho^+) = 0$ if $X_\infty(\rho,\rho^+)$ is not a torsion module.

Conjecture A. *Let p be an odd prime at which ρ is unramified. Fix a $G_\mathbb{Q}$-stable $\mathcal{O}_p$-lattice T_p of W_p and a p-stabilization (ρ^+, W_p^+) of W_p. Pick any T_p-optimal bases ω_p^+ and ω_∞^+ as before.*

EX$_{\rho,\rho^+}$ *There exists an element θ_{ρ,ρ^+} of the fraction field of Λ which has no pole outside the trivial character and which satisfies the following interpolation property: for all non-trivial characters $\eta \in \widehat{\Gamma}$ of exact conductor p^n, one has*

$$\eta(\theta_{\rho,\rho^+}) = \frac{\tau(\eta)^{d^-}}{\tau(\rho\otimes\eta)} \frac{\mathrm{Reg}_{\omega_p^+}(\rho\otimes\eta)}{\det(\rho^-)(p^n)} \frac{L^*\left((\rho\otimes\eta)^\vee, 0\right)}{\mathrm{Reg}_{\omega_\infty^+}(\rho\otimes\eta)},$$

where $\tau(-)$ is a Galois-Gauss sum and where $\det(\rho^-)$ is the Dirichlet character corresponding to the determinant of the Galois representation acting on $W_p^- = W_p/W_p^+$.

IMC$_{\rho,\rho^+}$: *The statement **EX**$_{\rho,\rho^+}$ holds, and θ_{ρ,ρ^+} is a generator of $\operatorname{char}_\Lambda X_\infty(\rho,\rho^+)$.*

EZC$_{\rho,\rho^+}$: *Let e be the dimension of $W_p^{-,0} = \mathrm{H}^0(\mathbb{Q}_p, W_p^-)$ and assume that ρ^+ is admissible. Then **EX**$_{\rho,\rho^+}$ holds, θ_{ρ,ρ^+} vanishes at $\mathbb{1}$ with multiplicity at least e, and one has*

$$\frac{1}{e!}\frac{\mathrm{d}^e}{\mathrm{d}s^e}\kappa^s(\theta_{\rho,\rho^+})\Big|_{s=0} = \tau(\rho)^{-1}\,(-1)^e\,\mathcal{L}(\rho,\rho^+)\,\mathcal{E}(\rho,\rho^+)\,\mathrm{Reg}_{\omega_p^+}(\rho)\,\frac{L^*\left(\rho^\vee,0\right)}{\mathrm{Reg}_{\omega_\infty^+}(\rho)},$$

where $\kappa^s \in \mathrm{Hom}_{\mathcal{O}-alg}(\Lambda, \overline{\mathbb{Q}}_p)$ is the homomorphism induced by the character $\langle\chi_{\mathrm{cyc}}\rangle : \Gamma \simeq 1 + p\mathbb{Z}_p \subseteq \overline{\mathbb{Q}}_p^\times$, raised to the power $s \in \mathbb{Z}_p$, where $\mathcal{L}(\rho,\rho^+) \in \overline{\mathbb{Q}}_p$ is the $\mathcal{L}$-invariant defined in Section 3.7, and where $\mathcal{E}(\rho,\rho^+)$ is a modified Euler factor given (in terms of the arithmetic Frobenius σ_p at p) by

$$\mathcal{E}(\rho,\rho^+) = \det(1 - p^{-1}\sigma_p | W_p^+)\det(1 - \sigma_p^{-1} | W_p^-/W_p^{-,0}).$$

Note that Conjecture A satisfies an obvious p-adic Artin formalism (Remark 3.9.2). Its truth is clearly independent from the choice of ω_∞^+ and ω_p^+, but its dependence on the choice of ρ^+ is somewhat subtle. Further, Conjecture A will follow immediately from Theorems A and B below in the hypothetical (but less interesting) case where $X_\infty(\rho,\rho^+)$ is not of Λ-torsion, giving $\theta_{\rho,\rho^+} = 0$.

We refer the reader to Section 6 for a detailed comparison of Conjecture A with various "main conjectures" and "p-adic Stark conjectures" that are already available in a multitude of special settings.

We now outline the main results of this paper.

1.2. **The main results.** We fix once and for all an odd prime p at which ρ is unramified. Our first theorem generalizes [Mak20, Théorème A] and yields an interpolation formula for a generator of $\mathrm{char}_\Lambda X_\infty(\rho,\rho^+)$ which is similar to $\mathbf{EX}_{\rho,\rho^+}$. As a piece of notation, we let $U_\infty = \varprojlim_n U_n$ be the Λ-module of all norm-coherent sequences of elements in the pro-p completion U_n of the group of units of $H\cdot\mathbb{Q}_n$.

Theorem A (=Theorem 3.6.4). *Fix a $G_\mathbb{Q}$-stable $\mathcal{O}_p$-lattice T_p of W_p, a p-stabilization ρ^+ of ρ and a T_p-optimal basis $\omega_p^+ = t_1\wedge\ldots\wedge t_{d^+}$ of W_p^+. The following assertions are equivalent:*

(1) the Selmer group $X_\infty(\rho,\rho^+)$ is of Λ-torsion,
(2) ρ^+ is η-admissible for some non-trivial character $\eta\in\widehat{\Gamma}$, and
(3) ρ^+ is η-admissible for all but finitely many characters $\eta\in\widehat{\Gamma}$.

Moreover, if these three equivalent conditions hold and if $d^+>0$, then there exist d^+ linearly independent $G_\mathbb{Q}$-equivariant homomorphisms $\Psi_1,\ldots,\Psi_{d^+}: T_p\longrightarrow U_\infty$ which only depend on T_p, and there exists a generator $\theta^{alg}_{\rho,\rho^+}$ of $char_\Lambda X_\infty(\rho,\rho^+)$ such that

$$\eta(\theta^{alg}_{\rho,\rho^+}) = \frac{\tau(\eta)^{d^-}}{\tau(\rho\otimes\eta)}\cdot\frac{p^{(n-1)\cdot d^+}}{\det(\rho^-)(p^n)}\cdot\det\left(\log_p|\Psi_j(t_i)|_\eta\right)_{1\le i,j\le d^+}$$

for all non-trivial characters $\eta\in\widehat{\Gamma}$ of conductor p^n, where $|\cdot|_\eta: U_\infty\longrightarrow U_{n-1}$ stands for a certain "η-projection" introduced in Section 3.6.

Theorem A implies that $X_\infty(\rho,\rho^+)$ is of Λ-torsion when $d^+=1$ and ρ^+ is motivic. Another simple consequence when $d^+=d$ (*i.e.*, when ρ is even) is the validity of the "$\rho\otimes\eta$-isotypic component of Leopoldt's conjecture for H_η and p" for all but finitely many $\eta\in\widehat{\Gamma}$ (see Section 3.1 for a precise statement of this conjecture). But of course, this does not suffice to prove the Leopoldt's conjecture for any of the fields H_η. The proof of Theorem A makes a critical use of Coleman's theory [Col79] and of classical results on the structure of U_∞. We first compare $X_\infty(\rho,\rho^+)$ with a Bloch-Kato-style Selmer group and we apply Poitou-Tate duality theorem. Some limits of unit groups and of ideal class groups will appear via an application of Hochschild-Serre's exact sequence. We then use classical results of Kuz'min [Kuz72] and Belliard [Bel02] on the Galois structure of global (p-)units, and also a construction of modified Coleman maps in order to recover some information on the structure of $X_\infty(\rho,\rho^+)$ as a Λ-module.

The Tamagawa number conjecture of Bloch and Kato [BK90] expresses special values of motivic L-functions in terms of arithmetic invariants. The following theorem is an instance of how one can tackle Bloch-Kato's conjecture via Iwasawa theory.

Theorem B (=Corollary 3.8.4). *Fix a $G_\mathbb{Q}$-stable $\mathcal{O}_p$-lattice T_p of W_p. Then $X_\infty(\rho,\rho^+)$ is of Λ-torsion for every admissible p-stabilization ρ^+ of ρ such that the $\mathcal{L}$-invariant $\mathcal{L}(\rho,\rho^+)$ is non-zero. Moreover, if there exists such a ρ^+ for which Conjecture A also holds, then the p-part of the Tamagawa Number Conjecture (in the formulation of Fontaine and Perrin-Riou* [FPR94]*) for ρ is valid. In particular, for p not dividing the order of the image of ρ, one has*

$$\frac{L^*\left(\rho^\vee,0\right)}{\mathrm{Reg}_{\omega^+_\infty}(\rho)}\sim_p \#\mathrm{Hom}_{\mathcal{O}_p[G_\mathbb{Q}]}(T_p,\mathcal{O}_p\otimes C\ell(H)),$$

where $a\sim_p b$ means that a and b are equal up to a p-adic unit, and where $C\ell(H)$ is the ideal class group of the field H cut out by ρ. Here, the bases ω_f and ω^+_∞ used to compute $\mathrm{Reg}_{\omega^+_\infty}(\rho)$ are chosen to be T_p-optimal.

The proof of Theorem B rests upon a comparison between $X_\infty(\rho,\rho^+)$ and Benois' definition of the Perrin-Riou's module of p-adic L-functions. The latter is defined in [Ben14] in terms of a Selmer complex attached to certain crystalline motives endowed with a "regular submodule" (see Section 3.8 for its definition). For the dual motive of ρ, the choice of a regular submodule amounts to the one of an admissible p-stabilization of ρ. Once the relation between $X_\infty(\rho,\rho^+)$ and Perrin-Riou's theory is fully established, Theorem B will follow from the main result of *loc. cit.*.

Our last result compares Conjecture A in the case where ρ is induced from a non-trivial character $\chi : G_k \longrightarrow E^\times$ of prime-to-p order with Iwasawa-theoretic conjectures of [BKS17, §3-4]. In this case, θ_{ρ,ρ^+} should interpolate leading terms of abelian L-functions for which Rubin formulated a Stark conjecture "over $\mathbb{Z}$" [Rub96]. It involves some Rubin-Stark elements $(\varepsilon_n^\chi)_n$ and u^χ for which we will assume the conjecture "over E" ($\mathbf{RS}_{\chi,E}$, Conjecture 4.1.3), so that the statement (2) holds. Under the conjecture "over $\mathcal{O}_p$" ($\mathbf{RS}_{\chi,p}$, Conjecture 4.1.4), which we will also assume, one may formulate a "higher rank Iwasawa Main Conjecture" for $\varepsilon_\infty^\chi = \varprojlim_n \varepsilon_n^\chi$ ($\mathbf{IMC}_\chi$, Conjecture 4.2.1) which generalizes the usual main conjecture, and an "Iwasawa-theoretic Mazur-Rubin-Sano conjecture" ($\mathbf{MRS}_\chi$, Conjecture 4.2.3) which connects u^χ and the bottom layer of ε_∞^χ.

Theorem C (=Theorems 5.3.1, 5.3.3 and 5.4.3)**.** *Suppose that $\rho = \mathrm{Ind}_k^{\mathbb{Q}}\chi$ and let $T_p = \mathrm{Ind}_k^{\mathbb{Q}}\mathcal{O}_p(\chi)$ be the "standard" lattice in W_p. Assume the Rubin-Stark conjectures $\boldsymbol{RS}_{\chi,E}$ and $\boldsymbol{RS}_{\chi,p}$.*

(1) $\boldsymbol{EX}_{\rho,\rho^+}$ holds true for every p-stabilization ρ^+.
(2) For every p-stabilization ρ^+ such that $X_\infty(\rho,\rho^+)$ is of Λ-torsion, $\boldsymbol{IMC}_{\rho,\rho^+}$ is equivalent to $\boldsymbol{IMC}_\chi$.
(3) $\boldsymbol{MRS}_\chi$ implies $\boldsymbol{EZC}_{\rho,\rho^+}$ for every p-stabilization ρ^+. Conversely, if $\boldsymbol{EZC}_{\rho,\rho^+}$ holds for every p-stabilization ρ^+, and if there exists at least one admissible p-stabilization ρ^+ such that $\mathcal{L}(\rho,\rho^+) \neq 0$, then $\boldsymbol{MRS}_\chi$ is valid.

Let us comment on the proof of Theorem C. The first claim mainly follows from the machinery of refined Coleman maps. Our second claim is much in the spirit of theorems which compare a main conjecture “with p-adic L-functions” with a main conjecture “without p-adic L-functions”. The main idea behind its proof is the use of "extended Coleman maps" as introduced in Section 2, together with a variant of the description of $X_\infty(\rho,\rho^+)$ used for Theorems A and B. For the last claim, we first need to compute the constant term of our extended Coleman maps. Roughly speaking, Conjecture $\mathbf{MRS}_\chi$ is an equality between elements in a vector space of dimension $\binom{d^++f}{d^+}$, and we reinterpret (a slightly stronger version of) $\mathbf{EZC}_{\rho,\rho^+}$ as the same equality, after having applied a certain linear form attached to ρ^+. To achieve the proof of the last claim, we then prove in Section 3.9 that the validity of $\mathbf{EZC}_{\rho,\rho^+}$ for enough choices of ρ^+ implies $\mathbf{EZC}_{\rho,\rho^+}$ for all ρ^+. We also would like to mention a side result which might be interesting in itself: we show that the family of Rubin-Stark elements ε_∞^χ is non-zero if there exists a non-trivial character $\eta \in \widehat{\Gamma}$ such that the $\chi\otimes\eta$-isotypic component of Leopoldt's conjecture holds for H_η and p. Lastly, one should be able to remove the mild hypothesis on the order of χ, which is mainly used to ease some algebraic computations.

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2. Coleman Maps

2.1. **Classical results on Coleman maps.** Fix an odd prime number p. Let K be an unramified finite extension of $\mathbb{Q}_p$ and let $\varphi \in \mathrm{Gal}(K/\mathbb{Q}_p)$ be the Frobenius automorphism. The Galois group $\mathrm{Gal}(K(\mu_{p^\infty})/\mathbb{Q}_p)$ splits into a product $\mathrm{Gal}(K/\mathbb{Q}_p)\times\Gamma_{\mathrm{cyc}} \simeq \mathrm{Gal}(K/\mathbb{Q}_p)\times\Gamma\times$

$\mathrm{Gal}(K(\mu_p)/K)$, where we have put $\Gamma_{\mathrm{cyc}} = \mathrm{Gal}(K(\mu_{p^\infty})/K)$ and where Γ is the Galois group of $\mathbb{Z}_p$-cyclotomic extension of K. The cyclotomic character induces an isomorphism $\chi_{\mathrm{cyc}} : \Gamma_{\mathrm{cyc}} \simeq \mathbb{Z}_p^\times$ and Γ can be identified with $1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$. Let us fix once and for all a system of compatible roots of unity $\tilde{\zeta} := (\zeta_n)_{n\geq 0}$ of p-power order (*i.e.*, $\zeta_{n+1}^p = \zeta_n$, $\zeta_0 = 1$ and $\zeta_1 \neq 1$). Note that $\tilde{\zeta}$ is also norm-coherent, which means that it lives in the inverse limit $\varprojlim_n K(\mu_{p^n})^\times$ where the transition maps are the local norms. Coleman proved the following theorem (see [Col79, Theorem A] or [dS87, Chapter I, §2]).

Theorem 2.1.1 (Coleman)**.** *Let $v = (v_n)_{n\geq 0} \in \varprojlim_n K(\mu_{p^n})^\times$. There exists a unique power series $f_v(T) \in T^{\mathrm{ord}(v_0)} \cdot \mathcal{O}_K[[T]]^\times$, called* Coleman's power series of v*, which satisfies the following properties:*

- *for all $n \geq 1$, one has $f_v(\zeta_n - 1) = \varphi^n(v_n)$,*
- *$\prod_{i=0}^{p-1} f_v(\zeta_1^i(1+T)-1) = \varphi(f_v)((1+T)^p - 1)$, where we let φ act on the coefficients of f_v.*

Furthermore, the map $v \mapsto f_v(T)$ is multiplicative and it is compatible with the action of Γ_{cyc}, i.e., one has $f_{\gamma(v)}(T) = f_v((1+T)^{\chi_{\mathrm{cyc}}(\gamma)} - 1)$ for all $\gamma \in \Gamma_{\mathrm{cyc}}$.

Note that $f_v(0)$ and v_0 are not equal but are simply related by the formula $v_0 = (1 - \varphi^{-1}) f_v(0)$ which is easily deduced from the second property satisfied by f_u. To better understand Coleman's power series, we consider the operator $\mathcal{L} : \mathcal{O}_K[[T]]^\times \longrightarrow K[[T]]$ given by

$$\mathcal{L}(f)(T) = \frac{1}{p} \log_p \left(\frac{f(T)^p}{\varphi(f)((1+T)^p - 1)} \right),$$

where the p-adic logarithm $\log_p : \mathcal{O}_K[[T]]^\times \longrightarrow K[[T]]$ is defined by $\log_p(\zeta) = 0$ for $\zeta \in \mu(K)$ and $\log_p(1 + Tg(T)) = Tg(T) - \frac{1}{2}T^2 g(T)^2 + \frac{1}{3}T^3 g(T)^3 + \ldots$ for $g(T) \in \mathcal{O}_K[[T]]$. Moreover, it is known that $\mathcal{L}$ takes values in $\mathcal{O}_K[[T]]$.

We can now define what is usually referred to as the Coleman map. One can associate to any $\mathcal{O}_K$-valued measure λ over $\mathbb{Z}_p$ its Amice transform $\mathcal{A}_\lambda(T) \in \mathcal{O}_K[[T]]$ given by

$$\mathcal{A}_\lambda(T) = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \binom{x}{n} T^n \lambda(x) = \int_{\mathbb{Z}_p} (1+T)^x \lambda(x).$$

This construction yields an isomorphism of $\mathcal{O}_K$-algebras (the product of measures being the convolution product) $\mathcal{O}_K[[\mathbb{Z}_p]] \simeq \mathcal{O}_K[[T]]$. When the Amice transform of a measure λ is equal to $\mathcal{L}(f_u(T))$ for some norm-coherent sequence of *units* $u \in \varprojlim_n \mathcal{O}_{K(\mu_{p^n})}^\times$, then one can show that λ is actually the extension by zero of a measure over $\mathbb{Z}_p^\times$. It will be convenient to see such a λ as a measure over Γ_{cyc} after taking a pull-back by χ_{cyc}. Let

$$\mathrm{Col} : \varprojlim_n \mathcal{O}_{K(\mu_{p^n})}^\times \longrightarrow \mathcal{O}_K[[\Gamma_{\mathrm{cyc}}]]$$

be the map sending u to the $\mathcal{O}_K$-valued measure λ over Γ_{cyc} whose Amice transform is $\mathcal{L}(f_u(T))$. Note that $\mathrm{Col}(uu') = \mathrm{Col}(u) + \mathrm{Col}(u')$ for norm-coherent sequences of units u and u'.

Proposition 2.1.2. *There is an exact sequence of $\mathrm{Gal}(K/\mathbb{Q}_p) \times \Gamma_{\mathrm{cyc}}$-modules*

$$0 \longrightarrow \mu(K) \times \mathbb{Z}_p(1) \longrightarrow \varprojlim_n \mathcal{O}_{K(\mu_{p^n})}^\times \xrightarrow{\mathrm{Col}} \mathcal{O}_K[[\Gamma_{\mathrm{cyc}}]] \longrightarrow \mathbb{Z}_p(1) \longrightarrow 0$$

Here, the first map sends $\xi \in \mu(K)$ to $(\xi)_n$ and $a \in \mathbb{Z}_p$ to $(\zeta_n^a)_n$, and the last one sends λ to $\mathrm{Tr}_{K/\mathbb{Q}_p} \int_{\Gamma_{\mathrm{cyc}}} \chi_{\mathrm{cyc}}(\sigma)\lambda(\sigma)$. Moreover, Γ_{cyc} acts on $\mathcal{O}_K[[\Gamma_{\mathrm{cyc}}]]$ via $\int_{\Gamma_{\mathrm{cyc}}} f(\sigma)(\gamma \cdot \lambda)(\sigma) = \int_{\Gamma_{\mathrm{cyc}}} f(\gamma\sigma)\lambda(\sigma)$ for all $\gamma \in \Gamma_{\mathrm{cyc}}$ and all $\mathbb{Q}_p$-valued continuous function f on Γ_{cyc}.

Proof. See [CS06, Theorem 3.5.1]. □

Now take invariants by $\Delta := \mathrm{Gal}(K(\mu_p)/K)$. As Δ is of prime-to-p order, one can identify $\mathcal{O}_K[[\Gamma_{\mathrm{cyc}}]]^\Delta$ with $\mathcal{O}_K[[\Gamma]]$ via the map sending λ to the measure μ defined by $\int_\Gamma g(\gamma)\mu(\gamma) = \frac{1}{p-1}\int_{\Gamma_{\mathrm{cyc}}} g(\sigma \bmod \Delta)\lambda(\sigma)$ for all continuous $g : \Gamma \longrightarrow \mathbb{Q}_p$. The inverse map is given by the formula $\int_{\Gamma_{\mathrm{cyc}}} f(\sigma)\lambda(\sigma) = \int_\Gamma Tf(\gamma)\mu(\gamma)$, where $f : \Gamma_{\mathrm{cyc}} \longrightarrow \mathbb{Q}_p$ is continuous and where $Tf : \Gamma \longrightarrow \mathbb{Q}_p$ is the sum $Tf(\gamma) = \sum_{\delta\in\Delta} f(\gamma\delta)$. Let us denote by $K_n = K(\mu_{p^{n+1}})^\Delta$ the n-th layer of the $\mathbb{Z}_p$-cyclotomic extension of K for all $n \geq 0$, and let $K_{-1} = K_0 = K$. The Coleman map thus restricts to a surjective map $\mathrm{Col} : \varprojlim_n \mathcal{O}_{K_{n-1}}^\times \longrightarrow \mathcal{O}_K[[\Gamma]]$ whose kernel is identified with $\mu(K)$. By restricting further to principal units $\mathcal{O}_{K_n}^{\times,1} \subseteq \mathcal{O}_{K_n}^\times$ (*i.e.*, units that are congruent to 1 modulo the maximal ideal of $\mathcal{O}_{K_n}$), one obtains an isomorphism

$$\mathrm{Col} : \varprojlim_n \mathcal{O}_{K_{n-1}}^{\times,1} \xrightarrow{\simeq} \mathcal{O}_K[[\Gamma]] \tag{5}$$

of $\mathbb{Z}_p[\mathrm{Gal}(K/\mathbb{Q}_p)][[\Gamma]]$-modules (or, equivalently, of φ-linear $\mathbb{Z}_p[[\Gamma]]$-modules) whose properties are now recalled. For $n \geq 2$, one may see via ι_p any non-trivial $\overline{\mathbb{Q}}$-valued Dirichlet character η of p-power order and of conductor p^n as a p-adic character of $\Gamma_{n-1} = \mathrm{Gal}(K_{n-1}/K)$ which does not factor through Γ_{n-2}. We will denote by $e_\eta := p^{1-n}\sum_{g\in\Gamma_{n-1}} \eta(g^{-1})g \in \mathbb{Q}_p(\mu_{p^n})[\Gamma_{n-1}]$ the idempotent attached to η, and by $\mathfrak{g}(\eta) = \sum_{a \bmod p^n} \eta(a)\zeta_n^a$ its usual Gauss sum.

Lemma 2.1.3. *Let $u = (u_n)_{n\geq1} \in \varprojlim_n \mathcal{O}_{K_{n-1}}^{\times,1}$ and let $\mu = \mathrm{Col}(u)$. For all non-trivial characters $\eta : \Gamma \longrightarrow \overline{\mathbb{Q}}^\times$ of conductor p^n, one has*

$$\int_\Gamma \eta(\gamma)\mu(\gamma) = \frac{p^{n-1}}{\mathfrak{g}(\eta^{-1})}\cdot\varphi^n\left(e_\eta \log_p(u_n)\right).$$

Moreover, if $u_0 \neq 1$, then $u_0 \notin 1 + p\mathbb{Z}_p$ and one has

$$\int_\Gamma \mu = \frac{1-p^{-1}\varphi}{1-\varphi^{-1}}\left(\log_p(u_1)\right).$$

Proof. When η is non-trivial, one checks that

$$\begin{aligned}
\mathfrak{g}(\eta^{-1})\int_\Gamma \eta(\gamma)\mu(\gamma) &= (p-1)^{-1}\sum_{a \bmod p^n} \eta^{-1}(a)\cdot\mathcal{L}(f_u)(\zeta_n^a - 1)\\
&= p^{n-1}\cdot e_\eta\,\mathcal{L}(f_u)(\zeta_n - 1)\\
&= p^{n-1}\cdot e_\eta\,\varphi^n\left(\log_p(u_n) - \frac{1}{p}\log_p(u_{n-1})\right).
\end{aligned}$$

Since η is of conductor p^n, the idempotent e_η kills u_{n-1}, so the first equality holds. Assume now that $u_0 \neq 1$. If $u_0 \in 1 + p\mathbb{Z}_p$, then $u_0^{[K:\mathbb{Q}_p]} = N_{K/\mathbb{Q}_p}(u_0)$ would be a universal norm in $\mathbb{Q}_{p,\infty}/\mathbb{Q}_p$. As it is also a principal unit, the exact sequence (6) below shows that it should be equal to 1, and so does u_0. Hence $u_0 \notin 1 + p\mathbb{Z}_p$, and we have

$$\int_\Gamma \mu = (p-1)^{-1}\cdot\mathcal{L}(f_u)(0) = (p-1)^{-1}\cdot(1-p^{-1}\varphi)\log_p(f_u(0)) = \frac{1-p^{-1}\varphi}{1-\varphi^{-1}}\left(\log_p(u_1)\right),$$

because of the relation $u_1^{p-1} = u_0 = (1-\varphi^{-1})f_u(0)$. □

2.2. **Isotypic components.** Let $\delta : G \longrightarrow \mathcal{O}^\times$ be any character of a finite group G which takes values in a finite flat extension $\mathcal{O}$ of $\mathbb{Z}_p$. There are two slightly different notions of δ-isotypic components for an $\mathcal{O}[G]$-module M, namely

$$M^\delta := \{m \in M \mid \forall g \in G,\ g.m = \delta(g)m\}, \quad \text{or,} \quad M_\delta := M \otimes_{\mathcal{O}[G]} \mathcal{O},$$

where the ring homomorphism $\mathcal{O}[G] \longrightarrow \mathcal{O}$ is the one induced by δ. The modules M^δ and M_δ are respectively the largest submodule and the largest quotient of M on which G acts via δ (see [Tsu99, §2]) and they will simply be called the δ-part and the δ-quotient of M. When p does not divide the order of G, the natural map $M^\delta \longrightarrow M_\delta$ is an isomorphism, and M^δ equals $e_\delta M$, where $e_\delta = \#(G)^{-1}\sum_{g\in G}\delta(g^{-1})g \in \mathcal{O}[G]$ is the usual idempotent attached to δ.

