ON IMMERSIONS OF SURFACES INTO $SL(2,\mathbb{C})$ AND GEOMETRIC CONSEQUENCES

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Abstract. We study immersions of smooth manifolds into holomorphic Riemannian space forms of constant sectional curvature -1, including $SL(2,\mathbb{C})$ and the space of geodesics of $\mathbb{H}^3$, and we prove a Gauss-Codazzi Theorem in this setting.

This approach has some interesting geometric consequences:

- it provides a model for the transitioning of hypersurfaces among $\mathbb{H}^n$, $\mathbb{A}\mathrm{d}\mathbb{S}^n$, $\mathrm{d}\mathbb{S}^n$, and $\mathbb{S}^n$;
- it provides an effective tool to construct holomorphic maps to the $\mathrm{SO}(n,\mathbb{C})$-character variety, bringing to a simpler proof of the holomorphicity of the complex landslide;
- it leads to a correspondence, under certain hypotheses, between complex metrics on a surface (i.e. complex bilinear forms of its complexified tangent bundle) and pairs of projective structures with the same holonomy. Through Bers Theorem, we prove a Uniformization Theorem for complex metrics.

Contents

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1. Introduction

It is a well known result in Riemannian and Pseudo-Riemannian Geometry that the theory of codimension-1 (admissible) immersions of a simply connected manifold M into a space form is completely described by the study of two tensors on M, the *first fundamental form* and the *shape operator* of the immersion, corresponding respectively to the pull-back metric and to the derivative of a local normal vector field.

On the one hand, for a given immersion the first fundamental form and the shape operator turn out to satisfy two equations known as Gauss and Codazzi equations. On the other hand, given a metric and a self-adjoint (1,1)-tensor on M satisfying these equations, there exists one equivariant immersion from the universal cover into the space form with that pull-back metric and that self-adjoint tensor respectively as first fundamental form and shape operator; moreover, such immersion is unique up to post-composition with ambient isometries of the space form. Results of this kind are often denoted as *Gauss-Codazzi Theorems*, *Bonnet Theorems* or as *Fundamental Theorem of hypersurfaces*. See for instance [17], [21].

In [5], the authors consider a generalization of this formalism in the holomorphic context, applying a similar approach to immersions of complex manifolds of dimension 2 into $\mathbb{C}^3$.

In this article we approach a different extension of the Gauss-Codazzi formalism to the complex setting, studying immersions of smooth manifolds into holomorphic Riemannian space forms of constant curvature -1. A proper definition of holomorphic Riemannian metric and sectional curvature will be given later in this paper (see Section 2), let us just mention that holomorphic Riemannian metrics are a natural analogue of Riemannian metrics in the complex setting and a good class of examples is given by complex semisimple Lie groups equipped with the Killing form (extended globally on the group via translations).

The study of this kind of immersions turns out to have some interesting consequences, including some remarks concerning geometric structures on surfaces. In order to give a general picture:

- it provides a formalism for the study of immersions of surfaces into $\mathrm{SL}(2,\mathbb{C})$ and into the space of geodesics of $\mathbb{H}^3$, that we will denote by $\mathbb{G}$;
- it generalizes the classical theory of immersions into non-zero curvature space forms, leading to a model to study the transitioning of hypersurfaces among $\mathbb{H}^n$, $\mathbb{A}\mathrm{d}\mathbb{S}^n$, $\mathrm{d}\mathbb{S}^n$ and $\mathbb{S}^n$;
- it furnishes a tool to construct holomorphic maps from complex manifolds to the character variety of $\mathrm{SO}(n,\mathbb{C})$ (including $\mathrm{PSL}(2,\mathbb{C}) \cong \mathrm{SO}(3,\mathbb{C})$), providing also an alternative proof for the holomorphicity of the complex landslide map (see [2]);
- it introduces a notion of *complex metrics* which extends Riemannian metrics and which turns out to be useful to describe the geometry of couples of projective structures with the same monodromy. We also show a uniformization theorem in this setting which is in a way equivalent to the classical Bers Theorem for quasi-Fuchsian representations.

1.1. **Holomorphic Riemannian manifolds.** The notion of *holomorphic Riemannian metrics on complex manifolds* can be seen as a natural analogue of Riemannian metrics in the complex setting: a holomorphic Riemannian metric on $\mathbb{M}$ is a holomorphic never degenerate section of $\mathrm{Sym}_{\mathbb{C}}(T^*\mathbb{M} \otimes T^*\mathbb{M})$, namely it consists pointwise of $\mathbb{C}$-bilinear inner products on the holomorphic tangent spaces whose variation on the manifold is holomorphic. Such structures turn out to have several interesting geometric properties and have been largely studied (e.g. see the works by LeBrun, Dumitrescu, Zeghib, Biswas as in [19], [20] [7], [8]).

In an attempt to provide a self-contained introduction to the aspects we will deal with, Section 2 starts with some basic results on holomorphic Riemannian manifolds. After a short overview in the general setting - where we recall the notions of Levi-Civita connections (with the corresponding curvature tensors) and sectional curvature in this setting - we will focus on holomorphic Riemannian space forms, namely geodesically-complete simply-connected holomorphic Riemannian manifolds with constant sectional curvature. We prove a theorem of existence and uniqueness of the holomorphic Riemannian space form of prescribed constant curvature and dimension and then focus on the ones of curvature -1, that we will denote as $\mathbb{X}_n$. The space $\mathbb{X}_n$ can be defined as

$$\mathbb{X}_n = \{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} \mid \sum_{i=1}^{n+1} z_1^2 + \dots + z_{n+1}^2 = -1\}$$

with the metric inherited as a complex submanifold on $\mathbb{C}^{n+1}$ equipped with the standard $\mathbb{C}$-bilinear inner product.

This quadric model of $\mathbb{X}_n$ may look familiar: it is definitely analogue to some models of $\mathbb{H}^n$, $\mathbb{A}\mathrm{d}\mathbb{S}^n$, $\mathrm{d}\mathbb{S}^n$ and $\mathbb{S}^n$. In fact, all the pseudo-Riemannian space forms of curvature -1 immerge isometrically in $\mathbb{X}_n$: as a result, $\mathbb{H}^n$, $\mathbb{A}\mathrm{d}\mathbb{S}^n$ embed isometrically while $\mathrm{d}\mathbb{S}^n$ and $\mathbb{S}^n$ embed anti-isometrically, i.e. $-\mathrm{d}\mathbb{S}^n$ and $-\mathbb{S}^n$ (namely, $\mathrm{d}\mathbb{S}^n$ and $\mathbb{S}^n$ equipped with the opposite of their standard metric) embed isometrically.

For $n = 1, 2, 3$, $\mathbb{X}_n$ turns out to be familiar also in another sense:

- $\mathbb{X}_1$ is isometric to $\mathbb{C}^*$ equipped with the metric given by the quadratic differential $\frac{dz^2}{z^2}$;
- $\mathbb{X}_2$ is isometric to the space $\mathbb{G}$ of oriented lines of $\mathbb{H}^3$, canonically identified with $\mathbb{CP}^1 \times \mathbb{CP}^1 \setminus \Delta$, equipped with the only $\mathrm{PSL}(2,\mathbb{C})$-invariant holomorphic Riemannian metric of curvature -1;
- $\mathbb{X}_3$ is isometric up to a scale to $\mathrm{SL}(2,\mathbb{C})$ equipped with the Killing form.

1.2. **Immersions of manifolds into $\mathbb{X}_n$.** In Section 3, we approach the study of immersions of smooth manifolds into $\mathbb{X}_n$. The idea is to try to emulate the usual approach as in the pseudo-Riemannian case, but things turn out to be slightly more complicated.

Given a smooth map $\sigma\colon M \to \mathbb{X}_n$, the pull-back of the holomorphic Riemannian metric is some $\mathbb{C}$-valued symmetric bilinear form, so one can extend it to a $\mathbb{C}$-bilinear inner product on $\mathbb{C}TM = TM \otimes \mathbb{C}$; it now makes sense to require it to be non-degenerate. This is the genesis of what we define as *complex (valued) metrics* on smooth manifolds, namely smoothly varying non-degenerate inner products on each $\mathbb{C}T_xM$, $x \in M$. We will say that an immersion $\sigma\colon M \to \mathbb{X}_n$ is *admissible* if the pull-back metric is a complex valued metric. By elementary linear algebra, σ is admissible only if $dim(M) \leq n$: the real codimension is n and, despite it seems high, it cannot be lower than that. It therefore makes sense to redifine the *codimension* of σ as $n - dim(M)$. In this paper we focus on immersions of codimension 1 and 0.

It turns out that this condition of admissibility is the correct one in order to have extrinsic geometric objects that are analogue to the ones of the pseudo-Riemannian case. Complex metrics come with some complex analogue of the Levi-Civita connection, which in turn allows to define a curvature tensor and a notion of sectional curvature. In codimension 1, admissible immersions come with a notion of local normal vector field (unique up to a sign) that allows to define a second fundamental form and a shape operator.

Section 3 ends with Theorem 3.4 in which we deduce some analogue of Gauss and Codazzi equations.

Section 4 is mainly devoted to the proof of a Gauss-Codazzi theorem for immersions into $\mathbb{X}_n$ in codimension 1, as stated in Theorem 4.1.

For immersions of surfaces into $\mathbb{X}_3 \cong \mathrm{SL}(2,\mathbb{C})$ Theorems 3.4 and 4.1 can be stated in the following way.

Theorem A. Let S be a smooth simply connected surface. Consider a complex metric g on S, with induced Levi-Civita connection ∇ and curvature K_g, and a g-self adjoint bundle homomorphism $\Psi\colon \mathbb{C}TS \to \mathbb{C}TS$.

The couple (g, Ψ) satisfies

$$1) d^\nabla \Psi \equiv 0; \tag{1}$$

$$2) K_g = -1 + det(\Psi). \tag{2}$$

if and only if there exists an isometric immersion $\sigma\colon (S, g) \to \mathrm{SL}(2,\mathbb{C})$ whose corresponding shape operator is Ψ.

Such immersion σ is unique up to post-composition with elements in $\mathrm{Isom}_0(\mathrm{SL}(2,\mathbb{C})) \cong \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \big/ \{\pm(I_2, I_2)\}$.

The almost uniqueness of the immersion grants that if Γ is a group acting on S preserving g and Ψ, then the immersion σ is $(\Gamma, \mathrm{Isom}_0(\mathrm{SL}(2,\mathbb{C})))$-equivariant, namely there exists a representation

$$mon_\sigma\colon \Gamma \to \mathrm{Isom}_0(\mathrm{SL}(2,\mathbb{C}))$$

such that for all $\gamma \in \Gamma$

$$mon_\sigma(\gamma)\circ\sigma = \sigma\circ\gamma.$$

As a result if S is not simply connected, solutions of Gauss-Codazzi equations correspond to $(\pi_1(S), \mathrm{Isom}_0(\mathrm{SL}(2,\mathbb{C})))$- equivariant immersions of its universal cover into $\mathrm{SL}(2,\mathbb{C})$.

1.3. **Immersions in codimension zero and into pseudo-Riemannian space forms.** The study of immersions into $\mathbb{X}_n$ in codimension zero leads to an interesting result.

Theorem B. Let $M = M^n$ be a smooth manifold, g a complex metric on M.

Then g has constant sectional curvature -1 if and only if there exists an isometric immersion $(M, g) \to \mathbb{X}_n$.

Such immersion is unique up to post-composition with elements in $\mathrm{Isom}(\mathbb{X}_n)$.

This result can be deduced by the Gauss-Codazzi Theorem: in fact immersions in $\mathbb{X}_n$ of codimension 0 correspond to codimension-1 totally geodesic immersions in $\mathbb{X}_{n+1}$, namely immersions with shape operator $\Psi = 0$. A full proof is in Section 4.

As a result, every pseudo-Riemannian space form of constant curvature -1 and dimension n admits an essentially unique isometric immersion into $\mathbb{X}_n$.

In fact, the last remark and the similar description of Gauss-Codazzi equations for immersions into pseudo-Riemannian space forms lead to Theorem 4.11: as regards the case of $\mathrm{SL}(2,\mathbb{C})$, the result can be stated in the following way.

Theorem C. Let $\sigma\colon S \to \mathrm{SL}(2,\mathbb{C})$ be an admissible immersion with pull-back metric g and shape operator Ψ.

- $\sigma(S)$ is contained in the image of an isometric embedding of $\mathbb{H}^3$ if and only if g is Riemannian and Ψ is real.
- $\sigma(S)$ is contained in the image of an isometric embedding of $\mathbb{A}\mathrm{d}\mathbb{S}^3$ if and only if either g is Riemannian and $i\Psi$ is real, or if g has signature $(1,1)$ and Ψ is real.
- $\sigma(S)$ is contained in the image of an isometric embedding of $-\mathrm{d}\mathbb{S}^3$ if and only if either g has signature $(1,1)$ and $i\Psi$ is real, or if g is negative definite and Ψ is real.
- $\sigma(S)$ is contained in the image of an isometric embedding of $-\mathbb{S}^3$ if and only if g is negative definite and $i\Psi$ is real.

1.4. **Holomorphic dependence of the monodromy on the immersion data.** Given a smooth manifold M of dimension n, we say that (g, Ψ) is a couple of *immersion data* for M if there exists a $\pi_1(M)$-equivariant immersion $\widetilde{M} \to \mathbb{X}_{n+1}$ with pull-back metric $\widetilde{g}$ and shape operator $\widetilde{\Psi}$. As a result of the essential uniqueness of the immersion, each immersion data comes with a monodromy map $mon_\sigma\colon \pi_1(M) \to \mathrm{Isom}_0(\mathbb{X}_{n+1})$.

In Section 5, we consider families of immersion data $\{(g_\lambda, \Psi_\lambda)\}_{\lambda\in\Lambda}$ for M.

Let Λ be a complex manifold. We say that the family $\{(g_\lambda, \Psi_\lambda)\}_{\lambda\in\Lambda}$ is a *holomorphic family of immersion data* if the functions

$$\begin{aligned}\Lambda &\to \mathrm{Sym}^2(\mathbb{C}T^*_x M)\\ \lambda &\mapsto g_\lambda(x)\end{aligned} \qquad \text{and} \qquad \begin{aligned}\Lambda &\to \mathrm{End}_{\mathbb{C}}(\mathbb{C}T_x M)\\ \lambda &\mapsto \Psi_\lambda(x)\end{aligned}$$

are holomorphic for all $x \in M$.

For a fixed hyperbolic Riemannian metric h on a surface S, an instructive example is given by the family $\{(g_z, \psi_z)\}_{z\in\mathbb{C}}$ defined by

$$\begin{cases} g_z = \cosh^2(z)h;\\ \psi_z = \tanh(z)id \end{cases}$$

whose monodromy is going to be the monodromy of an immersion into $\mathbb{H}^3$ for $z \in \mathbb{R}$ and the monodromy of an immersion into $\mathbb{A}\mathrm{d}\mathbb{S}^3$ for $z \in i\mathbb{R}$.

The main result of Section 5 is the following.

Theorem D. Let Λ be a complex manifold and M be a smooth manifold of dimension n.

Let $\{(g_\lambda, \Psi_\lambda)\}_{\lambda\in\Lambda}$ be a holomorphic family of immersion data for $\pi_1(M)$-equivariant immersions $\widetilde{M} \to \mathbb{X}_{n+1}$. Then there exists a smooth map

$$\sigma\colon \Lambda \times \widetilde{M} \to \mathbb{X}_{n+1}$$

such that, for all $\lambda \in \Lambda$ and $x \in M$:

- $\sigma_\lambda := \sigma(\lambda, \cdot)\colon \widetilde{M} \to \mathbb{X}_{n+1}$ is an admissible immersion with immersion data $(g_\lambda, \Psi_\lambda)$;
- $\sigma(\cdot, x)\colon \Lambda \to \mathbb{X}_{n+1}$ is holomorphic.

Moreover, the monodromy map

$$\begin{aligned}\Lambda &\to \mathcal{X}(\pi_1(M), \mathrm{SO}(n+2, \mathbb{C}))\\ \lambda &\mapsto mon(\sigma_\lambda)\end{aligned}$$

is holomorphic.

In Section 5.2 we show an alternative proof of the holomorphicity of the complex landslide using D.

1.5. **Uniformizing complex metrics and Bers Theorem.** In Section 6 we focus on complex metrics on surfaces.

Even in dimension 2, complex metrics can have a rather wild behaviour. Nevertheless, we point out a neighbourhood of the Riemannian locus whose elements have some nice features: we will call these elements *positive complex metrics* (Definition 6.3).

In Theorem 6.10 we prove that the standard Gauss-Bonnet Theorem also holds for positive complex metrics on closed surfaces.

The most relevant result in Section 6 is a version of the Uniformization Theorem for positive complex metrics.

Theorem E. Let S be a surface with $\chi(S) < 0$.

For any positive complex metric g on S there exists a smooth function $f\colon S \to \mathbb{C}^*$ such that the positive complex metric $f \cdot g$ has constant curvature -1 and has quasi-Fuchsian monodromy.

The proof of this result uses Bers Simultaneous Uniformization Theorem (Theorem 6.13 in this paper) and in a sense is equivalent to it.

Indeed, by Theorem B, complex metrics on S with constant curvature -1 correspond to equivariant isometric immersions of $\widetilde{S}$ into $\mathbb{G} = \mathbb{CP}^1 \times \mathbb{CP}^1 \setminus \Delta$: hence, they can be identified with some couple of maps $\widetilde{S} \to \mathbb{CP}^1$ with the same monodromy. In this sense, Bers Theorem provides a whole group of immersions into $\mathbb{CP}^1 \times \mathbb{CP}^1 \setminus \Delta$: such immersions correspond to complex metrics of curvature -1 which prove to be positive. The proof of the uniformization consists in showing that every complex positive metric is conformal to a metric constructed with this procedure.

2. Geometry of holomorphic Riemannian space forms

2.1. Holomorphic Riemannian metrics.

Let $\mathbb{M}$ denote a complex analytic manifold, with complex structure $\mathbb{J}\colon T\mathbb{M} \to T\mathbb{M}$, and $n = dim_{\mathbb{C}}\mathbb{M}$.

Definition 2.1. A *holomorphic Riemannian metric* $\langle \cdot, \cdot \rangle$ on $\mathbb{M}$ is a symmetric 2-form on $T\mathbb{M}$, i.e. a section of $\mathrm{Sym}^2(T^*\mathbb{M})$, such that:

- $\langle \cdot, \cdot \rangle$ is $\mathbb{C}$-bilinear, namely for all $X, Y \in T_p\mathbb{M}$ one has $\langle \mathbb{J}X, Y\rangle = \langle X, \mathbb{J}Y\rangle = i\langle X, Y\rangle$;
- it is a non-degenerate complex bilinear form at each point;
- for all X_1, X_2 local holomorphic vector fields, $\langle X_1, X_2\rangle$ is a holomorphic function.

We also denote $\|X\|^2 := \langle X, X\rangle \in \mathbb{C}$.

Observe that, for a given holomorphic Riemannian metric $\langle \cdot, \cdot \rangle$, both the real part $Re\langle \cdot, \cdot \rangle$ and the imaginary part $Im\langle \cdot, \cdot \rangle$ are pseudo-Riemannian metrics on $\mathbb{M}$ with signature (n, n).

Drawing inspiration from basic (Pseudo-)Riemannian Geometry, one can define several constructions associated to a holomorphic Riemannian metric, such as a Levi-Civita connection - leading to the notions of curvature tensor, (complex) geodesics and completeness - and sectional curvatures. We recall some basic notions, the reader may find a more detailed treatment in [20] and in the Thesis [11].

Let us start with an analogue of the Levi-Civita Theorem.

Proposition 2.2 (See [20])**.** Given a holomorphic Riemannian metric $\langle \cdot, \cdot \rangle$ on $\mathbb{M}$, there exists a unique connection D over $T\mathbb{M}$, that we will call *Levi-Civita connection*, such that for all $X, Y \in \Gamma(T\mathbb{M})$ the following conditions hold:

$$d\langle X, Y\rangle = \langle DX, Y\rangle + \langle X, DY\rangle \qquad (D \text{ is compatible with the metric}); \qquad (3)$$

$$[X, Y] = D_X Y - D_Y X \qquad (D \text{ is torsion free}). \qquad (4)$$

Such connection coincides with the Levi-Civita connections of $Re\langle \cdot, \cdot \rangle$ and $Im\langle \cdot, \cdot \rangle$ and $D\mathbb{J} = 0$.

