RIGIDITY OF MINIMAL LAGRANGIAN DIFFEOMORPHISMS BETWEEN SPHERICAL CONE SURFACES

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Abstract. We prove that any minimal Lagrangian diffeomorphism between two closed spherical surfaces with cone singularities is an isometry, without any assumption on the multiangles of the two surfaces. As an application, we show that every branched immersion of a closed surface of constant positive Gaussian curvature in Euclidean three-space is a branched covering onto a round sphere, thus generalizing the classical rigidity theorem of Liebmann to branched immersions.

1. Introduction

Minimal Lagrangian maps have played an important role in the study of hyperbolic structures on surfaces. As observed independently by Labourie [Lab92] and Schoen [Sch93], given two closed hyperbolic surfaces (Σ_1, h_1) and (Σ_2, h_2), there exists a unique minimal Lagrangian diffeomorphism in the homotopy class of every diffeomorphism $\Sigma_1 \to \Sigma_2$. See also [Lee94] and [TV95, Smi20] for extensions of this result. Alternative proofs have been provided later, in the context of Anti-de Sitter three-dimensional geometry (see [BBZ07] and [BS20, §7]), using higher codimension mean curvature flow (see [Wan01] and [LS11]). Using Anti-de Sitter geometry, the result of Labourie and Schoen has been generalized in various directions: in [BS10, BS18] in the setting of universal Teichmüller space; in [Tou16] for closed hyperbolic surfaces with cone singularities of angles in $(0, \pi)$, provided the diffeomorphism $\Sigma_1 \to \Sigma_2$ maps cone points to cone points of the same angles. Toulisse then proved in [Tou19] the existence of minimal maps between closed hyperbolic surface of different cone angles, by purely analytic methods. We remark that interesting results in a similar spirit have been obtained for minimal Lagrangian diffeomorphisms between bounded domains in the Euclidean plane ([Del91, Wol97]) and in a complete non-positively curved Riemannian surface ([Bre08]).

On the other hand, spherical metrics with cone singularities on a closed surfaces have been studied in [Tro86, McO88, Tro89, Tro91, LT92]. Very recently the works [MP16, MP19, AEP20], by geometric methods, and [MW17, MZ20, MZ19], by analytic methods, developed the study of the deformations spaces of spherical cone metric, highlighting their complexity.

1.1. Main statement.

It thus seems a natural question to ask whether one can find a minimal Lagrangian diffeomorphism between two spherical cone surfaces. In this paper we answer negatively to this question, without any assumption on the cone angles. We show that two spherical cone surfaces do not admit any minimal Lagrangian diffeomorphism unless they are isometric. When they are isometric, the only minimal Lagrangian diffeomorphisms are isometries. We summarize these statements as follows:

Theorem 1.1. *Given two closed spherical cone surfaces $(\Sigma_1, \mathfrak{p}_1, g_1)$ and $(\Sigma_2, \mathfrak{p}_2, g_2)$, any minimal Lagrangian diffeomorphism $\varphi : (\Sigma_1, \mathfrak{p}_1, g_1) \to (\Sigma_2, \mathfrak{p}_2, g_2)$ is an isometry.*

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We remark that, as part of our definition (Definition 2.2), a minimal Lagrangian diffeomorphism φ is a smooth diffeomorphism between $\Sigma_1 \setminus \mathfrak{p}_1$ and $\Sigma_2 \setminus \mathfrak{p}_2$ that extends *continuously* to the cone points. *A priori*, we do not assume that such a smooth map extends *smoothly* to the cone points. This subtlety is at the origin of an important technical point in our proof, which is summarized in Section 1.3 below.

1.2. **Surfaces of constant Gauss curvature.** We provide an application of our main result for branched immersions of constant Gaussian curvature in Euclidean three-space, generalizing the classical Liebmann's theorem which states that every closed immersed surface of positive constant Gaussian curvature in Euclidean space is a round sphere.

In [GHM13], Gálvez, Hauswirth and Mira classified the *isolated singularities* of surfaces of constant Gaussian curvature. According to their definition, isolated singularities of an immersion $\sigma : U \setminus \{p\} \to \mathbb{R}^3$, for U a disc, are those that extend continuously on U. Among these, they considered *extendable singularities*, namely those for which the normal vector extends smoothly at p, and showed that they are either *removable*, meaning that they extend to an immersion of U, or *branch points*, meaning that the Gauss map is locally expressed as the map $z \mapsto z^k$ with respect to some coordinates on U and on $\mathbb{S}^2$. Here we show a rigidity result for branched immersions of closed surfaces:

Corollary 1.2. *Every branched immersion in Euclidean three-space of a closed surface of constant positive Gaussian curvature is a branched covering onto a round sphere.*

As we mentioned, Corollary 1.2 can be regarded as a generalization of Liebmann's theorem, which we indeed recover by an independent proof when the immersion has no branch points. Roughly speaking, we prove Corollary 1.2 by applying Theorem 1.1 to the Gauss map of a branched immersion $\sigma : \Sigma \to \mathbb{R}^3$, which induces a minimal Lagrangian self-diffeomorphism of Σ with respect to the first and third fundamental form, both of which are spherical cone metrics.

Finally, we remark that the hypothesis that the surface Σ is closed is essential in Corollary 1.2, as well as the closedness of Σ_1 and Σ_2 in Theorem 1.1. Indeed one can find *local* deformations of spheres of constant Gaussian curvature, with branch points (see [Bra16] for many examples) or without (for instance by surfaces of revolution); their Gauss maps provide non-isometric minimal Lagrangian diffeomorphisms between open spherical surfaces (with or without cone points).

1.3. **Outline of the proof of Theorem 1.1.** A map $\varphi : (\Sigma_1, \mathfrak{p}_1, g_1) \to (\Sigma_2, \mathfrak{p}_2, g_2)$ is minimal Lagrangian if it is area-preserving and its graph (restricted to the nonsingular locus) is minimal in the product $\Sigma_1 \times \Sigma_2$. A useful characterization is that one can express (on the nonsingular locus) $\varphi^* g_2 = g_1(b\cdot, b\cdot)$ for b a (1,1) tensor which is self-adjoint with respect to g_1, positive definite, and satisfies the conditions $d^{\nabla^{g_1}} b = 0$ and $\det b = 1$. For the sake of completeness, we prove the equivalence of the two definitions in Appendix A. From this characterization, one sees that minimal Lagrangian maps are those that can be *locally* realized as the Gauss maps of surfaces of constant Gaussian curvature one in Euclidean three-space, as a consequence of the Gauss-Codazzi equations.

Starting by this characterization, using the spherical metric g_1 and the (1,1) tensor b, we produce a pair (G, B) where G is a Riemannian metric on Σ_1 and B a (1,1) G-self-adjoint traceless tensor, satisfying the equations

$$d^{\nabla^G} B = 0 \qquad \text{and} \qquad K_G = 1 + \det B \, .$$

Although we will not use spherical three-dimensional geometry in this paper, we remark that these are precisely the Gauss-Codazzi equations for a surface in $\mathbb{S}^3$, which is minimal since B is traceless. Equivalently, by the Lawson correspondence, the pair $(G, \mathbb{1} + B)$ satisfies the Gauss-Codazzi equations for a surface of constant mean curvature one in $\mathbb{R}^3$. Such constant mean curvature surface is realized (at least locally) as the parallel surface from the surface of constant Gaussian curvature mentioned above, which is determined by the pair (g_1, b). Assuming $\varphi : (\Sigma_1, \mathfrak{p}_1, g_1) \to (\Sigma_2, \mathfrak{p}_2, g_2)$ is a minimal Lagrangian diffeomorphism, the goal of the proof is to show that B vanishes identically, which is equivalent to φ being an isometry.