When $\mathcal{O}$ contains $\mathcal{O}_K$ and for $G = \mathrm{Gal}(K/\mathbb{Q}_p)$, $M = \mathcal{O}_K \otimes_{\mathbb{Z}_p} \mathcal{O}$ as in Section 2.1, the same argument as in the proof of [Mak20, Lemme 3.2.3] shows that the internal multiplication $M \longrightarrow \mathcal{O}$ induces by restriction an $\mathcal{O}$-linear isomorphism $M^\delta \simeq \mathcal{O}$. When one moreover fixes an isomorphism $\mathcal{O}_K \simeq \mathbb{Z}_p[G]$ (given by a normal integral basis of the unramified extension $K/\mathbb{Q}_p$) one also has $M_\delta \simeq \mathcal{O}[G] \otimes_{\mathcal{O}[G]} \mathcal{O} = \mathcal{O}$.

Definition 2.2.1. For any character δ of $G = \mathrm{Gal}(K/\mathbb{Q}_p)$ with values in a finite flat $\mathbb{Z}_p$-algebra $\mathcal{O}$ containing $\mathcal{O}_K$, we define δ-isotypic component of the Coleman map as to be the composite isomorphisms of $\mathcal{O}[[\Gamma]]$-modules

$$\mathrm{Col}^\delta : \varprojlim_n \left(\mathcal{O}^{\times,1}_{K_{n-1}} \otimes_{\mathbb{Z}_p} \mathcal{O}\right)^\delta \xrightarrow{\mathrm{Col}} \left(\mathcal{O}_K \otimes_{\mathbb{Z}_p} \mathcal{O}\right)^\delta[[\Gamma]] \simeq \mathcal{O}[[\Gamma]],$$

where Col is the restriction of isomorphism (5) to the δ-parts and where the last isomorphism is induced by the internal multiplication $\mathcal{O}_K \otimes_{\mathbb{Z}_p} \mathcal{O} \longrightarrow \mathcal{O}$.

When δ is trivial, it will be later convenient to consider a natural extension of the map $\mathrm{Col}^{\mathbb{1}}$ which we construct in the rest of the paragraph. Here, we may assume that $K = \mathbb{Q}_p$ and that $\mathcal{O} = \mathbb{Z}_p$, and we let $\widehat{\mathbb{Q}}^\times_{p,n}$ be the pro-p completion of $\mathbb{Q}^\times_{p,n}$ for all $n \geq 0$. Concretely, once we fix a uniformizer ϖ_n of $\mathbb{Z}_{p,n} := \mathcal{O}_{\mathbb{Q}_{p,n}}$, it is isomorphic to $\mathbb{Z}^{\times,1}_{p,n} \times \varpi_n^{\mathbb{Z}_p}$, while $\mathbb{Q}^\times_{p,n} = \mathbb{Z}^\times_{p,n} \times \varpi_n^{\mathbb{Z}}$. Let $\mathcal{A} := \ker\left(\mathbb{Z}_p[[\Gamma]] \longrightarrow \mathbb{Z}_p\right)$ be the augmentation ideal of the Iwasawa algebra $\mathbb{Z}_p[[\Gamma]]$. We still denote by γ the image of an element $\gamma \in \Gamma$ under the canonical injection $\Gamma \hookrightarrow \mathcal{O}[[\Gamma]]^\times$.

Lemma 2.2.2. *The multiplication map $m : \mathcal{A} \otimes_{\mathbb{Z}_p[[\Gamma]]} \varprojlim_n \widehat{\mathbb{Q}}^\times_{p,n-1} \longrightarrow \varprojlim_n \widehat{\mathbb{Q}}^\times_{p,n-1}$ is injective and has image $\varprojlim_n \mathbb{Z}^{\times,1}_{p,n-1}$.*

Proof. We first check that m is injective and we fix a topological generator γ of Γ. As $\mathcal{A}$ is $\mathbb{Z}_p[[\Gamma]]$-free and generated by $\gamma - 1$, any element of $\mathcal{A} \otimes_{\mathbb{Z}_p[[\Gamma]]} \varprojlim_n \widehat{\mathbb{Q}}^\times_{p,n-1}$ can be written as a pure tensor $(\gamma-1)\otimes v$ for some v. If $(\gamma-1)\cdot v = 1$, then $v_n \in \widehat{\mathbb{Q}}^\times_p$ for all $n \geq 1$ hence $v = 1$ since $\widehat{\mathbb{Q}}^\times_p$ has no non-trivial p-divisible element. Thus m is injective. Inflation and restriction maps in discrete group cohomology provide an exact sequence

$$0 \longrightarrow \mathrm{H}^1(\Gamma, \mathbb{Q}_p/\mathbb{Z}_p) \xrightarrow{\mathrm{inf}} \mathrm{H}^1(\mathbb{Q}_p, \mathbb{Q}_p/\mathbb{Z}_p) \xrightarrow{\mathrm{res}} \mathrm{H}^1(\mathbb{Q}_{p,\infty}, \mathbb{Q}_p/\mathbb{Z}_p)^\Gamma \longrightarrow \mathrm{H}^2(\Gamma, \mathbb{Q}_p/\mathbb{Z}_p).$$

As Γ is pro-cyclic, the last term vanishes. Moreover, the Galois action on $\mathbb{Q}_p/\mathbb{Z}_p$ being trivial, all the H^1's involved are Hom's and one easily deduces from local class field and from exactness of Pontryagin functor $\mathrm{Hom}(-, \mathbb{Q}_p/\mathbb{Z}_p)$ the following short exact sequence

$$(6) \qquad 0 \longrightarrow \left(\varprojlim_n \widehat{\mathbb{Q}}^\times_{p,n-1}\right)_\Gamma \xrightarrow{v \mapsto v_1} \widehat{\mathbb{Q}}^\times_p \xrightarrow{\mathrm{rec}} \Gamma \longrightarrow 0,$$

where the subscript Γ means that we took Γ-coinvariants and where rec is the local reciprocity map. Since $M_\Gamma = M/\mathcal{A}M$ for any $\mathbb{Z}_p[[\Gamma]]$-module M and since the map $\mathrm{rec}_{|1+p\mathbb{Z}_p}$ is an isomorphism, it follows that both $\varprojlim_n \mathbb{Z}_{p,n-1}^{\times,1}$ and the image of m coincide with the kernel of the map $\varprojlim_n \widehat{\mathbb{Q}}_{p,n-1}^\times \longrightarrow \widehat{\mathbb{Q}}_p^\times$ given by $v \mapsto v_1$, so they are equal. □

The following definition makes sense by Lemma 2.2.2.

Definition 2.2.3. Let $\mathcal{O}$ be any finite flat $\mathbb{Z}_p$-algebra and let $\mathcal{J}_\mathcal{O} \subseteq \mathrm{Frac}(\mathcal{O}[[\Gamma]])$ be the invertible ideal of $\mathcal{O}[[\Gamma]]$ consisting of quotients of p-adic measures on Γ with at most one simple pole at the trivial character. The extended $\mathbb{1}$-isotypic component of the Coleman map is the isomorphism

$$\widetilde{\mathrm{Col}}^{\mathbb{1}} : \varprojlim_n \widehat{\mathbb{Q}}_{p,n-1}^\times \otimes \mathcal{O} \xrightarrow{\simeq} \mathcal{J}_\mathcal{O}$$

given by $\widetilde{\mathrm{Col}}^{\mathbb{1}}(v) = \frac{1}{a}\mathrm{Col}^{\mathbb{1}}(av)$ for any choice of a non-zero element a in the augmentation ideal of $\mathcal{O}[[\Gamma]]$. For any non-trivial $\mathcal{O}$-valued character δ of $\mathrm{Gal}(K/\mathbb{Q}_p)$, we will also let $\widetilde{\mathrm{Col}}^{\delta} = \mathrm{Col}^\delta$.

2.3. **Constant term of Coleman maps.**

We keep the same notations as in Sections 2.1 and 2.2. Fix a finite flat extension $\mathcal{O}$ of $\mathbb{Z}_p$ which contains $\mathcal{O}_K$ and let δ be an $\mathcal{O}$-valued character of $\mathrm{Gal}(K/\mathbb{Q}_p)$. Then δ is the trivial character if and only if $\beta = \delta(\varphi) \in \mathcal{O}^\times$ is equal to 1. When δ is non-trivial, Lemma 2.1.3 shows that the constant term of $\mu = \mathrm{Col}^\delta(u)$ is given by

$$\int_\Gamma \mu = \frac{1-p^{-1}\beta}{1-\beta^{-1}} \log_p(u_1).$$

We now give a similar formula when δ is trivial.

Lemma 2.3.1. *Let $u \in \varprojlim_n \mathbb{Z}_{p,n-1}^{\times,1} \otimes \mathcal{O}$ and put $\mu = \mathrm{Col}^{\mathbb{1}}(u)$. Write u as $(\gamma-1)\cdot v$ for some $\gamma \in \Gamma$ and $v \in \varprojlim_n \widehat{\mathbb{Q}}_{p,n-1}^\times \otimes \mathcal{O}$. Then*

$$\int_\Gamma \mu = \left(1-p^{-1}\right)\cdot \log_p(\chi_{\mathrm{cyc}}(\gamma))\cdot \mathrm{ord}(v_1), \tag{7}$$

where $\mathrm{ord} : \widehat{\mathbb{Q}}_p^\times \otimes_{\mathbb{Z}_p} \mathcal{O} \longrightarrow \mathcal{O}$ is the usual p-valuation map.

Proof. We may take $\mathcal{O} = \mathbb{Z}_p$. Let us choose any $\pi = (\pi_n)_{n\geq 1} \in \varprojlim_n \widehat{\mathbb{Q}}_{p,n-1}^\times$ such that $\pi_1 = p$ (this is possible thanks to the short exact sequence (6)). Then $\pi \in \varprojlim_n \mathbb{Q}_{p,n-1}^\times$ and $f_\pi(T) = T^{p-1}U(T)$ for some $U(T) \in \mathbb{Z}_p[[T]]^\times$, so we have

$$f_{(\gamma-1)\cdot\pi}(T) = \left(\frac{(1+T)^{\chi_{cyc}(\gamma)}-1}{T}\right)^{p-1} \cdot \frac{U((1+T)^{\chi_{cyc}(\gamma)}-1)}{U(T)}.$$

Hence $\log_p(f_{(\gamma-1)\cdot\pi}(0)) = (p-1)\cdot\log_p(\chi_{\mathrm{cyc}}(\gamma))$. If we write v as $u'\cdot\pi^{\mathrm{ord}(v_1)}$ with $u' \in \varprojlim_n \mathbb{Z}_{p,n-1}^{\times,1}$, then one has by linearity

$$\begin{aligned}
\int_\Gamma \mu &= \int_\Gamma \mathrm{Col}^{\mathbb{1}}((\gamma-1)\cdot u') &&+ \mathrm{ord}(v_1)\int_\Gamma \mathrm{Col}^{\mathbb{1}}((\gamma-1)\cdot\pi)\\
&= \int_\Gamma \left(\gamma\cdot\mathrm{Col}^{\mathbb{1}}(u') - \mathrm{Col}^{\mathbb{1}}(u')\right) &&+ \mathrm{ord}(v_1)(p-1)^{-1}\cdot\mathcal{L}(f_{(\gamma-1)\cdot\pi})(0)\\
&= 0 &&+ \mathrm{ord}(v_1)(1-p^{-1})(p-1)^{-1}\cdot\log_p(f_{(\gamma-1)\cdot\pi}(0))\\
&= \left(1-p^{-1}\right)\cdot\log_p(\chi_{\mathrm{cyc}}(\gamma))\cdot\mathrm{ord}(v_1).
\end{aligned}$$

□

3. Cyclotomic Iwasawa theory for Artin motives

Thoughout this section we keep the notations of the introduction: in particular, ρ does not contain the trivial representation and is unramified at our fixed prime $p>2$. Without loss of generality, we will also assume that E contains the field H cut out by ρ, so the completion $E_p = \overline{\iota_p(E)}$ contains $K := \overline{\iota_p(H)}$.

3.1. p-stabilizations.

Definition 3.1.1. A *p-stabilization* (ρ^+, W_p^+) of (ρ, W_p) is an E_p-linear subspace W_p^+ of W_p of dimension d^+ which is stable under the action of the local Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. We will say that ρ^+ is:

(1) *motivic* if W_p^+ is of the form $E_p \otimes_{E,\iota_p} W^+$, where W^+ is an E-linear subspace of W, and
(2) *η-admissible* (for a given character $\eta \in \widehat{\Gamma}$) if the p-adic pairing (3) is non-degenerate.

To explain in greater details the η-admissibility property, consider a p-stabilization W_p^+ and fix a character $\eta \in \widehat{\Gamma}$. If we let $\omega_{\mathrm{f},\eta} = \psi_1 \wedge \ldots \wedge \psi_{d^+}$ be a basis of the motivic Selmer group $\mathrm{H}^1_{\mathrm{f}}((\rho\otimes\eta)^\vee(1)) = \mathrm{Hom}_{G_\mathbb{Q}}(W_\eta, E_\eta \otimes \mathcal{O}_{H_\eta}^\times)$ and $\omega_p^+ = w_1 \wedge \ldots \wedge w_{d^+}$ be a basis of W_p^+, then the determinant of the p-adic pairing computed in these bases is given by

$$\mathrm{Reg}_{\omega_p^+}(\rho\otimes\eta) = \det\left(\log_p(\psi_j(w_i))\right)_{1\le i,j\le d^+} \in E_{p,\eta}. \tag{8}$$

Here, we denoted by $E_{p,\eta} \subseteq \overline{\mathbb{Q}}_p$ the compositum of E_p with the image of η, and by

$$\log_p : \overline{\mathbb{Q}}_p \otimes_\mathbb{Z} \overline{\mathbb{Z}}^\times \longrightarrow \overline{\mathbb{Q}}_p \tag{9}$$

the map given by $\log_p(x\otimes a) = x \cdot \log_p^{\mathrm{Iw}}(\iota_p(a))$, where $\log_p^{\mathrm{Iw}} : \overline{\mathbb{Q}}_p^\times \longrightarrow \overline{\mathbb{Q}}_p$ is Iwasawa's p-adic logarithm. The p-regulator $\mathrm{Reg}_{\omega_p^+}(\rho\otimes\eta)$ does not vanish for some (hence, all) choices of bases if and only W_p^+ is η-admissible.

We now recall the (weak) p-adic Schanuel conjecture [CM09, Conjecture 3.10]:

Conjecture 3.1.2 (p-adic Schanuel conjecture)**.** *Let $a_1,\ldots,a_n$ be n non-zero algebraic numbers contained in a finite extension F of $\mathbb{Q}_p$. If $\log_p^{\mathrm{Iw}}(a_1),\ldots,\log_p^{\mathrm{Iw}}(a_n)$ are linearly independent over $\mathbb{Q}$, then the extension field $\mathbb{Q}(\log_p^{\mathrm{Iw}}(a_1),\ldots,\log_p^{\mathrm{Iw}}(a_n)) \subset F$ has transcendence degree n over $\mathbb{Q}$.*

Lemma 3.1.3. *Assume that W_p^+ is motivic. Then W_p^+ is η-admissible for all characters $\eta \in \widehat{\Gamma}$ if $d^+ = 1$, or if Conjecture 3.1.2 holds.*

Proof. Fix a character $\eta \in \widehat{\Gamma}$. Since W_p^+ is motivic, it admits a basis ω_p^+ which consists of vectors of W, so $\mathrm{Reg}_{\omega_p^+}(\rho\otimes\eta)$ is a polynomial expression in p-adic logarithms of $\mathbb{Q}$-linearly independent units in $\overline{\mathbb{Q}}\otimes\overline{\mathbb{Z}}^\times$. Therefore, $\mathrm{Reg}_{\omega_p^+}(\rho\otimes\eta)$ does not vanish if the p-adic Schanuel conjecture holds. This is also true when $d^+ = 1$ by the injectivity of the restriction of (9) to $\overline{\mathbb{Q}}\otimes\overline{\mathbb{Z}}^\times$ proven by Brumer [Bru67]. □

We end this section by exploring the link between the η-admissibility of a p-stabilization W_p^+ and the Leopoldt's conjecture for H_η and p. We first note that W_p^+ is η-admissible if and only if $E_{p,\eta}\otimes W_p^+$ does not meet the linear subspace

$$\widetilde{W}_p^- = \bigcap_{\psi} \ker\left[\log_p \circ \psi : W_{p,\eta} \longrightarrow E_{p,\eta}\otimes\mathcal{O}_{H_\eta}^\times \longrightarrow E_{p,\eta}\right],$$

where the intersection runs over all ψ in $E_{p,\eta}\otimes \mathrm{H}^1_{\mathrm{f}}((\rho\otimes\eta)^\vee(1))$. The dimension of $\widetilde{W}_p^-$ can be written as d^-+s, where s is the dimension of the kernel of the map $\alpha_\eta : E_{p,\eta}\otimes \mathrm{H}^1_{\mathrm{f}}((\rho\otimes\eta)^\vee(1)) \longrightarrow \mathrm{Hom}(W_{p,\eta},E_{p,\eta})$ induced by $\log_p$. By Kummer theory, the domain of α_η is canonically isomorphic to the global Bloch-Kato Selmer group $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q},W^*_{p,\eta}(1))$, where $W^*_{p,\eta}(1)$ is the arithmetic dual of the Galois representation $W_{p,\eta}$. One may also see $W^*_{p,\eta}(1)$ as $G_{\mathbb{Q}_p}$-representation and consider the local Bloch-Kato Selmer group $\mathrm{H}^1(\mathbb{Q}_p,W^*_{p,\eta}(1))$, which is, again by Kummer theory, canonically isomorphic to $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(W_{p,\eta},E_{p,\eta}\otimes\mathcal{O}^{\times,1}_{K_\eta})$, where $\mathcal{O}^{\times,1}_{K_\eta}$ is the $\mathbb{Z}_p$-module of principal units of the completion K_η at p of H_η. Under the above identifications, the map α_η is nothing but the composite map

$$\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q},W^*_{p,\eta}(1)) \xrightarrow{\mathrm{loc}_p} \mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}_p,W^*_{p,\eta}(1)) \xrightarrow{\simeq} \mathrm{Hom}(W_{p,\eta},E_{p,\eta}), \tag{10}$$

where loc_p is a localization map at p and where the second map is the isomorphism induced by the p-adic logarithm on K_η, $\log_p : E_{p,\eta}\otimes\mathcal{O}^{\times,1}_{K_\eta}\longrightarrow E_{p,\eta}$. Let $\mathcal{U}_{p,\eta}$ be the product of the principal units $\mathcal{O}^{\times,1}_{K'_\eta}$ of K'_η, where K'_η runs over all the completions of H_η at primes above p. There is an alternative description of $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}_p,W^*_{p,\eta}(1))$ in terms of semi-local Galois cohomology which identifies loc_p of (10) with the map

$$\mathrm{Hom}_{G_{\mathbb{Q}}}(W_{p,\eta},E_{p,\eta}\otimes\mathcal{O}^\times_{H_\eta}) \longrightarrow \mathrm{Hom}_{G_{\mathbb{Q}}}(W_{p,\eta},E_{p,\eta}\otimes\mathcal{U}_{p,\eta}) \tag{11}$$

induced by the diagonal embedding $\iota_{\mathrm{Leo}}:\mathcal{O}^\times_{H_\eta}\longrightarrow\mathcal{U}_{p,\eta}$. The injectivity of ι_{Leo}, known as Leopoldt's conjecture, implies the injectivity of the map in (11) which should be thought as the "$\rho\otimes\eta$-isotypic component of the Leopoldt's conjecture for H_η and p".

Lemma 3.1.4. *W_p admits at least one η-admissible p-stabilization if and only if the $\rho\otimes\eta$-isotypic component of the Leopoldt's conjecture for H_η and p holds, i.e., if the map in (11) is injective.*

Proof. There exists at least one η-admissible p-stabilization of W_p if and only if the linear subspace $\widetilde{W}_p^-$ has dimension d^-, that is, if the map α_η is injective. But we have seen that its injectivity is equivalent to the one of the map in (11). ☐

3.2. **Selmer groups.** A Galois-stable lattice of W_p is a free $\mathcal{O}_p$-submodule of W_p of rank d which is stable by the action of the global Galois group $G_{\mathbb{Q}}$. The pair (ρ,ρ^+) will always refer to the choice of:

(1) a Galois-stable lattice T_p of W_p,
(2) a p-stabilization W_p^+ of ρ,

which we will be fixed henceforth. Let $W_p^-=W_p/W_p^+$, $T_p^+=W_p^+\cap T_p$ and $T_p^-=T_p/T_p^+$. We also define $\mathcal{O}_p$-divisible Galois modules $D_p=W_p/T_p$ and $D_p^\pm=W_p^\pm/T_p^\pm$. Once we fix a generator of the different of $\mathcal{O}_p$ over $\mathbb{Z}_p$, one may identify $\mathcal{O}_p^\vee$ with $E_p/\mathcal{O}_p$ where $M^\vee:=\mathrm{Hom}_{\mathbb{Z}_p}(M,\mathbb{Q}_p/\mathbb{Z}_p)$ stands for the Pontryagin dual of a $\mathbb{Z}_p$-module M. This allows us to identify $D_p^\vee$ with T_p. Let $n\in\mathbb{N}\cup\{\infty\}$ and let I_ℓ be the inertia subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}_n)$ at the place above ℓ determined by ι_ℓ.

Definition 3.2.1. The Selmer group of level n attached to (ρ,ρ^+) is defined as to be

$$\mathrm{Sel}_n(\rho,\rho^+) \ := \ \ker\left[\mathrm{H}^1(\mathbb{Q}_n,D_p)\longrightarrow \mathrm{H}^1(I_p,D_p^-)\times\textstyle\prod_{\ell\neq p}\mathrm{H}^1(I_\ell,D_p)\right].$$

The strict Selmer group $\mathrm{Sel}_n^{\mathrm{str}}(\rho,\rho^+)$ of level n attached to (ρ,ρ^+) is the sub-$\mathcal{O}_p$-module of $\mathrm{Sel}_n(\rho,\rho^+)$ whose cohomology classes are trivial on the decomposition subgroup at p.

The dual Selmer group is defined as the Pontryagin dual of $\mathrm{Sel}_n(\rho,\rho^+)$, that is,

$$X_n(\rho,\rho^+) := \mathrm{Sel}_n(\rho,\rho^+)^\vee = \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Sel}_n(\rho,\rho^+),\mathbb{Q}_p/\mathbb{Z}_p).$$

We also define the strict dual Selmer group of level n by putting $X_n^{\mathrm{str}}(\rho,\rho^+) = \mathrm{Sel}_n^{\mathrm{str}}(\rho,\rho^+)^\vee$.

By standard properties of discrete cohomology groups, $X_\infty(\rho,\rho^+)$ can be identified with the inverse limit $\varprojlim_n X_n(\rho,\rho^+)$ and it is a finitely generated module over the Iwasawa algebra $\Lambda = \mathcal{O}_p[[\Gamma]]$.

Lemma 3.2.2. *If $X_\infty(\rho,\rho^+)$ is of Λ-torsion, then there is a short exact sequence of torsion Λ-modules*

$$0 \longrightarrow \mathrm{H}^0(\mathbb{Q}_p,T_p^-) \longrightarrow X_\infty(\rho,\rho^+) \longrightarrow X_\infty^{\mathrm{str}}(\rho,\rho^+) \longrightarrow 0,$$

where the Γ-action on the first term is trivial.

Proof. The first map is obtained by evaluating cocycles at σ_p and by applying Pontryagin duality, and the second one is the dual of the inclusion $\mathrm{Sel}_\infty^{\mathrm{str}}(\rho,\rho^+) \subseteq \mathrm{Sel}_\infty(\rho,\rho^+)$. The only non-obvious statement is the injectivity of the first map, which will follow from [GV00, Proposition (2.1)], but we must check that $\mathrm{H}^0(\mathbb{Q}_\infty,\breve{D}_p)$ is finite, where $\breve{D}_p = \mathrm{Hom}(T_p,\mu_{p^\infty})$. Since ρ is unramified at p and since it does not contain the trivial representation, $\mathrm{H}^0(\mathbb{Q}(\mu_{p^\infty}),D_p)$ is finite, so $\mathrm{H}^0(\mathbb{Q}_\infty,\breve{D}_p) \subseteq \mathrm{H}^0(\mathbb{Q}(\mu_{p^\infty}),\breve{D}_p) = \mathrm{Hom}(\mathrm{H}^0(\mathbb{Q}(\mu_{p^\infty}),D_p),\mu_{p^\infty})$ is also finite, as wanted. □

The study of the structure of $X_\infty(\rho,\rho^+)$ was initiated in [Mak20], where the unramifiedness assumption was partially released (only the quotient W_p^- was assumed to be unramified). However, ρ was taken irreducible and ρ^+ motivic. This last hypothesis implies the η-admissibility of ρ^+ for all characters η under the Weak p-adic Schanuel conjecture (or when $d^+=1$) as in Lemma 3.1.3. We recall the results obtained in *loc. cit.*.

Theorem 3.2.3. *Assume that ρ is irreducible and that ρ^+ is motivic. If $d^+=1$ or if Conjecture 3.1.2 holds, then the following four claims are true.*

(1) The Selmer groups $X_n(\rho,\rho^+)$ are finite for all $n\in\mathbb{N}$.
(2) The Λ-module $X_\infty(\rho,\rho^+)$ is torsion and has no non-trivial finite submodules.
(3) Let $\theta^{alg}_{\rho,\rho^+}\in\Lambda$ be a generator of its characteristic ideal. Then $\eta(\theta^{alg}_{\rho,\rho^+})$ does not vanish for all non-trivial finite order characters $\eta:\Gamma\longrightarrow\overline{\mathbb{Q}}^\times$.
(4) Let $e=\dim\mathrm{H}^0(\mathbb{Q}_p,W_p^-)$. Then $\theta^{alg}_{\rho,\rho^+}$ vanishes at the trivial character $\mathbb{1}$ if and only if $e=0$. Moreover, $\theta^{alg}_{\rho,\rho^+}$ has a zero of order $\geq e$ at $\mathbb{1}$, i.e.,

$$\theta^{alg}_{\rho,\rho^+}\in\mathcal{A}^e,$$

where $\mathcal{A}$ is the augmentation ideal of Λ.

