arXiv:2012.04398v1 [math.PR] 8 Dec 2020

A NEW CORRESPONDENCE BETWEEN TASEP AND BURGERS' EQUATION

FRANCO FLANDOLI, BENJAMIN GESS, AND FRANCESCO GROTTO

Abstract. We introduce a new particle model, which we dub Active Bi-Directional Flow, conjugated to the Totally Asymmetric Exclusion Process in discrete time. We then associate to our model intrinsically stochastic, non-entropic weak solutions of Burgers' equation on $\mathbb{R}$, thus linking the latter to the KPZ universality class.

1. Introduction

Among the most striking recent results on stochastic systems is the first quite complete understanding of the KPZ fixed point as a scaling limit of fluctuations of the height function associated to the totally asymmetric simple exclusion process (TASEP). Height functions of models in the KPZ universality class are conjectured to converge in the 1:2:3 scaling limit,

$$h(t,x) \mapsto h^\varepsilon(t,x) = \varepsilon^{1/2} h(\varepsilon^{-3/2}t, \varepsilon^{-1}x) - C_\varepsilon t, \quad \varepsilon \downarrow 0, C_\varepsilon \uparrow \infty, \tag{1}$$

to a universal, scale invariant limit process, characterized as a Markov process by its transition probabilities in [14].

The Kardar-Parisi-Zhang (KPZ) equation, introduced in [13],

$$\partial_t h = \nu \partial_x^2 h + \lambda(\partial_x h)^2 + \sigma\xi, \quad \nu, \lambda, \sigma > 0, \tag{2}$$

where ξ denotes space-time white noise, is not invariant under the 1:2:3 scaling. Its solution theory, initiated in [3], has been the starting point of recent breakthrough developments in stochastic PDE theory, [11, 10]. Solutions to the KPZ equation are special models in the KPZ universality class, as they are expected to describe the unique heteroclinic orbit between the KPZ fixed point and the Gaussian Edwards-Wilkinson fixed point, [19]. Under the 1:2:3 scaling, the diffusion and noise terms of (2) vanish, formally leading to the Hamilton-Jacobi equation

$$\partial_t h = \lambda(\partial_x h)^2. \tag{3}$$

This, informally, suggests that the KPZ fixed point can be understood as a (stochastic) solution to (3). However, as pointed out in [14], entropy solutions to (3) given by the Hopf-Lax formula,

$$h(t,x) = \sup_y \left(h(0,y) - \frac{(x-y)^2}{4\lambda t} \right),$$

are not suited to describe the KPZ fixed point. Indeed, entropy solutions would not preserve the regularity of Brownian motion, unlike the KPZ fixed point. In addition, since the KPZ fixed point has the space regularity of Brownian motion,

Scuola Normale Superiore, Piazza dei Cavalieri, 7, 56126 Pisa, Italia
Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany
& Fakultät für Mathematik, Universität Bielefeld, Bielefeld, Germany
Université du Luxembourg, 6 Avenue de la Fonte, 4364 Esch-sur-Alzette, Luxembourg
E-mail addresses: franco.flandoli@sns.it , benjamin.gess@mis.mpg.de , francesco.grotto@uni.lu.
Date: December 9, 2020.

the nonlinear term of (3) is ill-posed. The possibility of a different kind of weak solutions to (3) describing the KPZ fixed point was left open in [14].

The aim of this paper is to describe a link between discrete-time TASEP and a class of weak, non entropic, solutions to Burgers' equation. Burgers' equation takes the place of (3) by passing to the space derivative $u = \partial_x h$, which on the level of TASEP corresponds to consider particles rather than their height functions. The present work focuses on the particle level, and does not consider height functions. Let us stress that discrete-time TASEP (to which we will also refer simply as TASEP in the following) belongs to the KPZ universality class just as its more popular continuous-time counterpart, [1].

The main result of this work, see Theorem 27 below, yields a bijection, before passing to scaling limits, between samples of TASEP and a class of intrinsically stochastic, weak solutions to Burgers' equation,

$$\partial_t u + u \partial_x u = 0. \tag{4}$$

We note that (4) formally corresponds to (3) with $\lambda = -1$ and $u = \partial_x h$.

The link between TASEP and the Burgers' equation described here proceeds via the introduction of a new interacting particle process. Although, naturally, this process shares features with other models in the KPZ universality class, in particular the discrete-time polynuclear growth (PNG) process, to the best of our knowledge it has not been introduced before. We call it ABDF: Active Bi-Directional Flow, or more extensively the Bi-Directional Flow model with Active empty positions. It is an interesting model in itself, and arises from TASEP when considering pairs of occupied and empty positions.

We note that the choice of parameter $\lambda \neq 0$ in (3) does not play an important role in the present paper, see Remark 22, so we prefer to fix it for the sake of simplicity. It might, however, be an important feature in the study of the KPZ scaling limit. The investigation of the scaling limit of the weak solutions of Burgers' equation constructed in this work is expected to be highly non-trivial, and is left as an open problem. Our hope is that the link between TASEP and Burgers' equation identified here may shed some light on the open problem of finding an equation satisfied by the KPZ fixed point, [8, 17]. In particular, suitable non-entropic weak solutions might be the right objects to describe the KPZ fixed point (*cf.* [2] and remarks in the introduction of [8]).

Although we will not discuss scaling limits, it is essential to refer to the recent works [14, 15, 18] on transition probabilities of KPZ fixed point obtained as limits of the ones of TASEP. The $1:2:3$ scaling we referred to above identifies fluctuations, and it is worth recalling that macroscopic limits of models such as TASEP are classically known to be solutions of nonlinear conservation laws, [21, 20].

The paper is organized as follows. In section 2 we introduce the ABDF model; in Section 3 we link it to TASEP and finally, in Section 5 we link them to the Burgers' equations. Preliminarily, in section 4, we introduce the class of weak solutions of Burgers' equations involved in the conjugacy. Some ideas about such solutions have been identified previously [6], but the link with TASEP described here is new.

2. ABDF model: Active Bi-Directional Flow

We begin with a description of the model in plain words, and then provide rigorous definitions. A configuration of the ABDF model is made of particles and empty positions, with no more than one particle at each position. Particles are divided into two classes: *left and right particles.* Empty positions are also divided into two classes: *active and inert positions.*

Left particles are those which move to the left, right particles to the right (the motion is described below). The relative position of left and right particles is not arbitrary, as we now describe. We call *consecutive particles* two particles occupying positions $x_1 < x_2$ such that no particle is in between, namely the positions $x_1 + k$, $k \geq 1$, $x_1 + k < x_2$ are all empty. The rule which restricts particle occupancy is that two consecutive particles, of different type (one left and one right, independently of the order) are always separated by an odd number of empty positions. Two consecutive particles of the same kind (both left or both right) are always separated by an even number of empty positions.

Empty positions, as said above, can be active or inert, depending on the order number with respect to the first particle on the left or right of the empty position, and the class of such particles. Precisely, assume the empty position, say x_0, lies in between two consecutive particles at positions $x_1 < x_2$. Let k be the positive integer such that $x_0 = x_1 + k$. If the particle at x_1 is a left-particle and k is odd, then the empty position at x_0 is active, otherwise it is inert. If the particle at x_1 is right and k is odd, then the empty position is inert, otherwise it is active.

We may give the definition analogously, based on x_2. Call h the positive integer such that $x_0 = x_2 - h$. If the particle at x_2 is left and h is odd, then the empty position at x_0 is inert; if the particle at x_2 is right and h is odd, then it is active, otherwise the empty position is inert. The two definitions coincide because of the rule on the number of empty positions between consecutive particles; it is easy to see this case by case in some examples and it is formally checked below in Lemma 2.

Finally, if an empty position is not in between two consecutive particles, it means either that we deal with the configuration made of all empty positions, or it has just a first particle on the left, or a first particle on the right, namely it is part of an half line of empty particles. In the second case, the rule concerning active or inert property is thus the same described above, where we use the first particle at x_1 on the left of the empty position or the first particle at x_2 on the right, depending on the case.

The case of all empty positions is, on the contrary, a very special one. Indeed, active and inert positions should alternate; but there are two ways to alternate, differing for the property attributed to the site $x = 0$. It is important to distinguish between them, both possibly occurring during the evolution described in the next Definition 4. Therefore we introduce two "zero" configurations. Let us introduce them before the definition for simplicity of exposition. Call alt_0 and alt_1 the sequences (setting $(-1)^0 = 1$)

$$alt_0(x) = \frac{1 - (-1)^x}{2}, \qquad alt_1(x) = \frac{1 - (-1)^{x+1}}{2}$$

and call $\overline{alt}_0$ and $\overline{alt}_1$ the double sequences $\overline{alt}_0$, $\overline{alt}_1 : \mathbb{Z} \to \{-1, 0, 1\} \times \{0, 1\}$, defined as

$$\overline{alt}_\alpha(x) = (0, alt_\alpha(x)), \qquad \alpha = 0, 1$$

for all $x \in \mathbb{Z}$. The first coordinate declares that $\overline{alt}_0$ and $\overline{alt}_1$ are both zero sequences, namely they represent ABDF configurations with all empty sites; the second coordinate alternates 0 and 1, in the two possible different ways; the subscript 0 or 1 in the name alt_0, alt_1 refers to the value at $x = 0$: $alt_\alpha(0) = \alpha$.

The property of empty positions of being active or inert is a (nonlocal) *consequence* of the particle configuration. Therefore, in the formal definition of ABDF configurations we specify the particles, first component of the configuration and *deduce* the active or inert positions, second component of the configuration, by what we call *activation map*. The only exceptions are the ABDF configurations with all empty sites, where two different activation profiles are possible.

Let us give a rigorous definition of a configuration of the ABDF model and of its activated empty positions. We associate $+1$ to right particles, -1 to left particles, 0 to the empty positions; then we introduce the "activation record" which associates 0 to any position where a new pair (a concept introduced in the sequel) cannot arise (the positions occupied by particles are also of this type), and 1 to the empty positions that are "active", namely those which may give rise to a new pair of particles. We start from a preliminary definition introducing the main elements.

Definition 1. *Let Λ_0 be the set of sequences*

$$\theta : \mathbb{Z} \to \{-1, 0, 1\}$$

which are not identically zero (we write simply $\theta \neq 0$) such that

i) *if $x_1 < x_2 \in \mathbb{Z}$ have the properties $\theta(x_1)\theta(x_2) = -1$ and $\theta(x) = 0$ for all $x \in (x_1, x_2) \cap \mathbb{Z}$, then the cardinality of $(x_1, x_2) \cap \mathbb{Z}$ is odd;*
ii) *if $x_1 < x_2 \in \mathbb{Z}$ have the properties $\theta(x_1)\theta(x_2) = 1$ and $\theta(x) = 0$ for all $x \in (x_1, x_2) \cap \mathbb{Z}$, then the cardinality of $(x_1, x_2) \cap \mathbb{Z}$ is even.*

For every $\theta \in \Lambda_0$, introduce the activation record *sequence*

$$ar(\theta) : \mathbb{Z} \to \{0, 1\},$$

defined as:

iii) *if $\theta(x_0) \in \{-1, 1\}$ then $ar(\theta)(x_0) = 0$;*
iv) *if $\theta(x_0) = 0$ and the set*

$$L(x_0) := \{x < x_0 : x \in \mathbb{Z},\ \theta(x) \in \{-1, 1\}\}$$

is not empty, taking $x_1 = \max L(x_0)$ and $k \in \mathbb{N}$ such that $x_0 = x_1 + k$,

$$ar(\theta)(x_0) = \left| \frac{\theta(x_1) + (-1)^k}{2} \right|$$

v) *if $\theta(x_0) = 0$ and the set*

$$R(x_0) := \{x > x_0 : x \in \mathbb{Z},\ \theta(x) \in \{-1, 1\}\}$$

is not empty, taking $x_2 = \min R(x_0)$ and $h \in \mathbb{N}$ such that $x_0 = x_2 - h$,

$$ar(\theta)(x_0) = \left| \frac{\theta(x_2) + (-1)^{h+1}}{2} \right|.$$

Lemma 2. *If both $L(x_0)$ and $R(x_0)$ are not empty, points (iv-v) above give the same definition of $ar(\theta)$.*

Proof. Assume both $L(x_0)$ and $R(x_0)$ are not empty, let x_1, x_2, k, h be defined as above. The number of empty positions ($x \in \mathbb{Z}$ such that $\theta(x) = 0$) strictly between x_1 and x_2 is $n := k + h - 1$. We have to prove that

$$\left| \frac{\theta(x_1) + (-1)^k}{2} \right| = \left| \frac{\theta(x_2) + (-1)^{n-k+1}}{2} \right|,$$

which is true if either

$$\frac{\theta(x_1) + (-1)^k}{2} = \frac{-\theta(x_2) + (-1)^k(-1)^{n+1}}{2} \quad \text{or} \quad \frac{\theta(x_2) + (-1)^k(-1)^n}{2}.$$

In case (i), $\theta(x_1)\theta(x_2) = -1$, n is odd, hence the first identity is true. In case (ii), the second identity is true. □

Definition 3. *A configuration of the ABDF model is a map*

$$(\theta, act) : \mathbb{Z} \to \{-1, 0, 1\} \times \{0, 1\}$$

with the following properties:

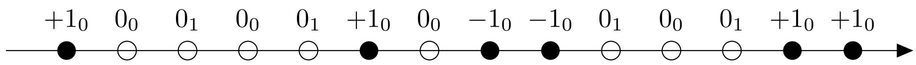

Figure 1. A piece of an ABDF configuration. Numbers ± 1 and 0 denote particle type or empty sites, subscripts are the values of activation record.

a) *if* $\theta = 0$*, then either* $act = alt_0$ *or* $act = alt_1$ *(in other words, either* $(\theta, act) = \overline{alt}_0$ *or* $(\theta, act) = \overline{alt}_1$*);*
b) *if* $\theta \neq 0$*, then* $\theta \in \Lambda_0$ *and* $act = ar(\theta)$*, where the set* Λ_0 *and the map* ar *are introduced in Definition 1.*

The set of all ABDF configurations will be denoted by Λ*.*

Figure 1 represents a piece of an ABDF configuration. The value of $ar(\theta)(x)$ is zero whenever $\theta(x) = 1$ or $\theta(x) = -1$. In the empty positions, namely those with $\theta(x) = 0$, the value of $ar(\theta)$ alternates, equal to 0 or 1, starting from zero at the right of a position with $\theta(x) = 1$, while starting from one at the right of a position with $\theta(x) = -1$. The example makes it apparent how the number of empty positions between non-empty ones is regulated by the concordance of signs of the extremes.

