

# KERNEL SELECTION IN NONPARAMETRIC REGRESSION

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ABSTRACT. In the regression model  $Y = b(X) + \varepsilon$ , where  $X$  has a density  $f$ , this paper deals with an oracle inequality for an estimator of  $bf$ , involving a kernel in the sense of Lerasle et al. (2016), selected via the PCO method. In addition to the bandwidth selection for kernel-based estimators already studied in Lacour, Massart and Rivoirard (2017) and Comte and Marie (2020), the dimension selection for anisotropic projection estimators of  $f$  and  $bf$  is covered.

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**MSC2010:** 62G05 ; 62G08.

## 1. INTRODUCTION

Consider  $n \geq 2$  independent  $\mathbb{R}^d$ -valued ( $d \geq 2$ ) random variables  $(X_1, Y_1), \dots, (X_n, Y_n)$ , having the same probability distribution assumed to be absolutely continuous with respect to Lebesgue's measure, and

$$\widehat{s}_{K,\ell}(n; x) := \frac{1}{n} \sum_{i=1}^n K(X_i, x) \ell(Y_i); \quad x \in \mathbb{R}^d,$$

where  $\ell: \mathbb{R} \rightarrow \mathbb{R}$  is a Borel function and  $K$  is a symmetric continuous map from  $\mathbb{R}^d \times \mathbb{R}^d$  into  $\mathbb{R}$ . This is an estimator of the function  $s: \mathbb{R}^d \rightarrow \mathbb{R}$  defined by

$$s(x) := \mathbb{E}(\ell(Y_1) | X_1 = x) f(x); \quad x \in \mathbb{R}^d,$$

where  $f$  is a density of  $X_1$ . For  $\ell = 1$ ,  $\widehat{s}_{K,\ell}(n; \cdot)$  coincides with the estimator of  $f$  studied in Lerasle et al. [11], but for  $\ell \neq 1$ , it covers estimators involved in nonparametric regression. Assume that for every  $i \in \{1, \dots, n\}$ ,

$$(1) \quad Y_i = b(X_i) + \varepsilon_i$$

where  $\varepsilon_i$  is a centered random variable, independent of  $X_i$ , and  $b: \mathbb{R}^d \rightarrow \mathbb{R}$  is a Borel function.

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If  $\ell = \text{Id}_{\mathbb{R}}$ ,  $k$  is a symmetric kernel and

$$(2) \quad K(x', x) = \prod_{q=1}^d \frac{1}{h_q} k\left(\frac{x'_q - x_q}{h_q}\right) \text{ with } h_1, \dots, h_d > 0$$

for every  $x, x' \in \mathbb{R}^d$ , then  $\widehat{s}_{K,\ell}(n; \cdot)$  is the numerator of Nadaraya-Watson's estimator of the regression function  $b$ . Precisely,  $\widehat{s}_{K,\ell}(n; \cdot)$  is an estimator of  $s = bf$ . If  $\ell \notin \text{Id}_{\mathbb{R}}$ , then  $\widehat{s}_{K,\ell}(n; \cdot)$  is the numerator of the estimator studied in Einmahl and Mason [5, 6].

If  $\ell = \text{Id}_{\mathbb{R}}$ ,  $\mathbf{B}_{m_q} = \{\varphi_1^{m_q}, \dots, \varphi_{m_q}^{m_q}\}$  ( $m_q \in \mathbb{N}^*$  and  $q \in \{1, \dots, d\}$ ) is an orthonormal family of  $\mathbb{L}^2(\mathbb{R})$  and

$$(3) \quad K(x', x) = \prod_{q=1}^d \sum_{j=1}^{m_q} \varphi_j^{m_q}(x_q) \varphi_j^{m_q}(x'_q)$$

for every  $x, x' \in \mathbb{R}^d$ , then  $\widehat{s}_{K,\ell}(n; \cdot)$  is the projection estimator on  $\mathbf{S} = \text{span}(\mathbf{B}_{m_1}, \dots, \mathbf{B}_{m_d})$  of  $s = bf$ .

Now, assume that for every  $i \in \{1, \dots, ng\}$ ,  $Y_i$  is defined by the heteroscedastic model

$$(4) \quad Y_i = \sigma(X_i) \varepsilon_i,$$

where  $\varepsilon_i$  is a centered random variable of variance 1, independent of  $X_i$ , and  $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}$  is a Borel function. If  $\ell(x) = x^2$  for every  $x \in \mathbb{R}$ , then  $\widehat{s}_{K,\ell}(n; \cdot)$  is an estimator of  $s = \sigma^2 f$ .

These ten last years, several data-driven procedures have been proposed in order to select the bandwidth of Parzen-Rosenblatt's estimator ( $\ell = 1$  and  $K$  defined by (2)). First, Goldenshluger-Lepski's method, introduced in [8], which reaches the adequate bias-variance compromise, but is not completely satisfactory on the numerical side (see Comte and Rebafka [4]). More recently, in [10], Lacour, Massart and Rivoirard proposed the PCO (Penalized Comparison to Overfitting) method and proved an oracle inequality for the associated adaptative Parzen-Rosenblatt's estimator by using a concentration inequality for the U-statistics due to Houdré and Reynaud-Bouret [9]. Together with Varet, they established the numerical efficiency of the PCO method in Varet et al. [13].

Comte and Marie [3] deals with an oracle inequality and numerical experiments for an adaptative Nadaraya-Watson's estimator with a numerator and a denominator having distinct bandwidths, both selected via the PCO method. Since the output variable in a regression model has no reason to be bounded, there were significant additional difficulties, bypassed in [3], to establish an oracle inequality for the numerator's adaptative estimator. Via similar arguments, the present article deals with an oracle inequality for  $\widehat{s}_{\widehat{K},\ell}(n; \cdot)$ , where  $\widehat{K}$  is selected via the PCO method in the spirit of Lerasle et al. [11]. In addition to the bandwidth selection for kernel-based estimators already studied in [10, 3], it covers the dimension selection for anisotropic projection estimators of  $f$ ,  $bf$  (when  $Y_1, \dots, Y_n$  are defined by Model (1)) and  $\sigma^2 f$  (when  $Y_1, \dots, Y_n$  are defined by Model (4)). As for the bandwidth selection for kernel based estimators, for  $d > 1$ , the PCO method allows to bypass the numerical difficulties generated by the Goldenshluger-Lepski type method involved in the anisotropic model selection procedures (see Chagny [1]).

In Section 2, some examples of kernels sets are provided and a risk bound for  $\widehat{s}_{K,\ell}(n; \cdot)$  is established. Section 3 deals with an oracle inequality for  $\widehat{s}_{\widehat{K},\ell}(n; \cdot)$ , where  $\widehat{K}$  is selected via the PCO method. Finally, Section 4 deals with a basic numerical study.

## 2. RISK BOUND

Throughout the paper,  $s \in \mathbb{L}^2(\mathbb{R}^d)$ . Let  $\mathbf{K}_n$  be a set of symmetric continuous maps from  $\mathbb{R}^d \times \mathbb{R}^d$  into  $\mathbb{R}$ , of cardinal less or equal than  $n$ , fulfilling the following assumption.

**Assumption 2.1.** *There exists a deterministic constant  $\mathfrak{m}_{\mathbf{K},\ell} > 0$ , not depending on  $n$ , such that*

(1) For every  $K \in \mathcal{K}_n$ ,

$$\sup_{x' \in \mathbb{R}^d} \|K(x', \cdot)\|_2^2 \leq \mathfrak{m}_{\mathcal{K}, \ell} n.$$

(2) For every  $K \in \mathcal{K}_n$ ,

$$\|s_{K, \ell}\|_2^2 \leq \mathfrak{m}_{\mathcal{K}, \ell}$$

with

$$s_{K, \ell} := \mathbb{E}(\widehat{s}_{K, \ell}(n; \cdot)) = \mathbb{E}(K(X_1, \cdot)\ell(Y_1)).$$

(3) For every  $K, K' \in \mathcal{K}_n$ ,

$$\mathbb{E}(\|hK(X_1, \cdot) - hK'(X_2, \cdot)\ell(Y_2)\|_2^2) \leq \mathfrak{m}_{\mathcal{K}, \ell} \mathfrak{s}_{K', \ell}$$

with

$$\mathfrak{s}_{K', \ell} := \mathbb{E}(\|K'(X_1, \cdot)\ell(Y_1)\|_2^2).$$

(4) For every  $K \in \mathcal{K}_n$  and  $\psi \in \mathbb{L}^2(\mathbb{R}^d)$ ,

$$\mathbb{E}(\|hK(X_1, \cdot) - \psi\|_2^2) \leq \mathfrak{m}_{\mathcal{K}, \ell} \|\psi\|_2^2.$$

The elements of  $\mathcal{K}_n$  are called kernels. Let us provide two natural examples of kernels sets.

**Proposition 2.2.** Consider

$$\mathcal{K}_k(h_{\min}) := \left\{ (x', x) \mapsto \prod_{q=1}^d \frac{1}{h_q} k\left(\frac{x'_q - x_q}{h_q}\right) ; h_1, \dots, h_d \in [h_{\min}, \dots, 1] \right\},$$

where  $k$  is a symmetric kernel (in the usual sense) and  $nh_{\min}^d \geq 1$ . The kernels set  $\mathcal{K}_k(h_{\min})$  fulfills Assumption 2.1 and, for any  $K \in \mathcal{K}_k(h_{\min})$  such that

$$K(x', x) = \prod_{q=1}^d \frac{1}{h_q} k\left(\frac{x'_q - x_q}{h_q}\right) ; \forall x, x' \in \mathbb{R}^d$$

with  $h_1, \dots, h_d \in [h_{\min}, \dots, 1]$ ,

$$\mathfrak{s}_{K, \ell} = \|k\|_2^{2d} \mathbb{E}(\ell(Y_1)^2) \prod_{q=1}^d \frac{1}{h_q}.$$

**Proposition 2.3.** Consider

$$\mathcal{K}_{\mathcal{B}_1, \dots, \mathcal{B}_n}(m_{\max}) := \left\{ (x', x) \mapsto \prod_{q=1}^d \sum_{j=1}^{m_q} \varphi_j^{m_q}(x_q) \varphi_j^{m_q}(x'_q) ; m_1, \dots, m_d \in [1, \dots, m_{\max}] \right\},$$

where  $m_{\max}^d \in [1, \dots, n]$  and, for every  $m \in [1, \dots, n]$ ,  $\mathcal{B}_m = \{\varphi_1^m, \dots, \varphi_m^m\}$  is an orthonormal family of  $\mathbb{L}^2(\mathbb{R})$  such that

$$\sup_{x' \in \mathbb{R}} \sum_{j=1}^m \varphi_j^m(x')^2 \leq \mathfrak{m}_{\mathcal{B}} m$$

with  $\mathfrak{m}_{\mathcal{B}} > 0$  not depending on  $m$  and  $n$ , and

$$(5) \quad \mathcal{B}_m \perp \mathcal{B}_{m+1} ; \forall m \in [1, \dots, n-1]$$

or

$$(6) \quad \bar{\mathfrak{m}}_{\mathcal{B}} := \sup_{j \in \mathbb{N}} \mathbb{E}(\|K(X_1, x)\|_j) ; K \in \mathcal{K}_{\mathcal{B}_1, \dots, \mathcal{B}_n}(m_{\max}) \text{ and } x \in \mathbb{R}^d \text{ is finite and doesn't depend on } n.$$

The kernels set  $\mathcal{K}_{\mathcal{B}_1, \dots, \mathcal{B}_n}(m_{\max})$  fulfills Assumption 2.1 and, for any  $K \in \mathcal{K}_{\mathcal{B}_1, \dots, \mathcal{B}_n}(m_{\max})$  such that

$$K(x', x) = \prod_{q=1}^d \sum_{j=1}^{m_q} \varphi_j^{m_q}(x_q) \varphi_j^{m_q}(x'_q) ; \forall x, x' \in \mathbb{R}^d$$

with  $m_1, \dots, m_n \geq 1, \dots, m_{\max} \mathbf{g}$ ,

$$\mathfrak{s}_{K,\ell} \leq m_{\mathcal{B}}^d \mathbb{E}(\ell(Y_1)^2) \prod_{q=1}^d m_q.$$

**Remark.** Note that Condition (5) (resp. (6)) is close to (resp. the same that) Condition (19) (resp. (20)) of Lerasle et al. [11], Proposition 3.2. See also Massart [12], Chapter 7 on these conditions. For instance, the trigonometric basis and Hermite's basis satisfy Condition (5). The regular histograms basis satisfy Condition (6). Indeed, by taking  $\varphi_j^m = \psi_j^m := \frac{1}{m} \mathbf{1}_{[(j-1)/m, j/m[}$  for every  $m \geq 1, \dots, n \mathbf{g}$  and  $j \geq 1, \dots, m \mathbf{g}$ ,

$$\begin{aligned} \left| \mathbb{E} \left[ \prod_{q=1}^d \sum_{j=1}^{m_q} \psi_j^{m_q}(X_{1,q}) \psi_j^{m_q}(x_q) \right] \right| &= \sum_{j_1=1}^{m_1} \sum_{j_d=1}^{m_d} \left( \prod_{q=1}^d m_q \mathbf{1}_{[(j_q-1)/m_q, j_q/m_q[}(x_q) \right) \\ &\quad \int_{(j_1-1)/m_1}^{j_1/m_1} \int_{(j_d-1)/m_d}^{j_d/m_d} f(x'_1, \dots, x'_d) dx'_1 \dots dx'_d \\ &\leq k f k_{\infty} \prod_{q=1}^d \sum_{j=1}^{m_q} \mathbf{1}_{[(j-1)/m_q, j/m_q[}(x) \leq k f k_{\infty} \end{aligned}$$

for every  $m_1, \dots, m_d \geq 1, \dots, n \mathbf{g}$  and  $x \in \mathbb{R}^d$ .

The following proposition provides a suitable control of the variance of  $\widehat{s}_{K,\ell}(n; \cdot)$ .