For this purpose, assuming by contradiction that B does not vanish identically, the next step consists in computing the Laplace-Beltrami operator of the function χ defined, in the complement of the zeros of B, as the logarithm of the positive eigenvalue of B (up to a certain constant). It turns out that $\Delta^G \chi$ equals the curvature of the metric G, which is positive (Corollary 3.2), hence χ is subharmonic and negative and the contradiction is then obtained by an application of the maximum principle.

However, it is essential to prove that the metric G has the conformal type of a punctured disc in a neighbourhood of every cone point of Σ_1. This would be automatically satisfied assuming some additional regularity on the minimal Lagrangian map φ: for instance, if φ is supposed quasiconformal, which is equivalent to boundedness of the (1,1) tensor b, then g_1 and G are quasiconformal, and therefore both g_1 and G have the conformal type of a punctured disc near the cone points. But, as we mentioned above, in our Theorem 1.1 we assume a weaker regularity on φ at the cone points, namely we only suppose that φ is continuous.

To prove that G has the conformal type of a punctured disc around the cone points, we apply the interpretation in terms of surfaces in Euclidean space, and we show that G can be realized in a punctured neighbourhood U^* of any cone point as the metric induced by the first fundamental form of an equivariant immersion of $\widetilde{U^*}$ in $\mathbb{R}^3$. We also prove that the normal vector of the equivariant immersion admits a limit, and the vertical projection induced a bi-Lipschitz equivalence between G and a flat metric on U^*. A complex analytic argument, based on Schwarz Reflection Principle, shows that this flat metric has the conformal type of $\mathbb{D}^*$ at the puncture, and this implies that G has the conformal type of $\mathbb{D}^*$ as well.

Contents

1. Introduction 1
1.1. Main statement 1
1.2. Surfaces of constant Gauss curvature 2
1.3. Outline of the proof of Theorem 1.1. 2
2. Definitions and setup 4
2.1. Conical metrics 4
2.2. Minimal Lagrangian maps 4
2.3. Defining the pair (G, B) 5
3. A maximum principle 6
3.1. A bounded subharmonic function 6
3.2. Proof of Theorem 1.1 7
4. Immersions in Euclidean space 8
4.1. Proof of Proposition 3.4 8
4.2. Proof of Corollary 1.2 11
4.3. Final remarks 12

Appendix A. Equivalent definitions 13
Appendix B. Alternative proof of Lemma 3.1 14
References 15

2. Definitions and setup

Let us start by introducing the fundamental definitions and some well-known properties.

2.1. **Conical metrics.** We give the general definition of cone Riemannian metric.

Definition 2.1. Given a smooth surface Σ and a discrete subset $\mathfrak{p} \subset \Sigma$, a *cone* metric on Σ is a Riemannian metric g on $\Sigma \setminus \mathfrak{p}$ such that can be written in a punctured neighbourhood $U \setminus \{p\}$ of every point $p \in \mathfrak{p}$ as:

$$g = e^{2f}|z|^{2\alpha-2}|dz|^2 \tag{1}$$

with respect to a coordinate $z : U \to \mathbb{C}$, for $f : U \setminus \{p\} \to \mathbb{R}$ a smooth bounded function and $\alpha \in (0,1) \cup (1, +\infty)$.

The subset $\mathfrak{p}$ is called the *singular locus*, and its complement the *regular locus*. It will be convenient to set $\theta := 2\pi\alpha$, which is called the *cone angle* at p.

A cone metric is called *spherical* when it has constant curvature $+1$ on the regular locus. In this cases one has the following explicit local expression for the metric tensor (1):

$$g = \frac{4\alpha^2|z|^{2\alpha-2}}{(1+|z|^{2\alpha})^2}|dz|^2 \ . \tag{2}$$

2.2. **Minimal Lagrangian maps.** Let us now move on to the definition of minimal Lagrangian maps.

Definition 2.2. Given two spherical cone surfaces $(\Sigma_1, \mathfrak{p}_1, g_1)$ and $(\Sigma_2, \mathfrak{p}_2, g_2)$, a *minimal Lagrangian diffeomorphism* is a diffeomorphism $\varphi : \Sigma_1 \setminus \mathfrak{p}_1 \to \Sigma_2 \setminus \mathfrak{p}_2$, that extends to a homeomorphism between Σ_1 and Σ_2, having the property that the unique g_1-self-adjoint, positive definite $(1,1)$ tensor b on $\Sigma_1 \setminus \mathfrak{p}_1$ such that $\varphi^* g_2 = g_1(b\cdot, b\cdot)$ satisfies the conditions:

$$\det b = 1 \qquad \text{and} \qquad d^{\nabla^{g_1}} b = 0 \ . \tag{3}$$

Here and in what follows, ∇^g denotes the Levi-Civita connection of a Riemannian metric g. We recall that, for a connection ∇ and a $(1,1)$ tensor A, the *exterior derivative* $d^\nabla A$ is defined as

$$d^\nabla A(v,w) = \nabla_v(A(w)) - \nabla_w(A(v)) - A([v,w]) \ .$$

A tensor satisfying $d^{\nabla^g} A = 0$ is called *Codazzi tensor* with respect to the metric g.

For the sake of completeness, in Appendix A we shall prove that Definition 2.2 is equivalent to the fact that the graph of φ is minimal Lagrangian in $\Sigma_1 \times \Sigma_2$, thus justifying the terminology.

Remark 2.3. It is natural to require that a minimal Lagrangian map maps cone points to cone points, as in Definition 2.2. Indeed (as we explain in Remark 4.4), if $\varphi : (U_1, p_1, g_1) \to (U_2, p_2, g_2)$ is minimal Lagrangian diffeomorphism between two punctured discs endowed with metrics of the form (2), then the cone angles of g_1 and g_2 are necessarily equal. In particular, if the "cone angle" is 2π for g_1 at p_1, meaning that the metric extends to a smooth spherical metric on the disc, then the same holds for g_2 at p_2, and moreover in this case φ extends smoothly to a minimal Lagrangian diffeomorphism between U_1 and U_2.

When a metric is written in the expression $g(A\cdot, A\cdot)$ for A an invertible Codazzi tensor with respect to g, its connection and curvature are easily related to those of g, as in the following well-known lemma.

Lemma 2.4 ([KS07, Proposition 3.12])**.** *Let g be a Riemannian metric on a surface Σ and let A a smooth $(1,1)$ tensor with $d^{\nabla^g}A = 0$ such that $\det A$ vanishes nowhere. Define $h = g(A\cdot, A\cdot)$. Then the Levi-Civita connections of g and h are related by:*

$$\nabla^h_v w = A^{-1}\nabla^g_v(A(w)) \ , \tag{4}$$

and their curvatures by:

$$K_h = \frac{K_g}{\det A} \ . \tag{5}$$

Remark 2.5. This lemma has two immediate consequences. First, it turns out that the condition $\det b = 1$ is actually redundant in Definition 2.2. Indeed, assuming $d^{\nabla_{g_1}} b = 0$, it follows from $K_{g_1} = K_{\varphi^* g_2} = 1$ and from Equation (5) that $\det b = 1$. Second, the inverse of a minimal Lagrangian diffeomorphism is minimal Lagrangian, since one can write $g_1 = \varphi^* g_2(b^{-1}\cdot, b^{-1}\cdot)$ and it is easily checked that b^{-1} is self-adjoint and Codazzi for $\varphi^* g_2$, using (4). Hence $\varphi_* b^{-1} := (d\varphi)\circ b^{-1}\circ(d\varphi)^{-1}$ satisfies the conditions in the Definition 2.2 for $\varphi^{-1}: \Sigma_2 \to \Sigma_1$.