Let us come to the description of ABDF dynamics, first at a heuristic level. All right particles move to the right by one position at every time step, all left particles to the left: no abortion of the move happens due to a possible occupancy of the arrival positions, like in exclusion processes, because *all* particles move.

All active empty positions x_0 may generate, at random with probability $1/2$, a pair of particles. Ideally, they are generated at time $t-1$, but they are not visible at such generation time (the time when the position is active); they are observed at the next time step t; one of the generated particles is a left-particles and it is observed, at time t, at position $x_0 - 1$; the other is a right-particle, observed at $x_0 + 1$.

It often happens that two particles meet at one position: a right particle which moved from $x-1$ to x and a left particle which moved from $x+1$ to x arrive at the same time t at x. In such a case, the two particles disappear, they annihilate each other and position x is empty at time t.

Obviously we have to check that these rules are not contradictory and that they give rise to configurations of the class described above, an ABDF configuration. We do it in the formal description.

Definition 4. *Let* $\Omega = \{0,1\}^{\mathbb{N}\times\mathbb{Z}}$*, with the* σ*-algebra* $\mathcal{F}$ *generated by cylinder sets, and the product probability measure* P *of Bernoulli* $p = \frac{1}{2}$ *random variables. Given* $\omega \in \Omega$*, we write* $\omega(t,x)$ *for its* (t,x)*-coordinate,* $(t,x) \in \mathbb{N}\times\mathbb{Z}$ *and write* $\varkappa(t,x) := 1 - \omega(t,x)$ *for the complementary value.*

Then, based on the probability space $(\Omega, \mathcal{F}, P)$*, we introduce a family of maps*

$$\mathcal{T}_{ABDF}(t,\omega,\cdot) : \Lambda \to (\{-1,0,1\} \times \{0,1\})^{\mathbb{Z}}$$

indexed by $t \in \mathbb{N}$ *and* $\omega \in \Omega$*, defined as follows. Denote*

$$\mathcal{T}_{ABDF}(t,\omega,(\theta,act)) = \left(\mathcal{T}_{ABDF}(t,\omega,(\theta,act))_1, \mathcal{T}_{ABDF}(t,\omega,(\theta,act))_2\right),$$

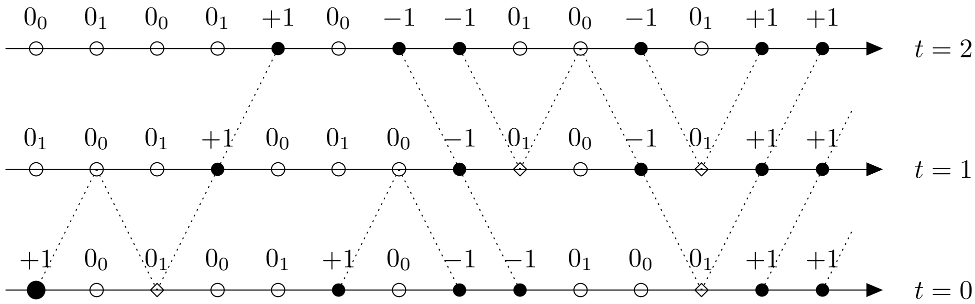

FIGURE 2. A sample of ABDF dynamics starting from the configuration of Figure 1. Dotted lines track movements of particles. "Activated" empty sites have the empty circle replaced by an empty square.

where we recall that $act = ar(\theta)$ *unless* $\theta = 0$. *The map* $\mathcal{T}_{ABDF}(0, \omega, \cdot)$ *is the identity. For* $t > 0$, *the first component is defined as*

$$\begin{aligned}(5) \qquad \mathcal{T}_{ABDF}(t, \omega, (\theta, act))_1(x) := &\max(\theta(x-1), 0) + act(x-1)\varkappa(t-1, x-1) \\ &+ \min(\theta(x+1), 0) - act(x+1)\varkappa(t-1, x+1)\end{aligned}$$

for every $x \in \mathbb{Z}$. *For* $t > 0$, *if* $\mathcal{T}_{ABDF}(t, \omega, (\theta, act))_1$ *is not the identically null sequence, the second component is defined by*

$$\mathcal{T}_{ABDF}(t, \omega, (\theta, act))_2 = ar(\mathcal{T}_{ABDF}(t, \omega, (\theta, act))_1)$$

where the activation record $ar(\cdot)$ *is given by Definition 1. If* $\mathcal{T}_{ABDF}(t, \omega, (\theta, act))_1 = 0$ *then* $\mathcal{T}_{ABDF}(t, \omega, (\theta, act))_2$ *is either* alt_0 *or* alt_1. *It is equal to* alt_1 *in each one of the following three cases:*

$$\begin{aligned} &\theta(0) = -1, \\ act(-1) = 1 \quad &and \quad \omega(t-1, -1) = 1, \\ act(0) = 1 \quad &and \quad \omega(t-1, 0) = 0, \end{aligned}$$

otherwise it is equal to alt_0.

It is not easy to see right away why we set $\mathcal{T}_{\mathrm{ABDF}}(t, \omega, (\theta, act))_2 = alt_1$ precisely in these three cases: this will become clear in the correspondence between a TASEP configuration η and θ, but this is an element introduced below, hence we cannot anticipate the intuition.

The quantity $\mathcal{T}_{\mathrm{ABDF}}(t, \omega, (\theta, act))_1(x)$ can only take values in $\{-1, 0, 1\}$. We shall prove that it satisfies Definition 1, (i)-(ii) and therefore $\mathcal{T}_{\mathrm{ABDF}}(t, \omega, \theta) \in \Lambda$. To minimize double proofs, we postpone this fact to the verification of the link with TASEP (see Theorem 9 below).

Once this is proved, one can introduce the ABDF random dynamical system, of which $\mathcal{T}_{\mathrm{ABDF}}(t, \omega, \cdot)$ is just the 1-step dynamics at time t. We let $\phi_{\mathrm{ABDF}}(0, \omega) = id$, and for $t > 0$, $t \in \mathbb{N}$,

$$\phi_{\mathrm{ABDF}}(t, \omega) := \mathcal{T}_{\mathrm{ABDF}}(t, \omega) \circ \mathcal{T}_{\mathrm{ABDF}}(t-1, \omega) \circ \cdots \circ \mathcal{T}_{\mathrm{ABDF}}(1, \omega),$$

so that it holds the random dynamical system property

$$\phi_{\mathrm{ABDF}}(t, \omega) \circ \phi_{\mathrm{ABDF}}(s, \omega) = \phi_{\mathrm{ABDF}}(t+s, \omega), \quad t, s \in \mathbb{N}, \omega \in \Omega.$$

3. TASEP, its pairs and ABDF

A TASEP configuration is a map

$$\eta : \mathbb{Z} \to \{0,1\}.$$

When $\eta(x) = 1$, we say that x is occupied by a particle; when $\eta(x) = 0$, we say that x is empty.

TASEP dynamics in discrete time $t \in \mathbb{N}$ consists in particles moving to the right by one position with probability $\frac{1}{2}$, with simultaneous independent jumps, aborted when the arrival position is occupied. More precisely, given a configuration η at time $t-1 \in \mathbb{N}$, a particle at position $x \in \mathbb{Z}$ (which means $\eta_{t-1}(x) = 1$) has probability $\frac{1}{2}$ to jump on the right at time t (namely $\eta_t(x+1) = 1$), but the jump is aborted if $\eta_{t-1}(x+1) = 1$.

Using the probability space $(\Omega, \mathcal{F}, P)$ defined above, when a particle is at time $t-1 \in \mathbb{N}$ at positions $x \in \mathbb{Z}$, it jumps if both $\omega(t-1,x) = 1$ and the position $x+1$ is free. Denote by $\mathcal{T}_{\text{TASEP}}(t,\omega,\cdot)$ the random map which associates to a given TASEP configuration η and a given random choice $\omega \in \Omega$ the subsequent, one-time step, TASEP configuration. Heuristic prescriptions are summarized in

$$\mathcal{T}_{\text{TASEP}}(t,\omega,\eta)(x) = \begin{cases} \eta(x) & \text{if} \quad \eta(x) = \eta(x+1) = 1 \\ \varkappa(t-1,x) & \text{if} \quad \eta(x) = 1, \eta(x+1) = 0 \\ \omega(t-1,x-1) & \text{if} \quad \eta(x) = 0, \eta(x-1) = 1 \\ \eta(x) & \text{if} \quad \eta(x) = \eta(x-1) = 0 \end{cases},$$

or equivalently

$$\mathcal{T}_{\text{TASEP}}(t,\omega,\eta)(x) \\ = \begin{cases} \varkappa(t-1,x)\,\eta(x) + \omega(t-1,x)\,\eta(x+1) & \text{if} \quad \eta(x) = 1 \\ \varkappa(t-1,x-1)\,\eta(x) + \omega(t-1,x-1)\,\eta(x-1) & \text{if} \quad \eta(x) = 0 \end{cases},$$

which gives rise to the following rigorous Definition.

Definition 5. *Let $(\Omega, \mathcal{F}, P)$ be the probability space of Definition 4. We define the family of maps $\mathcal{T}_{TASEP}(t,\omega,\cdot)$ on $\{0,1\}^{\mathbb{Z}}$, indexed by $t \in \mathbb{N}$ and $\omega \in \Omega$, by*

(6)
$$\mathcal{T}_{TASEP}(t,\omega,\eta)(x) = [\varkappa(t-1,x)\,\eta(x) + \omega(t-1,x)\,\eta(x+1)]\,\eta(x) \\ + [\varkappa(t-1,x-1)\,\eta(x) + \omega(t-1,x-1)\,\eta(x-1)]\,(1-\eta(x))$$

when $t > 0$, $\mathcal{T}_{TASEP}(0,\omega,\cdot) = id$.

As for ABDF above, we may introduce TASEP random dynamical system by setting $\phi_{\text{TASEP}}(0,\omega) = id$ and, for $t > 0$, $t \in \mathbb{N}$,

$$\phi_{\text{TASEP}}(t,\omega) := \mathcal{T}_{\text{TASEP}}(t,\omega) \circ \mathcal{T}_{\text{TASEP}}(t-1,\omega) \circ \cdots \circ \mathcal{T}_{\text{TASEP}}(1,\omega),$$

the latter satisfying the random dynamical system property

$$\phi_{\text{TASEP}}(t,\omega) \circ \phi_{\text{TASEP}}(s,\omega) = \phi_{\text{TASEP}}(t+s,\omega), \quad t,s \in \mathbb{N},\, \omega \in \Omega.$$

We now turn our attention to pairs in TASEP configurations: pairs of particles and pairs of empty positions.

Definition 6. *The pair operator*

$$\mathcal{P} : \{0,1\}^{\mathbb{Z}} \to (\{-1,0,1\} \times \{0,1\})^{\mathbb{Z}}, \quad \mathcal{P}(\eta)(x) = (\mathcal{P}(\eta)_1(x), \mathcal{P}(\eta)_2(x)), \quad x \in \mathbb{Z},$$

is the function defined as follows:

a) for every $\eta \in \{0,1\}^{\mathbb{Z}}$ and $x \in \mathbb{Z}$,

$$\mathcal{P}(\eta)_1(x) = 1 - \eta(x) - \eta(x+1)$$

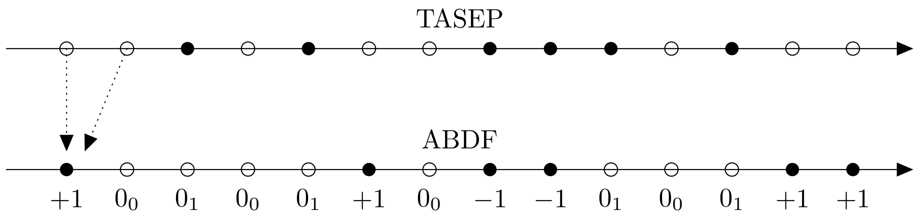

Figure 3. The TASEP configuration associated with the ABDF one of Figure 1. Dotted arrows show (for the right-most site) two TASEP site determining the state of a ABDF one.

b) if $\mathcal{P}(\eta)_1 \neq 0$, then $\mathcal{P}(\eta)_2 = ar(\mathcal{P}(\eta)_1)$, where the activation record ar is introduced in Definition 1,

c) if $\mathcal{P}(\eta)_1 = 0$, namely $\eta = alt_\alpha$ for $\alpha = 0$ or 1, then $\mathcal{P}(\eta)_2 = \eta$; in other words, $\mathcal{P}(alt_\alpha) = \overline{alt}_\alpha$, $\alpha = 0, 1$.

Remark 7. For geometric elegance it would be better to define

$$\mathcal{P}(\eta)_1(y) = 1 - \eta\left(y - \frac{1}{2}\right) - \eta\left(y + \frac{1}{2}\right)$$

for $y \in \mathbb{Z} + \frac{1}{2}$, the intermediate point of the pair, but notation would be burdened.

The link between TASEP pairs and ABDF configurations is simply:

Proposition 8. *The pair map $\mathcal{P}$ is a bijection of $\{0,1\}^{\mathbb{Z}}$ onto Λ.*

We shall prove this claim in subsection 3.1. Let us state the main result linking the TASEP and the ABDF model:

Theorem 9. *For every $\eta \in \{0,1\}^{\mathbb{Z}}$, $t \in \mathbb{N}$, $\omega \in \Omega$,*

$$\mathcal{T}_{ABDF}(t, \omega, \mathcal{P}(\eta)) = \mathcal{P}(\mathcal{T}_{TASEP}(t, \omega, \eta)),$$

thus the random dynamical systems $\phi_{ABDF}(t, \omega)$ and $\phi_{TASEP}(t, \omega)$ are conjugated.

In particular, $\mathcal{T}_{ABDF}(t, \omega, \Lambda) \subset \Lambda$.

The proof is given in subsection 3.3, after giving as a hint the simpler proof in the deterministic case in subsection 3.2.

3.1. Proof of Proposition 8.

The proof in itself can be made more concise than the following description, but we profit of the proof to introduce some more structure. The subsection thus has also a descriptive nature.