Proof. This follows from [Mak20, Théorème 2.1.5]. □

3.3. **Artin L-functions and Galois-Gauss sums.**

We review some classical results on Artin L-functions and on Galois-Gauss sums, and we give equivalent reformulations of Conjecture A. Our main reference is [Mar77]. Let (V,π) be an Artin representation of $G_\mathbb{Q}$ of dimension d and of Artin conductor $\mathfrak{f}(\pi)$. Put $d^+=\dim\mathrm{H}^0(\mathbb{R},V)$ and $d^-=d-d^+$. The Artin

L-function of π is the meromorphic continuation to $s \in \mathbb{C}$ of the infinite product (converging for $\Re(s) > 1$) over all rational primes

$$L(\pi, s) = \prod_{\ell} \det(1 - \ell^{-s}\sigma_{\ell} \mid V^{I_{\ell}}),$$

where σ_{ℓ} is the Frobenius substitution at ℓ and where $I_{\ell} \subseteq G_{\mathbb{Q}}$ is any inertia group at ℓ. It is known to satisfy a functional equation which can compactly be written $\Lambda(\pi, 1-s) = W(\pi)\Lambda(\pi^{\vee}, s)$, where $W(\pi)$ is Artin's root number and $\Lambda(\pi, s)$ is the "enlarged" L-function. By definition, $\Lambda(\pi, s)$ is equal to the product $\mathrm{f}(\pi)^{s/2}\Gamma(\pi, s)L(\pi, s)$, where $\Gamma(\pi, s) = \Gamma_{\mathbb{R}}(s)^{d^+}\Gamma_{\mathbb{R}}(s+1)^{d^-}$ (and $\Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(s/2)$) is the L-factor at ∞ of π. The Galois-Gauss sum of π is defined as

$$\tau(\pi) = i^{d^-}\sqrt{\mathrm{f}(\pi)}W(\pi^{\vee}) = i^{d^-}\frac{\sqrt{\mathrm{f}(\pi)}}{W(\pi)},$$

see *loc. cit.*, Chapter II, Definition 7.2 and the remark that follows. In particular, when $\mathrm{H}(\mathbb{Q}, V) = 0$, the functional equation yields

$$L^*(\pi^{\vee}, 0) := \lim_{s\to 0} L(\pi^{\vee}, s)/s^{d^+} = \frac{\tau(\pi)}{2^{d^+}}\frac{L(\pi, 1)}{(-i\pi)^{d^-}}. \tag{12}$$

Lemma 3.3.1. *Keep the notations of Conjecture A. The interpolation property of θ_{ρ,ρ^+} can be written as follows:*

$\boldsymbol{EX}_{\rho,\rho^+}$: *for all non-trivial characters $\eta \in \widehat{\Gamma}$ of exact conductor p^n, one has*

$$\eta(\theta_{\rho,\rho^+}) = \frac{\tau(\eta)^{d^-}}{2^{d^+}}\frac{\mathrm{Reg}_{\omega_p^+}(\rho\otimes\eta)}{\det(\rho^-)(p^n)}\frac{L\left(\rho\otimes\eta, 1\right)}{(-i\pi)^{d^-}\mathrm{Reg}_{\omega_\infty^+}(\rho\otimes\eta)}.$$

$\boldsymbol{EZC}_{\rho,\rho^+}$: *If W_p^+ is admissible, then θ_{ρ,ρ^+} has a trivial zero at the trivial character $\mathbb{1}$ of order at least e, and one has*

$$\frac{1}{e!}\frac{\mathrm{d}^e}{\mathrm{d}s^e}\kappa^s(\theta_{\rho,\rho^+})\Big|_{s=0} = 2^{-d^+}(-1)^e\,\mathscr{L}(\rho,\rho^+)\,\mathcal{E}(\rho,\rho^+)\,\mathrm{Reg}_{\omega_p^+}(\rho)\,\frac{L\left(\rho, 1\right)}{(-i\pi)^{d^-}\mathrm{Reg}_{\omega_\infty^+}(\rho)}.$$

Proof. This follows from Formula (12) applied to $\pi = \rho\otimes\eta$. ☐

Lemma 3.3.2. *(1) If χ is a Dirichlet character, then $\tau(\chi)$ is the usual Gauss sum of χ^{-1}, i.e., $\tau(\chi) = \mathfrak{g}(\chi^{-1})$.*

(2) We have $\tau(\pi) \in F^{\times}$ for any splitting field $F \subseteq \overline{\mathbb{Q}}$ of π.

(3) If π is unramified at p, then $\tau(\pi)$ is a p-adic unit.

(4) Take $\pi = \rho\otimes\eta$ with $\eta \in \widehat{\Gamma}$ and put $N = \mathrm{f}(\rho)$, $p^n = \mathrm{f}(\eta)$. Then

$$\tau(\rho\otimes\eta) = \tau(\rho)\mathfrak{g}(\eta^{-1})^d\det(\rho)^{-1}(p^n)\eta^{-1}(N).$$

Proof. The first statement follows from the well-known fact that $\mathfrak{g}(\chi^{-1}) = i^{d^-}W(\chi^{-1})\mathrm{f}(\chi)^{1/2}$, and the second statement from Fröhlich's theorem [Mar77, Chapter II, Theorem 7.2]. For the third statement, recall first that $\tau(\pi)$ is a product over all primes ℓ of local Galois-Gauss sums $\tau(\pi_{\ell}) \in \overline{\mathbb{Q}}^{\times}$ attached to the local representation π_{ℓ} over $\mathbb{Q}_{\ell}$ associated with π (see *loc.cit.*, Chapter II, Proposition 7.1). We claim that $\tau(\pi_{\ell})$ is only divisible by primes above ℓ and that $\tau(\pi_p) = 1$. Local Galois-Gauss sums are defined with the aid of Brauer induction from the case of multiplicative characters θ of $\mathrm{Gal}(\overline{\mathbb{Q}}_{\ell}/M)$ for a finite extension $M/\mathbb{Q}_{\ell}$ (see *loc.cit.*, Chapter II, §4. and §2.). It is known that $\tau(\theta)$ is an algebraic integer dividing the norm of the local conductor (which is a power of ℓ), and moreover that $\tau(\theta) = 1$ whenever both θ

and $M/\mathbb{Q}_\ell$ are unramified. This implies easily our claim and (3) as well. Since ρ and η have coprime conductors, the statement (4) follows from *loc. cit.*, Chapter IV, Exercise 3b). □

Proposition 3.3.3. *(1) The statement $\boldsymbol{EX}_{\rho,\rho^+}$ in Conjecture A is equivalent to the existence of an element $\theta'_{\rho,\rho^+} \in \mathrm{Frac}(\Lambda)$ which has at most a pole at $\mathbb{1}$ and which satisfies the following interpolation property: for all non-trivial characters $\eta \in \widehat{\Gamma}$ of conductor p^n, we have*

$$\eta(\theta'_{\rho,\rho^+}) = \frac{\det(\rho^+)(p^n)}{\mathfrak{g}(\eta^{-1})^{d^+}}\ \mathrm{Reg}_{\omega_p^+}(\rho\otimes\eta)\ \frac{L^*\left((\rho\otimes\eta)^\vee,0\right)}{\mathrm{Reg}_{\omega_\infty^+}(\rho\otimes\eta)}.$$

Moreover, if $\boldsymbol{EX}_{\rho,\rho^+}$ holds, then θ_{ρ,ρ^+} and θ'_{ρ,ρ^+} are equal up to multiplication by a unit of Λ.

(2) If $\boldsymbol{EX}_{\rho,\rho^+}$ holds and if W_p^+ is admissible, then $\boldsymbol{EZC}_{\rho,\rho^+}$ is equivalent to

$$\frac{1}{e!}\frac{\mathrm{d}^e}{\mathrm{d}s^e}\kappa^s(\theta'_{\rho,\rho^+})\Big|_{s=0} = \mathcal{L}(\rho,\rho^+)\ \mathcal{E}(\rho,\rho^+)\ \mathrm{Reg}_{\omega_p^+}(\rho)\ \frac{L^*\left(\rho^\vee,0\right)}{\mathrm{Reg}_{\omega_\infty^+}(\rho)}.$$

Proof. By Lemma 3.3.2 (4), the two quotients of p-adic measures θ_{ρ,ρ^+} and θ'_{ρ,ρ^+} are related (when they exist) by the formula $\theta_N\theta'_{\rho,\rho^+} = \tau(\rho)\theta_{\rho,\rho^+}$, where we have written $\theta_N = \prod_{\ell|N}\gamma_\ell^{\mathrm{ord}_\ell(N)} \in \Gamma \subseteq \Lambda^\times$ (and where $\gamma_\ell \in \Gamma$ is equal to the restriction to $\mathbb{Q}_\infty$ of σ_ℓ). By Lemma 3.3.2 (3), one has $\tau(\rho) \in \mathcal{O}_p^\times$, so θ_{ρ,ρ^+} and θ'_{ρ,ρ^+} are equal up to a unit of Λ, and the proposition follows easily. □

3.4. **Local and global duality.** In order to better describe our Selmer group we first need to introduce and compare the "unramified condition" and the "f-condition" of Bloch and Kato for (the dual of) a local Galois representation with finite image. Take any $\mathcal{O}_p$-representation $\mathbf{T}$ of the absolute Galois group G_F of a finite extension F of $\mathbb{Q}_\ell$ (including $\ell = p$). Assume that $\mathbf{T}$ is of finite image, *i.e.*, the action of G_F factors through the Galois group Δ of a finite extension L/F. Define $\mathbf{D} = \mathbf{T}\otimes\mathbb{Q}_p/\mathbb{Z}_p$ and $\check{\mathbf{T}} = \mathbf{T}^*(1)$ the arithmetic dual of $\mathbf{T}$. Recall that, if $I \subseteq G_F$ is the inertia subgroup of G_F and M is a G_F-module, then $\mathrm{H}^1_{\mathrm{ur}}(F,M)$ is the kernel of the restriction map $\mathrm{H}^1(F,M) \longrightarrow \mathrm{H}^1(I,M)$.

Lemma 3.4.1. *(1) If $\ell \neq p$, then we have $\mathrm{H}^1_{\mathrm{ur}}(F,\check{\boldsymbol{T}}) \subseteq \mathrm{H}^1_{\mathrm{f}}(F,\check{\boldsymbol{T}})$ and $\mathrm{H}^1_{\mathrm{f}}(F,\boldsymbol{D}) \subseteq \mathrm{H}^1_{\mathrm{ur}}(F,\boldsymbol{D})$.*

(2) If $\ell = p$ and if $\boldsymbol{T}$ is unramified, then $\mathrm{H}^1_{\mathrm{ur}}(F,\boldsymbol{D}) = \mathrm{H}^1_{\mathrm{f}}(F,\boldsymbol{D})$.

(3) Under the local Tate pairing $\mathrm{H}^1(F,\check{\boldsymbol{T}})\times\mathrm{H}^1(F,\boldsymbol{D}) \longrightarrow \mathcal{O}_p\otimes\mathbb{Q}_p/\mathbb{Z}_p$, the $\mathrm{H}^1_{\mathrm{f}}$'s are orthogonal complements for any prime ℓ and the $\mathrm{H}^1_{\mathrm{ur}}$'s are orthogonal complements for any prime $\ell \neq p$.

Proof. The first point follows from [Rub00, Lemma 3.5]. Let us prove the second statement and assume now that $\ell = p$ and that $\mathbf{T}$ is unramified. Recall that $\mathrm{H}^1_{\mathrm{f}}(F,\mathbf{D})$ is by definition the image of $\mathrm{H}^1_{\mathrm{f}}(F,\mathbf{W})$ under the map $\mathrm{H}^1(F,\mathbf{W}) \longrightarrow \mathrm{H}^1(F,\mathbf{D})$, where we have put $\mathbf{W} = \mathbf{T}\otimes\mathbb{Q}_p$. Since $\mathbf{W}$ is unramified, it is easy to see that $\mathrm{H}^1_{\mathrm{f}}(F,\mathbf{W}) = \mathrm{H}^1_{\mathrm{ur}}(F,\mathbf{W})$. Moreover, the map $\mathrm{H}^1_{\mathrm{ur}}(F,\mathbf{W}) \longrightarrow \mathrm{H}^1_{\mathrm{ur}}(F,\mathbf{D})$ is surjective because it coincides with the projection map $\mathbf{W}_\Delta \twoheadrightarrow \mathbf{D}_\Delta$ by [Rub00, Lemma 3.2.(i)], where $(-)_\Delta$ means that we took the Δ-coinvariants. Therefore, we have $\mathrm{H}^1_{\mathrm{ur}}(F,\mathbf{D}) = \mathrm{H}^1_{\mathrm{f}}(F,\mathbf{D})$. The third statement is standard (see [BK90, Proposition 3.8] for Bloch-Kato's condition and [Rub00, Proposition 4.3.(i)] for the unramified one). □

For $F = \mathbb{Q}, \mathbb{Q}_\ell$ (for any prime ℓ) and for any compact $\mathcal{O}_p[[G_F]]$-module $\mathbf{T}$, define the Iwasawa cohomology along the $\mathbb{Z}_p$-cyclotomic extension $F_\infty = \cup_n F_n$ of F by letting

$$\begin{aligned} \mathrm{H}^1_{\mathrm{Iw},*}(F,\mathbf{T}) &:= \varprojlim_n \mathrm{H}^1_*(F_n,\mathbf{T}) \qquad (* \in \{\emptyset, \mathrm{f}, \mathrm{ur}\}), \\ \mathrm{H}^1_{\mathrm{Iw,f},p}(\mathbb{Q},\mathbf{T}) &:= \varprojlim_n \mathrm{H}^1_{\mathrm{f},p}(\mathbb{Q}_n,\mathbf{T}), \end{aligned}$$

where the subscript f,p in the last global cohomology groups means that we relaxed the condition of being crystalline at p. We also use the standard notation $\mathrm{H}^1_{\mathrm{Iw},/\mathrm{f}}(\mathbb{Q}_\ell,\mathbf{T})$ for the quotient $\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_\ell,\mathbf{T})/\mathrm{H}^1_{\mathrm{Iw,f}}(\mathbb{Q}_\ell,\mathbf{T})$. All these cohomology groups are finitely generated modules over Λ by Shapiro's lemma.

We keep the notations of Section 3.2 and we fix a Galois-stable lattice T_p of W_p and any p-stabilization W_p^+ of W_p. By semisimplicity of linear representations of finite groups we may (and we will) identify W_p^- with a $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-stable complement of W_p^+ in W_p. Let $\breve{T}_p = T_p(1)^*$ (resp. $\breve{T}_p^\pm = T_p^\pm(1)^*$) be the arithmetic dual of T_p (resp. of $T_p^\pm$). In particular, $\breve{T}_p^\pm$ is a free $\mathcal{O}_p$-submodule of $\breve{T}_p$ of rank $d^\pm$. We will relate $\mathrm{Sel}_\infty^{\mathrm{str}}(\rho,\rho^+)$ and $\mathrm{Sel}_\infty(\rho,\rho^+)$ to the following three localization maps

$$\begin{aligned} \mathrm{Loc}_+^{\mathrm{str}}: &\quad \mathrm{H}^1_{\mathrm{Iw,f},p}(\mathbb{Q},\breve{T}_p) \longrightarrow \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p,\breve{T}_p^+) \\ \mathrm{Loc}_+: &\quad \mathrm{H}^1_{\mathrm{Iw,f}}(\mathbb{Q},\breve{T}_p) \longrightarrow \mathrm{H}^1_{\mathrm{Iw,f}}(\mathbb{Q}_p,\breve{T}_p^+), \\ \mathrm{Loc}'_+: &\quad \mathrm{H}^1_{\mathrm{Iw,f},p}(\mathbb{Q},\breve{T}_p) \longrightarrow \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p,\breve{T}_p^+) \bigoplus \mathrm{H}^1_{\mathrm{Iw},/\mathrm{f}}(\mathbb{Q}_p,\breve{T}_p^-). \end{aligned}$$

For $\ell \neq p$, it is known that the quotient of the absolute Galois group of $\mathbb{Q}_{\ell,\infty}$ by its inertia subgroup I_ℓ is of order prime to p, so the restriction map $\mathrm{H}^1(\mathbb{Q}_{\ell,\infty},D) \longrightarrow \mathrm{H}^1(I_\ell,D)$ is injective. This, together with Lemma 3.4.1, implies that

$$\varinjlim_n \mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}_{\ell,n},D_p) = \varinjlim_n \mathrm{H}^1_{\mathrm{ur}}(\mathbb{Q}_{\ell,n},D_p) = 0, \quad \text{and} \quad \mathrm{H}^1_{\mathrm{Iw,f}}(\mathbb{Q}_\ell,\breve{T}_p) = \mathrm{H}^1_{\mathrm{Iw,ur}}(\mathbb{Q}_\ell,\breve{T}_p) = \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_\ell,\breve{T}_p).$$

Hence, the Selmer group of (ρ,ρ^+) fits into an exact sequence

$$0 \longrightarrow Ш^1_\infty(D_p) \longrightarrow \mathrm{Sel}_\infty(\rho,\rho^+) \longrightarrow \varinjlim_n \left(\mathrm{H}^1(\mathbb{Q}_{p,n},D_p^+) \bigoplus \mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}_{p,n},D_p^-)\right)$$

where $Ш^1_\infty(D_p) = \ker\left[\mathrm{H}^1(\mathbb{Q}_\infty,D_p) \longrightarrow \prod_\ell \mathrm{H}^1(\mathbb{Q}_{\ell,\infty},D_p)\right]$ is the first Tate-Shafarevitch group. It follows from Poitou-Tate duality [Rub00, Corollary 7.5] that there is a commutative diagram of Λ-modules with short exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Ш^1_\infty(D_p) & \longrightarrow & \mathrm{Sel}_\infty(\rho,\rho^+) & \longrightarrow & \mathrm{coker}\left(\mathrm{Loc}'_+\right)^\vee & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{H}^1_{\mathrm{ur}}(\mathbb{Q}_\infty,D_p) & \longrightarrow & \mathrm{Sel}_\infty(\rho,\rho^+) & \longrightarrow & \mathrm{coker}(\mathrm{Loc}_+)^\vee & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & Ш^1_\infty(D_p) & \longrightarrow & \mathrm{Sel}^{\mathrm{str}}_\infty(\rho,\rho^+) & \longrightarrow & \mathrm{coker}\left(\mathrm{Loc}^{\mathrm{str}}_+\right)^\vee & \longrightarrow & 0, \end{array} \tag{13}$$

where $\mathrm{H}^1_{\mathrm{ur}}(\mathbb{Q}_\infty,D_p) = \ker\left[\mathrm{H}^1(\mathbb{Q}_\infty,D_p) \longrightarrow \prod_\ell \mathrm{H}^1(I_\ell,D_p)\right]$ (and where I_ℓ is the inertia subgroup of $\mathbb{Q}_{\ell,\infty}$).

We close this section by proving the Weak Leopoldt conjecture for both W_p and $\breve{W}_p = W_p(1)^*$. The proof is self-contained and it won't use the running assumption that ρ is unramified at p (but p is still be assumed to be odd). Let Σ be a finite set of places of $\mathbb{Q}$ containing p

and all the primes at which ρ is ramified. For $i \in \mathbb{N}$ and $\mathbf{T} = T_p$ or $\breve{T}_p$ define the cohomology groups $\mathrm{H}^i_{\mathrm{Iw},\Sigma}(\mathbb{Q},\mathbf{T}) = \varprojlim_n \mathrm{H}^i(\mathbb{Q}_\Sigma/\mathbb{Q}_n,\mathbf{T})$, where $\mathbb{Q}_\Sigma/\mathbb{Q}$ is the maximal extension of $\mathbb{Q}$ which is unramified outside Σ and ∞. We will also consider the second (compact) Tate-Shafarevich groups $Ш^2_\infty(\mathbf{T}) = \ker\left[\mathrm{H}^2_{\mathrm{Iw}}(\mathbb{Q},\mathbf{T}) \longrightarrow \prod_\ell \mathrm{H}^2_{\mathrm{Iw}}(\mathbb{Q}_\ell,\mathbf{T})\right]$.

Proposition 3.4.2. *(1) We have $Ш^2_\infty(\breve{T}_p) \simeq Ш^1_\infty(D_p)^\vee$ as Λ-modules.*

(2) The Weak Leopoldt conjecture along $\mathbb{Q}_\infty/\mathbb{Q}$ for W_p and $\breve{W}_p$ holds, that is, the Λ-modules $\mathrm{H}^2_{\mathrm{Iw},\Sigma}(\mathbb{Q},T_p)$ and $\mathrm{H}^2_{\mathrm{Iw},\Sigma}(\mathbb{Q},\breve{T}_p)$ are torsion.

Proof. The first statement follows from Poitou-Tate duality (see, for instance, [Mil86, Theorem 4.10.(a)]). Let us first prove the Weak Leopoldt conjecture for W_p. By [PR95, Proposition 1.3.2], it is equivalent to $\mathrm{H}^2(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty,D_p) = 0$. As $\mathrm{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty)$ has cohomological dimension 2, this module is p-divisible because D_p is. It is thus enough to show that $\mathrm{H}^2(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty,D_p)^\vee$ is a torsion Λ-module, which has already been shown in the proof of [Mak20, Proposition 2.5.5]. Consider now the Weak Leopoldt Conjecture for $\breve{W}_p$. Since the groups $\mathrm{H}^2_{\mathrm{Iw}}(\mathbb{Q}_\ell,\breve{T}_p)$ for $\ell \in \Sigma$ are torsion Λ-modules (for $\ell = p$ as well) by Tate's local duality, it is enough to show that $Ш^2_\infty(\breve{T}_p)$ is of Λ-torsion, or equivalently, that $Ш^1_\infty(D_p)$ is of Λ-cotorsion by the first point. Recall that we denoted by $G = \mathrm{Gal}(H/\mathbb{Q})$ the Galois group of the extension cut out by ρ and that we let $H_\infty = H\mathbb{Q}_\infty$. The inflation-restriction exact sequence shows that the kernel of the restriction map

$$\mathrm{H}^1(\mathbb{Q}_\infty,D_p) \longrightarrow \mathrm{H}^1(H_\infty,D_p) = \mathrm{Hom}(G_{H_\infty},D_p)$$

is killed by $\#G$, so it is the same for its Pontryagin dual. Therefore, it is enough to show that the image of $Ш^1_\infty(D_p)$ under this restriction map is Λ-cotorsion. But its image lies inside the submodule of morphisms $\sigma : G_{H_\infty} \longrightarrow D_p$ which factor through the Galois group A_∞ of the maximal abelian pro-p extension of H_∞ which is everywhere unramified. By Iwasawa's classical results [Iwa73], A_∞ is of Λ-torsion, so $\mathrm{Hom}(A_\infty,D_p) = \mathrm{Hom}(A_\infty,\mathbb{Q}_p/\mathbb{Z}_p)\otimes T_p$ is Λ-cotorsion as well as $Ш^1_\infty(D_p)$. □

3.5. **Limits of unit groups.** The various cohomology groups introduced in Section 3.4 can usefully be described in terms of ideal class groups and of unit groups.

Notation 3.5.1. Let $n \geq 0$ be an integer and let w be a p-adic place of H. We still denote by w the unique place of H_n above H, and we let

- A_n be the p-part of the ideal class group of H_n, and A'_n its quotient by all the classes of p-adic primes of H_n,
- $\mathcal{O}^\times_{H_n}$ (resp. $\mathcal{O}_{H_n}[\frac{1}{p}]^\times$) be the unit group (resp. the group of p-units) of H_n,
- U_n (resp. U'_n) be the pro-p completion of the unit group (resp. of the group of p-units) of H_n,
- $U_{n,w}$ (resp. $U'_{n,w}$) be the pro-p completion of the unit group (resp. of the group of non-zero elements) of $H_{n,w}$.

We also let $A_\infty, A'_\infty, U_\infty, U'_\infty, U_{\infty,w}, U'_{\infty,w}$ respectively be the projective limits of the preceding groups, where the transition maps are the (global or local) norm maps. All of these groups are $\mathbb{Z}_p$-modules but we will keep the same notations for the $\mathcal{O}_p$-modules that are obtained after tensoring with $\mathcal{O}_p$ by a slight abuse of notation.

Fix as in Section 3.2 a Galois-stable $\mathcal{O}_p$-lattice T_p and a p-stabilization W_p^+ of W_p. We also keep the notations of Section 3.4.