A direct computation shows that the Levi-Civita connection D for a holomorphic Riemannian metric $\langle\cdot,\cdot\rangle$ is explicitly described, for all $X,Y,Z\in\Gamma(T\mathbb{M})$, by

$$\begin{aligned}\langle D_X Y, Z\rangle = \frac{1}{2}\Big(X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle +\\ +\langle [X,Y],Z\rangle - \langle [Y,Z],X\rangle + \langle [Z,X],Y\rangle\Big).\end{aligned} \tag{5}$$

The notion of Levi-Civita connection D for the metric $\langle\cdot,\cdot\rangle$ leads to the standard definition of the $(1,3)$-type and $(0,4)$-type *curvature tensors*, that we will denote with R, defined by

$$R(X,Y,Z,T) := -\langle R(X,Y)Z,T\rangle := -\langle \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, T\rangle$$

for all $X,Y,Z,T\in\Gamma(T\mathbb{M})$.

Since D is the Levi-Civita connection for $Re\langle\cdot,\cdot\rangle$ and for $Im\langle\cdot,\cdot\rangle$, it is easy to check that all of the standard symmetries of curvature tensors for (the Levi-Civita connections of) pseudo-Riemannian metrics hold for (the Levi-Civita connections of) holomorphic Riemannian metrics, too. So, for instance,

$$R(X,Y,Z,T) = -R(X,Y,T,Z) = R(Z,T,X,Y) = -R(Z,T,Y,X).$$

Since the $(0,4)$-type R is obviously $\mathbb{C}$-linear on the last component, we conclude that it is $\mathbb{C}$-multilinear.

Example 2.3. Let $\mathbb{M}=G$ be a complex semisimple Lie group. Since its Killing form $\langle\cdot,\cdot\rangle\colon Lie(G)\times Lie(G)\to\mathbb{C}$ is non-degenerate and Ad-invariant, it can be extended to a holomorphic Riemannian metric on G equivalently by left or right multiplication.

In this case, the induced Levi-Civita connection admits a simple description. Indeed, applying Formula (5) to left-invariant vector fields, one gets that

$$D_{X_L}Y_L = \frac{1}{2}[X_L, Y_L]$$

for all X_L, Y_L left-invariant vector fields.

By a straightforward computation, one also gets (see [22]) that for all $X,Y,Z\in Lie(G)$ the curvature tensor R induced by D is described by

$$R(X,Y)Z = -\frac{1}{4}[[X,Y],Z].$$

Definition 2.4. A *non-degenerate plane* of $T_p\mathbb{M}$ is a complex vector subspace $\mathcal{V} < T_p\mathbb{M}$ with $dim_{\mathbb{C}}\mathcal{V}=2$ and such that $\langle\cdot,\cdot\rangle_{|\mathcal{V}}$ is a non degenerate bilinear form.

For holomorphic Riemannian metrics, we can define the complex sectional curvature of a nondegenerate complex plane $\mathcal{V}=Span_{\mathbb{C}}(V,W) < T_pM$ as

$$K(Span_{\mathbb{C}}(V,W)) = \frac{-\langle R(V,W)V,W\rangle}{\|V\|^2\|W\|^2 - \langle V,W\rangle^2}. \tag{6}$$

This definition of $K(Span_{\mathbb{C}}(V,W))$ is well-posed since R is $\mathbb{C}$-multilinear.

2.2. **Holomorphic Riemannian space forms.** We will say that a connected holomorphic Riemannian manifold $\mathbb{M}=(\mathbb{M},\langle\cdot,\cdot\rangle)$ is *complete* if its Levi-Civita connection is geodesically complete

We will call *holomorphic Riemannian space form* a complete, simply connected holomorphic Riemannian manifold with constant sectional curvature.

Theorem 2.5. For all integer $n \geq 2$ and $k \in \mathbb{C}$ there exists exactly one holomorphic Riemannian space form of dimension n with constant sectional curvature k up to isometry.

We first prove uniqueness, then existence will follow from an explicit description of the space forms.

Proof of Theorem 2.5 -Uniqueness. The proof follows the standard proof in the Riemannian case.

Let $(\mathbb{M}, \langle\cdot,\cdot\rangle_{\mathbb{M}}), (\mathbb{M}', \langle\cdot,\cdot\rangle_{\mathbb{M}'})$ be two holomorphic Riemannian space forms with the same dimension n and constant sectional curvature $k \in \mathbb{C}$. Fix any $p \in \mathbb{M}$ and $q \in \mathbb{M}'$. Since all the non-degenerate complex bilinear forms on a complex vector space are isomorphic, there exists a linear isometry $L\colon (T_p\mathbb{M}, \langle\cdot,\cdot\rangle_{\mathbb{M}}) \to (T_q\mathbb{M}', \langle\cdot,\cdot\rangle_{\mathbb{M}'})$.

For all $X \in T_pM$ and for all $t \in [0,1]$, the composition of L with the parallel transports via the geodesics $\gamma(t) = exp^{\mathbb{M}}(tX)$ and $\gamma'_L(t) = exp^{\mathbb{M}'}(tL(X))$ induces a linear isometry

$$L_\gamma\colon T_{exp_p^{\mathbb{M}}(X)}\mathbb{M} \to T_{exp_q^{\mathbb{M}'}(L(X))}\mathbb{M}'.$$

Applying the same proof as in the Riemannian case (see for instance [4]), the symmetries of the curvature tensor allow to say that

$$R^{\mathbb{M}}(X,Y)Z = -k(\langle X,Z\rangle_{\mathbb{M}}\, Y - \langle Y,Z\rangle_{\mathbb{M}}\, X), \tag{7}$$

for any $X, Y, Z \in \Gamma(T\mathbb{M})$, and similarly for $\mathbb{M}'$. As a result, L_γ being an isometry implies that

$$L^*_{\gamma(1)}(R^{\mathbb{M}'}) = R^{\mathbb{M}}$$

both when R is meant as a $(0,4)$-tensor and as a $(1,3)$ tensor.

By iteration, for any piecewise geodesic curve $\gamma\colon [0,1] \to \mathbb{M}$ with $\gamma(0) = p$, one has a well-defined notion of induced piecewise geodesic curve $\gamma'_L\colon [0,1] \to \mathbb{M}'$, which induces a linear isometry

$$L_\gamma\colon T_{\gamma(1)}\mathbb{M} \to T_{\gamma'_L(1)}\mathbb{M}'$$

such that $L^*_\gamma(R^{\mathbb{M}'}) = R^{\mathbb{M}}$.

The classical Cartan-Ambrose-Hicks Theorem for affine connections (e.g. see [23]) allows to conclude that there exist a diffeomorphism

$$f\colon \mathbb{M} \to \mathbb{M}'$$

such that $f^*(D^{\mathbb{M}'}) = D^{\mathbb{M}}$, $f(p) = q$ and such that, for every piecewise geodesic curve $\gamma\colon [0,1] \to \mathbb{M}$ with $\gamma(0) = p$, and $f_{*,\gamma(1)} = L_\gamma$. Since any point on $\mathbb{M}$ can be linked to p by a piecewise geodesic curve, f is an isometry. □

2.2.1. *Existence in Theorem 2.5 - the spaces* $\mathbb{X}_n$. The simplest example of complex manifold with a holomorphic Riemannian metric is $\mathbb{C}^n$ with the usual inner product

$$\langle \underline{z}, \underline{w}\rangle_0 = {}^t\underline{z} \cdot \underline{w} = \sum_{i=1}^n z_i w_i.$$

In this paper, we will focus on another important class of examples.

Consider the complex manifold

$$\mathbb{X}_n = \{\underline{z} \in \mathbb{C}^{n+1} \mid {}^t\underline{z} \cdot \underline{z} = -1\}.$$

The restriction to $\mathbb{X}_n$ of the metric $\langle \cdot, \cdot \rangle_0$ of $\mathbb{C}^{n+1}$ defines a holomorphic Riemannian metric. Indeed, the restriction of the bilinear form to each $p^\perp = T_p\mathbb{X}_n$ is non degenerate since $\langle p, p \rangle_{\mathbb{C}^{n+1}} \neq 0$; moreover, being the inclusion $\mathbb{X}_n \hookrightarrow \mathbb{C}^{n+1}$ holomorphic, the inherited metric is in fact holomorphic.

Since $\mathrm{SO}(n+1,\mathbb{C})$ acts by isometries on $\mathbb{C}^{n+1}$ with its inner product, it acts by isometries on $\mathbb{X}_n$ as well, and the action is trivially transitive. Moreover, for $e := (0, \dots, 0, i) \in \mathbb{X}_n$,

$$Stab(e) = \begin{pmatrix} \mathrm{SO}(n,\mathbb{C}) & \underline{0} \\ {}^t\underline{0} & 1 \end{pmatrix} \cong \mathrm{SO}(n,\mathbb{C});$$

we conclude that $\mathbb{X}_n$ has a structure of homogeneous space

$$\mathbb{X}_n \cong {}^{\mathrm{SO}(n+1,\mathbb{C})}\!\big/_{\mathrm{SO}(n,\mathbb{C})}.$$

Theorem 2.6. The $n-$dimensional space form with constant sectional curvature $k \in \mathbb{C}$ is:

- $\mathbb{C}^n$ with the usual inner product $\langle \cdot, \cdot \rangle_0$ for $k = 0$;
- $(\mathbb{X}_n, -\frac{1}{k}\langle \cdot, \cdot \rangle)$ for $k \in \mathbb{C}^*$.

It is clear that $\mathbb{C}^n$ is the flat space form. It is also clear that, if we prove that $\mathbb{X}_n = (\mathbb{X}_n, \langle \cdot, \cdot \rangle)$ is the space form for constant sectional curvature -1, then, for all $\alpha \in \mathbb{C}^*$, $(\mathbb{X}_n, -\frac{1}{\alpha}\langle \cdot, \cdot \rangle)$ has the same Levi-Civita connection and is the space form of constant sectional curvature α.

We give a proof of the fact that $\mathbb{X}_n = (\mathbb{X}_n, \langle \cdot, \cdot \rangle)$ is the space form of constant sectional curvature -1 among the following remarks on the geometry of this space.

Remark 2.7. (1) Define a *complex volume form* for $\mathbb{X}_n$ as a holomorphic non-zero n-form with the property of being $\mathrm{SO}(n+1,\mathbb{C})$-invariant. At least one complex volume form exists: any $\mathbb{C}$-multilinear n-form on $T_e\mathbb{X}_n$ is $\mathrm{SO}(n,\mathbb{C})$-invariant (indeed, it is $SL(n,\mathbb{C})$-invariant), so the action of $\mathrm{SO}(n+1,\mathbb{C})$ defines a well-posed $\mathrm{SO}(n+1,\mathbb{C})$-invariant n-form on $\mathbb{X}_n$.

Moreover, any automorphism of a holomorphic Riemannian manifold is uniquely identified by its 1-jet at any point (indeed this holds for the real part of the metric), hence we get that

$$\mathrm{SO}(n+1,\mathbb{C}) \cong \mathrm{Isom}_0(\mathbb{X}_n) \cong \mathrm{Isom}(\mathbb{X}_n, \omega_0).$$

Moreover, $\mathrm{Isom}_0(\mathbb{X}_n)$ is a subgroup of index 2 of $\mathrm{Isom}(\mathbb{X}_n) \cong O(n+1,\mathbb{C})$.

(2) By standard arguments on submanifolds, the Levi-Civita connection D for $\mathbb{X}_n$ is the tangent component of the canonical connection d for $\mathbb{C}^{n+1}$, so one can immediately

see that the exponential map at a point $p \in \mathbb{X}_n \subset \mathbb{C}^{n+1}$ is given by

$$\begin{aligned} \exp_p \colon T_p\mathbb{X}_n &\to \mathbb{X}_n \\ v &\mapsto \begin{cases} \cosh(\sqrt{\langle v,v\rangle})p + \frac{\sinh\left(\sqrt{\langle v,v\rangle}\right)}{\sqrt{\langle v,v\rangle}} v & \text{if } \langle v,v\rangle \neq 0 \\ p+v & \text{if } \langle v,v\rangle = 0 \end{cases} . \end{aligned} \tag{8}$$

Note that the description of the exponential map is independent of the choice of the square root.

(3) The space $\mathbb{X}_n$ is diffeomorphic to $T\mathbb{S}^n$.

Regard $T\mathbb{S}^n$ as $\{(\underline{u},\underline{v}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \|\underline{u}\|_{\mathbb{R}^{n+1}} = 1, \langle \underline{u},\underline{v}\rangle_{\mathbb{R}^{n+1}} = 0\}$. Then a diffeomorphism $\mathbb{X}_n \xrightarrow{\sim} T\mathbb{S}^n$ is given by

$$\underline{z} = \underline{x} + i\underline{y} \mapsto \left(\frac{1}{\|\underline{y}\|_{\mathbb{R}^{n+1}}}\underline{y}, \underline{x}\right),$$

which is well-posed since

$$\langle \underline{x}+i\underline{y}, \underline{x}+i\underline{y}\rangle = -1 \iff \begin{cases} \|\underline{y}\|^2_{\mathbb{R}^{n+1}} = \|\underline{x}\|^2_{\mathbb{R}^{n+1}} + 1 > 0 \\ \langle \underline{x},\underline{y}\rangle_0 = 0 \end{cases} .$$

In particular, $\mathbb{X}_n$ is simply connected for $n \geq 2$.

(4) For $n \geq 2$, $\mathbb{X}_n$ has constant sectional curvature -1.

It is clear that it has constant sectional curvature since $\mathrm{SO}(n,\mathbb{C})$ acts transitively on complex nondegenerate planes of $(T_e\mathbb{X}, \langle\cdot,\cdot\rangle)$. In order to compute the value of the sectional curvature, observe that the embedding

$$\begin{aligned} \mathbb{R}^{2,1} &\hookrightarrow \mathbb{C}^{n+1} \\ (x_1,x_2,x_3) &\mapsto (x_1,x_2,0,\dots,0,ix_3) \end{aligned}$$

induces an isometric embedding

$$\iota \colon \mathbb{H}^2 \hookrightarrow \mathbb{X}_n$$

which is totally geodesic by Formula (8). Computing the sectional curvature of $Span_{\mathbb{C}}(\iota_*(T_{(0,0,1)}\mathbb{H}^2))$ by taking generators in $\iota_*(T_{(0,0,1)}\mathbb{H}^2)$, one concludes that the sectional curvature is -1.

(5) Consider the projective quotient $\mathrm{p}_n \colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$.

Then $\mathrm{p}_{n|\mathbb{X}_n}$ is a two-sheeted covering on its image $\mathbb{PX}_n$, which corresponds to the complementary in $\mathbb{CP}^n$ of the non-degenerate hyperquadric

$$Q_n = \{z_1^2 + \dots + z_{n+1}^2 = 0\} =: \partial\mathbb{PX}_n$$

that we denote as the *boundary* of $\mathbb{PX}_n$.

The $\mathrm{SO}(n+1,\mathbb{C})$-invariancy of the the metric on $\mathbb{X}_n$ implies that the action of $\mathrm{SO}(n+1,\mathbb{C})$ on $\mathbb{CP}^n$ fixes $\mathbb{PX}_n$ globally (hence the complementary hyperquadric) and acts by isometries on it.

The group of isometries of $\mathbb{PX}_n$ is given by $\mathbb{P}O(n+1,\mathbb{C})$ acting on the whole $\mathbb{CP}^n$ as a subgroup of $PGL(n+1,\mathbb{C})$. Conversely, it is simple to check that $\mathbb{P}O(n+1,\mathbb{C})$

coincides exactly with the subgroup of elements of $PGL(n+1,\mathbb{C})$ that fix $\mathbb{P}\mathbb{X}_n$ globally (or, equivalently, that fix $\partial\mathbb{P}\mathbb{X}_n$ globally).

2.3. $\mathbb{X}_3$ **as** $\mathrm{SL}(2,\mathbb{C})$**.** We show that $\mathbb{X}_3$ is isometric (up to a scale) to the complex Lie group $\mathrm{SL}(2,\mathbb{C}) = \{A \in Mat(2,\mathbb{C}) \mid det(A) = 1\}$ equipped with the holomorphic Riemannian metric given by the Killing form, globally pushed forward from I_2 equivalently by left or right translation.

Consider on $Mat(2,\mathbb{C})$ the non-degenerate quadratic form given by $M \mapsto -det(M)$, which corresponds to the complex bilinear form

$$\langle M, N\rangle_{Mat_2} = \frac{1}{2}\Big(tr(M\cdot N) - tr(M)\cdot tr(N)\Big).$$

In the identification $TMat(2,\mathbb{C}) = Mat(2,\mathbb{C})\times Mat(2,\mathbb{C})$, this complex bilinear form induces a holomorphic Riemannian metric on $Mat(2,\mathbb{C})$.

Observe that the action of $\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})$ on $Mat(2,\mathbb{C})$ given by $(A,B)\cdot M := AMB^{-1}$ is by isometries, because it preserves the quadratic form.

Since all the non-degenerate complex bilinear forms on complex vector spaces of the same dimension are isomorphic, there exists a linear isomorphism $F\colon (\mathbb{C}^4, \langle\cdot,\cdot\rangle_0) \to (Mat(2,\mathbb{C}), \langle\cdot,\cdot\rangle_{Mat_2})$ which is also an isometry of holomorphic Riemannian manifolds: such isometry F restricts to an isometry between $\mathbb{X}_3$ and $SL(2,\mathbb{C})$, where $SL(2,\mathbb{C})$ is equipped with the submanifold metric.

For all $A \in \mathrm{SL}(2,\mathbb{C})$, $T_A\mathrm{SL}(2,\mathbb{C}) = A\cdot T_{I_2}SL(2,\mathbb{C})$ and

$$T_{I_2}SL(2,\mathbb{C}) = \mathfrak{sl}(2,\mathbb{C}) = \{M\in Mat(2,\mathbb{C}) \mid tr(M) = 0\}$$

can be endowed with the structure of complex Lie algebra associated to the complex Lie group $\mathrm{SL}(2,\mathbb{C})$.

The induced metric on $\mathrm{SL}(2,\mathbb{C})$ at a point A is given by

$$\langle AV, AW\rangle_A = \langle V, W\rangle_{I_2} = \langle V,W\rangle_{Mat_2} = \frac{1}{2}tr(V\cdot W),$$

as a consequence we have $\langle\cdot,\cdot\rangle_{I_2} = \frac{1}{8}Kill$ where $Kill$ is the Killing form of $\mathfrak{sl}(2,\mathbb{C})$, which is an Ad-invariant bilinear form (e.g see [3]) and $\langle\cdot,\cdot\rangle_A$ is the symmetric form on $T_A\mathrm{SL}(2,\mathbb{C})$ induced by pushing forward the Killing form equivalently by right or left translation by A.

We will often focus on $\mathrm{PSL}(2,\mathbb{C}) = {}^{\mathrm{SL}(2,\mathbb{C})}\!/\!_{\{\pm I_2\}} \cong \mathbb{P}\mathbb{X}_3$ too.

Proposition 2.8.

$$\mathrm{Isom}_0(\mathrm{SL}(2,\mathbb{C})) \cong {}^{\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})}\!/\!_{\{\pm(I_2,I_2)\}}$$

where the action of $\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})$ is given by

$$\begin{aligned} \mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C}) &\to \mathrm{SL}(2,\mathbb{C}) \\ (A,B)\cdot C &:= A\ C\ B^{-1} \end{aligned} \tag{9}$$

Proof. Since $\mathrm{Isom}_0(\mathbb{X}_3)\cong\mathrm{Isom}_0(\mathbb{C}^4,\langle\cdot,\cdot\rangle_0)\cong\mathrm{SO}(4,\mathbb{C})$, one has $\mathrm{Isom}_0(\mathrm{SL}(2,\mathbb{C}))\cong\mathrm{Isom}_0(Mat(2,\mathbb{C}))\cong \mathrm{SO}(4,C)$. Since, for all $A,B\in\mathrm{SL}(2,\mathbb{C})$, $det(A\ M\ B^{-1}) = det(M)$, the action above preserves the quadratic form, hence it is by isometries. We therefore have a homomorphism

$\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})\to\mathrm{Isom}(\mathrm{SL}(2,\mathbb{C}))$ whose kernel is $\{\pm(I_2,I_2)\}$. Finally, since

$$dim_{\mathbb{C}}\left(\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})\big/_{\{\pm(I_2,I_2)\}}\right)=6=dim_{\mathbb{C}}\mathrm{Isom}(\mathrm{SL}(2,\mathbb{C})),$$

we conclude that $\mathrm{Isom}_0(\mathrm{SL}(2,\mathbb{C}))\cong\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})\big/_{\{\pm(I_2,I_2)\}}$ □

2.4. $\mathbb{X}_2$ **as the space of geodesics of** $\mathbb{H}^3$**.** An interesting geometric description of $\mathbb{X}_2$ is given by the space of geodesics of $\mathbb{H}^3$. Let us look at it quite closely.