The first component of the map $\mathcal{P}$ is local; its effect is to associate (for alt_0, alt_1 we refer to the first component):

$$(0,0) \to 1, \qquad (1,0) \to 0,$$
$$(1,1) \to -1, \qquad (0,1) \to 0,$$

the first symbol being the TASEP pair at $x, x+1$ and the second its associated value at point x, under $\mathcal{P}$. The local action, originally defined on pairs of values, becomes relevant on special finite strings.

Definition 10. *a) A finite sequence in $\mathbb{Z}$ of the form*

$$x_1, x_1 + 1, ..., x_1 + n + 1 = x_2$$

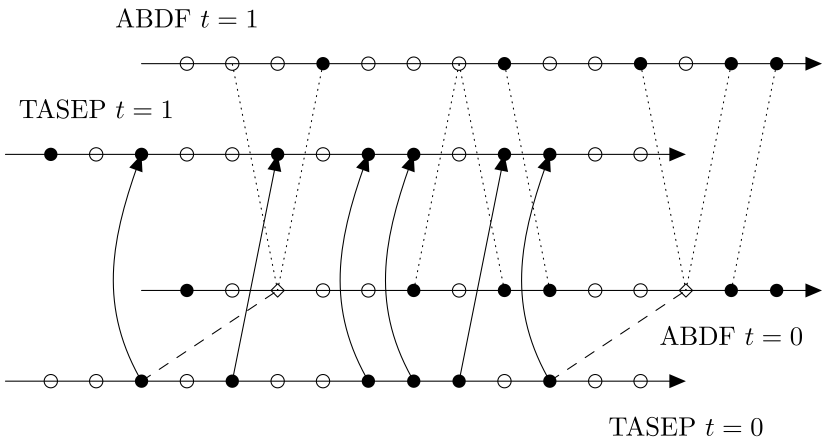

FIGURE 4. TASEP and ABDF evolutions, starting from Figure 3. Solid arrows denote trajectories of TASEP particles, dotted lines the ones of ABDF as above. Two dashed lines couple generations of ABDF particles to TASEP particles not jumping even if they can do so.

where n is a non-negative integer, will be denoted by $[x_1, x_2]_{\mathbb{Z}}$ and called a segment of cardinality $n+2$ (n is the number of "internal" points, namely it is the cardinality of $(x_1, x_2) \cap \mathbb{Z}$).

b) Given a segment $[x_1, x_2]_{\mathbb{Z}}$, we define

$$\mathcal{P}_{[x_1,x_2]_{\mathbb{Z}}} : \{0,1\}^{[x_1,x_2+1]_{\mathbb{Z}}} \to \{-1,0,1\}^{[x_1,x_2]_{\mathbb{Z}}}$$

by setting, for $\eta \in \{0,1\}^{[x_1,x_2+1]_{\mathbb{Z}}}$,

$$\mathcal{P}_{[x_1,x_2]_{\mathbb{Z}}}(\eta)(x) = 1 - \eta(x) - \eta(x+1)$$

for every $x \in [x_1, x_2]_{\mathbb{Z}}$.

The case $n = 0$ is when we look at the action on pairs. The following remark may be used directly to invert the pair map $\mathcal{P}$.

Remark 11. Call $\theta(x) = 1 - \eta(x) - \eta(x+1)$, $x \in \mathbb{Z}$. Then, for every $n \in \mathbb{N}$

$$\theta(x+n) = 1 - \eta(x+n) - \eta(x+n+1),$$

hence

$$\begin{aligned}\eta(x+n+1) &= 1 - \eta(x+n) - \theta(x+n) \\ &= \eta(x+n-1) + \theta(x+n-1) - \theta(x+n) \\ &= 1 - \eta(x+n-2) - \theta(x+n-2) + \theta(x+n-1) - \theta(x+n),\end{aligned}$$

and so on, which gives us

$$\eta(x+n+1) = \frac{1+(-1)^n}{2} - (-1)^n \eta(x) - \sum_{k=0}^{n} (-1)^k \theta(x+n-k). \tag{7}$$

A similar formula holds for negative integer n. Hence, we may reconstruct η from θ at the price of fixing one value of η, say $\eta(0)$.

The solution to invertibility of $\mathcal{P}$ described by this remark is however quite inefficient: if $\theta = \mathcal{P}(\eta)$ and $\theta(x) = 1$ for some $x \in \mathbb{Z}$, we already know that $\eta(x) = 0, \eta(x+1) = 0$. Similarly, if $\theta(x) = -1$, we already know that $\eta(x) = 1, \eta(x+1) = 1$. Only when $\theta(x) = 0$ we are in doubt about the inversion at x and $x+1$. But then, in such a case, we may use the previous remark. The idea is then to use formula (7) between consecutive points where $\theta(x) \neq 0$. This is formalized by the following arguments.

One problem is the reconstruction formula or algorithm, another is whether we can associate a TASEP configuration to every sequence of $-1, 0, 1$. A crucial property is that a right pair and a left pair are always separated by an odd number (= 1,3,5,...) of empty pairs; while two right pairs by an even number of empty pairs (= 0,2,4,...) and similarly two left pairs by an even number. See Figure 3 for an example.

Definition 12. *A segment $[x_1, x_2]_{\mathbb{Z}}$ of cardinality $n+2$ is called a maximal alternating segment of $\eta \in \{0,1\}^{\mathbb{Z}}$ if:*

- *$\eta(x_1) = \eta(x_1+1)$ and $\eta(x_2) = \eta(x_2+1)$ (namely $\mathcal{P}(\eta)_1(x_1), \mathcal{P}(\eta)_1(x_2) \in \{-1,1\}$)*
- *$\eta(x_1+k) \neq \eta(x_1+k+1)$ (namely $\mathcal{P}(\eta)_1(x_1+k) = 0$) for $k = 1, 2, ..., n$ (we understand that this condition is not imposed when $n = 0$).*

It is called a maximal alternating segment of $\theta \in \{-1,0,1\}^{\mathbb{Z}}$ if

- *$\theta(x_1), \theta(x_2) \in \{-1,1\}$,*
- *$\theta(x_1+k) = 0$ for $k = 1, 2, ..., n$ (not imposed when $n = 0$).*

We call maximal alternating half line the analogously defined sets $(-\infty, x_2]_{\mathbb{Z}}$ and $[x_1, \infty)_{\mathbb{Z}}$.

Clearly, if $[x_1, x_2]_{\mathbb{Z}}$ is a maximal alternating segment of $\eta \in \{0,1\}^{\mathbb{Z}}$, then it is a maximal alternating segment of $\theta := \mathcal{P}(\eta)_1$ (similarly for half lines). The two key facts on this concept are expressed by the following two lemmata.

Lemma 13. *Let $[x_1, x_2]_{\mathbb{Z}}$ be a maximal alternating segment of $\eta \in \{0,1\}^{\mathbb{Z}}$ (thus of $\theta := \mathcal{P}(\eta)_1$) with cardinality $n+2$. Then:*

- *if $\theta(x_1)\theta(x_2) = 1$, n is even;*
- *if $\theta(x_1)\theta(x_2) = -1$, n is odd.*

Proof. If $\theta(x_1)\theta(x_2) = 1$, then $\theta(x_1) = \theta(x_2) = \pm 1$. In the $+1$ case, $\eta(x_1) = \eta(x_1+1) = 0$, $\eta(x_2) = \eta(x_2+1) = 0$. Then, from identity (7),

$$\begin{aligned}\eta(x_1+n+1) &= \frac{1+(-1)^n}{2} - (-1)^n \eta(x_1) - \sum_{k=0}^{n} (-1)^k \theta(x_1+n-k) \\ &= \frac{1+(-1)^n}{2} - (-1)^n \theta(x_1) = \frac{1+(-1)^n}{2} - (-1)^n,\end{aligned}$$

where we have used $\theta(x_1+n-k) = 0$ for $k = 0, 1, ..., n-1$, in the second-last step. Since $\eta(x_1+n+1) = 0$, this implies n even. The other cases are completely similar. □

Lemma 13 imposes a restriction to the sequences of $\{-1,0,1\}^{\mathbb{Z}}$ which are in the range of the first component of $\mathcal{P}$. Recall the definition of Λ given above.

Lemma 14. *Let $\theta \in \Lambda \setminus \{\overline{alt}_0, \overline{alt}_1\}$ and let $[x_1, x_2]_{\mathbb{Z}}$ be a maximal alternating segment of θ of cardinality $n+2$. Then there exists one and only one $\eta \in \{0,1\}^{[x_1, x_2+1]_{\mathbb{Z}}}$ such that*

$$\mathcal{P}_{[x_1,x_2]_{\mathbb{Z}}}(\eta) = \theta|_{[x_1,x_2]_{\mathbb{Z}}}.$$

The string η satisfies $\eta(x_1) = \eta(x_1 + 1)$, $\eta(x_2) = \eta(x_2 + 1)$, with the unique values determined by the values $\theta(x_1), \theta(x_2)$ and in the middle it is given by formula (7), precisely

$$\eta(x_1 + j + 1) = \frac{1 + (-1)^j}{2} - (-1)^j \eta(x_1) - \sum_{k=0}^{j} (-1)^k \theta(x_1 + j - k)$$

for $j = 1, ..., n - 1$.

Proof. If $\eta \in \{0,1\}^{[x_1, x_2+1]_{\mathbb{Z}}}$ satisfies $\mathcal{P}_{[x_1,x_2]_{\mathbb{Z}}}(\eta) = \theta|_{[x_1,x_2]_{\mathbb{Z}}}$, then the properties of the values of η, including the formula for the intermediate values, are obvious or have been proved above. Thus uniqueness is clear. Proving the existence means proving that $\eta(x_1 + n + 1)$ given by the formula coincides with the values of $\eta(x_2) = \eta(x_2 + 1)$ prescribed by $\theta(x_2)$. This must be checked case by case. Let us do it only in one case - the others are the same - namely when $\theta(x_1) = \theta(x_2) = 1$, hence n is even, by the property $\theta \in \Lambda$; and $\eta(x_1) = \eta(x_1 + 1)$, $\eta(x_2) = \eta(x_2 + 1)$, all equal to zero. We have, from the formula,

$$\begin{aligned}\eta(x_1 + n + 1) &= \frac{1 + (-1)^n}{2} - (-1)^n \eta(x_1) - \sum_{k=0}^{n} (-1)^k \theta(x_1 + n - k) \\ &= \frac{1 + (-1)^n}{2} - (-1)^n = 0 = \eta(x_2).\end{aligned}$$

□

We are now ready to prove Proposition 8 which we restate.

Proposition (8)**.** *The pair map $\mathcal{P}$ is a bijection of $\{0,1\}^{\mathbb{Z}}$ onto Λ.*

Proof. It is sufficient to prove that, given $\theta \in \Lambda \setminus \{\overline{alt}_0, \overline{alt}_1\}$, there exists one and only one $\eta \in \{0,1\}^{\mathbb{Z}} \setminus \{alt_0, alt_1\}$ such that $\mathcal{P}(\eta) = \theta$.

Let $\{x_n\}$ be the strictly increasing possibly bi-infinite sequence of point of $\mathbb{Z}$ such that $[x_n, x_{n+1}]_{\mathbb{Z}}$ is an even or odd maximal segment of θ. There are four cases: $\{x_n\}$ is bi-infinite, or infinite only to the left, or infinite only to the right, or finite. The construction of $\{x_n\}$ may proceed from the origin of $\mathbb{Z}$: we denote by x_0 the first point ≥ 0 with $\theta(x_0) \neq 0$; by x_1 the first point $> x_0$ such that $\theta(x_1) \neq 0$; and so on, obviously if they exist. And we denote by x_{-1} the first point < 0 such that $\theta(x_{-1}) \neq 0$ and so on.

For every n such that x_n, x_{n+1} exists, we construct the corresponding values of $\eta(x_n)$, ..., $\eta(x_{n+1} + 1)$ as in (d) of the previous lemma. The construction is unique with the property that, locally, $\mathcal{P}(\eta)_1 = \theta$ on $[x_n, x_{n+1}]_{\mathbb{Z}}$. However, in principle the definition for $[x_n, x_{n+1}]_{\mathbb{Z}}$ may be in contradiction with the definition for $[x_{n+1}, x_{n+2}]_{\mathbb{Z}}$ because the points $x_{n+1}, x_{n+1} + 1$ are in common. But, based on $[x_n, x_{n+1}]_{\mathbb{Z}}$, we have defined $\eta(x_{n+1}) = \eta(x_{n+1} + 1)$, equal to 1 if $\theta(x_{n+1}) = -1$, equal to 0 if $\theta(x_{n+1}) = 1$. And based on $[x_{n+1}, x_{n+2}]_{\mathbb{Z}}$ we have given the same definition. Therefore there is no contradiction. The treatment of half lines is entirely similar and completes the constructions. □

3.2. **Proof of Theorem 9 in the deterministic case.** This section has only a pedagogical value. It illustrates some of the elements of the next proof in a simpler case, when the nonlocal map *ar* does not play any role. It is the case of ABDF and TASEP deterministic dynamics. By this we mean that we replace the probability 1/2 with probability 1 in the definition of P, the product measure on $(\Omega, \mathcal{F})$, see Definition 4. Precisely, we assume to have probability 1 that each $\omega(t, x) = 1$. Or, even more simply, we replace Ω by the new space

$$\Omega^* = \{1\}^{\mathbb{N} \times \mathbb{Z}}.$$

In this case time t plays no role, as well as ω (the cardinality of Ω^* is one) hence we drop (t,ω) from the notations. We call $\mathcal{T}^{\text{det}}_{\text{ABDF}}$ and $\mathcal{T}^{\text{det}}_{\text{TASEP}}$ the maps $\mathcal{T}_{\text{ABDF}}(t,\omega,\cdot)$ and $\mathcal{T}_{\text{TASEP}}(t,\omega,\cdot)$ in the deterministic case just defined. The result is:

Proposition 15. *For every $\eta\in\{0,1\}^{\mathbb{Z}}$*

$$\mathcal{T}^{\text{det}}_{ABDF}(\mathcal{P}(\eta))=\mathcal{P}\left(\mathcal{T}^{\text{det}}_{TASEP}(\eta)\right).$$

Proof. **Step 1**. In this step we prove

$$\mathcal{T}^{\text{det}}_{\text{ABDF}}(\mathcal{P}(\eta))_1=\mathcal{P}\left(\mathcal{T}^{\text{det}}_{\text{TASEP}}(\eta)\right)_1. \tag{8}$$