2.3. **Defining the pair (G, B).** We now introduce the fundamental construction for our proofs.

Definition 2.6. Given a minimal Lagrangian map $\varphi : (\Sigma_1, \mathfrak{p}_1, g_1) \to (\Sigma_2, \mathfrak{p}_2, g_2)$, we define on $\Sigma_1 \setminus \mathfrak{p}_1$ a Riemannian metric

$$G = \frac{1}{4} g_1((\mathbb{1} + b)\cdot, (\mathbb{1} + b)\cdot) \ ,$$

and a $(1,1)$-tensor

$$B = (\mathbb{1} + b)^{-1}(b - \mathbb{1}) \ ,$$

for b as in Definition 2.2.

Remark 2.7. Definition 2.6 has a symmetry with respect to g_1 and g_2. More precisely, the metric G' on $\Sigma_2 \setminus \mathfrak{p}_2$ associated to the map $\varphi^{-1} : (\Sigma_2, \mathfrak{p}_2, g_2) \to (\Sigma_1, \mathfrak{p}_1, g_1)$, which is again minimal Lagrangian (Remark 2.5), is isometric to the metric G on $\Sigma_1 \setminus \mathfrak{p}_1$.

To see this, we have observed in Remark 2.5 that the (1,1) tensor associated to the minimal Lagrangian map φ^{-1} in Definition 2.2 is $\varphi_* b^{-1}$. Hence G' is the metric on Σ_2 defined by $G' = (1/4) g_2((\mathbb{1}+\varphi_* b^{-1})\cdot, (\mathbb{1}+\varphi_* b^{-1})\cdot)$, and one sees immediately that $\varphi^* G' = G$. Similarly, one finds $B' = -\varphi_* B$.

It is immediate to check that B is G-self-adjoint, since b is g_1-self-adjoint. The following lemma is an immediate algebraic computation.

Lemma 2.8. *The eigenspaces of B coincide with those of b, and if we denote by λ and $1/\lambda$ the eigenvalues of b, then the eigenvalues of B are*

$$\Lambda = \frac{\lambda - 1}{1 + \lambda} \qquad \text{and} \qquad \Lambda' = \frac{1 - \lambda}{1 + \lambda} = -\Lambda \ .$$

In particular, B is traceless and $|\Lambda|, |\Lambda'| < 1$. Finally, at any point we have $B = 0$ if and only if $b = \mathbb{1}$.

We observe that the pair (G, B) satisfies the following important properties:

Proposition 2.9. *The following Codazzi equations is satisfied by the pair (G, B):*

$$d^{\nabla^G} B = 0 \ .$$

Moreover, the curvature of G is positive.

Proof. This is a straightforward verification using Lemma 2.4. Indeed, by Equation (4)

$$d^{\nabla^G} B = (\mathbb{1} + b)^{-1} d^{\nabla^{g_1}} (b - \mathbb{1}) = 0$$

since both $\mathbb{1}$ and b are g_1-Codazzi. For the curvature condition, using Equation (5) we have

$$K_G = \frac{4K_{g_1}}{\det(\mathbb{1} + b)} = \frac{4}{2 + \mathrm{tr} b} > 0 \ .$$

This concludes the proof. □

Remark 2.10. A simple computation shows moreover that

$$1 + \det B = 1 + \frac{2 - \mathrm{tr} b}{2 + \mathrm{tr} b} = \frac{4}{2 + \mathrm{tr} b} = K_G \ .$$

In other words, together with Proposition 2.9, we see that the pair (G, B) satisfies the Gauss-Codazzi equation when the ambient manifold is $\mathbb{S}^3$. However, in this paper we will not use spherical geometry; the construction of Section 4 is motivated by the observation that $(G, \mathbb{1} + B)$ satisfy the Gauss-Codazzi equations in Euclidean space $\mathbb{E}^3$.

This is the so-called *Lawson correspondence* introduced in [Law70]: as a consequence of $\mathrm{tr} B = 0$, the Gauss equation $K_G = 1 + \det B$ is equivalent to $K_G = \det(\mathbb{1} + B)$, namely the Gauss equation in $\mathbb{E}^3$; furthermore $\mathbb{1} + B$ is clearly Codazzi with respect to G. In summary, when lifted to the universal cover of $\Sigma_1 \setminus \mathfrak{p}_1$, (G, B) provide the immersion data of a minimal surface in $\mathbb{S}^3$, while $(G, \mathbb{1} + B)$ those of a constant mean curvature one surface in $\mathbb{E}^3$.

3. A maximum principle

The key idea in the proof of Theorem 1.1 is an application of the maximum principle to show that B is identically zero, that is b is the identity operator (by Lemma 2.8). This will show that any minimal Lagrangian map $\varphi : (\Sigma_1, \mathfrak{p}_1, g_1) \to (\Sigma_2, \mathfrak{p}_2, g_2)$ is an isometry.

3.1. A bounded subharmonic function.

The fundamental relation involved in our application of the maximum principle is a consequence of the following formula, presented in [KS07, Lemma 3.11]. Since this is a fundamental step, we provide a quick proof for convenience of the reader. In Appendix B we give another short proof, entirely based on the fact that $G(B\cdot, \cdot)$ is the real part of a holomorphic quadratic differential.

Lemma 3.1. *Let G be a Riemannian metric on a surface U and B a traceless G-self-adjoint, G-Codazzi $(1,1)$ tensor that does not vanish on U. Denote $\chi = (1/4)\log(-\det B)$. Then*

$$K_G = \Delta^G \chi \ .$$

Here we denote by Δ^G the Laplace-Beltrami operator of G, with negative spectrum.

Proof. Let e, e' be an oriented orthonormal frame of eigenvalues of B, so that $B(e) = \Lambda e$ and $B(e') = \Lambda' e' = -\Lambda e'$. Let us denote by ω the associated connection form, which satisfies $\nabla_v e = \omega(v) e'$ and $\nabla_v e' = -\omega(v) e$, where to simplify the notation we set $\nabla = \nabla^G$. Since B does not vanish on U by hypothesis, we can assume moreover that Λ is the positive eigenvalue of B, so that $\chi = (1/4)\log(\Lambda^2) = (1/2)\log\Lambda$.

First, let us compute the Codazzi condition applied to the frame $\{e, e'\}$:

$$\begin{aligned} 0 &= \nabla_e B(e') - \nabla_{e'} B(e) - B(\nabla_e e' - \nabla_{e'} e) \\ &= -(\partial_e \Lambda) e' - (\partial_{e'} \Lambda) e - 2\Lambda \nabla_e e' - 2\Lambda \nabla_{e'} e \ . \end{aligned}$$

Hence we get $\partial_e \Lambda = -2\Lambda\omega(e')$ and $\partial_{e'}\Lambda = 2\Lambda\omega(e)$. In terms of $\chi = (1/2)\log\Lambda$, we have $\partial_e \chi = -\omega(e')$ and $\partial_{e'}\chi = \omega(e)$.