**Proposition 2.4.** *Under Assumption 2.1.(1,2,3), if  $s \in \mathbb{L}^2(\mathbb{R}^d)$  and if there exists  $\alpha > 0$  such that  $\mathbb{E}(\exp(\alpha j \ell(Y_1))) < 1$ , then there exists a deterministic constant  $\mathfrak{c}_{2.4} > 0$ , not depending on  $n$ , such that for every  $\theta \in ]0, 1[$ ,*

$$\mathbb{E} \left( \sup_{K \in \mathcal{K}_n} \left\{ \left| k \widehat{s}_{K,\ell}(n; \cdot) - s_{K,\ell} k_2^2 - \frac{\mathfrak{s}_{K,\ell}}{n} \right| - \frac{\theta}{n} \mathfrak{s}_{K,\ell} \right\} \right) \leq \mathfrak{c}_{2.4} \frac{\log(n)^5}{\theta n}.$$

Finally, let us state the main result of this section.

**Theorem 2.5.** *Under Assumption 2.1, if  $s \in \mathbb{L}^2(\mathbb{R}^d)$  and if there exists  $\alpha > 0$  such that  $\mathbb{E}(\exp(\alpha j \ell(Y_1))) < 1$ , then there exists a deterministic constant  $\mathfrak{c}_{2.5}, \bar{\mathfrak{c}}_{2.5} > 0$ , not depending on  $n$ , such that for every  $\theta \in ]0, 1[$ ,*

$$\mathbb{E} \left( \sup_{K \in \mathcal{K}_n} \left\{ k \widehat{s}_{K,\ell}(n; \cdot) - s k_2^2 - (1 + \theta) \left( k s_{K,\ell} - s k_2^2 + \frac{\mathfrak{s}_{K,\ell}}{n} \right) \right\} \right) \leq \mathfrak{c}_{2.5} \frac{\log(n)^5}{\theta n}$$

and

$$\mathbb{E} \left( \sup_{K \in \mathcal{K}_n} \left\{ k s_{K,\ell} - s k_2^2 + \frac{\mathfrak{s}_{K,\ell}}{n} - \frac{1}{1 - \theta} k \widehat{s}_{K,\ell}(n; \cdot) - s k_2^2 \right\} \right) \leq \bar{\mathfrak{c}}_{2.5} \frac{\log(n)^5}{\theta(1 - \theta)n}.$$

**Remark.** Note that the first inequality in Theorem 2.5 gives a risk bound on the estimator  $\widehat{s}_{K,\ell}(n; \cdot)$ :

$$\mathbb{E}(k \widehat{s}_{K,\ell}(n; \cdot) - s k_2^2) \leq (1 + \theta) \left( k s_{K,\ell} - s k_2^2 + \frac{\mathfrak{s}_{K,\ell}}{n} \right) + \mathfrak{c}_{2.5} \frac{\log(n)^5}{\theta n}$$

for every  $\theta \in ]0, 1[$ . The second inequality is useful in order to establish a risk bound on the adaptive estimator defined in the next section (see Theorem 3.2).

### 3. KERNEL SELECTION

This section deals with a risk bound on the adaptive estimator  $\widehat{s}_{\widehat{K},\ell}(n; \cdot)$ , where

$$\widehat{K} \geq \arg \min_{K \in \mathcal{K}_n} \left( k \widehat{s}_{K,\ell}(n; \cdot) - \widehat{s}_{K_0,\ell}(n; \cdot) k_2^2 + \text{pen}(K) \mathbf{g} \right)$$

$K_0$  is an overfitting proposal for  $K$  and

$$(7) \quad \text{pen}(K) := \frac{2}{n^2} \sum_{i=1}^n \mathbb{h} K(\cdot, X_i), K_0(\cdot, X_i) i_2 \ell(Y_i)^2; \quad \forall K \in \mathcal{K}_n.$$

**Example.** For  $K_n = K_k(h_{\min})$ , one should take

$$K_0(x', x) = \frac{1}{h_{\min}^d} \prod_{q=1}^d k\left(\frac{x'_q - x_q}{h_{\min}}\right); \quad \forall x, x' \in \mathbb{R}^d,$$

and for  $K_n = K_{\mathcal{B}_1, \dots, \mathcal{B}_n}(m_{\max})$ , one should take

$$K_0(x', x) = \prod_{q=1}^d \sum_{j=1}^{m_{\max}} \varphi_j^{m_{\max}}(x_q) \varphi_j^{m_{\max}}(x'_q); \quad \forall x, x' \in \mathbb{R}^d.$$

In the sequel, in addition to Assumption 2.1, the kernel set  $K_n$  fulfills the following assumption.

**Assumption 3.1.** *There exists a deterministic constant  $\bar{m}_{\mathcal{K}, \ell} > 0$ , not depending on  $n$ , such that*

$$\mathbb{E} \left( \sup_{K, K' \in \mathcal{K}_n} \mathbb{h}K(X_1, \cdot), s_{K', \ell} \right)_2^2 \leq \bar{m}_{\mathcal{K}, \ell}.$$

The following theorem provides an oracle inequality for the adaptive estimator  $\widehat{s}_{\widehat{K}, \ell}(n; \cdot)$ .

**Theorem 3.2.** *Under Assumptions 2.1 and 3.1, if  $s \in \mathbb{L}^2(\mathbb{R}^d)$  and if there exists  $\alpha > 0$  such that  $\mathbb{E}(\exp(\alpha |j|)) < \infty$ , then there exists a deterministic constant  $c_{3.2} > 0$ , not depending on  $n$ , such that for every  $\vartheta \in ]0, 1[$ ,*

$$\mathbb{E}(\mathbb{h}\widehat{s}_{\widehat{K}, \ell}(n; \cdot) - s)_2^2 \leq (1 + \vartheta) \min_{K \in \mathcal{K}_n} \mathbb{E}(\mathbb{h}s_{K, \ell}(n; \cdot) - s)_2^2 + \frac{c_{3.2}}{\vartheta} \left( \mathbb{h}s_{K_0, \ell} - s \right)_2^2 + \frac{\log(n)^5}{n}.$$

Finally, let us discuss about Assumption 3.1. Note that if  $s$  is bounded and

$$\mathbf{m}_{\mathcal{K}} := \sup_{K \in \mathcal{K}_n} \mathbb{h}K(x', \cdot)_1^2; \quad K \in \mathcal{K}_n \text{ and } x' \in \mathbb{R}^d$$

doesn't depend on  $n$ , then  $K_n$  fulfills Assumption 3.1. Indeed,

$$\begin{aligned} \mathbb{E} \left( \sup_{K, K' \in \mathcal{K}_n} \langle K(X_1, \cdot), s_{K', \ell} \rangle_2^2 \right) &\leq \left( \sup_{K' \in \mathcal{K}_n} \|s_{K', \ell}\|_{\infty}^2 \right) \mathbb{E} \left( \sup_{K \in \mathcal{K}_n} \|K(X_1, \cdot)\|_1^2 \right) \\ &\leq \mathbf{m}_{\mathcal{K}} \sup \left\{ \left( \int_{-\infty}^{\infty} |K'(x', x) s(x)| dx \right)^2; \quad K' \in \mathcal{K}_n \text{ and } x' \in \mathbb{R}^d \right\} \leq \mathbf{m}_{\mathcal{K}}^2 \|s\|_{\infty}^2. \end{aligned}$$

In the nonparametric regression framework (see Model (1)), to assume  $s$  bounded means that  $bf$  is bounded. For instance, this condition is fulfilled by the linear regression models with Gaussian inputs. The following examples focus on the condition on  $\mathbf{m}_{\mathcal{K}}$ .

**Examples:**

- (1) Consider  $K \in K_k(h_{\min})$ . Then, there exists  $h_1, \dots, h_d \in ]h_{\min}, \infty[$  such that

$$K(x', x) = \prod_{q=1}^d \frac{1}{h_q} k\left(\frac{x'_q - x_q}{h_q}\right); \quad \forall x, x' \in \mathbb{R}^d.$$

Clearly,  $\mathbb{h}K(x', \cdot)_1 = \prod_{q=1}^d \frac{1}{h_q} k\left(\frac{x'_q - x_q}{h_q}\right)$  for every  $x' \in \mathbb{R}^d$ . So, for  $K_n = K_k(h_{\min})$ ,  $\mathbf{m}_{\mathcal{K}} \leq \prod_{q=1}^d \frac{1}{h_q^2}$ .

- (2) For  $K_n = K_{\mathcal{B}_1, \dots, \mathcal{B}_n}(m_{\max})$ , the condition on  $\mathbf{m}_{\mathcal{K}}$  seems harder to check in general. Let us show that it is satisfied for the regular histograms basis defined in Section 2. For every  $m_1, \dots, m_d \in \mathbb{N}$ ,

$$\left\| \prod_{q=1}^d \sum_{j=1}^{m_q} \psi_j^{m_q}(x'_q) \psi_j^{m_q}(\cdot) \right\|_1 \leq \prod_{q=1}^d \left( m_q \sum_{j=1}^{m_q} \mathbf{1}_{[(j-1)/m_q, j/m_q]}(x'_q) \int_{(j-1)/m_q}^{j/m_q} dx \right) \leq 1.$$

The following proposition shows that  $K_{\mathcal{B}_1, \dots, \mathcal{B}_n}(m_{\max})$  fulfills Assumption 3.1 for the trigonometric basis, even if the condition on  $\mathbf{m}_{\mathcal{K}}$  is not satisfied.

**Proposition 3.3.** Consider  $\chi_1 := \mathbf{1}_{[0,1]}$  and, for every  $j \in \mathbb{N}^*$ , the functions  $\chi_{2j}$  and  $\chi_{2j+1}$  defined on  $\mathbb{R}$  by

$$\chi_{2j}(x) := \frac{1}{2} \cos(2\pi j x) \mathbf{1}_{[0,1]}(x) \text{ and } \chi_{2j+1}(x) := \frac{1}{2} \sin(2\pi j x) \mathbf{1}_{[0,1]}(x); \quad \forall x \in \mathbb{R}.$$

If  $s \in C^2(\mathbb{R}^d)$  and  $\mathcal{B}_m = \langle \chi_1, \dots, \chi_m \rangle$  for every  $m \in \{1, \dots, n\}$ , then  $\mathcal{K}_{\mathcal{B}_1, \dots, \mathcal{B}_n}(m_{\max})$  fulfills Assumption 3.1.

#### 4. BASIC NUMERICAL EXPERIMENTS

Throughout this section,  $d = 1$ ,  $\ell \in \{1, \text{Id}_{\mathbb{R}}\}$  and  $Y_1, \dots, Y_n$  are defined by Model (1) with  $\varepsilon_1, \dots, \varepsilon_n \sim \mathcal{N}(0, 1)$ . Some numerical experiments on  $\widehat{s}_{K,1}(n; \cdot)$  (resp.  $\widehat{s}_{K, \text{Id}_{\mathbb{R}}}(n; \cdot)$ ) for  $K \in \mathcal{K}_k(h_{\min})$  have already been done in Varet et al. [13] (resp. Comte and Marie [3]). So, this section deals with basic numerical experiments on  $\widehat{s}_{K,1}(n; \cdot)$  and  $\widehat{s}_{K, \text{Id}_{\mathbb{R}}}(n; \cdot)$  for  $K \in \mathcal{K}_{\mathcal{B}_1, \dots, \mathcal{B}_n}(m_{\max})$  and  $\mathcal{B}_m = \langle \psi_1^m, \dots, \psi_m^m \rangle$  for every  $m = 1, \dots, n$ .

In this case,  $\widehat{K} = \mathcal{K}_{\widehat{m}(\ell)}$  where

$$\mathcal{K}_m(x', x) := \sum_{j=1}^m \psi_j^m(x') \psi_j^m(x); \quad \forall x, x' \in \mathbb{R}, \quad \forall m \in \mathcal{M} = \{1, \dots, m_{\max}\},$$

$\widehat{m}(\ell)$  is a solution of the minimization problem

$$\min_{m \in \mathcal{M}} \langle \widehat{s}_{K_m, \ell}(n; \cdot) - \widehat{s}_{K_{m_{\max}}, \ell}(n; \cdot) \rangle_2^2 + \text{pen}(m)$$

and

$$\text{pen}(m) := \frac{2}{n^2} \sum_{i=1}^n \langle \mathcal{K}_m(\cdot, X_i), \mathcal{K}_{m_{\max}}(\cdot, X_i) \rangle_2 \ell(Y_i)^2; \quad \forall m \in \mathcal{M}.$$

For  $\ell \in \{1, \text{Id}_{\mathbb{R}}\}$ ,  $n = 250$  and  $m_{\max} = 30$ ,  $m$  is selected in  $\mathcal{M}$  for two basic densities and two nonlinear regression functions:

- $f = f_1$  the density of  $\mathcal{E}(5)$ .
- $f = f_2$  the density of  $\mathcal{N}(1/2, (1/8)^2)$ .
- $b(x) = b_1(x) := 10(x^2 - 1/2)$  for every  $x \in [0, 1]$ .
- $b(x) = b_2(x) := \cos(5\pi x)$  for every  $x \in [0, 1]$ .