Recall that, in spite of the subdivision into cases of the definitions of $\mathcal{P}$ and $\mathcal{T}^{\text{det}}_{\text{ABDF}}$, the rules

$$\mathcal{P}(\eta)_1(x)=1-\eta(x)-\eta(x+1)$$

$$\mathcal{T}^{\text{det}}_{\text{ABDF}}(\theta)_1(x)=\max(\theta(x-1),0)+\min(\theta(x+1),0)$$

are general. On one side we have, by (6),

$$\begin{aligned}\mathcal{P}\left(\mathcal{T}^{\text{det}}_{\text{TASEP}}(\eta)\right)_1(x)&=1-\mathcal{T}^{\text{det}}_{\text{TASEP}}(\eta)(x)-\mathcal{T}^{\text{det}}_{\text{TASEP}}(\eta)(x+1)\\&=1-\eta(x+1)\eta(x)-\eta(x-1)(1-\eta(x))\\&-\eta(t,x+2)\eta(x+1)-\eta(x)\left(1-\eta(x+1)\right)\\&=\left(1-\eta(x-1)\right)\left(1-\eta(x)\right)-\eta(t,x+2)\eta(x+1).\end{aligned}$$

On the other side we have, from (5),

$$\begin{aligned}\mathcal{T}^{\text{det}}_{\text{ABDF}}(\mathcal{P}(\eta))_1(x)&=\max\left(1-\eta(x-1)-\eta(x),0\right)\\&\quad+\min\left(1-\eta(x+1)-\eta(t,x+2),0\right).\end{aligned}$$

We have $\max(1-\eta(x-1)-\eta(x),0)=0$ unless $\eta(x-1)=\eta(x)=0$, when it is equal to 1, hence

$$\max\left(1-\eta(x-1)-\eta(x),0\right)=\left(1-\eta(x-1)\right)\left(1-\eta(x)\right).$$

Similarly $\min(1-\eta(x+1)-\eta(t,x+2),0)=0$ unless $\eta(x+1)=\eta(t,x+2)=1$, when it is -1, hence

$$\min\left(1-\eta(x+1)-\eta(t,x+2),0\right)=-\eta(x+1)\eta(t,x+2).$$

Therefore

$$\mathcal{T}^{\text{det}}_{\text{pairs}}(\mathcal{P}(\eta))_1(x)=\left(1-\eta(x-1)\right)\left(1-\eta(x)\right)-\eta(x+1)\eta\left(x+2\right),$$

which is equal to $\mathcal{P}\left(\mathcal{T}^{\text{det}}_{\text{TASEP}}(\eta)\right)_1(x)$. This proves identity (8). The statement of the Theorem is then proved when the second component is determined by the first one, namely when the terms in (8) are not zero.

Step 2. It remains to prove

$$\mathcal{T}^{\text{det}}_{\text{ABDF}}(\mathcal{P}(\eta))_2=\mathcal{P}\left(\mathcal{T}^{\text{det}}_{\text{TASEP}}(\eta)\right)_2 \tag{9}$$

assuming $\mathcal{T}^{\text{det}}_{\text{ABDF}}(\mathcal{P}(\eta))_1=0$ and $\mathcal{P}\left(\mathcal{T}^{\text{det}}_{\text{TASEP}}(\eta)\right)_1=0$. The first assumption implies that $\mathcal{T}^{\text{det}}_{\text{ABDF}}(\mathcal{P}(\eta))_2$ is either alt_0 or alt_1. The second assumption, that is $\mathcal{P}\left(\mathcal{T}^{\text{det}}_{\text{TASEP}}(\eta)\right)_1=0$, implies that $\mathcal{T}^{\text{det}}_{\text{TASEP}}(\eta)=alt_\alpha$ for some $\alpha=0,1$ and $\mathcal{P}\left(\mathcal{T}^{\text{det}}_{\text{TASEP}}(\eta)\right)_2=alt_\alpha$. Therefore we have to prove

$$\mathcal{T}^{\text{det}}_{\text{ABDF}}(\mathcal{P}(\eta))_2=alt_\alpha$$

where α is the value such that $\mathcal{T}^{\text{det}}_{\text{TASEP}}(\eta)=alt_\alpha$.

Assume $\alpha = 1$. We have to prove that one of the following two conditions hold (see the three conditions at the end of Definition 4, where we have eliminated the last one, impossible because $\omega(t-1,0) = 1$):

$$\mathcal{P}(\eta)_1(0) = -1$$
$$\mathcal{P}(\eta)_2(-1) = 1.$$

If $\mathcal{P}(\eta)_1(0) = 1$, then $\eta(0) = \eta(1) = 0$, hence we need $\eta(-1) = 1$ to get $\mathcal{T}^{\text{det}}_{\text{TASEP}}(\eta) = alt_1$. But then $\mathcal{P}(\eta)_1(-1) = 0$, $\mathcal{P}(\eta)_2(-1) = 1$, so the second condition hold. If $\mathcal{P}(\eta)_1(0) = 0$, then $\eta(0) \neq \eta(1)$. It cannot be $\eta(0) = 1$, $\eta(1) = 0$, otherwise $\mathcal{T}^{\text{det}}_{\text{TASEP}}(\eta)(1) = 1$, incompatible with $\mathcal{T}^{\text{det}}_{\text{TASEP}}(\eta) = alt_1$. Thus $\eta(0) = 0$, $\eta(1) = 1$, hence we must have $\eta(-1) = 1$ to get $\mathcal{T}^{\text{det}}_{\text{TASEP}}(\eta) = alt_1$. But then $\mathcal{P}(\eta)_1(-1) = 0$, $\mathcal{P}(\eta)_2(-1) = 1$, as above. The case $\alpha = 1$ is solved.

Assume $\alpha = 0$, namely $\mathcal{T}^{\text{det}}_{\text{TASEP}}(\eta) = alt_0$. We have to prove that no one of the following two conditions hold:

$$\mathcal{P}(\eta)_1(0) = -1$$
$$\mathcal{P}(\eta)_2(-1) = 1.$$

If by contradiction $\mathcal{P}(\eta)_1(0) = -1$, then $\eta(0) = \eta(1) = 1$, $\mathcal{T}^{\text{det}}_{\text{TASEP}}(\eta)(0) = 1$, incompatible with $\mathcal{T}^{\text{det}}_{\text{TASEP}}(\eta) = alt_0$. If, again by contradiction, $\mathcal{P}(\eta)_2(-1) = 1$, then $\eta(-1) = 1$, $\eta(0) = 0$. Hence again $\mathcal{T}^{\text{det}}_{\text{TASEP}}(\eta)(0) = 1$, incompatible with $\mathcal{T}^{\text{det}}_{\text{TASEP}}(\eta) = alt_0$. □

3.3. **Proof of Theorem 9.** We already stressed the drawback that $ar(\theta)$ is non local. But when $\theta = \mathcal{P}(\eta)$, the expression of $ar(\theta)(x)$ becomes local again when written in terms of η (which depends non-locally on θ). This is a key fact for the proof of Theorem 9. Moreover, we have an expression which unifies the case when η is alternating.

Lemma 16. *If $(\theta, act) = \mathcal{P}(\eta)$, then*

$$act(x) = \eta(x)(1 - \eta(x+1)).$$

In plain words, $act(x)$ is equal to one if and only if $\eta(x) = 1$, $\eta(x+1) = 0$, namely there is a particle at x and the position $x+1$ is free (hence the particle may jump).

Proof. Let us treat separately the case when $\eta = alt_\alpha$, $\alpha = 0, 1$. In this case $act = \eta$; and also $\eta(x)(1 - \eta(x+1)) = \eta(x)$, because if $\eta(x+1) = 0$ it is true, while if $\eta(x+1) = 1$ we necessarily have $\eta(x) = 0$ by alternation, which coincides with $\eta(x)(1 - \eta(x+1))$. The formula of the lemma is proved in this particular case.

Assume now η different from alt_α, $\alpha = 0, 1$, so that $act = ar(\theta)$. Recall the definition of $ar(\theta)(x)$ in Definition 3, points (iii)-(vi). Let x_0 be such that $\theta(x_0) \neq 0$. It means that $\eta(x_0) = \eta(x_0+1)$, both equal to 0 or 1. In both cases $\eta(x_0)(1 - \eta(x_0+1)) = 0$, hence equal to $ar(\theta)(x_0)$ as defined in Definition 3 point (iii).

Assume now $\theta(x_0) = 0$, from which $\eta(x_0) \neq \eta(x_0+1)$ and thus the pair $(\eta(x_0), \eta(x_0+1))$ is either $(1,0)$ or $(0,1)$. Assume that the set $L(x_0)$ is non empty and let x_1 be its maximum and let $k > 0$ be such that $x_1 + k = x_0$. The proof can be divided into several cases depending on the value of $\theta(x_1)$ and the parity of k. For instance, assume $\theta(x_1) = 1$, k odd. Thus $\eta(x_1) = \eta(x_1+1) = 0$, $\eta(x_1+2) = 1$, $\eta(x_1+3) = 0$, and so on, hence $\eta(x_0) = \eta(x_1+k) = 0$, and $\eta(x_0+1) = 1$. In this case

$$\eta(x_0)(1 - \eta(x_0+1)) = 0$$

and (from Definition 3 point (iv))

$$ar(\theta)(x_0) = \left| \frac{\theta(x_1) + (-1)^k}{2} \right| = 0$$

so they coincide. If $\theta(x_1) = 1$, k even,

$$\eta(x_0)(1 - \eta(x_0 + 1)) = 1, \quad ar(\theta)(x_0) = \left|\frac{\theta(x_1) + (-1)^k}{2}\right| = 1,$$

so they coincide. The reader can check the two cases with $\theta(x_1) = 0$. If $L(x_0)$ is empty and $R(x_0)$ is non empty, the arguments are similar. □

Proof of Theorem 9. **Step 1**. In this step, as in the deterministic case above, we prove the identity between the first components:

$$\mathcal{T}_{\text{ABDF}}(t, \omega, \mathcal{P}(\eta))_1 = \mathcal{P}(\mathcal{T}_{\text{TASEP}}(t, \omega, \eta))_1 . \tag{10}$$

Let $\eta \in \{0,1\}^{\mathbb{Z}}$, $t \in \mathbb{N}$, $\omega \in \Omega$, be given and write $\theta := \mathcal{P}(\eta)_1$, $\widehat{\eta} := \mathcal{T}_{\text{TASEP}}(t, \omega, \eta)$, $\widehat{\theta} := \mathcal{P}(\widehat{\eta})_1$, $\widetilde{\theta} := \mathcal{T}_{\text{ABDF}}(t, \omega, \theta)_1$. We have to prove $\widehat{\theta} = \widetilde{\theta}$.

From Definition 6 and Definition 5,

$$\begin{aligned}\widehat{\theta}(x) = 1 - \widehat{\eta}(x) - \widehat{\eta}(x+1) &= 1 - [\varkappa(t,x)\eta(x) + \omega(t,x)\eta(x+1)]\eta(x)\\ &- [\varkappa(t,x-1)\eta(x) + \omega(t,x-1)\eta(x-1)](1-\eta(x))\\ &- [\varkappa(t,x+1)\eta(x+1) + \omega(t,x+1)\eta(t,x+2)]\eta(x+1)\\ &- [\varkappa(t,x)\eta(x+1) + \omega(t,x)\eta(x)](1-\eta(x+1)).\end{aligned}$$

It simplifies, for instance, to

$$\begin{aligned}\widehat{\theta}(x) &= [1 - \omega(t,x-1)\eta(x-1)](1-\eta(x))\\ &\quad - [\varkappa(t,x+1) + \omega(t,x+1)\eta(t,x+2)]\eta(x+1),\end{aligned}$$

because $\eta(x)\eta(x) = \eta(x)$, $\eta(x)(1-\eta(x)) = 0$, $\varkappa(t,x) + \omega(t,x) = 1$.

Concerning $\widetilde{\theta}$, from Definitions 4 and 6,

$$\begin{aligned}\widetilde{\theta}(x) &= \max(\theta(x-1), 0) + act(x-1)\varkappa(t,x-1)\\ &\quad + \min(\theta(x+1), 0) - act(x+1)\varkappa(t,x+1),\end{aligned}$$

where, by definition of $\mathcal{P}(\eta)_1$ and from Lemma 16,

$$\theta(x) = 1 - \eta(x) - \eta(x+1), \quad act(x) = \eta(x)(1 - \eta(x+1)).$$

We have, as in the deterministic case,

$$\begin{aligned}\max(1 - \eta(x-1) - \eta(x), 0) &= (1 - \eta(x-1))(1-\eta(x)),\\ \min(1 - \eta(x+1) - \eta(t,x+2), 0) &= -\eta(x+1)\eta(t,x+2).\end{aligned}$$

Moreover,

$$act(x-1) = \eta(x-1)(1-\eta(x)), \quad act(x+1) = \eta(x+1)(1-\eta(x+2)).$$

Hence

$$\begin{aligned}\widetilde{\theta}(x) &= (1-\eta(x-1))(1-\eta(x)) + \eta(x-1)(1-\eta(x))\varkappa(t,x-1)\\ &\quad - \eta(x+1)\eta(t,x+2) - \eta(x+1)(1-\eta(t,x+2))\varkappa(t,x+1)\\ &= 1 - \eta(x) - \eta(x-1)(1-\eta(x))\omega(t,x-1)\\ &\quad - \eta(x+1)\varkappa(t,x+1) - \eta(x+1)\eta(t,x+2)\omega(t,x+1)\end{aligned}$$

which is equal to $\widehat{\theta}(x)$.

Step 2. We now prove the identity between second components:

$$\mathcal{T}_{\text{ABDF}}(t, \omega, \mathcal{P}(\eta))_2 = \mathcal{P}(\mathcal{T}_{\text{TASEP}}(t, \omega, \eta))_2 . \tag{11}$$

This identity is obviously true when the two elements of (10) are not zero, because both the terms of (11) are defined as the activation map of the corresponding terms of (10). Thus it remains to prove identity (11) when

$$\mathcal{T}_{\text{ABDF}}(t, \omega, \mathcal{P}(\eta))_1 = 0, \quad \mathcal{P}(\mathcal{T}_{\text{TASEP}}(t, \omega, \eta))_1 = 0.$$

Condition $\mathcal{P}\left(\mathcal{T}_{\text{TASEP}}\left(t,\omega,\eta\right)\right)_1 = 0$ implies $\mathcal{T}_{\text{TASEP}}\left(t,\omega,\eta\right) = alt_\alpha$ for some $\alpha = 0, 1$ and $\mathcal{P}\left(\mathcal{T}_{\text{TASEP}}\left(t,\omega,\eta\right)\right)_2 = alt_\alpha$. Therefore we have to prove

$$\mathcal{T}_{\text{ABDF}}\left(t,\omega,\mathcal{P}(\eta)\right)_2 = alt_\alpha.$$

The proof is similar to the deterministic case but more intricate, hence we split it in two more steps.

