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On Reductions of Local and Global Galois Representations Modulo Prime Powers

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Introduction

In this thesis, we investigate some local and global phenomena related to reductions modulo prime powers of Galois representations of dimension two. In order to be precise, we will first clarify what we mean by reductions modulo prime powers.

Let p be a rational prime and let $\mathbb{L}$ be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_\mathbb{L}$, a fixed choice for a uniformizer π and with residue field $k_\mathbb{L}$. Let G be a profinite group (in what will follow, it will usually be the absolute Galois group of $\mathbb{Q}$ or $\mathbb{Q}_p$) and let V be a $\mathbb{L}$-linear representation of G of dimension 2. Since G is profinite (and in particular compact), there exists a G-stable $\mathcal{O}_\mathbb{L}$-lattice inside V. By a reduction of V modulo π^n for some $n \geq 1$, we simply mean the $\mathcal{O}_\mathbb{L}[G]$-module $T/\pi^n T$. If $n = 1$, one usually consider the semi-simple residual reduction $\overline{V}$ obtained via the semi-simplification of $T \otimes_{\mathcal{O}_\mathbb{L}} k_\mathbb{L}$. By the Brauer-Nesbitt theorem, the representation $\overline{V}$ does not depend on the choice of a G-stable lattice T. In the general case, i.e. when $n \geq 2$, such process is not available anymore a priori and an implicit choice of a G-stable lattice is expected when we consider general reductions modulo prime powers. If V and V' are two $\mathbb{L}$-linear representations of G of dimension two, we will say that V and V' are congruent modulo π^n for some $n \geq 1$ if there exist G-stable lattices T inside V and T' inside V' such that $T/\pi^n T \cong T'/\pi^n T'$ as $\mathcal{O}_\mathbb{L}[G]$-modules.

The thesis is essentially divided in two distinct parts. In the first part, we deal with congruences between modular forms modulo prime powers and their attached Galois representations. In particular, we will study the problem of proving the existence of congruences in a level raising setting. In the second part, we deal with reductions modulo prime powers of irreducible, two-dimensional crystalline representations of the local absolute Galois group of $\mathbb{Q}_p$. Now, we will describe in more depth the settings for both problems and we will present the main results of this thesis.

First, we want to describe the problem of raising the level of newforms modulo some power of a prime p, which is part of the study of congruences between modular forms of different levels. This will be the content of the first chapter of this thesis. By congruences between modular forms we mean congruences between the l-th coefficient in the q-expansion where l runs over all rational primes except a finite number.

The level raising phenomenon (modulo a prime p) was extensively studied in the past thirty years and it was definitely understood in the classical case thanks to the work of Ribet (see [Rib90]) in weight two and trivial character, Diamond (see [Dia91]) for weight $k \geq 2$ and general characters, and Diamond-Taylor (see [DT94]).

In particular, we are interested in the following:

Theorem 0.0.1. *(Ribet, Diamond)*

Let f be a newform of weight $k \geq 2$, level N, character χ and let p be a prime not dividing N. If l is a prime not dividing pN and $p \nmid \frac{1}{2}\varphi(N)Nl(k-2)!$, then the following are equivalent:

(a) there exists a l-newform of level lN, say g, such that $f \equiv g \mod \mathfrak{p}$

(b) $a_l^2 \equiv \chi(l)l^{k-2}(l+1)^2 \mod \mathfrak{p}$.

Here, the symbol $\mathfrak{p}$ denotes a prime ideal of the coefficient field of f lying above the rational prime p, the symbol φ denotes Euler's totient function and a_l denotes the l-th coefficient of the q-expansion of f.

The fundamental idea of the existence of a non-trivial congruence module introduced and developed by Ribet in a geometric context (considering Jacobians attached to modular curves of the form $X_0(N)$) was refined in cohomological terms by Diamond (see [Dia89]). Proving the existence of a congruence module, whose non-triviality is granted by the level raising condition (i.e. the condition (b) in the above theorem), is the heart of both the proofs of Ribet and Diamond, and its cohomological construction (together with a cohomological version of Ihara's lemma) will allow us to apply Ribet's analysis in a slightly more general context.

As a natural extension of studying level raising modulo p, one could ask if the same holds modulo p^n for some positive integer n and if the natural generalization of the level raising condition is the suitable one in the prime powers setting. More specifically, given a newform of level N whose l-th coefficient (for a prime l not dividing pN) satisfies a certain level raising condition modulo p^n, one could ask if it is it true that there exists a newform (at l) of level lN which is congruent modulo p^n to the original newform f of level N. We will refer to this question as full level raising problem.

It turns out that for classical newforms satisfying Ribet-Diamond's level raising condition modulo prime powers, this is not always the case and we will present a counterexample in the last section. Nevertheless, there is a result that partially answers the question, namely Diamond proved (see Thm. 2, [Dia91]) that if the level raising condition holds modulo some power of p, then this gives rise to a family of congruences modulo lower powers of p between the original newform and possibly different newforms (new at l) of level lN. We will state the theorem in precise terms in the last section (see Thm. 1.2.1).

This leads to ask if it is possible to prove the full level raising modulo some prime power if we weaken the definition of a reduced cusp form modulo some prime power. Indeed, if p is a prime and $\mathbb{K}$ is a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_{\mathbb{K}}$, there is a natural arithmetic definition of cuspidal eigenform with coefficients in a complete Noetherian local $\mathcal{O}_{\mathbb{K}}$-algebra with finite residue characteristic, say A.

Definition 0.0.1. *Let $k \geq 2$ and $N \geq 5$ be positive integers. A cuspidal eigenform of weight k, level N (coprime with p) and coefficients in A is an element of the set $Hom_{\mathcal{O}_{\mathbb{K}}\text{-}Alg}(\mathbb{T}, A)$, i.e. it is an $\mathcal{O}_{\mathbb{K}}$-algebra homomorphism from the Hecke algebra $\mathbb{T}$ defined over $\mathcal{O}_{\mathbb{K}}$ acting faithfully on the space of classical cusp forms of weight k and level $\Gamma_1(N)$ to A.*

According to our understanding, this definition goes back to Carayol (see [Car94]) and it strictly depends on the fixed coefficient ring A. A motivation for this definition

comes from the classical case (see sec.2.1 in [Car94]). Indeed, assuming $p \nmid N$, there exists a perfect pairing of $\mathcal{O}_{\mathbb{K}}$-modules between $S_k(\Gamma_1(N), \mathcal{O}_{\mathbb{K}})$ and the Hecke $\mathcal{O}_{\mathbb{K}}$-algebra $\mathbb{T}_N$ given by $\varphi : \mathbb{T}_N \times S_k(\Gamma_1(N), \mathcal{O}_{\mathbb{K}}) \to \mathcal{O}_{\mathbb{K}}$ where $\varphi(T, f) = a_1(Tf)$ (the first coefficient in the q-expansion of Tf). Moreover, assuming $k \geq 2$ and $N \geq 5$, this definition is natural in the sense that it coincides with the definition of Katz cuspidal form with coefficients in A (see sec. 2.1.4 in [Car94]).
Now, by a cuspidal eigenform modulo some prime power (in the literature, it is also called "weak" cuspidal eigenform, see [CKW13], [TW17]) we simply mean a cuspidal eigenform defined as above with coefficients in the $\mathcal{O}_{\mathbb{K}}$-algebra $A = \mathcal{O}_{\mathbb{K}}/(\pi^r)$ where π is a uniformizer and r is a positive integer. A generalized definition, independent of the chosen ring of coefficients, is given by Chen, Kiming and Wiese (see [CKW13]). A natural notion of being new for a cuspidal eigenform modulo prime powers will consist of taking only those homomorphism which factors through the new quotient of the Hecke algebra considered. We will present a precise definition at the beginning of the next section.
A cuspidal eigenform modulo some prime power does not lift in general to an eigenform in characteristic zero once we fix the level and the weight, as it is observed by Calegari and Emerton (see [CE04]). More explicitly, Chen, Kiming and Wiese presented a general construction for non-liftable cuspidal eigenforms modulo p^2 and an explicit example for $p = 3$ (see sec. 5.3, [CKW13]). According to our understanding, there is no known characterization which determines whether or not a cuspidal eigenforms modulo some prime power comes from the reduction of an eigenform in characteristic zero. The only known case is the one of cuspidal eigenforms modulo p, thanks to the Deligne-Serre lifting lemma (see Lemma 6.11, [DS74], see also Prop. 1.10 [Edi97]).
The cuspidal eigenforms modulo modulo prime powers are of particular arithmetic interest. Naively speaking, they represent possible systems of modulo p^n eigenvalues of the Hecke operators. Level raising, level lowering and weight lowering modulo prime powers are phenomenona that represent the dependencies which can occur among these possible systems of eigenvalues. Understanding such dependencies is the motivation for some interesting open conjectures made by Kiming, Rustom and Wiese (see [KRW16]), also related to a conjecture of Buzzard on bounds for the degrees of the coefficient fields of classical Hecke eigenforms (see [Buz05]).
Coming back to the level raising problem for cuspidal eigenforms modulo prime powers, the case of weight two and trivial character was studied by several authors (see sec. 5.6, [BD05], and sec. 1.4, [BBV16]). In particular, the most general result has been proved by Tsaknias and Wiese with different techniques than the previously mentioned articles (see Thm. 5 in [TW17]). A partial result was known in the case of higher weights and trivial character (see Thm. 6.3, [Chi17]).
We will extend Tsaknias-Wiese's result proving the following:

Theorem 0.0.2. *Let $f : \mathbb{T}_N \to \mathcal{O}_{\mathbb{K}}/(\pi^r)$ be a cusp eigenform of level N, weight k and character χ. Assume that its associated residual Galois representation $\bar{\rho}_f$ is absolutely irreducible. Suppose that p does not divide $\varphi(N)N(k-2)!$ and the field $\mathbb{K}$ is sufficiently big.*
If $k = p$ or $k = p+1$, assume that the localized Hecke algebra $\mathbb{T}_{\mathcal{M}}$ is Gorenstein, where $\mathcal{M}$ is the kernel of the reduction of f modulo π.

Let l be a prime which does not divide pN. Then the two following statements are equivalent:

(i) $T_l - \epsilon(l+1)R_l \in Ker(f)$ *for some* $\epsilon \in \{\pm 1\}$, *where* $R_l \in \mathbb{T}_N$ *and* $R_l^2 = l^{k-2}\chi(l)$

(ii) *there exists a cusp eigenform* $g : \mathbb{T}_{N,l} \to \mathcal{O}_{\mathbb{K}}/(\pi^r)$ *of weight k and character χ such that:*

(a) $f(T_q) = g(T_q)$ *for all primes* $q \nmid lpN$,

(b) *g is new at l.*

Here, the symbol $\mathbb{T}_{N,l}$ denotes the Hecke $\mathcal{O}_{\mathbb{K}}$-algebra acting on the space of classical cusp forms of weight k and level $\Gamma_1(N) \cap \Gamma_0(l)$.