Second, we compute

$$\begin{aligned} K_G &= -d\omega(e, e') = -\partial_e \omega(e') + \partial_{e'}\omega(e) + \omega(\nabla_e e' - \nabla_{e'} e) \\ &= \partial_e \partial_e \chi + \partial_{e'}\partial_{e'}\chi - \omega((\partial_e \chi) e') - \omega((\partial_{e'}\chi) e) \\ &= \mathrm{Hess}\chi(e, e) + \mathrm{Hess}\chi(e', e') = \Delta^G \chi \ , \end{aligned}$$

where from the second to the third line we used

$$\begin{aligned} \mathrm{Hess}\chi(e, e) &= \partial_e \partial_e \chi - \partial_{\nabla_e e}\chi = \partial_e \partial_e \chi - \partial_{\omega(e) e'}\chi \\ &= \partial_e \partial_e \chi - \omega((\partial_{e'}\chi) e) = \partial_e \partial_e \chi - \omega(\omega(e) e) \\ &= \partial_e \partial_e \chi + \omega(\nabla_e e') \ , \end{aligned}$$

and similarly for $\mathrm{Hess}\chi(e', e')$. □

Proposition 2.9 and Lemma 3.1 show that, on the subset of $\Sigma_1 \setminus \mathfrak{p}_1$ where $B \neq 0$,

$$\Delta^G \chi > 0 \ .$$

We have thus shown that χ is subharmonic. Moreover χ is negative, because $|\Lambda| < 1$ by Lemma 2.8. We summarize these facts as follows:

Corollary 3.2. *The function $\chi = (1/2)\log|\Lambda|$ is negative and satisfies $\Delta^G \chi > 0$ on the complement of the zeros of B in $\Sigma_1 \setminus \mathfrak{p}_1$.*

Remark 3.3. Although not essential in the proof, we remark that given a smooth $(1,1)$ tensor A, A is g-self-adjoint and traceless if and only if $g(A\cdot, \cdot)$ is the real part of a holomorphic quadratic differential ([Hop51], see Proposition A.2). Hence either $B \equiv 0$ or B vanishes on a discrete subset of $\Sigma_1 \setminus \mathfrak{p}_1$.

3.2. **Proof of Theorem 1.1.** The main idea of the proof is to apply a maximum principle argument to the function χ of the previous section. To control the behaviour of χ at the singularities, we need the following statement on the conformal type of the metric G. Its proof is postponed to Section 4.

Proposition 3.4. *Let U_i be a disc endowed with a spherical metric g_i with a cone point at p_i, for $i = 1, 2$, let $\varphi : (U_1, p_1, g_1) \to (U_2, p_2, g_2)$ be a minimal Lagrangian diffeomorphism, and let b the $(1,1)$ tensor as in Definition 2.2. Then the conformal structure induced by the metric $G = (1/4) g_1((\mathbb{1} + b)\cdot, (\mathbb{1} + b)\cdot)$ on a neighbourhood of p_1 is biholomorphic to $\mathbb{D}^*$.*

Assuming Proposition 3.4, we now conclude the proof of Theorem 1.1.

Proof of Theorem 1.1. Assume by contradiction that B does not vanish identically. Recall that the function $\chi = (1/4)\log(-\det B)$ is negative and subharmonic by Corollary 3.2. We can extend χ to a function on $\Sigma_1 \setminus \mathfrak{p}_1$, with values in $[-\infty, +\infty)$.

Now, pick a cone point $p \in \mathfrak{p}_1$ and a neighbourhood U of p in which the metric G is biholomorphic to $\mathbb{D}^*$. We claim that

$$\sup_U \chi = \max_{\partial U} \chi \ . \tag{6}$$

This will conclude the proof, since it implies that χ has a maximum point in $\Sigma_1 \setminus \mathfrak{p}_1$, and this contradicts Corollary 3.2.

Let us prove (6). Let us pick a biholomorphic chart $z : U \to \mathbb{D}^*$, and consider χ as a function of z. The metric G is expressed in the z-coordinate as $e^{2f}|dz|^2$, hence the Laplace-Beltrami operator Δ^G equals $e^{-2f}\Delta$, where Δ is the flat Laplacian on the disc. By Corollary 3.2, this implies $\Delta\chi > 0$ on $\mathbb{D}^*$. Now choose any $\epsilon > 0$. The function $\chi_\epsilon(z) := \chi(z) + \epsilon \log|z|$ still satisfies $\Delta\chi_\epsilon > 0$ because $\log|z|$ is harmonic, and coincides with χ on $\mathbb{S}^1$. Moreover χ_ϵ tends to $-\infty$ at 0, because χ is bounded above. By the maximum principle, χ_ϵ cannot have an interior maximum point, hence

$$\chi_\epsilon(z) \leq \max_{\mathbb{S}^1} \chi_\epsilon = \max_{\mathbb{S}^1} \chi$$

for any $z \in \mathbb{D}^*$. It follows that

$$\chi(z) \leq \max_{\mathbb{S}^1} \chi - \epsilon \log|z| \ .$$

Since ϵ was chosen arbitrarily, this shows (6) and concludes the proof. □

4. Immersions in Euclidean space

In order to complete the proof of Theorem 1.1 it only remains to prove Proposition 3.4. We then prove Corollary 1.2. The relation between minimal Lagrangian diffeomorphisms and immersions in Euclidean space will play an essential role for both results.

4.1. **Proof of Proposition 3.4.** The guiding idea towards Proposition 3.4 is that, given the tensor b as in Definition 2.2, the pair (g_1, b) represents locally the embedding data of an immersed surface of constant Gaussian curvature one in Euclidean space, by the fundamental theorem of surfaces; moreover, there is a parallel constant mean curvature surface whose first fundamental form is G up to a factor (and whose shape operator is $\mathbb{1} + B$, compare Remark 2.10). However, since Σ_1 is not simply connected, we will need to lift (g_1, b) to its universal cover, and refine this approach in order to deal with the equivariance of the obtained immersion. Moreover, in the proof we find convenient to switch the roles of g_1 and g_2, namely we apply the above guiding idea to φ^{-1}, see Remark 2.7.

We will apply the following result.

Lemma 4.1 ([Fer81],[OS83, Proposition 1.3.3])**.** *Given a simply connected Riemannian manifold (M, g) of constant sectional curvature K and a self-adjoint $(1,1)$ tensor A satisfying the Codazzi equation $d^{\nabla^g} A = 0$, there exists a smooth function $u : M \to \mathbb{R}$ such that*

$$A = \nabla^g_\bullet \mathrm{grad}^g u + Ku\mathbb{1} \ .$$

We observe that the term $\nabla^g_\bullet \mathrm{grad}^g u$ is the Hessian of u as a $(1,1)$ tensor, i.e. $\mathrm{Hess}^g u(v, w) = g(\nabla^g_v \mathrm{grad}^g u, w)$.

Proof of Proposition 3.4. To simplify the notation, let us denote $U_i^* := U_i \setminus \{p_i\}$. Lift g_1 and b to the universal cover $\widetilde{U_1^*}$. Then $\widetilde{b}$ still satisfies the Codazzi equation with respect to $\widetilde{g}_1$. By Lemma 4.1 there exists a function $u : \widetilde{U_1^*} \to \mathbb{R}$ such that

$$\widetilde{b} = \nabla^{\widetilde{g}_1}_\bullet \mathrm{grad}^{\widetilde{g}_1} u + u\mathbb{1} \ .$$

Pick also a developing map $\mathrm{dev} : \widetilde{U_1^*} \to \mathbb{S}^2 \subset \mathbb{E}^3$ for the spherical structure of g_1 on U_1^*, namely dev is a local isometry with respect to the metric $\widetilde{g}_1$ on $\widetilde{U_1^*}$. We define the maps

$\sigma, \varsigma : \widetilde{U_1^*} \to \mathbb{E}^3$ by

$$\sigma(x) = \mathrm{dev}_*(\mathrm{grad}^{\widetilde{g}_1} u) + u\mathrm{dev}(x) \qquad \varsigma(x) = \frac{1}{2}\left(\mathrm{dev}_*(\mathrm{grad}^{\widetilde{g}_1} u) + (u+1)\mathrm{dev}(x)\right) .$$

Observe that $\varsigma = (1/2)(\sigma + \mathrm{dev})$.