On the one hand, on the four following figures, one can see the beam of all possible estimations of  $f$  and  $bf$  (i.e. for each  $m \in \mathcal{M}$ ) at left, the PCO criteria for  $\widehat{s}_{K,1}(n; \cdot)$  and  $\widehat{s}_{K, \text{Id}_{\mathbb{R}}}(n; \cdot)$  for each  $m \in \mathcal{M}$  at the middle, and the PCO estimations of  $f$  and  $bf$  (i.e. for  $m = \widehat{m}(1)$  and  $m = \widehat{m}(\text{Id}_{\mathbb{R}})$ ) at right:

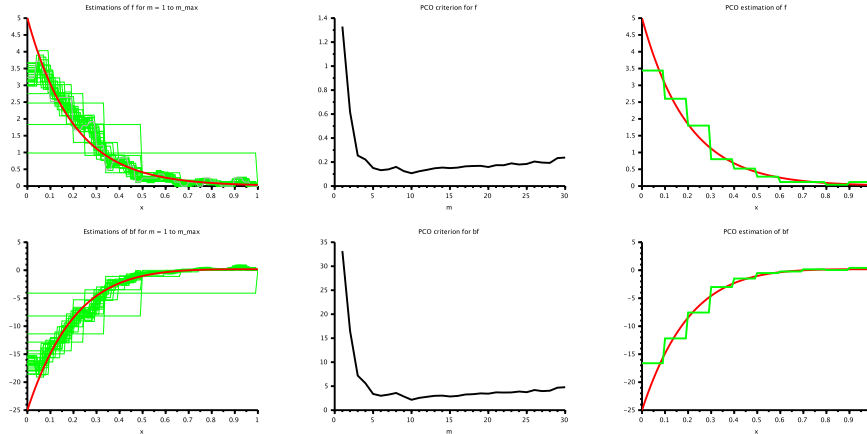


FIGURE 1.  $f = f_1$ ,  $b = b_1$ ,  $\widehat{m}(1) = 10$  and  $\widehat{m}(\text{Id}_{\mathbb{R}}) = 10$ .

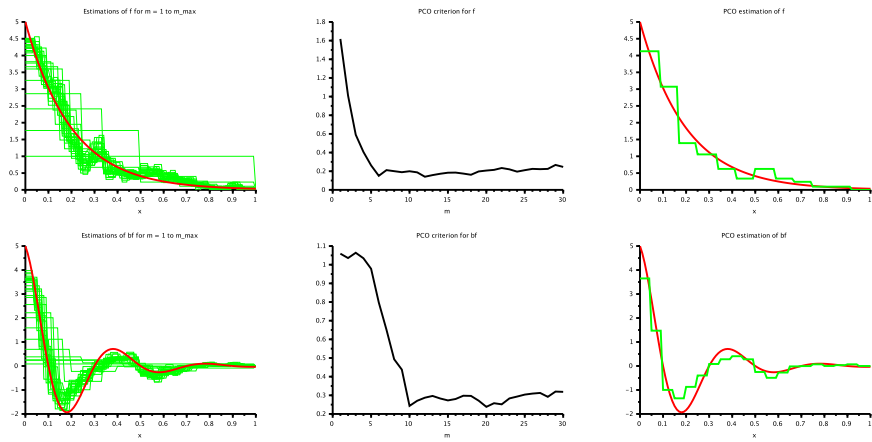


FIGURE 2.  $f = f_1$ ,  $b = b_2$ ,  $\hat{m}(1) = 12$  and  $\hat{m}(\text{Id}_{\mathbb{R}}) = 20$ .

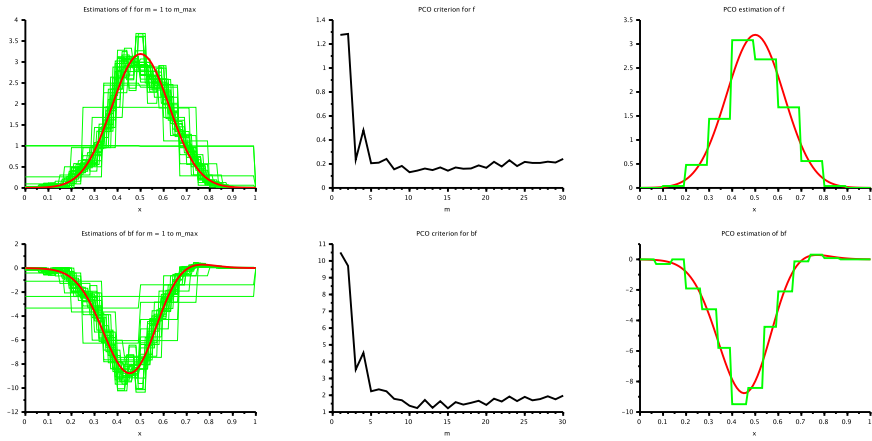


FIGURE 3.  $f = f_2$ ,  $b = b_1$ ,  $\hat{m}(1) = 10$  and  $\hat{m}(\text{Id}_{\mathbb{R}}) = 15$ .

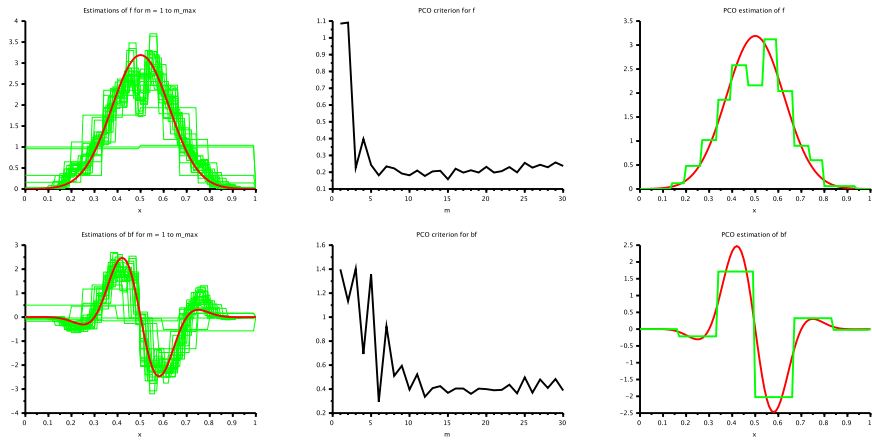
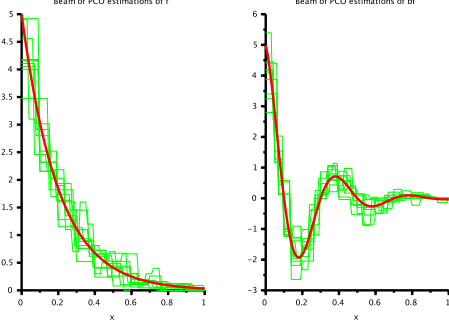
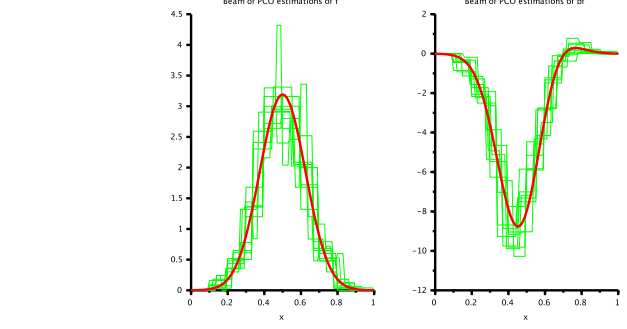


FIGURE 4.  $f = f_2$ ,  $b = b_2$ ,  $\hat{m}(1) = 15$  and  $\hat{m}(\text{Id}_{\mathbb{R}}) = 6$ .

On the other hand, for  $(f, b) = (f_1, b_2)$  and  $(f, b) = (f_2, b_1)$ , let us generate 10 datasets of  $n = 250$  observations of  $(X_1, Y_1)$  and, for each of these, select  $m \leq M$  via the PCO criterion introduced previously. On the two following figures, the beam of all PCO estimations of  $f$  (resp.  $bf$ ) is plotted at left (resp. at right):

FIGURE 5.  $f = f_1$  and  $b = b_2$ .FIGURE 6.  $f = f_2$  and  $b = b_1$ .

#### APPENDIX A. DETAILS ON KERNELS SETS: PROOFS OF PROPOSITIONS 2.2, 2.3 AND 3.3

**A.1. Proof of Proposition 2.2.** Consider  $K, K' \in \mathcal{K}_k(h_{\min})$ . Then, there exist  $h, h' \in [h_{\min}, \dots, 1] \mathfrak{g}^d$  such that

$$K(x', x) = \prod_{q=1}^d \frac{1}{h_q} k\left(\frac{x'_q - x_q}{h_q}\right) \text{ and } K'(x', x) = \prod_{q=1}^d \frac{1}{h'_q} k\left(\frac{x'_q - x_q}{h'_q}\right)$$

for every  $x, x' \in \mathbb{R}^d$ .

(1) For every  $x' \in \mathbb{R}^d$ ,

$$\|K(x', \cdot)\|_2^2 = \|k\|_2^{2d} \prod_{q=1}^d \frac{1}{h_q} \leq \|k\|_2^{2d} n.$$

(2) Since  $s_{K, \ell} = \|K\|_2$ ,  $\|s_{K, \ell}\|_2^2 \leq \|k\|_1^{2d} \|k\|_2^2$ .

(3) First,

$$s_{K', \ell} = \|k\|_2^{2d} \mathbb{E}(\ell(Y_1)^2) \prod_{q=1}^d \frac{1}{h'_q}.$$

Then,

$$\begin{aligned} \mathbb{E}(\|hK(X_1, \cdot) - K'(X_2, \cdot)\ell(Y_2)\|_2^2) &= \mathbb{E}(\|(K - K')(X_1 - X_2)^2 \ell(Y_2)^2\|_2^2) \\ &\leq \|f\|_\infty \|K - K'\|_2^2 \mathbb{E}(\ell(Y_1)^2) \\ &\leq \|f\|_\infty \|k\|_1^{2d} s_{K', \ell}. \end{aligned}$$

(4) For every  $\psi \in \mathbb{L}^2(\mathbb{R}^d)$ ,

$$\begin{aligned} \mathbb{E}(\|hK(X_1, \cdot) - \psi\|_2^2) &= \mathbb{E}(\|(K - \psi)(X_1)^2\|_2^2) \\ &\leq \|f\|_\infty \|K - \psi\|_2^2 \leq \|f\|_\infty \|k\|_1^{2d} \|k\|_2^2 \|\psi\|_2^2. \end{aligned}$$

**A.2. Proof of Proposition 2.3.** Consider  $K, K' \in \mathcal{K}_{\mathcal{B}_1, \dots, \mathcal{B}_n}(m_{\max})$ . Then, there exist  $m, m' \in [1, \dots, m_{\max}] \mathfrak{g}^d$  such that

$$K(x', x) = \prod_{q=1}^d \sum_{j=1}^{m_q} \varphi_j^{m_q}(x_q) \varphi_j^{m_q}(x'_q) \text{ and } K'(x', x) = \prod_{q=1}^d \sum_{j=1}^{m'_q} \varphi_j^{m'_q}(x_q) \varphi_j^{m'_q}(x'_q)$$

for every  $x, x' \in \mathbb{R}^d$ .



(1) For every  $x' \in \mathbb{R}^d$ ,

$$\begin{aligned} \|K(x', \cdot)\|_2^2 &= \prod_{q=1}^d \sum_{j, j'=1}^{m_q} \varphi_{j'}^{m_q}(x'_q) \varphi_j^{m_q}(x'_q) \int_{-\infty}^{\infty} \varphi_{j'}^{m_q}(x) \varphi_j^{m_q}(x) dx \\ &= \prod_{q=1}^d \sum_{j=1}^{m_q} \varphi_j^{m_q}(x'_q)^2 \leq \mathbf{m}_{\mathcal{B}}^d \prod_{q=1}^d m_q \leq \mathbf{m}_{\mathcal{B}}^d n. \end{aligned}$$

(2) Since

$$s_{K, \ell}(\cdot) = \sum_{j_1=1}^{m_1} \sum_{j_d=1}^{m_d} \mathfrak{h} s, \varphi_{j_1}^{m_1} \quad \varphi_{j_d}^{m_d} \mathfrak{i}_2(\varphi_{j_1}^{m_1} \quad \varphi_{j_d}^{m_d})(\cdot),$$

by Pythagore's theorem,  $\|s_{K, \ell}\|_2^2 \leq \|s\|_2^2$ .

(3) First,

$$s_{K', \ell} = \mathbb{E} \left[ \ell(Y_1)^2 \prod_{q=1}^d \sum_{j=1}^{m'_q} \varphi_j^{m'_q}(X_{1,q})^2 \right] \leq \mathbf{m}_{\mathcal{B}}^d \mathbb{E}(\ell(Y_1)^2) \prod_{q=1}^d m'_q.$$

On the one hand, if  $\mathcal{B}_1, \dots, \mathcal{B}_n$  satisfy Condition (5), then

$$\begin{aligned} \mathbb{E}(\langle K(X_1, \cdot), K'(X_2, \cdot) \ell(Y_2) \rangle_2^2) &= \int_{\mathbb{R}^d} \mathbb{E} \left[ \left( \prod_{q=1}^d \sum_{j=1}^{m_q \wedge m'_q} \varphi_j^{m'_q}(x'_q) \varphi_j^{m_q}(X_{2,q}) \right)^2 \ell(Y_2)^2 \right] f(x') \lambda_d(dx') \\ &\leq \|f\|_{\infty} \mathbb{E} \left[ \ell(Y_2)^2 \prod_{q=1}^d \sum_{j, j'=1}^{m_q \wedge m'_q} \varphi_{j'}^{m'_q}(X_{2,q}) \varphi_j^{m_q}(X_{2,q}) \int_{-\infty}^{\infty} \varphi_{j'}^{m'_q}(x') \varphi_j^{m_q}(x') dx' \right] \\ &\leq \|f\|_{\infty} s_{K', \ell}. \end{aligned}$$

On the other hand, if  $\mathcal{B}_1, \dots, \mathcal{B}_n$  satisfy Condition (6), then

$$\begin{aligned} \mathbb{E}(\mathfrak{h} K(X_1, \cdot), K'(X_2, \cdot) \ell(Y_2) \mathfrak{i}_2^2) &\leq \mathbb{E}(\|K(X_1, \cdot)\|_2^2 \|K'(X_2, \cdot)\|_2^2 \ell(Y_2)^2) \\ &= \mathbb{E}(K(X_1, X_1)) \mathbb{E}(\|K'(X_2, \cdot)\|_2^2 \ell(Y_2)^2) \leq \bar{\mathbf{m}}_{\mathcal{B}} s_{K', \ell}. \end{aligned}$$