Remark 0.0.1. As we will clarify later, by $\mathbb{K}$ sufficiently big, we mean a finite extension of $\mathbb{Q}_p$ which contains a square root of $\chi(l)$ and, if k is odd, it contains a square root of l.

The first chapter of this thesis is organized as follows. In the first section, we will see how to associate to each cuspidal eigenform modulo π^r (whose residual Galois representation is absolutely irreducible) a Galois representation with coefficients in $\mathcal{O}_{\mathbb{K}}/(\pi^r)$. This was done by Carayol (see Thm. 3, [Car94]) using deformation theory. Carayol's construction will allow us to transpose properties of classical newforms to properly defined newforms modulo prime powers and this will lead us to determine the "correct" level raising condition for the level raising problem in this setting. In the second section, we will study such a condition, comparing it with the one considered by Ribet and Diamond in the classical case. Finally, we will show the necessity of this condition if we assume the existence of a congruence coming from level raising. In the third section, we state the main result of this chapter and we give a proof based on the cohomological version of Ihara's lemma proven by Diamond (see [Dia91]). Finally, the end of chapter 1 will be dedicated to present a family of examples of general interest for the level raising phenomenon.
The second problem that we would like to describe concerns reductions modulo prime powers of irreducible, two-dimensional representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. This is the content of the second chapter of this thesis. Crystalline representations play a central role in the study of p-adic representations of the local absolute Galois group $G_{\mathbb{Q}_p} := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ (see, for example, some density results due to Berger (see Thm. IV.2.1 in [Ber04]), Chenevier (see Thm. A in [Che13]), Colmez (see sec. 5.1 in [Col08]), and Kisin (see Thm. 0.3 in [Kis10])). We are interested in studying irreducible crystalline representations of $G_{\mathbb{Q}_p}$ of dimension two and in particular their reductions modulo prime powers.
Let p be an odd prime, let $k \geq 2$ be an integer and $a_p \in \mathfrak{m}_{\mathbb{E}}$ where $\mathbb{E}$ is a finite extension of $\mathbb{Q}_p$, $\mathfrak{m}_{\mathbb{E}}$ denotes the maximal ideal of the ring of integers $\mathcal{O}_{\mathbb{E}}$ with residue field $k_{\mathbb{E}}$. Fix once and for all a choice of a uniformizer, say $\pi_{\mathbb{E}}$. Let $D_{k,a_p} := \mathbb{E}e_1 \oplus \mathbb{E}e_2$ be the filtered φ-module whose structure is given by:

$$\varphi = \begin{pmatrix} 0 & -1 \\ p^{k-1} & a_p \end{pmatrix} \quad \text{and a filtration } \mathrm{Fil}^i(D_{k,a_p}) = \begin{cases} D_{k,a_p} & \text{if } i \leq 0 \\ \mathbb{E}e_1 & \text{if } 1 \leq i \leq k-1 \\ 0 & \text{if } i \geq k \end{cases}$$

By a theorem of Colmez and Fontaine (see Thm. A in [CF00]), there exists a unique crystalline irreducible $\mathbb{E}$-linear representation V_{k,a_p} of dimension two, with Hodge-Tate weights $\{0, k-1\}$ such that $D_{\mathrm{cris}}(V^*_{k,a_p}) = D_{k,a_p}$, where V^*_{k,a_p} denotes the $\mathbb{E}$-linear dual representation of V_{k,a_p}. By a result of Breuil (see Prop. 3.1.1 in [Bre03]), up to twist, any irreducible two-dimensional crystalline representation is isomorphic to V_{k,a_p} for some $k \geq 2$ and $a_p \in \mathfrak{m}_{\mathbb{E}}$.
These results give rise to the natural questions of whether it is possible to completely classify V_{k,a_p} in terms of k and a_p, and how V_{k,a_p} varies when the parameters k and a_p vary p-adically. In general, classifying the representations V_{k,a_p} in characteristic zero turns out to be an hard problem even though some progress have been made in particular cases via the local Langlands correspondence (e.g. see [Pas09]). Nevertheless, much progress has been made in describing the semi-simple residual reductions of the representations V_{k,a_p} using different approaches. We will briefly recall the state of art in the residual case.
Consider the $\mathbb{E}$-linear representation V_{k,a_p} and let T_{k,a_p} be a $G_{\mathbb{Q}_p}$-stable lattice inside V_{k,a_p}, we have an isomorphism $T_{k,a_p} \otimes_{\mathcal{O}_{\mathbb{E}}} \mathbb{E} \cong V_{k,a_p}$ of $G_{\mathbb{Q}_p}$-modules. Denote by $\overline{V}_{k,a_p}$ the semi-simplification of $T_{k,a_p} \otimes_{\mathcal{O}_{\mathbb{E}}} k_{\mathbb{E}}$; by the Brauer-Nesbitt's theorem, the representation $\overline{V}_{k,a_p}$ does not depend on the chosen $G_{\mathbb{Q}_p}$-stable lattice T_{k,a_p}. The problem of describing the representations $\overline{V}_{k,a_p}$ has been deeply studied by many authors via the p-adic and mod p Langlands correspondence (see for example [Bre03] and [BG09]), via Fontaine's theory of (φ, Γ)-modules and its crystalline refinement via Wach modules (see for example [BLZ04]) or via deformation theory (see for example [Roz17])). However, the problem of classifying them is still open and only partial results are known (see for example [BLZ04], [BG15], [BGR18]) although partial conjectures have been formulated (see Conj. 1.5 in [Bre03] and Conj. 1.1 in [Gha19]).
In order to try to describe the reductions $\overline{V}_{k,a_p}$, one different approach consists in finding isomorphisms between different residual representations of the form $\overline{V}_{k,a_p}$ when we let k and a_p vary p-adically. This approach has been developed by Berger et al. with the so-called local constancy results both in the trace and in the weight (see Thm. A and Thm. B in [Ber12] and for for the case $a_p = 0$ see Thm. 1.1.1 in [BLZ04]).
The purpose of this article is to extend Berger's result to a prime power setting. The main difficulty lies in keeping track of the Galois stable lattices involved in the congruences because no semi-simplification process is, a priori, allowed (or defined) for general reductions modulo prime powers. Hence, in proving the existence of such congruences, a dependency on a choice of the Galois stable lattices is expected; but as we will see later, in some cases, the result will be independent of such choice.
The first result of the second chapter of this thesis is the following local constancy result with respect to the trace, i.e. we fix the weight and we let the trace of the crystalline Frobenius vary p-adically:

Theorem 0.0.3. (Local constancy with respect to a_p)
Let $a_p, a'_p \in \mathfrak{m}_{\mathbb{E}}$ and $k \geq 2$ be an integer. Let $m \in \frac{1}{e}(\mathbb{Z}_{\geq 1})$ such that $v(a_p - a'_p) \geq 2 \cdot v(a_p) + \alpha(k-1) + m$, then for every $G_{\mathbb{Q}_p}$-stable lattice T_{k,a_p} inside V_{k,a_p} there exists a $G_{\mathbb{Q}_p}$-stable lattice T_{k,a'_p} inside V_{k,a'_p} such that

$$T_{k,a_p} \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{O}_{\mathbb{E}}/(p^m) \cong T_{k,a'_p} \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{O}_{\mathbb{E}}/(p^m) \text{ as } G_{\mathbb{Q}_p}\text{-modules;}$$

where $\alpha(k-1) = \sum_{n\geq 1} \lfloor \frac{k-1}{p^{n-1}(p-1)} \rfloor$.

This result will be proven in two steps. First we will prove that if a_p and a'_p are sufficiently p-adically close then it is possible to deform p-adically the Wach module attached to the representation T_{k,a_p} into a new Wach module which will correspond to the representation T_{k,a'_p}. Afterwards, we will prove that if the two Wach modules involved are p-adically close (in some sense that will be clarified later) then the corresponding representations T_{k,a_p} and T_{k,a'_p} will be p-adically close as well. We will refer to this feature as continuity property of the Wach modules.
The second and last result of the second chapter is the following local constancy result with respect to the weight, i.e. we fix the trace of the crystalline Frobenius and we let the weight vary in a neighborhood of the weight space $\mathcal{W}$:

Theorem 0.0.4. (Local constancy with respect to k)
Let $a_p \in m_{\mathbb{E}} - \{0\}$ *for some finite extension* $\mathbb{E}$ *of* $\mathbb{Q}_p$*. Let* $k \geq 2$ *and* $m \in \frac{1}{e}(\mathbb{Z}_{\geq 1})$ *be fixed. Assume that*

$$k \geq (3v(a_p) + m) \cdot \left(1 - \frac{p}{(p-1)^2}\right)^{-1} + 1.$$

There exists an integer $r = r(k, a_p) \geq 1$ *such that if* $k' - k \in p^{r+m}(p-1)\mathbb{Z}_{\geq 0}$ *then there exist* $G_{\mathbb{Q}_p}$*-stable lattices* $T_{k,a_p} \subset V_{k,a_p}$ *and* $T_{k',a_p} \subset V_{k',a_p}$ *such that*

$$T_{k,a_p} \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{O}_{\mathbb{E}}/(p^m) \cong T_{k',a_p} \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{O}_{\mathbb{E}}/(p^m) \text{ as } G_{\mathbb{Q}_p}\text{-modules.}$$

The idea is to prove that the representations V_{k,a_p} and V_{k',a_p} are respectively congruent modulo p^m to two representations $V_{k,a_p+\frac{p^{k-1}}{a_p}}$ and $V_{k',a_p+\frac{p^{k'-1}}{a_p}}$ which fit into an analytic family of trianguline representations in the sense of Berger and Colmez (see [BC08]); as a consequence, the claims will follow from proving that if k and k' are sufficiently close in the weight space $\mathcal{W}$ then the representations $V_{k,a_p+\frac{p^{k-1}}{a_p}}$ and $V_{k',a_p+\frac{p^{k'-1}}{a_p}}$ are p-adically close as well (in a sense that will be clarified precisely later in the article). The result constitutes a converse (in a particular crystalline case) to a non-published theorem of Wintenberger, also proven by Berger and Colmez (see Thm 7.1.1 and Cor. 7.1.2 in [BC08]), concerning the continuity property of the Sen periods and the Hodge-Tate weights.
Specializing the above theorems to the case $m = 1/e$, we get a slightly stronger result than the known local constancy results in the semi-simple residual case (see Thm. A and Thm. B in [Ber12]); indeed our conclusions do not involve any semi-simplification process, so for example, being residually decomposable (i.e. direct sum of two characters modulo p) for some choice of lattice is also a locally constant phenomenon.
The motivation behind the study of local constancy phenomena modulo prime powers is two-fold. From a purely representation theoretical point of view, the interest in understanding reductions modulo prime powers of crystalline representations lies in the result of Berger on limits of crystalline representations (see [Ber04]). To be more precise, Berger's result implies that if V is any p-adic representation of $G_{\mathbb{Q}_p}$ with Hodge-Tate weights in a bounded interval I and if $\{V_i\}_{i\in I}$ is a countable family of crystalline representations with HT weights in I such that $T \equiv T_i \mod p^i$, where T is a fixed $G_{\mathbb{Q}_p}$-stable lattice in V and