The restriction of $\langle\cdot,\cdot\rangle_{Mat_2}$ to $\mathfrak{sl}(2,\mathbb{C})$ being a non degenerate $\mathbb{C}$-bilinear form, one has that $\mathrm{SL}(2,\mathbb{C})\cap\mathfrak{sl}(2,\mathbb{C})$, with the induced holomorphic Riemannian metric, is isometric to $\mathbb{X}_2$. Under the natural action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{H}^3$, one can see that $\mathrm{SL}(2,\mathbb{C})\cap\mathfrak{sl}(2,\mathbb{C})$ corresponds to rotations of angle π around some axis.

Define $\mathbb{G}:=\overline{\mathbb{C}}\times\overline{\mathbb{C}}\setminus\Delta$ and its 2-sheeted quotient $\mathbb{G}_\sim$ defined by $(z,w)\sim(w,z)$. Under the usual identification of $\overline{\mathbb{C}}$ as the visual boundary of $\mathbb{H}^3$, one can see $\mathbb{G}$ (resp $\mathbb{G}_\sim$) as the space of *maximal oriented* (resp. *unoriented*) *unparametrized geodesics* of $\mathbb{H}^3$.

Consider the map

$$Axis_\sim\colon\mathbb{X}_2\cong\mathrm{SL}(2,\mathbb{C})\cap\mathfrak{sl}(2,\mathbb{C})\to\mathbb{G}_\sim$$

that sends each rotation of angle π to its unoriented axis. One can immediately see that this map is a $2:1$ covering map, hence it lifts to a homeomorphism

$$Axis\colon\mathrm{SL}(2,\mathbb{C})\cap\mathfrak{sl}(2,\mathbb{C})\to\mathbb{G}.$$

Explicitly, one can define Axis as

$$Axis\left(\begin{pmatrix}a & b\\ c & -a\end{pmatrix}\right)=\Big(\frac{a+i}{c}=-\frac{b}{a-i},\frac{a-i}{c}=-\frac{b}{a+i}\Big),^1 \tag{10}$$

from which one deduces that Axis is a biholomorphism.

Another interesting remark is that *Axis* is $\mathrm{PSL}(2,\mathbb{C})$-equivariant, where $\mathrm{PSL}(2,\mathbb{C})$ acts on $\mathrm{SL}(2,\mathbb{C})\cap\mathfrak{sl}(2,\mathbb{C})$ by conjugation and on $\mathbb{G}$ via the natural diagonal action via Möbius maps.

Define $\langle\cdot,\cdot\rangle_{\mathbb{G}}$ as the push-forward of the holomorphic Riemannian metric of $\mathrm{SL}(2,\mathbb{C})\cap\mathfrak{sl}(2,\mathbb{C})$ via *Axis*. The metric on $\mathbb{G}$ can be computed explicitly.

Let (U,z) be an affine chart for $\mathbb{CP}^1$, so $(U\times U\setminus\Delta,z\times z)$ is a holomorphic chart for $\mathbb{G}$, set $z\times z=:(z_1,z_2)$

Proposition 2.9. The pull-back Riemannian holomorphic metric on $\mathbb{G}$ is locally described by

$$-\frac{4}{(z_1-z_2)^2}dz_1dz_2,$$

so $\mathbb{G}$ endowed with this Riemannian holomorphic metric is isometric to $\mathbb{X}_2$.

[1]This notation means that, on each component, at least one between RHS and LHS makes sense and they coincide if they both make sense.

Moreover, $\mathrm{Isom}_0(\mathbb{G}) \cong \mathrm{PSL}(2,\mathbb{C})$ acting diagonally on $\mathbb{G}$ via Möbius maps and the immersion

$$\begin{aligned} \mathbb{H}^2 &\to \mathbb{G} \\ z &\mapsto (z, \overline{z}), \end{aligned}$$

with $\mathbb{H}^2$ in the upper half-plane model, is an equivariant isometric embedding.

Proof.

- An isotropic vector in $T_{(z_1,z_2)}\mathbb{G} \cong \mathbb{C}\times\mathbb{C}$ is such that, for all $\psi \in Stab_0((z_1, z_2)) \subset \mathrm{PSL}(2,\mathbb{C})$, the differential of ψ sends a vector (w_1, w_2) to a multiple of its. By taking ψ as any purely hyperbolic map with axis (z_1, z_2), one gets that a tangent vector is isotropic if and only if it is of the form $(w, 0)$ or $(0, w)$.

  As a consequence, isotropic complex geodesics in $\mathbb{G}$ correspond to the factor components, i.e. they are of the form $(\mathbb{CP}^1 \setminus \{z_2\}) \times \{z_2\}$ or $\{z_1\} \times (\mathbb{CP}^1 \setminus \{z_1\})$.
- We fix an affine chart (U, z) for $\mathbb{CP}^1$ and we want to write the metric on the chart $(U \times U \setminus \Delta, z \times z)$ for $\mathbb{G}$.

  Since the isotropic directions for $\langle\cdot,\cdot\rangle_{\mathbb{G}}$ are the factor components, the metric at the point $(1, 0)$ is of the form

  $$\langle\cdot,\cdot\rangle_{1,0} = \lambda_0 dz_1 dz_2$$

  for some $\lambda_0 \in \mathbb{C}^*$.

  Since the metric is invariant by the action of $\mathrm{PSL}(2,\mathbb{C})$, consider for all $(z_1, z_2) \in \mathbb{C} \times \mathbb{C} \setminus \Delta$ the morphism $\psi\colon w \mapsto z_2 w + z_1(1 - w)$ and use it to deduce that

  $$\langle\cdot,\cdot\rangle_{(z_1,z_2)} = \psi_*^{-1}\langle\cdot,\cdot\rangle_{0,1} = \lambda_0 \frac{1}{(z_1 - z_2)^2} dz_1 dz_2.$$
- Let us compute λ_0.

  Each of the two connected components of $SL(2,\mathbb{R}) \cap \mathfrak{sl}(2,\mathbb{R})$ is isometric to $\mathbb{H}^2$: it can be seen in fact as $\mathbb{H}^2$ in the usual hyperboloid model. The image of one of these connected components goes into $\{(z, \overline{z})\}$ where z lies in the upper half plane H of $\overline{\mathbb{C}}$. As a result, by taking $\mathbb{H}^2$ in the upper half plane model, the immersion

  $$\begin{aligned} \sigma\colon H &\to \mathbb{G} \\ z &\mapsto (z, \overline{z}) \end{aligned}$$

  is a totally geodesic isometric immersion.

  The pull-back metric of $\langle\cdot,\cdot\rangle_{\mathbb{G}}$ is $\frac{\lambda_0}{-4(Im(z))^2} dz \cdot d\overline{z}$, hence it coincides with the usual hyperbolic metric if and only if $\lambda_0 = -4$.

□

3. Immersed hypersurfaces in $\mathbb{X}_{n+1}$

In this section we study the geometry of smooth immersions of the form

$$M \to \mathbb{X}_{n+1}$$

where M is a smooth manifold of (real) dimension n and $\mathbb{X}_{n+1}$ is the Riemannian holomorphic space form of constant sectional curvature -1 and complex dimension $n+1$.

As an immersion between smooth manifolds, it has very high codimension. Nevertheless, we can define a suitable class of immersions for which we can translate in this setting some aspects of the classical theory of immersions of hypersurfaces. In order to do it, we will introduce a new structure on manifolds that extends the notion of Riemannian metric: *complex valued metrics*.

We will use $\mathtt{X}, \mathtt{Y}, \mathtt{Z}$ to denote elements (and sections) of TM and X, Y, Z to denote elements (and sections) of the complexified tangent bundle $\mathbb{C}TM := TM \oplus iTM = \mathbb{C} \otimes_{\mathbb{R}} TM$ whose elements can be seen as complex derivations of germs of complex-valued functions.

Let M be a smooth manifold of (real) dimension m and $\sigma\colon M \to \mathbb{X}_{n+1}$, with $n+1 \geq m$, be a smooth immersion. Since $\mathbb{X}_{n+1}$ is a complex manifold, the differential map σ_* extends by $\mathbb{C}$-linearity to a map

$$\begin{aligned} \sigma_*\colon \mathbb{C}TM &\to T\mathbb{X}_n \\ X = \mathtt{X} + i\mathtt{Y} &\mapsto \sigma_*(\mathtt{X}) + \mathbb{J}\sigma_*(\mathtt{Y}) =: \sigma_*(X). \end{aligned}$$

Now, consider the $\mathbb{C}-$bilinear pull-back form $\sigma^*\langle \cdot, \cdot \rangle$ on $\mathbb{C}TM$ defined by

$$\begin{aligned} \sigma^*\langle \cdot, \cdot \rangle_p \colon \mathbb{C}T_pM \times \mathbb{C}T_pM &\to \mathbb{C} \\ (X, Y) &\mapsto \langle \sigma_* X, \sigma_* Y \rangle. \end{aligned}$$

Note that $\sigma^*\langle \cdot, \cdot \rangle_p$ is $\mathbb{C}$-bilinear and symmetric because $\langle \cdot, \cdot \rangle_{\sigma(p)}$ is.

Definition 3.1.
- A *complex (valued)* metric g on M is a non-degenerate smooth section of the bundle $\mathrm{Sym}(\mathbb{C}T^*M \otimes \mathbb{C}T^*M)$, i.e. it is a smooth choice at each point $p \in M$ of a non-degenerate symmetric complex bilinear form
$$g_p\colon \mathbb{C}T_pM \times \mathbb{C}T_pM \to \mathbb{C}.$$
- A smooth immersion $\sigma\colon M \to \mathbb{X}_{n+1}$ is *admissible* if $g = \sigma^*\langle \cdot, \cdot \rangle$ is a complex valued metric for M, i.e. if $\sigma^*\langle \cdot, \cdot \rangle_p$ is non-degenerate.
- If g is a complex metric on M, an immersion $\sigma\colon (M, g) \to \mathbb{X}_{n+1}$ is *isometric* if $\sigma^*\langle \cdot, \cdot \rangle = g$.

Remark 3.2. If σ is an admissible immersion, then $\sigma_{*p}\colon \mathbb{C}T_pM \to T_{\sigma(p)}\mathbb{X}_{n+1}$ is injective. Indeed, if $\sigma_{*p}(X) = 0$ then clearly $\sigma^*\langle X, \cdot \rangle \equiv 0$, hence $X = 0$.

3.1. **Levi-Civita connection and curvature for complex metrics.** Let M be a manifold of dimension m and g be a complex valued metric on $\mathbb{C}TM$. Recall that for sections X, Y of $\mathbb{C}TM$ we have a well-posed Lie bracket $[X, Y]$ which coincides with the $\mathbb{C}$-bilinear extension of the usual Lie bracket for vector fields.

We define a *connection on* $\mathbb{C}TM$ as the $\mathbb{C}$-linear application

$$\begin{aligned} \nabla\colon \Gamma(\mathbb{C}TM) &\to \Gamma(\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}TM, \mathbb{C}TM)) \\ \alpha &\mapsto \nabla\alpha(\colon X \mapsto \nabla_X \alpha) \end{aligned}$$

such that, for all $f \in C^\infty(M, \mathbb{C})$, $\nabla_X(f\alpha) = f\nabla_X \alpha + X(f)\alpha$.

In a similar way as in classical Riemannian geometry, a complex valued metric g induces a canonical choice of a connection on $\mathbb{C}TM$.

Proposition 3.3. For every complex valued metric g on M, there exists a unique connection ∇ on $\mathbb{C}TM$, that we will call *Levi-Civita connection*, such that for all $X, Y \in \Gamma(\mathbb{C}TM)$ the following conditions stand:

$$d(g(X,Y)) = g(\nabla X, Y) + g(X, \nabla Y) \qquad (\nabla \text{ is compatible with the metric});$$
$$[X,Y] = \nabla_X Y - \nabla_Y X \qquad (\nabla \text{ is torsion free}).$$

Proof. Exactly as in the Riemannian case. □

Observe that if g is obtained as a $\mathbb{C}$-bilinear extension of some (pseudo-)Riemannian metric, then the induced Levi-Civita connection is the complex extension of the Levi-Civita connection for the (pseudo-)Riemannian metric on M.

We can also define the $(0,4)$-type and the $(1,3)$-type *curvature tensors* for g defined on $\mathbb{C}TM$ by

$$R(X,Y,Z,T) := -g(R(X,Y)Z,T) = -g\bigg(\nabla_X\nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z, T\bigg)$$

with $X, Y, Z, T \in \Gamma(\mathbb{C}TS)$. The curvature tensor is $\mathbb{C}$-multilinear and has all of the standard symmetries of the curvature tensors induced by Riemannian metrics.

Finally, for every complex plane $Span_{\mathbb{C}}(X,Y) \in \mathbb{C}T_pM$ such that $g|_{Span_{\mathbb{C}}(X,Y)}$ is non-degenerate, we can define the sectional curvature $K(X,Y) := K(Span_{\mathbb{C}}(X,Y))$ as

$$K(X,Y) = \frac{-g(R(X,Y)X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2} \tag{11}$$

where the definition of $K(X,Y)$ is independent from the choice of the basis $\{X,Y\}$ for $Span_{\mathbb{C}}(X,Y)$.

It is simple to check, via the Gram-Schmidt algorithm, that in a neighbourhood of any point $p \in M$ it is possible to construct a local $g-$orthonormal frame $(X_j)_{j=1}^m$ on M. We show it explicitly.

Fix a orthonormal basis $(W_j(p))_{j=1}^m$ for $\mathbb{C}T_pM$ and locally extend, in a neighbourhood of p, each $W_j(p)$ to a complex vector field W_j. Up to shrinking the neighbourhood in order to make the definition well-posed, define by iteration the local vector fields Y_j by

$$Y_j := W_j - \sum_{k=1}^{j-1} \frac{g(W_j, Y_k)}{g(Y_k, Y_k)} Y_k$$

which are such that $Y_j(p) = W_j(p)$ and by construction the Y_j's are pairwise orthogonal. Finally, up to shrinking the neighbourhood again, the vectors $X_j = \frac{Y_j}{\sqrt{g(Y_j, Y_j)}}$ (defined for any local choice of the square root) determine a local g-orthonormal frame around p. Similarly, every set of orthonormal vector fields can be extended to a orthonormal frame.

Let $(X_j)_{j=1}^m$ be a local orthonormal frame for g, with $X_j \in \mathbb{C}TM$. Let $(\theta^j)_{j=1}^m$ be the correspondent coframe, $\theta^i \in \mathbb{C}T^*M$, defined by $\theta^i = g(X_i, \cdot)$.

We can define the *Levi-Civita connection forms* θ^i_j for the frame $(X_i)_i$ by

$$\nabla X_i = \sum_h \theta^h_i \otimes X_h.$$

or, equivalently, by the equations

$$\begin{cases} d\theta^i = -\sum_j \theta^i_j \wedge \theta^j \\ \theta^i_j = -\theta^j_i \end{cases}.$$

3.2. **Extrinsic geometry of hypersurfaces in $\mathbb{X}_{n+1}$.** From now on, assume $dim(M) = n$.

Let $\sigma\colon M \to \mathbb{X}_{n+1}$ be an admissible immersion and $g = \sigma^*\langle\cdot,\cdot\rangle$ be the induced complex metric. Denote with D be the Levi-Civita connection on $\mathbb{X}_{n+1}$ and with ∇ be the Levi-Civita connection for g. We want to adapt the usual extrinsic theory for immersed hypersurfaces to our setting. The following ideas are in fact quite classical but the complex context requires some additional caution.

Define the pull-back vector bundle $\Lambda := \sigma^*(T\mathbb{X}_{n+1}) \to M$, which has a structure of complex vector bundle given by $i\sigma^*(v) := \sigma^*(\mathbb{J}v)$ and which can be endowed on each fiber with the pull-back complex bilinear form, that we will still denote with $\langle\cdot,\cdot\rangle$. If U is an open subset of M over which σ restricts to an embedding, then $\Lambda_{|U} \cong T\mathbb{X}_{n+1|\sigma(U)}$.

The tangent map $\sigma_*\colon \mathbb{C}TM \to \mathbb{X}_{n+1}$ being injective, $\mathbb{C}TM$ can be seen canonically as a complex sub-bundle of Λ via the correspondence

$$\begin{aligned} \mathbb{C}TM &\hookrightarrow \Lambda \\ X &\mapsto \sigma^*(\sigma_* X), \end{aligned}$$

Moreover, the complex bilinear form on Λ corresponds to the one on $\mathbb{C}TM$ when restricted to it, since σ is an isometric immersion.

We pull back the Levi-Civita connection D of $T\mathbb{X}_{n+1}$ in order to get a $\mathbb{R}$-linear connection $\overline{\nabla}$ on Λ,

$$\overline{\nabla} = \sigma^* D\colon \Gamma(\Lambda) \to \Gamma\Big(\mathrm{Hom}_{\mathbb{R}}(TM, \Lambda)\Big).$$

Observe that $\overline{\nabla}$ is completely defined by the Leibniz rule and by the condition

$$\overline{\nabla}_{\mathtt{X}}\ \sigma^*\xi := \sigma^*\Big(D_{\sigma_*\mathtt{X}}\ \xi\Big) \qquad \forall \xi \in \Gamma(T\mathbb{X}_{n+1}), \mathtt{X} \in \Gamma(TM).$$

By $\mathbb{C}$-linearity, we can see $\overline{\nabla}$ as a map $\overline{\nabla}\colon \Gamma(\Lambda) \to \Gamma\Big(\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}TM, \Lambda)\Big)$ by defining $\overline{\nabla}_{\mathtt{X}_1 + i\mathtt{X}_2}\hat{\xi} := \overline{\nabla}_{\mathtt{X}_1}\hat{\xi} + i\overline{\nabla}_{\mathtt{X}_2}\hat{\xi}$.

Via the canonical immersion of bundles $\mathbb{C}TM \hookrightarrow \Lambda$, it makes sense to consider the vector field $\overline{\nabla}_X Y$ with $X, Y \in \Gamma(\mathbb{C}TM)$.

Since $D\mathbb{J} = 0$ on $\mathbb{X}_{n+1}$, $\overline{\nabla}$ is $\mathbb{C}$-bilinear. Indeed, for all $\mathtt{X} \in \Gamma(TM)$ and $Y \in \Gamma(\mathbb{C}TM)$,

$$\overline{\nabla}_{\mathtt{X}}(i\sigma^*(\xi)) = \sigma^*\Big(D_{\sigma_*\mathtt{X}}(\mathbb{J}\xi)\Big) = \sigma^*\Big(D_{\sigma_*\mathtt{X}}\mathbb{J}\ \xi\Big) = \sigma^*\Big(\mathbb{J}D_{\sigma_*\mathtt{X}}\ \xi\Big) = i\overline{\nabla}_{\mathtt{X}}\sigma^*\xi\ .$$

We observed that $\mathbb{C}TM$ is a sub-bundle of Λ over which the restriction of the complex bilinear form of Λ is non-degenerate. Hence, we can consider the *normal bundle* $\mathcal{N} = \mathbb{C}TM^\perp$ over M defined as the orthogonal complement of $\mathbb{C}TM$ in Λ. $\mathcal{N}$ is a rank-1 complex bundle on M. For all local fields $X, Y \in \Gamma(\mathbb{C}TM)$ we can define $\underline{\mathrm{II}}(X,Y)$ as the component in $\mathcal{N}$ of $\overline{\nabla}_X Y$. Similarly to the Riemannian case, one has that $\overline{\nabla}$ decomposes on $\mathbb{C}TM$ as

$$\overline{\nabla}_X Y = \nabla_X Y + \underline{\mathrm{II}}(X,Y).$$

where we recall that ∇ is the Levi-Civita connection of g. In particular, one gets that $\underline{\mathrm{II}}$ is a symmetric $\mathbb{C}$-bilinear tensor.

For all $p \in M$, consider on a suitable neighbourhood $U_p \subset M$ a norm-1 section ν of $\mathcal{N}$: we call such ν a local *normal vector field* for σ. A local normal vector field fixed, we can locally define the *second fundamental form* of the immersion σ as the tensor

$$\mathrm{II} := \langle \underline{\mathrm{II}}, \nu \rangle = \langle \overline{\nabla}, \nu \rangle.$$

Since there are two opposite choices for the section ν, II is defined up to a sign.

We define the *shape operator* Ψ associated to the immersion $\sigma\colon M \to \mathbb{X}_{n+1}$ as the tensor

$$\Psi \in \Gamma\Big(\mathrm{Sym}(\mathbb{C}T^*M \otimes_{\mathbb{C}} \mathbb{C}TM)\Big)$$

such that, $\forall p \in M$ and $\forall \mathtt{X}, \mathtt{Y} \in T_pM$, $g(\Psi(\mathtt{X}), \mathtt{Y}) = -\langle \underline{\mathrm{II}}(\mathtt{X}, \mathtt{Y}), \nu \rangle = -\mathrm{II}(\mathtt{X}, \mathtt{Y})$. As II is defined up to a sign, Ψ is defined up to a sign as well.