(4) For every  $\psi \in \mathbb{L}^2(\mathbb{R}^d)$ ,

$$\begin{aligned} \mathbb{E}(\mathfrak{h} K(X_1, \cdot), \psi \mathfrak{i}_2^2) &= \mathbb{E} \left[ \left| \sum_{j_1=1}^{m_1} \sum_{j_d=1}^{m_d} \mathfrak{h} \psi, \varphi_{j_1}^{m_1} \quad \varphi_{j_d}^{m_d} \mathfrak{i}_2(\varphi_{j_1}^{m_1} \quad \varphi_{j_d}^{m_d})(X_1) \right|^2 \right] \\ &\leq \|f\|_{\infty} \left\| \sum_{j_1=1}^{m_1} \sum_{j_d=1}^{m_d} \mathfrak{h} \psi, \varphi_{j_1}^{m_1} \quad \varphi_{j_d}^{m_d} \mathfrak{i}_2(\varphi_{j_1}^{m_1} \quad \varphi_{j_d}^{m_d})(\cdot) \right\|_2^2 \leq \|f\|_{\infty} \| \psi \|_2^2. \end{aligned}$$

**A.3. Proof of Proposition 3.3.** For the sake of readability, assume that  $d = 1$ . Consider  $K, K' \in \mathcal{K}_{\mathcal{B}_1, \dots, \mathcal{B}_n}(m_{\max})$ . Then, there exist  $m, m' \in \mathbb{F}_1, \dots, m_{\max} \mathfrak{g}$  such that

$$K(x', x) = \sum_{j=1}^m \chi_j(x) \chi_j(x') \text{ and } K'(x', x) = \sum_{j=1}^{m'} \chi_j(x) \chi_j(x') ; \forall x, x' \in \mathbb{R}.$$

First, there exist  $\mathbf{m}_1(m, m') \in \mathbb{N}$  and  $\mathbf{c}_1 > 0$ , not depending on  $n$ ,  $K$  and  $K'$ , such that for any  $x' \in [0, 1]$ ,

$$\begin{aligned} |hK(x', \cdot), s_{K', \ell}|_2 &= \left| \sum_{j=1}^{m \wedge m'} \mathbb{E}(\ell(Y_1) \chi_j(X_1)) \chi_j(x') \right| \\ &\leq \mathbf{c}_1 + 2 \left| \sum_{j=1}^{\mathbf{m}_1(m, m')} \mathbb{E}(\ell(Y_1) (\cos(2\pi j X_1) \cos(2\pi j x') + \sin(2\pi j X_1) \sin(2\pi j x'))) \mathbf{1}_{[0,1]}(X_1) \right| \\ &= \mathbf{c}_1 + 2 \left| \sum_{j=1}^{\mathbf{m}_1(m, m')} \mathbb{E}(\ell(Y_1) \cos(2\pi j (X_1 - x'))) \mathbf{1}_{[0,1]}(X_1) \right|. \end{aligned}$$

Moreover, for any  $j \in \mathbb{N}$ ,

$$\begin{aligned} \mathbb{E}(\ell(Y_1) \cos(2\pi j (X_1 - x'))) \mathbf{1}_{[0,1]}(X_1) &= \int_0^1 \cos(2\pi j (x - x')) s(x) dx \\ &= \frac{1}{j} \left[ \frac{\sin(2\pi j (x - x'))}{2\pi} s(x) \right]_0^1 \\ &\quad + \frac{1}{j^2} \left[ \frac{\cos(2\pi j (x - x'))}{4\pi^2} s'(x) \right]_0^1 - \frac{1}{j^2} \int_0^1 \frac{\cos(2\pi j (x - x'))}{4\pi^2} s''(x) dx \\ &= \frac{s(0) - s(1)}{2\pi} - \frac{\alpha_j(x')}{j} + \frac{\beta_j(x')}{j^2} \end{aligned}$$

where  $\alpha_j(x') := \sin(2\pi j x')$  and

$$\beta_j(x') := \frac{1}{4\pi^2} \left( (s'(1) - s'(0)) \cos(2\pi j x') - \int_0^1 \cos(2\pi j (x - x')) s''(x) dx \right).$$

Then, there exists a deterministic constant  $\mathbf{c}_2 > 0$ , not depending on  $n$ ,  $K$ ,  $K'$  and  $x'$ , such that

$$(8) \quad |hK(x', \cdot), s_{K', \ell}|_2^2 \leq \mathbf{c}_2 \left[ 1 + \left( \sum_{j=1}^{\mathbf{m}_1(m, m')} \frac{\alpha_j(x')}{j} \right)^2 + \left( \sum_{j=1}^{\mathbf{m}_1(m, m')} \frac{\beta_j(x')}{j^2} \right)^2 \right].$$

Let us show that each term of the right-hand side of Inequality (8) are uniformly bounded in  $x'$ ,  $m$  and  $m'$ . On the one hand,

$$\left| \sum_{j=1}^{\mathbf{m}_1(m, m')} \frac{\beta_j(x')}{j^2} \right| \leq \max_{j \in \{1, \dots, n\}} |\beta_j| \sum_{j=1}^n \frac{1}{j^2} \leq \frac{1}{24} (2Ks'K_\infty + Ks''K_\infty).$$

On the other hand, for every  $x \in [0, \pi]$  such that  $[\pi/x] + 1 \leq \mathbf{m}_1(m, m')$  (without loss of generality),

$$(9) \quad \begin{aligned} \left| \sum_{j=1}^{\mathbf{m}_1(m, m')} \frac{\sin(jx)}{j} \right| &\leq \left| \sum_{j=1}^{[\pi/x]} \frac{\sin(jx)}{j} \right| + \left| \sum_{j=[\pi/x]+1}^{\mathbf{m}_1(m, m')} \frac{\sin(jx)}{j} \right| \\ &\leq x \left[ \frac{\pi}{x} \right] + \frac{2}{(1 + [\pi/x]) \sin(x/2)} \leq \pi + 2. \end{aligned}$$

Since  $x \mapsto \sin(x)$  is continuous, odd and  $2\pi$ -periodic, Inequality (9) holds true for every  $x \in \mathbb{R}$ . So,

$$\left| \sum_{j=1}^{\mathbf{m}_1(m, m')} \frac{\alpha_j(x')}{j} \right| \leq \pi + 2.$$

Therefore,

$$\mathbb{E} \left[ \sup_{K, K' \in \mathcal{K}_{\mathcal{B}_1, \dots, \mathcal{B}_n}(m_{\max})} \mathfrak{h}K(X_1, \cdot), s_{K', \ell} \mathfrak{i}_2^2 \right] \leq \mathfrak{c}_2 \left( 1 + (\pi + 2)^2 + \frac{1}{24^2} (2ks'k_\infty + ks''k_\infty)^2 \right).$$

## APPENDIX B. PROOFS OF RISK BOUNDS

In this section, the proofs follow the same pattern as in Comte and Marie [2, 3].

**B.1. Preliminary results.** This subsection provides three lemmas used several times in the sequel.

**Lemma B.1.** *Consider*

$$U_{K, K', \ell}(n) := \sum_{i \neq j} \mathfrak{h}K(X_i, \cdot) \ell(Y_i) \quad s_{K, \ell}, K'(X_j, \cdot) \ell(Y_j) \quad s_{K', \ell} \mathfrak{i}_2 ; \mathfrak{8}K, K' \mathfrak{2} \mathfrak{K}_n.$$

Under Assumption 2.1.(1,2,3), if  $s \mathfrak{2} \mathbb{L}^2(\mathbb{R}^d)$  and if there exists  $\alpha > 0$  such that  $\mathbb{E}(\exp(\alpha \mathfrak{j} \ell(Y_1) \mathfrak{j})) < \mathfrak{1}$ , then there exists a deterministic constant  $\mathfrak{c}_{B.1} > 0$ , not depending on  $n$ , such that for every  $\theta \mathfrak{2} ]0, 1[$ ,

$$\mathbb{E} \left( \sup_{K, K' \in \mathcal{K}_n} \left\{ \frac{\mathfrak{j} U_{K, K', \ell}(n) \mathfrak{j}}{n^2} \quad \frac{\theta}{n} s_{K', \ell} \right\} \right) \leq \mathfrak{c}_{B.1} \frac{\log(n)^5}{\theta n}.$$

**Lemma B.2.** *Consider*

$$V_{K, \ell}(n) := \frac{1}{n} \sum_{i=1}^n \mathfrak{k}K(X_i, \cdot) \ell(Y_i) \quad s_{K, \ell} \mathfrak{k}_2^2 ; \mathfrak{8}K \mathfrak{2} \mathfrak{K}_n.$$

Under Assumption 2.1.(1,2), if  $s \mathfrak{2} \mathbb{L}^2(\mathbb{R}^d)$  and if there exists  $\alpha > 0$  such that  $\mathbb{E}(\exp(\alpha \mathfrak{j} \ell(Y_1) \mathfrak{j})) < \mathfrak{1}$ , then there exists a deterministic constant  $\mathfrak{c}_{B.2} > 0$ , not depending on  $n$ , such that for every  $\theta \mathfrak{2} ]0, 1[$ ,

$$\mathbb{E} \left( \sup_{K \in \mathcal{K}_n} \left\{ \frac{1}{n} \mathfrak{j} V_{K, \ell}(n) \quad s_{K, \ell} \mathfrak{j} \quad \frac{\theta}{n} s_{K, \ell} \right\} \right) \leq \mathfrak{c}_{B.2} \frac{\log(n)^3}{\theta n}.$$

**Lemma B.3.** *Consider*

$$W_{K, K', \ell}(n) := \mathfrak{h} \widehat{s}_{K, \ell}(n; \cdot) \quad s_{K, \ell}, s_{K', \ell} \quad s \mathfrak{i}_2 ; \mathfrak{8}K, K' \mathfrak{2} \mathfrak{K}_n.$$

Under Assumption 2.1.(1,2,4), if  $s \mathfrak{2} \mathbb{L}^2(\mathbb{R}^d)$  and if there exists  $\alpha > 0$  such that  $\mathbb{E}(\exp(\alpha \mathfrak{j} \ell(Y_1) \mathfrak{j})) < \mathfrak{1}$ , then there exists a deterministic constant  $\mathfrak{c}_{B.3} > 0$ , not depending on  $n$ , such that for every  $\theta \mathfrak{2} ]0, 1[$ ,

$$\mathbb{E} \left( \sup_{K, K' \in \mathcal{K}_n} \left\{ \mathfrak{j} W_{K, K', \ell}(n) \mathfrak{j} \quad \theta \mathfrak{k} s_{K', \ell} \quad s \mathfrak{k}_2^2 \mathfrak{g} \right\} \right) \leq \mathfrak{c}_{B.3} \frac{\log(n)^4}{\theta n}.$$

**B.1.1. Proof of Lemma B.1.** Consider  $\mathfrak{m}(n) := 8 \log(n) / \alpha$ . For any  $K, K' \mathfrak{2} \mathfrak{K}_n$ ,

$$U_{K, K', \ell}(n) = U_{K, K', \ell}^1(n) + U_{K, K', \ell}^2(n) + U_{K, K', \ell}^3(n) + U_{K, K', \ell}^4(n)$$

where

$$U_{K, K', \ell}^l(n) := \sum_{i \neq j} g_{K, K', \ell}^l(n; X_i, Y_i, X_j, Y_j) ; l = 1, 2, 3, 4$$

with, for every  $(x', y), (x'', y') \mathfrak{2} E = \mathbb{R}^d \times \mathbb{R}$ ,

$$\begin{aligned} g_{K, K', \ell}^1(n; x', y, x'', y') &:= \mathfrak{h}K(x', \cdot) \ell(y) \mathbf{1}_{|\ell(y)| \leq \mathfrak{m}(n)} \quad s_{K, \ell}^+(n; \cdot), K'(x'', \cdot) \ell(y') \mathbf{1}_{|\ell(y')| \leq \mathfrak{m}(n)} \quad s_{K', \ell}^+(n; \cdot) \mathfrak{i}_2, \\ g_{K, K', \ell}^2(n; x', y, x'', y') &:= \mathfrak{h}K(x', \cdot) \ell(y) \mathbf{1}_{|\ell(y)| > \mathfrak{m}(n)} \quad s_{K, \ell}^-(n; \cdot), K'(x'', \cdot) \ell(y') \mathbf{1}_{|\ell(y')| \leq \mathfrak{m}(n)} \quad s_{K', \ell}^+(n; \cdot) \mathfrak{i}_2, \\ g_{K, K', \ell}^3(n; x', y, x'', y') &:= \mathfrak{h}K(x', \cdot) \ell(y) \mathbf{1}_{|\ell(y)| \leq \mathfrak{m}(n)} \quad s_{K, \ell}^+(n; \cdot), K'(x'', \cdot) \ell(y') \mathbf{1}_{|\ell(y')| > \mathfrak{m}(n)} \quad s_{K', \ell}^-(n; \cdot) \mathfrak{i}_2, \\ g_{K, K', \ell}^4(n; x', y, x'', y') &:= \mathfrak{h}K(x', \cdot) \ell(y) \mathbf{1}_{|\ell(y)| > \mathfrak{m}(n)} \quad s_{K, \ell}^-(n; \cdot), K'(x'', \cdot) \ell(y') \mathbf{1}_{|\ell(y')| > \mathfrak{m}(n)} \quad s_{K', \ell}^-(n; \cdot) \mathfrak{i}_2 \end{aligned}$$

and, for every  $k \mathfrak{2} \mathfrak{K}_n$ ,

$$s_{k, \ell}^+(n; \cdot) := \mathbb{E}(\mathfrak{k}(X_1, \cdot) \ell(Y_1) \mathbf{1}_{|\ell(Y_1)| \leq \mathfrak{m}(n)}) \quad \text{and} \quad s_{k, \ell}^-(n; \cdot) := \mathbb{E}(\mathfrak{k}(X_1, \cdot) \ell(Y_1) \mathbf{1}_{|\ell(Y_1)| > \mathfrak{m}(n)}).$$