***Step 1*:** Let us show that the first fundamental forms of σ and ς are the lifts to the universal cover of $\varphi^* g_2$ and G, respectively. Being a local statement, we can isometrically identify an open neighbourhood of any point of $\widetilde{U_1^*}$ with a subset of $\mathbb{S}^2$, so that dev is the identity. By differentiating σ with this identification, we see that

$$\begin{aligned} D_v\sigma_x &= D_v \mathrm{grad}^{\mathbb{S}^2} u + \langle \mathrm{grad}^{\mathbb{S}^2} u, v\rangle x + uv \\ &= \nabla^{\mathbb{S}^2}_v \mathrm{grad}^{\mathbb{S}^2} u + uv = \widetilde{b}(v) \ , \end{aligned}$$

where we used D to denote the ambient derivative in $\mathbb{E}^3$, $\langle\cdot,\cdot\rangle$ the metric of $\mathbb{E}^3$ (which restricts to the metric of $\mathbb{S}^2$), and from the first to the second line we applied the fact that the second fundamental form of $\mathbb{S}^2$ equals $-\langle\cdot,\cdot\rangle$ with respect to the outer unit normal. This computation has several consequences, namely:

(1) The map σ is an immersion, since its differential is nonsingular.
(2) The normal vector of σ at a point $x \in \widetilde{U_1^*}$ is $\mathrm{dev}(x)$. Indeed, in the above identification, x itself is orthogonal to the image of the differential of σ at x. In other words, the Gauss map of σ is dev.
(3) The first fundamental form of σ equals $\langle D_v\sigma_x, D_w\sigma_x\rangle = \widetilde{g}_1(\widetilde{b}(v), \widetilde{b}(w)) = \widetilde{\varphi^* g_2}(v, w)$.
(4) The shape operator of σ is $\widetilde{b}^{-1}$, since (performing again the computation locally) the normal vector is $N(x) = x$, hence its derivative is $dN(v) = v$ and this equals $\widetilde{b}^{-1}$ applied to $D_v\sigma_x$.

The computation for ς is completely analogous, implying (under the same identification as above):

(1') Its differential equals $(1/2)(\mathbb{1} + \widetilde{b})$ and is nonsingular.
(2') Its Gauss map is again dev.
(3') Its first fundamental form is $(1/4)\widetilde{g}_1((\mathbb{1} + \widetilde{b})\cdot, (\mathbb{1} + \widetilde{b})\cdot) = \widetilde{G}$.

***Step 2*:** The immersions σ and ς are equivariant with respect to a representation ρ of $\pi_1(U_1^*) \cong \mathbb{Z}$ into the isometry group of $\mathbb{E}^3$. Indeed, by construction $\widetilde{g}_1$ and $\widetilde{b}$ are preserved by the action of $\mathbb{Z}$ by deck transformations of $\widetilde{U_1^*}$, hence so are the first fundamental form and the shape operator of σ, by the items (3) and (4) of the list above. By the uniqueness part of the fundamental theorem of surfaces, there exists a representation

$$\rho : \mathbb{Z} \to \mathrm{Isom}(\mathbb{R}^3) \cong \mathrm{SO}(3) \ltimes \mathbb{R}^3$$

such that $\sigma \circ \gamma = \rho(\gamma) \circ \sigma$ for every $\gamma \in \mathbb{Z}$.

Now, observe that the developing map $\mathrm{dev} : \widetilde{U_1^*} \to \mathbb{S}^2$ is also equivariant with respect to a rotation of angle θ_1 (modulo 2π), as a consequence of the definition of spherical cone metric. That is, if we denote by γ_1 the standard generator of $\pi_1(U_1^*)$, then

$$\mathrm{dev}(\gamma_1 \cdot \widetilde{q}) = R_{\theta_1}(\mathrm{dev}(\widetilde{q}))$$

for every $\widetilde{q} \in \widetilde{U_1^*}$. Up to composing σ and ς with an element of $\mathrm{SO}(3)$, we can assume that R_{θ_1} is the rotation fixing $(0,0,1)$. By item (2) above, the Gauss map of σ coincides with dev, hence the linear part of ρ is the holonomy of dev. Concretely, we have for every $\widetilde{q} \in \widetilde{U_1^*}$.

$$\sigma(\gamma_1 \cdot \widetilde{q}) = R_{\theta_1}(\sigma(\widetilde{q})) + \tau$$

for some $\tau \in \mathbb{E}^3$. Since $\varsigma = (1/2)(\sigma + \mathrm{dev})$, it turns out that ς satisfies the equivariance:

$$\varsigma(\gamma_1 \cdot \widetilde{q}) = R_{\theta_1}(\sigma(\widetilde{q})) + \frac{1}{2}\tau \ .$$

As a consequence of the next step, we will see that, up to composing σ with a translation, we can assume $\tau = 0$.

Step 3: In this step we will show that, roughly speaking, the immersions σ and ς admit a limit in correspondence of the puncture of U_1^*. Let us denote by $\mathcal{D}$ a fundamental domain for the action of $\pi_1(U_1^*)$ on $\widetilde{U_1^*}$. We now claim that there exists a point $\xi \in \mathbb{E}^3$ having the property that $\sigma(\widetilde{q}_n) \to \xi$ for every sequence $\widetilde{q}_n \in \mathcal{D}$ such that $\Pi(\widetilde{q}_n) \to p_1$, where $\Pi : \widetilde{U_1^*} \to U_1^*$ is the covering projection. To see this, recall that the first fundamental form of σ is the lift to the universal cover of the spherical cone metric $\varphi^* g_2$. Let us first fix *one* sequence $\widetilde{q}_n$ as above. Since the metric completion of $(U_1^*, \varphi^* g_2)$ is obtained by adding the cone point p_1, $\widetilde{q}_n$ is a Cauchy sequence for the first fundamental form of σ, which is $\widetilde{\varphi^* g_2}$. Hence $\sigma(\widetilde{q}_n)$ is a Cauchy sequence in $\mathbb{R}^3$, and it converges. Let us call its limit point ξ.

Now pick any other sequence $\widetilde{q}'_n$ contained in $\mathcal{D}$ such that $\Pi(\widetilde{q}'_n)$ converges to p_1. The distance between $\widetilde{q}_n$ and $\widetilde{q}'_n$ for the first fundamental form of σ tends to zero. Hence also the Euclidean distance $\|\sigma(\widetilde{q}_n) - \sigma(\widetilde{q}'_n)\|$ tends to zero, and therefore the limit of $\sigma(\widetilde{q}'_n)$ is ξ again. One can in fact repeat the same argument only assuming that $\widetilde{q}'_n$ is contained in the union

$$\bigcup_{i \in I} \gamma_i \cdot \mathcal{D}$$

for I a finite subset of $\pi_1(U_1^*) \cong \mathbb{Z}$. This observation also shows that the representation ρ introduced in the previous step fixes ξ, by applying the above argument to $\widetilde{q}'_n = \rho(\gamma)\widetilde{q}_n$. Up to composing with a translation, we will assume $\xi = 0$, which shows that ρ is a linear representation, or in other words, $\tau = 0$ in the previous step.

We also obtain an analogous property for ς, namely that for any sequence $\widetilde{q}_n \in \mathcal{D}$ such that $\Pi(\widetilde{q}_n) \to p_1$, $\varsigma(\widetilde{q}_n)$ converges to a point in the axis fixed by R_{θ_1}. But in this case the proof does *not* follow from the same argument. Indeed we do not know that the metric G on U_1^* has a cone singularity at p_1, hence we cannot repeat the above argument wordly. (This is indeed the reason why so far we dealt with σ and ς simultaneously, although we are only interested in the final statement for ς.) Nevertheless, since $\varsigma = (1/2)(\sigma + \mathrm{dev})$ and dev converges to $(0, 0, 1)$ on any sequence $\widetilde{q}_n$ as above, the conclusion for ς follows immediately, and the limit of ς is in fact $(0, 0, 1)$ under our assumptions.