We will say that σ is *totally geodesic* if and only if $\underline{\mathrm{II}} \equiv 0$, i.e. $\Psi \equiv 0$.

Exactly as in the Riemannian case, one gets that

$$\Psi = \overline{\nabla}\nu. \tag{12}$$

Indeed, for all $\mathtt{X} \in \Gamma(TM)$, since $\langle \overline{\nabla}_{\mathtt{X}}\nu, \nu \rangle = \frac{1}{2} d(\langle \nu, \nu \rangle)(\mathtt{X}) = 0$, $\overline{\nabla}_{\mathtt{X}}\nu \in \mathbb{C}TM$; moreover, for all $Y \in \Gamma(\mathbb{C}TM)$,

$$g(\Psi(\mathtt{X}), Y) = -\mathrm{II}(\mathtt{X}, Y) = -\langle \overline{\nabla}_{\mathtt{X}} Y, \nu \rangle = \langle Y, \overline{\nabla}_{\mathtt{X}}\nu \rangle - d(\langle Y, \nu \rangle)(\mathtt{X}) = \langle Y, \overline{\nabla}_{\mathtt{X}}\nu \rangle \ :$$

as a result, Equation (12) holds on TM, therefore on $\mathbb{C}TM$ by $\mathbb{C}$-bilinearity.

We are finally able to show that Gauss and Codazzi equations hold as well

Consider the exterior covariant derivative d^∇ associated to the Levi-Civita connection ∇ on $\mathbb{C}TM$, namely

$$(d^\nabla \Psi)(X,Y) = \nabla_X(\Psi(Y)) - \nabla_Y(\Psi(X)) - \Psi([X,Y])$$

for all $X, Y \in \Gamma(\mathbb{C}TM)$.

Fix a local orthonormal frame $(X_i)_{i=i}^n$ with coframe $(\theta^i)_{i=1}^n$. We can see the shape operator Ψ in coordinates:

$$\Psi =: \Psi^i_j \cdot \theta^j \otimes X_i =: \Psi^i \otimes X_i,$$

where $\Psi^i_j \in \mathbb{C}^\infty(U, \mathbb{C})$, with $\Psi^i_j = \Psi^j_i$, and $\Psi^i \in \Omega^1(\mathbb{C}TM)$. Notice that

$$\mathrm{II}(X_i, \cdot) = -\langle \Psi(X_i), \cdot \rangle = -\langle \Psi^j_i X_j, \cdot \rangle = -\Psi^j_i \theta^j = -\Psi^i_j \theta^j = -\Psi^i \tag{13}$$

Theorem 3.4 (Gauss-Codazzi, first part)**.** Let $\sigma\colon M^n \to \mathbb{X}_{n+1}$ be an admissible immersion and $g = \sigma^*\langle\cdot,\cdot\rangle$ be the induced complex metric.

Let ∇ be the Levi-Civita connection on M, R^M the curvature tensor of g and Ψ the shape operator associated to σ.

Fix a g-orthonormal frame $(X_i)_i$ with corresponding coframe $(\theta^i)_i$.

Then the following equations hold:

$$1) d^\nabla \Psi \equiv 0 \qquad \text{(Codazzi equation);} \qquad (14)$$

$$2) R^M(X_i, X_j, \cdot,\cdot) - \Psi^i \wedge \Psi^j = -\theta^i \wedge \theta^j \qquad \text{(Gauss equation).} \qquad (15)$$

Proof. Once again, the proof follows the standard well-known proof in the Riemannian case. We recall the essential steps and skip the full computations (for which one can refer to the thesis [11]).

In a neighbourhood U of a point $p \in M$, fix a local normal vector field ν. Let $Z \in \Gamma(\mathbb{C}TU)$, let $\mathtt{X}(p), \mathtt{Y}(p) \in T_pM$ and let $\mathtt{X}, \mathtt{Y} \in \Gamma(TU)$ be local extensions of $\mathtt{X}(p)$ and $\mathtt{Y}(p)$ such that $[\mathtt{X},\mathtt{Y}] = 0$.

By definition, $\overline{\nabla}_\mathtt{Y} Z = \nabla_\mathtt{Y} Z + \mathrm{II}(Z,\mathtt{Y})\nu$. Hence, by second derivation, one gets

$$\overline{\nabla}_\mathtt{X}\overline{\nabla}_\mathtt{Y} Z = \nabla_\mathtt{X}\nabla_\mathtt{Y} Z + \mathrm{II}(Z,\mathtt{Y})\Psi(\mathtt{X}) + \Big(\mathrm{II}(\mathtt{X}, \nabla_\mathtt{Y} Z) + d(\mathrm{II}(Z,\mathtt{Y}))(\mathtt{X})\Big)\nu.$$

We can therefore compute $\overline{R} = \sigma^*(R^{\mathbb{X}_{n+1}})$ in terms of g and II. Since $[\mathtt{X},\mathtt{Y}] = 0$, one gets

$$\overline{R}(\mathtt{X},\mathtt{Y})Z = R^M(\mathtt{X},\mathtt{Y})Z - g(\Psi(\mathtt{Y}), Z)\Psi(\mathtt{X}) + g(\Psi(\mathtt{X}), Z)\Psi(\mathtt{Y}) + \Big((\nabla_\mathtt{Y}\mathrm{II})(Z,\mathtt{X}) - (\nabla_\mathtt{X}\mathrm{II})(Z,\mathtt{Y})\Big)\nu.$$

Now, by Equation 7, for all $V_1, V_2, V_3 \in T_{\underline{z}}\mathbb{X}_{n+1}$

$$R^{\mathbb{X}_{n+1}}(V_1,V_2)V_3 \in Span_\mathbb{C}(V_1, V_2)$$

thus $\langle \overline{R}(\mathtt{X},\mathtt{Y})Z, \nu\rangle = 0$. As a result, we have the equalities

$$\text{(a)} : (\nabla_\mathtt{X}\mathrm{II})(Z,\mathtt{Y}) - (\nabla_\mathtt{Y}\mathrm{II})(Z,\mathtt{X}) = 0;$$

$$\text{(b)} : \overline{R}(\mathtt{X},\mathtt{Y})Z = R^M(\mathtt{X},\mathtt{Y})Z - g(\Psi(\mathtt{Y}), Z)\Psi(\mathtt{X}) + g(\Psi(\mathtt{X}), Z)\Psi(\mathtt{Y}).$$

We deduce (14) from (a) and (15) from (b).

By manipulation of (a), we get:

$$\begin{aligned}(\nabla_\mathtt{Y}\mathrm{II})(Z,\mathtt{X}) - (\nabla_\mathtt{X}\mathrm{II})(Z,\mathtt{Y}) = &- d(g(\Psi(\mathtt{X}), Z))(\mathtt{Y}) + g(\nabla_\mathtt{Y} Z, \Psi(\mathtt{X})) + g(Z, \Psi(\nabla_\mathtt{Y}\mathtt{X})) + \\ &+ d(g(\Psi(\mathtt{Y}), Z))(\mathtt{X}) - g(\nabla_\mathtt{X} Z, \Psi(\mathtt{Y})) - g(Z, \Psi(\nabla_\mathtt{X}\mathtt{Y})) = \\ =& g(Z, \nabla_\mathtt{X}\Psi(\mathtt{Y})) - g(Z, \nabla_\mathtt{Y}\Psi(\mathtt{X})) = \\ =& g(Z, d^\nabla\Psi(\mathtt{X},\mathtt{Y}))\end{aligned}$$

Since this holds for all Z in $\Gamma(\mathbb{C}TM)$, $d^\nabla\Psi(\mathtt{X},\mathtt{Y}) = 0$ for all $\mathtt{X},\mathtt{Y} \in T_pM$, hence $d^\nabla\Psi = 0$.

In order to prove (15), observe that by $\mathbb{C}$-linearity (b) is equivalent to

$$\text{(b')} : \overline{R}(X,Y)Z = R^M(X,Y)Z - g(\Psi(Y), Z)\Psi(X) + g(\Psi(X), Z)\Psi(Y)$$

for all $X, Y, Z \in \Gamma(\mathbb{C}TM)$.

Recalling Equation 7, in the orthonormal frame one gets

$$-\big(g(X_i,X_k)g(X_j,X_h)-g(X_i,X_h)g(X_j,X_k)\big)=R^M(X_i,X_j,X_k,X_h)+(\Psi^i\wedge\Psi^j)(X_h,X_k)$$

and Equation 15 follows. ☐

For $n=2$, the Gauss equation (15) can be written in a simpler way. Fixed an orthonormal frame $\{X_1,X_2\}$ on the surface M, the curvature tensor R is completely determined by its value $R(X_1,X_2,X_1,X_2)$ which is the curvature of M. Similarly, $\Psi^1\wedge\Psi^2=(\Psi^1_1\Psi^2_2-\Psi^2_1\Psi^1_2)\theta^1\wedge\theta^2=det(\Psi)\theta^1\wedge\theta^2$. Therefore, equation (15) is equivalent to

$$\text{Gauss equation for surfaces in } \mathrm{PSL}(2,\mathbb{C}): \qquad K-det(\Psi)=-1 \tag{16}$$

4. Integration of Gauss-Codazzi equations and immersion data

The main aim of this section is to show that the converse of Theorem 3.4 is also true for simply connected manifolds, in the way explained in the following theorem.

Theorem 4.1 (Gauss-Codazzi, second part)**.** Let M be a smooth simply connected manifold of dimension n. Consider a complex metric g on M with induced Levi-Civita connection ∇, and a g-symmetric bundle-isomorphism $\Psi\colon \mathbb{C}TM\to\mathbb{C}TM$.

Assume g and Ψ satisfy

$1) d^\nabla\Psi\equiv 0$;

$2) R(X_i,X_j,\cdot,\cdot)-\Psi^i\wedge\Psi^j=-\theta^i\wedge\theta^j$

for every local g-frame $(X_i)_{i=1}^n$ with corresponding coframe $(\theta^i)_{i=1}^n$ and with $\Psi=\Psi^i\otimes X_i$.

Then, there exists an isometric immersion $\sigma\colon (M,g)\to\mathbb{X}_{n+1}$ with shape operator Ψ.

Such σ is unique up to post-composition with an element in $\mathrm{Isom}_0(\mathbb{X}_{n+1})\cong\mathrm{SO}(n+2,\mathbb{C})$. More precisely, if σ' is another isometric immersion with the same shape operator, there exists a unique $\phi\in\mathrm{Isom}_0(\mathbb{X}_{n+1})$ such that $\sigma'(x)=\phi\cdot\sigma(x)$ for all $x\in M$

We state the case of surfaces in $\mathrm{SL}(2,\mathbb{C})$ as a corollary.

Corollary 4.2. Let S be a smooth simply connected surface. Consider a complex metric g on S, with induced Levi-Civita connection ∇, and a g-symmetric bundle-isomorphism $\Psi\colon\mathbb{C}TS\to\mathbb{C}TS$.

Assume g and Ψ satisfy

$$1) d^\nabla\Psi\equiv 0; \tag{17}$$

$$2) K=-1+det(\Psi). \tag{18}$$

Then, there exists an isometric immersion $\sigma\colon S\to\mathrm{SL}(2,\mathbb{C})$ whose corresponding shape operator is Ψ.

Moreover, such σ is unique up to post-composition with elements in $\mathrm{Isom}_0(\mathrm{SL}(2,\mathbb{C}))=\mathbb{P}(\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C}))$.

4.1. **Proof of Theorem 4.1.** We provide a proof of Theorem 4.1 through a moving frame argument, following the well-known proof of the analogous statement for hypersurfaces in pseudo-Riemannian space forms. We will skip some details and technicalities, recalling the essential steps and paying more attention to the main differences that the complex approach involves. A full proof can be found in the thesis [11].

A main tool from Lie Theory.

Let G be a Lie group. We recall that the *Maurer-Cartan form* of G is the $1-$form $\omega_G \in \Omega^1(G, Lie(G))$ defined by

$$(\omega_G)_g(\dot{g}) = (L_g^{-1})_*(\dot{g}),$$

hence it is invariant by left translations. Moreover, it is completely characterized by the differential equation

$$d\omega_G + [\omega_G, \omega_G] = 0.$$

The proof of both existence and uniqueness of Theorem 4.1 are based on the following result on Lie groups.

Lemma 4.3 (See [16])**.** Let M be a simply connected manifold and G be a Lie group.

Let $\omega \in \Omega^1(M, Lie(G))$.

Then

$$d\omega + [\omega, \omega] = 0 \tag{19}$$

if and only if there exists a smooth $\Phi\colon M \to G$ such that $\omega = \Phi^*\omega_G$, where ω_G denotes the Maurer-Cartan form of G.

Moreover, such Φ is unique up to post-composition with some L_g, $g \in G$.

A proof of Lemma 4.3 follows by constructing a suitable Cartan connection - depending on ω - on the trivial principal bundle $\pi_M\colon M \times G \to M$ so that its curvature being zero is equivalent to condition (19). The solutions Φ correspond to flat sections of the bundle.

In Section 5 we will prove Proposition 5.7 which can be seen as a more general version of Lemma 4.3.

Lifting to immersions into $\mathrm{SO}(n+2,\mathbb{C})$.

Let $dim(M) = n$ and let $\sigma\colon (M, g) \to \mathbb{X}_{n+1}$ be an isometric immersion with shape operator Ψ. Let $(X_i)_{i=1}^n$ be a local orthonormal frame for $\mathbb{C}TM$ on some open subset $U \subset M$ and ν be a normal vector field for the immersion σ.

Let us constract a natural lift of $\sigma|_U$ to an immersion into $\mathrm{SO}(n+2,\mathbb{C}) \cong \mathrm{Isom}_0(\mathbb{X}_{n+1})$.

Under the canonical inclusion $T_\sigma(x)\mathbb{X}_{n+1} < \mathbb{C}^{n+2}$ for all $x \in U$, $(\sigma_*(X_1(x)), \dots \sigma_*(X_n(x)), \sigma_*\nu(x), -i\sigma(x))$ can be seen as an orthonormal basis for $\mathbb{C}^{n+2}$. Up to switching ν with $-\nu$, we can assume that this basis of $\mathbb{C}^{n+2}$ lies in the same $\mathrm{SO}(n+2,\mathbb{C})$-orbit as the canonical basis $(\underline{v}_1^0, \dots, \underline{v}_{n+2}^0)$

of $\mathbb{C}^{n+2}$. Recall that we defined

$$\underline{e} = \begin{pmatrix} 0 \\ \dots \\ 0 \\ i \end{pmatrix} = i\underline{v}^0_{n+2} \in \mathbb{X}_{n+1}.$$

Given an immersion $\sigma\colon U \to \mathbb{X}_{n+1}$, we can therefore construct the smooth map

$$\Phi\colon U \to \mathrm{SO}(n+2, \mathbb{C})$$

defined, for all $x \in U$, by

$$\begin{aligned} \Phi(x)(\underline{v}^0_i) &= \sigma_{*x}(X_i) \qquad\qquad i = 1, \dots n \\ \Phi(x)(\underline{v}^0_{n+1}) &= \sigma_{*x}(\nu), \\ \Phi(x)(\underline{v}^0_{n+2}) &= -i\Phi(x)(\underline{e}) = -i\sigma(x)\ . \end{aligned} \tag{20}$$

Note that the lift Φ depends on the choice of a local g-orthonormal frame.

Uniqueness.

Proposition 4.4 (Uniqueness in Theorem 4.1)**.** *Let M be a connected smooth manifold of dimension n. Let $\sigma, \sigma'\colon M \to \mathbb{X}_{n+1}$ be two admissible immersions of a hypersurface with the same induced complex metric $g = \sigma^*\langle\cdot,\cdot\rangle = (\sigma')^*\langle\cdot,\cdot\rangle$ and the same shape operator Ψ. Then, there exists a unique $\phi \in \mathrm{Isom}_0(\mathbb{X}_{n+1})$ such that $\sigma'(x) = \phi\cdot\sigma(x)$ for all $x \in M$.*

Proof. We first prove that ϕ is unique. Assume $\phi_1 \circ \sigma = \phi_2 \circ \sigma$ for some $\phi_1, \phi_2 \in \mathrm{SO}(n+2, \mathbb{C})$, then, for any $x \in U$, $\phi_2^{-1} \circ \phi_1$ coincides with the identity on $Span_{\mathbb{C}}(\sigma(x), \sigma_*(T_xM)) < \mathbb{C}^{n+2}$ which is a complex vector subspace of dimension $n+1$, so $\phi_1 = \phi_2$.

Clearly the main obstacle is to prove that ϕ exists. Observe that it is enough to show that the statement holds for any open subset of M that admits a global g-orthonormal frame, then the thesis follows by the uniqueness of ϕ and by the fact that M is connected. We therefore assume in the proof that M admits a global ortonormal frame without loss of generality.

Consider an isometric immersion $\sigma\colon M \to \mathbb{X}_{n+1}$ with immersion data (g, Ψ), fix a g-orthonormal frame $(X_i)_i$ and construct the lifting $\Phi\colon M \to \mathrm{Isom}_0(\mathbb{X}_{n+1})$ as in (20). Define $\omega_\Phi = \Phi^*(\omega_G)$, where ω_G denotes the Maurer-Cartan form of $\mathrm{SO}(n+2, \mathbb{C})$, therefore $\omega_\Phi \in \Gamma(\mathbb{C}TM, \mathfrak{o}(n+2, \mathbb{C})) = Skew(n, \Gamma(\mathbb{C}TM))$. An explicit computation (see [11]) shows that

$$\omega_\Phi = \begin{pmatrix} & & & -\Psi^1 & -i\theta^1 \\ & \Theta & & \dots & \dots \\ & & & -\Psi^n & -i\theta^n \\ \Psi^1 & \dots & \Psi^n & 0 & 0 \\ i\theta^1 & \dots & i\theta^n & 0 & 0 \end{pmatrix}$$

where $\Theta = (\theta^i_j)_{i,j}$ and Ψ^i is defined by $\Psi = \Psi^i \otimes X_i$: as a consequence, ω_Φ depends on Φ only through g, Ψ, and through the choice of the global g-orthonormal frame $(X_i)_i$. Assume that σ and σ' are two isometric immersions with the same immersion data (g, Ψ), the induced lifts $\Phi, \Phi'\colon M \mapsto G = \mathrm{SO}(n+2, \mathbb{C})$ from the frame $(X_i)_i$ would be such that $\Phi^*\omega_G = (\Phi')^*\omega_G$:

by Lemma 4.3, there exists $\phi \in \mathrm{SO}(n+2,\mathbb{C})$ such that $\Phi'(x) = \phi \cdot \Phi(x)$ for all $x \in M$, hence $\sigma'(x) = \phi \cdot \sigma(x)$. □

Corollary 4.5. Let $\sigma, \sigma' \colon (M^n, g) \to \mathbb{X}_{n+1}$ be two isometric immersions with the same shape operator Ψ. Assume $\sigma(x) = \sigma'(x)$ and $\sigma_{*x} = \sigma'_{*x}$, then $\sigma \equiv \sigma'$.

Proof. By Proposition 4.4, there exists $\phi \in \mathrm{SO}(n+2,\mathbb{C})$ such that $\phi \circ \sigma = \sigma'$. Since there exists at most one matrix in $\mathrm{SO}(n+2,\mathbb{C})$ sending $(n+1)$ linearly independent vectors in $\mathbb{C}^{n+2}$ into other $(n+1)$ given vectors, we have $d_{\sigma(x)}\phi = id$, hence $\phi = id$. □

Local existence.

To prove the existence part of Theorem 4.1 we start by assuming that there exists a global g-orthonormal frame on M: this holds up to replacing M with some small open subset U.

Let $(X_i)_{i=1}^n$ in $\mathbb{C}TU$ be a g-orthonormal frame with dual frame $(\theta^i)_i$, and let $\Theta = (\theta^i_j)$ be the skew-symmetric matrix of the Levi-Civita connection forms for $(X_i)_i$.

Our aim is to construct a suitable form $\omega \in \Omega^1(U, \mathfrak{o}(n+2,\mathbb{C}))$ satisfying (19) in order to be able to apply Lemma 4.3. The proof in Proposition 4.4 suggests to define ω as

$$\omega = \begin{pmatrix} & \Theta & & -\Psi^1 & -i\theta^1 \\ & & & \dots & \dots \\ & & & -\Psi^n & -i\theta^n \\ \Psi^1 & \dots & \Psi^n & 0 & 0 \\ i\theta^1 & \dots & i\theta^n & 0 & 0 \end{pmatrix}$$

In fact, assuming that Gauss-Codazzi equations hold, one gets explicitly that $d\omega + [\omega, \omega] = 0$ (see [11]).