On the one hand, since  $\mathbb{E}(g_{K,K',\ell}^1(n; x', y, X_1, Y_1)) = 0$  for every  $(x', y) \in E$ , by Giné and Nickl [7], Theorem 3.4.8, there exists a universal constant  $\mathfrak{m} \geq 1$  such that for any  $\lambda > 0$ , with probability larger than  $1 - 5.4e^{-\lambda}$ ,

$$\frac{|U_{K,K',\ell}^1(n)|}{n^2} \leq \frac{\mathfrak{m}}{n^2} (\mathfrak{c}_{K,K',\ell}(n)\lambda^{1/2} + \mathfrak{d}_{K,K',\ell}(n)\lambda + \mathfrak{b}_{K,K',\ell}(n)\lambda^{3/2} + \mathfrak{a}_{K,K',\ell}(n)\lambda^2)$$

where the constants  $\mathfrak{a}_{K,K',\ell}(n)$ ,  $\mathfrak{b}_{K,K',\ell}(n)$ ,  $\mathfrak{c}_{K,K',\ell}(n)$  and  $\mathfrak{d}_{K,K',\ell}(n)$  are defined and controlled later. First, note that

$$(10) \quad U_{K,K',\ell}^1(n) = \sum_{i \neq j} (\varphi_{K,K',\ell}(n; X_i, Y_i, X_j, Y_j) - \psi_{K,K',\ell}(n; X_i, Y_i) - \psi_{K',K,\ell}(n; X_j, Y_j) + \mathbb{E}(\varphi_{K,K',\ell}(n; X_i, Y_i, X_j, Y_j))),$$

where

$$\varphi_{K,K',\ell}(n; x', y, x'', y'') := \mathfrak{h}K(x', \cdot)\ell(y)\mathbf{1}_{|\ell(y)| \leq \mathfrak{m}(n)}, K'(x'', \cdot)\ell(y'')\mathbf{1}_{|\ell(y'')| \leq \mathfrak{m}(n)} \mathfrak{i}_2$$

and

$$\psi_{k,k',\ell}(n; x', y) := \mathfrak{h}k(x', \cdot)\ell(y)\mathbf{1}_{|\ell(y)| \leq \mathfrak{m}(n)}, s_{k',\ell}^+(n; \cdot) \mathfrak{i}_2 = \mathbb{E}(\varphi_{k,k',\ell}(n; x', y, X_1, Y_1))$$

for every  $k, k' \in \mathcal{K}_n$  and  $(x', y), (x'', y'') \in E$ . Let us now control  $\mathfrak{a}_{K,K',\ell}(n)$ ,  $\mathfrak{b}_{K,K',\ell}(n)$ ,  $\mathfrak{c}_{K,K',\ell}(n)$  and  $\mathfrak{d}_{K,K',\ell}(n)$ :

**The constant  $\mathfrak{a}_{K,K',\ell}(n)$ .** Consider

$$\mathfrak{a}_{K,K',\ell}(n) := \sup_{(x', y), (x'', y'') \in E} |jg_{K,K',\ell}^1(n; x', y, x'', y'')|.$$

By (10), Cauchy-Schwarz's inequality and Assumption 2.1.(1),

$$\begin{aligned} \mathfrak{a}_{K,K',\ell}(n) &\leq 4 \sup_{(x', y), (x'', y'') \in E} |j\mathfrak{h}K(x', \cdot)\ell(y)\mathbf{1}_{|\ell(y)| \leq \mathfrak{m}(n)}, K'(x'', \cdot)\ell(y'')\mathbf{1}_{|\ell(y'')| \leq \mathfrak{m}(n)} \mathfrak{i}_2| \\ &\leq 4\mathfrak{m}(n)^2 \left( \sup_{x' \in \mathbb{R}^d} \|K(x', \cdot)\|_2 \right) \left( \sup_{x'' \in \mathbb{R}^d} \|K'(x'', \cdot)\|_2 \right) \leq 4\mathfrak{m}_{\mathcal{K},\ell} \mathfrak{m}(n)^2 n. \end{aligned}$$

So,

$$\frac{1}{n^2} \mathfrak{a}_{K,K',\ell}(n)\lambda^2 \leq \frac{4}{n} \mathfrak{m}_{\mathcal{K},\ell} \mathfrak{m}(n)^2 \lambda^2.$$

**The constant  $\mathfrak{b}_{K,K',\ell}(n)$ .** Consider

$$\mathfrak{b}_{K,K',\ell}(n)^2 := n \sup_{(x', y) \in E} \mathbb{E}(g_{K,K',\ell}^1(n; x', y, X_1, Y_1)^2).$$

By (10), Jensen's inequality, Cauchy-Schwarz's inequality and Assumption 2.1.(1),

$$\begin{aligned} \mathfrak{b}_{K,K',\ell}(n)^2 &\leq 16n \sup_{(x', y) \in E} \mathbb{E}(\mathfrak{h}K(x', \cdot)\ell(y)\mathbf{1}_{|\ell(y)| \leq \mathfrak{m}(n)}, K'(X_1, \cdot)\ell(Y_1)\mathbf{1}_{|\ell(Y_1)| \leq \mathfrak{m}(n)} \mathfrak{i}_2^2) \\ &\leq 16n\mathfrak{m}(n)^2 \sup_{x' \in \mathbb{R}^d} \|K(x', \cdot)\|_2^2 \mathbb{E}(\|K'(X_1, \cdot)\ell(Y_1)\mathbf{1}_{|\ell(Y_1)| \leq \mathfrak{m}(n)}\|_2^2) \leq 16\mathfrak{m}_{\mathcal{K},\ell} n^2 \mathfrak{m}(n)^2 \mathfrak{s}_{K',\ell}. \end{aligned}$$

So, for any  $\theta \in ]0, 1[$ ,

$$\begin{aligned} \frac{1}{n^2} \mathfrak{b}_{K,K',\ell}(n)\lambda^{3/2} &\leq 2 \left( \frac{3\mathfrak{m}}{\theta} \right)^{1/2} \frac{2}{n^{1/2}} \mathfrak{m}_{\mathcal{K},\ell}^{1/2} \mathfrak{m}(n)\lambda^{3/2} \left( \frac{\theta}{3\mathfrak{m}} \right)^{1/2} \frac{1}{n^{1/2}} \mathfrak{s}_{K',\ell}^{1/2} \\ &\leq \frac{\theta}{3\mathfrak{m}n} \mathfrak{s}_{K',\ell} + \frac{12\mathfrak{m}\lambda^3}{\theta n} \mathfrak{m}_{\mathcal{K},\ell} \mathfrak{m}(n)^2. \end{aligned}$$

**The constant  $\mathfrak{c}_{K,K',\ell}(n)$ .** Consider

$$\mathfrak{c}_{K,K',\ell}(n)^2 := n^2 \mathbb{E}(g_{K,K',\ell}^1(n; X_1, Y_1, X_2, Y_2)^2).$$

By (10), Jensen's inequality and Assumption 2.1.(3),

$$\begin{aligned} \mathfrak{c}_{K,K',\ell}(n)^2 &\leq 16n^2 \mathbb{E}(\mathfrak{h}K(X_1, \cdot)\ell(Y_1)\mathbf{1}_{|\ell(Y_1)| \leq \mathfrak{m}(n)}, K'(X_2, \cdot)\ell(Y_2)\mathbf{1}_{|\ell(Y_2)| \leq \mathfrak{m}(n)} \mathfrak{i}_2^2) \\ &\leq 16n^2 \mathfrak{m}(n)^2 \mathbb{E}(\mathfrak{h}K(X_1, \cdot), K'(X_2, \cdot)\ell(Y_2) \mathfrak{i}_2^2) \leq 16\mathfrak{m}_{\mathcal{K},\ell} n^2 \mathfrak{m}(n)^2 \mathfrak{s}_{K',\ell}. \end{aligned}$$

So,

$$\frac{1}{n^2} \mathbf{c}_{K,K',\ell}(n) \lambda^{1/2} \leq \frac{\theta}{3mn} \mathfrak{s}_{K',\ell} + \frac{12m\lambda}{\theta n} \mathbf{m}_{\mathcal{K},\ell} \mathbf{m}(n)^2.$$

**The constant  $\mathfrak{d}_{K,K',\ell}(n)$ .** Consider

$$\mathfrak{d}_{K,K',\ell}(n) := \sup_{(a,b) \in \mathcal{A}} \mathbb{E} \left[ \sum_{i < j} a_i(X_i, Y_i) b_j(X_j, Y_j) g_{K,K',\ell}^1(n; X_i, Y_i, X_j, Y_j) \right],$$

where

$$\mathcal{A} := \left\{ (a, b) : \sum_{i=1}^{n-1} \mathbb{E}(a_i(X_i, Y_i)^2) \leq 1 \text{ and } \sum_{j=2}^n \mathbb{E}(b_j(X_j, Y_j)^2) \leq 1 \right\}.$$

By (10), Jensen's inequality, Cauchy-Schwarz's inequality and Assumption 2.1.(3),

$$\begin{aligned} \mathfrak{d}_{K,K',\ell}(n) &\leq 4 \sup_{(a,b) \in \mathcal{A}} \mathbb{E} \left[ \sum_{i=1}^{n-1} \sum_{j=i+1}^n j a_i(X_i, Y_i) b_j(X_j, Y_j) \varphi_{K,K',\ell}(n; X_i, Y_i, X_j, Y_j) \right] \\ &\leq 4nm(n) \mathbb{E}(\mathfrak{h}K(X_1, \cdot), K'(X_2, \cdot)) \ell(Y_2) i_2^2)^{1/2} \leq 4m_{\mathcal{K},\ell}^{1/2} nm(n) \mathfrak{s}_{K',\ell}^{1/2}. \end{aligned}$$

So,

$$\frac{1}{n^2} \mathfrak{d}_{K,K',\ell}(n) \lambda \leq \frac{\theta}{3mn} \mathfrak{s}_{K',\ell} + \frac{12m\lambda^2}{\theta n} \mathbf{m}_{\mathcal{K},\ell} \mathbf{m}(n)^2.$$

Then, since  $m \geq 1$  and  $\lambda > 0$ , with probability larger than  $1 - 5.4e^{-\lambda}$ ,

$$\frac{jU_{K,K',\ell}^1(n)}{n^2} \leq \frac{\theta}{n} \mathfrak{s}_{K',\ell} + \frac{40m^2}{\theta n} \mathbf{m}_{\mathcal{K},\ell} \mathbf{m}(n)^2 (1 + \lambda)^3.$$

So, with probability larger than  $1 - 5.4j\mathbf{K}_n j e^{-\lambda}$ ,

$$S_{\mathcal{K},\ell}(n, \theta) := \sup_{K, K' \in \mathcal{K}_n} \left\{ \frac{jU_{K,K',\ell}^1(n)}{n^2} - \frac{\theta}{n} \mathfrak{s}_{K',\ell} \right\} \leq \frac{40m^2}{\theta n} \mathbf{m}_{\mathcal{K},\ell} \mathbf{m}(n)^2 (1 + \lambda)^3.$$

For every  $t \in \mathbb{R}_+$ , consider

$$\lambda_{\mathcal{K},\ell}(n, \theta, t) := 1 + \left( \frac{t}{\mathbf{m}_{\mathcal{K},\ell}(n, \theta)} \right)^{1/3} \text{ with } \mathbf{m}_{\mathcal{K},\ell}(n, \theta) = \frac{40m^2}{\theta n} \mathbf{m}_{\mathcal{K},\ell} \mathbf{m}(n)^2.$$

Then, for any  $T > 0$ ,

$$\begin{aligned} \mathbb{E}(S_{\mathcal{K},\ell}(n, \theta)) &\leq T + \int_T^\infty \mathbb{P}(S_{\mathcal{K},\ell}(n, \theta) \geq (1 + \lambda_{\mathcal{K},\ell}(n, \theta, t))^3 \mathbf{m}_{\mathcal{K},\ell}(n, \theta)) dt \\ &\leq 2T + 5.4\mathbf{c}_1 j\mathbf{K}_n j \mathbf{m}_{\mathcal{K},\ell}(n, \theta) \exp\left( -\frac{T^{1/3}}{2\mathbf{m}_{\mathcal{K},\ell}(n, \theta)^{1/3}} \right) \text{ with } \mathbf{c}_1 = \int_0^\infty e^{1-r^{1/3}/2} dr. \end{aligned}$$

Moreover,

$$\mathbf{m}_{\mathcal{K},\ell}(n, \theta) \leq \mathbf{c}_2 \frac{\log(n)^2}{\theta n} \text{ with } \mathbf{c}_2 = \frac{40}{\alpha^2} 8^2 m^2 \mathbf{m}_{\mathcal{K},\ell}.$$

So, by taking

$$T = 2^3 \mathbf{c}_2 \frac{\log(n)^5}{\theta n},$$

and since  $j\mathbf{K}_n j \leq n$ ,

$$\mathbb{E}(S_{\mathcal{K},\ell}(n, \theta)) \leq 2^4 \mathbf{c}_2 \frac{\log(n)^5}{\theta n} + 5.4\mathbf{c}_1 \mathbf{m}_{\mathcal{K},\ell}(n, \theta) \frac{j\mathbf{K}_n j}{n} \leq (2^4 + 5.4\mathbf{c}_1) \mathbf{c}_2 \frac{\log(n)^5}{\theta n}.$$

On the other hand, by Assumption 2.1.(1), Cauchy-Schwarz's inequality and Markov's inequality,

$$\begin{aligned} \mathbb{E} \left( \sup_{K, K' \in \mathcal{K}_n} |g_{K, K', \ell}^2(n; X_1, Y_1, X_2, Y_2)| \right) &\leq 4\mathfrak{m}(n) \sum_{K, K' \in \mathcal{K}_n} \mathbb{E}(|\ell(Y_1)| \mathbf{1}_{|\ell(Y_1)| > \mathfrak{m}(n)} | \langle K(X_1, \cdot), K'(X_2, \cdot) \rangle_2 |) \\ &\leq 4\mathfrak{m}(n) \mathfrak{m}_{\mathcal{K}, \ell} n |\mathcal{K}_n|^2 \mathbb{E}(\ell(Y_1)^2)^{1/2} \mathbb{P}(|\ell(Y_1)| > \mathfrak{m}(n))^{1/2} \leq \mathfrak{c}_3 \frac{\log(n)}{n} \end{aligned}$$

with

$$\mathfrak{c}_3 = \frac{32}{\alpha} \mathfrak{m}_{\mathcal{K}, \ell} \mathbb{E}(\ell(Y_1)^2)^{1/2} \mathbb{E}(\exp(\alpha j \ell(Y_1) j))^{1/2}.$$