T_i is a $G_{\mathbb{Q}_p}$-stable lattice in V_i, then V is also crystalline.
Moreover, we observe that a good source of examples for the crystalline representations of the form V_{k,a_p} comes from restriction at $G_{\mathbb{Q}_p}$ of Galois representations attached to classical modular forms of tame level. To be precise, let f be a classical normalized cuspidal eigenform in $S_k(\Gamma_0(N))$ where N is a positive integer prime with p and denote by ρ_f its attached p-adic Galois representation constructed by Deligne and Shimura. We define by $V_p(f) := \rho_f\big|_{G_{\mathbb{Q}_p}}$ the restriction of ρ_f at the decomposition group at p. It is well-known (see [Sch90]), that under the mild hypothesis $a_p^2 \neq 4p^{k-1}$ (see [CE98]), the $G_{\mathbb{Q}_p}$-representation $V_p(f)$ is crystalline and moreover we have that $D_{\text{cris}}(V_p(f)^*) = D_{k,a_p}$ and so $V_p(f) \cong V_{k,a_p}$ where a_p is the p-th coefficient of the q-expansion of f. A straightforward application of the results in this article consists in using the explicit local constancy results in the trace (see Thm. 0.0.3 and Cor. 2.3.6) to find upper and lower bounds for the number of non-isomorphic classes of reductions modulo prime powers of modular crystalline representations of $G_{\mathbb{Q}_p}$ coming from classical modular forms of tame level.
The second chapter of this thesis is organized as follows. In the first section, we will recall the notions of (φ, Γ)-module of Fontaine and of Wach module and their main properties which will be used later in the article. In the second section, we will recall the continuity property of Wach modules which will play a key role in proving the local constancy result in the trace. In the third section, we will show how to p-adically deform Wach modules and we will state and prove the explicit local constancy result when the weight k is fixed and we let the trace of the crystalline Frobenius a_p vary. Finally, in the last section, we are going to state and prove the local constancy result when we fix the trace of the crystalline Frobenius a_p and we let the weight k vary.

Chapter 1

Level raising of cusp eigenforms modulo prime powers

1.1 Raising the level modulo prime powers

In order to generalize Tsaknias-Wiese's level raising result (see [TW17]) to cuspidal eigenforms modulo prime powers of weight greater than 2, we first need to establish a necessary condition for the level raising. We will then compare it with the one considered in the classical case by Ribet (see [Rib90], see also [Dia89]) and Diamond (see [Dia91]).
As in the previous sections, $\mathbb{K}$ will denote a finite extension of $\mathbb{Q}_p$ for a fixed prime $p \geq 5$, $\mathcal{O}_{\mathbb{K}}$ will denote its ring of integers and π will denote a uniformizer. We will assume that p is different from 2.
Let $N \geq 5$ and $k \geq 2$ two positive integers. Throughout the paper, we consider them fixed. From now on, we assume that the prime p does not divide $\varphi(N)$, where φ is the Euler's totient function.
We fix once and for all a Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathcal{O}_{\mathbb{K}}^\times$. We will denote by $\mathbb{T}_N$ the $\mathcal{O}_{\mathbb{K}}$-subalgebra of $\mathrm{End}_{\mathbb{K}}(S_k(\Gamma_1(N), \chi, \mathbb{K}))$ generated by the Hecke operators T_n for $n \geq 1$. If $m \in \mathbb{N}$ is coprime with pN, the $\mathcal{O}_{\mathbb{K}}$-algebra $\mathbb{T}_N$ contains the modified diamond operators $S_m := m^{k-2}\langle m\rangle = m^{k-2}\chi(m)$; they are scalar operators.
As we will see in detail later on, the condition that p does not divide $\varphi(N)$ ensures us that $S_k(\Gamma_1(N), \chi, \mathcal{O}_{\mathbb{K}})$ is a direct summand in $S_k(\Gamma_1(N), \mathcal{O}_{\mathbb{K}})$.
Similarly, if l is a prime which does not divide pN, we define the Hecke algebra $\mathbb{T}_{N,l}$ of level lN as the $\mathcal{O}_{\mathbb{K}}$-subalgebra of $\mathrm{End}_{\mathbb{K}}(S_k(\Gamma_1(N) \cap \Gamma_0(l), \chi, \mathbb{K}))$ generated by the Hecke operators. Note that $S_l \in \mathbb{T}_{N,l}$ since we are considering the congruence subgroup $\Gamma_1(N) \cap \Gamma_0(l)$.

1.1.1 Galois representations attached to cuspidal eigenforms modulo prime powers

Let A be a complete Noetherian local $\mathcal{O}_{\mathbb{K}}$-algebra with finite residue field of characteristic p. Let f be an element in the set $\mathrm{Hom}_{\mathcal{O}_{\mathbb{K}}\text{-alg}}(\mathbb{T}_N, A)$, or in other words, a cuspidal eigenform with coefficients in A and character χ.
First of all, we will start to associate to f a residual Galois representation $\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to$

$\mathrm{Gl}_2(\bar{\mathbb{F}}_p)$.
Let $\bar{f}$ be the reduction of f modulo the unique maximal ideal $\mathcal{M}_A$ of A, i.e. the composition of f with the natural projection map from A to the quotient $A/\mathcal{M}_A$. Now, $\bar{f}$ is a cusp eigenform whose coefficients lie in a finite extension of $\mathbb{F}_p$.
By a lemma of Carayol (see Prop. 1.10 [Edi97]) and recalling the assumption $p \geq 5$, we obtain that $\bar{f}$ comes from the reduction of a classical (i.e. holomorphic) Hecke eigenform $g \in S_k(\Gamma_1(N), \chi)$ with coefficients in an order of a number field.
By a theorem of Deligne and Shimura, we can associate to g, and so to $\bar{f}$, a semisimple residual Galois representation $\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gl}_2(\bar{\mathbb{F}}_p)$ which is unramified outside the primes which divide pN and such that $\mathrm{Trace}(\rho(\mathrm{Frob}_q)) = \bar{f}(T_q)$ and $\mathrm{Det}(\rho(\mathrm{Frob}_q)) = q\bar{f}(S_q) = q^{k-1}\chi(q)$ for any prime q which does not divide pN. We will denote this residual representation associated to $\bar{f}$ by $\rho_{\bar{f}}$.
Assuming that $\rho_{\bar{f}}$ is absolutely irreducible, a theorem of Carayol (Thm. 3, [Car94]), allows us to associate to each cusp eigenform with coefficients in A, a Galois representations ρ_f with coefficient in A whose reduction modulo the maximal ideal $\mathcal{M}_A$ coincides with $\rho_{\bar{f}}$. In other words, the representation ρ_f is a deformation of $\rho_{\bar{f}}$.
In other terms, the following result holds:

Theorem 1.1.1. *(Carayol)*
Let $f : \mathbb{T}_N \to A$ be a cusp eigenform of level N, weight k, character χ and coefficients in A such that its residual associated Galois representation is absolutely irreducible. Then there exists a unique (up to isomorphism) continuous representation:

$$\rho_f : Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow Gl_2(A)$$

which is unramified at primes not dividing pN, and which satisfies the relations:

$$\begin{cases} Trace(\rho_f(Frob_q)) = f(T_q) \\ Det(\rho_f(Frob_q)) = f(qS_q) = qf(S_q) = q^{k-1}\chi(q) \end{cases} \quad \textit{for all primes } q \nmid pN.$$

Hence, let $r \geq 1$ be an integer and let f be a cusp eigenform on $\Gamma_1(N)$ of character χ, weight $k \in \mathbb{N}_{\geq 2}$, level $N \in \mathbb{N}$ and coefficients in $A := \mathcal{O}_{\mathbb{K}}/(\pi^r)$ whose attached residual Galois representation is absolutely irreducible.
Using Carayol's theorem, we associated to f a Galois representation

$$\rho_f : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Gl}_2(\mathcal{O}_{\mathbb{K}}/(\pi^r))$$

such that:

$$\begin{cases} \mathrm{Trace}(\rho_f(\mathrm{Frob}_q)) = f(T_q) \\ \mathrm{Det}(\rho_f(\mathrm{Frob}_q)) = f(qS_q) = qf(S_q) = q^{k-1}\chi(q) \end{cases} \quad \text{for all primes } q \nmid pN$$

and which is unramified outside the set of primes dividing pN.
Now, since we are interested in congruences between cusp eigenforms of level N and level lN modulo some power of p , we need to understand what happens to these representations at the (possibly bad) prime l.