By Lemma 4.3 there exists $\Phi \colon U \to \mathrm{SO}(n+2,\mathbb{C})$ such that $\omega = \omega_\Phi = \Phi^*\omega_G$ where ω_G is the Maurer-Cartan form of $\mathrm{SO}(n+2,\mathbb{C})$, and we can define $\sigma \colon M \to \mathbb{X}_{n+1}$ as $\sigma(x) := \Phi(x) \cdot \underline{e} = i\Phi(x)\underline{v}^0_{n+2}$.

Observe that

$$\begin{aligned}\sigma_{*x}(X_j) = i\big(\Phi(\cdot)\underline{v}^0_{n+2}\big)_*(X_j) = i\Phi_{*x}(X_j)\underline{v}^0_{n+2} = \\ = i\Phi(x)\omega(X_j)\underline{v}^0_{(n+2)} = \theta^k(X_j)\Phi(x)\underline{v}^0_k = \Phi(x)\underline{v}^0_j.\end{aligned}$$

We conclude that σ is an immersion at every point and that it is an isometric immersion since its differential sends an orthonormal basis into orthonormal vectors.

We get that this construction is in fact inverse to construction (20): as a result, one gets *a posteriori* that the shape operator of σ is Ψ by the description of $\omega = \Phi^*\omega_G$.

4.1.1. *Global existence.*

To prove that local existence of the immersion involves global existence, we use that M is simply connected together with a classical analytic continuation argument. Here we give a sketch, see [11] for further details.

Say that an open subset of M is *immersible* if it admits an isometric immersion into $\mathbb{X}_{n+1}$ with immersion data (g, Ψ).

Starting from a base point $x \in M$ and from an immersion $\sigma\colon U \to \mathbb{X}_{n+1}$ with immersion data (g, Ψ) and with U being an open neighbourhood of x, for every $y \in M$ and for every simple path α connecting x and y, one can cover the image of α with a suitable finite collection of open immersible subsets: by the essentially uniqueness part of the statement, one can find for each immersible subset an immersion with data (g, Ψ) so that they coincide on the intersections, providing an extension of σ over the union of these subsets. This extension furnishes a value of $\sigma(y)$ and one can prove, using that M is simply connected, that such value is independent of the collection of open subsets and of the path α. By proceeding for all y, one gets a global immersion $\sigma\colon M \to \mathbb{X}_{n+1}$ with immersion data (g, Ψ).

Remark 4.6. Let (M, g) be a manifold with a complex metric g and a symmetric $(1, 1)$ form Ψ such that (g, Ψ) satisfy the Gauss-Codazzi equation.

Consider its universal cover $(\widetilde{M}, \widetilde{g})$, over which $\pi_1(M)$ acts by isometries, and the lifting $\widetilde{\Psi}$ of Ψ, which is $\pi_1(S)$-invariant; by the previous result, there exists an isometric immersion

$$\sigma\colon (\widetilde{M}, \widetilde{g}) \to \mathbb{X}_{n+1}$$

with shape operator $\widetilde{\Psi}$, unique up to an ambient isometry. It is now trivial to check that σ is $(\pi_1(M), \mathrm{SO}(n + 2, \mathbb{C}))$-equivariant. Indeed, for all $\alpha \in \pi_1(M)$, $\sigma \circ \alpha$ is a new isometric embedding, hence there exists a unique $\phi =: mon(\alpha) \in \mathrm{SO}(n + 2, \mathbb{C})$ such that

$$\sigma \circ \alpha = mon(\alpha) \circ \sigma.$$

We will call such a pair (g, Ψ) *immersion data* for S.

4.2. **Totally geodesic hypersurfaces in $\mathbb{X}_{n+1}$.** A particular case of immersions $M \to \mathbb{X}_{n+1}$ is given by totally geodesic immersions, namely immersions with $\Psi = 0$. The study of this case leads to several interesting results.

First of all, if g is a complex metric on a smooth manifold M with constant sectional curvature $k \in \mathbb{C}$, then, for any $X, Y, Z, W \in \Gamma(\mathbb{C}TM)$,

$$R(X, Y, Z, W) = k(g(X, Z)g(Y, W) - g(Y, Z)g(X, W)).$$

An analogous result was mentioned previously for holomorphic Riemannian manifolds in Equation 7, and, also for this statement, the proof works exactly as the standard proof for Riemannian metrics (see [4]).

In particular, one gets that $R(X, Y)Z \in Span_{\mathbb{C}}(X, Y)$.

Theorem 4.7. Let M be a smooth manifold of dimension n.

Then, g is a complex metric for M with constant sectional curvature -1 if and only if there exists an isometric immersion

$$(\widetilde{M}, \widetilde{g}) \to \mathbb{X}_n$$

which is unique up to post-composition with elements in $\mathrm{Isom}(\mathbb{X}_n)$ and therefore is $(\pi_1(M), O(n+1, \mathbb{C}))$-equivariant.

Proof. Let $\iota\colon \mathbb{C}^{n+1} \hookrightarrow \mathbb{C}^{n+2}$ be the immersion $\iota(z_1, \dots z_{n+1}) = (z_1, \dots, z_{n+1}, 0)$.

Assume there exists an isometric immersion $\sigma\colon (M,g)\to \mathbb{X}_n$, then $\overline{\sigma}=\iota\circ\sigma\colon (M,g)\to \mathbb{X}_{n+1}$ is another immersion and it has $\overline{\sigma}^*\underline{v}_0^{n+2}$ as a global normal vector field. The induced shape operator is therefore

$$\Psi=\overline{\nabla}(\overline{\sigma}^*\underline{v}_0^{n+2})=0.$$

By Gauss equation, this means that, for every local orthonormal frame $(X_i)_i$, $R(X_i,X_j,X_i,X_j)=-1$, i.e. (M,g) has constant sectional curvature -1.

Conversely, assume (M,g) has constant sectional curvature -1. Then, by taking $\Psi\equiv 0$, the couple (g,Ψ) trivially satisfies the Codazzi equation and, by the previous lemma, we have $R(X_i,X_j,\cdot,\cdot)=-\theta^1\wedge\theta^j$, so the Gauss equation holds as well. By Theorem 4.1, there exists an isometric $\pi_1(M)$-equivariant immersion $\overline{\sigma}\colon (M,g)\to\mathbb{X}_{n+1}$. Let $\nu=\overline{\sigma}^*\nu_0$ be a normal local vector field w.r.t. $\overline{\sigma}$, with $\nu_0(x)\in T_{\sigma(x)}\mathbb{X}_{n+1}$. As ν_0 is unitary it turns out that $\dot{\nu}_0$ is orthogonal to ν_0. On the other hand, differentiating $0=\langle\nu_0(t),\gamma(t)\rangle$ and using that ν_0 is orthogonal to $\dot{\gamma}$, we deduce that $\dot{\nu}_0$ is orthogonal to the vector $\gamma(t)$. Thus $\dot{\nu}_0$ is contained in $\overline{\sigma}_*(\mathbb{C}TM)$, and

$$\dot{\nu}_0=\sigma_*\Psi(\dot{\gamma})=0.$$

We conclude that $\dot{\nu}_0(t)\equiv 0$, hence ν_0 is a constant vector and $Im(\overline{\sigma})\subset\nu_0^{\perp}$. Up to composition with elements in $\mathrm{SO}(n+2,\mathbb{C})$, we can assume $\nu_0=(0,\dots,0,1)$, so $Im(\overline{\sigma})\subset\mathbb{X}_n$.

Finally, an isometric immersion of $(\widetilde{M},\widetilde{g})$ in $\mathbb{X}_n$ is unique up to composition with elements in $\mathrm{SO}(n+2,\mathbb{C})$ that stabilize $\mathbb{X}_n$, namely up to elements in $O(n,\mathbb{C})$. □

An interesting case we are going to treat in Section 6 is the case $n=2$. In this setting, the previous theorem can be stated in the following way.

Proposition 4.8. Let (S,g) be a surface equipped with a complex metric, denote with $(\widetilde{S},\widetilde{g})$ the universal covering. Then:

- (S,g) has constant curvature -1 if and only if there exists an isometric immersion
$$\sigma=(f_1,f_2)\colon(\widetilde{S},\widetilde{g})\to\mathbb{G}=(\mathbb{CP}^1\times\mathbb{CP}^1\setminus\Delta,-\frac{4}{(z_1-z_2)^2}dz_1dz_2)$$
which is $(\pi_1(S),\mathrm{Isom}(\mathbb{G}))$-equivariant. In particular, g induces a monodromy map
$$mon_g\colon\pi_1(S)\to\mathrm{PSL}(2,\mathbb{C})\times\mathbb{Z}_2$$
defined up to conjugation. Being σ admissible, the maps $f_j\colon\widetilde{S}\to\mathbb{CP}^1$ are such that $rk((f_j)_*)\geq 1$.
- By composing with some affine chart $(U\times U\setminus\Delta, z\times z)$ of $\mathbb{G}$, a complex metric g with constant curvature -1 can be locally expressed as
$$g=-\frac{4}{(f_1-f_2)^2}df_1df_2.$$
- The maps f_1, f_2 are local diffeomorphisms if and only if no real vector $\mathtt{X}\in TS\setminus\{0\}$ is isotropic for $g=\sigma^*\langle\cdot,\cdot\rangle=(f_1,f_2)^*\langle\cdot,\cdot\rangle$.

Proof. We only need to prove the last part of the statement. There exists $\mathtt{X}\in T_xS$ such that $g(\mathtt{X},\mathtt{X})=0$ if and only if $df_1(\mathtt{X})\cdot df_2(\mathtt{X})=0$ for some v, which holds if and only if one between f_1 and f_2 is not a local diffeomorphism. □

Example 4.9. The hyperbolic plane $\mathbb{H}^2$ in the upper half-plane model admits the isometric immersion $z \mapsto (z, \overline{z})$ as described in Section 2.

Consider the immersion of S^2 given by

$$(f_1, f_2)\colon S^2 \approx \overline{\mathbb{C}} \to \mathbb{G}$$
$$z \mapsto (z, -\frac{1}{\overline{z}})$$

which in fact embeds S^2 into the graph of the antipodal map. The pull-back metric is given by

$$(f_1, f_2)^* \langle \cdot, \cdot \rangle = -\frac{4}{(1+|z|^2)^2} dz d\overline{z}$$

which coincides with the negative definite space form of curvature -1, namely $-\mathbb{S}^2$, the sphere equipped with the opposite of the standard elliptic metric.

Another example is given by the Anti-de Sitter 2-space $\mathbb{A}\mathrm{d}\mathbb{S}^2$, in its model homeomorphic to the cylinder, which is isometric to $\mathbb{RP}^1 \times \mathbb{RP}^1 \setminus \Delta$ ¶.

4.3. **Connections with immersions into pseudo-Riemannian space forms.** For all $m_1, m_2 \in \mathbb{Z}_{\geq 0}$ with $n := m_1 + m_2 \geq 2$, denote with $\mathbb{H}^{m_1, m_2}$ the n-dimensional pseudo-Riemannian space form of signature (m_1, m_2) and constant sectional curvature -1. One has:

$\mathbb{H}^{n,0} \cong \mathbb{H}^n$	the hyperbolic space
$\mathbb{H}^{n-1,1} \cong \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^n}$	the universal Anti-de Sitter space
$\mathbb{H}^{1,n-1} \cong -\widetilde{\mathrm{d}\mathbb{S}^n}$	the de Sitter space with opposite metric ($\widetilde{\mathrm{d}\mathbb{S}^n} = \mathrm{d}\mathbb{S}^n$ for $n \geq 3$)
$\mathbb{H}^{0,n} \cong -\mathbb{S}^n$	the Riemannian sphere with opposite metric.

It is well known that $\mathbb{H}^{m_1,m_2}$ is isometric to the universal covering of

$$Q^{m_1,m_2} = \{(x_1, \dots, x_{n+1}) \mid x_1^2 + \dots + x_{m_1}^2 - x_{m_1+1}^2 - \dots - x_{n+1}^2 = -1\} \subset \mathbb{R}^{m_1, m_2+1}$$

where $\mathbb{R}^{m_1,m_2+1}$ is the Minkowski space of signature (m_1, m_2+1).

By Theorem 4.7, one has a unique isometric immersion of $\mathbb{H}^{m_1,m_2} \to \mathbb{X}_n$ up to composition with ambient isometries. In fact, one can explicitly check that the embedding

$$Q^{m_1,m_2} \hookrightarrow \mathbb{X}_n \subset \mathbb{C}^{n+1}$$
$$(x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_{m_1}, i x_{m_1+1}, \dots, i x_{n+1})$$

is isometric, so its lifting to the universal covering provides an isometric immersion of $\mathbb{H}^{m_1,m_2}$ into $\mathbb{X}_n$.

There exists a general theory of immersions of hypersurfaces into $\mathbb{H}^{m_1,m_2}$, see for instance [21]. One can define an immersion $\sigma\colon M^{n-1} \to \mathbb{H}^{m_1,m_2}$ to be *admissible* if the pull-back metric is a (non-degenerate) pseudo-Riemannian metric. By composition with the isometric immersion $\iota\colon \mathbb{H}^{m_1,m_2} \to \mathbb{X}_n$, one can see an immersion σ into $\mathbb{H}^{m_1,m_2}$ as an immersion $\overline{\sigma} = \iota \circ \sigma$ into $\mathbb{X}_n$, and the extrinsic geometry of the latter extends the one of the former.

Indeed:

- The map ι_* induces a canonical bundle inclusion of $\sigma^*T\mathbb{H}^{m_1,m_2}$ into $\overline{\sigma}^*T\mathbb{X}_n$: indeed, since $Span_{\mathbb{C}}(\iota_*(T_{\underline{x}}\mathbb{H}^{m_1,m_2})) = T_{\iota(\underline{x})}\mathbb{X}_n$, one has
$$\sigma^*T\mathbb{H}^{m_1,m_2}\otimes_{\mathbb{R}}\mathbb{C}\cong\overline{\sigma}^*T\mathbb{X}_n.$$
Since ι is isometric, the pull-back Levi-Civita connections coincide on $\sigma^*T\mathbb{H}^{m_1,m_2}$.
- If $\overline{\sigma}^*\langle\cdot,\cdot\rangle$ has signature (m_1-1,m_2), then one can define a local normal vector field ν as a local section of $\sigma^*T\mathbb{H}^{m_1,m_2}$ with $\langle\nu,\nu\rangle=1$; we denote this case as case a). Conversely, if $\overline{\sigma}^*\langle\cdot,\cdot\rangle$ has signature (m_1,m_2-1), then any local section of $\sigma^*T\mathbb{H}^{m_1,m_2}$ orthogonal to TM is timelike, so one can define a local normal vector field ν as a local section of $\sigma^*T\mathbb{H}^{m_1,m_2}$ with $\langle\nu,\nu\rangle=-1$; denote this case as case b).
In case a), ν is also a normal vector field for $\overline{\sigma}$ as we defined in Section 3; in case b), $i\nu$ is norm-1 and is a normal vector field for $\overline{\sigma}$.
- The exterior derivative of the local normal vector field defines a shape operator ψ, which coincides in case a) with the shape operator Ψ induced by $\overline{\sigma}$, while in case b) one has $\Psi=i\psi$.
- For immersions into $\mathbb{H}^{m_1,m_2}$ there exists a Gauss-Codazzi theorem analogous to Theorem 4.1.

Proposition 4.10. Let h be a pseudo-Riemannian metric on M with signature $(m_1-1+\delta, m_2-\delta)$ for some $\delta\in\{0,1\}$. Let $\psi\colon TM\to TM$ be a h-self adjoint $(1,1)$-form. The data (h,ψ) satisfy

$$1) d^{\nabla_h}\psi\equiv 0;$$
$$2) R(X_i,X_j,\cdot,\cdot)-(2\delta-1)\psi^i\wedge\psi^j=-\theta^i\wedge\theta^j$$
for every local h-frame $(X_i)_{i=1}^n$ with corresponding coframe $(\theta^i)_{i=1}^n$ and with $\psi=\psi^i\otimes X_i$.

if and only if there exists a $\pi_1(M)$-equivariant isometric immersion $\sigma\colon(\widetilde{M},\widetilde{h})\to\mathbb{H}^{m_1,m_2}$ with shape operator ψ.

If $\sigma\colon M\to\mathbb{H}^{m_1,m_2}$ is an admissible immersion with data (h,ψ), then $\iota\circ\sigma\colon M\to\mathbb{X}_n$ is an admissible immersion with immersion data $(g,\Psi)=(h,(i\delta+1-\delta)\psi)$. On the other hand, the pair (g,Ψ) uniquely determines the immersion of $\widetilde{M}$ into $\mathbb{X}_n$ up to ambient isometry. Since also ι is the unique isometric immersion of $\mathbb{H}^{m_1,m_2}$ into $\mathbb{X}_n$ up to ambient isometry, one can conclude the following.

Theorem 4.11. Let $M=M^{n-1}$, (g,Ψ) be immersion data for a $\pi_1(M)$-equivariant immersion of $\widetilde{M}$ into $\mathbb{X}_n$. Assume that g is real, namely that $g_{|TM}$ is pseudo-Riemannian, and that it has signature (m_1-1,m_2), $0\le p\le n-1$.

Then, if Ψ is real, i.e. if Ψ restricts to a bundle homomorphism $\Psi\colon TM\to TM$, there exists an isometric $\pi_1(M)$-equivariant immersion $\sigma\colon\widetilde{M}\to\mathbb{X}_n$ such that $\sigma(\widetilde{M})\subset\iota(\mathbb{H}^{m_1,m_2})$.

Similarly, if $i\Psi$ is real, then there exists an isometric $\pi_1(M)$-equivariant immersion $\sigma\colon\widetilde{M}\to\mathbb{X}_n$ such that $\sigma(\widetilde{M})\subset\iota(\mathbb{H}^{m_1-1,m_2+1})$.

4.4. **From immersions into $\mathbb{H}^3$ to immersions into $\mathbb{G}$.** Given $\sigma\colon \widetilde{S} \to \mathbb{H}^3$ a $\pi_1(S)$-equivariant immersion with immersion data (h, ψ) and normal field ν, one can define the immersion

$$\widehat{\sigma}\colon \widetilde{S} \to \mathbb{G}$$

where $\widehat{\sigma}(x)$ is the oriented maximal geodesic of $\mathbb{H}^3$ tangent to $\nu(x)$. In the identification $\mathbb{G} = \mathbb{CP}^1 \times \mathbb{CP}^1 \setminus \Delta$, one has $\widehat{\sigma} = (\sigma_{+\infty}, \sigma_{-\infty})$ corresponding to the endpoints of the geodesic rays starting at $\sigma(x)$ with tangent directions respectively $\nu(x)$ and $-\nu(x)$. We will call $\widehat{\sigma}$ the *Gauss map* of σ

The map $\widehat{\sigma}$ is $\pi_1(S)$-equivariant with the same monodromy as σ.

We want to prove the following formula for the pull-back metric for $\widehat{\sigma}$.

Proposition 4.12. Let J be the complex structure induced by h and inducing the same orientation as ν. Under the notation above, the pull-back $g = \widehat{\sigma}^*\langle\cdot,\cdot\rangle$ of the metric is the $\pi_1(S)$-invariant complex bilinear form on $\mathbb{C}TS$ given by

$$g = h((id + iJ\psi)\cdot, (id + iJ\psi)\cdot) = h - h(\psi\cdot, \psi\cdot) + ih((\psi J - J\psi)\cdot, \cdot)$$

which is non degenerate (i.e. a complex metric) in x if and only if $K_h(x) \neq 0$

In order to prove the proposition, we regard $\widehat{\sigma}$ as the composition of the normal section $\nu : \widetilde{S} \to T^1\mathbb{H}^3$ with the natural projection $\Pi : T^1\mathbb{H}^3 \to \mathbb{G}$, whose fibers are the leaves of the geodesic flow.

First of all let us recall that the tangent space $T_{(p,v)}(T^1\mathbb{H}^3)$ is naturally identified with $T_p\mathbb{H}^3 \oplus v^\perp$, where $v^\perp$ is the orthogonal space to v in $T_p\mathbb{H}^3$. The identification works as follows: given a path $\alpha : (-\epsilon, \epsilon) \to T^1\mathbb{H}^3$, which can be written as $\alpha(t) = (p(t), v(t))$ with $v(t)$ being a unit vector field along the path $p(t)$, the identification is given by

$$\dot{\alpha}(0) = \Big(\dot{p}(0), \frac{Dv}{dt}(0)\Big).$$

Notice that, since v is unitary, by differentiating $g_{\mathbb{H}^3}(v(t), v(t)) = 1$ one gets that the vector $\frac{Dv}{dt}(0)$ is orthogonal to v, so the correspondence is well-posed.