So,

$$\mathbb{E} \left( \sup_{K, K' \in \mathcal{K}_n} \frac{jU_{K, K', \ell}^2(n)j}{n^2} \right) \leq \mathfrak{c}_3 \frac{\log(n)}{n}$$

and, symmetrically,

$$\mathbb{E} \left( \sup_{K, K' \in \mathcal{K}_n} \frac{jU_{K, K', \ell}^3(n)j}{n^2} \right) \leq \mathfrak{c}_3 \frac{\log(n)}{n}.$$

By Assumption 2.1.(1), Cauchy-Schwarz's inequality and Markov's inequality,

$$\begin{aligned} \mathbb{E} \left( \sup_{K, K' \in \mathcal{K}_n} |g_{K, K', \ell}^4(n; X_1, Y_1, X_2, Y_2)| \right) &\leq 4 \sum_{K, K' \in \mathcal{K}_n} \mathbb{E}(|\ell(Y_1)\ell(Y_2)| \mathbf{1}_{|\ell(Y_1)|, |\ell(Y_2)| > \mathfrak{m}(n)} | \langle K(X_1, \cdot), K'(X_2, \cdot) \rangle_2 |) \\ &\leq 4\mathfrak{m}_{\mathcal{K}, \ell} n |\mathcal{K}_n|^2 \mathbb{E}(\ell(Y_1)^2) \mathbb{P}(|\ell(Y_1)| > \mathfrak{m}(n)) \leq \frac{\mathfrak{c}_4}{n^5} \end{aligned}$$

with

$$\mathfrak{c}_4 = 4\mathfrak{m}_{\mathcal{K}, \ell} \mathbb{E}(\ell(Y_1)^2) \mathbb{E}(\exp(\alpha j \ell(Y_1) j)).$$

So,

$$\mathbb{E} \left( \sup_{K, K' \in \mathcal{K}_n} \frac{jU_{K, K', \ell}^4(n)j}{n^2} \right) \leq \frac{\mathfrak{c}_4}{n^5}.$$

Therefore,

$$\mathbb{E} \left( \sup_{K, K' \in \mathcal{K}_n} \left\{ \frac{jU_{K, K', \ell}(n)j}{n^2} \quad \frac{\theta}{n} \mathfrak{s}_{K', \ell} \right\} \right) \leq (2^4 + 5.4\mathfrak{c}_1) \mathfrak{c}_2 \frac{\log(n)^5}{\theta n} + 2\mathfrak{c}_3 \frac{\log(n)}{n} + \frac{\mathfrak{c}_4}{n^5}.$$

B.1.2. *Proof of Lemma B.2.* First, the two following results are used several times in the sequel:

$$\begin{aligned} \mathfrak{k}s_{K, \ell} \mathfrak{k}_2^2 &\leq \mathbb{E}(\ell(Y_1)^2) \int_{\mathbb{R}^d} f(x') \int_{\mathbb{R}^d} K(x', x)^2 \lambda_d(dx) \lambda_d(dx') \\ (11) \quad &\leq \mathbb{E}(\ell(Y_1)^2) \mathfrak{m}_{\mathcal{K}, \ell} n \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}(V_{K, \ell}(n)) &= \mathbb{E}(\mathfrak{k}K(X_1, \cdot)\ell(Y_1) \quad \mathfrak{s}_{K, \ell} \mathfrak{k}_2^2) \\ (12) \quad &= \mathbb{E}(\mathfrak{k}K(X_1, \cdot)\ell(Y_1)\mathfrak{k}_2^2) + \mathfrak{k}s_{K, \ell} \mathfrak{k}_2^2 \quad 2 \int_{\mathbb{R}^d} \mathfrak{s}_{K, \ell}(x) \mathbb{E}(K(X_1, x)\ell(Y_1)) \lambda_d(dx) = \mathfrak{s}_{K, \ell} \quad \mathfrak{k}s_{K, \ell} \mathfrak{k}_2^2. \end{aligned}$$

Consider  $\mathfrak{m}(n) := 2 \log(n)/\alpha$  and

$$v_{K, \ell}(n) := V_{K, \ell}(n) \quad \mathbb{E}(V_{K, \ell}(n)) = v_{K, \ell}^1(n) + v_{K, \ell}^2(n),$$

where

$$v_{K, \ell}^j(n) = \frac{1}{n} \sum_{i=1}^n (g_{K, \ell}^j(n; X_i, Y_i) \quad \mathbb{E}(g_{K, \ell}^j(n; X_i, Y_i))) ; j = 1, 2$$

with, for every  $(x', y) \in E$ ,

$$g_{K, \ell}^1(n; x', y) := \mathfrak{k}K(x', \cdot)\ell(y) \quad \mathfrak{s}_{K, \ell} \mathfrak{k}_2^2 \mathbf{1}_{|\ell(y)| \leq \mathfrak{m}(n)}$$

and

$$g_{K,\ell}^2(n; x', y) := \mathbb{K}K(x', \cdot)\ell(y) - s_{K,\ell}k_2^2\mathbf{1}_{|\ell(y)| > \mathfrak{m}(n)}.$$

On the one hand, by Bernstein's inequality, for any  $\lambda > 0$ , with probability larger than  $1 - 2e^{-\lambda}$ ,

$$|jv_{K,\ell}^1(n)j| \leq \sqrt{\frac{2\lambda}{n}\mathfrak{v}_{K,\ell}(n)} + \frac{\lambda}{n}\mathfrak{c}_{K,\ell}(n)$$

where

$$\mathfrak{c}_{K,\ell}(n) = \frac{\mathbb{K}g_{K,\ell}^1(n; \cdot)\mathbb{K}_\infty}{3} \text{ and } \mathfrak{v}_{K,\ell}(n) = \mathbb{E}(g_{K,\ell}^1(n; X_1, Y_1)^2).$$

Moreover,

$$\begin{aligned} \mathfrak{c}_{K,\ell}(n) &= \frac{1}{3} \sup_{(x', y) \in E} \mathbb{K}K(x', \cdot)\ell(y) - s_{K,\ell}k_2^2\mathbf{1}_{|\ell(y)| \leq \mathfrak{m}(n)} \\ &\leq \frac{2}{3} \left( \mathfrak{m}(n)^2 \sup_{x' \in \mathbb{R}^d} \mathbb{K}K(x', \cdot)\mathbb{K}_2^2 + \mathfrak{K} s_{K,\ell} k_2^2 \right) \leq \frac{2}{3} (\mathfrak{m}(n)^2 + \mathbb{E}(\ell(Y_1)^2)) \mathfrak{m}_{\mathcal{K},\ell} n \end{aligned}$$

by Inequality (11), and

$$\begin{aligned} \mathfrak{v}_{K,\ell}(n) &\leq \mathbb{K}g_{K,\ell}^1(n; \cdot)\mathbb{K}_\infty \mathbb{E}(V_{K,\ell}(n)) \\ &\leq 2(\mathfrak{m}(n)^2 + \mathbb{E}(\ell(Y_1)^2)) \mathfrak{m}_{\mathcal{K},\ell} n (\mathfrak{s}_{K,\ell} + \mathfrak{K} s_{K,\ell} k_2^2) \end{aligned}$$

by Inequality (11) and Equality (12). Then, for any  $\theta \in ]0, 1[$ ,

$$\begin{aligned} |jv_{K,\ell}^1(n)j| &\leq 2\sqrt{\lambda(\mathfrak{m}(n)^2 + \mathbb{E}(\ell(Y_1)^2)) \mathfrak{m}_{\mathcal{K},\ell} (\mathfrak{s}_{K,\ell} + \mathfrak{K} s_{K,\ell} k_2^2)} + \frac{2\lambda}{3} (\mathfrak{m}(n)^2 + \mathbb{E}(\ell(Y_1)^2)) \mathfrak{m}_{\mathcal{K},\ell} \\ &\leq \theta \mathfrak{s}_{K,\ell} + \frac{5\lambda}{3\theta} (1 + \mathbb{E}(\ell(Y_1)^2)) \mathfrak{m}_{\mathcal{K},\ell} \mathfrak{m}(n)^2 \end{aligned}$$

with probability larger than  $1 - 2e^{-\lambda}$ . So, with probability larger than  $1 - 2\mathbb{K}n_j e^{-\lambda}$ ,

$$S_{\mathcal{K},\ell}(n, \theta) := \sup_{K \in \mathcal{K}_n} \left\{ \frac{|jv_{K,\ell}^1(n)j|}{n} - \frac{\theta}{n} \mathfrak{s}_{K,\ell} \right\} \leq \frac{5\lambda}{3\theta n} (1 + \mathbb{E}(\ell(Y_1)^2)) \mathfrak{m}_{\mathcal{K},\ell} \mathfrak{m}(n)^2.$$

For every  $t \in \mathbb{R}_+$ , consider

$$\lambda_{\mathcal{K},\ell}(n, \theta, t) := \frac{t}{\mathfrak{m}_{\mathcal{K},\ell}(n, \theta)} \text{ with } \mathfrak{m}_{\mathcal{K},\ell}(n, \theta) = \frac{5}{3\theta n} (1 + \mathbb{E}(\ell(Y_1)^2)) \mathfrak{m}_{\mathcal{K},\ell} \mathfrak{m}(n)^2.$$

Then, for any  $T > 0$ ,

$$\begin{aligned} \mathbb{E}(S_{\mathcal{K},\ell}(n, \theta)) &\leq T + \int_T^\infty \mathbb{P}(S_{\mathcal{K},\ell}(n, \theta) \geq \lambda_{\mathcal{K},\ell}(n, \theta, t) \mathfrak{m}_{\mathcal{K},\ell}(n, \theta)) dt \\ &\leq 2T + 2\mathfrak{c}_1 \mathbb{K} n_j \mathfrak{m}_{\mathcal{K},\ell}(n, \theta) \exp\left(-\frac{T}{2\mathfrak{m}_{\mathcal{K},\ell}(n, \theta)}\right) \text{ with } \mathfrak{c}_1 = \int_0^\infty e^{-r/2} dr = 2. \end{aligned}$$

Moreover,

$$\mathfrak{m}_{\mathcal{K},\ell}(n, \theta) \leq \mathfrak{c}_2 \frac{\log(n)^2}{\theta n} \text{ with } \mathfrak{c}_2 = \frac{10}{3\alpha^2} (1 + \mathbb{E}(\ell(Y_1)^2)) \mathfrak{m}_{\mathcal{K},\ell}.$$

So, by taking

$$T = 2\mathfrak{c}_2 \frac{\log(n)^3}{\theta n},$$

and since  $\mathbb{K}n_j \leq n$ ,

$$\mathbb{E}(S_{\mathcal{K},\ell}(n, \theta)) \leq 4\mathfrak{c}_2 \frac{\log(n)^3}{\theta n} + 4\mathfrak{m}_{\mathcal{K},\ell}(n, \theta) \frac{\mathbb{K}n_j}{n} \leq 8\mathfrak{c}_2 \frac{\log(n)^3}{\theta n}.$$

On the other hand, by Inequality (11) and Markov's inequality,

$$\begin{aligned} \mathbb{E} \left[ \sup_{K \in \mathcal{K}_n} \frac{jv_{K,\ell}^2(n)j}{n} \right] &\leq \frac{2}{n} \mathbb{E} \left( \sup_{K \in \mathcal{K}_n} \mathfrak{k}K(X_1, \cdot)\ell(Y_1) \quad s_{K,\ell} \mathfrak{k}_2^2 \mathbf{1}_{|\ell(Y_1)| > m(n)} \right) \\ &\leq \frac{4}{n} \mathbb{E} \left[ \left| \ell(Y_1)^2 \sup_{K \in \mathcal{K}_n} \mathfrak{k}K(X_1, \cdot)\mathfrak{k}_2^2 + \sup_{K \in \mathcal{K}_n} \mathfrak{k}s_{K,\ell}\mathfrak{k}_2^2 \right|^2 \right]^{1/2} \mathbb{P}(j\ell(Y_1)j > m(n))^{1/2} \leq \frac{\mathfrak{c}_3}{n} \end{aligned}$$

with

$$\mathfrak{c}_3 = 8\mathfrak{m}_{\mathcal{K},\ell} \mathbb{E}(\ell(Y_1)^4)^{1/2} \mathbb{E}(\exp(\alpha j\ell(Y_1)j))^{1/2}.$$

Therefore,

$$\mathbb{E} \left( \sup_{K \in \mathcal{K}_n} \left\{ \frac{jv_{K,\ell}(n)j}{n} \quad \frac{\theta}{n} \mathfrak{s}_{K,\ell} \right\} \right) \leq 8\mathfrak{c}_2 \frac{\log(n)^3}{\theta n} + \frac{\mathfrak{c}_3}{n}$$

and, by Equality (12), the definition of  $v_{K,\ell}(n)$  and Assumption 2.1.(2),

$$\mathbb{E} \left( \sup_{K \in \mathcal{K}_n} \left\{ \frac{1}{n} jV_{K,\ell}(n) \quad \mathfrak{s}_{K,\ell} j \quad \frac{\theta}{n} \mathfrak{s}_{K,\ell} \right\} \right) \leq 8\mathfrak{c}_2 \frac{\log(n)^3}{\theta n} + \frac{\mathfrak{c}_3 + \mathfrak{m}_{\mathcal{K},\ell}}{n}.$$