The strategy will be the following: first, we will briefly recall the construction of the universal representation in Carayol's proof of the above theorem, then we will consider a cusp eigenform modulo prime powers g of level lN (i.e. with respect to the congruence subgroup $\Gamma_1(N) \cap \Gamma_0(l)$ where l does not divide N) which is new at l, and we will use the Carayol's universal representation construction to deduce some properties on the Galois representation ρ_g associated to g.
As long as it is possible, we will work in the general context where coefficients of the cusp eigenforms lie in a complete Noetherian local $\mathcal{O}_{\mathbb{K}}$-algebra A with finite residue field of characteristic p.
Let f be a cusp eigenform defined as in the above theorem. We define $\bar{f}$ the reduction of f modulo the maximal ideal $\mathcal{M}_A$ of A. We denote by $\mathcal{M}$ the kernel of $\bar{f} : \mathbb{T}_N \to A/\mathcal{M}_A \hookrightarrow \bar{\mathbb{F}}_p$; it is a maximal ideal.
Since A is complete, by the universal property of completion, we have that f factors through the completion of the Hecke algebra at $\mathcal{M}$, i.e. we have the following commutative diagram:

$$\begin{array}{ccc} & & \mathbb{T}_{\mathcal{M}} \\ & \overset{\lambda^{\mathrm{univ}}}{\nearrow} & \downarrow \tilde{f} \\ \mathbb{T} & \xrightarrow[f]{} & A, \end{array}$$

where λ^{univ} is the natural ring homomorphism from $\mathbb{T} := \mathbb{T}_N$ to its completion at $\mathcal{M}$. Note that, when it is possible (i.e. clear from the context), we will drop the subscript denoting the level considered in order to simplify the notation.
This implies that if we are able to construct a Galois representation of the so-called (e.g. [Car94]) universal Hecke eigenform λ^{univ}, then we can associate to f the Galois representation given by the composition between the universal one and the homomorphism $\mathrm{Gl}_2(\mathbb{T}_{\mathcal{M}}) \to \mathrm{Gl}_2(A)$ induced by $\tilde{f}$.
Following Carayol, we proceed as follows: first we consider the $\mathcal{O}_{\mathbb{K}}$-subalgebra of $\mathbb{T}_N$ generated by all the Hecke operators T_n with $\gcd(n, pN) = 1$ and we denote it by $\mathbb{T}' \subset \mathbb{T}$. Note that in this context, the diamond operators are just scalars. We denote by $\hat{\mathbb{T}}'_{\mathcal{M}}$ the integral closure of $\mathbb{T}'_{\mathcal{M}}$ in $\mathbb{T}'_{\mathcal{M}} \otimes \mathbb{K}$, where $\mathbb{T}'_{\mathcal{M}}$ is the image of $\mathbb{T}'$ in $\mathbb{T}_{\mathcal{M}}$ (we drop the subscript that denotes the level N of the Hecke algebra). Since the Hecke operators T_n with $\gcd(n, N) = 1$ are simultaneously diagonalizable, the ring $\hat{\mathbb{T}}'_{\mathcal{M}}$ is isomorphic to the finite product of some rings of integers of finite extensions of $\mathbb{K}$ which we denote by $\mathbb{E}_j$ where the index j runs over a finite subset J of the positive integers. The finite set J is in bijection with the set of classical normalized eigenforms, up to $\mathrm{Gal}(\bar{\mathbb{K}}/\mathbb{K})$-conjugacy, whose residual Galois representation is isomorphic to $\bar{\rho}_f$. Hence, we have an injection:

$$\mathbb{T}'_{\mathcal{M}} \hookrightarrow \hat{\mathbb{T}}'_{\mathcal{M}} \cong \prod_{j \in J} \mathcal{O}_{\mathbb{E}_j}.$$

Now, fixing an $i \in J$, we can consider the composition with the standard projection on the i-th component Pr_i and so we get the ring homomorphism g_i:

$$g_i : \mathbb{T}' \to \mathbb{T}'_{\mathcal{M}} \hookrightarrow \hat{\mathbb{T}}'_{\mathcal{M}} \cong \prod_{j \in J} \mathcal{O}_{\mathbb{E}_j} \xrightarrow{Pr_i} \mathcal{O}_{\mathbb{E}_i}.$$

The above homomorphism g_i can be extended to a homomorphism $g_i : \mathbb{T} \to \tilde{\mathbb{E}}_i$ where $\tilde{\mathbb{E}}_i$ is a finite extension of $\mathbb{E}_i$. In order to do so, we need to define g_i on the operators T_q where q is a prime dividing the level N. Since all these operators satisfy their characteristic polynomials, it is enough to define $\tilde{\mathbb{E}}_i$ as the smallest field in which these characteristic polynomials split completely. Moreover since we have a finite number of these polynomials, we get a finite degree extension of $\mathbb{E}_i$. For simplicity, we keep denoting these extensions by $\mathbb{E}_i$.
Now, the ring homomorphism g_i is a classical normalized eigenform with coefficients in a finite extension of $\mathbb{Q}_p$ whose residual Galois representation is isomorphic to $\bar{\rho}_f$. By a theorem of Deligne and Shimura, we can associate to each of the g_i a Galois representation $\rho_{g_i} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gl}_2(\mathbb{E}_i)$ such that the traces of the images of Frobenius elements at primes different from p and not dividing the level are the coefficients of the q-expansion associated to the cusp form g_i. Finally, it is enough to define the representation of the universal modular form as the product of the ρ_{g_i} for all the finite indexes i, so we get:

$$\rho^{\mathrm{univ}} := \prod_{i \in J} \rho_{g_i} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gl}_2(\hat{\mathbb{T}}'_{\mathcal{M}}) \cong \mathrm{Gl}_2\Big(\prod_{i \in J} \mathcal{O}_{\mathbb{E}_i}\Big) \cong \prod_{i \in J} \mathrm{Gl}_2(\mathcal{O}_{\mathbb{E}_i}).$$

Now, a priori we know that the image of the universal representation is contained in $\mathrm{Gl}_2(\hat{\mathbb{T}}'_{\mathcal{M}})$, but it actually lies inside $\mathrm{Gl}_2(\mathbb{T}'_{\mathcal{M}})$. In order to show this, it is sufficient to observe that, since we are assuming that the residual Galois representation attached to f is absolutely irreducible, the claim follows from a theorem of Carayol (see Thm.2 in [Car94]; or sec. 6 in [Maz97]), Chebotarev's density theorem and from the fact that the traces at Frobenius elements lie in $\mathbb{T}'_{\mathcal{M}}$. The existence of the representation ρ^{univ} allows us to define ρ_f as the Galois representation attached to f via the following commutative diagram:

$$\begin{array}{ccc}
 & & \mathrm{Gl}_2(\mathbb{T}'_{\mathcal{M}}) \hookrightarrow \prod_{i \in J} \mathrm{Gl}_2(\mathcal{O}_{\mathbb{E}_i}) \\
 & \overset{\rho^{\mathrm{univ}}}{\nearrow} & \downarrow \tilde{f}_\star \\
\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) & \xrightarrow[\rho_f := \tilde{f}_\star \circ \rho^{\mathrm{univ}}]{} & \mathrm{Gl}_2(A)
\end{array}$$

where the homomorphism $\tilde{f}_\star$ is the one induced functorially by $\tilde{f} : \mathbb{T}_{\mathcal{M}} \to A$ on the general linear groups.
Now, we want to introduce a notion of newform for cuspidal eigenforms with coefficients in a complete Noetherian local $\mathcal{O}_{\mathbb{K}}$-algebra A, and deduce useful properties of its associated Galois representations. This will allow us to find a necessary congruence condition for the level raising problem in the next section.
Let l be a prime not dividing pN and let $\mathbb{T}^{l\text{-new}}_{N,l}$ be the $\mathcal{O}_{\mathbb{K}}$-algebra generated by all Hecke operators inside the endomorphism ring of the $\mathbb{K}$-vector space $S_k(\Gamma_1(N) \cap \Gamma_0(l), \chi, \mathbb{K})^{l\text{-new}}$. It is a natural quotient of the Hecke algebra $\mathbb{T}_{N,l}$. Note that when it is possible (i.e. clear from the context) we will drop the subscript that denotes the level in the symbol $\mathbb{T}^{l\text{-new}}_{N,l}$. We give the following:

Definition 1.1.1. *Let $g : \mathbb{T}_{N,l} \to A$ be a cuspidal eigenform of level lN weight k, character χ. The cuspidal eigenform g is l-new if it factors (as $\mathcal{O}_{\mathbb{K}}$-algebra homomorphism) via the quotient $\mathbb{T}^{l\text{-new}}_{N,l}$.*

A careful analysis of Carayol's construction of the universal Galois representation will allow us to translate properties of classical l-newforms to cuspidal eigenforms which are l-new in the above sense.
Fix a cuspidal eigenform $g : \mathbb{T}_{N,l} \to A$ of weight k, character χ which is new at l. Let $\mathcal{N}$ be the kernel of the reduction of g modulo the maximal ideal of A. Localizing $\mathbb{T}_{N,l}$ at the maximal ideal $\mathcal{N}$ and denoting by $\mathcal{N}^{l\text{-new}}$ the image of the maximal ideal $\mathcal{N}$ via the natural projection $\mathbb{T}_{N,l} \twoheadrightarrow \mathbb{T}_{N,l}^{l\text{-new}}$, we have the following commutative diagram (note that $\mathcal{N}^{l\text{-new}}$ is still a maximal ideal):

$$\begin{array}{ccc} & & \mathbb{T}^{l\text{-new}}_{\mathcal{N}^{l\text{-new}}} \\ & \overset{\lambda^{\text{univ}}_{l\text{-new}}}{\nearrow} & \big\downarrow \tilde{g} \\ \mathbb{T}^{l\text{-new}} & \underset{g}{\longrightarrow} & A \end{array}$$

where $\lambda^{\text{univ}}_{l\text{-new}}$ is the natural homomorphism from $\mathbb{T}^{l\text{-new}}$ to its localization at $\mathcal{N}^{l\text{-new}}$.
Hence we can consider the "universal" Galois representation given by the product of all the Galois representations associated to the classical Hecke eigenforms which are new at l:

$$\rho^{\text{univ}}_{l\text{-new}} : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \text{Gl}_2(\mathbb{T}^{l\text{-new}}_{\mathcal{N}^{l\text{-new}}}) \hookrightarrow \text{Gl}_2\Big(\prod_{i\in\tilde{J}} \mathcal{O}_{\mathbb{E}_i}\Big) \cong \prod_{i\in\tilde{J}} \text{Gl}_2(\mathcal{O}_{\mathbb{E}_i})$$

where $\tilde{J}$ is a finite set in bijection with the classical l-newforms of level N whose residual Galois representation is isomorphic to the residual Galois representation of g.
Now, recalling that we are always assuming that l does not divide pN, the representation $\rho^{\text{univ}}_{l\text{-new}}$ has an explicit description at the prime l:

Lemma 1.1.2. *Consider the universal Galois representation*

$$\rho^{univ}_{l\text{-}new} : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \text{Gl}_2(\mathbb{T}^{l\text{-}new}_{\mathcal{N}^{l\text{-}new}})$$

associated to the newforms at l constructed above. Then the trace map is well defined at $Frob_l$ and it satisfies:

$$Trace(\rho^{univ}_{l\text{-}new})(Frob_l) = (l+1)U_l$$

where $U_l \in \mathbb{T}^{l\text{-}new}_{\mathcal{N}^{l\text{-}new}}$. Equivalently, since by definition the finite set $\tilde{J}$ defined above is in bijection with the (finite) set of classical Hecke newforms at l of level lN (denoted by h_i for i in $\tilde{J}$), we have that:

$$Trace(\rho^{univ}_{l\text{-}new})(Frob_l) = ((l+1)h_i(U_l))_{i\in\tilde{J}} \in \prod_{i\in\tilde{J}} \mathcal{O}_{\mathbb{E}_i}.$$