Lemma 3.5.2. *Let G_p be the decomposition subgroup of G at the place w determined by ι_p and let $\bullet \in \{\emptyset,+,-\}$.*

(1) *The restriction maps on cohomology groups induce the following natural isomorphisms:*

$$\mathrm{H}^1_{\mathrm{Iw,f},p}(\mathbb{Q},\breve{T}_p)\simeq \mathrm{Hom}_G(T_p,U'_\infty),\qquad \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p,\breve{T}^\bullet_p)\simeq \mathrm{Hom}_{G_p}(T^\bullet_p,U'_{\infty,w}),$$

$$\mathrm{H}^1_{\mathrm{Iw,f}}(\mathbb{Q},\breve{T}_p)\simeq \mathrm{Hom}_G(T_p,U_\infty),\qquad \mathrm{H}^1_{\mathrm{Iw,f}}(\mathbb{Q}_p,\breve{T}^\bullet_p)\simeq \mathrm{Hom}_{G_p}(T^\bullet_p,U_{\infty,w}).$$

(2) *The Λ-modules $\text{Ш}^1_\infty(D_p)^\vee$ and $\mathrm{Hom}_G(T_p,A'_\infty)$ are isomorphic after tensoring with $\mathbb{Q}_p$ (as $\Lambda\otimes\mathbb{Q}_p$-modules). They are isomorphic as Λ-modules if we assume that p does not divide the order of G.*

(3) *The Λ-modules $\text{Ш}^1_\infty(D_p)$ and $\mathrm{Hom}_G(A'_\infty,D_p)$ are pseudo-isomorphic.*

Proof. We will derive the isomorphisms from Kummer theory and from the injectivity (resp. bijectivity) of restriction maps between certain local (resp. global) cohomology groups. Fix an integer $n\geq 0$, a prime number ℓ (including $\ell=p$) and let $\lambda|\ell$ be the prime of H_n determined by ι_ℓ. As in Section 3.4, let $\mathbb{Q}_\Sigma$ be the largest extension of $\mathbb{Q}$ (or, equivalently, the largest extension of H_n) which is unramified outside Σ and ∞. Consider the finite field extension F'/F and the Galois groups $\mathcal{G}'\subseteq\mathcal{G}$ given either by

- $H_n/\mathbb{Q}_n$ and $\mathcal{G}'=\mathrm{Gal}(\mathbb{Q}_\Sigma/H_n)$, $\mathcal{G}=\mathrm{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q}_n)$ (first case),
- or by $H^{\mathrm{ur}}_{n,w}/\mathbb{Q}^{\mathrm{ur}}_{n,w}$ and $\mathcal{G}'=\mathrm{Gal}(\overline{\mathbb{Q}}_p/H^{\mathrm{ur}}_{n,w})$, $\mathcal{G}=\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}^{\mathrm{ur}}_{n,w})$ (second case),
- or by $H_{n,\lambda}/\mathbb{Q}_{n,\lambda}$ and $\mathcal{G}'=\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/H_{n,\lambda})$, $\mathcal{G}=\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_{n,\lambda})$ (third case).

Note that $\mathcal{G}/\mathcal{G}'$ can be identified with G in the first case (because H in unramified at p) and with G_p in the third case when $\ell=p$. Since F' never contains $\mathbb{Q}(\mu_{p^\infty})$, we have in all three cases $\mathrm{H}^0(\mathcal{G}',\breve{T}_p)=T_p^*\otimes\mathrm{H}^0(\mathcal{G}',\mathbb{Z}_p(1))=0$. Moreover, Hochschild-Serre's spectral sequence applies to $\breve{T}_p$ in the first case and the third case with $\ell\neq p$ by [Rub00, Appendix B, Proposition 2.7], so the restriction map

$$\mathrm{H}^i(\mathcal{G},\breve{T}_p)\longrightarrow \mathrm{H}^0(\mathcal{G}/\mathcal{G}',\mathrm{H}^i(\mathcal{G}',\breve{T}_p)),\qquad (i=1,2)$$

is bijective when $i=1$, and surjective when $i=2$. It is also injective when $i=1$ in all three cases by the inflation-restriction exact sequence.

Let us first prove the first isomorphism of (1). By taking inverse limits in the first case with $i=1$, the restriction map gives an isomorphism $\mathrm{H}^1_{\mathrm{Iw},\Sigma}(\mathbb{Q},\breve{T}_p)\simeq\mathrm{H}^1_{\mathrm{Iw},\Sigma}(H,\breve{T}_p)^G$. Moreover, this isomorphism sends $\mathrm{H}^1_{\mathrm{Iw,f},p}(\mathbb{Q},\breve{T}_p)$ onto $\mathrm{H}^1_{\mathrm{Iw,f},p}(H,\breve{T}_p)^G$ by the injectivity of the local restriction maps (third case, $\ell\in\Sigma-\{p\}$, $i=1$). We compute this last module as follows: since G_H acts trivially on T_p, it is equal to $\mathrm{Hom}_G(T_p,\mathrm{H}^1_{\mathrm{Iw,f},p}(H,\mathcal{O}_p(1)))$. But Kummer theory naturally identifies $\mathrm{H}^1_{\mathrm{Iw,f},p}(H,\mathcal{O}_p(1))$ with U'_∞, so our claim follows. The three other isomorphisms in (1) are proven with similar arguments.

We now study $\text{Ш}^1_\infty(D_p)^\vee$, which is known to be isomorphic to $\text{Ш}^2_\infty(\breve{T}_p)$ by Proposition 3.4.2. The Hochschild-Serre spectral sequence provides in our setting a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{H}^1(G,\mathrm{H}^1_{\mathrm{Iw},\Sigma}(H,\breve{T}_p)) & \longrightarrow & \mathrm{H}^2_{\mathrm{Iw},\Sigma}(\mathbb{Q},\breve{T}_p) & \longrightarrow & \mathrm{H}^2_{\mathrm{Iw},\Sigma}(H,\breve{T}_p)^G & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\alpha} & & \\
0 & \longrightarrow & \prod_{\ell\in\Sigma}\mathrm{H}^1(G,\prod_{\lambda|\ell}\mathrm{H}^1_{\mathrm{Iw}}(H_\lambda,\breve{T}_p)) & \longrightarrow & \prod_{\ell\in\Sigma}\mathrm{H}^2_{\mathrm{Iw}}(\mathbb{Q}_\ell,\breve{T}_p) & \longrightarrow & \left(\prod_{\lambda|\ell\in\Sigma}\mathrm{H}^2_{\mathrm{Iw}}(H_\lambda,\breve{T}_p)\right)^G & \longrightarrow & 0.
\end{array}$$

The cohomology groups $\mathrm{H}^1(G,-)$ on the left are killed by the order $\#G$ of G, so they vanish after tensoring with $\mathbb{Q}_p$ or whenever p is coprime to $\#G$. Therefore, in order to prove the claim (2) it suffices to check that the module $Ш^2_\infty(H,\breve{T}_p)^G = \ker\alpha$ can be identified with $\mathrm{Hom}_G(T_p, A'_\infty)$. But again, G_H acts trivially on T_p so it is enough to see that $Ш^2_\infty(H,\mathcal{O}_p(1)) \simeq A'_\infty$, which is classical (see for instance [Nek06, (9.2.2.2)]).

The proof of last statement also uses inflation-restriction and is nearly identical to the proof of [Mak20, Lemme 2.2.2], once we identify A'_∞ with the Galois group of the maximal abelian pro-p extension of H_∞ which is unramified everywhere and in which all primes above p split completely. □

Lemma 3.5.3. *Assume that U is a free Λ-module of finite rank endowed with a Λ-linear action of G. Then the module $Z = \mathrm{Hom}_G(T_p, U)$ is also Λ-free.*

Proof. First recall that any finitely generated Λ-module V is free if and only if V^Γ vanishes and V_Γ is $\mathcal{O}_p$-free (see for example [Bel02, Lemme 1.1]). We already have $Z^\Gamma = \mathrm{Hom}_G(T_p, U^\Gamma) = 0$. It is thus enough to check that Z_Γ has no $\mathcal{O}_p$-torsion, and this will follow from the fact that Z_Γ injects into the torsion-free $\mathcal{O}_p$-module $\mathrm{Hom}_G(T_p, U_\Gamma)$. Let us first check that $(\gamma-1)\mathrm{Hom}_G(T_p,U) = \mathrm{Hom}_G(T_p,(\gamma-1)U)$, where γ is a topological generator of Γ. The inclusion $\subseteq$ is obvious, so we only consider the reverse inclusion $\supseteq$ and we take $\alpha \in \mathrm{Hom}_G(T_p,(\gamma-1)U)$. As T_p is $\mathcal{O}_p$-free, one may write $\alpha = (\gamma-1)\beta$ for some $\mathcal{O}_p$-linear map $\beta : T_p \longrightarrow U$. As the G-action is assumed to be Λ-linear, for all $g \in G$ and $t \in T_p$ we have $\beta(g.t) - g.\beta(t) \in U^\Gamma = 0$, so $\beta \in \mathrm{Hom}_G(T_p,U)$ and $\alpha \in (\gamma-1)\mathrm{Hom}_G(T_p,U)$ as claimed. Therefore, we have

$$\begin{aligned} Z_\Gamma = \mathrm{Hom}_G(T_p,U)/(\gamma-1)\mathrm{Hom}_G(T_p,U) &= \mathrm{Hom}_G(T_p,U)/\mathrm{Hom}_G(T_p,(\gamma-1)U) \\ &\hookrightarrow \mathrm{Hom}_G(T_p,U/(\gamma-1)U) = \mathrm{Hom}_G(T_p,U_\Gamma), \end{aligned}$$

as wanted. □

Lemma 3.5.4. *(1) The Λ-modules $\mathrm{H}^1_{\mathrm{Iw,f},p}(\mathbb{Q},\breve{T}_p)$ and $\mathrm{H}^1_{\mathrm{Iw,f}}(\mathbb{Q},\breve{T}_p)$ are both free of rank d^+.*

(2) If $\mathrm{coker}(\mathrm{Loc}_+)$ is of Λ-torsion, then the modules $\mathrm{coker}\left(\mathrm{Loc}'_+\right)$, $\mathrm{coker}\left(\mathrm{Loc}'_+\right)$ and the Selmer groups $X_\infty(\rho,\rho^+)$ and $X^{\mathrm{str}}_\infty(\rho,\rho^+)$ are all of Λ-torsion, and moreover $\ker(\mathrm{Loc}_+) = \ker\left(\mathrm{Loc}'_+\right) = \ker\left(\mathrm{Loc}^{\mathrm{str}}_+\right) = 0$.

Proof. Since H is unramified at p, we have $\mu_p \not\subseteq H$ so the Λ-modules U_∞ and U'_∞ are free by [Bel02, Théorème 1.5 and Corollaire 1.6]. Thus, both Iwasawa cohomology groups are Λ-free by Lemmas 3.5.2 and 3.5.3. Moreover, the validity of the Weak Leopoldt conjecture (Proposition 3.4.2 (2)) implies that they are both of rank d^+ over Λ. Let us treat (2) and assume that $\mathrm{coker}(\mathrm{Loc}_+)$ is of Λ-torsion. As in the proof of Proposition 3.4.2, one checks that the Λ-module $\mathrm{H}^1_{\mathrm{ur}}(\mathbb{Q}_\infty, D_p)$ is co-torsion because A_∞ is of Λ-torsion. Thus, the claims of torsionness of (2) all follow from the commutative diagram (13). Moreover, the source and the target of Loc_+ have the same rank and the source is torsion-free, so $\ker(\mathrm{Loc}_+)$ must vanish. Since the two other kernels are submodules of $\ker(\mathrm{Loc}_+)$, they must be trivial as well. □

3.6. **Torsionness of Selmer groups.** For any character $\eta \in \widehat{\Gamma}$ factoring through Γ_m but not through Γ_{m-1} for some $m \geq 0$, let $M[\eta] = M \otimes_{\Lambda,\eta} \overline{\mathbb{Q}}_p$ be the η-isotypic component of a Λ-module M, and let

$$|\cdot|_\eta : \left\{ \begin{array}{ccc} U'_\infty & \longrightarrow & U'_m[\eta] \\ (u_n)_{n\geq 0} & \mapsto & e_\eta . u_m, \end{array} \right.$$

where $e_\eta \in \overline{\mathbb{Q}}_p[\Gamma_m]$ is the idempotent attached to η (see Section 2.1), and where U_m is seen as a Λ-module via the projection map $\Lambda \twoheadrightarrow \mathcal{O}_p[\Gamma_m]$.

Lemma 3.6.1. *Let η be a non-trivial character of Γ of conductor p^n. Consider the following commutative diagram*

$$\begin{array}{ccc} \mathrm{Hom}_G(T_p,U_\infty)[\eta] & \longrightarrow & \mathrm{Hom}_G(T_p,U_{n-1})[\eta] \\ \downarrow & & \downarrow \\ \mathrm{Hom}_G(T_p,U'_\infty)[\eta] & \longrightarrow & \mathrm{Hom}_G(T_p,U'_{n-1})[\eta], \end{array}$$

where the horizontal maps are induced by $|\cdot|_\eta$. Then all the four maps are isomorphisms.

Proof. Since Γ acts trivially on the quotient U'_∞/U_∞ (resp. U'_{n-1}/U_{n-1}) and since η is non-trivial by assumption, we have $U_\infty[\eta]=U'_\infty[\eta]$ (resp. $U_{n-1}[\eta]=U'_{n-1}[\eta]$), so the two vertical maps are isomorphisms. Hence, we only have to show that the bottom horizontal map is an isomorphism. By Lemmas 3.5.2 (1) and 3.5.4 (1), its domain has dimension d^+ over $\overline{\mathbb{Q}}_p$, as well as for its codomain because $\eta\neq\mathbb{1}$. Therefore, it is enough to check its injectivity, which easily follows from the fact that the projection map injects $(U'_\infty)_{\Gamma^{p^{n-1}}}$ into U'_{n-1} by [Kuz72, Theorem 7.3]. □

Remark 3.6.2. The $\overline{\mathbb{Q}}_p$-vector space $\mathrm{Hom}_G(T_p,U_{n-1})[\eta]$ in Lemma 3.6.1 can be identified with $\mathrm{Hom}_{G_\mathbb{Q}}(W_{p,\eta},U_{n-1}\otimes\overline{\mathbb{Q}}_p)$, and hence with $\overline{\mathbb{Q}}_p\otimes_E \mathrm{H}^1_{\mathrm{f}}((\rho\otimes\eta)^\vee(1))$.

Fix a eigenbasis $\omega_p^+=t_1\wedge\ldots\wedge t_{d^+}$ of T_p^+ for the action of G_p and let $\delta_1,\ldots,\delta_{d^+}:G_p\longrightarrow\mathcal{O}_p^\times$ be the corresponding characters. The number of δ_i's which are trivial is $f-e$, where $f=\dim\mathrm{H}^0(\mathbb{Q}_p,W_p)$ and $e=\dim\mathrm{H}^0(\mathbb{Q}_p,W_p^-)$. We define two composite maps
(14)

$$\mathscr{C}_{\omega_p^+}:\bigwedge^{d^+}\mathrm{Hom}_G(T_p,U_\infty)\xrightarrow{\wedge\mathrm{Loc}_+}\bigwedge^{d^+}\mathrm{Hom}_{G_p}(T_p,U_{\infty,w})\xrightarrow[\simeq]{\wedge\mathrm{ev}_{\omega_p^+}}\bigwedge^{d^+}\left(\oplus_{i=1}^{d^+}U_{\infty,w}^{\delta_i}\right)\xrightarrow[\simeq]{\wedge_i\mathrm{Col}^{\delta_i}}\Lambda,$$

$$\mathscr{C}^{\mathrm{str}}_{\omega_p^+}:\bigwedge^{d^+}\mathrm{Hom}_G(T_p,U'_\infty)\xrightarrow{\wedge\mathrm{Loc}_+^{\mathrm{str}}}\bigwedge^{d^+}\mathrm{Hom}_{G_p}(T_p,U'_{\infty,w})\xrightarrow[\simeq]{\wedge\mathrm{ev}_{\omega_p^+}}\bigwedge^{d^+}\left(\oplus_{i=1}^{d^+}(U'_{\infty,w})^{\delta_i}\right)\xrightarrow[\simeq]{\wedge_i\widetilde{\mathrm{Col}}^{\delta_i}}\mathcal{J}^{f-e},$$

where the map $\mathrm{ev}_{\omega_p^+}$ is the natural map induced by the evaluation at $t_1,\ldots,t_{d^+}$, where the last maps are the one of Definitions 2.2.1 and 2.2.3 and where $\mathcal{J}$ is the invertible ideal of Λ introduced in Definition 2.2.3 for $\mathcal{O}=\mathcal{O}_p$. Once we fix a topological generator of Γ and thus an isomorphism $\Lambda\simeq\mathcal{O}_p[[T]]$, $\mathcal{J}^{f-e}$ is nothing but $T^{-(f-e)}\Lambda\subseteq\mathrm{Frac}(\Lambda)$.

Lemma 3.6.3. *Fix an eigenbasis ω_p^+ of T_p^+ for G_p and let $\omega=\Psi_1\wedge\ldots\wedge\Psi_{d^+}$ be an element of $\bigwedge^{d^+}\mathrm{Hom}_G(T_p,U_\infty)$ (resp. of $\bigwedge^{d^+}\mathrm{Hom}_G(T_p,U'_\infty)$). Then for all non-trivial characters $\eta\in\widehat{\Gamma}$ of conductor p^n, the image θ of ω under $\mathscr{C}_{\omega_p^+}$ (resp. under $\mathscr{C}^{\mathrm{str}}_{\omega_p^+}$) satisfies*

$$\eta(\theta)=\left(\frac{p^{n-1}}{\mathfrak{g}(\eta^{-1})}\right)^{d^+}\det(\rho^+)(p^n)\det\left(\log_p|\Psi_j(t_i)|_\eta\right)_{1\le i,j\le d^+}.$$

Proof. Let $\beta_i=\delta_i(\sigma_p)$ for all $1\le i\le d^+$. Note that $\det(\rho^+)(p)=\prod_{i=1}^{d^+}\beta_i$. In the case where $\theta=\mathscr{C}_{\omega_p^+}(\omega)$, Lemma 2.1.3 shows that

$$\begin{aligned}\eta(\theta)&=\det\left(\frac{p^{n-1}}{\mathfrak{g}(\eta^{-1})}\beta_i^n\log_p|\Psi_j(t_i)|_\eta\right)_{1\le i,j\le d^+}\\&=\left(\frac{p^{n-1}}{\mathfrak{g}(\eta^{-1})}\right)^{d^+}\det(\rho^+)(p^n)\det\left(\log_p|\Psi_j(t_i)|_\eta\right)_{1\le i,j\le d^+}\end{aligned}$$

for any $\eta \in \widehat{\Gamma}$ of conductor $p^n > 1$. Since $\mathscr{C}^{\rm str}_{\omega_p^+}$ extends $\mathscr{C}_{\omega_p^+}$, the case where $\theta = \mathscr{C}^{\rm str}_{\omega_p^+}(\omega)$ follows from the first one, noting that for non-trivial $\eta \in \widehat{\Gamma}$, the maps $\eta \circ \mathscr{C}_{\omega_p^+}$ and $\eta \circ \mathscr{C}^{\rm str}_{\omega_p^+}$ both factor through $\mathrm{Hom}_G(T_p, U_\infty)[\eta] = \mathrm{Hom}_G(T_p, U'_\infty)[\eta]$ and that they clearly coincide on it. □

Theorem 3.6.4. *Fix a basis ω_p^+ of T_p^+. The following conditions are equivalent:*

(i) $X_\infty(\rho,\rho^+)$ *is a torsion Λ-module,*
(ii) $\mathrm{coker}(\mathrm{Loc}_+)$ *is a torsion Λ-module,*
(iii) there exists a non-trivial character η of Γ of finite order such that $\mathrm{Reg}_{\omega_p^+}(\rho \otimes \eta) \neq 0$,
(iv) for all but finitely many characters η of Γ of finite order, one has $\mathrm{Reg}_{\omega_p^+}(\rho \otimes \eta) \neq 0$.

Moreover, if these equivalent conditions hold and if $d^+ > 0$, then there exists linearly independent elements $\Psi_1, \ldots, \Psi_{d^+} \in \mathrm{Hom}_G(T_p, U_\infty)$ which only depend on T_p and there exists a generator $\theta^{alg}_{\rho,\rho^+}$ of the characteristic ideal of $X_\infty(\rho,\rho^+)$ such that

$$\eta(\theta^{alg}_{\rho,\rho^+}) = \frac{\tau(\eta)^{d^-}}{\tau(\rho\otimes\eta)}\,\frac{p^{(n-1)\cdot d^+}}{\det(\rho^-)(p^n)} \cdot \det\left(\log_p |\Psi_j(t_i)|_\eta\right)_{1\le i,j\le d^+}, \tag{15}$$

for all non-trivial characters $\eta \in \widehat{\Gamma}$ of conductor p^n, where we have written $\omega_p^+ = t_1 \wedge \ldots \wedge t_{d^+}$.

Proof. We may assume without loss of generality that the basis $\omega_p^+ = t_1 \wedge \ldots \wedge t_{d^+}$ of T_p^+ is an eigenbasis for G_p. The equivalence of (i) and (ii) follows from Lemma 3.5.4. We now show the equivalence of the three last statements and we will use Lemma 3.5.2 to identify the source and the target of Loc_+ with the respective Hom's, which are known to both be free of rank d^+ over Λ by Lemma 3.5.4. The statement (ii) is equivalent to the injectivity of Loc_+, which in turn is equivalent to the injectivity of $\mathscr{C}_{\omega_p^+}$, *i.e.*, to the non-vanishing of $\theta_1^{\rm alg} := \mathscr{C}_{\omega_p^+}(\widetilde{\omega})$, where $\widetilde{\omega} = \tilde{\Psi}_1 \wedge \ldots \wedge \tilde{\Psi}_{d^+}$ is a Λ-basis of $\mathrm{Hom}_G(T_p, U_\infty)$. Moreover, if (ii) holds, then the characteristic ideal of coker(Loc_+) is generated by $\theta_1^{\rm alg}$. By Lemma 3.6.3, for any character $\eta \in \widehat{\Gamma}$ of conductor $p^n > 1$, one has

$$\eta(\theta_1^{\rm alg}) = \left(\frac{p^{n-1}}{\mathfrak{g}(\eta^{-1})}\right)^{d^+} \det(\rho^+)(p^n) \det\left(\log_p |\tilde{\Psi}_j(t_i)|_\eta\right)_{1\le i,j\le d^+}.$$

Since the image of the basis $\tilde{\Psi}_1, \ldots, \tilde{\Psi}_{d^+}$ under $|\cdot|_\eta$ is a $\overline{\mathbb{Q}}_p$-basis of $\overline{\mathbb{Q}}_p \otimes_E \mathrm{H}^1_{\rm f}((\rho\otimes\eta)^\vee(1))$ by Lemma 3.6.1 and Remark 3.6.2, the last $d^+ \times d^+$-sized determinant is a non-zero multiple of $\mathrm{Reg}_{\omega_p^+}(\rho\otimes\eta)$. Therefore, by Weierstrass preparation theorem, one has $\theta_1^{\rm alg} \neq 0$ if and only if $\mathrm{Reg}_{\omega_p^+}(\rho\otimes\eta) \neq 0$ for some character $\eta \neq \mathbb{1}$, if and only if $\mathrm{Reg}_{\omega_p^+}(\rho\otimes\eta) \neq 0$ for all but finitely many characters $\eta \neq \mathbb{1}$. This shows the equivalence (ii)⇔(iii)⇔(iv).

Let us now assume (i)-(iv) and let $\theta_2^{\rm alg}$ be a generator of $\mathrm{char}_\Lambda \mathrm{H}^1_{\rm ur}(\mathbb{Q}_\infty, D_p)^\vee$. By the exactness of the second row of the diagram (13) and by multiplicativity of characteristic ideals, the p-adic measure $\theta^{\rm alg}_{\rho,\rho^+} := \theta_1^{\rm alg}\theta_2^{\rm alg}$ is a generator of $\mathrm{char}_\Lambda X_\infty(\rho,\rho^+)$. If one moreover assumes $d^+ > 0$, then one can simply set $\Psi_1 = \tau(\rho)^{-1}\cdot\theta_N\cdot\theta_2^{\rm alg}\tilde{\Psi}_1, \Psi_2 = \tilde{\Psi}_2, \ldots, \Psi_{d^+} = \tilde{\Psi}_{d^+}$, where θ_N is as in the proof of Proposition 3.3.3, and Formula (15) follows by linearity from the above expression of $\eta(\theta_1^{\rm alg})$. □

3.7. **An $\mathcal{L}$-invariant for** (ρ,ρ^+). We define in this section an $\mathcal{L}$-invariant for ρ which will depend on the choice of an admissible p-stabilization W_p^+ of W_p. Its definition still makes sense for ρ ramified at p and it generalizes Gross's $\mathcal{L}$-invariant (see Section 6.3). We put

$\mathcal{U} = E_p \otimes U_0$ and $\mathcal{U}' = E_p \otimes U_0'$, where U_0 (resp. U_0') is the formal p-adic completion of the group of global units (resp. of p-units) of H (see Notation 3.5.1 for $n = 0$). Recall that the extension K of $\mathbb{Q}_p$ cut out by $\rho_{|G_{\mathbb{Q}_p}}$ is assumed to be contained in E_p. Recall also that $\mathrm{H}^1_{f,p}(\mathbb{Q}, \breve{W}_p) \simeq \mathrm{Hom}_G(W_p, \mathcal{U}')$. The p-adic valuation map and the p-adic logarithm map define two $G_{\mathbb{Q}_p}$-equivariant maps $\mathrm{ord}_p : \mathcal{U}' \longrightarrow E_p$ and $\log_p : \mathcal{U}' \longrightarrow K \otimes E_p$ where the $G_{\mathbb{Q}_p}$-action on the target of the first map is trivial. These maps give rise to two linear maps

$$\mathrm{ord}_p^0 : \mathrm{Hom}_G(W_p, \mathcal{U}') \longrightarrow \mathrm{Hom}_{G_p}(W_p, E_p) = \mathrm{Hom}(W_p^0, E_p)$$
$$\log_p : \mathrm{Hom}_G(W_p, \mathcal{U}') \longrightarrow \mathrm{Hom}_{G_p}(W_p, K \otimes E_p) \simeq \mathrm{Hom}(W_p, E_p),$$

where we have put $W_p^0 = \mathrm{H}^0(\mathbb{Q}_p, W_p)$ and where the last isomorphism is induced by the internal multiplication $K \otimes E_p \to E_p$. Choose any $G_{\mathbb{Q}_p}$-stable complement W_p^- of W_p^+ and let $W_p^{\pm,0} = \mathrm{H}^0(\mathbb{Q}_p, W_p^{\pm})$. The composition with restriction maps yields the following maps

$$\mathrm{ord}_p^{\pm,0} = \mathrm{res}_{W_p^{\pm,0}} \circ \mathrm{ord}_p^0, \quad \log_p^+ = \mathrm{res}_{W_p^+} \circ \log_p, \quad \log^{-,0} = \mathrm{res}_{W_p^{-,0}} \circ \log_p,$$

which can be combined in order to obtain two linear maps

$$\mathrm{Hom}_G(W_p, \mathcal{U}') \xrightarrow[\mathrm{ord}_p^{+,0} \oplus \log_p^{-,0} \oplus \log_p^+]{\mathrm{ord}_p^0 \oplus \log_p^+} \mathrm{Hom}(W_p^0, E_p) \bigoplus \mathrm{Hom}(W_p^+, E_p) =: Z.$$

Note that W_p^+ is admissible if and only if the restriction of $\log_p^+$ to $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}, \breve{W}_p) = \mathrm{Hom}_G(W_p, \mathcal{U})$ is an isomorphism onto $\mathrm{Hom}(W_p^+, E_p)$. Therefore, the map $\mathrm{ord}_p^0 \oplus \log_p^+$ is an isomorphism.