This identification allows us to get a simple expression for the differential of the normal section $\nu : \widetilde{S} \to T^1\mathbb{H}^3$.

Remark 4.13. Up to the above identification, for any $x \in \widetilde{S}$ and $v \in T_x\widetilde{S}$ we have

$$\nu_*(\mathtt{X}) = (\sigma_*\mathtt{X}, \sigma_*(\psi(\mathtt{X}))) \,.$$

Proof. The proof is trivial by definition of the shape operator. □

Lemma 4.14. Let us fix $(p, v) \in T^1\mathbb{H}^3$ and $w_1, w_2 \in v^\perp$. Then

$$\begin{aligned}\langle d_{(p,v)}\Pi(w_1, w_2), &d_{(p,v)}\Pi(w_1, w_2)\rangle_{\mathbb{G}} = \\ &= g_{\mathbb{H}^3}(w_1, w_1) - g_{\mathbb{H}^3}(w_2, w_2) + i(g_{\mathbb{H}^3}(w_1, v \times w_2) - g_{\mathbb{H}^3}(w_2, v \times w_1)),\end{aligned}$$

where $\times$ is the vector product on $T\mathbb{H}^3$.

Proof. We consider the half-space model of $\mathbb{H}^3 = \mathbb{C} \times \mathbb{R}_+$. Set $p = (0, 1)$ and, in the identification $T_p\mathbb{H}^3 = \mathbb{C} \times \mathbb{R}$, set $v = (1, 0)$. As a result, $v^\perp = Span_{\mathbb{R}}((i, 0), (0, 1)) < T_p\mathbb{H}^3$, we can canonically see $\partial\mathbb{H}^3 = \overline{\mathbb{C}}$ and $\Pi(p,v)$ is the geodesic with endpoints $\Pi_+(p,v) = 1$ and $\Pi_-(p,v) = -1$. The tangent space $T_{(1,-1)}\mathbb{G}$ can be trivially identified with $T_1\mathbb{C} \times T_{-1}\mathbb{C} \cong \mathbb{C} \times \mathbb{C}$.

Let us consider the following 1-parameter groups of isometries of $\mathbb{H}^3$

$$\begin{aligned} a(t) &= \exp(tX), \quad b(t) = \exp(itX), \\ c(t) &= \exp(tY), \quad d(t) = \exp(itY)\,. \end{aligned}$$

where $X, Y \in \mathfrak{sl}(2,\mathbb{C})$ are defined as $X = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$, and $Y = \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}$.

Notice that $a(t)$ and $c(t)$ are groups of hyperbolic transformations with axis respectively $(0,\infty)$ and $(i,-i)$. On the other hand, $b(t)$ and $d(t)$ are pure rotations around the corresponding axes. We have chosen the normalization so that the translation lengths of $a(t)$ and $c(t)$ equal t, and so that the rotation angle of $b(t)$ and $d(t)$ is t.

Observe that p lies on all the axes of $a(t), b(t), c(t), d(t)$. It follows that $t \to a(t)\cdot v$ is a parallel vector field along the axis of $a(t)$, so the derivative of $a(t)\cdot(p,v)$ corresponds under the natural identification to the vector $((0,1),(0,0))$. On the other hand, the variation of the endpoints of the family of geodesics $a(t)\cdot(-1,1) \in \mathbb{G}$ is given by $(-1,1) \in T_{(-1,1)}\mathbb{G}$. Using that the map Π is equivariant under the action of $\mathrm{PSL}(2,\mathbb{C})$, we conclude that

$$d_{(p,v)}\Pi((0,1),(0,0)) = (-1,1)\,.$$

In the same fashion, using $c(t)$ we deduce that

$$d_{(p,v)}\Pi((-i,0),(0,0)) = (-i,-i)\,.$$

On the other side, $b(t)p = p$ for all t, therefore one can explcitly compute that

$$b(t)\cdot v = d_p b(t) v = \cos t v + \sin t (0,1) \times v \in T_p\mathbb{H}^3.$$

It follows that the derivative at $t = 0$ of $b(t)\cdot(p,v)$ corresponds to $((0,0),(i,0))$. We conclude as above that

$$d_{(p,v)}\Pi((0,0),(i,0)) = (-i,i)$$

and analogously for $d(t)$ we get

$$d_{(p,v)}\Pi((0,0),(0,1)) = (1,1).$$

Finally, we can explcitly compute $d_{(p,v)}\Pi_{|v^\perp \oplus v^\perp} : v^\perp \oplus v^\perp \cong i\mathbb{R} \times \mathbb{R}_+ \to T_{(1,-1)}\mathbb{G} \cong \mathbb{C} \times \mathbb{C}$:

$$d\Pi((i\alpha,\beta),(i\gamma,\delta)) = ((\delta-\beta) + i(\alpha-\gamma), (\delta+\beta) + i(\alpha+\gamma)).$$

Using the description of the metric as in Proposition 2.9, we get that

$$\begin{aligned} ||d\Pi((i\alpha,\beta),(i\gamma,\delta))||^2 &= -4\frac{[(\delta-\beta)+i(\alpha-\gamma)][(\delta+\beta)+i(\alpha+\gamma)]}{(1-(-1))^2} = \\ &= \alpha^2 + \beta^2 - \gamma^2 - \delta^2 - 2i(\alpha\delta - \beta\gamma) = \\ &= ||(i\alpha,\beta)||^2 - ||(i\gamma,\delta)||^2 + 2ig_{\mathbb{H}^3}((i\alpha,\beta),(1,0)\times(i\gamma,\delta)) \end{aligned}$$

and the thesis follows. □

Proof of Proposition 4.12. The proof follows directly by Remark 4.13 and Lemma 4.14.

If $(\mathtt{X}_1, \mathtt{X}_2)$ is a h-orthonormal frame of eigenvectors for ψ, with corresponding eigenvalues λ_1 and λ_2 respectively, the pull-back bilinear form via $\widehat{\sigma}$ is described by

$$g \leftrightarrow \begin{pmatrix} 1-\lambda_1^2 & i(\lambda_1-\lambda_2) \\ i(\lambda_1-\lambda_2) & 1-\lambda_2^2 \end{pmatrix}$$

whose determinant is $(1-\lambda_1\lambda_2)^2$: hence, by Gauss equation, g is a complex metric at x if and only if $K_h(x) \neq 0$. □

5. On holomorphic dependence on the initial data

5.1. **The main result.** In this section we discuss holomorphic dependence on the immersion data for immersions into $\mathbb{X}_{n+1}$ and their monodromy.

Let M be a smooth n-manifold and Λ a complex manifold. We will say that the family of immersion data $\{(g_\lambda, \psi_\lambda)\}_{\lambda\in\Lambda}$ for $\pi_1(M)$-equivariant immersions of $\widetilde{M}$ into $\mathbb{X}_{n+1}$ is *holomorphic* if, for all $x \in M$, the maps

$$\begin{aligned} \Lambda &\to \mathrm{Sym}^2(\mathbb{C}T_x^*M) \\ \lambda &\mapsto g_\lambda(x) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \Lambda &\to \mathrm{End}_{\mathbb{C}}(\mathbb{C}T_xM) \\ \lambda &\mapsto \Psi_\lambda(x) \end{aligned}$$

are both holomorphic. We remark that this definition does not require any holomorphic structure on M.

This section is devoted to the proof of the following theorem.

Theorem 5.1. Let Λ be a complex manifold and M be a smooth manifold of dimension n.

Let $\{(g_\lambda, \Psi_\lambda)\}_{\lambda\in\Lambda}$ be a holomorphic family of immersion data for $\pi_1(M)$-equivariant immersions $\widetilde{M} \to \mathbb{X}_{n+1}$. Then there exists a smooth map

$$\sigma\colon \Lambda \times \widetilde{M} \to \mathbb{X}_{n+1}$$

such that, for all $\lambda \in \Lambda$ and $x \in M$:

- $\sigma_\lambda := \sigma(\lambda, \cdot)\colon \widetilde{M} \to \mathbb{X}_{n+1}$ is an admissible immersion with immersion data $(g_\lambda, \Psi_\lambda)$;
- $\sigma(\cdot, x)\colon \Lambda \to \mathbb{X}_{n+1}$ is holomorphic.

Moreover, defined the character variety

$$\mathcal{X}(\pi_1(M), \mathrm{SO}(n+2,\mathbb{C})) = \mathrm{Hom}\big(\pi_1(M), \mathrm{SO}(n+2,\mathbb{C})\big)//\mathrm{SO}(n+2,\mathbb{C}),$$

the monodromy map

$$\begin{aligned} \Lambda &\to \mathcal{X}(\pi_1(M), \mathrm{SO}(n+2,\mathbb{C})) \\ \lambda &\mapsto mon(\sigma_\lambda) \end{aligned}$$

is holomorphic.

As a corollary one gets the result for families of immersions of $\widetilde{M^n}$ into $\mathbb{X}_n$.

Corollary 5.2. Under the notations of Theorem 5.2, if $\Psi_\lambda = 0$ for all λ, then σ can be constructed so that its image lies in $\mathbb{X}_n$. Moreover, the induced monodromy map

$$\begin{aligned} \Lambda &\to \mathcal{X}(\pi_1(M), \mathrm{SO}(n+1,\mathbb{C})) \\ \lambda &\mapsto mon(\sigma_\lambda) \end{aligned}$$

is holomorphic.

Example 5.3. Let h be a hyperbolic metric on a closed surface S and let $b\colon TS \to TS$ be a h-self-adjoint (1,1)-form such that $det(b) = 1$ and $d^\nabla b = 0$, ∇ being the Levi-Civita connection of h; one may choose for instance $b = id$. Then, the family $\{(g_z, \psi_z)\}_{z\in\mathbb{C}}$ defined by

$$\begin{cases} g_z = \cosh^2(z)\widetilde{h}; \\ \psi_z = -\tanh(z)\widetilde{b} \end{cases}$$

is a holomorphic family of $\pi_1(S)$-equivariant immersion data for constant curvature immersions $\widetilde{S} \to \mathrm{SL}(2,\mathbb{C})$, with $K_{g_z} = -\frac{1}{\cosh(z)^2}$. Observe that $z \in \mathbb{R}$ corresponds to an immersion data into $\mathbb{H}^3$, while $z \in i\mathbb{R}$ corresponds to an immersion data into $\mathbb{A}\mathrm{d}\mathbb{S}^3$.

By Theorem 5.1, there exists a family of immersions $\sigma_z\colon \widetilde{S} \to \mathrm{SL}(2,\mathbb{C})$ with data (g_z, ψ_z) whose monodromy is a holomorphic function in z.

5.2. **An application: the complex landslide is holomorphic.** We postpone the proof of Theorem 5.1 to Sections 5.3 and 5.4.

In this section, we provide a direct application of Theorem 5.1: we show an alternative proof for the fact that the holonomy of the complex landslide, defined in [2], is holomorphic.

We briefly recall some basic notions on projective structures, the notions of landslide, smooth grafting and complex landslide. One can use as main references [6] and [2].

Let S be an oriented closed surface of genus $g \geq 2$.

Denote the character variety into $\mathrm{PSL}(2,\mathbb{C})$ by

$$\mathcal{X}(S) := \mathrm{Hom}\big(\pi_1(S), \mathrm{PSL}(2,\mathbb{C})\big) \ /\!/ \ \mathrm{PSL}(2,\mathbb{C}).$$

We recall that a *(complex) projective structure* on S is a $(\mathrm{PSL}(2,\mathbb{C}), \mathbb{CP}^1)$-structure on S. A projective structure induces a complex structure and an orientation, we will stick to projective structures compatible with the orientation on S. As a $(\mathrm{PSL}(2,\mathbb{C}), \mathbb{CP}^1)$-structure, a projective structure is determined by a $(\pi_1(S), \mathrm{PSL}(2,\mathbb{C}))$-equivariant developing map $\widetilde{S} \to \mathbb{CP}^1$, which is unique up to post-composition with elements in $\mathrm{PSL}(2,\mathbb{C})$. We therefore define

$$\begin{aligned} \widetilde{\mathcal{P}}(S) &:= \{\text{projective structures on } S\} = \\ &= \left\{ (f,\rho) \ \middle| \ \begin{array}{l} \rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C}), \\ f\colon \widetilde{S} \to \mathbb{CP}^1 \ \rho\text{-equiv. orientation-preserving local diffeo} \end{array} \right\} \Big/ \mathrm{PSL}(2,\mathbb{C}) \\ \mathcal{P}(S) &:= \widetilde{\mathcal{P}}(S) \big/ \mathrm{Diff}_0(S) \end{aligned}$$

Equip the set of pairs $\{(f,\rho)\}$ with the compact-open topology, and $\widetilde{\mathcal{P}}(S)$ and $\mathcal{P}(S)$ with the quotient topology. The space $\mathcal{P}(S)$ is a topological manifold and one can define on it a smooth and complex structures - compatible with the topology - via the Schwarzian parametrization (e.g. see [6]).

Also define the holonomy map

$$\widetilde{Hol}\colon \widetilde{\mathcal{P}}(S)\to \mathcal{X}(S),$$

which passes to the quotient to a map

$$Hol\colon \mathcal{P}(S)\to \mathcal{X}(S).$$

Theorem 5.4. (Hejhal [13], Hubbard [15], Earle [10]) The holonomy map Hol is a local biholomorphism.

Let $\mathfrak{T}(S)$ be the Teichmüller space of S.

In [2], the authors define two smooth functions satisfying several interesting properties: the *Landslide map*

$$L\colon \mathfrak{T}(S)\times\mathfrak{T}(S)\times\mathbb{R}\to\mathfrak{T}(S)\times\mathfrak{T}(S)$$

and the *Smooth Grafting map*

$$SGr\colon \mathfrak{T}(S)\times\mathfrak{T}(S)\times\mathbb{R}^+\to\mathcal{P}(S)$$

which turn out to have several interesting geometric properties and are related respectively to the earthquake map and to the grafting map.

There are several equivalent definitions of L and SGr, we mention the ones most fitted to our pourposes.

Let h be a hyperbolic metric. As a consequence of the works of Schoen [24] and Labourie [18], it has been proved that the map

$$\left\{\begin{array}{l|l} b\colon TS\to TS & h\text{-self adjoint,}\\ \text{bundle isomorphism} & d^{\nabla_h}b=0,\\ \text{such that:} & det(b)=1\end{array}\right\}\xrightarrow{\sim}\mathfrak{T}(S)$$

$$b\mapsto[h(b\cdot,b\cdot)]$$

is well-posed and bijective. We will denote the tensors in the domain as *h-regular*.

- Given $([h],[h(b\cdot,b\cdot)])\in\mathfrak{T}(S)\times\mathfrak{T}(S)$ with b h-regular, define for all $t\in\mathbb{R}$ the operator $\beta_t=\cos(\frac{t}{2})id+\sin(\frac{t}{2})Jb$ where J is the complex structure induced by h. The landslide map is defined as

$$L([h],[h(b\cdot,b\cdot)],t)=\Big(\Big[h\Big(\beta_t\ \cdot,\beta_t\ \cdot\Big)\Big],\Big[h\Big(\beta_{t+\pi}\ \cdot,\beta_{t+\pi}\ \cdot\Big)\Big]\Big)$$

- Given $([h],[h(b\cdot,b\cdot)])\in\mathfrak{T}(S)\times\mathfrak{T}(S)$, the data (h,b) satisfy Gauss-Codazzi equations for $\mathbb{H}^3$ and so does the data

$$(h_s,b_s):=(\cosh^2(\frac{s}{2})h,\tanh(\frac{s}{2})b)$$

for all $s \in \mathbb{R}$. As a result, for all s there exists a unique $\pi_1(S)$-equivariant isometric immersion

$$\sigma_s \colon (\widetilde{S}, \widetilde{h}_s) \to \mathbb{H}^3$$

with shape operator b_s (here the notation is $\sigma^*(D^{\mathbb{H}^3}) - \nabla^{h_s} = \mathrm{II}_{\sigma_s}\nu_s = -h_s(b_s\cdot,\cdot)\nu_s$, where II is the second fundamental form of σ_s and ν_s is the normal vector field). The tensor b_s having positive determinant for all $s \neq 0$, σ_s is convex for all $s \neq 0$ and the normal is oriented towards the concave side for $s > 0$. As a result, for $s > 0$ the map σ_s induces a local diffeomorphism

$$\sigma_{s,+\infty} \colon \widetilde{S} \to \partial\mathbb{H}^3 \cong \mathbb{CP}^1$$

defined by $\sigma_{s,+\infty}(x)$ being the endpoint of the geodesic ray starting at $\sigma(x)$ with tangent direction $\nu(x)$. The map $\sigma_{s,+\infty}$ is equivariant inducing the same monodromy into $\mathrm{PSL}(2,\mathbb{C})$ as σ_s, thus it defines a projective structure on S. The smooth grafting is defined by

$$SGr([h],[h(b\cdot,b\cdot)],s) = [\sigma_{s,+\infty}].$$

In the same article, for every hyperbolic metric h and h-regular tensor b, they define

$$P_{h,b} \colon H \to \mathcal{P}(S) \tag{21}$$

$$z = t + is \mapsto SGr\Big(L\Big([h],[h(b\cdot,b\cdot)],-t\Big),s\Big) \tag{22}$$

where $H \subset \mathbb{C}$ is the upper half-plane.

Theorem 5.5 (Bonsante - Mondello - Schlenker [2])**.** $P_{h,b}$ is holomorphic.

Here we provide an alternative proof for this statement, quite different from the one in [2], using Corollary 5.2 and Proposition 4.12.

Theorem 5.6. The holonomy of the projective structure $P_{h,b}(z)$ is equal to the monodromy of the complex metric

$$g_z = h\Big((\cos(z)id - \sin(z)Jb)\,\cdot, (\cos(z)id - \sin(z)Jb)\,\cdot\Big)\,,$$

where J is the complex structure of g.

The complex metric g_z has constant curvature -1, and, as a consequence of Corollary 5.2 and Theorem 5.4, $P_{h,b}$ is holomorphic.

Proof of Theorem 5.6. Let $z = t + is$ and recall $P_{h,b}(z) = SGr\Big(L\Big([h],[h(b\cdot,b\cdot)],-t\Big),s\Big)$.

Defining $\bar{h}_x := h\Big(\beta_x\,\cdot, \beta_x\,\cdot\Big)$ for all $x \in \mathbb{R}$, one has

$$L\Big([h],[h(b\cdot,b\cdot)],-t\Big) = \Big([\bar{h}_{-t}],[\bar{h}_{-t+\pi}]\Big).$$

It is easy to convince oneself that

$$\bar{h}_{-t+\pi} = \bar{h}_{-t}(\beta_t b \beta_{-t}\cdot, \beta_t b\beta_{-t}\cdot) =: \bar{h}_{-t}(\bar{b}_{-t}, \bar{b}_{-t})$$

and that the complex structure induced by $\bar{h}_{-t}$ is

$$\bar{J}_{-t} = \beta_t J \beta_{-t}.$$

In order to compute the monodromy of $SGr([\bar{h}_{-t}], [\bar{h}_{-t+\pi}], s)$, consider the immersion into $\mathbb{H}^3$ with immersion data $(cosh^2(s)\bar{h}_{-t}, \tanh(s)\bar{b}_{-t})$ and apply Proposition 4.12 to conclude that it has the same monodromy of its Gauss map, whose pull-back complex metric is

$$\cosh^2(s)\bar{h}_{-t}\bigg((id + i\tanh(s)\bar{J}_{-t}\bar{b}_{-t})\cdot, (id + i\tanh(s)\bar{J}_{-t}\bar{b}_{-t})\cdot \bigg) =$$

$$=\bar{h}_{-t}\bigg((\cosh(s)id + i\sinh(s)\beta_t Jb\beta_{-t})\cdot, (\cosh(s)id + i\sinh(s)\beta_t Jb\beta_{-t})\cdot \bigg) =$$

$$=h\bigg(\big((\cos(is)id - \sin(is)Jb\big) \circ \beta_{-t}\big)\cdot, \big((\cos(is)id - i\sinh(s)Jb\big) \circ \beta_{-t}\big)\cdot \bigg) =$$

$$=h\bigg((\cos(is)id - \sin(is)Jb)(\cos(t)id - \sin(t)Jb)\cdot, (\cos(is)id - \sin(is)Jb)(\cos(t)id - \sin(t)Jb)\cdot \bigg) =$$

$$=h\bigg((\cos(z) - \sin(z)Jb)\cdot, (\cos(z) - \sin(z)Jb)\cdot \bigg) = g_z$$

where in the last step we used - as one can check explicitly - that $JbJb = -id$.

By construction, $(g_z, 0)$ is an immersion data for an immersion into $\mathbb{G}$ for all z.