Proof. Let h be a classical Hecke newform at l of level lN, and consider the Galois representation ρ_h attached to h by Deligne and Shimura. Since h is new at l, it is well known (e.g sec. 3.3, [Wes05]) that ρ_h is ordinary at the prime l. We should recall that being ordinary is a deformation condition (sec. 30, [Maz97]). Moreover, we have a very explicit description of ρ_h when restricted to the decomposition group $G_{\mathbb{Q}_l}$ (see Lemma

2.6.1, [EPW06]). Indeed, let V_{I_l} be the largest quotient of the Galois module V given by the representation ρ_h, on which the inertia group I_l acts trivially. Then there exists an unramified character $\eta : G_{\mathbb{Q}_l} \to \bar{\mathbb{Q}}_p$ such that:

$$\begin{aligned}&(1)\ \eta(\mathrm{Frob}_l) = a_l(h) = h(U_l)\\ &(2)\ G_{\mathbb{Q}_l} \text{ acts on } V_{I_l} \text{ via the character } \eta\\ &(3)\ \rho_h\Big|_{G_{\mathbb{Q}_l}} = \begin{pmatrix} \omega_p\eta & \star \\ 0 & \eta \end{pmatrix}\end{aligned}$$

where ω_p denotes the p-adic cyclotomic character. Here, the symbol $a_l(h)$ denotes the l-coefficient of the q-expasion attached to h. Now, since $l \neq p$, we deduce that the trace is well defined at the Frobenius at l and then:

$$\mathrm{Trace}(\rho_h)(\mathrm{Frob}_l) = (\omega_p(\mathrm{Frob}_l) + 1)\eta(\mathrm{Frob}_l) = (l+1)h(U_l).$$

Since the representation $\rho^{\mathrm{univ}}_{l\text{-new}}$ is defined as the product $\prod_{i \in \tilde{J}} \rho_{h_i}$, it follows that the semi-simplification of its restriction to the decomposition group $G_{\mathbb{Q}_l}$ is unramified, and so the claim follows. □

Remark 1.1.1. The key role in the above proof is played by the local property (at l) of the Galois representation attached to a newform (at l) which consists of being ordinary at l. We refer the reader to section 5 in [Rib94] for a very clear explanation on the reason why the property holds.

Now, from the above lemma we deduce that if $g : \mathbb{T}_{N,l} \to A$ is a cusp eigenform of level lN, new at l with weight k, character χ and coefficients in A, then, as already mentioned above, its Galois representation ρ_g factors through the universal representation $\rho^{\mathrm{univ}}_{l\text{-new}}$. Hence, denoting by $\tilde{g}$ the map $\mathbb{T}^{l\text{-new}}_{\mathcal{N}^{l\text{-new}}} \to A$ induced by g, and denoting by $\tilde{g}_\star$ the map induced on the general linear groups by $\tilde{g}$, we can apply the trace map and get:

$$\mathrm{Trace}(\rho_g)(\mathrm{Frob}_l) = \mathrm{Trace}(\tilde{g}_\star(\rho^{\mathrm{univ}}_{l\text{-new}}))(\mathrm{Frob}_l) = \tilde{g}((l+1)U_l) = (l+1)g(U_l).$$

Note that even though ρ_g is a priori not unramified at l, the image of the restriction at the l-decomposition group is contained in a Borel subgroup and the two characters on the main diagonal are unramified at l, hence its semisimplification is unramified as well and so the trace and the determinant are well defined at Frob_l.
A key fact in proving the necessity of the level raising condition lies in the explicit description of the l-th coefficient of a classical newform of level lN , and now we will show how this knowledge can be extended in the more general case of cuspidal eigenforms modulo prime powers. More specifically, Hida proved (see Lemma 3.2, [Hid85]) that if g is a classical newform (at l) of level lN (as usual, $l \nmid N$), weight $k \geq 2$, and character χ of conductor which divides N, then $a_l(g)^2 = l^{k-2}\chi(l)$, where as usual $a_l(g)$ denotes the l-coefficient of the q-expansion of g. In other words, every $\mathcal{O}_{\mathbb{K}}$-algebra homomorphism $g : \mathbb{T}_{N;l} \to \mathcal{O}_{\mathbb{K}}$ which is new at l satisfies $g(U_l^2 - S_l) = 0$. We observe that we can apply Hida's lemma because we are working with the congruence subgroup $\Gamma_1(N) \cap \Gamma_0(l)$ and so the conductor of χ is coprime to l.

This result can be easily generalized to cuspidal eigenforms with coefficients in A of level lN which are new at l by observing that, thanks to the result of Hida mentioned before, the operator $U_l^2 - S_l$ is in the kernel of the natural projection $\mathbb{T}_{N,l} \twoheadrightarrow \mathbb{T}_{N,l}^{l\text{-new}}$.
Hence, we have the following lemma:

Lemma 1.1.3. *Let $g : \mathbb{T}_{N,l} \to A$ a cuspidal eigenform of weight $k \geq 2$, level lN, character χ and coefficients in A, which is new at l. Then g satisfies $g(U_l^2 - S_l) = 0$.*

The proof is immediate from the definition of g being new at l since the operator $U_l^2 - S_l$ belongs to the kernel of the natural projection $\mathbb{T}_{N,l} \twoheadrightarrow \mathbb{T}_{N,l}^{l\text{-new}}$. It is also worth to mention that this result can be expressed in terms of properties of the Galois representation attached to a cuspidal eigenform new at l with coefficients in the $\mathcal{O}_{\mathbb{K}}$-algebra A. Indeed, it is enough to observe that by construction:

$$\mathrm{Det}(\rho_{l\text{-new}}^{\mathrm{univ}})(\mathrm{Frob}_l) = lU_l^2 = lS_l$$

where the last equality holds because of Hida's result. We recall that the determinant is well defined at Frob_l since the semi-simplification of the restriction of $\rho_{l\text{-new}}^{\mathrm{univ}}$ to the decomposition group $G_{\mathbb{Q}_l}$ is unramified at l. Hence, Lemma 1.1.2 and Lemma 1.1.3 give the explicit formula for the characteristic polynomial at Frob_l:

$$\mathrm{charpol}((\rho_{l\text{-new}}^{\mathrm{univ}})(\mathrm{Frob}_l))(x) = x^2 - (l+1)U_l x + lS_l.$$

We recall that, for the sake of simplicity, we are making an abuse of notation in using the same symbol U_l for the l-th Hecke operator in the Hecke algebra $\mathbb{T}$ of level lN, and for the images of U_l in the completion $\mathbb{T}_{\mathcal{N}}$ and $\mathbb{T}_{\mathcal{N}^{l\text{-new}}}^{l\text{-new}}$.
Finally, we ready to use the theory of Galois representations to compare the coefficients of cuspidal eigenforms modulo prime powers in a context of level raising.

1.1.2 The level raising condition

In this section we will find a necessary condition for two cusp eigenforms modulo π^r (one of level N and one of level lN, new at l where l does not divide pN) to be equal (modulo π^r) almost everywhere. We will do it by comparing the associated Galois representations constructed in the previous subsection.
Let $A := \mathcal{O}_{\mathbb{K}}/(\pi^r)$ for some positive integer r. Let f be a cusp eigenform of level N, weight k, character χ and coefficients in A. As usual let l be a prime not dividing pN. We will always assume that the semisimple residual Galois representation $\rho_{\bar{f}}$ attached to f is absolutely irreducible. Then, by Carayol's theorem (see Thm. 3, [Car94]) mentioned above, we have that ρ_f is unramified at l and it satisfies:

$$\mathrm{Trace}(\rho_f(\mathrm{Frob}_l)) = f(T_l).$$

As in the previous section, let g be a cusp eigenform of level lN, new at l, same weight and character of f with coefficients in A. By the discussion in the previous section, we know that the Galois representation ρ_g attached to g through Carayol's theorem satisfies the well-defined property:

$$\mathrm{Trace}(\rho_g)(\mathrm{Frob}_l) = (l+1)g(U_l).$$

Hence, if we assume that f is congruent to g, i.e. $f(T_n) = g(T_n)$ for all positive integers n coprime with lpN, we have that in terms of Galois representations:

$$\mathrm{Trace}(\rho_f(\mathrm{Frob}_q)) = \mathrm{Trace}(\rho_g(\mathrm{Frob}_q))$$

for all primes q not dividing plN.
Hence, by the Chebotarev's density theorem, the representations ρ_f and ρ_g agree on a subset of density one of the set of generators of the absolute Galois group, hence they agree at all the traces, in particular at l. Thanks to a theorem of Carayol (see Thm.1, [Car94]), the two representations ρ_f and ρ_g are isomorphic. In particular,

$$\mathrm{Trace}(\rho_f(\mathrm{Frob}_l)) = \mathrm{Trace}(\rho_g(\mathrm{Frob}_l)),$$

where $\mathrm{Trace}(\rho_g(\mathrm{Frob}_l))$ is well defined since the characters on the diagonal of the restriction of ρ_g at the decomposition group at l are unramified. More explicitly, we have

$$f(T_l) = (l+1)g(U_l) \text{ in } \mathcal{O}_{\mathbb{K}}/(\pi^r),$$

where $T_l \in \mathbb{T}_N$ and $U_l \in \mathbb{T}_{N;l}$. Moreover, we can make the above condition independent of g and get the necessary so-called level raising condition. In order to do so, we first need to introduce a special operator in the Hecke algebra $\mathbb{T}_{N;l}$. Assume that the field $\mathbb{K}$ is sufficiently big in the sense that it contains the roots of the polynomial $x^2 - \chi(l)$ and, if the weight k is odd, it contains a square root of l, then the following lemma holds:

Lemma 1.1.4. *There exists an operator R_l in the $\mathcal{O}_{\mathbb{K}}$-algebra $\mathbb{T}_{N;l}$ such that $R_l^2 = S_l$.*