Step 4: Let us now consider the vertical projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2$, namely $\pi(x, y, z) = (x, y)$. A fundamental consequence of item (2') is that the Gauss map of ς (which coincides with dev) tends to $(0, 0, 1)$ along any sequence $\widetilde{q}_n$ such that $\Pi(\widetilde{q}_n) \to p_1$. It follows that, up to restricting U_1, we can assume that

$$\langle \mathrm{dev}(q), (0, 0, 1) \rangle > \epsilon > 0 \ . \tag{7}$$

In other words, the normal vector of ς is never horizontal. This implies that $\pi \circ \varsigma : \widetilde{U_1^*} \to \mathbb{R}^2$ is an immersion. Since ς is equivariant with respect to the representation ρ sending a generator to the rotation R_{θ_1} around the vertical axis, also $\pi \circ \varsigma$ is equivariant with a representation, which with a little abuse of notation we still denote by ρ, sending the same generator to the rotation in $\mathbb{R}^2$ of angle θ_1 (modulo 2π). By this equivariance, the first fundamental forms of the immersions ς and $\pi \circ \varsigma$ induce two Riemannian metrics on U_1^*: the metric induced by ς is G as a consequence of item (3') in Step 1; the metric induced by $\pi \circ \varsigma$, which we call H,

is flat. The two metrics G and H are bi-Lipschitz, i.e. there exists a constant C such that

$$\frac{1}{C}H(v,w) \leq G(v,w) \leq CH(v,w) \ , \tag{8}$$

for all v, w tangent to U_1^*. Indeed, it follows from (7) that that is there exists a constant $C > 1$ such that $(1/C)G(v,v) \leq ||d\pi \circ d\varsigma(v)||^2 \leq G(v,v)$ for any vector v tangent to $\widetilde{U_1^*}$.

Step 5: We now claim that the conformal structure on U_1^* induced by the metric H is biholomorphic to $\mathbb{D}^*$ around the point p_1. Recall that any conformal structure on U_1^* is biholomorphic to $\mathbb{D}^*$, $\mathbb{C}^*$ or $\mathbb{A}_{r_0} = \{z \in \mathbb{C} \,|\, 1 < |z| < r_0\}$ for some $r_0 > 1$. Let us show that (U_1^*, H) cannot be biholomorphic to any $\mathbb{A}_{r_0}$. Up to restricting the neighbourhood U_1^*, we then rule out the case $\mathbb{C}^*$ and conclude the claim.

Suppose by contradiction that there exists a biholomorphism $\psi : \mathbb{A}_{r_0} \to (U_1^*, H)$. Clearly ψ extends to one of the two boundary components, having limit p_1 therein; it is harmless to assume that such boundary component is $\{|z| = 1\}$. Considering the (holomorphic) universal covering map $z \mapsto \exp(-iz)$ of $\mathbb{A}_{r_0}$, defined on

$$\widetilde{\mathbb{A}_{r_0}} := \{z \in \mathbb{C} \,|\, 0 < \mathrm{Im}(z) < \log(r_0)\} \ , \tag{9}$$

we can lift ψ to a biholomorphism $\widetilde{\psi} : \widetilde{\mathbb{A}_{r_0}} \to (\widetilde{U_1^*}, \widetilde{H})$, where by construction $\widetilde{H}$ is the pull-back metric of the immersion $\pi \circ \varsigma : \widetilde{U_1^*} \to \mathbb{R}^2$. Hence $f := \pi \circ \varsigma \circ \widetilde{\psi} : \widetilde{\mathbb{A}_{r_0}} \to \mathbb{R}^2 \cong \mathbb{C}$ is a holomorphic map. In Steps 2 and 3 we showed that $\varsigma(\widetilde{q}_n)$ tends to the origin for any sequence $\widetilde{q}_n \in \widetilde{U_1^*}$ such that $\Pi(\widetilde{q}_n) \to p_1$. Together with our assumption on ψ, it follows that f extends continuously to the real line $\{\mathrm{Im}(z) = 0\}$, which is mapped to $0 \in \mathbb{C}$.

Now pick any point z_0 on the real line, and pick any $0 < \epsilon_0 < \log(r_0)$. By Schwarz Reflection Principle, we can extend f on $\widetilde{\mathbb{A}_{r_0}} \cap B(z_0, \epsilon_0)$ to a holomorphic map $F : B(z_0, \epsilon_0) \to \mathbb{C}$, which is defined on $\{\mathrm{Im}(z) < 0\} \cap B(z_0, \epsilon_0)$ by $F(z) = \overline{f(\overline{z})}$. Since f maps the real line to $0 \in \mathbb{C}$, the holomorphic extension F is constant. But this contradicts the fact that $\pi \circ \varsigma$ is an immersion.

Step 6: Finally, we can conclude that G has also the conformal type of $\mathbb{D}^*$. Indeed, the identity $\mathrm{id} : (U_1^*, G) \to (U_1^*, H)$ is bi-Lipschitz by (8), hence it is quasiconformal. Since H is biholomorphic to $\mathbb{D}^*$ by the previous step, this implies that (U_1^*, G) cannot have the conformal type of $\mathbb{A}_{r_0}$ or $\mathbb{C}^*$ and is therefore conformal to $\mathbb{D}^*$, see for instance [Hub06, §4.3] □

4.2. **Proof of Corollary 1.2.** Having proved Proposition 3.4, our proof of Theorem 1.1 is complete. We now provide the proof of Corollary 1.2.

Proof of Corollary 1.2. Suppose $\iota : \Sigma_1 \to \mathbb{E}^3$ is a branched immersion, where $\mathfrak{p}_1$ is the (discrete) branching set, and let $\nu : \Sigma_1 \setminus \mathfrak{p}_1 \to \mathbb{S}^2$ be its Gauss map. It follows from the definition of branched immersion and the compactness of Σ_1 that ν extends to a branched covering $\Sigma_1 \to \mathbb{S}^2$. Up to applying a homothety, it is harmless to assume that the (constant) curvature of ι is one.

Let g_1 be the first fundamental form of ι and b its shape operator. Since $b = -d\nu$, we see immediately that $\nu^* g_{\mathbb{S}^2} = g_1(b\cdot, b\cdot)$, where of course $g_{\mathbb{S}^2}$ denotes the spherical metric. The $(1,1)$ tensor b is g_1-self-adjoint and positive definite (up to changing the sign of ν). Moreover, by the Gauss-Codazzi equations, b satisfies $\det b = 1$ and $d^{\nabla^{g_1}} b = 0$. This shows that $\mathrm{id} : (\Sigma_1, \mathfrak{p}_1, g_1) \to (\Sigma_1, \mathfrak{p}_1, \nu^* g_{\mathbb{S}^2})$ is a minimal Lagrangian diffeomorphism.

We claim that both g_1 and $\nu^* g_{\mathbb{S}^2}$, which are clearly spherical metrics on $\Sigma_1 \setminus \mathfrak{p}_1$, have cone singularities at the points of $\mathfrak{p}_1$. For $\nu^* g_{\mathbb{S}^2}$, this is clear since ν is a local isometry for $\nu^* g_{\mathbb{S}^2}$

and behaves, in a neighbourhood of any $p \in \mathfrak{p}_1$, like a degree d covering onto a punctured disc in $\mathbb{S}^2$. Hence $\nu^* g_{\mathbb{S}^2}$ has a cone point of cone angle $2d\pi$.