Definition 3.7.1. Let $W_p^+ \subseteq W_p$ be an admissible p-stabilization of ρ. We define the $\mathcal{L}$-invariant attached to (ρ, ρ^+) as to be

$$\mathcal{L}(\rho, \rho^+) = \det\left(\left(\mathrm{ord}_p^{+,0} \oplus \log_p^{-,0} \oplus \log_p^+\right) \circ \left(\mathrm{ord}_p^0 \oplus \log_p^+\right)^{-1} \Big| Z \right) \in E_p.$$

We now give equivalent and more useful definitions of $\mathcal{L}(\rho, \rho^+)$. Fix bases $\omega_p^+ = t_1^+ \wedge \ldots \wedge t_{d^+}^+$ and $\omega_p^{-,0} = t_1^- \wedge \ldots \wedge t_e^-$ of W_p^+ and of $W_p^{-,0}$ respectively. Then one may identify Z with $d^+ + f$ copies of E_p and $\mathcal{L}(\rho, \rho^+)$ will satisfy

$$(\wedge \mathrm{ord}_p^{+,0}) \wedge (\wedge \log_p^{-,0}) \wedge (\wedge \log_p^+) = \mathcal{L}(\rho, \rho^+) \cdot (\wedge \mathrm{ord}_p^0) \wedge (\wedge \log_p^+)$$

in $\bigwedge^{d^+ + f} \mathrm{Hom}(\mathrm{Hom}(W_p, \mathcal{U}'), E_p) = \det_{E_p} \mathrm{Hom}(\mathrm{H}^1_{\mathrm{f},p}(\mathbb{Q}, \breve{W}_p), E_p)$.

The kernel of $\mathrm{ord}_p^{+,0}$ contains $\mathrm{Hom}_G(W_p, \mathcal{U})$ and is of dimension $d^+ + e$ because W_p^+ is admissible. Choose any basis $\psi_1 \wedge \ldots \wedge \psi_{d^+} \wedge \psi_1' \wedge \ldots \wedge \psi_e'$ of $\ker\left(\mathrm{ord}_p^{+,0}\right)$ such that $\psi_1 \wedge \ldots \wedge \psi_{d^+}$ is a basis of $\mathrm{Hom}_G(W_p, \mathcal{U})$, and define the following matrices with coefficients in E_p:

$$A^{\pm} = \left[\log_p\left(\psi_j(t_i^{\pm})\right)\right]_{i,j}, \quad B^{\pm} = \left[\log_p\left(\psi_j'(t_i^{\pm})\right)\right]_{i,j}, \quad O^- = \left[\mathrm{ord}_p\left(\psi_j'(t_i^-)\right)\right]_{i,j}.$$

The square matrices A^+, B^- and O^- have respective sizes d^+, e and e. The determinant of A^+ is non-zero and it is equal to $\mathrm{Reg}_{\omega_p^+}(\rho)$ modulo $E^\times$ if the basis $\psi_1 \wedge \ldots \wedge \psi_{d^+}$ is taken E-rational (see (8)). Also, O^- is invertible.

Lemma 3.7.2. *$\mathcal{L}(\rho, \rho^+)$ can be expressed as a quotient of determinants as follows:*

$$\mathcal{L}(\rho, \rho^+) = \frac{\det\begin{pmatrix} A^+ & B^+ \\ A^- & B^- \end{pmatrix}}{\det(A^+) \cdot \det(O^-)}.$$

Proof. It is a simple computation of linear algebra. □

3.8. **Relation to Perrin-Riou's theory.** Let M be the Artin motive attached to ρ and let $\breve{M} = M^*(1)$ be its dual. Then $\breve{M}$ is a pure motive of weight -2 over $\mathbb{Q}$ which is crystalline at p and whose p-adic realization is the arithmetic dual $\breve{W}_p = W_p^*(1)$ of W_p. We relate our Selmer group defined in Section 3.2 to Perrin-Riou's definition of the module of p-adic L-functions attached to $\breve{M}$ given in [PR95] by using Benois' interpretation in terms of Selmer complexes. It depends on the choice of a Galois stable $\mathcal{O}_p$-lattice T_p of W_p and of a regular subspace D of $\mathbf{D}_{\mathrm{crys}}(\breve{W}_p)$ whose definition is first recalled.

Let $G_p = \mathrm{Gal}(K/\mathbb{Q}_p)$ and assume as before that $K \subseteq E_p$. Let $t \in \mathbf{B}_{\mathrm{crys}}$ be Fontaine's p-adic period. The Dieudonné module $\mathbf{D}_{\mathrm{crys}}(\breve{W}_p) = \left(\breve{W}_p \otimes \mathbf{B}_{\mathrm{crys}}\right)^{G_{\mathbb{Q}_p}}$ can be described as

$$\mathbf{D}_{\mathrm{crys}}(\breve{W}_p) \simeq \mathrm{Hom}_{G_p}\left(W_p, t^{-1}K \otimes E_p\right) \simeq \mathrm{Hom}\left(W_p, E_p\right), \tag{16}$$

where K is seen as a G_p-module for the obvious action and where the second isomorphism is induced by the internal multiplication $t^{-1}K \otimes E_p \simeq K \otimes E_p \longrightarrow E_p$. The action of the crystalline Frobenius φ is given on $\mathrm{Hom}\left(W_p, E_p\right)$ by $\varphi(f)(w) = p^{-1}f(\sigma_p^{-1}.w)$, where $\sigma_p \in G_p$ is the arithmetic Frobenius at p.

Any φ-submodule D of $\mathbf{D}_{\mathrm{crys}}(\breve{W}_p)$ gives rise to a regulator map r_D given by the composition

$$r_D:\ \mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}, \breve{W}_p) \xrightarrow{\mathrm{loc}_p} \mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}_p, \breve{W}_p) \xrightarrow{\log_{\mathrm{BK}}} \mathbf{D}_{\mathrm{crys}}(\breve{W}_p) \twoheadrightarrow \mathbf{D}_{\mathrm{crys}}(\breve{W}_p)/D,$$

where loc_p is the localization at p, where $\log_{\mathrm{BK}}$ is Bloch-Kato's logarithm. The φ-module D is called regular whenever r_D is an isomorphism (see [Ben14, §4.1.3]).

Lemma 3.8.1. *Under the identification (16) any φ-submodule D of $\mathbf{D}_{\mathrm{crys}}(\breve{W}_p)$ of E_p-dimension d^- can be uniquely written as $D = \mathrm{Hom}\left(W_p/W_p^+, E_p\right)$ where W_p^+ is a p-stabilization of W_p, and any p-stabilization W_p^+ of W_p defines a φ-submodule in this way. It is moreover regular if and only if W_p^+ is admissible.*

Proof. The first claim is obvious. Let us prove the second claim and put $D = \mathrm{Hom}(W_p/W_p^+, E_p)$, where W_p^+ is a p-stabilization of W_p. Under the identification (16) the composite map $\log_{\mathrm{BK}} \circ \mathrm{loc}_p$ coincides with the composite map given in (10). Therefore, r_D coincides with the map $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}, \breve{W}_p) \longrightarrow \mathrm{Hom}(W_p^+, E_p)$ induced by the p-adic pairing (3), which, by definition, is an isomorphism if and only if W_p^+ is admissible. □

Given a pure motive of weight -2 whose p-adic realization V satisfies conditions **(C1-C5)** of [Ben14, §4.1.2] and given a regular submodule D of $D_{\mathrm{crys}}(V)$, Benois has defined an $\mathcal{L}$-invariant $\mathcal{L}(V, D)$ [Ben14, §4.1.4]. It is not hard to see that $V = \breve{W}_p$ satisfies the above-mentioned conditions: the first one follows from the finiteness of the ideal class group of H, the second one from the running assumption $\mathrm{H}^0(\mathbb{Q}, W) = 0$, the third and fourth ones from the unramifiedness assumption at p and from the semi-simplicity of $\rho(\sigma_p)$, and the last one is true whenever there exists at least one regular submodule D of $D_{\mathrm{crys}}(\breve{W}_p)$.

Lemma 3.8.2. *Let W_p^+ be an admissible p-stabilization of W_p, let $V = \breve{W}_p$ be the p-adic realization of $\breve{M}$ and let D be the regular submodule of $\mathbf{D}_{\mathrm{crys}}(V)$ defined as in (16). Then $\mathcal{L}(\rho, \rho^+) = (-1)^e \mathcal{L}(V, D)$, where $\mathcal{L}(V, D)$ is Benois' $\mathcal{L}$-invariant for V and D as defined in* [Ben14, §4.1.4].

Proof. Unwinding Benois' definition shows that $\mathcal{L}(V,D)$ is $(-1)^e$ times the quotient of determinants appearing in Lemma 3.7.2, so it is equal to $(-1)^e\mathcal{L}(\rho,\rho^+)$ by the same lemma. □

The main algebraic object in Perrin-Riou's formulation of Iwasawa theory for a motive that is crystalline at p is the module of p-adic L-functions, introduced and studied in [PR95, Chapter 2] and later interpreted (and generalized) in [Ben14, §6.2.3] in terms of Selmer complexes. Its definition only makes sense when $\mathcal{L}(V,D)\neq 0$ and under the Weak Leopoldt conjecture for V and for $\breve{V}=V^*(1)$ together with conditions **(C1-C5)** of *loc. cit.*. It is denoted by $\mathbf{L}^{(\eta_0)}_{\mathrm{Iw},h}(N,T)$ in *loc. cit.* and it depends on the choice of a $G_{\mathbb{Q}}$-stable lattice T of V, on the choice of a $\mathcal{O}_p$-lattice N of a regular submodule D of $D_{\mathrm{crys}}(V)$ and on a parameter $h>0$.

We consider here the case of the dual motive $\breve{M}$ of ρ. More precisely, let T_p be a $G_{\mathbb{Q}}$-stable lattice of W_p and let W_p^+ be an admissible p-stabilization of W_p. We put $V=\breve{W}_p$, $T=\breve{T}_p$ and $V^-=\breve{W}_p^-=(\widecheck{W_p/W_p^+})$. Under the identification (16), we define a regular submodule of $D_{\mathrm{crys}}(V)$ by letting $D=D_{\mathrm{crys}}(V^-)=\mathrm{Hom}_{G_p}(W_p^-,t^{-1}K\otimes E_p)$ (see Lemma 3.8.1). Explicitly, Bloch-Kato's logarithm map for V^- is the isomorphism

$$\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}_p,V^-)=\mathrm{Hom}_{G_p}(W_p^-,\mathcal{O}_K^{\times,1}\otimes E_p)\xrightarrow{\sim}\mathrm{Hom}_{G_p}(W_p^-,t^{-1}K\otimes E_p)=D \tag{17}$$

induced by the p-adic logarithm $\log_p:\mathcal{O}_K^{\times,1}\xrightarrow{\sim}p\mathcal{O}_K\subseteq K\simeq t^{-1}K$, where $\mathcal{O}_K^{\times,1}$ is the group of principal units of K. We define $N\subseteq D$ as to be $\mathrm{Hom}_{G_p}(T_p^-,t^{-1}p\mathcal{O}_K\otimes\mathcal{O}_p)$, so that the map in (17) sends $\mathrm{H}^1_{\mathrm{f}}(\mathbb{Q}_p,\breve{T}_p^-)$ onto N. We also may set $h=1$ in the definition of the module of p-adic L-functions for $\breve{M}$ since the $G_{\mathbb{Q}_p}$-representation V is the Tate twist of an unramified representation.

Proposition 3.8.3. *Assume that W_p^+ is admissible and that $\mathcal{L}(\rho,\rho^+)\neq 0$. Then $X_\infty(\rho,\rho^+)$ is of Λ-torsion, and we have $\mathbf{L}^{(\eta_0)}_{\mathrm{Iw},h}(N,T)=char_\Lambda X_\infty(\rho,\rho^+)$.*

Proof. Fix a topological generator γ of Γ and let

$$\mathcal{H}=\left\{f(\gamma-1)\mid f(X)\in E_p[[X]]\text{ is holomorphic on the open unit disc}\right\}$$

be the large Iwasawa algebra. Consider the complex of $\mathcal{H}$-modules $\mathbf{R}\Gamma^{(\eta_0)}_{\mathrm{Iw},h}(D,V)$ defined in [Ben14, §6.1.2]. Note that it is possible because we already checked conditions **(C1-5)** and because the Weak Leopoldt conjecture for V and $V^*(1)$ holds by Proposition 3.4.2. It is a Selmer complex in the sense of [Nek06, (6.1)] given by the following local conditions: at finite primes $\ell\neq p$ we take the unramified condition, and at p we take the derived version of Perrin-Riou's exponential map $\mathrm{Exp}_{V,h}:(N\otimes\Lambda)\otimes_\Lambda\mathcal{H}\longrightarrow\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p,\breve{T}_p^-)\otimes_\Lambda\mathcal{H}\subseteq\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p,T)\otimes_\Lambda\mathcal{H}$. As explained in [PR94, §4.1.3-5], the map $\mathrm{Exp}_{V,h}$ is induced by the inverse of Coleman's isomorphism $\mathrm{Col}_N:\mathrm{H}^1_{\mathrm{Iw,f}}(\mathbb{Q}_p,\breve{T}_p^-)\xrightarrow{\sim}N\otimes\Lambda$. Since $\mathcal{H}$ is flat over Λ, the complex $\mathbf{R}\Gamma^{(\eta_0)}_{\mathrm{Iw},h}(D,V)$ is a base change to $\mathcal{H}$ of a Selmer complex $\mathbf{R}\Gamma_{\mathrm{Iw}}(\rho,\rho^+)$ over Λ given by the unramified condition at $\ell\neq p$, and at p by the morphism of complexes $N\otimes\Lambda[-1]\longrightarrow\mathbf{R}\Gamma_{\mathrm{Iw}}(\mathbb{Q}_p,T)$ induced by $(\mathrm{Col}_N)^{-1}$ in degree 1. By [Ben14, Theorem 4], the Λ-module $\mathbf{R}^i\Gamma_{\mathrm{Iw}}(\rho,\rho^+)$ vanishes when $i\neq 2$ and it is of Λ-torsion for $i=2$. Moreover, as in [Ben14, §6.1.3.3] we have a short exact sequence

$$0\longrightarrow\mathrm{coker}(\mathrm{Loc}'_+)\longrightarrow\mathbf{R}^2\Gamma_{\mathrm{Iw}}(\rho,\rho^+)\longrightarrow Ш^2_\infty(\breve{T}_p)\longrightarrow 0,$$

where Loc'_+ is the localization map introduced in Section 3.4. It follows easily from the exactness of the first row of (13) and from Proposition 3.4.2 that $X_\infty(\rho,\rho^+)$ is also of Λ-torsion, and that it shares the same characteristic ideal with $\mathbf{R}^2\Gamma_{\mathrm{Iw}}(\rho,\rho^+)$, the latter being equal to $\mathbf{L}^{(\eta_0)}_{\mathrm{Iw},h}(N,T)$ by construction. □

Corollary 3.8.4. *Assume that W_p^+ is admissible and that $\mathcal{L}(\rho,\rho^+)\neq 0$. If Conjecture A holds, then the p-part of Bloch-Kato's conjecture (in the formulation of Fontaine and Perrin-Riou,* [FPR94, III, 4.5.2]*) holds for the Artin motive associated with ρ, that is,*

$$\frac{L^*\left(\rho^\vee,0\right)}{\mathrm{Reg}_{\omega_\infty^+}(\rho)}\sim_p \frac{\#\text{Ш}(T_p)\cdot\prod_{\ell\neq p}\#\left(\mathrm{H}^1(I_\ell,\breve{T}_p)^{G_{\mathbb{Q}_\ell}}\right)_{tors}}{\#\mathrm{H}^0(\mathbb{Q},D_p)},$$

where $a\sim_p b$ means that a and b are equal up to a p-adic unit, where $\mathrm{Reg}_{\omega_\infty^+}(\rho)$ is computed with respect to T_p-optimal bases ω_∞^+ and ω_{f} of $\mathrm{H}^0(\mathbb{R},W)$ and $\mathrm{H}^1_{\mathrm{f}}(\rho^\vee(1))$ respectively, where $\text{Ш}(T_p)$ is the Tate-Shafarevitch group of T_p [FPR94, II,5.3.4] *and where I_ℓ is the absolute inertia group at ℓ. In particular, when p does not divide the order of the image of ρ, one has*

$$\frac{L^*\left(\rho^\vee,0\right)}{\mathrm{Reg}_{\omega_\infty^+}(\rho)}\sim_p \#\mathrm{Hom}_{\mathcal{O}_p[G]}(T_p,\mathcal{O}_p\otimes_{\mathbb{Z}} C\ell(H)),$$

where $C\ell(H)$ is the ideal class group of the field H cut out by ρ.

Proof. Assume Conjecture A for (ρ,ρ^+) and consider the p-adic measure θ'_{ρ,ρ^+} of Proposition 3.3.3. Fix a T_p-optimal basis ω_p^+ of W_p^+. By $\mathbf{IMC}_{\rho,\rho^+}$, the p-adic analytic function $L_p: s\mapsto \kappa^s(\theta'_{\rho,\rho^+})$ is equal (up to a unit) to the one denoted $L_{\mathrm{Iw},h}(T,N,s)$ in [Ben14, §6.2.3]. Therefore, [Ben14, Corollary 2], together with a comparison of the p-adic regulators and of the modified Euler factors and with a straightforward computation of the local Tamagawa numbers (as in [FPR94, I, 4.2]), shows that

$$\frac{1}{e!}\cdot\frac{L_p^{(e)}(0)}{\mathrm{Reg}_{\omega_p^+}(\rho)}\sim_p \mathcal{L}(\rho,\rho^+)\cdot\mathcal{E}(\rho,\rho^+)\cdot\frac{\#\text{Ш}(T_p)\cdot\prod_{\ell\neq p}\#\left(\mathrm{H}^1(I_\ell,\breve{T}_p)^{G_{\mathbb{Q}_\ell}}\right)_{\mathrm{tors}}}{\#\mathrm{H}^0(\mathbb{Q},D_p)},$$

so we obtain the desired formula from $\mathbf{EZC}_{\rho,\rho^+}$ after simplification by $\mathcal{L}(\rho,\rho^+)\cdot\mathcal{E}(\rho,\rho^+)\neq 0$.

We now explain how to simplify the formula in the case where p does not divide the order of G. It is plain that $\mathrm{H}^0(\mathbb{Q},D_p)\hookrightarrow \mathrm{H}^1(G,T_p)=0$. To see that the local Tamagawa numbers are all trivial, let us fix any prime $\ell\neq p$. By [Rub00, Lemma 3.2 (ii) and Lemma 3.5 (ii-iii)], one has $\left(\mathrm{H}^1(I_\ell,\breve{T}_p)^{G_{\mathbb{Q}_\ell}}\right)_{\mathrm{tors}}\simeq \mathcal{W}^{\sigma_\ell=1}$, where $\mathcal{W}$ is the quotient of $\breve{D}_p^{I_\ell}$ by its divisible part. But the action of I_ℓ on $\breve{D}_p$ factors through a finite group of prime-to-p order, so we must have $\mathcal{W}=0$. Finally, the description of $\text{Ш}(T_p)$ in terms of class groups directly follows from the inflation-restriction exact sequence. □

We end this section with some applications of our results to a generalization of the ρ-isotypic part of "Gross's finiteness conjecture" appearing in [BKS17, Theorem 1.1]:

Conjecture 3.8.5. *Let $A'_\infty=\varprojlim_n A'_n$ be the inverse limit over n of the p-split ideal class group of H_n (see Notation 3.5.1). Then the module of Γ-coinvariants of $\mathrm{Hom}_G(T_p,A'_\infty)$ is finite.*

Theorem 3.8.6. *Let $f=\dim\mathrm{H}^0(\mathbb{Q}_p,W_p)$.*

(1) *Let W_p^+ be any admissible p-stabilization of W_p such that $\mathcal{L}(\rho,\rho^+)\neq 0$. Any generator of the characteristic ideal of $X_\infty(\rho,\rho^+)$ belongs to $\mathcal{A}^e\backslash\mathcal{A}^{e+1}$, where $\mathcal{A}\subseteq\Lambda$ is the augmentation ideal and where $e=\dim \mathrm{H}^0(\mathbb{Q}_p,W_p^-)$.*

(2) *If there exists at least one admissible p-stabilization W_p^+ of W_p such that $\mathcal{L}(\rho,\rho^+)\neq 0$, then Conjecture 3.8.5 holds.*

(3) *If $f=0$ and if the ρ-isotypic component of Leopoldt's conjecture for H and p holds (see (11)), then Conjecture 3.8.5 holds as well.*

(4) *If $f,d^+\leq 1$, then Conjecture 3.8.5 holds.*

Proof. The first statement follows from Proposition 3.8.3 and from [Ben14, Theorem 5 (i)]. For the three other statements, first note that, by the exactness of the third row of (13), by Proposition 3.4.2 (1) by Lemma 3.5.2 (ii), the existence of a p-stabilization ρ^+ such that $X_\infty^{\mathrm{str}}(\rho,\rho^+)$ has finite Γ-coinvariants immediately implies Conjecture 3.8.5. Therefore, claim (2) follows from (1) and from Lemma 3.2.2. Consider (3) and assume that $f=0$ and that the map in (11) is injective for $\eta=\mathbb{1}$. By Lemma 3.1.4, there exists an admissible p-stabilization ρ^+ of ρ, and since $f=0$, one must have $e=0$ as well. Therefore, $\mathcal{L}(\rho,\rho^+)=1$ by Lemma 3.7.2, so (3) follows from (2). Let us prove (4), and assume that $f,d^+\leq 1$. When $d^+=1$, it is easy to produce a motivic p-stabilization ρ^+ such that $e=0$, so $\mathcal{L}(\rho,\rho^+)=1$ (take W_p^+ containing $\mathrm{H}^0(\mathbb{Q}_p,W_p)$). Since every motivic p-stabilization is automatically admissible by Lemma 3.1.3, Conjecture 3.8.5 follows in this case from (2). The case where $d^+=0$ follows from [Gro81, Proposition 2.13], once we have checked that $\mathcal{L}(\rho,\rho^+)$ generalizes Gross's regulator for $\rho^+=0$ (see Section 6.3 for details). ☐

3.9. **Changing the p-stabilization.** Let $t_1,\dots,t_d$ be an eigenbasis of T_p for σ_p. We may define a basis of $\bigwedge^{d^+}W_p$ by letting $\omega_{p,\alpha}^+=t_{i_1}\wedge\ldots\wedge t_{i_{d^+}}$, where $\alpha=(1\leq i_1<\ldots<i_{d^+}\leq d)$ runs over the set I of strictly increasing sequences of d^+ integers between 1 and d. For each $\alpha\in I$, $\omega_{p,\alpha}^+$ defines a T_p-optimal basis of a p-stabilization $(\rho_\alpha^+,W_{p,\alpha}^+)$ of W_p. Let $\omega_p^+\in\bigwedge^{d^+}W_p$ be a T_p-optimal eigenbasis of a given p-stabilization (ρ^+,W_p^+) of W_p. Write ω_p^+ as $\sum_{\alpha\in I}c_\alpha\cdot\omega_{p,\alpha}^+$ for $c_\alpha\in\mathcal{O}_p$. Writing ω_p^+ as a pure tensor and expanding in the eigenbasis $t_1,\dots,t_d$ shows that, for any $\alpha\in I$, we have $c_\alpha=0$ unless $\rho^+(\sigma_p)$ and $\rho_\alpha^+(\sigma_p)$ share the same list of eigenvalues. Thus, we have in particular $\mathcal{E}(\rho,\rho^+)=\mathcal{E}(\rho,\rho_\alpha^+)$, $\det(\rho^\pm)(p)=\det(\rho_\alpha^\pm)(p)$ and $e:=\dim\mathrm{H}^0(\mathbb{Q}_p,W_p^-)=\dim\mathrm{H}^0(\mathbb{Q}_p,W_{p,\alpha}^-)$ for all $\alpha\in I_{\rho^+}=\{\alpha\in I/c_\alpha\neq 0\}$. Consider the following strengthening of $\mathbf{EZC}_{\rho,\rho^+}$:

$\mathbf{sEZC}_{\rho,\rho^+}$: $\mathbf{EZC}_{\rho,\rho^+}$ holds, and if W_p^+ is not admissible, then θ_{ρ,ρ^+} has an order of vanishing greater than or equal to $e+1$ at the trivial character.

Proposition 3.9.1. *(1) If $\boldsymbol{EX}_{\rho,\rho_\alpha^+}$ holds for all $\alpha\in I_{\rho^+}$, then $\boldsymbol{EX}_{\rho,\rho^+}$ holds as well, and $\theta_{\rho,\rho^+}=\sum_{\alpha\in I_{\rho^+}}c_\alpha\cdot\theta_{\rho,\rho_\alpha^+}$.*

(2) If $\boldsymbol{sEZC}_{\rho,\rho_\alpha^+}$ holds for all $\alpha\in I_{\rho^+}$, then $\boldsymbol{EZC}_{\rho,\rho^+}$ holds as well.