B.1.3. *Proof of Lemma B.3.* Consider  $m(n) = 12 \log(n)/\alpha$ . For any  $K, K' \in \mathcal{K}_n$ ,

$$W_{K,K',\ell}(n) = W_{K,K',\ell}^1(n) + W_{K,K',\ell}^2(n)$$

where

$$W_{K,K',\ell}^j(n) := \frac{1}{n} \sum_{i=1}^n (g_{K,K',\ell}^j(n; X_i, Y_i) - \mathbb{E}(g_{K,K',\ell}^j(n; X_i, Y_i))) ; j = 1, 2$$

with, for every  $(x', y) \in E$ ,

$$g_{K,K',\ell}^1(n; x', y) := \mathfrak{h}K(x', \cdot)\ell(y), s_{K',\ell} \quad \mathfrak{s}i_2 \mathbf{1}_{|\ell(y)| \leq m(n)}$$

and

$$g_{K,K',\ell}^2(n; x', y) := \mathfrak{h}K(x', \cdot)\ell(y), s_{K',\ell} \quad \mathfrak{s}i_2 \mathbf{1}_{|\ell(y)| > m(n)}.$$

On the one hand, by Bernstein's inequality, for any  $\lambda > 0$ , with probability larger than  $1 - 2e^{-\lambda}$ ,

$$jW_{K,K',\ell}^1(n)j \leq \sqrt{\frac{2\lambda}{n} \mathfrak{v}_{K,K',\ell}(n)} + \frac{\lambda}{n} \mathfrak{c}_{K,K',\ell}(n)$$

where

$$\mathfrak{c}_{K,K',\ell}(n) = \frac{\mathfrak{k}g_{K,K',\ell}^1(n; \cdot)\mathfrak{k}_\infty}{3} \quad \text{and} \quad \mathfrak{v}_{K,K',\ell}(n) = \mathbb{E}(g_{K,K',\ell}^1(n; X_1, Y_1)^2).$$

Moreover,

$$\begin{aligned} \mathfrak{c}_{K,K',\ell}(n) &= \frac{1}{3} \sup_{(x', y) \in E} j\mathfrak{h}K(x', \cdot)\ell(y), s_{K',\ell} \quad \mathfrak{s}i_2 j \mathbf{1}_{|\ell(y)| \leq m(n)} \\ &\leq \frac{1}{3} m(n) \mathfrak{k}s_{K',\ell} \quad \mathfrak{s}k_2 \sup_{x' \in \mathbb{R}^d} \mathfrak{k}K(x', \cdot)\mathfrak{k}_2 \leq \frac{1}{3} \mathfrak{m}_{\mathcal{K},\ell}^{1/2} n^{1/2} m(n) \mathfrak{k}s_{K',\ell} \quad \mathfrak{s}k_2 \end{aligned}$$

by Assumption 2.1.(1), and

$$\mathfrak{v}_{K,\ell}(n) \leq \mathbb{E}(\mathfrak{h}K(X_1, \cdot)\ell(Y_1), s_{K',\ell} \quad \mathfrak{s}i_2 \mathbf{1}_{|\ell(Y_1)| \leq m(n)}) \leq m(n)^2 \mathfrak{m}_{\mathcal{K},\ell} \mathfrak{k}s_{K',\ell} \quad \mathfrak{s}k_2^2$$

by Assumption 2.1.(4). Then, since  $\lambda > 0$ , for any  $\theta \in ]0, 1[$ ,

$$\begin{aligned} jW_{K,K',\ell}^1(n)j &\leq \sqrt{\frac{2\lambda}{n} m(n)^2 \mathfrak{m}_{\mathcal{K},\ell} \mathfrak{k}s_{K',\ell} \quad \mathfrak{s}k_2^2} + \frac{\lambda}{3n^{1/2}} \mathfrak{m}_{\mathcal{K},\ell}^{1/2} m(n) \mathfrak{k}s_{K',\ell} \quad \mathfrak{s}k_2 \\ &\leq \theta \mathfrak{k}s_{K',\ell} \quad \mathfrak{s}k_2^2 + \frac{\mathfrak{m}_{\mathcal{K},\ell}}{2\theta n} m(n)^2 (1 + \lambda)^2 \end{aligned}$$

with probability larger than  $1 - 2e^{-\lambda}$ . So, with probability larger than  $1 - 2j\mathcal{K}_n j e^{-\lambda}$ ,

$$S_{\mathcal{K},\ell}(n, \theta) := \sup_{K, K' \in \mathcal{K}_n} jW_{K,K',\ell}^1(n)j \quad \theta \mathfrak{k}s_{K',\ell} \quad \mathfrak{s}k_2^2 \leq \frac{\mathfrak{m}_{\mathcal{K},\ell}}{2\theta n} m(n)^2 (1 + \lambda)^2.$$



For every  $t \in \mathbb{R}_+$ , consider

$$\lambda_{\mathcal{K},\ell}(n, \theta, t) := 1 + \left( \frac{t}{\mathbf{m}_{\mathcal{K},\ell}(n, \theta)} \right)^{1/2} \quad \text{with } \mathbf{m}_{\mathcal{K},\ell}(n, \theta) = \frac{\mathbf{m}_{\mathcal{K},\ell} \mathbf{m}(n)^2}{2\theta n}.$$

Then, for any  $T > 0$ ,

$$\begin{aligned} \mathbb{E}(S_{\mathcal{K},\ell}(n, \theta)) &\leq T + \int_T^\infty \mathbb{P}(S_{\mathcal{K},\ell}(n, \theta) \geq (1 + \lambda_{\mathcal{K},\ell}(n, \theta, t))^2 \mathbf{m}_{\mathcal{K},\ell}(n, \theta)) dt \\ &\leq 2T + 2\mathbf{c}_1 \mathbf{jK}_n \mathbf{j} \mathbf{m}_{\mathcal{K},\ell}(n, \theta) \exp\left( -\frac{T^{1/2}}{2\mathbf{m}_{\mathcal{K},\ell}(n, \theta)^{1/2}} \right) \quad \text{with } \mathbf{c}_1 = \int_0^\infty e^{-r^{1/2}/2} dr. \end{aligned}$$

Moreover,

$$\mathbf{m}_{\mathcal{K},\ell}(n, \theta) \leq \mathbf{c}_2 \frac{\log(n)^2}{\theta n} \quad \text{with } \mathbf{c}_2 = \frac{12^2}{2\alpha^2} \mathbf{m}_{\mathcal{K},\ell}.$$

So, by taking

$$T = 2^2 \mathbf{c}_2 \frac{\log(n)^4}{\theta n},$$

and since  $\mathbf{jK}_n \mathbf{j} \leq n$ ,

$$\mathbb{E}(S_{\mathcal{K},\ell}(n, \theta)) \leq 2^3 \mathbf{c}_2 \frac{\log(n)^4}{\theta n} + 2\mathbf{c}_1 \mathbf{m}_{\mathcal{K},\ell}(n, \theta) \frac{\mathbf{jK}_n \mathbf{j}}{n} \leq (2^3 + 2\mathbf{c}_1) \mathbf{c}_2 \frac{\log(n)^4}{\theta n}.$$

On the other hand, by Assumption 2.1.(2,4), Cauchy-Schwarz's inequality and Markov's inequality,

$$\begin{aligned} \mathbb{E}\left( \sup_{K, K' \in \mathcal{K}_n} \mathbf{j} W_{K, K', \ell}^2 \mathbf{j} \right) &\leq 2\mathbb{E}(\ell(Y_1)^2 \mathbf{1}_{|\ell(Y_1)| > \mathbf{m}(n)})^{1/2} \sum_{K, K' \in \mathcal{K}_n} \mathbb{E}(\mathbf{h}K(X_1, \cdot), s_{K', \ell} \quad \mathbf{s}i_2^2)^{1/2} \\ &\leq 2\mathbf{m}_{\mathcal{K},\ell}^{1/2} \mathbf{k} s_{K', \ell} \quad \mathbf{s}k_2 \mathbb{E}(\ell(Y_1)^4)^{1/4} \mathbf{jK}_n \mathbf{j}^2 \mathbb{P}(\mathbf{j}\ell(Y_1)\mathbf{j} > \mathbf{m}(n))^{1/4} \leq \frac{\mathbf{c}_3}{n} \end{aligned}$$

with

$$\mathbf{c}_3 = 2\mathbf{m}_{\mathcal{K},\ell}^{1/2} (\mathbf{m}_{\mathcal{K},\ell}^{1/2} + \mathbf{k} s_{K', \ell} \quad \mathbf{s}k_2) \mathbb{E}(\ell(Y_1)^4)^{1/4} \mathbb{E}(\exp(\alpha \mathbf{j}\ell(Y_1)\mathbf{j}))^{1/4}.$$

Therefore,

$$\mathbb{E}\left( \sup_{K, K' \in \mathcal{K}_n} \mathbf{j} W_{K, K', \ell}(n) \mathbf{j} \quad \theta \mathbf{k} s_{K', \ell} \quad \mathbf{s}k_2^2 \mathbf{g} \right) \leq (2^3 + 2\mathbf{c}_1) \mathbf{c}_2 \frac{\log(n)^4}{\theta n} + \frac{\mathbf{c}_3}{n} \leq \mathbf{c}_4 \frac{\log(n)^4}{\theta n}$$

with  $\mathbf{c}_4 = (2^3 + 2\mathbf{c}_1) \mathbf{c}_2 + \mathbf{c}_3$ .

**B.2. Proof of Proposition 2.4.** For any  $K \in \mathcal{K}_n$ ,

$$(13) \quad \mathbf{k} \widehat{s}_{K, \ell}(n; \cdot) \quad \mathbf{s}k_{2, \ell}^2 = \frac{U_{K, \ell}(n)}{n^2} + \frac{V_{K, \ell}(n)}{n}$$

with  $U_{K, \ell}(n) = U_{K, K, \ell}(n)$  and  $V_{K, \ell}(n) = V_{K, K, \ell}(n)$ . Then, by Lemmas B.1 and B.2,

$$\mathbb{E}\left( \sup_{K \in \mathcal{K}_n} \left\{ \left| \mathbf{k} \widehat{s}_{K, \ell}(n; \cdot) \quad \mathbf{s}k_{2, \ell}^2 \quad \frac{\mathbf{s}K_{K, \ell}}{n} \right| \quad \frac{\theta}{n} \mathbf{s}K_{K, \ell} \right\} \right) \leq \mathbf{c}_{2.4} \frac{\log(n)^5}{\theta n}$$

with  $\mathbf{c}_{2.4} = \mathbf{c}_{B.1} + \mathbf{c}_{B.2}$ .

**B.3. Proof of Theorem 2.5.** On the one hand, for every  $K \in \mathcal{K}_n$ ,

$$\mathbf{k} \widehat{s}_{K, \ell}(n; \cdot) \quad \mathbf{s}k_2^2 \quad (1 + \theta) \left( \mathbf{k} s_{K, \ell} \quad \mathbf{s}k_2^2 + \frac{\mathbf{s}K_{K, \ell}}{n} \right)$$

can be written

$$\mathbf{k} \widehat{s}_{K, \ell}(n; \cdot) \quad \mathbf{s}k_{2, \ell}^2 \quad (1 + \theta) \frac{\mathbf{s}K_{K, \ell}}{n} + W_{K, K, \ell}(n) \quad \theta \mathbf{k} s_{K, \ell} \quad \mathbf{s}k_2^2.$$

Then, by Proposition 2.4 and Lemma B.3,

$$\mathbb{E}\left( \sup_{K \in \mathcal{K}_n} \left\{ \mathbf{k} \widehat{s}_{K, \ell}(n; \cdot) \quad \mathbf{s}k_2^2 \quad (1 + \theta) \left( \mathbf{k} s_{K, \ell} \quad \mathbf{s}k_2^2 + \frac{\mathbf{s}K_{K, \ell}}{n} \right) \right\} \right) \leq \mathbf{c}_{2.5} \frac{\log(n)^5}{\theta n}$$

with  $\mathfrak{c}_{2.5} = \mathfrak{c}_{2.4} + \mathfrak{c}_{B.3}$ . On the other hand, for any  $K \in \mathcal{K}_n$ ,

$$\mathfrak{k}s_{K,\ell} \quad \mathfrak{s}k_2^2 = \mathfrak{k}\widehat{s}_{K,\ell}(n; \cdot) \quad \mathfrak{s}k_2^2 \quad \mathfrak{k}\widehat{s}_{K,\ell}(n; \cdot) \quad \mathfrak{s}_{K,\ell}\mathfrak{k}_2^2 \quad W_{K,\ell}(n).$$

Then,

$$(1 - \theta) \left( \mathfrak{k}s_{K,\ell} \quad \mathfrak{s}k_2^2 + \frac{\mathfrak{s}_{K,\ell}}{n} \right) \quad \mathfrak{k}\widehat{s}_{K,\ell}(n; \cdot) \quad \mathfrak{s}k_2^2 \leq |W_{K,\ell}(n)| \quad \theta \mathfrak{k}s_{K,\ell} \quad \mathfrak{s}k_2^2 + \Lambda_{K,\ell}(n) \quad \theta \frac{\mathfrak{s}_{K,\ell}}{n}$$

where

$$\Lambda_{K,\ell}(n) := \left| \mathfrak{k}\widehat{s}_{K,\ell} \quad \mathfrak{s}_{K,\ell}\mathfrak{k}_2^2 \quad \frac{\mathfrak{s}_{K,\ell}}{n} \right|.$$

By Equalities (13) and (12),

$$\Lambda_{K,\ell}(n) = \left| \frac{U_{K,\ell}(n)}{n^2} + \frac{v_{K,\ell}(n)}{n} \quad \frac{\mathfrak{k}s_{K,\ell}\mathfrak{k}_2^2}{n} \right|.$$

By Lemmas B.2 and B.1, there exists a deterministic constant  $\mathfrak{c}_1 > 0$ , not depending  $n$  and  $\theta$ , such that

$$\mathbb{E} \left( \sup_{K \in \mathcal{K}_n} \left\{ \Lambda_{K,\ell}(n) \quad \theta \frac{\mathfrak{s}_{K,\ell}}{n} \right\} \right) \leq \mathfrak{c}_1 \frac{\log(n)^5}{\theta n}.$$

By Lemma B.3,

$$\mathbb{E} \left( \sup_{K \in \mathcal{K}_n} |W_{K,\ell}(n)| \quad \theta \mathfrak{k}s_{K,\ell} \quad \mathfrak{s}k_2^2 \right) \leq \mathfrak{c}_{B.3} \frac{\log(n)^3}{\theta n}.$$

Therefore,

$$\mathbb{E} \left( \sup_{K \in \mathcal{K}_n} \left\{ \mathfrak{k}s_{K,\ell} \quad \mathfrak{s}k_2^2 + \frac{\mathfrak{s}_{K,\ell}}{n} \quad \frac{1}{1 - \theta} \mathfrak{k}\widehat{s}_{K,\ell}(n; \cdot) \quad \mathfrak{s}k_2^2 \right\} \right) \leq \bar{\mathfrak{c}}_{2.5} \frac{\log(n)^5}{\theta(1 - \theta)n}$$

with  $\bar{\mathfrak{c}}_{2.5} = \mathfrak{c}_{B.3} + \mathfrak{c}_1$ .