For g_1, pick any point $p \in \mathfrak{p}_1$. We know from the definition that both ι and ν admit a limit point at p. We can assume that the limit of ι is the origin of $\mathbb{R}^3$, and the limit of ν is $(0,0,1)$. We can now repeat Steps 4, 5 and 6 of the proof of Proposition 3.4 to show that g_1 has the conformal type of a punctured disc around p. Let us briefly summarize these steps for easiness of the reader. As in Step 4, for q in a small neighbourhood U of p, we have $\langle \nu(q), (0,0,1)\rangle > \epsilon > 0$. Hence $\pi \circ \iota$ is a locally bi-Lipschitz immersion with respect to the first fundamental form g_1 of ι and the flat metric of $\mathbb{R}^2$. More precisely $\mathrm{id} : (U, g_1) \to (U, h_1)$, where h_1 is the pull-back metric of $\pi \circ \iota$, is bi-Lipschitz. Repeating Step 5, we show that (U, h_1) is biholomorphic to $\mathbb{D}^*$ around p (up to restricting U to rule out the case $\mathbb{C}^*$): indeed, by lifting a biholomorphism $\psi : \mathbb{A}_{r_0} \to (U, h_1)$ to the universal cover (9) and applying Schwarz Reflection Principle to $\pi \circ \iota \circ \widetilde{\psi}$, one would obtain a contradiction. As in Step 6, one then shows that g_1 has the conformal type of $\mathbb{D}^*$ as well.

Moreover g_1 has finite area around p. Indeed h_1 has finite area: to see this, let $\psi : \mathbb{D}^* \to (U^*, h_1)$ be a biholomorphism, and consider the holomorphic map $\pi \circ \iota \circ \psi : \mathbb{D}^* \to \mathbb{R}^2$, that extends to a holomorphic map on $\mathbb{D}$, hence has finite degree if restricted to a compact neighbourhood of 0. Hence the area of h_1 is finite around p, and since g_1 is bi-Lipschitz to h_1, it has finite area as well. Then we can apply [Bry88, Proposition 4] and deduce that g_1 has a cone point at p, because it is a spherical metric, it is biholomorphic to $\mathbb{D}^*$ in a neighbourhood of p and has finite area.

Having showed that g_1 and $\nu^* g_{\mathbb{S}^2}$ are spherical cone metrics, it follows from Theorem 1.1 that $\mathrm{id} : (\Sigma_1, \mathfrak{p}_1, g_1) \to (\Sigma_1, \mathfrak{p}_1, \nu^* g_{\mathbb{S}^2})$ is an isometry, i.e. $g_1 = \nu^* g_{\mathbb{S}^2}$. Hence b is the identity operator. This means that, considering ν as an immersion in $\mathbb{R}^3$, its first fundamental form is g_1. Moreover its shape operator is the identity, since its image is a subset of $\mathbb{S}^2$. Hence ι and ν are immersions of $\Sigma_1 \setminus \mathfrak{p}_1$ having the same first fundamental form (namely g_1) and the same shape operator (namely the identity). By the uniqueness part of the fundamental theorem of surfaces, there exists an isometry A of $\mathbb{R}^3$ such that $\iota = A \circ \nu$. Since ν is a branched covering of $\mathbb{S}^2$, this implies that ι is a branched covering of some round sphere. □

We observe that in the proof of Corollary 1.2 we are making use of Theorem 1.1 in full strength, namely under the sole assumption that the minimal Lagrangian diffeomorphism φ extends *continuously* at the cone points. Indeed, by the definition of branch point for spherical surfaces, the Gauss map ν of the immersion ι is supposed to extend continuously at the branch points, without any additional regularity assumption.

4.3. **Final remarks.** In conclusion, we would like to add some related remarks.

Remark 4.2. Let (G, B) be the pair constructed in Definition 2.6. Recalling that the (1,1) tensor B is bounded (Lemma 2.8), and $G(B\cdot, \cdot)$ is the real part of a holomorphic quadratic differential q for the conformal structure induced by G (see Proposition A.2), a simple computation shows that if $\alpha \in (0, 1/2]$ (i.e. cone angle in $(0, \pi]$), then q has at most simple poles at p. If the cone angle is in $(\pi, 2\pi]$ then q does not have a pole, and if the cone angle is in $(k\pi, (k+1)\pi])$ for $k \geq 2$ then q has a zero of order at least $k-1$.

Remark 4.3. With some more technicalities, similar to those pursued in [BS16, §4] in a similar Lorentzian setting, one can strengthen the arguments in the proof of Proposition 3.4 and show that the metric G has a cone point of angle θ_1 at p. Moreover the immersion ς induces an embedding of a punctured disc in a *singular Euclidean space*, namely the singular

Riemannian manifold

$$|z|^{2\alpha_1-2}|dz|^2 + dt^2 \ ,$$

which is the product of the standard flat metric on $\mathbb{R}^2$ with cone angle $\theta_1 = 2\pi\alpha_1$ and $\mathbb{R}$. This induced embedding is orthogonal to the singular locus $\{z = 0\}$ and its first fundamental form of is precisely G. We stress that this follows from a local analysis, that is under the sole assumption that $\varphi : (U_1, p_1, g_1) \to (U_2, p_2, g_2)$ is a minimal Lagrangian diffeomorphism between discs endowed with spherical cone metrics.

Remark 4.4. A consequence of Remark 4.3 is that, if $\varphi : (U_1, p_1, g_1) \to (U_2, p_2, g_2)$ is a minimal Lagrangian diffeomorphism between discs endowed with spherical cone metrics, then the cone angles of g_1 and g_2 at p_1 and p_2 are necessarily equal. Indeed, Remark 4.3 shows that the metric G has a cone point of angle equal θ_1 at a point p_1; applying the same remark to φ^{-1} (Remark 2.7), one shows that this cone angle equals also θ_2.

This statement holds as well when the "cone angle" equals 2π, in the following sense. Suppose φ is a minimal Lagrangian diffeomorphism between $(U_1 \setminus \{p_1\}, g_1)$ and $(U_2 \setminus \{p_2\}, g_2)$ where *at least one* between g_1 and g_2 extends to a nonsingular spherical metric on the whole U_i. Then *both* g_1 *and* g_2 are nonsingular spherical metrics on U_1 and U_2.

In this setting, the arguments of the proof of Proposition 3.4 (up to switching φ and φ^{-1}, which does not affect the conclusion) can be adapted to show that φ can be realized as the Gauss map of an embedded surface of constant Gaussian curvature one in $\mathbb{R}^3$, and moreover the embedding extends continuously to p_1. Classical regularity for Monge-Ampère equations then implies (again only by a local analysis) that the embedding extend smoothly at p_1. Therefore the minimal Lagrangian map φ extends smoothly to a minimal Lagrangian diffeomorphism between U_1 and U_2.

Remark 4.5. As mentioned in the end of the introduction, our Theorem 1.1 and Corollary 1.2 (unlike Remarks 4.3 and 4.4) cannot be improved to purely local statements, and necessarily require some topological assumption, for instance closedness of the surface. Indeed, examples of non-isometric minimal Lagrangian diffeomorphisms between domains of $\mathbb{S}^2$ can be found as the Gauss maps of surfaces of constant Gaussian curvature in $\mathbb{R}^3$, for instance as surfaces of revolution. There are also many examples of branched immersions of (necessarily non-closed) surfaces with constant Gaussian curvature one, whose image is not contained in a round sphere (see for instance [Bra16]). Following the proof of Corollary 1.2, their Gauss maps induce minimal Lagrangian diffeomorphisms between open spherical surfaces with cone angles $2n\pi$.

Appendix A. Equivalent definitions

In this appendix we show that Definition 2.2 is equivalent to the condition that the graph of φ is a minimal Lagrangian submanifold in the product $\Sigma_1 \times \Sigma_2$, endowed with the Riemannian metric $g_1 \oplus g_2$ and with the symplectic form $\pi_1^* dA_{g_1} - \pi_2^* dA_{g_2}$. This is in fact a local computation.