Proof. Let us begin with (1), and assume that $\mathbf{EX}_{\rho,\rho_\alpha^+}$ holds for all $\alpha\in I_{\rho^+}$. For every character $\eta\in\widehat{\Gamma}$, the rule $\omega_p^+\mapsto\mathrm{Reg}_{\omega_p^+}(\rho\otimes\eta)$ (where the p-adic regulator is computed in a fixed basis $\omega_{\mathrm{f},\eta}$ of $\mathrm{H}^1_{\mathrm{f}}((\rho\otimes\eta)^\vee(1))$) defines a E_p-linear map $\bigwedge^{d^+}W_p\longrightarrow E_{p,\eta}$, so we have

$$\mathrm{Reg}_{\omega_p^+}(\rho\otimes\eta)=\sum_{\alpha\in I_{\rho^+}}c_\alpha\cdot\mathrm{Reg}_{\omega_{p,\alpha}^+}(\rho\otimes\eta).$$

Therefore, the element $\theta_{\rho,\rho^+} \in \mathrm{Frac}(\Lambda)$ defined as $\sum_{\alpha\in I_{\rho^+}} c_\alpha \cdot \theta_{\rho,\rho^+_\alpha}$ will satisfy

$$\begin{aligned}\eta(\theta_{\rho,\rho^+}) &= \sum_{\alpha\in I_{\rho^+}} c_\alpha \cdot \eta(\theta_{\rho,\rho^+_\alpha}) \\ &= M_{\rho,\eta} \cdot \sum_{\alpha\in I_{\rho^+}} c_\alpha \cdot \frac{\mathrm{Reg}_{\omega^+_{p,\alpha}}(\rho\otimes\eta)}{\det(\rho^-_\alpha)(p^n))} \\ &= M_{\rho,\eta} \cdot \frac{\mathrm{Reg}_{\omega^+_p}(\rho\otimes\eta)}{\det(\rho^-)(p^n))}\end{aligned}$$

for all non-trivial characters $\eta \in \widehat{\Gamma}$ of conductor p^n, where we have put $M_{\rho,\eta} = \frac{\tau(\eta)^{d^-}}{\tau(\rho\otimes\eta)} \frac{L^*((\rho\otimes\eta)^\vee,0)}{\mathrm{Reg}_{\omega^+_\infty}(\rho\otimes\eta)}$. Therefore, θ_{ρ,ρ^+} satisfies the interpolation property of $\mathbf{EX}_{\rho,\rho^+}$. Since it has no pole outside $\mathbb{1}$ by construction, we have shown that $\mathbf{EX}_{\rho,\rho^+}$ is valid.

Assume for (2) that $\mathbf{sEZC}_{\rho,\rho^+_\alpha}$ holds for all $\alpha \in I_{\rho^+}$. If ρ^+ is not admissible, then the only statement to prove is $\mathbf{EX}_{\rho,\rho^+}$, which follows from (1). Assume that ρ^+ is admissible, and denote by $I^{\mathrm{adm}}_{\rho^+}$ the subset I_{ρ^+} consisting of the elements $\alpha \in I_{\rho^+}$ for which $W^+_{p,\alpha}$ is admissible. Since $\mathrm{Reg}_{\omega^+_p}(\rho) \neq 0$, the formula for $\mathrm{Reg}_{\omega^+_p}(\rho)$ proven earlier shows that $I^{\mathrm{adm}}_{\rho^+}$ is non-empty, and a direct computation shows that

$$\mathcal{L}(\rho,\rho^+) = \sum_{\alpha\in I^{\mathrm{adm}}_{\rho^+}} c_\alpha \cdot \frac{\mathrm{Reg}_{\omega^+_{p,\alpha}}(\rho)}{\mathrm{Reg}_{\omega^+_p}(\rho)} \mathcal{L}(\rho,\rho^+_\alpha).$$

It follows easily from (1) and from this identity that θ_{ρ,ρ^+} has an order of vanishing at $\mathbb{1}$ greater than or equal to e, and that the e-th derivative of $s \mapsto \kappa^s(\theta_{\rho,\rho^+})$ at $s=0$ satisfies the formula predicted by $\mathbf{EZC}_{\rho,\rho^+}$. □

Remark 3.9.2. Conjecture A satisfies the following "p-adic Artin formalism": if $\rho = \rho_1 \bigoplus \rho_2$, and if ρ^+ is a p-stabilization of ρ which splits into a sum of two p-stabilizations ρ^+_1 and ρ^+_2 of ρ_1 and ρ_2 respectively, then the validity of **IMC** for any two pairs in $\{(\rho,\rho^+),(\rho_1,\rho^+_1),(\rho_2,\rho^+_2)\}$ implies the validity of **IMC** for the third pair, in which case $\theta_{\rho,\rho^+} = \theta_{\rho_1,\rho^+_1} \cdot \theta_{\rho_2,\rho^+_2}$. Also, either **EX** or **EZC** for both (ρ_1,ρ^+_1) and (ρ_2,ρ^+_2) implies the same statement for (ρ,ρ^+). However, ρ^+ needs not split in general even if ρ is reducible. Therefore, Conjecture A for ρ (and varying ρ^+) appears to be stronger than Conjecture A for ρ_1 and ρ_2 taken together.

4. Conjectures on Rubin-Stark elements

4.1. **The Rubin-Stark conjecture.** Let $H/\mathbb{Q}$ be a Galois extension which is unramified at p and let χ be a non-trivial E-valued character of $\mathrm{Gal}(H/k)$, where $k/\mathbb{Q}$ is an intermediate extension of $H/\mathbb{Q}$. Denote by $L = H^{\ker\chi}$ be the field cut out by χ and by Δ the Galois group of the abelian extension L/k. We fix for the moment an integer $n \geq 0$, and we put $L_n = L\mathbb{Q}_n$ and $\Delta_n = \mathrm{Gal}(L_n/k) \simeq \Delta \times \Gamma_n$. Consider the following finite sets of places of k:

$$\begin{aligned} S &= S_\infty(k) \cup S_{\mathrm{ram}}(L/k), \\ S' &= S \cup S_p(k), \\ V' &= \{v \in S' \mid \chi(\Delta_v) = 1\} = \{v_{\infty,1},\dots,v_{\infty,d^+},v_{p,1},\dots,v_{p,f}\}, \\ V &= V' \backslash S_p(k) = \{v_{\infty,1},\dots,v_{\infty,d^+}\}. \end{aligned}$$

Fix once and for all a place $w_{\infty,i}$ (resp. $w_{p,i}$) of L_n above $v_{\infty,i}$ (resp. above $v_{p,i}$) for all index i. Let $Y_{L_n,S'}$ be the free abelian group on the set of places of L_n above S' and define the subgroup

$$X_{L_n,S'} := \left\{ \sum_w a_w \cdot w \in Y_{L_n,S'} \;\middle|\; \sum_w a_w = 0 \right\}.$$

Let $\mathcal{O}_{L_n,S'}$ be the ring of S'_{L_n} integers of L_n. We know by Dirichlet's unit theorem that the regulator map

$$\lambda_{L_n,S'} : \mathbb{R}\mathcal{O}^\times_{L_n,S'} \xrightarrow{\sim} \mathbb{R}X_{L_n,S'}, \qquad a \mapsto -\sum_{w|v\in S'} \log|a|_w w,$$

is a Δ_n-equivariant isomorphism (see [Rub96, §1.1]). For any character $\eta \in \widehat{\Gamma}_n = \mathrm{Hom}(\Gamma_n, \overline{\mathbb{Q}}^\times)$, the order of vanishing of the S'-truncated L-function of $(\chi\otimes\eta)^{-1}$ is, by [Tat84, Chapter I, Proposition 3.4],

$$r := \mathrm{ord}_{s=0} L_{S'}((\chi\otimes\eta)^{-1}, s) = \dim_{\mathbb{C}}(e_{\chi\otimes\eta}\mathbb{C}\mathcal{O}^\times_{L_n,S'}) = \dim_{\mathbb{C}}(e_{\chi\otimes\eta}\mathbb{C}X_{L_n,S'}) = \begin{cases} d^+ & \text{if } \eta \neq 1 \\ d^+ + f & \text{if } \eta = 1, \end{cases}$$

where $e_{\chi\otimes\eta} = (\#\Delta_n)^{-1}\sum_{\delta\in\Delta_n}(\chi\otimes\eta)^{-1}(\delta)\delta = e_\chi \cdot e_\eta$ denotes the idempotent associated with $\chi\otimes\eta$. Thus, the limit $L^*_{S'}((\chi\otimes\eta)^{-1}, 0) := \lim_{s\to 0} L_{S'}(\chi, s)/s^r \in \mathbb{C}$ is well-defined and non-zero.

Definition 4.1.1. The χ-part of the Rubin-Stark elements

$$\varepsilon^\chi_n \in {\bigwedge}^{d^+}_{\mathbb{C}[\Gamma_n]} e_\chi \mathbb{C}\mathcal{O}^\times_{L_n,S'} \quad (n \geq 1), \qquad \text{resp. } \; u^\chi \in {\bigwedge}^{d^++f}_{\mathbb{C}} e_\chi \mathbb{C}\mathcal{O}^\times_{L,S'} \quad (n = 0),$$

is defined to be the inverse image under $\lambda_{L_n,S'}$ of

$$\left(\sum_{\eta\in\widehat{\Gamma}_n} L^*_{S'}((\chi\otimes\eta)^{-1}, 0) e_{\chi\otimes\eta} \right) \cdot \bigwedge_w w \in {\bigwedge}^r_{\mathbb{C}[\Gamma_n]} e_\chi \mathbb{C}Y_{L_n,S'} = {\bigwedge}^r_{\mathbb{C}[\Gamma_n]} e_\chi \mathbb{C}X_{L_n,S'},$$

where w runs through $\{w_{\infty,1}, \ldots, w_{\infty,d^+}\}$ (resp. through $\{w_{\infty,1}, \ldots, w_{p,f}\}$). Note that the last equality follows from our assumption that χ is non-trivial.

Remark 4.1.2. It will be convenient to see the χ-part of the Rubin-Stark elements as p-units of H via the equality $e_\chi\mathbb{C}\mathcal{O}^\times_{L_n,S'} = e_\chi\mathbb{C}\mathcal{O}_{H_n}[\frac{1}{p}]^\times$. On the other hand, the L-series $L_{S'}((\chi\otimes\eta)^{-1}, s)$ coincides with $L_{\{p\}}((\chi\otimes\eta)^{-1}, s)$ for $\chi = \mathbb{1}$ and with $L((\chi\otimes\eta)^{-1}, s)$ for $\chi \neq \mathbb{1}$.

The Rubin-Stark conjecture over $\mathbb{Q}$ [Rub96, Conjecture A'] implies the following conjecture.

Conjecture 4.1.3 (Rubin-Stark conjecture for χ: algebraicity statement)**.** *One has*

$$\varepsilon^\chi_n \in {\bigwedge}^{d^+}_{E[\Gamma_n]} e_\chi E\mathcal{O}_{H_n}[\tfrac{1}{p}]^\times \quad (n \geq 1), \qquad \textit{resp. } \; u^\chi \in {\bigwedge}^{d^++f}_{E} e_\chi E\mathcal{O}_H[\tfrac{1}{p}]^\times \quad (n = 0).$$

By means of the isomorphism $j : \mathbb{C} \simeq \overline{\mathbb{Q}}_p$, one may see the χ-part of Rubin-Stark elements as living in the top exterior algebra of $e_\chi\overline{\mathbb{Q}}_p\mathcal{O}_{H_n}[\frac{1}{p}]^\times$. For $R = \mathcal{O}_p[\Gamma_n]$ or $R = \Lambda$ and for any finitely generated $\mathcal{O}_p$-free R-module M, let

$${\bigcap}^r_R M := \left({\bigwedge}^r_R M^*\right)^* \hookrightarrow {\bigwedge}^r_{\overline{\mathbb{Q}}_p\otimes R} \overline{\mathbb{Q}}_p \otimes M$$

be the (r-th order) exterior bi-dual of M, where we have put $(-)^* = \mathrm{Hom}_R(-, R)$ (see [BKS16, §4] and [BS19, Appendix B] for its basic properties). Note that the canonical map ${\bigwedge}^r_R M \longrightarrow {\bigcap}^r_R M$ is an isomorphism when M is R-projective.

Recall that we denoted by U_n (resp. U'_n) the $\mathcal{O}_p$-span of the pro-p completion of the group of units (resp. of p-units) of H_n (Notation 3.5.1). We omit the index when $n = 0$. Note that

U'_n is torsion-free because H is unramified at p. The Rubin-Stark conjecture over $\mathbb{Z}$ [Rub96, Conjecture B'] implies the following conjecture.

Conjecture 4.1.4 (Rubin-Stark conjecture for χ: p-integrality statement)**.** *One has*

$$\varepsilon_n^\chi \in \bigcap\nolimits^{d^+}_{\mathcal{O}_p[\Gamma_n]} U'_n \quad (n \geq 1), \qquad \textit{resp.}\ \ u^\chi \in \bigwedge\nolimits^{d^++f}_{\mathcal{O}_p} U' \quad (n=0).$$

Recall that, if φ is a linear form on a R-module M (for a commutative ring R), and if $t \geq 1$ is an integer, then φ induces a R-linear map $\varphi : \wedge^t_R M \longrightarrow \wedge^{t-1}_R M$ which sends $m_1 \wedge \ldots \wedge m_t$ to $\sum_{i=1}^t (-1)^{i-1} m_1 \wedge \ldots \wedge m_{i-1} \wedge m_{i+1} \wedge \ldots \wedge m_t$. More generally, s linear forms $\varphi_1, \ldots \varphi_s$ on M with $s \leq t$ induce a R-linear map

$$\bigwedge_{1 \leq i \leq s} \varphi_i : \bigwedge\nolimits^t_R M \longrightarrow \bigwedge\nolimits^{t-s}_R M$$

given by $m \mapsto \varphi_s \circ \ldots \circ \varphi_1(m)$.

We take $n=0$ for the rest of this section and for $1 \leq i \leq f$ we consider the p-adic valuation $\mathrm{ord}_{w_{p,i}} : \mathbb{C}\mathcal{O}_H[\frac{1}{p}]^\times \longrightarrow \mathbb{C}$ induced by the place $w_{p,i}$. By [San14, Proposition 3.6], the induced map

$$\bigwedge_{1 \leq i \leq f} \mathrm{ord}_{w_{p,i}} : \bigwedge\nolimits^{d^++f}_{\mathbb{C}} \mathbb{C}\mathcal{O}_H[\tfrac{1}{p}]^\times \longrightarrow \bigwedge\nolimits^{d^+}_{\mathbb{C}} \mathbb{C}\mathcal{O}_H[\tfrac{1}{p}]^\times$$

sends u^χ on the Rubin-Stark element

$$\xi^\chi \in \bigwedge\nolimits^{d^+}_{\mathbb{C}} \mathbb{C}\mathcal{O}_H[\tfrac{1}{p}]^\times$$

defined as the inverse image under $\lambda_{L,S}$ of $L^*(\chi^{-1},0)e_\chi \cdot w_{\infty,1} \wedge \ldots \wedge w_{\infty,d^+}$. Note that, if Conjecture 4.1.3 or Conjecture 4.1.4 holds for u^χ, then the corresponding statement for ξ^χ is also true.

4.2. **Iwasawa-theoretic conjectures.** We assume in this section that χ is of prime-to-p order. The idempotent e_χ has coefficients in $\mathcal{O}_p$ and the χ-part M^χ and the χ-quotient M_χ of an $\mathcal{O}_p$-module M (see Section 2.2) both coincide with $e_\chi M$. We let the integer $n \geq 0$ of last section vary and we assume Conjecture 4.1.4 for every n. As explained in [BKS17, 3B2] the family $(\varepsilon_n^\chi)_{n \geq 1}$ is norm-compatible, so it defines an element

$$\varepsilon_\infty^\chi \in \varprojlim_n \bigcap\nolimits^{d^+}_{\mathcal{O}_p[\Gamma_n]} (U'_n)^\chi = \bigcap\nolimits^{d^+}_{\Lambda} (U'_\infty)^\chi = \bigwedge\nolimits^{d^+}_{\Lambda} (U'_\infty)^\chi.$$

Here, the first identification follows easily from [BS19, Corollary B.5] and the second one from the fact that $(U'_\infty)^\chi = \mathrm{H}^1_{\mathrm{Iw,f},p}(\mathbb{Q}, \breve{T}_p)$ is free (of rank d^+) over Λ by the results of Section 3.5. The following conjecture is taken from [BKS17, Conj. 3.14] and should be thought as a cyclotomic Iwasawa main conjecture for χ. Let us mention that this conjecture may also be formulated for other $\mathbb{Z}_p$-extensions of k than the cyclotomic one.

Conjecture 4.2.1 (IMC$_\chi$). *We have*

$$char_\Lambda\left(\bigwedge\nolimits^{d^+} (U'_\infty)^\chi\right) / \left(\Lambda \cdot \varepsilon_\infty^\chi\right) = \mathcal{A}^f \cdot char_\Lambda(A'_\infty)_\chi,$$

where $\mathcal{A}$ is the augmentation ideal of Λ and where A'_∞ is the inverse limit over $n \geq 0$ of the p-split ideal class groups of H_n (see Notation 3.5.1).

Since $\wedge^{d^+}(U'_\infty)^\chi$ is free of rank one over Λ, Conjecture 4.2.1 implies immediately the non-vanishing of ε_∞^χ, as well as the following conjecture:

Conjecture 4.2.2 (wEZC$_\chi$). *We have $\varepsilon_\infty^\chi \in \mathcal{A}^f \cdot \wedge^{d^+}(U'_\infty)^\chi$.*

This is [BS19, Conj. 2.7] for the cyclotomic extension (and a more general number field k), where it is referred to as the Exceptional Zero Conjecture for Rubin-Stark elements. Assume Conjecture 4.2.2 and fix γ a topological generator of Γ. Following *loc. cit.* we now reformulate the (cyclotomic) Iwasawa-theoretic Mazur-Rubin-Sano Conjecture for (χ, S, V') in terms of the element $\kappa_{\infty,\gamma} \in \bigwedge^{d^+}(U'_\infty)^\chi$ which satisfies

$$\varepsilon_\infty^\chi = (\gamma - 1)^f \cdot \kappa_{\infty,\gamma}.$$

For all $1 \le i \le f$, let $\mathrm{rec}_{w_{p,i}} : L^\times \longrightarrow \mathrm{Gal}((\mathbb{Q}_\infty L)_{w_{p,i}}/L_{w_{p,i}}) \simeq \Gamma$ be the local reciprocity map for L at $w_{p,i}$. We still denote by $\mathrm{rec}_{w_{p,i}}$ the induced $\mathcal{O}_p$-homomorphism

$$\mathrm{rec}_{w_{p,i}} : (U')^\chi = e_\chi(\mathcal{O}_p \otimes \mathcal{O}_{L,S'}^\times) \longrightarrow \mathcal{O}_p \otimes \Gamma \simeq \mathcal{A}/\mathcal{A}^2.$$

Conjecture 4.2.3 (MRS$_\chi$). *Conjecture 4.2.2 holds true, and if we let $\kappa_\gamma \in \bigwedge_{\mathcal{O}_p}^{d^+}(U')^\chi$ be the bottom layer of $\kappa_{\infty,\gamma}$, then the map*

$$\bigwedge_{1\le i\le f} \mathrm{rec}_{w_{p,i}} : \bigwedge_{\mathcal{O}_p}^{d^+ + f}(U')^\chi \longrightarrow \mathcal{A}^f/\mathcal{A}^{f+1} \otimes_{\mathcal{O}_p} \bigwedge_{\mathcal{O}_p}^{d^+}(U')^\chi$$

sends u^χ to $(-1)^{d^+\cdot f} \cdot (\#\Delta)^{-f} \cdot (\gamma-1)^f \otimes \kappa_\gamma$.

Remark 4.2.4. Assuming Conjecture 4.2.2, one can check (as noted in [BS19, Rem. 2.10 (i)]) that Conjecture 4.2.3 is equivalent to [BKS17, Conj. 4.2, MRS($H_\infty/k, S, \emptyset, \chi, V'$)]. As for Conjecture 4.1.4, taking $T = \emptyset$ (in the notations of [Rub96, BKS17]) is allowed because the $\mathbb{Z}_p$-module U'_n is torsion-free for all $n \ge 0$. Lastly, note that, while the definition of the Rubin-Stark elements depends on how we ordered the places $v_{p,1}, \ldots, v_{p,f}$ of $V' \backslash V$ and on the choice of $w_{p,i}$ above $v_{p,i}$, the validity of all the conjectures of this section does not depend on these choices.

5. Monomial representations

5.1. **Induced representations.** Let ρ be a monomial representation and fix an isomorphism $\rho \simeq \mathrm{Ind}_k^{\mathbb{Q}} \chi$ over E, where $\chi : \mathrm{Gal}(H/k) \longrightarrow E^\times$ is a non-trivial character. We do not assume yet that χ has order prime to p. The underlying space of ρ is then equal to $W = E[G] \otimes_{E[\mathrm{Gal}(H/k)]} E(\chi)$, where $E(\chi)$ is a E-line on which $\mathrm{Gal}(H/k)$ acts via χ and where the tensor product follows the rule $gh \otimes 1 = g \otimes \chi(h)$ for all $g \in G, h \in \mathrm{Gal}(H/k)$. The (left) G-action on W is given by $g \cdot (g' \otimes 1) = gg' \otimes 1$ for all $g, g' \in G$. By Frobenius reciprocity we have $\rho \otimes \eta \simeq \mathrm{Ind}_k^{\mathbb{Q}}(\chi \otimes \eta)$ for any $\eta \in \widehat{\Gamma}$, where we still denoted by η its restriction to G_k. We will assume throughout Section 5 that the $\mathcal{O}_p$-lattice T_p of W_p is

$$T_p = \mathcal{O}[G] \otimes_{\mathcal{O}[\mathrm{Gal}(H/k)]} \mathcal{O}(\chi),$$

so that the family $(g \otimes 1)_{g \in G}$ generates T_p over $\mathcal{O}_p$.

Lemma 5.1.1. *(1) Given any $\mathcal{O}_p[G]$-module M, there is a canonical isomorphism*

$$\mathrm{Hom}_G(T_p, M) \xrightarrow{\sim} M^\chi, \qquad \psi \mapsto \psi(1 \otimes 1),$$

where M^χ denotes the χ-part of M, seen as a $\mathrm{Gal}(H/k)$-module. In the same fashion, for any $\eta \in \widehat{\Gamma}$ and for any $E_\eta[G_{\mathbb{Q}}]$-module M, there is a canonical isomorphism $\mathrm{Hom}_{G_{\mathbb{Q}}}(W_\eta, M) \simeq e_{\chi\otimes\eta} M = M^{\chi\otimes\eta}$.

(2) Given any $\mathcal{O}_p[G]$-module M, the module $\mathrm{Hom}_G(M, D_p)$ is canonically isomorphic to the Pontryagin dual $(M_\chi)^\vee$ of the χ-quotient of M.

Proof. The first statement is straightforward to check, so we only prove the second one. Once we have fixed a generator of the different of $\mathcal{O}_p$ over $\mathbb{Z}_p$, the $\mathcal{O}_p$-module $\left(M_\chi\right)^\vee$ can be identified with $\mathrm{Hom}_{\mathcal{O}_p}(M_\chi, E_p/\mathcal{O}_p)$ as in Section 3.2. On the other hand, since induction and co-induction functors over finite groups coincide, D_p can be described as the $\mathcal{O}_p$-module of maps $f: G \longrightarrow E_p/\mathcal{O}_p \otimes \mathcal{O}_p(\chi)$ satisfying $f(hg) = h \cdot f(g)$ for all $h \in \mathrm{Gal}(H/k)$ and $g \in G$, the left G-action being $(g \cdot f)(g') = f(g'g)$. Therefore, the map $F \mapsto (m \mapsto F(m)(1 \otimes 1))$ identifies $\mathrm{Hom}_G(M, D_p)$ with $\mathrm{Hom}_{\mathcal{O}_p}(M_\chi, E_p/\mathcal{O}_p)$. □

Notation 5.1.2. For any $\eta \in \widehat{\Gamma}$, for any module M as in Lemma 5.1.1 (1) and for any $m \in M$, we let ψ_m be the element of $\mathrm{Hom}_G(T_p, M)$ (resp. of $\mathrm{Hom}_{G_\mathbb{Q}}(W_\eta, M)$) which satisfies $\psi_m(1 \otimes 1) = m$. More generally, for any $\omega \in \bigwedge^r M$ (and $r \geq 0$) we let ψ_ω be the element of $\bigwedge^r_{\mathcal{O}_p} \mathrm{Hom}_G(T_p, M)$ (resp. of $\bigwedge^r_{E_\eta} \mathrm{Hom}_{G_\mathbb{Q}}(W_\eta, M)$) corresponding to ω under the induced isomorphism on exterior products.

5.2. **Complex regulators.** We first define a natural basis ω_∞^+ of $\mathrm{H}^0(\mathbb{R}, W)$ in which we will compute all the complex regulators. The embedding $\iota_\infty : \overline{\mathbb{Q}} \subseteq \mathbb{C}$ defines a place w_∞ (resp. v_∞) of $\overline{\mathbb{Q}}$ (resp. of k) as well as a complex conjugation which will be denoted σ_∞. As in Section 4.1, we denote by $V = \{v_{\infty,1}, \ldots, v_{\infty,d^+}\}$ the set of archimedean places of k which split completely in $L = H^{\ker \chi}$. We choose for each $i = 1, \ldots, d^+$ an automorphism $\tau_{\infty,i} \in G_\mathbb{Q}$ which sends $v_{\infty,i}$ onto v_∞ and we put $w_{\infty,i} = \tau_{\infty,i}^{-1}(w_\infty)$. For simplicity we still write $\tau_{\infty,i}$ and $w_{\infty,i}$ for their restrictions to finite extensions of L. We obtain a basis $\omega_\infty^+ = \{t_{\infty,1}, \ldots, t_{\infty,d^+}\}$ of $H^0(\mathbb{R}, W) = W^{\sigma_\infty = 1}$ by letting

$$t_{\infty,i} = \begin{cases} \tau_{\infty,i} \otimes 1 & \text{if } v_i \text{ is real,} \\ \tau_{\infty,i} \otimes 1 + \sigma_\infty \cdot \tau_{\infty,i} \otimes 1 & \text{if } v_i \text{ is complex.} \end{cases}$$

Note that it is moreover T_p-optimal for our fixed choice of T_p.