Step 3. We continue the proof of Step 2 assuming $\alpha = 1$. We have to prove that one of the following three conditions hold (see the three conditions at the end of Definition 4):

$$\begin{aligned} \mathcal{P}(\eta)_1(0) &= -1; \\ \mathcal{P}(\eta)_2(-1) = 1 \quad &\text{and} \quad \omega\left(t-1,-1\right) = 1; \\ \mathcal{P}(\eta)_2(0) = 1 \quad &\text{and} \quad \omega\left(t-1,0\right) = 0. \end{aligned} \tag{12}$$

If the first one is true, the proof is complete. Otherwise we have $\mathcal{P}(\eta)_1(0) = 1$ or $\mathcal{P}(\eta)_1(0) = 0$. Let us prove that $\mathcal{P}(\eta)_1(0) = 1$ implies the second condition; and that $\mathcal{P}(\eta)_1(0) = 0$ implies either the second or third conditions.

Thus assume $\mathcal{P}(\eta)_1(0) = 1$. In this case $\eta(0) = \eta\left(1\right) = 0$, hence we need $\eta(-1) = 1$ and $\omega\left(t-1,-1\right) = 1$ to get $\mathcal{T}_{\text{TASEP}}\left(t,\omega,\eta\right) = alt_1$. But then, from $\eta(-1) = 1$, $\eta(0) = 0$ and $\omega\left(t-1,-1\right) = 1$ we deduce $\mathcal{P}(\eta)_1(-1) = 0$ and $\mathcal{P}(\eta)_2\left(-1\right) = 1$ (Lemma 16), so the second condition hold true.

If $\mathcal{P}(\eta)_1(0) = 0$, then $\eta(0) \neq \eta\left(1\right)$. It cannot be $\eta\left(0\right) = 1, \eta\left(1\right) = 0, \omega\left(t-1,0\right) = 1$, otherwise $\mathcal{T}_{\text{TASEP}}\left(t,\omega,\eta\right)\left(1\right) = 1$, incompatible with $\mathcal{T}_{\text{TASEP}}\left(t,\omega,\eta\right) = alt_1$. Thus: i) either $\eta(0) = 0$, $\eta\left(1\right) = 1$; ii) or $\eta(0) = 1$, $\eta\left(1\right) = 0$, $\omega\left(t-1,0\right) = 0$. In case (i), we must have $\eta(-1) = 1$ and $\omega\left(t-1,-1\right) = 1$ to get $\mathcal{T}_{\text{TASEP}}\left(t,\omega,\eta\right) = alt_1$; in this case the conclusion is, as above, that the second condition holds true. In case (ii) we have $\mathcal{P}(\eta)_1(0) = 0$, $\mathcal{P}(\eta)_2(0) = 1$ (Lemma 16) and of course $\omega\left(t-1,0\right) = 0$, hence the last of the three conditions hold. The case $\alpha = 1$ is solved.

Step 4. We finally continue the proof of Step 2 assuming $\alpha = 0$, namely $\mathcal{T}_{\text{TASEP}}\left(t,\omega,\eta\right) = alt_0$. We have to prove that no one of the three conditions (12) hold. We argue by contradiction, observing that

$$\mathcal{P}(\eta)_1(0) = -1 \;\Rightarrow\; \eta(0) = \eta\left(1\right) = 1 \;\Rightarrow\; \mathcal{T}_{\text{TASEP}}\left(t,\omega,\eta\right)\left(0\right) = 1,$$

$$\begin{cases} \mathcal{P}(\eta)_2(-1) = 1 \\ \omega\left(t-1,-1\right) = 1 \end{cases} \;\Rightarrow\; \eta(-1) = 1, \eta(0) = 0 \;\Rightarrow\; \mathcal{T}_{\text{TASEP}}\left(t,\omega,\eta\right)\left(0\right) = 1,$$

$$\begin{cases} \mathcal{P}(\eta)_2(0) = 1 \\ \omega\left(t-1,0\right) = 0 \end{cases} \;\Rightarrow\; \eta(0) = 1, \eta\left(1\right) = 0 \;\Rightarrow\; \begin{cases} \mathcal{T}_{\text{TASEP}}\left(t,\omega,\eta\right)\left(0\right) = 1 \\ \mathcal{T}_{\text{TASEP}}\left(t,\omega,\eta\right)\left(1\right) = 0 \end{cases},$$

where conditions on the right are incompatible with $\mathcal{T}_{\text{TASEP}}\left(t,\omega,\eta\right) = alt_0$. This completes the proof in case $\alpha = 0$, the last one to be checked. □

We complete this section with a simple corollary of Theorem 9 which is not easy to prove directly on ABDF dynamics. In plain words it says that a point x_0 where a pair coalesces, cannot be the origin of a pair, at the same time of the coalescence.

Corollary 17. *Let $x_0 \in \mathbb{Z}$, $(\theta, act) \in \Lambda$ be such that*

$$\theta\left(x_0 - 1\right) = 1, \theta\left(x_0\right) = 0, \theta\left(x_0 + 1\right) = -1.$$

Then, for every $(t,\omega) \in \mathbb{N} \times \Omega$

$$\mathcal{T}_{ABDF}\left(t,\omega,(\theta,act)\right)_1\left(x_0\right) = 0, \quad \mathcal{T}_{ABDF}\left(t,\omega,(\theta,act)\right)_2\left(x_0\right) = 0.$$

The same result holds if one or both $x_0 - 1, x_0 + 1$ are arising pair point for θ.

Proof. Let η be such that $(\theta, act) = \mathcal{P}(\eta)$. By hypothesis

$$\eta(x_0 - 1) = 0, \eta(x_0) = 0, \eta(x_0 + 1) = 1, \eta(x_0 + 2) = 1.$$

TASEP dynamics cannot change the values at x_0 and $x_0 + 1$, hence

$$\mathcal{T}_{\text{TASEP}}(t, \omega, \eta)(x_0) = 0, \mathcal{T}_{\text{TASEP}}(t, \omega, \eta)(x_0 + 1) = 1.$$

This implies the result in the first case.

Now, assume $x_0 - 1$ is an arising pair point for $((\theta, act), \omega)$, and $\theta(x_0) = 0$, $\theta(x_0 + 1) = -1$. We now have

$$\eta(x_0 - 1) = 1, \quad \eta(x_0) = 0, \quad \eta(x_0 + 1) = 1, \quad \eta(x_0 + 2) = 1$$
$$\omega(t - 1, x_0 - 1) = 0.$$

Again TASEP dynamics does not change the values at x_0 and $x_0 + 1$. The same argument applies to the case when x_0+1 is an arising pair point for $((\theta, act), \omega)$. □

4. Pure-Jump Weak Solutions of Burgers' Equation

We consider in this section Burgers' equation (4),

$$\partial_t u + u\partial_x u = 0.$$

We are interested in bounded (non entropic!) weak solutions, so we restrict the definition to bounded functions, although it could be more general.

Definition 18. *We say that a bounded measurable function $u : [t_0, t_1] \times \mathbb{R} \to \mathbb{R}$ is a weak solution on $[t_0, t_1]$ if:*

i) *for every smooth test function $\varphi : \mathbb{R} \to \mathbb{R}$ with compact support in $\mathbb{R}$ the function $t \mapsto \int_{\mathbb{R}} u(t, x) \varphi(x)dx$ is continuous on $[t_0, t_1]$;*
ii) *for every smooth test function $\phi : \mathbb{R}^2 \to \mathbb{R}$ with compact support in $(t_0, t_1) \times \mathbb{R}$, we have*

$$\int_{t_0}^{t_1} \int_{\mathbb{R}} \left(u(t, x) \partial_t \phi(t, x) + \frac{1}{2} u^2(t, x) \partial_x \phi(t, x) \right) dxdt = 0.$$

Given a test function φ, the function $t \mapsto \int_{\mathbb{R}} u(t, x) \varphi(x)dx$ is always defined almost everywhere, by Fubini-Tonelli theorem. We require that it is continuous, for a minor reason appearing in the next Proposition. It is not restrictive in our examples.

In the sequel we shall piece together weak solutions defined on different space-time domains. Let us see two rules. When we say that $u(\bar{t}, \cdot) = v(\bar{t}, \cdot)$ for a certain $\bar{t} \in [t_0, t_1]$ we mean that $\int_{\mathbb{R}} u(\bar{t}, x) \varphi(x)dx = \int_{\mathbb{R}} v(\bar{t}, x) \varphi(x)dx$ for all test functions φ of the class above.

Proposition 19. *Assume $u(t, x)$ is a weak solution on $[t_0, t_1]$ and $v(t, x)$ a weak solution on $[t_1, t_2]$, with $u(t_1, \cdot) = v(t_1, \cdot)$. Then the function w, defined on $[t_0, t_2]$, equal to u on $[t_0, t_1]$ and v on $[t_1, t_2]$, is a weak solution on $[t_0, t_2]$.*

Proof. Let $\varphi : \mathbb{R} \to \mathbb{R}$ with compact support in $\mathbb{R}$. Consider the function $t \mapsto \int_{\mathbb{R}} w(t, x) \varphi(x)dx$, defined a.s. by Fubini-Tonelli theorem. By the continuity of the function $t \mapsto \int_{\mathbb{R}} u(t, x) \varphi(x)dx$ on $[t_0, t_1]$ and of $t \mapsto \int_{\mathbb{R}} v(t, x) \varphi(x)dx$ on $[t_1, t_2]$ and by the property $u(t_1, \cdot) = v(t_1, \cdot)$, we deduce that $t \mapsto \int_{\mathbb{R}} w(t, x) \varphi(x)dx$ is continuous.

Given ϕ as in the definition, part (ii), introduce

$$\phi_n(t, x) = \phi(t, x)(1 - \chi_n(t - t_1))$$

where $\chi_n(s) = \chi(ns)$ and χ is smooth, $\chi(x) = \chi(-x)$, with values in $[0,1]$, equal to 1 in $[-1,1]$, to zero outside $[-2,2]$; and take n large enough. The function $\phi_n(t,x)$, restricted to $(t_0,t_1)\times\mathbb{R}$, is a good test function for u, hence

$$\int_{t_0}^{t_1}\int_{\mathbb{R}}\left(u(t,x)\,\partial_t\phi_n(t,x)+\frac{1}{2}u^2(t,x)\,\partial_x\phi_n(t,x)\right)dxdt=0.$$

Similarly on $(t_1,t_2)\times\mathbb{R}$ for v, hence

$$\int_{t_0}^{t_2}\int_{\mathbb{R}}\left(w(t,x)\,\partial_t\phi_n(t,x)+\frac{1}{2}w^2(t,x)\,\partial_x\phi_n(t,x)\right)dxdt=0.$$

The same identity holds for ϕ, completing the proof, if we show that

$$\lim_{n\to\infty}\int_{t_0}^{t_2}\int_{\mathbb{R}}w(t,x)\,\phi(t,x)\,\partial_t\chi_n(t-t_1)\,dxdt=0,$$
$$\lim_{n\to\infty}\int_{t_0}^{t_2}\int_{\mathbb{R}}w(t,x)\,\partial_t\phi(t,x)\,\chi_n(t-t_1)\,dxdt=0,$$
$$\lim_{n\to\infty}\int_{t_0}^{t_2}\int_{\mathbb{R}}w^2(t,x)\,\chi_n(t-t_1)\,\partial_x\phi(t,x)\,dxdt=0.$$

The second and third limits are clear. The first claim is equivalent to

$$\lim_{n\to\infty}\int_{t_1-\frac{2}{n}}^{t_1+\frac{2}{n}}n\chi'(n(t-t_1))\left(\int_{\mathbb{R}}w(t,x)\,\phi(t,x)\,dx\right)dt=0.$$

It is easy, using also the boundedness of w, to show that the function

$$t\mapsto\int_{\mathbb{R}}w(t,x)\,\phi(t,x)\,dx$$

is continuous (approximate ϕ by functions piecewise constant in t). With a similar argument we can replace this function by a constant in the previous limit and thus reduce us to check the property

$$\lim_{n\to\infty}\int_{-\frac{2}{n}}^{\frac{2}{n}}n\chi'(nt)\,dt=0$$

(we have also changed variables). But this means $\lim_{n\to\infty}\int_{-2}^{2}\chi'(s)\,ds=0$, which is true by symmetry of χ. □

Proposition 20. *Assume that u,v are two weak solutions, on $[t_0,t_1]$, with disjoint supports, namely there are sets $S_u,S_v\subset[t_0,t_1]\times\mathbb{R}$, disjoint, Borel measurable, such that $u=0$ a.s. outside S_u and $v=0$ a.s. outside S_v. Define*

$$w=u+v.$$

Then w is a weak solution. The result remains true when the intersection of the supports has Lebesgue measure zero.