Proof. Indeed, fix an odd positive integer m and fix once and for all a root $\zeta \in \mathbb{K}$ of the polynomial $x^2 - \chi(l)$.
We define the operator $R_l := (l^{\frac{2-k}{2}}\zeta^{-1})^m S_l^{\frac{m+1}{2}}$ (see also sec. 4, [Dia89] for a similar definition in weight $k = 2$). A straightforward computation shows that $R_l^2 = S_l$. Note that if $g : \mathbb{T}_{N;l} \to A$ is a cusp eigenform then $g(R_l) = l^{\frac{k-2}{2}}\zeta$, and so the operator R_l does not depend on the choice of m. We need to prove that R_l lies in the $\mathcal{O}_{\mathbb{K}}$-algebra $\mathbb{T}_{N;l}$.
Since m is odd and $S_l \in \mathbb{T}_{N;l}$ we deduce that $S_l^{\frac{m+1}{2}} \in \mathbb{T}_{N;l}$. Since $\chi(l)$ is a root of unity and $\zeta \in \mathbb{K}$ we deduce that $\zeta \in \mathcal{O}_{\mathbb{K}}^{\times}$, so same holds for ζ^{-m}. Since $l^{\frac{k-2}{2}} \in \mathcal{O}_{\mathbb{K}}^{\times}$ we conclude that $R_l \in \mathbb{T}_{N;l}$.
Indeed, since $g(U_l)^2 = g(S_l)$, the assumption $p \neq 2$ and Hensel's lemma ensure that there exists an $\epsilon_{l,g} \in \{\pm 1\}$ such that $g(U_l) = \epsilon_{l,g}g(R_l) = \epsilon_{l,g}l^{\frac{k-2}{2}}\zeta$. We recall that $\zeta \in \mathcal{O}_{\mathbb{K}}^{\times}$ satisfies $\zeta^2 = \chi(l)$ and it exists because, by assumption, $\mathbb{K}$ contains the splitting field of $p_{l,\chi}(x) = x^2 - \chi(l) = 0$. □

Since g has the same weight k, and same character χ of f, we finally conclude that:

$$f(T_l) = \epsilon_{l,g}(l+1)f(R_l) = \epsilon_{l,g}(l+1)l^{\frac{k-2}{2}}\zeta \qquad \text{in } \mathcal{O}_{\mathbb{K}}/(\pi^r).$$

Denoting the kernel of $f : \mathbb{T}_N \to \mathcal{O}_{\mathbb{K}}/(\pi^r)$ by I, we then restate the above necessary property for the level raising as: for $\epsilon \in \{\pm 1\}$

$$T_l - \epsilon(l+1)R_l \in I \qquad\qquad (\mathrm{LRC})_\epsilon$$

We will refer to the above condition by "*level raising condition of parameter* ϵ", or shortly $(\mathrm{LRC})_\epsilon$.
Then we have proven the following necessary condition for the level raising in the context of newforms modulo prime powers:

Proposition 1.1.5. *Let $f : \mathbb{T}_N \to \mathcal{O}_{\mathbb{K}}/(\pi^r)$ be a cusp eigenform of weight $k \geq 2$, level $N \in \mathbb{Z}_{>0}$, character χ where $\mathbb{K}$ is a sufficiently big, finite extension of $\mathbb{Q}_p$. Assume that $\rho_{\bar{f}}$ is absolutely irreducible. Let l be a prime such that $l \nmid pN$. Suppose there exists a cusp eigenform $g : \mathbb{T}_{N,l} \to \mathcal{O}_{\mathbb{K}}/(\pi^r)$ of level lN, same weight and character as f, which is new at l and that satisfies:*

$$f(T_n) = g(T_n) \text{ for all } n \in \mathbb{N} \text{ such that } gcd(n, plN) = 1$$

then f satisfies $(\mathrm{LRC})_\epsilon$ for some $\epsilon \in \{\pm 1\}$.

Remark 1.1.2. By $\mathbb{K}$ sufficiently big we mean a finite extension of $\mathbb{Q}_p$ which contains a square root of $\chi(l)$ and, if k is odd, it contains a square root of l.

We want to remark that in the literature (e.g. [Rib90], [Dia91], [Wes05]) the level raising condition is usually expressed as follows:

$$T_l^2 - (l+1)^2 S_l \in I \qquad\qquad (\mathrm{LRC})^2.$$

We will refer to the above condition as $(\mathrm{LRC})^2$. Clearly, if f is a cusp eigenform modulo π^r which satisfies $(\mathrm{LRC})_\epsilon$, then it satisfies also $(\mathrm{LRC})^2$. The converse does not hold in general. Before stating the result connecting the two level raising conditions we need some definitions. First, we will denote by v_π the π-adic valuation on $\mathcal{O}_{\mathbb{K}}$. Now, let $f : \mathbb{T}_N \to \mathcal{O}_{\mathbb{K}}/(\pi^r)$ be a cusp eigenform of weight k and character χ. Let l be a prime which does not divide pN such that f satisfies the level raising condition at l given by $(\mathrm{LRC})^2$. Let $s \leq r-1$ be a positive integer. As usual we denote by I the kernel of f. We define I_s as the kernel of the composition $f_s : \mathbb{T}_N \to \mathcal{O}_{\mathbb{K}}/(\pi^r) \twoheadrightarrow \mathcal{O}_{\mathbb{K}}/(\pi^s)$ where the first arrow is given by f and the second one is given by the natural projection. We say that f satisfies s-$(\mathrm{LRC})_\epsilon$ if there exists an $\epsilon \in \{\pm 1\}$ such that the reduction of f mod π^s, i.e. f_s, satisfies $f_s(T_l) - \epsilon(l+1)f_s(R_l) \in I_s$. For instance, the conditions r-$(\mathrm{LRC})_\epsilon$ and $(\mathrm{LRC})_\epsilon$ are the same. The condition 0-$(\mathrm{LRC})_\epsilon$ is the empty condition. Now we are ready to present the following result which states a more precise connection between the level raising conditions. We recall that we are assuming that p is an odd prime.

Lemma 1.1.6. *Let $f : \mathbb{T}_N \to \mathcal{O}_{\mathbb{K}}/(\pi^r)$ a cusp eigenform of weight k and character χ. Let l be a prime which does not divide pN such that f satisfies the level raising condition at l given by* $(\mathrm{LRC})^2$. *Let v be the value of the π-adic valuation at $l+1$, i.e. $v = v_\pi(l+1)$. Then there exists an $\epsilon \in \{\pm 1\}$ such that f satisfies the following level raising conditions* $(r-v)$-$(\mathrm{LRC})_\epsilon$. *In particular, if the prime l satisfies $v_\pi(l+1) = 0$, then the conditions* $(\mathrm{LRC})^2$ *and* $(\mathrm{LRC})_\epsilon$ *for some $\epsilon \in \{\pm 1\}$ are equivalent.*

Proof. The result comes from straightforward computations modulo prime powers. Consider the identity $f(T_l)^2 - (l+1)^2 f(S_l) = (f(T_l) - (l+1)f(R_l))(f(T_l) + (l+1)f(R_l))$ and define $a := v_\pi(f(T_l) - (l+1)f(R_l))$ and $b := v_\pi(f(T_l) + (l+1)f(R_l))$. We can assume, without loss of generality, that $a \le b$. Since $v_\pi(2f(R_l)) = 0$, we have that

$$v = v_\pi(l+1) = v_\pi(f(T_l) - (l+1)f(R_l) - f(T_l) - (l+1)f(R_l)) \ge \min\{a, b\} = a,$$

and so we deduce that $b = r - a \ge r - v$. In other words, the cuspidal eigenform f satisfies $(r-v)$-$(\mathrm{LRC})_-$.

□

1.1.3 The level raising congruence

As in the previous section, let $N \ge 5$, r and $k \ge 2$ be positive integers. Fix an odd prime p not dividing N. Let $\mathbb{K}$ be a sufficiently big finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_{\mathbb{K}}$ and a uniformizer π.
We will prove the following:

Theorem 1.1.7. *Let $f : \mathbb{T}_N \to \mathcal{O}_{\mathbb{K}}/(\pi^r)$ be a cusp eigenform of level N, weight k and character χ. Assume that its associated residual Galois representation $\bar{\rho}_f$ is absolutely irreducible. Suppose that p does not divide $\varphi(N)N(k-2)!$.*
If $k = p$ or $k = p+1$, assume that the localized Hecke algebra $\mathbb{T}_{\mathcal{M}}$ is Gorenstein, where $\mathcal{M}$ is the kernel of the reduction of f modulo π.
Let l be a prime which does not divide pN. Then the two following statements are equivalent:

(i) f satisfies the level raising condition $(\mathrm{LRC})_\epsilon$ *at the prime l for some $\epsilon \in \{\pm 1\}$*
(ii) there exists a cusp eigenform $g : \mathbb{T}_{N,l} \to \mathcal{O}_{\mathbb{K}}/(\pi^r)$ of level lN, weight k and character χ which sat

(ii.1) $f(T_q) = g(T_q)$ for all primes $q \nmid lpN$,
(ii.2) g is new at l, i.e. it factors through the quotient $\mathbb{T}_{N,l}^{l\text{-new}}$.

Remark 1.1.3. The theorem specializes to Ribet's and Diamond's results (for $p \ne 2, 3$ and assuming the Gorenstein condition for $k = p$ and $k = p+1$ for the localized Hecke algebra) when $k \ge 2$ and $r = 1$ (see Thm. 1 in [Rib90], and Thm. 1 in [Dia91]) thanks to the Deligne-Serre's lifting lemma (see Lemma 6.11, [DS74]) and Carayol's lemma (see Prop. 1.10 [Edi97]). As a consequence of Carayol's lemma, we can still recover Diamond's theorem in the case $p = 3$ and non-trivial character if some extra conditions on χ hold. Moreover, under the restriction $p \ne 2$, it specializes to Tsaknias-Wiese's result when $k = 2$ and $r \ge 1$ (see Thm. 5 in [TW17]).

Remark 1.1.4. The Gorenstein property for the localized Hecke algebra at some maximal ideal of characteristic p (odd prime) arises naturally in the context of residual modular Galois representations to study multiplicity one questions and it was extensively studied by several authors.
Being Gorenstein is a property of Noetherian local rings and as such it can be studied from a purely algebraic point of view (see [Til97]). Concerning Hecke algebras, for weight $k = 2$, it is always true that the localization at a maximal ideal (whose characteristic p does not divide the level N) is Gorenstein (see Thm. 9.2, [Edi92]).
The case of general weight $2 \leq k \leq p-1$, was settled by a theorem of Faltings and Jordan (see [FJ95]).
The cases corresponding to weights $k = p$ or $k = p+1$ are more complicated. One can still find sufficient conditions for the Gorenstein property to hold in the article of Edixhoven (see [Edi92]); but there are known counterexamples due to Kilford and Wiese (see [KW08]) and theoretical results, due to Wiese (see [Wie07]), that establish the existence of counterexamples in general. Moreover, in [KW08], a notion of Gorenstein defect is defined and studied.