**B.4. Proof of Theorem 3.2.** The proof of Theorem 3.2 is dissected in three steps.

**Step 1.** This first step is devoted to provide a suitable decomposition of

$$\mathfrak{k}\widehat{s}_{\widehat{K},\ell}(n; \cdot) \quad \mathfrak{s}k_2^2.$$

First,

$$\mathfrak{k}\widehat{s}_{\widehat{K},\ell}(n; \cdot) \quad \mathfrak{s}k_2^2 = \mathfrak{k}\widehat{s}_{\widehat{K},\ell}(n; \cdot) \quad \widehat{s}_{K_0,\ell}(n; \cdot) \mathfrak{k}_2^2 + \mathfrak{k}\widehat{s}_{K_0,\ell}(n; \cdot) \quad \mathfrak{s}k_2^2 \quad 2\mathfrak{h}\widehat{s}_{K_0,\ell}(n; \cdot) \quad \widehat{s}_{\widehat{K},\ell}(n; \cdot), \widehat{s}_{K_0,\ell}(n; \cdot) \quad \mathfrak{s}i_2$$

From (7), it follows that for any  $K \in \mathcal{K}_n$ ,

$$\begin{aligned} \mathfrak{k}\widehat{s}_{\widehat{K},\ell}(n; \cdot) \quad \mathfrak{s}k_2^2 &\leq \mathfrak{k}\widehat{s}_{K,\ell}(n; \cdot) \quad \mathfrak{s}k_2^2 + \text{pen}(K) \quad \text{pen}(\widehat{K}) + \mathfrak{k}\widehat{s}_{K_0,\ell}(n; \cdot) \quad \mathfrak{s}k_2^2 \\ &\quad 2\mathfrak{h}\widehat{s}_{K_0,\ell}(n; \cdot) \quad \widehat{s}_{\widehat{K},\ell}(n; \cdot), \widehat{s}_{K_0,\ell}(n; \cdot) \quad \mathfrak{s}i_2 \\ (14) \quad &= \mathfrak{k}\widehat{s}_{K,\ell}(n; \cdot) \quad \mathfrak{s}k_2^2 + \psi_n(K) \quad \psi_n(\widehat{K}) \end{aligned}$$

where

$$\psi_n(K) := 2\mathfrak{h}\widehat{s}_{K,\ell}(n; \cdot) \quad \mathfrak{s}, \widehat{s}_{K_0,\ell}(n; \cdot) \quad \mathfrak{s}i_2 \quad \text{pen}(K).$$

Let's complete the decomposition of  $\mathfrak{k}\widehat{s}_{\widehat{K},\ell}(n; \cdot) \quad \mathfrak{s}k_2^2$  by writing

$$\psi_n(K) = 2(\psi_{1,n}(K) + \psi_{2,n}(K) + \psi_{3,n}(K)),$$

where

$$\psi_{1,n}(K) := \frac{U_{K,K_0,\ell}(n)}{n^2},$$

$$\psi_{2,n}(K) := \frac{1}{n^2} \left( \sum_{i=1}^n \ell(Y_i) \mathfrak{h}K_0(X_i, \cdot), \mathfrak{s}_{K,\ell} \mathfrak{i}_2 + \sum_{i=1}^n \ell(Y_i) \mathfrak{h}K(X_i, \cdot), \mathfrak{s}_{K_0,\ell} \mathfrak{i}_2 \right) + \frac{1}{n} \mathfrak{h}\mathfrak{s}_{K_0,\ell}, \mathfrak{s}_{K,\ell} \mathfrak{i}_2 \text{ and}$$

$$\psi_{3,n}(K) := W_{K,K_0,\ell}(n) + W_{K_0,K,\ell}(n) + \mathfrak{h}\mathfrak{s}_{K,\ell} \quad \mathfrak{s}, \mathfrak{s}_{K_0,\ell} \quad \mathfrak{s}i_2.$$

**Step 2.** In this step, we give controls of the quantities

$$\mathbb{E}(\psi_{i,n}(K)) \text{ and } \mathbb{E}(\psi_{i,n}(\widehat{K})) ; i = 1, 2, 3.$$

By Lemma B.1, for any  $\theta \in ]0, 1[$ ,

$$\mathbb{E}(\mathbf{j}\psi_{1,n}(K)\mathbf{j}) \leq \frac{\theta}{n} \mathfrak{s}_{K,\ell} + \mathfrak{c}_{B.1} \frac{\log(n)^5}{\theta n}$$

and

$$\mathbb{E}(\mathbf{j}\psi_{1,n}(\widehat{K})\mathbf{j}) \leq \frac{\theta}{n} \mathbb{E}(\mathfrak{s}_{\widehat{K},\ell}) + \mathfrak{c}_{B.1} \frac{\log(n)^5}{\theta n}.$$

On the one hand, for any  $K, K' \in \mathcal{K}_n$ , consider

$$\Psi_{2,n}(K, K') := \frac{1}{n} \sum_{i=1}^n \ell(Y_i) \mathbf{h}K(X_i, \cdot), \mathfrak{s}_{K',\ell} \mathbf{i}_2.$$

Then, by Assumption 3.1,

$$\begin{aligned} \mathbb{E} \left( \sup_{K, K' \in \mathcal{K}_n} \mathbf{j}\Psi_{2,n}(K, K')\mathbf{j} \right) &\leq \mathbb{E}(\ell(Y_1)^2)^{1/2} \mathbb{E} \left( \sup_{K, K' \in \mathcal{K}_n} \mathbf{h}K(X_1, \cdot), \mathfrak{s}_{K',\ell} \mathbf{i}_2^2 \right)^{1/2} \\ &\leq \overline{\mathfrak{m}}_{\mathcal{K},\ell}^{1/2} \mathbb{E}(\ell(Y_1)^2)^{1/2}. \end{aligned}$$

On the other hand, by Assumption 2.1.(2),

$$\mathbf{j}\mathfrak{h}\mathfrak{s}_{K,\ell}, \mathfrak{s}_{K_0,\ell} \mathbf{i}_2 \mathbf{j} \leq \mathfrak{m}_{\mathcal{K},\ell}.$$

Then, there exists a deterministic constant  $\mathfrak{c}_1 > 0$ , not depending on  $n$  and  $K$ , such that

$$\mathbb{E}(\mathbf{j}\psi_{2,n}(K)\mathbf{j}) \leq \frac{\mathfrak{c}_1}{n} \text{ and } \mathbb{E}(\mathbf{j}\psi_{2,n}(\widehat{K})\mathbf{j}) \leq \frac{\mathfrak{c}_1}{n}.$$

By Lemma B.3,

$$\begin{aligned} \mathbb{E}(\mathbf{j}\psi_{3,n}(K)\mathbf{j}) &\leq \frac{\theta}{4} (\mathfrak{k}\mathfrak{s}_{K,\ell} \quad \mathfrak{s}\mathfrak{k}_2^2 + \mathfrak{k}\mathfrak{s}_{K_0,\ell} \quad \mathfrak{s}\mathfrak{k}_2^2) + 8\mathfrak{c}_{B.3} \frac{\log(n)^4}{\theta n} \\ &\quad + \left(\frac{\theta}{2}\right)^{1/2} \mathfrak{k}\mathfrak{s}_{K,\ell} \quad \mathfrak{s}\mathfrak{k}_2 \quad \left(\frac{2}{\theta}\right)^{1/2} \mathfrak{k}\mathfrak{s}_{K_0,\ell} \quad \mathfrak{s}\mathfrak{k}_2 \\ &\leq \frac{\theta}{2} \mathfrak{k}\mathfrak{s}_{K,\ell} \quad \mathfrak{s}\mathfrak{k}_2^2 + \left(\frac{\theta}{4} + \frac{1}{\theta}\right) \mathfrak{k}\mathfrak{s}_{K_0,\ell} \quad \mathfrak{s}\mathfrak{k}_2^2 + 8\mathfrak{c}_{B.3} \frac{\log(n)^4}{\theta n} \end{aligned}$$

and

$$\mathbb{E}(\mathbf{j}\psi_{3,n}(\widehat{K})\mathbf{j}) \leq \frac{\theta}{2} \mathbb{E}(\mathfrak{k}\mathfrak{s}_{\widehat{K},\ell} \quad \mathfrak{s}\mathfrak{k}_2^2) + \left(\frac{\theta}{4} + \frac{1}{\theta}\right) \mathfrak{k}\mathfrak{s}_{K_0,\ell} \quad \mathfrak{s}\mathfrak{k}_2^2 + 8\mathfrak{c}_{B.3} \frac{\log(n)^4}{\theta n}.$$

**Step 3.** By the previous step, there exists a deterministic constant  $\mathfrak{c}_2 > 0$ , not depending on  $n, \theta, K$  and  $K_0$ , such that

$$\mathbb{E}(\mathbf{j}\psi_n(K)\mathbf{j}) \leq \theta \left( \mathfrak{k}\mathfrak{s}_{K,\ell} \quad \mathfrak{s}\mathfrak{k}_2^2 + \frac{\mathfrak{s}_{K,\ell}}{n} \right) + \left(\frac{\theta}{2} + \frac{2}{\theta}\right) \mathfrak{k}\mathfrak{s}_{K_0,\ell} \quad \mathfrak{s}\mathfrak{k}_2^2 + \mathfrak{c}_2 \frac{\log(n)^5}{\theta n}$$

and

$$\mathbb{E}(\mathbf{j}\psi_n(\widehat{K})\mathbf{j}) \leq \theta \mathbb{E} \left( \mathfrak{k}\mathfrak{s}_{\widehat{K},\ell} \quad \mathfrak{s}\mathfrak{k}_2^2 + \frac{\mathfrak{s}_{\widehat{K},\ell}}{n} \right) + \left(\frac{\theta}{2} + \frac{2}{\theta}\right) \mathfrak{k}\mathfrak{s}_{K_0,\ell} \quad \mathfrak{s}\mathfrak{k}_2^2 + \mathfrak{c}_2 \frac{\log(n)^5}{\theta n}.$$

Then, by Theorem 2.5,

$$\mathbb{E}(\mathbf{j}\psi_n(K)\mathbf{j}) \leq \frac{\theta}{1} \mathbb{E}(\mathfrak{k}\widehat{\mathfrak{s}}_{K,\ell}(n; \cdot) \quad \mathfrak{s}\mathfrak{k}_2^2) + \left(\frac{\theta}{2} + \frac{2}{\theta}\right) \mathfrak{k}\mathfrak{s}_{K_0,\ell} \quad \mathfrak{s}\mathfrak{k}_2^2 + \left(\frac{\mathfrak{c}_2}{\theta} + \frac{\mathfrak{c}_{2.5}}{1} \frac{1}{\theta}\right) \frac{\log(n)^5}{n}$$

and

$$\mathbb{E}(\mathbf{j}\psi_n(\widehat{K})\mathbf{j}) \leq \frac{\theta}{1} \mathbb{E}(\mathfrak{k}\widehat{\mathfrak{s}}_{\widehat{K},\ell}(n; \cdot) \quad \mathfrak{s}\mathfrak{k}_2^2) + \left(\frac{\theta}{2} + \frac{2}{\theta}\right) \mathfrak{k}\mathfrak{s}_{K_0,\ell} \quad \mathfrak{s}\mathfrak{k}_2^2 + \left(\frac{\mathfrak{c}_2}{\theta} + \frac{\mathfrak{c}_{2.5}}{1} \frac{1}{\theta}\right) \frac{\log(n)^5}{n}.$$

By decomposition (14), there exist two deterministic constants  $\mathfrak{c}_3, \mathfrak{c}_4 > 0$ , not depending on  $n, \theta, K$  and  $K_0$ , such that

$$\begin{aligned} \mathbb{E}(\widehat{\mathfrak{ks}}_{\widehat{K}, \ell}(n; \cdot) \quad \mathfrak{sk}_2^2) &\leq \mathbb{E}(\widehat{\mathfrak{ks}}_{K, \ell}(n; \cdot) \quad \mathfrak{sk}_2^2) + \mathbb{E}(\mathfrak{j}\psi_n(K)\mathfrak{j}) + \mathbb{E}(\mathfrak{j}\psi_n(\widehat{K})\mathfrak{j}) \\ &\leq \left(1 + \frac{\theta}{1 - \theta}\right) \mathbb{E}(\widehat{\mathfrak{ks}}_{K, \ell}(n; \cdot) \quad \mathfrak{sk}_2^2) + \frac{\theta}{1 - \theta} \mathbb{E}(\widehat{\mathfrak{ks}}_{\widehat{K}, \ell}(n; \cdot) \quad \mathfrak{sk}_2^2) \\ &\quad + \frac{\mathfrak{c}_3}{\theta} \mathfrak{ks}_{K_0, \ell} \quad \mathfrak{sk}_2^2 + \frac{\mathfrak{c}_4}{\theta(1 - \theta)} \frac{\log(n)^5}{n}. \end{aligned}$$

This concludes the proof.

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