Proposition A.1. *Let (U_1, g_1) and (U_2, g_2) be spherical surfaces, $\varphi : U_1 \to U_2$ a diffeomorphism, and b the unique positive definite, g_1-self-adjoint $(1,1)$ tensor on U_1 such that $\varphi^* g_2 = g_1(b\cdot, b\cdot)$. Then the graph of φ is minimal Lagrangian if and only if $d^{\nabla^{g_1}} b = 0$ and $\det b = 1$.*

The proof uses the following well-known characterization of Codazzi tensors, which is also applied in Appendix B:

Proposition A.2 ([Hop51]). *Given a Riemannian metric G on a surface Σ and a smooth $(1,1)$ tensor B, B is G-self-adjoint and traceless if and only if $G(B\cdot,\cdot)$ is the real part of a quadratic differential q. Moreover B is G-Codazzi if and only if q is holomorphic.*

Proof of Proposition A.1. First, let us observe that the area forms dA_{g_1} and $\varphi^* dA_{g_2}$ differ by the factor $\det b$, hence the graph of φ is Lagrangian (i.e. φ is area-preserving) if and only if $\det b = 1$. In the rest of the proof, we will always assume that both these conditions hold.

It is known ([ES64, §4]) that an immersion $F : U \hookrightarrow U_1 \times U_2$ is minimal if and only if it is conformal and harmonic, that is, if and only if it is harmonic with respect to the conformal class of $F^*(g_1 \oplus g_2)$ on U. Applying this to the graphical immersion $x \mapsto (x, \varphi(x))$, minimality of the graph of φ is equivalent to harmonicity of $\mathrm{id} : (U_1, g_1 \oplus \varphi^* g_2) \to (U_1, g_1)$ and of $\varphi : (U_1, g_1 \oplus \varphi^* g_2) \to (U_2, g_2)$.

So let us fix φ, take b as in the statement, and assume moreover $\det b = 1$. As in Definition 2.6, we introduce the metric $G = (1/4) g_1((\mathbb{1} + b)\cdot, (\mathbb{1} + b)\cdot)$ and the $(1,1)$ tensor $B = (\mathbb{1} + b)^{-1}(b - \mathbb{1})$. One sees immediately that B is G-self-adjoint and traceless.

It turns out that the metric G is conformal to $g_1 \oplus \varphi^* g_2$. Indeed $b^2 - \mathrm{tr}(b) b + (\det b)\mathbb{1} = 0$ by the Cayley-Hamilton theorem, hence $\mathbb{1} + b^2 = \mathrm{tr}(b) b$, which implies $g_1 + \varphi^* g_2 = \mathrm{tr}(b) g_1(b\cdot, \cdot)$ and

$$G = \frac{1}{4}\left(1 + \frac{2}{\mathrm{tr}(b)}\right)(g_1 + \varphi^* g_2) \ .$$

Since harmonicity only depends on the conformal class on the source, we will conclude the proof by showing that $d^{\nabla^{g_1}} b = 0$ if and only if $\mathrm{id} : (U_1, G) \to (U_1, g_1)$ and $\varphi : (U_1, G) \to (U_2, g_2)$ are harmonic.

As a last preliminary step, a direct computation shows

$$g_1 = G((\mathbb{1} - B)\cdot, (\mathbb{1} - B)\cdot) \tag{10}$$

for $B = (\mathbb{1} + b)^{-1}(b - \mathbb{1})$ as in Definition 2.6. Using again Cayley-Hamilton, since B is traceless, $B^2 = -(\det B)\mathbb{1}$, and it follows that

$$g_1 = (1 - \det B)\, G - 2G(B\cdot, \cdot) \ .$$

Since B is G-self-adjoint and traceless, by Proposition A.2 $G(B\cdot,\cdot)$ is the real part of a quadratic differential q. Then by definition the Hopf differential of $\mathrm{id} : (U_1, G) \to (U_1, g_1)$ is $-q$. Similarly, one writes

$$\varphi^* g_2 = G((\mathbb{1} + B)\cdot, (\mathbb{1} + B)\cdot) = (1 - \det B)\, G + 2G(B\cdot, \cdot) \ ,$$

hence the Hopf differential of $\varphi : (U_1, G) \to (U_2, g_2)$ equals q.

Now, Proposition 2.9 shows that if $d^{\nabla^{g_1}} b = 0$, then $d^{\nabla^G} B = 0$, which by Proposition A.2 is equivalent to q being holomorphic, and therefore that $\mathrm{id} : (U_1, G) \to (U_1, g_1)$ and $\varphi : (U_1, G) \to (U_2, g_2)$ are harmonic (since id and φ are diffeomorphisms, see [Sam78, §9]).

Conversely, suppose that $\mathrm{id} : (U_1, G) \to (U_1, g_1)$ is harmonic, hence q is holomorphic and thus by Proposition A.2 B satisfies the Codazzi condition $d^{\nabla^G} B = 0$. Using (10) and observing that $b = (\mathbb{1} - B)^{-1}(\mathbb{1} + B)$, one repeats exactly the same proof as in Proposition 2.9 to show that $d^{\nabla^{g_1}} b = 0$. This concludes the desired equivalence. □

Appendix B. Alternative proof of Lemma 3.1

In this appendix we provide a new proof of Lemma 3.1. Denote $\chi = (1/4)\log(-\det B)$, which is defined in the complement of zeros of B. We will show that

$$K_G = \Delta^G \chi \ . \tag{11}$$

New proof of Lemma 3.1. Let q be as in Proposition A.2, and assume that q does not vanish identically. Since (11) is a local statement, we will work on a local isothermal coordinate $z = x + iy$. Let us first prove the formula for a flat metric $G_0 = |dz|^2$ and a G_0-Codazzi tensor B_0. In this case we have:

$$B_0 = G_0^{-1}\mathrm{Re}(q) = \begin{pmatrix} \mathrm{Re}(\phi) & -\mathrm{Im}(\phi) \\ -\mathrm{Im}(\phi) & -\mathrm{Re}(\phi) \end{pmatrix} .$$

where $q = \phi(z)dz^2$ is the holomorphic quadratic differential as in Proposition A.2. Hence

$$\chi_0 = \frac{1}{4}\log(-\det B_0) = \frac{1}{4}\log(|\phi|^2) .$$

To show (11) in this case, it suffices to observe that χ_0 is harmonic with respect to G_0. Indeed,

$$\partial_z \chi_0 = \frac{1}{4}\partial_z \log(\phi\overline{\phi}) = \frac{1}{4\phi\overline{\phi}}\overline{\phi}\partial_z\phi = \frac{\partial_z\phi}{4\phi} .$$

Hence $\partial_{\overline{z}}\partial_z\chi_0 = 0$ and χ_0 is harmonic.

For the general case, given a Riemannian metric $G = e^{2f}|dz|^2$ and a G-Codazzi tensor B, as a consequence of Proposition A.2 we have that $B_0 := e^{2f}B$ is Codazzi with respect to the flat metric $G_0 = e^{-2f}G = |dz|^2$. Observe that $\det B_0 = e^{4f}\det B$, hence $\chi_0 = (1/4)\log(-\det B_0) = \chi + f$, and by the flat case discussed in the first part of the proof,

$$\Delta^{G_0}\chi_0 = \Delta^{G_0}\chi + \Delta^{G_0} f = 0 .$$

Using the formula for the curvature and Laplacian of a conformal metric, we have

$$\Delta^G\chi = e^{-2f}\Delta^{G_0}\chi = -e^{-2f}\Delta^{G_0} f = K_G$$

as claimed. □

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