Remark 1.1.5. Here we summarize the key points of the proof of Theorem 1.1.7, which will be given in details in the next section.
In order to raise the level of a cusp eigenform $f : \mathbb{T}_N \to \mathcal{O}_{\mathbb{K}}/(\pi^r)$ we first construct a $\mathbb{T}_N$-module W_f such that $\mathrm{Ann}_{\mathbb{T}}(W_f) = \mathrm{Ker}(f)$ and the action is given by $T \cdot w = f(T)w$ for all $w \in W_f$ and for all $T \in \mathbb{T}_N$. The $\mathbb{T}_N$-module structure of W_f determines uniquely the cuspidal eigenform f (this will be a consequence of lemma 1.1.10).
By a (slightly adapted) result of Diamond (see Lemma 3.2, [Dia91]) and assuming the level raising condition $(LRC)_\epsilon$ for some $\epsilon \in \{\pm 1\}$, it is possible to associate to W_f a $\mathbb{T}_{N,l}$-module V_f such that $\mathbb{T}_{N;l}/\mathrm{Ann}_{\mathbb{T}_{N,l}}(V_f)$ is isomorphic (as $\mathcal{O}_{\mathbb{K}}$-algebra) to $\mathcal{O}_{\mathbb{K}}/(\pi^r)$. The structure homomorphism of V_f as $\mathbb{T}_{N;l}$-module is given by a $\mathcal{O}_{\mathbb{K}}$-algebra homomorphism $g : \mathbb{T}_{N,l} \to \mathrm{End}_{\mathcal{O}_{\mathbb{K}}}(V_f)$ which factors via the natural projection (we keep calling it g) $g : \mathbb{T}_{N,l} \to \mathbb{T}_{N,l}/\mathrm{Ann}_{\mathbb{T}_{N,l}}(V_f)$. By construction, the $\mathcal{O}_{\mathbb{K}}$-algebra homomorphism $g : \mathbb{T}_{N,l} \to \mathcal{O}_{\mathbb{K}}/(\pi^r)$ (i.e. a cuspidal eigenform modulo π^r of level lN) will satisfy $f(T_n) = g(T_n)$ for all positive integers n coprime with lpN.
Finally, we will prove that g is new at l, i.e. it factors through $\mathbb{T}_{N,l}^{l\text{-new}}$. Indeed, assuming the level raising condition $(LRC)_\epsilon$, we prove that $U_l - \epsilon R_l$ is contained in $\mathrm{Ann}_{\mathbb{T}_{N,l}}(V_f)$ and, as a consequence, V_f is a $\mathbb{T}_{N,l}$-submodule of a non-trivial congruence module Ω, i.e. a $\mathbb{T}_{N;l}$-module whose annihilator $\mathrm{Ann}_{\mathbb{T}_{N,l}}(\Omega)$ contains both the kernels of the natural projections of the Hecke algebra $\mathbb{T}_{N,l}$ onto its l-old and l-new parts. The existence of such congruence module is granted by the work of Diamond (see [Dia91]). Hence, we have that $\mathrm{Ann}_{\mathbb{T}_{N,l}}(\Omega) \subseteq \mathrm{Ann}_{\mathbb{T}_{N,l}}(V_f)$. It follows that $\mathrm{Ker}(\mathbb{T}_{N,l} \twoheadrightarrow \mathbb{T}_{N,l}^{l\text{-new}})$ is contained in $\mathrm{Ann}_{\mathbb{T}_{N,l}}(V_f)$, and so in particular g factors through the quotient $\mathbb{T}_{N,l}^{l\text{-new}}$, i.e. it is new at l.

1.1.4 Proof of Theorem 1.1.7

The necessity of the level raising condition is exactly the content of Proposition 1.1.5. Hence, we need to prove that if f satisfies $(LRC)_\epsilon$ for some $\epsilon \in \{\pm 1\}$ then there exists a cusp eigenform $g : \mathbb{T}_{N,l} \to \mathcal{O}_{\mathbb{K}}/(\pi^r)$ such that $f(T_n) = g(T_n)$ for every n such that

$(n, lN) = 1$ and such that g factors through $\mathbb{T}_{N,l}^{l\text{-new}}$.
Denote the kernel of $f : \mathbb{T}_N \to \mathcal{O}_{\mathbb{K}}/(\pi^r)$ by I. Assume that

$$T_l - \epsilon(l+1)R_l \in I \qquad (\mathrm{LRC})_\epsilon$$

for some $\epsilon \in \{\pm 1\}$. Denote by $\bar{f}$ the reduction of f modulo π and denote by $\mathcal{M}$ its kernel. We have that $\mathcal{M}$ is a maximal ideal and $I \subseteq \mathcal{M}$.
For $k \geq 2$, and for Γ any congruence subgroup of $\mathrm{Sl}_2(\mathbb{Z})$, let $L_{k-2}(\mathcal{O}_{\mathbb{K}})$ denote the Γ-module $\mathrm{Sym}^{k-2}(\mathcal{O}_{\mathbb{K}}^2)$. The Γ-module structure of $L_{k-2}(\mathcal{O}_{\mathbb{K}})$ is induced by the symmetric power of the module structure of $\mathcal{O}_{\mathbb{K}}^2$ given by standard matrix multiplication.
The parabolic cohomology group $H^1_P(\Gamma_1(N), L_{k-2}(\mathcal{O}_{\mathbb{K}}))$ is obtained as a subgroup of the standard cohomology group $H^1(\Gamma_1(N), L_{k-2}(\mathcal{O}_{\mathbb{K}}))$ by considering the cocycles w satisfying $w(\gamma) \in (\gamma - 1)L_{k-2}(\mathcal{O}_{\mathbb{K}})$ for all parabolic elements $\gamma \in \Gamma_1(N)$, i.e. all the matrices conjugate to $\pm\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$ for some integer d (see sec. 1.4 [Shi94]). Recalling that l does not divide pN, we define:

$$W(\mathcal{O}_{\mathbb{K}}) = \mathrm{Image}(H^1_P(\Gamma_1(N), L_{k-2}(\mathcal{O}_{\mathbb{K}})) \xrightarrow{j_1} H^1_P(\Gamma_1(N), L_{k-2}(\mathbb{K})))$$

$$V(\mathcal{O}_{\mathbb{K}}) = \mathrm{Image}(H^1_P(\Gamma_1(N) \cap \Gamma_0(l), L_{k-2}(\mathcal{O}_{\mathbb{K}}))) \xrightarrow{j_2} H^1_P(\Gamma_1(N) \cap \Gamma_0(l), L_{k-2}(\mathbb{K})))$$

where the maps j_1 and j_2 are the ones induced on cohomology by the natural injection of $\mathcal{O}_{\mathbb{K}}$ in $\mathbb{K}$.
Consider the $\mathcal{O}_{\mathbb{K}}$-module $\mathbb{K}/\mathcal{O}_{\mathbb{K}}$, we define: $W(\mathbb{K}/\mathcal{O}_{\mathbb{K}}) := W(\mathcal{O}_{\mathbb{K}}) \otimes_{\mathcal{O}_{\mathbb{K}}} \mathbb{K}/\mathcal{O}_{\mathbb{K}}$.
By Hida (see Thm. 3.2, [Hid81], we recall that we are under the assumption $N \geq 5$), assuming that $p \nmid N(k-2)!$ (or equivalently, $p \nmid N$ and $k \leq p+1$) the $\mathcal{O}_{\mathbb{K}}$-module $W(\mathcal{O}_{\mathbb{K}})$ (resp. $V(\mathcal{O}_{\mathbb{K}})$) is finite, free and self-dual with respect to the perfect pairing given by the cup product.
The double coset operators of $\mathrm{Sl}_2(\mathbb{Z})$ act on the $\mathcal{O}_{\mathbb{K}}$-module $W(\mathcal{O}_{\mathbb{K}})$ and this action is compatible with the action of the double coset operators on the space of classical cuspidal eigenforms of weight k and level N (see sec. 8.3 in [Shi94]). In other words, Eichler and Shimura (see sec. 8.3, [Shi94]) proved that the $\mathcal{O}_{\mathbb{K}}$-module $W(\mathcal{O}_{\mathbb{K}})$ (resp. $V(\mathcal{O}_{\mathbb{K}})$) is invariant under the action of the full Hecke algebra acting on the space of cusp forms of weight $k \geq 2$ and level $\Gamma_1(N)$ (resp. level $\Gamma_1(N) \cap \Gamma_0(l)$). Now, let α be the map

$$\alpha : W(\mathbb{K}/\mathcal{O}_{\mathbb{K}})^{\oplus 2} \to V(\mathbb{K}/\mathcal{O}_{\mathbb{K}})$$

induced by the inclusions of $\Gamma_1(N) \cap \Gamma_0(l)$ in $\Gamma_1(N)$ via the identity and the conjugation by $\begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}$. As a generalization of Ihara's lemma (see lemma 3.2, [Iha75]), Diamond proved (see lemma 3.2, [Dia91]) the following:

Lemma 1.1.8. *(Diamond) The map α is injective.*

Diamond proved this lemma, which is a fundamental step in the proof of level raising for classical cusp eigenforms of weight $k \geq 2$, to deduce that if the level raising condition holds then there exists a non-trivial congruence module Ω in $V(\mathbb{K}/\mathcal{O}_{\mathbb{K}})$, i.e. Ω is a non-trivial $\mathbb{T}_{N,l}$-submodule of $V(\mathbb{K}/\mathcal{O}_{\mathbb{K}})$ whose action of $\mathbb{T}_{N,l}$ factors through $\mathbb{T}_{N,l}^{l\text{-new}}$ and $\mathbb{T}_{N,l}^{l\text{-old}}$.