Proof. All properties are easily checked. Notice that $w^2=u^2+v^2$ almost everywhere. □

Let us recall the definition of the Heaviside function and its formal weak derivative

$$H(x)=1_{[0,\infty)}(x)=1_{\{x\geq 0\}},\quad H'(x)=\delta_0(x)=\delta(x=0).$$

Given $t_0\in\mathbb{R}$, the simplest example of a pure-jump weak solution —which is not an entropy solution— of Burgers' equation on $[t_0,t_1]$, for any $t_1>t_0$, is

$$u(t,x)=1_{\{x\geq x_0+v(t-t_0)\}}w=1_{\{x-x_0-v(t-t_0)\geq 0\}}w=H(x-x_0-v(t-t_0))\,w$$

for $t \in [t_0, t_1]$, and with $v > 0$ and $w = 2v$. Part (i) of Definition 18 comes from

$$\int_{\mathbb{R}} u(t,x)\,\varphi(x)dx = \int_{x_0+v(t-t_0)}^{\infty} w\varphi(x)dx,$$

and this will be the case in all examples below, hence we shall not repeat it. The rigorous proof of (ii) of Definition 18 is:

$$\begin{aligned}
&\int_{t_0}^{t_1}\int_{\mathbb{R}}\left(u(t,x)\,\partial_t\phi(t,x) + \frac{1}{2}u^2(t,x)\,\partial_x\phi(t,x)\right)dxdt\\
&= \int_{t_0}^{t_1}\int_{\mathbb{R}}\left(1_{\{x\geq x_0+v(t-t_0)\}}w\partial_t\phi(t,x) + \frac{1}{2}1_{\{x\geq x_0+v(t-t_0)\}}w^2\partial_x\phi(t,x)\right)dxdt\\
&= w\int_{\mathbb{R}}\int_{t_0}^{\left(t_0+\frac{x-x_0}{v}\right)\wedge t_1}\partial_t\phi(t,x)\,dtdx + \frac{w^2}{2}\int_{t_0}^{t_1}\int_{x_0+v(t-t_0)}^{\infty}\partial_x\phi(t,x)\,dxdt\\
&= w\int_{\mathbb{R}}\phi\left(\left(t_0+\frac{x-x_0}{v}\right)\wedge t_1, x\right)dx - \frac{w^2}{2}\int_{t_0}^{t_1}\phi(t, x_0+v(t-t_0))\,dt = 0,
\end{aligned}$$

(making use of $\operatorname{supp}\phi \subseteq (t_0,t_1)\times\mathbb{R}$ in the second step to ignore the case in which $\left(t_0+\frac{x-x_0}{v}\right)\wedge t_1 = t_0+\frac{x-x_0}{v} < t_0$), since it holds

$$\begin{aligned}
w\int_{\mathbb{R}}\phi\left(\left(t_0+\frac{x-x_0}{v}\right)\wedge t_1, x\right)dx &= w\int_{x_0}^{x_0+v(t_1-t_0)}\phi\left(t_0+\frac{x-x_0}{v}, x\right)dx\\
&= wv\int_{t_0}^{t_1}\phi(t, x_0+v(t-t_0))\,dt,
\end{aligned}$$

where the first equality uses $\operatorname{supp}\phi \subset (t_0,t_1)\times\mathbb{R}$, the second one is obtained by substituting $x = x_0 + v(t-t_0)$, and it holds $vw = \frac{w^2}{2}$. The above computation is quite long already in the case of the Heaviside solution, not to mention the next cases below, let us show how to verify a little bit formally the same result in a few, more transparent, lines (we use the representation by Heaviside function to take the derivative):

$$\begin{gathered}
\partial_t u = -\delta_0(x-x_0-v(t-t_0))\,wv,\\
u^2(t,x) = u(t,x)\,w, \qquad \partial_x u^2 = \delta_0(x-x_0-v(t-t_0))\,w^2,
\end{gathered}$$

which, by $w^2 = 2vw$, implies $\partial_x u^2 = -2\partial_t u$.

4.1. **Isolated quasi-particles.** Given $h > 0$ (it will be typically small in our main results), $v > 0$, $t_1 > t_0$, we call *right-quasi-particle* on $[t_0,t_1]$ a function of the following form: for $(t,x) \in \mathbb{R}\times[t_0,t_1]$, and $w = 2v$,

$$\begin{aligned}
u(t,x) &= 1_{\{x_0+v(t-t_0)-h\leq x<x_0+v(t-t_0)\}}w\\
&= 1_{\{x\geq x_0+v(t-t_0)-h\}}w - 1_{\{x\geq x_0+v(t-t_0)\}}w\\
&= H(x-x_0-v(t-t_0)+h)\,w - H(x-x_0-v(t-t_0))\,w.
\end{aligned}$$

The latter is a weak solution of Burgers' equation on $[t_0,t_1]$:

$$\begin{aligned}
\partial_t u &= -\delta_0(x-x_0-v(t-t_0)+h)\,wv + \delta_0(x-x_0-v(t-t_0))\,wv,\\
u^2(t,x) &= u(t,x)\,w,\\
\partial_x u^2 &= \delta_0(x-x_0-v(t-t_0)+h)\,w^2 - \delta_0(x-x_0-v(t-t_0))\,w^2,
\end{aligned}$$

hence $\partial_x u^2 = -2\partial_t u$. Seen as a soliton, a right-quasi-particle moves to the right with velocity v.

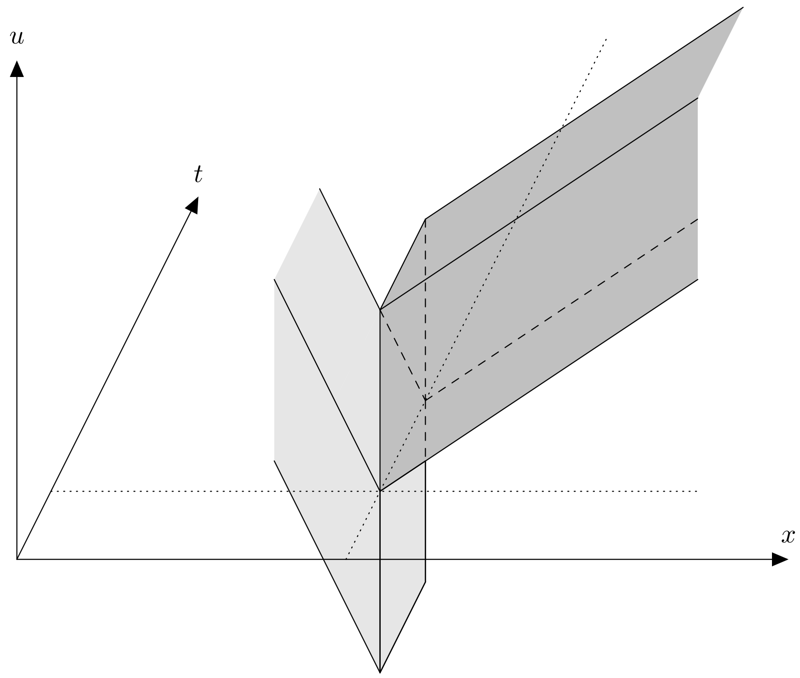

FIGURE 5. Generation of a couple of right and left quasi-particles, in two different shades of gray.

A *left-quasi-particle* on $[t_0, t_1]$ has the form (with $v = w/2 > 0$ as above)

$$\begin{aligned} u(t,x) &= -1_{\{x_0 - v(t-t_0) \le x < x_0 - v(t-t_0)+h\}} w, \\ &= -\left(1_{\{x \ge x_0 - v(t-t_0)\}} - 1_{\{x \ge x_0 - v(t-t_0)+h\}}\right) w, \\ &= -\left(H\left(x - x_0 + v\left(t - t_0\right)\right) - H\left(x - x_0 + v\left(t - t_0\right) - h\right)\right) w. \end{aligned}$$

Seen as a soliton, a left-quasi-particle moves to the left, again with velocity v.

4.2. **Arising pairs of quasi-particles.** In our construction, the traveling solitons defined in the previous subsection usually emerge somewhere and disappear somewhere else, except for a few extremal cases. Thus they have the above form only on certain intervals $[t_0, t_1]$; this is why we stressed everywhere the presence of such an interval.

In this subsection we describe the typical mechanism of emergence of such solitons: they appear in pairs, one positive and one negative, moving in opposite directions. After a short time of order h they become isolated solitons of the form described in the previous subsection. But at the beginning, when they emerge and develop, they are made of two pieces with increasing size smaller than h, as described below, see also Figure 5.

Remark 21. An arising pair comes from the identically zero solution. Hence, before the birth time t_0, we have $u = 0$, which is a weak solution. In the time interval $\left[t_0, t_0 + \frac{h}{v}\right]$ the pair develops. After time $t_0 + \frac{h}{v}$ we have two disjoint isolated quasi-particles, which is again a weak solution, by Proposition 20 and the examples of the previous subsection. Thus, thanks to Proposition 19, it is sufficient to define the arising pair in $\left[t_0, t_0 + \frac{h}{v}\right]$ and to prove that it is a weak solution there.

Given $h>0$, $v>0$ and t_0, we call *arising pair of quasi-particles at* (t_0,x_0), the function defined for $(t,x)\in\left[t_0,t_0+\frac{h}{v}\right]\times\mathbb{R}$ by

$$u(t,x)=1_{\{x_0\le x<x_0+v(t-t_0)\}}w-1_{\{x_0-v(t-t_0)\le x<x_0\}}w$$

with $w=2v$. An expression for continuing the motion of the pair also for times larger than $t_0+\frac{h}{v}$ is

$$u(t,x)=1_{\{x_0+v(t-t_0)-\min(h,v(t-t_0))\le x<x_0+v(t-t_0)\}}w \\ -1_{\{x_0-v(t-t_0)\le x<x_0-v(t-t_0)+\min(h,v(t-t_0))\}}w.$$

Let us treat only the case $t\in\left[t_0,t_0+\frac{h}{v}\right]$. It can also be written as

$$\begin{aligned}u(t,x)&=1_{\{x\ge x_0\}}w-1_{\{x\ge x_0+v(t-t_0)\}}w-\left(1_{\{x\ge x_0-v(t-t_0)\}}-1_{\{x\ge x_0\}}\right)w\\&=H(x-x_0)w-H(x-x_0-v(t-t_0))w\\&\quad-(H(x-x_0+v(t-t_0))-H(x-x_0))w.\end{aligned}$$

It is a weak solution of Burgers' equation: indeed it holds

$$\partial_t u=\delta_0(x-x_0-v(t-t_0))wv-\delta_0(x-x_0+v(t-t_0))wv$$

and, using the fact that the two pieces have disjoint support,

$$\begin{aligned}u^2(t,x)&=1_{\{x_0\le x\le x_0+v(t-t_0)\}}w^2+1_{\{x_0-v(t-t_0)\le x\le x_0\}}w^2\\&=H(x-x_0)w^2-H(x-x_0-v(t-t_0))w^2\\&\quad+H(x-x_0+v(t-t_0))w^2-H(x-x_0)w^2,\\ \partial_x u^2&=\delta_0(x-x_0)w^2-\delta_0(x-x_0-v(t-t_0))w^2\\&\quad+\delta_0(x-x_0+v(t-t_0))w^2-\delta_0(x-x_0)w^2=-2\partial_t u.\end{aligned}$$

4.3. **Coalescing pairs of quasi-particles.** In our model, as anticipated above, usually a quasi-particle meets after a short time another quasi-particle traveling in the opposite direction: in this case they annihilate each other. This process is described by the following solution: given $h>0$, $v=w/2>0$ and t_1, we call *coalescing pair of quasi-particles at* (t_1,x_0), the function defined for $(t,x)\in\left[t_1-\frac{h}{v},t_1\right]\times\mathbb{R}$ by

$$u(t,x)=1_{\{x_0-v(t_1-t)\le x<x_0\}}w-1_{\{x_0\le x<x_0+v(t_1-t)\}}w. \tag{13}$$

The proof that it is a weak solution is the same as for arising quasi-particles: in fact one can also argue by time-reversal of Burgers' equation. Let us check directly:

$$\begin{aligned}u(t,x)&=1_{\{x\ge x_0-v(t_1-t)\}}w-1_{\{x\ge x_0\}}w-\left(1_{\{x\ge x_0\}}-1_{\{x\ge x_0+v(t_1-t)\}}\right)w\\&=H(x-x_0+v(t_1-t))w-H(x-x_0)w\\&\quad-(H(x-x_0)-H(x-x_0-v(t_1-t)))w\\ \partial_t u&=-\delta_0(x-x_0+v(t_1-t))wv+\delta_0(x-x_0-v(t_1-t))wv,\end{aligned}$$

and, using the fact that the two pieces have disjoint support

$$\begin{aligned}u^2(t,x)&=1_{\{x_0-v(t_1-t)\le x<x_0\}}w^2+1_{\{x_0\le x<x_0+v(t_1-t)\}}w^2\\&=H(x-x_0+v(t_1-t))w^2-H(x-x_0)w^2\\&\quad+H(x-x_0)w^2-H(x-x_0-v(t_1-t))w^2,\end{aligned}$$

so that finally

$$\begin{aligned}\partial_x u^2&=\delta_0(x-x_0+v(t_1-t))w^2-\delta_0(x-x_0)w^2\\&\quad+\delta_0(x-x_0)w^2-\delta_0(x-x_0-v(t_1-t))w^2=-2\partial_t u.\end{aligned}$$

Remark 22. The content of the present section is easily adapted to produce weak solutions of

$$\partial_t u = \lambda \partial_x u^2$$

with $\lambda \neq 0$, the latter being the formal derivative of (3). Indeed, it suffices to replace the relation $w = 2v$ between parameters v, w with $w = -\lambda v$.

5. TASEP pairs, ABDF and Burgers quasi-particles

In this section we describe a bijection between TASEP realizations (and therefore ABDF realizations) and realizations of a random weak solution of Burgers' equation. The idea is that we associate special configurations of Burgers solutions to ABDF configurations, using integer times $t_0 \in \mathbb{N}$ for this correspondence; then we interpolate for $t \in [t_0, t_0+1]$ using the special quasi-particle weak solutions defined in the previous section.

This idea however is complicated by a tricky detail. In the ABDF model, creation of new pairs happens at integer times $t_0 - 1$ (without being visible) and is observed only at time t_0: discrete time allows to do so. We refer to Figure 2 for an example: new pairs arise from empty sites (diamond-shaped in the picture). On the contrary, for Burgers' equation, due to continuous time, we need to create new pairs before integer times, so that the pair is fully formed at integer time. The creation instant of a new pair will be at times $t_0 - 1 - \frac{1}{2}$, $t_0 \in \mathbb{N}_0$.

Strictly speaking, the correspondence between TASEP and Burgers' samples is not a conjugation of random dynamical systems, as opposed to the conjugation between TASEP and ABDF, because in order to define the configuration at time $t_0 \in \mathbb{N}$ of the random weak solution of Burgers' equation we need two pieces of information: the configuration of TASEP – or ABDF – at time t_0 and the noise values $\{\omega(t_0, x); x \in \mathbb{Z}\}$. Nevertheless, it is a bijection of stochastic processes, or even more a bijection between their realizations, and thus it may allow to study the behavior of each one of the two processes starting from the other one.