Lemma 5.2.1. *Assume Conjecture 4.1.3. Let $\eta \in \widehat{\Gamma}$ be a character of order p^n. Put $\omega_{\mathrm{f},\eta} = \psi_{e_\eta \cdot \varepsilon_n^\chi}$ if $\eta \neq \mathbb{1}$ and $\omega_{\mathrm{f},\eta} = \psi_{\xi^\chi}$ if $\eta = \mathbb{1}$ (see Notation 5.1.2). The complex regulator of $\rho \otimes \eta$ computed in the bases ω_∞^+ and $\omega_{\mathrm{f},\eta}$ is equal to*

$$\mathrm{Reg}_{\omega_\infty^+}(\rho \otimes \eta) = p^{-n \cdot d^+} \cdot L^*((\rho \otimes \eta)^\vee, 0).$$

Proof. For $a \in L_n$ which is seen in $L_{n,w_{\infty,i}}$, put

$$|a|_{w_i} = \begin{cases} \mathrm{sgn}(a)a & \text{if } v_i \text{ is real,} \\ a \cdot \bar{a} & \text{if } v_i \text{ is complex,} \end{cases}$$

where sgn is the sign function when $L_{n,w_{\infty,i}} = \mathbb{R}$ and $a \mapsto \bar{a}$ is the complex conjugation when $L_{n,w_{\infty,i}} = \mathbb{C}$. Write $e_\eta \cdot \varepsilon_n^\chi$ (or ξ^χ if η is trivial) as $\mu_1 \wedge \ldots \wedge \mu_{d^+}$ and write $\omega_{\mathrm{f},\eta} = \psi_{\mu_1} \wedge \ldots \wedge \psi_{\mu_{d^+}}$ accordingly. Then by construction of the $t_{\infty,i}$'s, one has $1 \otimes (\iota_\infty)(\psi_{\mu_j}(t_{\infty,i})) = |\mu_j|_{w_{\infty,i}} \in E_\eta \otimes \mathbb{R}^\times$ for all $1 \leq i, j \leq d^+$, so we have

$$\mathrm{Reg}_{\omega_\infty^+}(\rho \otimes \eta) = \det\left(\log_\infty |\mu_j|_{w_{\infty,i}}\right)_{1 \leq i,j \leq d^+}.$$

Since $L(\rho \otimes \eta)^\vee, s) = L((\chi \otimes \eta)^{-1}, s)$ and since $e_\eta \cdot \varepsilon_n^\chi = p^{-n} \sum_{g \in \Gamma_n} \eta^{-1}(g) g(\varepsilon_n^\chi)$ by definition, the result follows directly from [Rub96, Lemma 2.2] and Remark 4.1.2. □

5.3. **Iwasawa main conjectures.** In the two last sections we explore the relation between Conjecture A and the various conjectures on Rubin-Stark elements of Section 4.1. We henceforth assume that χ is of prime-to-p order. In what follows, the basis $\omega_p^+ = t_1 \wedge \ldots \wedge t_{d^+}$ of a given p-stabilization W_p^+ of W_p is always assumed to be T_p-optimal and to be an eigenbasis of σ_p as in Lemma 3.6.3.

Theorem 5.3.1. *Assume Conjectures 4.1.3 and 4.1.4 and pick any p-stabilization W_p^+ of W_p.*

1. *The statement $\boldsymbol{EX}_{\rho,\rho^+}$ in Conjecture A is true, and the element θ'_{ρ,ρ^+} of Proposition 3.3.3 coincides with $\mathscr{C}^{\mathrm{str}}_{\omega_p^+}(\psi_{\varepsilon^\chi_\infty})$, where $\mathscr{C}^{\mathrm{str}}_{\omega_p^+}$ is the operator introduced in Section 3.6.*
2. *Conjecture 4.2.2 implies that θ_{ρ,ρ^+} has an order of vanishing at $\mathbb{1}$ greater than or equal to e. The converse implication also holds if we moreover assume that W_p^+ is admissible and that $\mathcal{L}(\rho,\rho^+)$ does not vanish.*

Proof. Put $\theta = \mathscr{C}^{\mathrm{str}}_{\omega_p^+}(\psi_{\varepsilon^\chi_\infty})$ and fix a non-trivial character $\eta \in \widehat{\Gamma}$ of conductor p^n. By Lemma 3.6.3 and Lemma 5.2.1, we have

$$\begin{aligned}\eta(\theta) &= p^{(1-n)\cdot d^+}\cdot\frac{\det(\rho^+)(p^n)}{\mathfrak{g}(\eta^{-1})^{d^+}}\mathrm{Reg}_{\omega_p^+}(\rho\otimes\eta)\\ &= \frac{\det(\rho^+)(p^n)}{\mathfrak{g}(\eta^{-1})^{d^+}}\frac{\mathrm{Reg}_{\omega_p^+}(\rho\otimes\eta)}{\mathrm{Reg}_{\omega_\infty^+}(\rho\otimes\eta)}L^*((\rho\otimes\eta)^\vee,0),\end{aligned}$$

where the regulators are computed with respect to the basis $\omega_{\mathfrak{f}_\eta}$ defined in Lemma 5.2.1. A comparison with the interpolation property of θ'_{ρ,ρ^+} and Weierstrass' preparation theorem then shows that $\theta'_{\rho,\rho^+} = \theta$; hence, $\mathbf{EX}_{\rho,\rho^+}$ is true by Proposition 3.3.3, and (1) follows. The first implication of (2) is obvious, since $\mathscr{C}^{\mathrm{str}}_{\omega_p^+}$ is Λ-linear and since θ has at most a pole of order $f-e$ at $\mathbb{1}$ by construction. For the converse implication, assume that $\theta'_{\rho,\rho^+} \in \mathcal{A}^e$ (so θ is also in $\mathcal{A}^e$), that W_p^+ is admissible and that $\mathcal{L}(\rho,\rho^+) \neq 0$. Then, we know by Theorem 3.8.6 (1), by Lemma 3.2.2 and by the exactness of the last row of (13) that the cokernel of $\mathscr{C}^{\mathrm{str}}_{\omega_p^+}$ has finite Γ-coinvariants, so its image is generated over Λ by an element in $\mathcal{J}^{f-e}\backslash\mathcal{J}^{f-e-1}$. We may then write θ as $(\gamma-1)^f\cdot\theta_\gamma$ for some topological generator γ of Γ and some $\theta_\gamma \in \mathrm{im}(\mathscr{C}^{\mathrm{str}}_{\omega_p^+})$; hence, Conjecture 4.2.2 follows from the injectivity of $\mathscr{C}^{\mathrm{str}}_{\omega_p^+}$. ☐

Corollary 5.3.2. *Assume that there exists a non-trivial character $\eta \in \widehat{\Gamma}$ such that the $\chi\otimes\eta$-part of Leopoldt's conjecture for H_η holds, that is, the $\chi\otimes\eta$-part of the linear extension of the diagonal embedding*

$$e_{\chi\otimes\eta}\cdot\left(E_{p,\eta}\otimes\mathcal{O}^\times_{H_\eta}\right)\longrightarrow e_{\chi\otimes\eta}\cdot\left(E_{p,\eta}\otimes\prod_{K_\eta}\mathcal{O}^{\times,1}_{K_\eta}\right),$$

where K_η runs over the all the p-adic completions of H_η, is injective. Then $\varepsilon^\chi_\infty \neq 0$.

Proof. By Lemma 3.1.4, the $\chi\otimes\eta$-part of Leopoldt's conjecture for H_η implies the existence of a η-admissible p-stabilization W_p^+ of W_p. For such a W_p^+ we know that the map $\mathscr{C}^{\mathrm{str}}_{\omega_p^+}$ is injective and that $\theta'_{\rho,\rho^+} \neq 0$ by Theorem 3.6.4 and by Lemma 3.5.4. Therefore, Theorem 5.3.1 implies that ε^χ_∞ does not vanish. ☐

Theorem 5.3.3. *Assume Conjectures 4.1.3 and 4.1.4. Let W_p^+ be a p-stabilization of W_p such that $X_\infty(\rho,\rho^+)$ is of Λ-torsion.*

(1) If either $\mathbf{IMC}_{\rho,\rho^+}$ or $\mathbf{IMC}_\chi$ are true, then Conjecture 4.2.2 is also true.
(2) $\mathbf{IMC}_{\rho,\rho^+}$ and $\mathbf{IMC}_\chi$ are equivalent.

Proof. We have already seen that $\mathbf{IMC}_\chi$ implies Conjecture 4.2.2 because $\bigwedge^{d^+}(U'_\infty)^\chi$ is free of rank one over Λ. Thus, the first claim is implied by the second one, which we prove now. By Theorem 3.6.4 and by Lemma 3.5.4 we know that the map $\mathrm{Loc}_+^{\mathrm{str}}$ of Section 3.4 is injective. Since its domain and codomain are free, $\mathrm{coker}(\mathrm{Loc}_+^{\mathrm{str}})$ and $\mathrm{coker}(\wedge^{d^+}\mathrm{Loc}_+^{\mathrm{str}})$ have the same characteristic ideal. On the other hand, recall that we may identify $(U'_\infty)^\chi$ with $\mathrm{Hom}_G(T_p,U'_\infty)$ via Lemma 5.1.1 and $\Lambda\cdot\mathscr{C}^{\mathrm{str}}_{\omega_p^+}(\varepsilon_\infty^\chi)$ with $\Lambda\cdot\theta_{\rho,\rho^+}$ via Theorem 5.3.1. By Lemma 3.2.2 and by (13) we have exact three short exact sequences

$$0 \longrightarrow \underbrace{\left(\bigwedge^{d^+}(U'_\infty)^\chi\right)/\left(\Lambda\cdot\varepsilon_\infty^\chi\right)}_{B} \xrightarrow{\mathscr{C}^{\mathrm{str}}_{\omega_p^+}} \underbrace{\mathcal{I}^{e-f}/\left(\Lambda\cdot\theta_{\rho,\rho^+}\right)}_{C} \longrightarrow \underbrace{\mathrm{coker}(\wedge^{d^+}\mathrm{Loc}_+^{\mathrm{str}})}_{D} \longrightarrow 0$$

$$0 \longrightarrow \mathrm{H}^0(\mathbb{Q}_p,T_p^-) \longrightarrow X_\infty(\rho,\rho^+) \longrightarrow X_\infty^{\mathrm{str}}(\rho,\rho^+) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{coker}(\mathrm{Loc}_+^{\mathrm{str}}) \longrightarrow X_\infty^{\mathrm{str}}(\rho,\rho^+) \longrightarrow Ш^1_\infty(D_p)^\vee \longrightarrow 0.$$

The Γ-action on $\mathrm{H}^0(\mathbb{Q}_p,T_p^-)$ being trivial, its characteristic ideal is $\mathcal{A}^e$. Moreover, $Ш^1_\infty(D_p)^\vee$ and $(A'_\infty)_\chi$ are pseudo-isomorphic Λ-modules by Lemmas 3.5.2 (3) and 5.1.1 (2). Therefore, by multiplicativity of characteristic ideals, we may conclude that

$$\begin{aligned}
\mathbf{IMC}_\chi &\Longleftrightarrow \mathrm{char}_\Lambda(B)=\mathcal{A}^f\cdot\mathrm{char}_\Lambda(A'_\infty)_\chi\\
&\Longleftrightarrow \mathrm{char}_\Lambda(C)=\mathcal{A}^f\cdot\mathrm{char}_\Lambda X_\infty^{\mathrm{str}}(\rho,\rho^+)\\
&\Longleftrightarrow \mathcal{A}^{f-e}\cdot(\Lambda\cdot\theta_{\rho,\rho^+})=\mathcal{A}^{f-e}\cdot\mathrm{char}_\Lambda X_\infty(\rho,\rho^+)\\
&\Longleftrightarrow (\Lambda\cdot\theta_{\rho,\rho^+})=\mathrm{char}_\Lambda X_\infty(\rho,\rho^+)\\
&\Longleftrightarrow \mathbf{IMC}_{\rho,\rho^+}.
\end{aligned}$$

□

5.4. **Extra zeros at the trivial character.** We first construct an $\mathcal{O}_p$-basis of $\mathrm{H}^0(\mathbb{Q}_p,T_p)$ as follows. Let w_p be the p-adic place of H defined by ι_p, and denote by $v_{p,1},\dots,v_{p,f}$ the p-adic places of k which totally split in L as in Section 4.1. Fix also a place $w_{p,i}$ of H above $v_{p,i}$, and let

$$t_{p,i}=[H_{w_{p,i}}:k_{v_{p,i}}]^{-1}\cdot\sum_{\substack{g\in G,\\ g(w_{p,i})=w_p}} g\otimes 1 \quad \in T_p, \qquad (1\le i\le f).$$

This defines an $\mathcal{O}_p$-basis $t_{p,1},\dots,t_{p,f}$ of $\mathrm{H}^0(\mathbb{Q}_p,T_p)$. For any $t\in T_p$, consider the following two composite maps

$$\mathrm{ord}_p^{(t)}\;:\;\mathrm{Hom}_G(T_p,U')\xrightarrow{\iota_p\circ\mathrm{ev}_t}\mathcal{O}_p\otimes\widehat{K}^\times\xrightarrow{\mathrm{ord}_p}\mathcal{O}_p,$$

$$\log_p^{(t)} : \mathrm{Hom}_G(T_p, U') \xrightarrow{\iota_p \circ \mathrm{ev}_t} \mathcal{O}_p \otimes \widehat{K}^\times \xrightarrow{\log_p} \mathcal{O}_p,$$

where ev_t is the evaluation map at t and where $K = H_{w_p}$ as in Section 3.

Lemma 5.4.1. *For all $1 \le i \le f$ and $u \in (U')^\chi$, we have*

$$\mathrm{ord}_p^{(t_{p,i})}(\psi_u) = [k_{v_{p,i}} : \mathbb{Q}_p] \cdot \mathrm{ord}_{w_{p,i}}(u), \qquad \log_p^{(t_{p,i})}(\psi_u) = -\log_p \circ \chi_{\mathrm{cyc}} \circ \mathrm{rec}_{w_{p,i}}(u).$$

Proof. Fix $1 \le i \le f$ and $u \in (U')^\chi = (\mathcal{O}_p \otimes \mathcal{O}_L[\frac{1}{p}]^\times)^\chi$. Since ψ_u is G-equivariant, we have

$$\begin{aligned}\iota_p(\psi_u(t_{p,i})) &= [H_{w_{p,i}} : k_{v_{p,i}}]^{-1} \cdot \iota_p\left(\prod_{\substack{g\in G,\\ g(w_{p,i})=w_p}} g(u)\right)\\ &= N_i(\iota_{p,i}(u)),\end{aligned}$$

where N_i is the norm map of the extension $L_{w_{p,i}} = k_{v_{p,i}}$ over $\mathbb{Q}_p$, and where $\iota_{p,i}$ is the p-adic embedding of L defined by $w_{p,i}$. Since $\mathrm{ord}_p(N_i(\iota_{p,i}(u))) = [k_{v_{p,i}} : \mathbb{Q}_p] \cdot \mathrm{ord}_{w_{p,i}}(u)$ and since $\chi_{\mathrm{cyc}} \circ \mathrm{rec}_{w_{p,i}}(u) = N_i(\iota_{p,i}(u))^{-1}$, the lemma follows easily. □

Proposition 5.4.2. *(1) Let $W_p^0 = \mathrm{H}^0(\mathbb{Q}_p, W_p)$. Then,*

$$\left(\bigwedge_{1\le i\le f} \mathrm{ord}_p^{(t_{p,i})}\right)(\psi_{u^\chi}) = (-1)^{d^+\cdot f} \cdot \frac{\det(1-\sigma_p^{-1} \mid W_p/W_p^0)}{(\#\Delta)^f} \cdot \psi_{\xi^\chi}$$

in $\bigwedge_{\mathbb{C}}^{d^+} (\mathbb{C} \otimes \mathcal{O}_H^\times)^\chi$.

*(2) Assume Conjectures 4.1.3 and 4.1.4 and 4.2.2. Fix a topological generator γ of Γ and put $\varpi_\gamma = \log_p \circ \chi_{\mathrm{cyc}}(\gamma) \in p\mathbb{Z}_p$. Then, **MRS**$_\chi$ is equivalent to the equality*

$$\left(\bigwedge_{1\le i\le f} \log_p^{(t_{p,i})}\right)(\psi_{u^\chi}) = (-1)^{d^+\cdot f} \cdot \left(-\frac{\varpi_\gamma}{\#\Delta}\right)^f \cdot \psi_{\kappa_\gamma}$$

in $\bigwedge_{\mathcal{O}_p}^{d^+} (U')^\chi$.

Proof. Let us prove (1). First of all, a direct computation gives

$$\det(1-\sigma_p^{-1} \mid W_p/W_p^0) = \prod_{v\in S_p^{00}(k)} (1-\chi^{-1}(v)) \cdot \prod_{1\le i\le f} [k_{v_{p,i}} : \mathbb{Q}_p],$$

where $S_p^{00}(k) = S_p(k) - \{v_{p,1}, \ldots, v_{p,f}\}$ and where we saw χ as a Hecke character over k. Consider the operator $\Phi_{V',V}$ of [San14, Proposition 3.6] (extended by linearity to a $\mathbb{C}[\Delta]$-linear map), where V and V' are as in Section 4.1. Its χ-part is given by

$$e_\chi \cdot \Phi_{V',V} = (-1)^{d^+\cdot f} \cdot (\#\Delta)^f \cdot \bigwedge_{1\le i\le f} \mathrm{ord}_{w_{p,i}} = (-1)^{d^+\cdot f} \frac{(\#\Delta)^f}{\prod_{1\le i\le f} [k_{v_{p,i}} : \mathbb{Q}_p]} \cdot \bigwedge_{1\le i\le f} \mathrm{ord}_p^{(t_{p,i})},$$

the last equality being a consequence of Lemme 5.4.1 (1). On the other hand, by Propositions 3.5 and 3.6 of *loc. cit.*, we know that

$$\left(e_\chi \cdot \Phi_{V',V}\right)(\psi_{u^\chi}) = \prod_{v\in S_p^{00}(k)} (1-\chi^{-1}(v)) \cdot \psi_{\xi^\chi}.$$

The claim (1) then follows from the above three equations. As for the second claim, it follows immediately from the formula for $\log_p^{(t_{p,i})}$ of Lemma 5.4.1. □

We are now in a position to state and prove a theorem comparing $\mathbf{MRS}_\chi$ and $\mathbf{EZC}_{\rho,\rho^+}$. Let $0 \le e \le f$ be an integer and take any p-stabilization $(\widetilde{\rho}^+, \widetilde{W}_p^+)$ of W_p such that $\mathrm{H}^0(\mathbb{Q}_p, \widetilde{W}_p^+)$ is generated by $t_{p,i_1}, \ldots, t_{p,i_{f-e}}$, for some $1 \le i_1 < \ldots < i_{f-e} \le f$ (this condition is empty when $e = f$). Note that the $t_{p,i}$'s live in W, so many choices of $\widetilde{\rho}^+$ are motivic, hence η-admissible for all $\eta \in \widehat{\Gamma}$ under the hypotheses of Lemma 3.1.3.

Theorem 5.4.3. *Assume Conjectures 4.1.3 and 4.1.4.*

(1) $\mathbf{MRS}_\chi$ implies $\mathbf{sEZC}_{\rho,\widetilde{\rho}^+}$ for all $\widetilde{\rho}^+$ as described above (see Section 3.9 for the formulation of $\mathbf{sEZC}_{\rho,\rho^+}$). More generally, $\mathbf{MRS}_\chi$ implies $\mathbf{EZC}_{\rho,\rho^+}$ for every p-stabilization ρ^+ of W_p.

(2) Conversely, if $\mathbf{EZC}_{\rho,\widetilde{\rho}^+}$ holds for all $\widetilde{\rho}^+$ as described above, and if there exists at least one admissible p-stabilization ρ^+ such that $\mathcal{L}(\rho,\rho^+) \neq 0$, then $\mathbf{MRS}_\chi$ holds true.

Proof. First take any p-stabilization W_p^+ of W_p. From the first part of Theorem 5.3.1 we already know that $\mathbf{EX}_{\rho,\rho^+}$ holds. From its second part, we may also assume without loss of generality that $\mathbf{wEZC}_\chi$ holds and that θ_{ρ,ρ^+} vanishes at $\mathbb{1}$ with multiplicity $\ge e$.

We start by assuming $\mathbf{MRS}_\chi$ and we consider $W_p^+ = \widetilde{W}_p^+$ as above. Since $\mathbf{MRS}_\chi$ does not depend on the ordering of the $v_{p,i}$'s, we may assume that $i_1 = 1, \ldots, i_{f-e} = f-e$. Choose an eigenbasis $t_1, \ldots, t_d$ of T_p for σ_p such that: (a) $t_1 = t_{p,1}, \ldots, t_{f-e} = t_{p,f-e}$, (b) the element $\widetilde{\omega}_p^+ = t_1 \wedge \ldots \wedge t_{d^+}$ is a basis of $\widetilde{W}_p^+$, and (c) $\{t_{d^++1}, \ldots, t_{d^++e}\} = \{t_{p,f-e+1}, \ldots, t_{p,f}\}$. For simplicity, we put

$$I^{+,0} = \{1, \ldots, f-e\},\ I^{+,00} = \{f-e+1, \ldots, d^+\},\ I^{-,0} = \{d^++1, \ldots, d^++e\},\ I^{-,00} = \{d^++e+1, \ldots, d\},$$

so we obtain a partition of $\{1, \ldots, d\}$. For $1 \le i \le d$, we also write $\mathrm{ord}_p^{(i)}$ (resp. $\log_p^{(i)}$) for $\mathrm{ord}_p^{(t_{p,i})}$ (resp. for $\log_p^{(t_{p,i})}$). By Proposition 5.4.2, $\mathbf{MRS}_\chi$ is equivalent to

$$\left(\bigwedge_{i \in I^0} \log_p^{(i)}\right)(\psi_{u^\chi}) = (-1)^{d^+ \cdot f} \cdot \left(-\frac{\varpi_\gamma}{\#\Delta}\right)^f \cdot \psi_{\kappa_\gamma}, \tag{18}$$

where $I^0 = I^{+,0} \bigcup I^{-,0}$, where γ is a fixed generator of Γ, where $\varpi_\gamma = \log_p \circ \chi_{\mathrm{cyc}}(\gamma)$ and where κ_γ is the bottom layer of the element $\kappa_{\infty,\gamma} \in \bigwedge^{d^+}(U'_\infty)^\chi$ satisfying $\varepsilon_\infty^\chi = (\gamma-1)^f \cdot \kappa_{\infty,\gamma}$. For $i \in I^+ = I^{+,0} \bigcup I^{+,00}$, define $\nu^{(i)}$ as $\mathrm{ord}_p^{(i)}$ if $i \in I^{+,0}$ and as $\log_p^{(i)}$ if $i \in I^{+,00}$, and apply the homomorphism

$$\nu = \bigwedge_{i \in I^+} \nu^{(i)} : \bigwedge^{d^+} \mathrm{Hom}_G(T_p, U') \longrightarrow \mathcal{O}_p$$

to the equality (18). We explain now how to compute the left-hand side (LHS) and the right-hand side (RHS) of the resulting equality. Taking into account the sign rule, we have, by definition of $\mathcal{L}(\rho, \widetilde{\rho}^+)$, by Proposition 5.4.2 (1) and by Lemma 5.2.1:

$$\begin{aligned}
\mathrm{LHS} &= (-1)^{f-e} \cdot \mathcal{L}(\rho, \widetilde{\rho}^+) \cdot \left(\bigwedge_{i \in I^0} \mathrm{ord}_p^{(i)}\right) \wedge \left(\bigwedge_{i \in I^+} \log_p^{(i)}\right)(\psi_{u^\chi}) \\
&= (-1)^{f-e} \cdot \mathcal{L}(\rho, \widetilde{\rho}^+) \cdot \left(\bigwedge_{i \in I^+} \log_p^{(i)}\right)\left((-1)^{d^+ \cdot f} \cdot \frac{\det(1-\sigma_p^{-1} \mid W_p/W_p^0)}{(\#\Delta)^f} \cdot \psi_{\xi^\chi}\right) \\
&= (-1)^{f-e+d^+ \cdot f} \cdot \mathcal{L}(\rho, \widetilde{\rho}^+) \cdot \frac{\det(1-\sigma_p^{-1} \mid W_p/W_p^0)}{(\#\Delta)^f} \cdot \frac{\mathrm{Reg}_{\widetilde{\omega}_p^+}(\rho)}{\mathrm{Reg}_{\omega_\infty^+}(\rho)} \cdot L^*(\rho^\vee, 0).
\end{aligned}$$

Let us explicit the relation between RHS and the e-th derivative of $L_p(s) := \kappa^s(\theta'_{\rho,\rho^+})$ at $s=0$. By Theorem 5.3.1, we may write $L_p(s) = (\kappa(\gamma)^s - 1)^e \cdot \kappa^s(\theta_\gamma)$, where $\theta_\gamma = \mathscr{C}_{\widetilde{\omega}_p^+}((\gamma-1)^{f-e} \cdot \kappa_{\infty,\gamma}) \in \Lambda$. Therefore, $\frac{1}{e!}L_p^{(e)}(0) = \varpi_\gamma^e \cdot \mathbb{1}(\theta_\gamma)$. On the other hand, if we write $(\gamma-1)^{f-e} \cdot \kappa_{\infty,\gamma}$ as a wedge product $(\gamma-1)\cdot v_1 \wedge \ldots (\gamma-1)\cdot v_{f-e} \wedge v_{f-e+1} \wedge \ldots \wedge v_{d^+}$, then Lemmas 2.1.3 and 2.3.1 together show that

$$\mathbb{1}(\theta_\gamma) = \varpi_\gamma^{f-e} \cdot (1-p^{-1})^{f-e} \cdot \prod_{i \in I^{+,00}} \frac{1-p^{-1}\beta_i}{1-\beta_i^{-1}} \cdot \nu(\kappa_\gamma),$$

where β_i is the eigenvalue of σ_p acting on t_i. Since $\mathcal{E}(\rho, \widetilde{\rho}^+) = \prod_{i \in I^+}(1-p^{-1}\beta_i) \cdot \prod_{i \in I^{-,00}}(1-\beta_i^{-1})$, we may conclude that

$$\begin{aligned}
\frac{1}{e!}L_p^{(e)}(0) &= \varpi_\gamma^f \cdot (1-p^{-1})^{f-e} \cdot \prod_{i \in I^{+,00}} \frac{1-p^{-1}\beta_i}{1-\beta_i^{-1}} \cdot (-1)^{d^+ \cdot f} \cdot \left(-\frac{\#\Delta}{\varpi_\gamma}\right)^f \cdot \mathrm{RHS} \\
&= (-1)^{f+d^+\cdot f} \cdot \frac{\mathcal{E}(\rho, \widetilde{\rho}^+)}{\det(1-\sigma_p^{-1} \mid W_p/W_p^0)} \cdot (\#\Delta)^f \cdot \mathrm{LHS} \\
&= (-1)^e \cdot \mathcal{L}(\rho, \widetilde{\rho}^+) \cdot \mathcal{E}(\rho, \widetilde{\rho}^+) \cdot \frac{\mathrm{Reg}_{\widetilde{\omega}_p^+}(\rho)}{\mathrm{Reg}_{\omega_\infty^+}(\rho)} \cdot L^*(\rho^\vee, 0).
\end{aligned}$$

Hence, $\mathbf{sEZC}_{\rho,\widetilde{\rho}^+}$ holds if $\widetilde{\rho}^+$ is admissible by Proposition 3.3.3. It also holds if it is not admissible, since the vanishing of $\mathrm{Reg}_{\widetilde{\omega}_p^+}(\rho)$ implies the vanishing of $L_p^{(e)}(0)$ as well. An application of Proposition 3.9.1 ends the proof of (1).