Clearly, for any $X, Y \in \Gamma(TS)$,

$$g_z(X,Y) = \cos^2(z)h(X,X) - \sin^2(z)h(bX,bY) - \sin z \cos z(h(X,JbY) + h(JbX,Y))$$

is holomorphic in z. By Theorem 5.1, the monodromy of g_z, hence the holonomy of $P_{h,b}$, is holomorphic in z; by Theorem 5.4, the projective structure $P_{h,b}(z)$ depends on z holomorphicly. □

5.3. **Proof of Theorem 5.1.** A proof of Theorem 5.1 relies on some general techniques which are well-known to experts and have been used widely (e.g. see [14]). Here we provide the sketch of a proof, the reader might refer to the thesis [11] for additional details.

Generalizing Lemma 4.3.

The first step consists in improving Lemma 4.3, in the way expressed in the following proposition.

Proposition 5.7. Let M and Λ, with M simply connected, and let G be a Lie group with Lie algebra $\mathfrak{g}$.

Consider a smooth family of forms $\{\omega_\lambda\}_{\lambda\in\Lambda} \subset \Omega^1(M,\mathfrak{g})$, namely the parametrization provided by a smooth map $\omega_\bullet \colon \Lambda \to \Omega^1(M,\mathfrak{g})$.

Assume for all $\lambda \in \Lambda$ that

$$d\omega_\lambda + [\omega_\lambda, \omega_\lambda] = 0 \ . \tag{23}$$

Then, for every fixed $x_0 \in M$, smooth $\phi_0 \colon \Lambda \to G$ and for every smooth collection of $\mathfrak{g}$-valued 1-forms $\{\omega_\lambda\}_{\lambda\in\Lambda}$, there exists a unique smooth solution $\Phi\colon \Lambda \times M \to G$ to

$$\begin{cases} \omega_\lambda = \Phi(\lambda,\cdot)^*\omega_G \\ \Phi(\lambda, x_0) = \phi_0 \end{cases} \tag{24}$$

Note that For $\Lambda = \{pt\}$ one has Lemma 4.3.

Here is an idea of the proof of Proposition 5.7. To prove the existence part of the statement, one first shows that Equation 23 implies that the distribution

$$D_{(\lambda,x,g)} = \{(\dot\lambda, \dot x, \dot g) \in T_{(\lambda,x,g)}(\Lambda \times M \times G) \mid \omega_\lambda(\dot x) = \omega_G(\dot g)\} = T_{(\lambda,x,g)}\Lambda \oplus D^0_{(\lambda,x,g)}.$$

is integrable. Then one deduces that each leaf of the corresponding foliation can be seen as the graph of a smooth function $\Phi' \colon \Lambda \times M \to G$, and that construction satisfies $\omega_\lambda = \Phi'(\lambda,\cdot)^*\omega_G$. Finally, by left-invariance of the Maurer-Cartan form, one concludes that $\Phi(\lambda,x) := \Phi'(\lambda,x) \circ L_{\psi(\lambda)(\Phi(\lambda,x_0))^{-1}}$ satisfies Equation (24). Uniqueness of Φ follows by uniqueness of the integrating foliation.

Holomorphic dependence of Φ *on* $\omega_\bullet$.

Assume, under the notations above, that G is a complex Lie group and that $\omega_\bullet \colon \Lambda \to \Omega^1(M,\mathfrak{g})$ is a holomorphic parametrization, namely that, for all $X \in TM$, $\omega_\bullet(X) \colon \Lambda \to \mathfrak{g}$ is holomorphic. Then, for $\phi_0 = e_G$, one gets that Φ is holomorphic in λ as well.

Indeed, for all $\lambda_0 \in \Lambda$, $\dot\lambda \in T_{\lambda_0}\Lambda$, both

$$f_1(x) = \partial_{i\dot\lambda}\big(\Phi(\lambda,x)\cdot(\Phi(\lambda_0,x))^{-1}\big) \qquad \text{and} \qquad f_2(x) = i\partial_{\dot\lambda}\big(\Phi(\lambda,x)\cdot(\Phi(\lambda_0,x))^{-1}\big)$$

satisfy the Cauchy problem

$$\begin{cases} f_*(\cdot) = Ad(\Phi_{\lambda_0}) \circ \partial_{i\dot\lambda}\big(\omega(\cdot)\big) \colon TM \to \mathfrak{g} \\ f(x_0) = 0 \end{cases} \tag{25}$$

which trivially has a unique solution.

5.3.1. *Proof of the main statement.* To prove Theorem 5.1, it is sufficient to retrace the proof of Theorem 4.1 in Subsection 4.1, and to check that, starting from a holomorphic family of immersion data $(g_\lambda, \Psi_\lambda)_\lambda$, the holomorphicity is preserved at each step.

For all fixed $(\bar\lambda, \bar x) \in \Lambda \times M$ one can consider a local neighbourhood $U \times V$ such that for all $\lambda \in U$ there exists a g_λ-frame $(e_{k;\lambda})_k$ such that $\lambda \mapsto e_{k;\lambda}(x)$ is holomorphic for all k and $x \in V$. From each $(g_\lambda, \Psi_\lambda)$, locally construct the form $\omega_\lambda \in \Omega^1(V, \mathfrak{o}(n+2,\mathbb{C}))$

$$\omega_\lambda = \begin{pmatrix} & \Theta_\lambda & & -\Psi^1_\lambda & -i\theta^1_\lambda \\ & & & \dots & \dots \\ & & & -\Psi^n_\lambda & -i\theta^n_\lambda \\ \Psi^1_\lambda & \dots & \Psi^n_\lambda & 0 & 0 \\ i\theta^1_\lambda & \dots & i\theta^n_\lambda & 0 & 0 \end{pmatrix}$$

where $(\theta^k_\lambda)_k$ is the dual of $(e_{k;\lambda})_k$, while Θ_λ and the Ψ^k_λ's are respectively the matrix of the Levi-Civita connection forms and the shape operator components with respect to the adapted frame.

By Proposition 5.7 and by the subsequent remark, there exists a smooth $\Phi \colon U \times V \to \mathrm{SO}(n+2,\mathbb{C})$ such that $\Phi(\cdot,x)$ is holomorphic for all x. As we observed in Subsection 4.1, $\sigma := \Phi \cdot \underline{e} \colon U \times V \to \mathbb{X}_{n+1}$ is now such that $\sigma(\lambda,\cdot)$ has immersion data $(g_\lambda, \Psi_\lambda)$ for all $\lambda \in U$, while clearly $\sigma(\cdot,x)$ is holomorphic for all $x \in V$.

One can directly check, using Proposition 5.7, that if $\sigma' : U \times V'$ is another function with the sought properties such that $V \cap V' \neq \emptyset$, then there exists a holomorphic $\psi\colon U \to \mathrm{SO}(n+2,\mathbb{C})$ such that $\psi(\lambda) \cdot \sigma'(\lambda, x) = \sigma(\lambda, x)$ for all $x \in U \cap U'$: as a result one gets that there exists a global $\sigma\colon U \times \widetilde{M} \to \mathrm{SO}(n+2,\mathbb{C})$.

By choosing $\Phi(\lambda, x_0) = id$ for some fixed x_0 and for all λ, one has that the 1-jet of $\sigma(\lambda, \cdot)$ at x_0 does not depend on λ. Via this observation, one gets that different σ's with domain of the form $V' \times \widetilde{M}$ constructed this way glue to one another, proving that there exists such a σ with domain on the whole $\Lambda \times \widetilde{M}$.

It is now simple to check that the monodromy of $\sigma(\lambda, \cdot)$ is holomorphic with respect to λ. Indeed, for all $\gamma \in \pi_1(M)$, the map $mon_\gamma\colon \Lambda \to \mathrm{SO}(n+2,\mathbb{C})$ is defined by

$$\sigma(\lambda, \gamma(x)) = mon_\gamma(\lambda) \circ \sigma(\lambda, x) :$$

since both σ and $\sigma \circ \gamma$ satisfy the conditions in the statement of the theorem for the data (g, Ψ), hence mon_γ is holomorphic in λ.

If $\Psi_\lambda = 0$ for all λ, then by constructing σ so that the 1-jet of $\sigma(\lambda, \cdot)$ at some x_0, one gets immediately that the image of σ lies in some isometric copy of $\mathbb{X}_n$ inside $\mathbb{X}_{n+1}$ as in the proof of Theorem 4.7, leading to Corollary 5.2.

It is now simple to check that the monodromy of $\sigma(\lambda, \cdot)$ is holomorphic with respect to λ. Indeed, for all $\gamma \in \pi_1(M)$, the map $mon_\gamma\colon \Lambda \to \mathrm{SO}(n+2,\mathbb{C})$ is defined by

$$\sigma(\lambda, \gamma(x)) = mon_\gamma(\lambda) \circ \sigma(\lambda, x) :$$

since both σ and $\sigma \circ \gamma$ satisfy the conditions in the statement of the theorem for the data (g, Ψ), hence mon_γ is holomorphic in λ.

6. A Gauss-Bonnet Theorem and a Uniformization Theorem for complex metrics

6.1. Positive complex metrics.

6.1. Positive complex metrics. In this section we suggest an intrinsic study of complex metrics on surfaces. To this aim, we will introduce a natural generalization of the concept of complex structure for surfaces and we will present a uniformization theorem in this setting.

Given an *oriented* surface S, the natural inclusion $TS \hookrightarrow \mathbb{C}TS$ factors to a bundle inclusion

$$\begin{array}{ccc} \mathbb{P}_{\mathbb{R}}(TS) & \hookrightarrow & \mathbb{P}_{\mathbb{C}}(\mathbb{C}TS) \\ & \searrow \quad \swarrow & \\ & S & \end{array} .$$

Fiberwise, for every $x \in S$, $\mathbb{P}_{\mathbb{R}}(T_xS)$ is mapped homeomorphicly into a circle in $\mathbb{P}_{\mathbb{C}}(\mathbb{C}T_xS)$ whose complementary is the disjoint union of two open discs. The conjugation map on $\mathbb{C}TS$ descends to a bundle isomorphism on $\mathbb{P}_{\mathbb{C}}(\mathbb{C}TS)$ that fixes $\mathbb{P}_{\mathbb{R}}(TS)$ and that swaps the two discs fiberwise.

Definition 6.1. A *bicomplex structure* on a surface S is a tensor $\mathbf{J} \in \Gamma(\mathbb{C}T^*S \otimes \mathbb{C}TS)$ such that

- for each $x \in S$ the endomorphism is $\mathbf{J}_x$ diagonalizable with eigenvalues $\pm i$ and eigenspaces $V_i(\mathbf{J}), V_{-i}(\mathbf{J})$ with complex dimension 1;
- for all $x \in S$, the eigenspaces $V_i(\mathbf{J}_x)$ and $V_{-i}(\mathbf{J}_x)$ of $\mathbf{J}_x$ have trivial intersection with T_xS and are such that the points $\mathbb{P}_{\mathbb{C}}(V_i(\mathbf{J}_x))$ and $\mathbb{P}_{\mathbb{C}}(V_{-i}(\mathbf{J}_x))$ lie in different connected components of $\mathbb{P}_{\mathbb{C}}(\mathbb{C}T_xS) \setminus \mathbb{P}_{\mathbb{R}}(T_xS)$.

As a result if $\mathbf{J}$ is a bicomplex structure then $\mathbf{J}^2 = \mathbf{J} \circ \mathbf{J} = -id_{\mathbb{C}TS}$.

Clearly complex structures extend to bicomplex structures. Observe that a bicomplex structure $\mathbf{J}$ is not a complex structure for S in general, since it may not restrict to a section of $TS \otimes T^*S$. In other words, $\mathbf{J}(\overline{X}) \neq \overline{\mathbf{J}(X)}$ in general.

Nevertheless, $\mathbf{J}$ induces two complex structures J_1, J_2 defined by the conditions

$$V_i(\mathbf{J}) = V_i(J_2) = \overline{V_{-i}(J_2)} \qquad \text{and} \qquad V_{-i}(\mathbf{J}) = V_{-i}(J_1) = \overline{V_i(J_1)}$$

which totally characterize J_1 and J_2.

Also observe that J_1 and J_2 induce the same orientation. Indeed, representing the set of complex structures on S as $C(S) \sqcup C(\overline{S})$, the map

$$\begin{aligned} C(S) \sqcup C(\overline{S}) &\to \mathbb{P}_{\mathbb{C}}(\mathbb{C}T_xS) \setminus \mathbb{P}_{\mathbb{R}}(T_xS) \\ J_0 &\mapsto \mathbb{P}_{\mathbb{C}}(V_i(J_0)_x) \end{aligned}$$

induces a bijection between the connected components.

As a result, every bicomplex structure induces an orientation. An orientation of S fixed, denote with $BC(S)$ the set of orientation-consistent bicomplex structures. With the notations above, we therefore have a bijection

$$\begin{aligned} BC(S) &\xrightarrow{\sim} C(S) \times C(S) \\ \mathbf{J} &\mapsto (J_1, J_2). \end{aligned} \tag{26}$$

Endow $BC(S)$ with the pull-back topology for which, in particular, it is connected.

Remark 6.2. More geometrically, the orientation of T_xS induces an orientation of $\mathbb{P}_{\mathbb{R}}(T_xS)$. As $\mathbb{P}_{\mathbb{C}}(\mathbb{C}T_xS)$ is naturally oriented, there is an induced orientation on the normal bundle of $\mathbb{P}_{\mathbb{R}}(T_xS)$ in $\mathbb{P}_{\mathbb{C}}(\mathbb{C}T_xS)$, which allows to distinguish the two components of $\mathbb{P}_C(\mathbb{C}T_xS) \setminus \mathbb{P}_R(T_xS)$. With respect to the oriented curve $\mathbb{P}_R(T_xS)$, the component *on the left* of $\mathbb{P}_R(T_xS)$ is called the upper component, the component *on the right* will be called the lower component. More explicit if v_1, v_2 is a positive basis of T_xS, elements of the upper components are of the form $[av_1 + bv_2]$ such that $Im(a/b) > 0$. It turns out that $\mathbf{J}$ is positive if $V_i(\mathbf{J})$ is a point in the upper component of $\mathbb{P}_{\mathbb{C}}(T_xS) \setminus \mathbb{P}_R(T_xS)$.

Definition 6.3. Let g be a complex metric on S.

- Denote the set of isotropic vectors of g as
$$ll(g) := \{v \in \mathbb{C}TS \mid g(v,v) = 0\} \setminus \{0_S\}$$
and define $ll(g_x) := ll(g) \cap \mathbb{C}T_xS$. Notice that $ll(g_x)$ is the union of two complex lines with trivial intersection.

- A *positive complex metric* on a surface S is a complex metric g such that the two points $\mathbb{P}_{\mathbb{C}}(ll(g_x))$ lie in different connected components of $\mathbb{P}_{\mathbb{C}}(\mathbb{C}T_xS) \setminus \mathbb{P}_{\mathbb{R}}(T_xS)$.
- Denote with $CM^+(S)$ the space of positive complex metrics on S with the usual C^∞-topology for spaces of sections.

If g is a positive complex metric on S then for every $x \in S$ we denote by $ll_+(g_x)$ the isotropic line contained in the upper component of $\mathbb{P}_{\mathbb{C}}(\mathbb{C}T_xS) \setminus \mathbb{P}_{\mathbb{R}}(T_xS)$, and by $ll_-(g_x)$ the isotropic line contained in the lower component. Let us remark that the sets

$$ll_+(g) = \bigcup_{x\in S} ll_+(g_x) \qquad ll_-(g) = \bigcup_{x\in S} ll_-(g_x)$$

are complex subbundles of $\mathbb{C}TS$ of dimension 1. Indeed if (X_1, X_2) is a local oriented orthonormal frame than $X_1 + iX_2$ and $X_1 - iX_2$ locally span those bundles. Clearly we have that $\mathbb{C}TS = ll_+(g) \oplus ll_-(g)$.

Proposition 6.4. Let g be a positive complex metric on a surface S. There exists a unique bicomplex structure $\mathbf{J}$ compatible with the orientation of S such that for all $x \in S$ we have

$$g_x(\mathbf{J}_x\cdot, \mathbf{J}_x\cdot) = g_x(\cdot,\cdot) \tag{27}$$

Proof. Condition (27) forces $ll_+(g_x)$ and $ll_-(g_x)$ to be eigenspaces of $\mathbf{J}$. Since we require $\mathbf{J}$ to be compatible with the orientation of S we must set $V_i(\mathbf{J}) = ll_+(g_x)$ and $V_-i(\mathbf{J}) = ll_-(g_x)$. On the other hand defining $\mathbf{J}$ in this way, (27) is clearly satisfied. □

Let us give a more geometric and motivating interpretation of positive complex metrics.

Proposition 6.5. Let g a complex metric on $\mathbb{C}TS$. Then it is positive if and only if there exists a function $\alpha : S \to \mathbb{C}^*$ such that $Re(\alpha \cdot g)|_{TS}$ is a Riemannian metric on TS.

ho chiamato la matrice B perchè poco prima J era una struttura complessa

We postpone the proof of this Proposition for a simple lemma of linear algebra.

Lemma 6.6. Let $B \in SL_2(\mathbb{C})$ such that $B^2 = -I_2$. Consider the action of B on $\mathbb{CP}^1$ and assume that B fixes two points, one in each component of $\mathbb{CP}^1 \setminus \mathbb{RP}^1$. Then there are two linearly independent vectors $X, Y \in \mathbb{R}^2$ and a complex number $\mu \neq 0$ such that $BX = \mu Y$ and $BY = \frac{1}{\mu} X$.

Proof. The matrix B induces an isometry on the hyperbolic space $\mathbb{H}^3$, which is a rotation by π around the axis ℓ with endpoints the fixed points of B in $\mathbb{CP}^1 \cong \partial\mathbb{H}^3$. Let Σ be the totally geodesic plane in $\mathbb{H}^3$ whose asymptotic boundary is $\mathbb{RP}^1$. By the assumption, ℓ intersect Π at a point. Consider a geodesic line m contained in Σ and orthogonal to ℓ: if p, q are the end-points of m then $Bp = q$ and $Bq = p$. Finally, real vectors X and Y representing p and q satisfy the statement. □

Proof of Proposition 6.5. First we show that if g is a complex metric such that $Re(g)|_{TS}$ is Riemannian, then g is positive. As being positive is a conformal invariant condition, this will prove one implication of the statement.

Let g_0 denote both the Riemannian metric $Re(g)|_{TS}$ and its complex bilinear extension on $\mathbb{C}TS$. It is simple to check that the complex bilinear form $g_t = (1-t)g_0 + tg$ is non degenerate and such that $Re(g_t) = g_0$. Note that the isotropic directions of $g_0|_x$ are two conjugate points of $\mathbb{P}_{\mathbb{C}}(\mathbb{C}T_xS)$, so they lie in different components of $\mathbb{P}_{\mathbb{C}}(\mathbb{C}T_xS) \setminus \mathbb{P}_{\mathbb{R}}(T_xS)$. On the other hand for every $t \in [0,1]$ we have that $Re(g_t) = g_0$ is positive definite on TS, so isotropic directions of g_t never meet $\mathbb{P}_{\mathbb{R}}(TS)$. So a simple continuity argument shows that g is positive.

On the other hand let us assume that g is positive. First let us prove that for every $x \in S$ there is a complex number $\alpha_0 \in \mathbb{C}$ such that $Re(\alpha_0 g_x)|_{TS}$ is positive definite.

Notice that, fixing a positive basis of T_xS, we may identify $\mathbb{C}T_xS$ with $\mathbb{C}^2$ so that T_xS is sent to $\mathbb{R}^2 \subset \mathbb{C}^2$. Through this identification the bicomplex structure $\mathbf{J}_x$ is sent to an element $B \in SL_2(\mathbb{C})$ and the condition of the positiveness of the metric implies that B satisfy the assumption in Lemma 6.6. We deduce that there are $\mathtt{X}, \mathtt{Y} \in T_xS$ and a complex number $\mu \neq 0$ such that $\mathbf{J}(\mathtt{X}) = \mu\mathtt{Y}$. In particular $g(\mathtt{X}, \mathtt{Y}) = 0$.

Let us distinguish two cases. If $Re(g_x)(\mathtt{X}, \mathtt{X}) \cdot Re(g_x)(\mathtt{Y}, \mathtt{Y}) > 0$, then either $Re(g_x)|_{TS}$ or $-Re(g_x)|_{TS}$ is positive definite, so $\alpha_0 = \pm 1$ works. Otherwise, if there are isotropic real vectors for $Re(g)$, denote $g(\mathtt{X}, \mathtt{X}) =: a + ib$ and $g(\mathtt{Y}, \mathtt{Y}) =: c + id$ it turns out that the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible, otherwise any isotropic vector for $\Re(g)$ would be isotropic for g, contradicting that g is positive. Therefore, there are real numbers k, h such that $ka + hb = 1$ and $kc + hd = 1$. Setting $\alpha_0 = (k - ih)$, it turns out that $Re(\alpha_0 g)(\mathtt{X}, \mathtt{X}) = Re(\alpha_0 g)(\mathtt{Y}, \mathtt{Y}) = 1$ an $Re(\alpha_0 g)(\mathtt{X}, \mathtt{Y}) = 0$. So α_0 works.