Definition 23. *Given $t_0 \in \mathbb{N}$, $(\theta, act) \in \Lambda$, $\omega \in \Omega$, let us introduce the following sets:*

$$\begin{aligned}
M^+(\theta) &= \{z \in \mathbb{Z} : \theta(z) = 1\} \\
M^-(\theta) &= \{z' \in \mathbb{Z} : \theta(z') = -1\}, \\
A(\theta, act, \omega, t_0) &= \{z \in \mathbb{Z} : act(z) = 1, \omega(t_0, z) = 0\}, \\
MA^+(\theta, act, \omega, t_0) &= M^+(\theta) \cup A(\theta, act, \omega, t_0), \\
MA^-(\theta, act, \omega, t_0) &= M^-(\theta) \cup A(\theta, act, \omega, t_0), \\
C(\theta, act, \omega, t_0) &= \big\{z \in \mathbb{Z} : z - 1 \in MA^+(\theta, act, \omega, t_0), \\
&\qquad z + 1 \in MA^-(\theta, act, \omega, t_0)\big\}.
\end{aligned}$$

(In plain words they are the sets of, respectively, particles Moving to the right, particles Moving to the left, sites of Arising pairs, sites of Moving or Arising, sites of Coalescence).

Given $(\theta, act) \in \Lambda$, $t_0 \in \mathbb{N}$ and $\omega \in \Omega$, we define, for $x \in \mathbb{R}$,

$$\begin{aligned}
u(t_0, x, \omega) = &\ 2 \sum_{z \in M^+(\theta)} 1_{\left\{z \le x < z + \frac{1}{2}\right\}} - 2 \sum_{z \in M^-(\theta)} 1_{\left\{z - \frac{1}{2} \le x < z\right\}} \\
&+ 2 \sum_{z \in A(\theta, act, \omega, t_0)} \left(1_{\left\{z \le x < z + \frac{1}{2}\right\}} - 1_{\left\{z - \frac{1}{2} \le x < z\right\}}\right).
\end{aligned}$$

To explain the definition, assume $(\theta, act) = \mathcal{P}(\eta)$. The formula for $u(t_0, x)$ includes three summands:

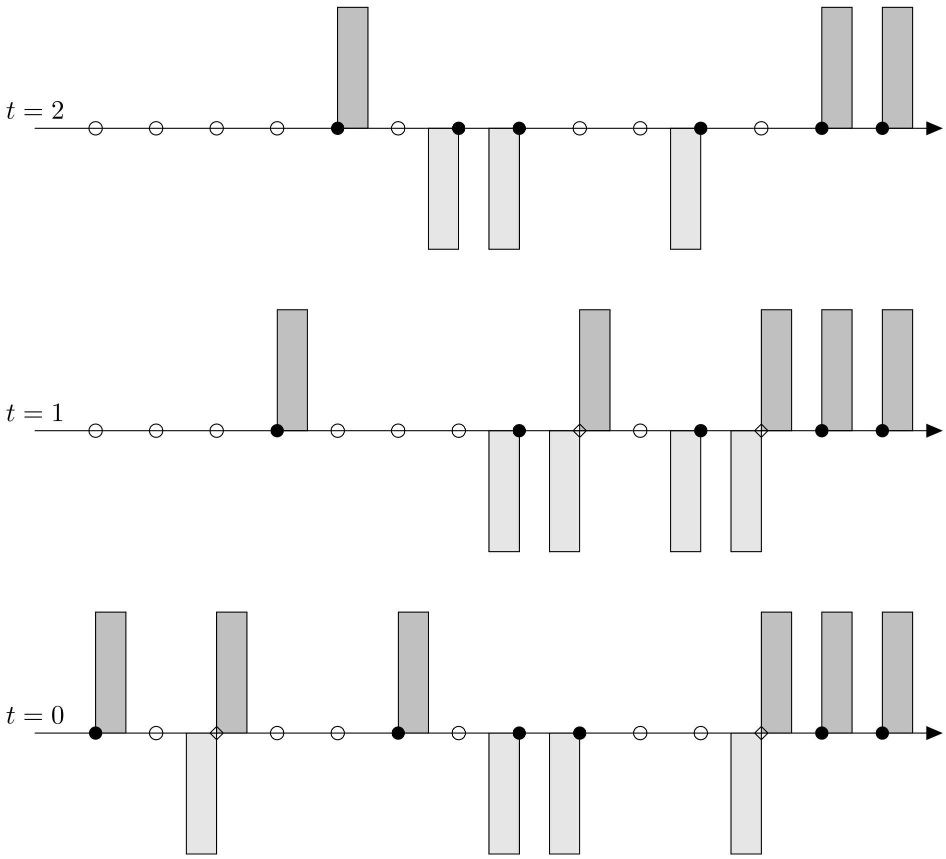

FIGURE 6. Profile of u defined by Definition 23, starting from ABDF configuration at times $t = 0, 1, 2$ of Figure 2.

- a right-quasi-particle
$$2 \cdot 1_{\left\{z \leq x < z + \frac{1}{2}\right\}}$$
at each point z where $\theta(z) = 1$, namely where there is a right-particle of ABDF, or equivalently where TASEP has two consecutive empty spaces;
- a left-quasi-particle
$$-2 \cdot 1_{\left\{z - \frac{1}{2} \leq x < z\right\}}$$
at each point z where $\theta(z) = -1$, namely where there is a left-particle of ABDF, or equivalently where TASEP has two consecutive particles;
- an arising pair of quasi-particles
$$2\left(1_{\left\{z \leq x < z + \frac{1}{2}\right\}} - 1_{\left\{z - \frac{1}{2} \leq x < z\right\}}\right)$$
at each point z where $\theta(z) = 0$, $act(z) = 1$ (or equivalently $\eta(z) = 1$, $\eta(z+1) = 0$) and $\omega(t_0, x) = 0$, namely at each active empty position of ABDF, where the noise prescribes the creation of two particles.

We have chosen $h = \frac{1}{2}$, $v = 1$, and thus $w = 2$, in the definitions of quasi-particles of section 4; the value of h can be changed to any value in $(0, 1)$ without consequences, while the value of v is coordinated with the scheme.

5.1. **Bijection with TASEP at Integer Times.** In the remainder of this section we need to show several facts. The first one is the bijection property between TASEP (or ABDF) realizations and these particular functions $u(t_0, x, \omega)$. This can be formalized in different ways; we limit ourselves to state that, given the function $u(t_0, x, \omega)$, above, we can reconstruct the values of (θ, act). The proof is straightforward, just noticing that each $z \in \mathbb{Z}$ appears at most in one of the sums defining $u(t_0, x, \omega)$.

Proposition 24. *Let $u(t_0, x, \omega)$ be given by Definition 23, with respect to $(\theta, act) \in \Lambda$ and $\omega \in \Omega$. Then*

$$\theta(z) = \int_{z-\frac{1}{2}}^{z+\frac{1}{2}} u(t_0, x, \omega)\, dx.$$

If $\theta \neq 0$, act is $ar(\theta)$ and thus can be reconstructed from u. If $\theta = 0$, for a.e. ω there are infinitely many active points z where $\omega(t_0, z) = 0$; this means that $u(t_0, x, \omega)$ contains infinitely many points z where the jump

$$[u(t_0, \cdot, \omega)]_z := \lim_{x \to z^+} u(t_0, x, \omega) - \lim_{x \to z^-} u(t_0, x, \omega)$$

is equal to 4. If these points are even, $act = alt_1$, otherwise $act = alt_0$.

5.2. **Continuation Shortly After Integer Times.** Next, we have to construct solutions of Burgers'equation interpolating the functions $u(t_0, x, \omega)$ between integer times. The particular weak solutions introduced in section 4 prescribe a unique continuation of $u(t_0, x, \omega)$ from the "initial" value at time $t_0 \in \mathbb{N}$ to all values of t in $\left[t_0, t_0 + \frac{1}{2}\right]$ (the value $\frac{1}{2}$ is related to the choice $h = \frac{1}{2}$):

$$\begin{aligned} u(t, x, \omega) = {} & 2 \sum_{z \in MA^+(\theta, act, \omega, t_0)} 1_{\left\{z + (t - t_0) \leq x < z + (t - t_0) + \frac{1}{2}\right\}} \\ & - 2 \sum_{z' \in MA^-(\theta, act, \omega, t_0)} 1_{\left\{z' - (t - t_0) - \frac{1}{2} \leq x < z' - (t - t_0)\right\}}. \end{aligned}$$

Proposition 25. *The function thus defined for $t \in \left[t_0, t_0 + \frac{1}{2}\right]$, $x \in \mathbb{R}$ is a weak solution of Burgers' equation.*

Proof. Coincidence of the last formula at time t_0 with the initial condition $u(t_0, x, \omega)$ above is obvious. The statement is a consequence of a simple fact: every pair of terms taken from the two sums defining $u(t, x, \omega)$ is made of quasi-particles with disjoint supports on $\left[t_0, t_0 + \frac{1}{2}\right]$, and thus the sum solves Burgers' equation in weak sense –by Proposition 20 – on the interval $\left[t_0, t_0 + \frac{1}{2}\right]$.

Let us check that supports are disjoint. Quasi-particles moving to the right (those corresponding to the first sum) are clearly isolated between themselves, having a "support" of size $\frac{1}{2}$ of the form $[x(t), x(t) + \frac{1}{2})$ with $x(t)$ of the form $z + (t - t_0)$ with z of distance at least one from each other. The same holds for left-quasi-particles, among themselves. Thus the problem is only about the interaction between a right-quasi-particle

$$2 \cdot 1_{\left\{z + (t - t_0) \leq x < z + (t - t_0) + \frac{1}{2}\right\}}$$

and a left-quasi-particle

$$-2 \cdot 1_{\left\{z' - (t - t_0) - \frac{1}{2} \leq x < z' - (t - t_0)\right\}}$$

with z in the first sum and z' in the second one. The supports of these two solitons have size $\frac{1}{2}$ and are of the form $[x(t), x(t) + \frac{1}{2})$ with $x(t) = z + (t - t_0)$ and $[x'(t) - \frac{1}{2}, x'(t))$ with $x'(t) = z' - (t - t_0)$, respectively. We claim that these supports are disjoint, for $t \in \left[t_0, t_0 + \frac{1}{2}\right]$. If $z' \leq z$ this is clear, since $x'(t)$ is decreasing and $x(t)$ is increasing. When $z' > z$ we claim that sets $[x(t), x(t) + \frac{1}{2})$

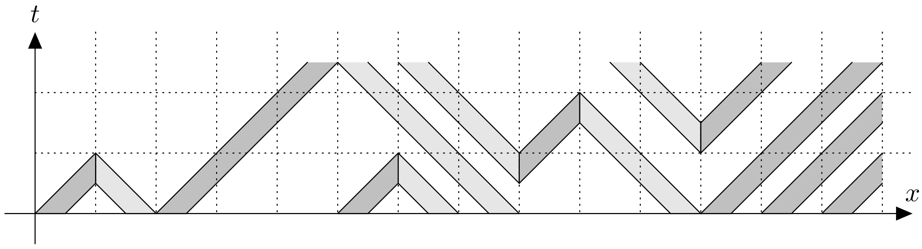

FIGURE 7. Evolution of u considered in Proposition 25 and Proposition 26 built upon ABDF evolution of Figure 2. Two shades of gray denote, as above, right and left quasi-particles. The dotted grid has side length 1, so $u = \pm 2$ respectively on dark and light gray areas.

and $\left[x'(t) - \frac{1}{2}, x'(t)\right)$ are disjoint because $(t - t_0) + \frac{1}{2} \leq 1$ and $z' \geq z + 2$ (to be shown below) and thus

$$z' - (t - t_0) - \frac{1}{2} \geq z + (t - t_0) + \frac{1}{2}.$$

The key fact $z' \geq z+2$ requires inspection into the conditions that z and z' belong to two different sums. Recall we are treating the case $z' > z$, hence the two solitons are not the result of an arising pair.

We have $z \in MA^+(\theta, act, \omega, t_0)$. This is the union of two cases. Consider the case $\theta(z) = 1$ and assume by contradiction that $z' = z+1$. By the rules of Λ, $\theta(z')$ cannot be -1 (because the number of integer points strictly between z and z' is even); by the rules of the map ar, if $\theta(z') = 0$, $ar(\theta)(z')$ is zero. Hence we have found a contradiction.

Consider the case $act(z) = 1$ and assume by contradiction that $z' = z+1$. Again by the rules of Λ we cannot have $\theta(z') = -1$ and we cannot have $act(z') = 1$. Hence, we get a contradiction also in this case, and this proves $z' \geq z + 2$. □

5.3. **Continuation Shortly Before Integer Times.**

To continue the solution in time intervals $t \in \left[t_0 + \frac{1}{2}, t_0 + 1\right]$ is somewhat trickier because of two phenomena: coalescence of quasi-particles, and growth of new pairs. We are now at time $t_0+1/2$, namely

$$u\left(t_0 + \frac{1}{2}, x, \omega\right) = 2 \sum_{z \in MA^+(\theta, act, \omega, t_0)} 1_{\left\{z+\frac{1}{2} \leq x < z+1\right\}} - 2 \sum_{z' \in MA^-(\theta, act, \omega, t_0)} 1_{\left\{z'-1 \leq x < z' - \frac{1}{2}\right\}}.$$

The continuation depends on this configuration and on the section of the noise

$$\{\omega(t_0 + 1, x)\,; x \in \mathbb{Z}\}.$$

Indeed, at time $t_0 + 1$ we could observe the result of arising pairs, as it was above at time t_0. These pairs started existing at time $t_0 + \frac{1}{2}$.