Assume now that $\mathbf{EZC}_{\rho,\widetilde{\rho}^+}$ for all p-stabilization $\widetilde{W}_p^+$ as described above and let us prove (18). It is an equality in $Y := \bigwedge^{d^+} \mathrm{Hom}_G(T_p, U')$, so it is enough to prove it after applying any linear form ν_0 over Y. Keeping the same notations as above, a basis of linear forms over Y is given by the family $\nu_\beta = \mathrm{ord}_p^{(i_1)} \wedge \ldots \mathrm{ord}_p^{(i_k)} \wedge \log_p^{(i_{k+1})} \wedge \ldots \wedge \log_p^{(i_{d^+})}$, where $\beta = (i_j)_j$ is such that $i_1 < \ldots < i_k$ are in I^0 and $i_{k+1} < \ldots < i_{d^+}$ are in I^+. In fact, any such β defines a p-stabilization $W_{p,\beta}^+$ by taking the linear span over E_p of $t_{i_1}, \ldots, t_{i_{d^+}}$, and the computation performed above shows that applying ν_β to (18) is equivalent to $\mathbf{EZC}_{\rho,\rho_\beta^+}$; hence, $\mathbf{MRS}_\chi$ holds under our assumption, and this concludes the proof of the theorem. ☐

6. Examples

6.1. **Deligne-critical motives.** The complex L-functions of motives and the p-adic interpolation of their special values are better understood for motives that admit critical points in the sense of Deligne. With the notations of the introduction, the Artin motives which have this property are precisely those which satisfy $d^+ = d$ (the even ones) or $d^+ = 0$ (the odd ones). In particular, all Dirichlet motives fall in this category. There exists nowadays an extensive literature on the construction and on the properties of p-adic L-functions in this context, and we will simply recall what is needed.

Keep the notations of the introduction and assume that ρ is even, that $\mathrm{H}^0(\mathbb{Q}, \rho) = 0$ and that ρ is of type S at $p > 2$, which means that the extension $H/\mathbb{Q}$ cut out by ρ is linearly disjoint to the $\mathbb{Z}_p$-cyclotomic extension. Note this last condition is weaker than being unramified at p, and if we denote by ω the Teichmüller character, then the odd representation $\rho \otimes \omega^{-1}$ is still of type S. There exists a p-adic measure $\theta_\rho^{\mathrm{DR}} \in \Lambda$ satisfying the following interpolation property:

$$(19) \qquad \forall\ n \geq 2, \quad \forall\ \eta \in \widehat{\Gamma}, \qquad \eta \cdot \kappa^n(\theta_\rho^{\mathrm{DR}}) = L_{\{p\}}(\rho \otimes \eta\omega^{-n}, 1-n),$$

where the subscript $\{p\}$ means that we removed the Euler factor at p. The interpolation property is first proven by the work of Deligne-Ribet [DR80] for monomial representations and includes $n=1$, and it is proven in general by a simple application of Brauer's induction theorem (see [Gre14]). The fact that $\theta_\rho^{\mathrm{DR}} \in \Lambda$ and not only in $\mathrm{Frac}(\Lambda)$, the so-called "p-adic Artin conjecture" follows from [Wil90, Theorem 1.1] as a consequence of the classical Iwasawa Main conjecture. The p-adic L-function attached to ρ is the p-adic analytic function given by

$$L_p(\rho,s)=\kappa^{1-s}(\theta_\rho^{\mathrm{DR}}) \qquad\qquad (s\in\mathbb{Z}_p).$$

Note that since ρ is of type S, the above-mentioned Euler factor at p is in fact equal to 1 at $n=1$, provided that $\eta\neq\mathbb{1}$, and the corresponding L-value is non-zero. Therefore, the formula (19) still holds when $n=1$ and $\eta\neq\mathbb{1}$.

6.2. **Even Artin motives.** We expand here upon the relation between Conjecture A, the classical Iwasawa Main Conjecture over totally real fields and "p-adic Stark conjectures at $s=1$", when $d^+=d$. The only choice of p-stabilization is then $W_p^+=W_p$, and there is no extra zeros, so $e=0$ and $\mathcal{L}(\rho,\rho^+)=1$. On the algebraic side, the corresponding Selmer groups $X_\infty(\rho,\rho^+)$ coincides with the one studied by Greenberg in [Gre14]. It is shown in *loc. cit.* that $X_\infty(\rho,\rho^+)$ is of Λ-torsion by using the validity of the weak Leopoldt conjecture, and that its characteristic ideal does not depend on the choice of T_p. It is also shown that $\mathrm{char}_\Lambda X_\infty(\rho,\rho^+)$ is generated by $\theta_\rho^{\mathrm{DR}}$ by invoking Wiles' theorem [Wil90, Theorem 1.3]. To make the connection with Conjecture A, one needs to compare $\theta_\rho^{\mathrm{DR}}$ with θ_{ρ,ρ^+}, thus, to understand the values $\eta(\theta_\rho^{\mathrm{DR}})$ for $\eta\in\widehat{\Gamma}$. It is precisely the content of "p-adic Stark conjectures at $s=1$" attributed to Serre by Tate in [Tat84, Chapitre VI, §5] and we refer the reader to [JN20, §4.4] for more detail. With the notations of Conjecture 4.9 of *loc. cit.*, it asserts for the representation $\rho\otimes\eta$ that

$$\eta(\theta_\rho^{\mathrm{DR}}) \overset{?}{=} \Omega_j(\rho\otimes\eta)\cdot j\left(L_{\{p\}}(\rho\otimes\eta,1)\right) = \begin{cases} \Omega_j(\rho\otimes\eta)\cdot j\left(L(\rho\otimes\eta,1)\right) & \text{if } \eta\neq\mathbb{1} \\ \Omega_j(\rho)\cdot\mathcal{E}(\rho,\rho^+)\cdot j\left(L(\rho,1)\right) & \text{if } \eta=\mathbb{1}.\end{cases} \tag{20}$$

Moreover, [JN20, Lemma 4.20] implies easily that the quantity $(-1)^d\cdot\Omega_j(\rho\otimes\eta)$ coincides with the quotient of regulators (4) for any choice of bases ω_∞^+ and ω_p^+ of $\mathrm{H}^0(\mathbb{R},W_p)=W_p^+=W_p$ such that $\omega_\infty^+=\omega_p^+$. Therefore, under the validity of (20) for all $\eta\in\widehat{\Gamma}$, the p-adic measure θ_{ρ,ρ^+} exists, it satisfies

$$\theta_{\rho,\rho^+}=(-2)^{-d}\theta_\rho^{\mathrm{DR}},$$

and the full Conjecture A holds for any choice of T_p. As an example, when ρ is a one-dimensional even character whose associated Dirichlet character χ has prime-to-p conductor $d\neq1$, for all characters $\eta\in\widehat{\Gamma}$ of conductor p^n we have

$$\Omega_j(\rho\otimes\eta)=\frac{\log_p(\varepsilon_{\mathrm{cyc}}^{\chi\otimes\eta})}{-j\left(\log_\infty(\varepsilon_{\mathrm{cyc}}^{\chi\otimes\eta})\right)},$$

where we have put

$$\varepsilon_{\mathrm{cyc}}^{\chi\otimes\eta}=\prod_{a \mod dp^n}\left(e^{\frac{2i\pi a}{dp^n}}-1\right)^{(\chi\otimes\eta)^{-1}(a)}\in\left(\mathbb{Z}[\mu_{dp^n}]^\times\otimes_{\mathbb{Z}}\mathbb{Q}(\chi\otimes\eta)\right)^{\chi\otimes\eta}.$$

The conjectural equality (20) directly follows from the well-known formula $L(\chi\otimes\eta,1)=\mathfrak{g}(\chi^{-1}\otimes\eta^{-1})^{-1}\cdot\log_\infty(\varepsilon_{\mathrm{cyc}}^{\chi\otimes\eta})$ and from its p-adic analogue (known as Leopoldt's formula), see [Col00, §1.3].

6.3. **Odd Artin motives.** We focus here on the connection between Conjecture A and the "p-adic Stark conjecture at $s=0$" (that is, the Gross-Stark conjecture) when $d^+=0$. The only choice of p-stabilization is then $W_p^+=0$, and both regulators are equal to 1. The representation $\widetilde{\rho}=\rho^\vee\otimes\omega$ is even and of type S, and we may consider $\theta=\mathrm{Tw}_{-1}(\theta_{\widetilde{\rho}}^{\mathrm{DR},\iota})$, where ι is the involution of Λ induced by $\gamma\mapsto\gamma^{-1}$ and where Tw_{-1} is the twist by κ^{-1}. We thus have

$$\kappa^s(\theta)=L_p(\rho^\vee\otimes\omega,s),$$

for all $s\in\mathbb{Z}_p$ and moreover,

$$\eta(\theta)=L((\rho\otimes\eta)^\vee,0)$$

holds for all non-trivial characters $\eta\in\widehat{\Gamma}$ by Section 6.1. Therefore, one already can conclude to the existence of the p-adic measure θ_{ρ,ρ^+} and to the equality $\theta=\theta'_{\rho,\rho^+}$, where θ'_{ρ,ρ^+} is the renormalization of θ_{ρ,ρ^+} appearing in Proposition 3.3.3. On the algebraic side, a duality theorem for Selmer groups [Gre89, Theorem 2] and Wiles' theorem for $\widetilde{\rho}$ proves that θ is a generator of the characteristic ideal of the Selmer group $X_\infty(\rho,\rho^+)$; hence the first part of Conjecture A is valid. Consider now the extra zeros conjecture for (ρ,ρ^+). The number $e=\dim\mathrm{H}^0(\mathbb{Q}_p,W^-)=\dim\mathrm{H}^0(\mathbb{Q}_p,W)$ is nothing but the order of vanishing of the $\{p\}$-truncated Artin L-function $L_{\{p\}}(\rho^\vee,s)$ at $s=0$. The "Weak p-adic Gross-Stark Conjecture" for $\rho^\vee$ as formulated by Gross in [Gro81, Conjecture 2.12b)] states that

$$\frac{1}{e!}L_p^{(e)}(\rho^\vee\otimes,0)\overset{?}{=}R_p(W_p^\vee)\cdot\frac{L^*_{\{p\}}(\rho^\vee,0)}{(-\log(p))^e}=(-1)^e\cdot R_p(W_p^\vee)\cdot\mathcal{E}(\rho,\rho^+)\cdot L(\rho^\vee,0),$$

where $R_p(W_p^\vee)$ is Gross's p-adic regulator defined in *loc. cit.*, (2.10) and computed with respect to the set of places $\{p,\infty\}$ of $\mathbb{Q}$. It is not hard to see that $R_p(W_p^\vee)=\mathcal{L}(\rho,\rho^+)$; hence, $\mathbf{EZC}_{\rho,\rho^+}$ is here equivalent to the Weak p-adic Gross-Stark Conjecture. It has already been proven true by Dasgupta, Kakde and Ventullo in [DKV18] for monomial representations. Moreover, it holds true for any ρ under Gross's "Order of Vanishing Conjecture" [Gro81, Conj. 2.12a)] for all monomial representation cutting out the same field extension as ρ (see [Bur20, Theorem 2.6]). This last conjecture states that the p-adic L-function $L_p(\rho^\vee\otimes\omega,s)$ has exact vanishing order e at $s=0$. It is known to be equivalent to the non-vanishing of $R_p(W_p^\vee)$ by [Bur20, Theorem 3.1 (i) and (iii)].

6.4. **Restriction to the cyclotomic line of Katz's p-adic L-function.** It was first shown in [BS19] that the Iwasawa-theoretic properties of Rubin-Stark elements imply p-adic Beilinson type formulae for Katz p-adic L-functions. We follow the notations of *loc. cit.*. Let k be a CM field of degree $2g$ and let k^+ be its maximal totally real subfield. Assume as in Section 5 that ρ is induced from a non-trivial character $\chi:\mathrm{Gal}(H/k)\longrightarrow E^\times$ of prime-to-p order and that $T_p=\mathcal{O}_p[G]\otimes_{\mathcal{O}_p[\mathrm{Gal}(H/k)]}\mathcal{O}_p(\chi)$. Hence, $d=2g$ and $d^+=g$. Under the assumption that every prime of k^+ above p splits in k (known as Katz's p-ordinarity condition), one may pick a p-adic CM type of k. This amounts to choosing a subset $\Sigma\subseteq S_p(k)$ such that Σ and its complex conjugate form a partition of $S_p(k)$. To any p-adic CM type Σ one can attach a (motivic) p-stabilization $(\rho^+_\Sigma,W^+_{p,\Sigma})$ of W_p by taking the linear span of $\{g\otimes 1\mid g\in G,\ g^{-1}(v_p)\in\Sigma\}$, where v_p is the place of k defined by ι_p. By Shapiro lemma, one can show that $X_\infty(\rho,\rho^+)$ is isomorphic over Λ to the χ-isotypic component of the Galois group $\mathrm{Gal}(M_{\infty,\Sigma}/H_\infty)$ of the maximal pro-p abelian extension of $H\cdot\mathbb{Q}_\infty$ which is Σ-ramified (as in [Mak20, Section 4.2]). In particular, it is of Λ-torsion by [HT94, Theorem 1.2.2 (iii)], and the cyclotomic part of the

Iwasawa main conjecture for CM fields asserts that

$$\mathrm{char}_\Lambda X_\infty(\rho,\rho^+) \stackrel{?}{=} \Lambda\cdot L_{p,\chi,\Sigma}^{\mathrm{cyc},\iota},$$

where $L_{p,\chi,\Sigma}^{\mathrm{cyc}} \in \Lambda$ is the restriction to the cyclotomic extension of Katz's p-adic L-function for χ and Σ [HT94]. This latter is a p-adic measure on the Galois group Γ_∞ of the compositum of all $\mathbb{Z}_p$-extensions of k which p-adically interpolates the algebraic part of critical Hecke L-values for χ^{-1} twisted by characters of Γ_∞ [Kat78, HT93]. The L-values $L((\chi\otimes\eta)^{-1},0)$ do not belong to its range of interpolation, for they are not-critical. However, the (cyclotomic part of the) explicit reciprocity conjecture [BS19, Conj. 4.7] states that

$$\mathscr{C}^{\mathrm{str}}_{\omega_p^+}(\varepsilon_\infty^\chi) \stackrel{?}{=} L_{p,\Sigma}^{\mathrm{cyc},\iota}, \tag{21}$$

where $\mathscr{C}^{\mathrm{str}}_{\omega_p^+}$ is the operator introduced in Section 3.6. Since $\mathscr{C}^{\mathrm{str}}_{\omega_p^+}(\varepsilon_\infty^\chi) = \theta'_{\rho,\rho^+}$ by Theorem 5.3.1, we see that $\mathbf{IMC}_{\rho,\rho_\Sigma^+}$ is equivalent to the cyclotomic Iwasawa main conjecture for k and Σ conditionally on (21). As for $\mathbf{EZC}_{\rho,\rho_\Sigma^+}$, one can check that the admissibility of $W_{p,\Sigma}^+$ is equivalent to Σ-Leopoldt's conjecture and that $\mathcal{L}(\rho,\rho_\Sigma^+)$ is equal to the cyclotomic $\mathcal{L}$-invariant defined in [BS19, Section 5]; accordingly, the implication $\mathbf{MRS}_\chi \implies \mathbf{EZC}_{\rho,\rho_\Sigma^+}$ of Theorem C can be seen as a reformulation of the main theorem of [BS19] in the cyclotomic case.

In the particular case where k is an imaginary quadratic field in which p splits as $p\mathcal{O}_k = \mathfrak{p}\bar{\mathfrak{p}}$, the full Conjecture A is almost completely proven. Indeed, as noted in [BS19, Remark 4.8], the equality (21) is valid for both choices $\Sigma=\{\mathfrak{p}\}$ and $\Sigma=\{\bar{\mathfrak{p}}\}$ thanks to Yager [Yag82]. The Iwasawa main conjecture for χ and Σ also holds by Rubin's famous work [Rub91], and $\mathbf{MRS}_\chi$ is a consequence of the equivariant Tamagawa number conjecture for χ, which was proven by Bley [Ble06], at least when p does not divide the class number h_k of k. Therefore, Conjecture A is valid for both stabilizations $\rho^+_{\{\mathfrak{p}\}}$ and $\rho^+_{\{\bar{\mathfrak{p}}\}}$ under the same assumption. The statement $\mathbf{EZC}_{\rho,\rho^+}$ even holds for any p-stabilization ρ^+ by Proposition 3.9.1 when $p\nmid h_k$, since W_p is the direct sum of $W^+_{p,\{\mathfrak{p}\}}$ and $W^+_{p,\{\bar{\mathfrak{p}}\}}$. Note nevertheless that this last argument would break down when $[k:\mathbb{Q}]=2g>2$ since there are at most 2^g p-adic CM types, whereas $\bigwedge^{d^+}W_p$ has dimension $(2g)!(g!)^{-2}>2^g$, so Conjecture A predicts the existence of more p-adic L-functions for χ than the ones constructed by Katz.

6.5. **Weight one modular forms.** The simplest examples of non-critical Artin motives are the two-dimensional irreducible representations ρ satisfying $d^+=1$. The proof by Khare and Wintenberger of Serre's modularity conjecture implies that the Artin representation associated with ρ is the Deligne-Serre representation of a newform f of weight 1 [KW09, Corollary 10.2 (ii)]. The newform $f(q)=\sum_{n\geq 1}a_nq^n \in E[[q]]$ has level equal to the Artin conductor N of ρ, and for all primes ℓ not dividing N, the ℓ-th Hecke eigenvalue a_ℓ of f is the trace of $\rho(\sigma_\ell)$, where σ_ℓ is the Frobenius substitution at ℓ (see [DS74]). Some aspects of Iwasawa theory for f and an odd prime p were studied in [Mak20] under the hypothesis that $p\nmid N$ (*i.e.*, ρ is unramified at p) and that f is regular at p. The last assumption means that the eigenvalues α and β of σ_p are distinct so there exist exactly two choices of p-stabilization of ρ, namely the eigenspaces $W_\alpha = W_p^{\sigma_p=\alpha}$ and $W_\beta = W_p^{\sigma_p=\beta}$ of σ_p. Note that both p-stabilizations are motivic, hence η-admissible for all $\eta\in\widehat{\Gamma}$ by Lemma 3.1.3. Under the extra assumption that p does not divide the order of the image of ρ, there is only one choice of $G_\mathbb{Q}$-stable lattice T_p up to isomorphism; hence, the residual representation $\overline{\rho}$ is irreducible and p-distinguished.

Let us consider $W_p^+ = W_\beta$ (so $W_p^- = W_\alpha$), that we let correspond to the p-stabilization $f_\alpha(q) = f(q) - \beta f(q^p)$ of U_p-eigenvalue α of f. Under the above assumptions on f and ρ, [Mak20, Conjecture 4.5.4] (labeled $\mathbf{IMC}_{f_\alpha}$ below) proposes a potential candidate $\theta_{f_\alpha} \in \Lambda$ (well-defined up to multiplication by a p-adic unit) for a generator of the characteristic ideal of the torsion Λ-module $X_\infty(f_\alpha) := X_\infty(\rho, \rho^+)$ by means of Hida theory. Roughly speaking, it is the weight one specialization of a two-variable p-adic L-function attached to the unique Hida family $\mathbf{f}$ passing through f_α. A careful study of the structure of Selmer groups and a local parametrization of $\mathbf{f}$ around f_α carried out in *loc. cit.* shows that the Iwasawa main conjecture for members of $\mathbf{f}$ in classical weight ≥ 2 implies $\mathbf{IMC}_{f_\alpha}$. Moreover, an application of a theorem due to Kato shows that one has $p^m \theta_{f_\alpha} \in \mathrm{char}_\Lambda X_\infty(f_\alpha)$ for m big enough (see [Mak20, Theorem C]). It is thus natural to expect that θ_{f_α} computes the L-values of $f \otimes \eta$ for $\eta \in \widehat{\Gamma}$, leading to a simple relationship between θ_{f_α} and the conjectural p-adic measures θ_{ρ,ρ^+} or θ'_{ρ,ρ^+}.

6.6. **Adjoint of a weight one modular form.** Here we consider a "variant of Gross-Stark conjecture" formulated in [DLR16] and proven in [RR21] in the adjoint setting. We take ρ as to be the traceless adjoint of the Deligne-Serre representation ρ_g attached to a weight one newform g (so ρ does not contain the trivial representation and it is three-dimensional with $d^+ = 1$). Assume that ρ_g is residually irreducible and p-distinguished and that it is not induced from a character of a real quadratic field in which p splits. Assume also for simplicity that $\alpha \neq -\beta$, where α and β are the roots of the p-th Hecke polynomial of g; hence the Frobenius substitution σ_p at p acts on W with distincts eigenvalues $1, \alpha/\beta, \beta/\alpha$, and there are accordingly three choices $W_1, W_{\alpha/\beta}, W_{\beta/\alpha}$ of p-stabilizations of W_p. Note that these three lines are motivic, hence η-admissible for all $\eta \in \widehat{\Gamma}$ by Lemma 3.1.3. Taking W_p^+ as to be $E_p \otimes_E W_{\beta/\alpha}$, we have then $\mathrm{H}^0(\mathbb{Q}_p, W_p^-) = W_1$, and it follows from Lemma 3.7.2 that [RR21, Theorem A] is equivalent to

$$I_p(g) \equiv \mathcal{L}(\rho, \rho^+) \mod E^\times, \tag{22}$$

where $I_p(g) \in E_p$ is the p-adic period attached to g_α (see [RR21, §1] and [DLR16, §6] for its definition). By [DLR16, Lemma 4.2] and [RR21, Proposition 2.5], the quantity $I_p(g)$ can be recast modulo $E^\times$ as the value at $s = 1$ of the derivative of Hida-Schmidt's p-adic L-function (denoted by $L_p(\mathrm{ad}^0(g_\alpha), s)$ in *loc. cit.*) attached to ρ and the eigenvalue α. This p-adic L-function is defined as the quotient of the specialization at $\mathrm{ad}(g_\alpha)$ of Hida's p-adic Rankin L-function by the p-adic zeta function $\zeta_p(s)$, so that it satisfies the same "p-adic Artin formalism" as the one proven for higher weights in [Das16]. In light of (22) and Conjecture A, one might hope to find a simple relationship between $L_p(\mathrm{ad}^0(g_\alpha), s)$ and the conjectural p-adic L-function $L_p(\rho, \rho^+, s) = \kappa^{1-s}(\theta'_{\rho,\rho^+})$, computed with respect to a (E-rational) basis ω_p^+ of $W_{\beta/\alpha} \subseteq W_p^+$. For instance, assuming $\mathbf{EZC}_{\rho,\rho^+}$ we know that both functions have a simple zero at $s = 1$ (at least if $\mathcal{L}(\rho, \rho^+) \neq 0$) and their leading terms satisfy $L'_p(\mathrm{ad}^0(g_\alpha), 1) \equiv \log_p(u_{\beta/\alpha})^{-1} \cdot L'_p(\rho, \rho^+, 1) \bmod E^\times$, where $u_{\beta/\alpha}$ is any element in the ρ-isotypic component of $E \otimes \mathcal{O}_H^\times$ on which σ_p acts with eigenvalue β/α. In the p-ordinary CM case, that is, when g has CM by an imaginary quadratic field in which p splits, we can say more. Assume that ρ_g is induced by a finite-order character ψ of G_k. Then we have

$$\rho \simeq \varepsilon_k \bigoplus \mathrm{Ind}_k^{\mathbb{Q}} \psi_{\mathrm{ad}},$$

where ε_k is the nontrivial (odd) character of $\mathrm{Gal}(k/\mathbb{Q})$ and $\psi_{\mathrm{ad}} = \psi/\overline{\psi}$, where $\overline{\psi}$ is the complex conjugate of ψ. Let us write $p\mathcal{O}_k$ as $\mathfrak{p}\bar{\mathfrak{p}}$, where $\mathfrak{p}$ is the p-adic prime of k defined by ι_p and where $\bar{\mathfrak{p}}$ is its complex conjugate, and take $\beta = \psi(\mathfrak{p})$, so that $\alpha = \psi(\bar{\mathfrak{p}})$ and $\psi_{\mathrm{ad}}(\mathfrak{p}) = \beta/\alpha$. The p-stabilization $W_p^+ = W_{\beta/\alpha}$ then lies in $\mathrm{Ind}_k^{\mathbb{Q}}\psi_{\mathrm{ad}}$, and with the notations of Section 6.4, it is in fact equal to $W_{p,\Sigma}^+$, where $\Sigma = \{\mathfrak{p}\}$. Therefore, the p-adic Artin formalism for θ'_{ρ,ρ^+}, together with the discussions of Sections 6.3 and 6.4 enables us to reinterpret [RR21, Theorem 6.2] as

$$L_p(\mathrm{ad}^0(g_\alpha), s) = \frac{\mathfrak{f}(s)}{\log(u_{\beta/\alpha})} \cdot L_p(\rho, \rho^+, s)$$

in the p-ordinary CM case, where $\mathfrak{f}(s)$ is called a E-rational fudge factor in *loc. cit.*, that is, a rational function with coefficients in E which extends to an Iwasawa function with neither poles nor zeroes at crystalline classical points. Unfortunately, it seems rather hard to make $\mathfrak{f}(s)$ explicit. A similar formula in the general case would provide strong evidence in support of $\mathbf{EX}_{\rho,\rho^+}$ and $\mathbf{EZC}_{\rho,\rho^+}$. A promising approach would perhaps be to prove an explicit reciprocity law for the specialization in weight one of the Euler system of Beilinson-Flach elements.

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