To conclude let us notice that being positive definite is an open condition, and so is each of the two cases shown above. As a result, for every $x \in S$ there is a neighborhood U of x and a function $\alpha_U : U \to \mathbb{C}^*$ such that $Re(\alpha_U \cdot g)|_{TS} > 0$. Using a partition of unity, we can then define a global function $\alpha : S \to C$, which satisfy the statement. □

Remark 6.7. Given a positive complex metric g, one can define an *area form* $dA \in \Omega^2(\mathbb{C}TS, \mathbb{C})$ for g as a complex 2-form such that $\|dA\|_g^2 = 1$, i.e. for all local $g-$orthonormal frame (X_1, X_2) one has $dA(X_1, X_2) \in \{-1, 1\}$. The area form is unique up to a sign.

An orientation of S fixed, there exists a canonical area form for g compatible with the orientation: if $\mathbf{J}$ is the bicomplex structure compatible with the orientation and with g in the sense of Equation 27, then we define the area form compatible with the orientation as

$$dA = g(\mathbf{J}\cdot, \cdot) \ .$$

We will say that (X, Y) is a *positive orthonormal frame* for $\mathbb{C}TS$ with respect to the orientation induced by $\mathbf{J}$ if $\mathbf{J}(X) = Y$.

With the language of fiber bundles, a positive complex metric on a surface S provides a reduction of the frame bundle of $\mathbb{C}TS \to S$ to the structure group $O(2, \mathbb{C})$, while the pair of a positive complex metric and an orientation on S provides a reduction to $\mathrm{SO}(2, \mathbb{C})$.

Corollary 6.8. Given an oriented surface S, the map that sends a positive complex metric g to the orientation-consistent bicomplex structure $\mathbf{J}$ compatible with g induces a homeomorphism

$$\begin{aligned} CM^+(S)\Big/_{C^\infty(S,\mathbb{C}^*)} &\xrightarrow{\sim} BC(S) \cong C(S)\times C(S) \\ C^\infty(S,\mathbb{C}^*)\cdot g &\mapsto \mathbf{J}_g \end{aligned} \tag{28}$$

Conformal structures of Riemannian metrics correspond to elements of the diagonal of $C(S)\times C(S)$.

Proof. We showed that the map is continuous. In order to construct an inverse, fix a real volume form $\alpha\in\Omega^2(TS)$, which extends by $\mathbb{C}$-bilinearity to an element in $\Omega^2(\mathbb{C}TS,\mathbb{C})$, and define for any bicomplex structure $\mathbf{J}$ the metric

$$g := \alpha(\cdot, \mathbf{J}\cdot) :$$

Clearly g is non-degenerate and $\mathbb{C}$-bilinear, and it is symmetric since $g(v,w)=\alpha(v,\mathbf{J}w)=-\alpha(\mathbf{J}(\mathbf{J}v),\mathbf{J}w)=(det\mathbf{J})\alpha(w,\mathbf{J}v)=g(w,v)$. Moreover, g induces $\mathbf{J}$ in turn, hence it is a positive complex metric.

By choosing α as a real volume form for S, one gets an explicit correspondence between the diagonal of $C(S)\times C(S)$ and the set of conformal structures of Riemannian metrics. □

Finally, we remark that Proposition 4.8 can be restated in an interesting way for positive complex metrics in terms of developing maps for projective structures.

Theorem 6.9. Let S be an oriented surface and g a positive complex metric on S. Denote with $(\widetilde{S},\widetilde{g})$ the universal cover.

(S,g) has constant curvature -1 if and only if there exists a smooth map

$$(f_1,f_2)\colon \widetilde{S}\to \mathbb{G}=\mathbb{CP}^1\times\mathbb{CP}^1\setminus\Delta$$

such that:

- $\widetilde{g}=(f_1,f_2)^*\langle\cdot,\cdot\rangle=-\frac{4}{(f_1-f_2)^2}df_1\cdot df_2$
- f_1 and f_2 are local diffeomorphisms, respectively preserving and reversing the orientation.
- f_1 and f_2 are $(\pi_1(S),\mathrm{PSL}(2,\mathbb{C}))$-equivariant local diffeomorphisms with the same monodromy

$$mon_{f_1}=mon_{f_2}=mon_{(f_1,f_2)}\colon \pi_1(S)\to\mathrm{PSL}(2,\mathbb{C})$$

Moreover, if $\mathbf{J}$ is the bicomplex structure on S induced by g, then, in identification (26),

$$\mathbf{J}\equiv(f_1^*(\mathbb{J}^{\mathbb{CP}^1}), f_2^*(-\mathbb{J}^{\mathbb{CP}^1}))$$

where $\mathbb{J}^{\mathbb{CP}^1}$ is the standard complex structure on $\mathbb{CP}^1$.

Proof. As we observed in Proposition 4.8, the fact that the complex metric g is positive implies that f_1 and f_2 are local diffeomorphisms.

Let w_k be a local holomorphic chart on $x \in S$ for the pull-back complex structure $f_k^* \mathbb{J}^{\mathbb{CP}^1}$. Then, by Cauchy-Riemann equations,

$$Ker(df_k \colon \mathbb{C}T_xS \to \mathbb{C}) = Span_{\mathbb{C}}\Big(\frac{\partial}{\partial \overline{w_k}}_{|x}\Big)$$

so by the explicit description of the metric one has

$$ll(g_x) = \Big(Span_{\mathbb{C}}\Big(\frac{\partial}{\partial \overline{w_1}}_{|x}\Big) \oplus Span_{\mathbb{C}}\Big(\frac{\partial}{\partial \overline{w_2}}_{|x}\Big)\Big) \setminus \{0_x\}$$

and the two points of $\mathbb{P}_{\mathbb{C}}(ll(g_x))$ lie in two different components of $\mathbb{P}_{\mathbb{C}}(\mathbb{C}T_xS) \setminus \mathbb{P}_{\mathbb{C}}(T_xS)$ if and only if the two pull-back complex structures via f_1 and f_2 induce opposite orientations.

The choice of (f_1, f_2) inducing the same $\widetilde{g}$ is unique up to post-composition with elements of $\mathrm{PSL}(2,\mathbb{C})$, since the composition with the diagonal swap $s : (x, y) \mapsto (y, x)$ on $\mathbb{G}$ gives an immersion with orientation-reversing first component.

All the elements of $\pi_1(S)$ are orientation-preserving, hence σ is $(\pi_1(S), \mathrm{PSL}(2,\mathbb{C}))$-equivariant and, since $\mathrm{PSL}(2,\mathbb{C})$ acts diagonally on $\mathbb{G}$, both f_1 and f_2 are equivariant with the same holonomy.

From the description of $ll(g)$ above, one can conclude the last part of the theorem. □

6.2. **A Gauss-Bonnet Theorem for positive complex metrics.** The aim of this subsection is to prove a generalization of Gauss-Bonnet Theorem in the setting of positive complex metrics.

Let g be a positive complex metric on an oriented surface. Let $\mathbf{J}$ be the bi-complex structure associated to g and let dA be the corresponding area form for g as in Remark 6.7.

Theorem 6.10. Let S be an oriented surface and g a positive complex metric of curvature K. Then

$$\int_S K dA = 2\pi\chi(S)$$

where $\chi(S)$ is the Euler-Poincaré characteristic of S.

In order to prove Theorem 6.10, the first step consists in showing that KdA can be regarded as the curvature form of a linear bundle.

Notice that the line bundles $ll_+(g)$ and $ll_-(g)$ are parallel for the Levi Civita connection. This comes from the fact that if η is an isotropic section for g then, by metric compatibility, $\nabla\eta$ is orthogonal to η, so it is in fact a multiple of η. So ∇ restricts to connections on $ll_\pm(g)$, whose curvature forms are complex valued 2-forms, say $\Omega_\pm$, on S. Classically $\Omega_\pm$ is regarded as a complex valued form on TS, but it can be clearly extended to a complex bilinear alternate form on $\mathbb{C}TS$. Under this identification we have the following statement:

Lemma 6.11. With the notation above

$$\Omega_+ = -iKdA.$$

Proof. Thus it is sufficient to prove that if X is a local norm-1 section of $\mathbb{C}TS$, and $Y = \mathbf{J}X$, then $\Omega_+(X,Y) = -iK$. Now observe that the metric compatibility of the connection implies that there is a 1-form ω such that $\nabla X = \omega \otimes Y$. On the other hand $\eta := X - i\mathbf{J}X$ is a section of $ll_+(g)$, and one sees that $\nabla\eta = i\omega \otimes \eta$, so $\Omega_+ = id\omega$.

By Cartan's Formula $\Omega_+(X,Y) = i(X(\omega(Y) - Y(\omega(X)) - \omega([X,Y])X)$. By a similar computation

$$\nabla_X\nabla_Y X - \nabla_Y\nabla_X X - \nabla_{[X,Y]}X = (X(\omega(Y) - Y(\omega(X)) - \omega([X,Y]))Y$$

and we conclude. □

Proof of Theorem 6.10. Recall that the first Chern class of $ll_+(g)$ is represented by the closed form $-\frac{1}{2\pi i}\Omega_+$ so by Lemma 6.11 the integral of the statement is equal to the $2\pi\deg(ll_+(g))$.

Notice that the morhism $F : \mathbb{C}TS \to ll_+(g)$ defined by $X \mapsto X - i\mathbf{J}X$, induces an isomorphism between TS and $ll_+(g)$ so we deduce that $|\deg(ll_+(g))| = \chi(S)$. In order to conclude we need to prove that the isomorphism induced by F preserve the orientation, i.e. that, if $(\mathtt{X}, \mathtt{Y})$ is any oriented basis of TS, then $F(\mathtt{Y}) = \alpha F(\mathtt{X})$ with $Im(\alpha) > 0$.

Recall that, by Lemma 6.6, there is a positive basis $(\mathtt{X}, \mathtt{Y})$ of TS such that $\mathbf{J}\mathtt{X} = \mu\mathtt{Y}$ and $\mathbf{J}\mathtt{Y} = -\frac{1}{\mu}\mathtt{X}$. Since $F(\mathtt{X}) = \mathtt{X} - i\mu\mathtt{Y}$ is in the eigenspace V_i, which is contained in the upper component of $\mathbb{P}_{\mathbb{C}}(\mathbb{C}TS) \setminus \mathbb{P}_R(TS)$, we deduce that $Im(-\frac{1}{i\mu}) > 0$ (see Remark 6.2). On the other hand we simply see that $F(\mathtt{Y}) = -\frac{1}{i\mu}F(\mathtt{X})$ and we conclude.

□

Remark 6.12. The form KdA can be defined any time a bi-complex structure $\mathbf{J}$ compatible with the metric exists, that is any time the set of isotropic directions can be written as the union of two linear subbundles. The proof above shows that in general its integral computes the degree of the bundle $ll_+(g)$. Notice however that this degree is not necessarily $\chi(S)$. In fact it is not hard to construct examples of complex metrics such that $ll_+(g)$ is trivial.

6.3. **A Uniformization Theorem through Bers Theorem.** The aim of this subsection is to prove a generalization of the classic Riemann's Uniformization Theorem for bicomplex structures and complex metrics.

Before stating the main theorem, we recall some notions about quasi-Fuchsian representations.

A representation $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ is *quasi-Fuchsian* if its limit set $\Lambda_\rho \subset \mathbb{CP}^1$, i.e. the set of accumulation points of any $\rho(\pi_1(S))-$orbit in $\mathbb{CP}^1$, is a Jordan curve in $\mathbb{CP}^1$. Since this condition is preserved by the action of $\mathrm{PSL}(2,\mathbb{C})$, the set of quasi-Fuchsian representations defines an open subset of the character variety $QF(S) \subset \mathcal{X}(S)$.

Theorem 6.13 (Bers Simultaneous Uniformization Theorem, [1])**.** Let S be an oriented surface with $\chi(S) < 0$. For all J_1, J_2 complex structures over S, there exists a unique quasi-Fuchsian representation $\rho = \rho(J_1, J_2)\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ such that, defined $\Lambda_\rho \subset \mathbb{CP}^1$ as the limit set of ρ and Ω_+, Ω_- as the connected components of $\mathbb{CP}^1 \setminus \Lambda_\rho$,

- there exists a unique diffeomorphism $f_1\colon \widetilde{S} \to \Omega_+$, which is J_1-holomorphic and ρ-equivariant;
- there exists a unique diffeomorphism $f_2\colon \widetilde{S} \to \Omega_-$, which is J_2-antiholomorphic and ρ-equivariant.

Moreover, (ρ, f_1, f_2) are continuous functions of $(J_1, J_2) \in BC(S)$.

This correspondence determines a homeomorphism in the quotient

$$\mathfrak{B}\colon \boldsymbol{\mathfrak{T}}(S)\times \boldsymbol{\mathfrak{T}}(S) \xrightarrow{\sim} QF(S) \tag{29}$$

where $\boldsymbol{\mathfrak{T}}(S)$ is the Teichmüller space of S.

Define

$$\widetilde{\mathcal{P}}_{QF}(S) = \widetilde{Hol}^{-1}(QF(S)),$$
$$\mathcal{P}_{QF}(S) = Hol^{-1}(QF(S))$$

which, by continuity of the holonomy maps, are open subsets of $\widetilde{\mathcal{P}}(S)$ and of $\mathcal{P}(S)$.

Theorem 6.14 (W. Goldman, [12])**.** Let S be a compact oriented surface of genus $g(S) \geq 2$.
An open connected component of the space $\widetilde{\mathcal{P}}_{QF}(S)$ is

$$\widetilde{\mathcal{P}}_0(S) := \left\{ \begin{matrix} f\colon \widetilde{S} \to \mathbb{CP}^1,\ \rho\text{-equiv. with quasi-Fuchsian} \\ \text{mondromy } \rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C}), \text{ with } f(\widetilde{S}) \cap \Lambda_\rho = \emptyset \end{matrix} \right\}$$

Moreover, the restriction

$$Hol\colon \widetilde{\mathcal{P}}_0(S) \big/ \mathrm{Diff}_0(S) =: \mathcal{P}_0(S) \to QF(S)$$

is a homeomorphism.

With the notations of Theorem 6.13 and equation (29), the inverse is given by

$$[\rho] \mapsto [f_1(\rho)] = \Big[f_1\Big(\mathfrak{B}^{-1}([\rho])\Big)\Big].$$

We are now able to state a generalization of the Uniformization Theorem.

For an oriented surface S, define

$$CM^+_{-1}(S) := \{\text{Positive complex metrics with constant curvature } -1\}$$

endowed with the subspace topology from $CM^+(S)$.

Theorem 6.15. Let S be an oriented surface with $\chi(S) < 0$.

1) For all $g \in CM^+(S)$, there exists a smooth $f\colon S \to \mathbb{C}^*$ such that $f \cdot g$ has constant curvature -1 and quasi-Fuchsian holonomy.

   More precisely, the map

$$\mathfrak{J}\colon CM^+(S) \to BC(S)$$
$$g \mapsto [g] = \mathbf{J}_g$$

   admits a continuous right inverse

$$\mathfrak{U}\colon C(S)\times C(S) \cong BC(S) \to CM^+_{-1}(S)$$

such that the diagram

$$\begin{array}{ccc} C(S)\times C(S)\cong BC(S) & \xrightarrow{\mathfrak{U}} & CM^+_{-1}(S) \\ \downarrow & & \downarrow{\scriptstyle mon} \\ \mathfrak{T}(S)\times\mathfrak{T}(S) & \xrightarrow{\mathfrak{B}} & \mathcal{X}(S) \end{array}$$

commutes, where $\mathfrak{B}$ is defined as in (29).

2) If S is closed, the image of $\mathfrak{U}$ is the connected component of $CM^+_{-1}(S)$ containing Riemannian metrics.

Proof of Theorem 6.15. 1) Consider $\mathbf{J}\in BC(S)$, which corresponds to two complex structures $(J_1,J_2)\in C(S)\times C(S)$.

Applying Theorem 6.13 to the couple (J_2,J_1), there exist:

– $\rho=\rho(J_1,J_2)\colon \pi_1(S)\to \mathrm{PSL}(2,\mathbb{C})$ quasi-Fuchsian;
– $f_1\colon \widetilde{S}\to\mathbb{CP}^1$ J_1-antiholomorphic and ρ-equivariant embedding;
– $f_2\colon \widetilde{S}\to\mathbb{CP}^1$ J_2-holomorphic and ρ-equivariant embedding

with $Im(f_1)\cap Im(f_2)=\emptyset$. Hence, we have an admissible, ρ-equivariant embedding

$$\sigma=(f_1,f_2)\colon \widetilde{S}\to\mathbb{G}=\mathbb{CP}^1\times\mathbb{CP}^1\setminus\Delta.$$

By Theorem 6.9, the pull-back complex metric on $\widetilde{S}$ is positive and projects to a positive complex metric g on S.

After we show that the constructed g is compatible with $\mathbf{J}$, the thesis follows by defining $\mathfrak{U}(\mathbf{J}):=g$. Let w_1 and w_2 be the complex coordinates on $\widetilde{S}$ induced by J_1 and J_2 respectively. Then,

$$g=\frac{4}{(f_1-f_2)^2}f_1^*dz_1\cdot f_2^*dz_2=\frac{4}{(f_1-f_2)^2}df_1\cdot df_2=$$
$$=\frac{4}{(f_1-f_2)^2}\frac{\partial f_1}{\partial\overline{w_1}}\frac{\partial f_2}{\partial w_2}d\overline{w_1}\cdot dw_2;$$

hence $V_i(J_1)=V_i(\mathbf{J})$ and $V_{-i}(J_2)=V_{-i}(\mathbf{J})$ are the isotropic directions of g.

2) Since one can see

$$Im(\mathfrak{U})=\{g\mid(\mathfrak{U}\circ\mathfrak{J})(g)=g\}\subset CM^+_{-1}(S),$$

$Im(\mathfrak{U})$ is a closed connected subset of $\mathbb{C}M^+_{-1}(S)$. We need to prove that it is also open.

As we stated in Theorem 6.9, for any $g\in CM^+_{-1}$ there exist two developing maps for projective structures f_1 and f_2 on S and $\overline{S}$ respectively, uniquely defined up to post-composition with elements in $\mathrm{PSL}(2,\mathbb{C})$, such that $\sigma=(f_1,f_2)\colon(\widetilde{S},\widetilde{g})\to\mathbb{G}$ is an isometric immersion. We therefore have a map

$$proj\colon CM^+_{-1}(S)\to\widetilde{\mathcal{P}}(S)\times\widetilde{\mathcal{P}}(\overline{S})$$
$$g=(f_1,f_2)^*\langle\cdot,\cdot\rangle_{\mathbb{G}}\mapsto([f_1],[f_2]).$$

The thesis follows after we show that

$$Im(\mathfrak{U})=proj^{-1}(\widetilde{\mathcal{P}}_0(S)\times\widetilde{\mathcal{P}}_0(\overline{S}))$$

since $\widetilde{\mathcal{P}}_0(S)$ is an open subset of $\widetilde{\mathcal{P}}_{QF}(S)$, hence of $\widetilde{\mathcal{P}}(S)$.

By construction of $\mathfrak{U}$, clearly

$$Im(\mathfrak{U}) \subseteq proj^{-1}(\widetilde{\mathcal{P}}_0(S) \times \widetilde{\mathcal{P}}_0(\overline{S})).$$

Conversely, assume $g = (f_1, f_2)^*\langle\cdot,\cdot\rangle_{\mathbb{G}} \in proj^{-1}(\widetilde{\mathcal{P}}_0(S) \times \widetilde{\mathcal{P}}_0(\overline{S}))$. Since $hol_{f_1} = hol_{f_2} \colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$, f_1 and f_2, by Theorem 6.14, the maps f_1 and f_2 correspond exactly to the maps one gets through Theorem 6.13 from the couple of complex structures $(f_1^*(\mathbb{J}^{\mathbb{CP}^1}), f_2^*(-\mathbb{J}^{\mathbb{CP}^1})$, hence

$$(f_1, f_2)^*\langle\cdot,\cdot\rangle_{\mathbb{G}} = \mathfrak{U}(f_1^*(\mathbb{J}^{\mathbb{CP}^1}), f_2^*(-\mathbb{J}^{\mathbb{CP}^1})).$$

□

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