Let us also write explicitly where we want to arrive at at time $t_0 + 1$: called

$$(\theta', act') = \mathcal{T}_{\mathrm{ABDF}}(t_0, \omega, (\theta, act))$$

we want to have

$$u(t_0+1,x,\omega) := 2\sum_{z\in M^+(\theta')} 1_{\left\{z\le x<z+\frac{1}{2}\right\}} - 2\sum_{z\in M^-(\theta')} 1_{\left\{z-\frac{1}{2}\le x<z\right\}} + 2\sum_{z\in A(\theta',act',\omega,t_0+1)} \left(1_{\left\{z\le x<z+\frac{1}{2}\right\}} - 1_{\left\{z-\frac{1}{2}\le x<z\right\}}\right).$$

Proposition 26. *Given the sets defined above, let us introduce also*

$$MA^+_{iso}(\theta,act,\omega,t_0) = \left\{z\in MA^+(\theta,act,\omega,t_0) : z+1\notin C(\theta,act,\omega,t_0)\right\},$$
$$MA^-_{iso}(\theta,act,\omega,t_0) = \left\{z\in MA^-(\theta,act,\omega,t_0) : z-1\notin C(\theta,act,\omega,t_0)\right\}.$$

Consider the functions $u\left(t_0+\frac{1}{2},x,\omega\right)$ *and* $u(t_0+1,x,\omega)$ *we just defined. The following function* $u(t,x,\omega)$, $t\in\left[t_0+\frac{1}{2},t_0+1\right]$, *interpolates between them and is a weak solution of Burgers' equation: we set, for* $t\in\left[t_0+\frac{1}{2},t_0+1\right]$,

$$u(t,x,\omega) = u_{iso}(t,x,\omega) + u_{coa}(t,x,\omega) + u_{ari}(t,x,\omega),$$

where u_{iso} *collects* isolated *quasi-particles,*

$$u_{iso}(t,x,\omega) = \sum_{z\in MA^+_{iso}(\theta,act,\omega,t_0)} u^{(z,+)}_{iso}(t,x,\omega) + \sum_{z'\in MA^-_{iso}(\theta,act,\omega,t_0)} u^{(z',-)}_{iso}(t,x,\omega),$$

$$u^{(z,+)}_{iso}(t,x,\omega) = 2\cdot 1_{\left\{z+(t-t_0)\le x<z+(t-t_0)+\frac{1}{2}\right\}},$$

$$u^{(z',-)}_{iso}(t,x,\omega) = -2\cdot 1_{\left\{z'-(t-t_0)-\frac{1}{2}\le x<z'-(t-t_0)\right\}},$$

u_{coa} *the* coalescing *quasi-particles,*

$$u_{coa}(t,x,\omega) = \sum_{x_0\in C(\theta,act,\omega,t_0)} u^{(x_0)}_{coa}(t,x,\omega),$$

$$u^{(x_0)}_{coa}(t,x,\omega) = 2\cdot 1_{\{x_0-(t_0+1-t)\le x<x_0\}} - 2\cdot 1_{\{x_0\le x<x_0+(t_0+1-t)\}},$$

and finally, with $(\theta',act') = \mathcal{T}_{ABDF}(t_0,\omega,(\theta,act))$, u_{ari} *corresponds to* arising *pairs,*

$$u_{ari}(t,x,\omega) = \sum_{x_0\in A(\theta',act',\omega,t_0+1)} u^{(x_0)}_{ari}(t,x,\omega),$$

$$u^{(x_0)}_{ari}(t,x,\omega) = 2\cdot 1_{\left\{x_0\le x<x_0+\left(t-t_0-\frac{1}{2}\right)\right\}} - 2\cdot 1_{\left\{x_0-\left(t-t_0-\frac{1}{2}\right)\le x<x_0\right\}}.$$

Proof. The fact that, for almost all x, $(u_{iso}+u_{coa}+u_{ari})(t,x,\omega)$ coincides with the functions $u\left(t_0+\frac{1}{2},x,\omega\right)$ and $u(t_0+1,x,\omega)$ defined above can be easily checked – notice that $u_{coa}(t_0+1,x,\omega)=0$ and $u_{ari}\left(t_0+\frac{1}{2},x,\omega\right)=0$. Considered by itself, each term of u_{iso} is a weak solution on $\left[t_0+\frac{1}{2},t_0+1\right]$; similarly, each $u^{(x_0)}_{coa}(t,x,\omega)$ is a coalescing pair on $\left[t_0+\frac{1}{2},t_0+1\right]$ and each $u^{(x_0)}_{ari}(t,x,\omega)$ is an arising pair on $\left[t_0+\frac{1}{2},t_0+1\right]$. Thus the sum of all these functions is a weak solution if they have disjoint supports.

Elements $u^{(z,+)}_{iso}(t,x,\omega)$ have supports of the form $[x(t),x(t)+\frac{1}{2})$ with $x(t)=z+(t-t_0)$; and $[x'(t)-\frac{1}{2},x'(t))$ with $x'(t)=z'-(t-t_0)$ for $u^{(z',-)}_{iso}(t,x,\omega)$. Supports of functions $u^{(z,+)}_{iso}(t,x,\omega)$ cannot intersect each other, since quasi-particles move in parallel; the same holds for $u^{(z',-)}_{iso}(t,x,\omega)$, among themselves.

The support of a function $u^{(z,+)}_{iso}(t,x,\omega)$ cannot intersect the support of a function $u^{(z',-)}_{iso}(t,x,\omega)$ for the following reason. It is not possible if $z\ge z'$, because they move in opposite directions. Keeping in mind that we consider $t \in \left[t_0 + \frac{1}{2}, t_0 + 1\right]$, the same argument applies when $z = z' - 1$. If $z = z' - 2$, then $x_0 := z + 1$ is of class $C(\theta, act, \omega, t_0)$, hence z and z' cannot be in $MA^+_{iso}(\theta, act, \omega, t_0)$ and $MA^-_{iso}(\theta, act, \omega, t_0)$ respectively. It remains to discuss the case $z \le z' - 3$. But now the supports $[x(t), x(t) + \frac{1}{2})$ and $[x'(t) - \frac{1}{2}, x'(t))$ do not have sufficient time to meet, for $t \in \left[t_0 + \frac{1}{2}, t_0 + 1\right]$. Summarizing, we have proved that all terms of $u_{iso}(t, x, \omega)$ have disjoint supports.

Let $x_0 \in C(\theta, act, \omega, t_0)$. Its corresponding coalescing pair $u^{(x_0)}_{coa}(t, x, \omega)$; its support has the form $[x_0 - (t_0 + 1 - t), x_0 + (t_0 + 1 - t))$, contained in $[x_0 - \frac{1}{2}, x_0 + \frac{1}{2})$ for $t \in \left[t_0 + \frac{1}{2}, t_0 + 1\right]$. These supports are clearly disjoint when x_0 varies in $C(\theta, act, \omega, t_0)$. They are also disjoint from any element of $u_{iso}(t, x, \omega)$: let us see why, in the case of a function $u^{(z,+)}_{iso}(t, x, \omega)$. Since $x_0 \in C(\theta, act, \omega, t_0)$, $x_0 - 1$ cannot be of class $MA^+_{iso}(\theta, act, \omega, t_0)$. Thus we need to have $z < x_0 - 1$ and $u^{(z,+)}_{iso}(t, x, \omega)$ cannot reach the coalescing pair in the time interval $\left[t_0 + \frac{1}{2}, t_0 + 1\right]$, for the same reason why different points of $MA^+_{iso}(\theta, act, \omega, t_0)$ cannot lead to intersections.

Finally, let us consider a point $x_0 \in A(\theta', act', \omega, t_0 + 1)$ where a new arising pair starts to exists at time $t_0 + \frac{1}{2}$, and the associated function $u^{(x_0)}_{ari}(t, x, \omega)$. Let us first discuss the case of $z \in MA^+_{iso}(\theta, act, \omega, t_0)$. If $z \ge x_0$ or $z \le x_0 - 2$ there is no intersection: the difficult case is $z = x_0 - 1$. But in such a case, having excluded by $MA^+_{iso}(\theta, act, \omega, t_0)$ the possibility of coalescing points, we should have $\theta'(x_0) = 1 \neq 0$, in contradiction with $x_0 \in A(\theta', act', \omega, t_0 + 1)$. Hence this case does not exist. Points $z \in MA^-_{iso}(\theta, act, \omega, t_0)$ are similar.

The most difficult case is when $x_0 \in A(\theta', act', \omega, t_0 + 1)$ is also an element of $C(\theta, act, \omega, t_0)$. In plain words, the question is whether an arising pair may arise in a point of coalescence. This case however is solved by Corollary 17, since the latter gives us that if $x_0 \in C(\theta, act, \omega, t_0)$, then $x_0 \notin A(\theta', act', \omega, t_0 + 1)$. Indeed, assuming the former, by definition $x_0 - 1, x_0 + 1 \in A(\theta, act, \omega, t_0)$, and by Corollary 17 this implies $act'(x_0) = 0$, so $x_0 \notin A(\theta', act', \omega, t_0 + 1)$. This rules out the last possible intersection of supports, and the proof is complete. □

5.4. **Conclusion and Main Result.** Merging the statements of Proposition 25 and Proposition 26, along with the simple claim of Proposition 24, we finally get the main result of this work:

Theorem 27. *Given, at time $t_0 = 0$, an element $(\theta, act) \in \Lambda$ and the section $\{\omega(0, x); x \in \mathbb{Z}\}$, define $u_0(x, \omega) := u(0, x, \omega)$, following Definition 23.*

Construct the stochastic process $(\theta(t_0, \omega), act(t_0, \omega))$, $t_0 \in \mathbb{N}$, by setting

$$(\theta(t_0, \omega), act(t_0, \omega)) := \phi_{ABDF}(t_0, \omega)(\theta, act)$$

namely by performing the ABDF random dynamics.

Define the stochastic process $u(t, x, \omega)$, $t \in [0, \infty)$, $x \in \mathbb{R}$ as follows. For every $t_0 \in \mathbb{N}$, define $u(t_0, x, \omega)$ from Definition 23 with respect to (θ, act) given by $(\theta(t_0, \omega), act(t_0, \omega))$; define $u(t, x, \omega)$ for $t \in \left[t_0, t_0 + \frac{1}{2}\right]$ by Proposition 25; finally define $u(t, x, \omega)$ for $t \in \left[t_0 + \frac{1}{2}, t_0 + 1\right]$ by Proposition 26.

Then $u(t, x, \omega)$ is a weak solution of Burgers' equation.

We have thus shown that, given a realization of the ABDF process, we construct a weak solution of Burgers' equation; for almost every ω, from this weak solution it is possible to reconstruct the underlying ABDF realization, by Proposition 24.

Remark 28. The stochastic process u so defined is adapted to the noise filtration shifted by $\frac{1}{2}$. Namely, if $\mathcal{F}_t$ is the natural filtration of the noise, $u(t, \cdot)$ is $\mathcal{F}_{t+\frac{1}{2}}$-adapted, due to the creation mechanism that starts at half-integer times. This anticipation is just instrumental, and not a deep phenomenon. One can develop an alternative construction such that $u(t,\cdot)$ is $\mathcal{F}_t$-adapted, just shifting time by $\frac{1}{2}$, or more precisely starting to create new particles at integer times and completing annihilation at half-integer times. However, we deem the construction just described more elegant.

Acknowledgements

BG acknowledges support by the Max Planck Society through the Max Planck Research Group *Stochastic partial differential equations* and by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through the CRC 1283 *Taming uncertainty and profiting from randomness and low regularity in analysis, stochastics and their applications.*

References

[1] Yuta Arai. The KPZ fixed point for discrete time TASEPs. *Journal of Physics A: Mathematical and Theoretical*, 53(41):415202, Sep 2020.

[2] Yuri Bakhtin and Konstantin Khanin. On global solutions of the random Hamilton-Jacobi equations and the KPZ problem. *Nonlinearity*, 31(4):R93–R121, 2018.

[3] Lorenzo Bertini and Giambattista Giacomin. Stochastic Burgers and KPZ equations from particle systems. *Comm. Math. Phys.*, 183(3):571–607, 1997.

[4] Alexei Borodin, Patrik L. Ferrari, Michael Prähofer, and Tomohiro Sasamoto. Fluctuation properties of the TASEP with periodic initial configuration. *J. Stat. Phys.*, 129(5-6):1055–1080, 2007.

[5] Alexei Borodin, Patrik L. Ferrari, and Tomohiro Sasamoto. Large time asymptotics of growth models on space-like paths. II. PNG and parallel TASEP. *Comm. Math. Phys.*, 283(2):417–449, 2008.

[6] A. Boritchev, C.-E. Bréhier, M. Mariani, personal communication.

[7] Ivan Corwin, Zhipeng Liu, and Dong Wang. Fluctuations of TASEP and LPP with general initial data. *Ann. Appl. Probab.*, 26(4):2030–2082, 2016.

[8] Ivan Corwin, Jeremy Quastel, and Daniel Remenik. Renormalization fixed point of the KPZ universality class. *J. Stat. Phys.*, 160(4):815–834, 2015.

[9] Patrik L. Ferrari and Herbert Spohn. On time correlations for KPZ growth in one dimension. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 12:Paper No. 074, 23, 2016.

[10] Massimiliano Gubinelli and Nicolas Perkowski. KPZ reloaded. *Comm. Math. Phys.*, 349(1):165–269, 2017.

[11] Martin Hairer. Solving the KPZ equation. *Ann. of Math. (2)*, 178(2):559–664, 2013.

[12] Kurt Johansson. Discrete polynuclear growth and determinantal processes. *Comm. Math. Phys.*, 242(1-2):277–329, 2003.

[13] Mehran Kardar, Giorgio Parisi, and Yi-Cheng Zhang. Dynamic scaling of growing interfaces. *Physical Review Letters*, 56(9):889, 1986.

[14] Konstantin Matetski, Jeremy Quastel, and Daniel Remenik. The kpz fixed point. *arXiv preprint arXiv:1701.00018*, 2016.

[15] Mihai Nica, Jeremy Quastel, and Daniel Remenik. Solution of the Kolmogorov equation for TASEP. *Ann. Probab.*, 48(5):2344–2358, 2020.

[16] Michael Prähofer and Herbert Spohn. Scale invariance of the PNG droplet and the Airy process. *J. Statist. Phys.*, 108(5-6):1071–1106, 2002. Dedicated to David Ruelle and Yasha Sinai on the occasion of their 65th birthdays.

[17] Jeremy Quastel. Introduction to KPZ. In *Current developments in mathematics, 2011*, pages 125–194. Int. Press, Somerville, MA, 2012.

[18] Jeremy Quastel and Konstantin Matetski. From the totally asymmetric simple exclusion process to the KPZ fixed point. In *Random matrices*, volume 26 of *IAS/Park City Math. Ser.*, pages 251–301. Amer. Math. Soc., Providence, RI, 2019.

[19] Jeremy Quastel and Sourav Sarkar. The kpz equation converges to the kpz fixed point. *arXiv preprint arXiv:2008.06584*, 2020.

[20] Fraydoum Rezakhanlou. Hydrodynamic limit for attractive particle systems on 417-1417-1417-1. *Communications in mathematical physics*, 140(3):417–448, 1991.

[21] Hermann Rost. Non-equilibrium behaviour of a many particle process: Density profile and local equilibria. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 58(1):41–53, 1981.