Moreover, there is an isomorphism $\Omega \cong \mathrm{Ker}(\beta \circ \alpha)$ where β is the adjoint map of α given by the standard cup product on the cohomology groups.
Since we want to prove that the level raising congruence can be obtained preserving the Dirichlet character χ associated to the diamond operators, we will first restrict ourself to the χ-invariant submodules of the parabolic cohomology groups defined above.
Indeed, the $\mathcal{O}_{\mathbb{K}}$-module $W(\mathcal{O}_{\mathbb{K}})$ has a natural action of the group associated to diamond operators $\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^\times$ (via double coset operators, see chap. 8 in [Shi94]). For any Dirichlet character $\psi : (\mathbb{Z}/N\mathbb{Z})^\times \to (\mathcal{O}_{\mathbb{K}})^\times$, we define

$$W(\mathcal{O}_{\mathbb{K}})^{(\psi)} := \{v \in W(\mathcal{O}_{\mathbb{K}}) : h \cdot v = \psi(h)v \quad \text{for all } \ h \in (\mathbb{Z}/N\mathbb{Z})^\times\}.$$

Morever, we define $W(\mathbb{K}/\mathcal{O}_{\mathbb{K}})^{(\psi)} := W(\mathcal{O}_{\mathbb{K}})^{(\psi)} \otimes_{\mathcal{O}_{\mathbb{K}}} \mathbb{K}/\mathcal{O}_{\mathbb{K}}$. The $\mathcal{O}_{\mathbb{K}}$-submodule $W(\mathbb{K}/\mathcal{O}_{\mathbb{K}})^{(\psi)}$ (resp. $V(\mathbb{K}/\mathcal{O}_{\mathbb{K}})^{(\psi)}$) inherits a natural structure of $\mathbb{T}_N$-module (resp. $\mathbb{T}_{N,l}$-module).
The assumption that the odd prime p does not divide $\varphi(N)$ ensures us that the Hecke submodule $W(\mathcal{O}_{\mathbb{K}})^{(\psi)}$ of cocycle classes on which the group $(\mathbb{Z}/N\mathbb{Z})^\times$ acts via the Dirichlet character ψ is a direct summand of $W(\mathcal{O}_{\mathbb{K}})$. In order to see this, it is enough to observe that that there exist, in the $\mathcal{O}_{\mathbb{K}}$-algebra $\mathbb{T}_N$, the idempotent $e_\psi : W(\mathcal{O}_{\mathbb{K}}) \to W(\mathcal{O}_{\mathbb{K}})^{(\psi)}$ given by $e_\psi = \frac{1}{\varphi(N)} \sum_{g \in (\mathbb{Z}/N\mathbb{Z})^\times} \psi(g^{-1})g$.
The theory of cohomological congruence modules developed by Diamond (see [Dia91]) works with fixed character once assumed that p does not divide $\varphi(N)$.
Indeed, under the extra hypothesis $p \nmid \varphi(N)$, the restriction of α to $(W(\mathbb{K}/\mathcal{O}_{\mathbb{K}})^{(\chi)})^{\oplus 2}$ gives us an injective map

$$\alpha^{(\chi)} : (W(\mathbb{K}/\mathcal{O}_{\mathbb{K}})^{(\chi)})^{\oplus 2} \to V(\mathbb{K}/\mathcal{O}_{\mathbb{K}})^{(\chi)}.$$

As for α, we can associate to the map $\alpha^{(\chi)}$ a well defined congruence module $\Omega^{(\chi)} \cong \mathrm{Ker}(\beta^{(\chi)} \circ \alpha^{(\chi)})$, where $\beta^{(\chi)}$ is the adjoint map associated to $\alpha^{(\chi)}$ with respect to the cup product (see also sec. 3, [Dia89]).
Now, we define $W_f := W(\mathbb{K}/\mathcal{O}_{\mathbb{K}})[I]$. We recall that if R is a commutative ring, I an ideal and M is an R-module, we define $M[I] = \{m \in M : \ Im = 0\}$.
The $\mathcal{O}_{\mathbb{K}}$-module W_f has a natural structure of $\mathbb{T}_N$-module, and it is contained in $W(\mathbb{K}/\mathcal{O}_{\mathbb{K}})^{(\chi)}$. Hence, the assumption $p \nmid \varphi(N)$ allows us to restrict ourself to work with Hecke submodules of $W(\mathbb{K}/\mathcal{O}_{\mathbb{K}})$ on which the diamond operators act via the character χ.
By these considerations, from now on, we will simplify the notation by dropping the superscript for the fixed character χ; in other words, from now on $\alpha = \alpha^{(\chi)}$ and similar for $W(\mathbb{K}/\mathcal{O}_{\mathbb{K}})^{(\chi)}$, $V(\mathbb{K}/\mathcal{O}_{\mathbb{K}})^{(\chi)}$and $\Omega^{(\chi)}$. We can now continue the proof.
The chain of inclusion $(\pi^r) \subseteq I \subset \mathcal{M}$ induces:

$$W(\mathbb{K}/\mathcal{O}_{\mathbb{K}})[\pi^r] \supseteq W_f \supseteq W(\mathbb{K}/\mathcal{O}_{\mathbb{K}})[\mathcal{M}].$$

The characteristic of $\mathbb{T}_N/I$ is a power of p, in particular we have the following chain of inclusions of ideals in $\mathbb{T}_N$:

$$(\pi^r) \subseteq I \subseteq \mathcal{M}.$$

The action of $\mathbb{T}_N$ on W_f factors through the quotient $\mathbb{T}_N/I$. Because of the Gorenstein assumption, a result in commutative algebra (see Lemma 1.1.9) will ensure us that W_f is a faithful $\mathbb{T}_N/I$-module, or in other words $\mathrm{Ann}_{\mathbb{T}_N}(W_f) = I$. We will prove this in the

next section, now we will complete the proof of the main theorem.
We can finally apply Ribet's analysis in the cohomological context. We recall that we dropped the notation for the fixed character but we are always working with scalar diamond operators coming from the definition of the Hecke algebras $\mathbb{T}_N$ and $\mathbb{T}_{N,l}$ acting faithfully on the space of cusp forms of character χ and respective level N and lN.
Hence, in order to get a $\mathbb{T}_{N,l}$-module which will lead to the level raising of f, we will embed W_f in $W(\mathbb{K}/\mathcal{O}_{\mathbb{K}})^{\oplus 2}$ with a slightly modified antidiagonal embedding "$\mathrm{ad}^{\epsilon}_{k,\chi}$" which will depend on $(\mathrm{LRC})_\epsilon$ (in the sense that it will depend on ϵ) and then we will apply α.
Since R_l is invertible, we define the embedding:

$$\begin{aligned}\mathrm{ad}^{(\epsilon)}_{k,\chi} : W(\mathbb{K}/\mathcal{O}_{\mathbb{K}}) &\hookrightarrow W(\mathbb{K}/\mathcal{O}_{\mathbb{K}})^{\oplus 2}\\ x &\mapsto (x, -\epsilon l^{2-k} R_l x)\end{aligned}$$

Note that if $k = 2$ and $\chi = 1$, the embedding $\mathrm{ad}^{\epsilon}_{2,1}$ (which sends x to $(x, -\epsilon x)$) coincides with the embedding considered by Ribet (see [Rib90]) and by Tsaknias and Wiese (see [TW17]) in the context of analogous modular Jacobian varieties (we recall anyway that Ribet already observed in his article (see [Rib90]) that his argument is entirely cohomological).
Now, applying α we can define $V_f := \alpha(\mathrm{ad}^{\epsilon}_{k,\chi}(W_f)) \subseteq V(\mathbb{K}/\mathcal{O}_{\mathbb{K}})$. First, we want to show that V_f is Hecke stable, i.e. it inherits the structure of $\mathbb{T}_{N,l}$-module from the standard Hecke action of $\mathbb{T}_{N,l}$ on $V(\mathbb{K}/\mathcal{O}_{\mathbb{K}})$. Since α commutes with the action of the Hecke operators T_n with $(n, l) = 1$ and with the automorphism R_l, we have that V_f is $\mathbb{T}_{N,l}$-stable if it is stable under the action of the operator U_l.
The action of the operator U_l on $W(\mathbb{K}/\mathcal{O}_{\mathbb{K}})^{\oplus 2}$ coincides with the action of the operator U_l on the l-old space of classical cusp forms of level lN, i.e. it is given by the 2×2 matrix (see [Dia91]):

$$U_l|_{W(\mathbb{K}/\mathcal{O}_{\mathbb{K}})^{\oplus 2}} = \begin{pmatrix} T_l & l^{k-1} \\ -l^{2-k} S_l & 0 \end{pmatrix}.$$

Finally, we can check that V_f is U_l-invariant:
fix $x \in W_f$ and let $y = \alpha(ad^{(\epsilon)}_{k,\chi}(x)) = \alpha((x, -\epsilon l^{2-k} R_l x))$, we have that (since α is injective)

$$U_l(y) = U_l(\alpha(ad^{\epsilon}_{k,\chi})) = \alpha\left(\begin{pmatrix} T_l & l^{k-1} \\ -l^{2-k} S_l & 0 \end{pmatrix}\begin{pmatrix} x \\ -\epsilon l^{2-k} R_l x \end{pmatrix}\right) = \alpha\left(\begin{pmatrix} T_l x - \epsilon l R_l x \\ -l^{2-k} S_l x \end{pmatrix}\right)$$

Since $(\mathrm{LRC})_\epsilon$ holds, we have that:

$$\begin{aligned}\alpha\left(\begin{pmatrix} T_l x - \epsilon l R_l x \\ -l^{2-k} S_l x \end{pmatrix}\right) &= \alpha\left(\begin{pmatrix} \epsilon(l+1) R_l x - \epsilon l R_l x \\ -l^{2-k} {R_l}^2 \epsilon^2 x \end{pmatrix}\right)\\ &= \alpha\left(\begin{pmatrix} \epsilon R_l & 0 \\ 0 & \epsilon R_l \end{pmatrix}\begin{pmatrix} x \\ -\epsilon l^{2-k} R_l x \end{pmatrix}\right)\\ &= \epsilon R_l y.\end{aligned}$$

The operator R_l is a scalar, and so V_f is a well-defined $\mathbb{T}_{N,l}$-module.
Hence, the action of $\mathbb{T}_{N,l}$ on V_f is represented by a $\mathcal{O}_{\mathbb{K}}$-algebra homomorphism $g : \mathbb{T}_{N,l} \to \mathcal{O}_{\mathbb{K}}/(\pi^r)$. The last thing to prove is that g factors through $\mathbb{T}^{l\text{-new}}_{N,l}$. Since the raising level condition is satisfied, there exists a non-trivial congruence module Ω inside $V(\mathbb{K}/\mathcal{O}_{\mathbb{K}})$, i.e. a $\mathbb{T}_{N,l}$-module whose $\mathbb{T}_{N,l}$-action factors both through $\mathbb{T}^{l\text{-old}}_{N,l}$ and $\mathbb{T}^{l\text{-new}}_{N,l}$. By construction, we have that the action of $\mathbb{T}_{N,l}$ on V_f factors through $\mathbb{T}^{l\text{-old}}_{N,l}$, hence we will prove directly that V_f is contained in Ω, which concludes the proof. Indeed, since $\mathrm{Ker}(\mathbb{T}_{N,l} \to \mathbb{T}^{l\text{-new}}_{N,l})$ is contained in the annihilator of Ω (as $\mathbb{T}_{N,l}$ -module), if $V_f \subseteq \Omega$ then the corresponding contravariant inclusion of annihilators proves that the action factors via $\mathbb{T}^{l\text{-new}}_{N,l}$.
The above claim follows directly from the injectivity of α. In particular, it follows from lemma 1.2.1 that $\Omega = \mathrm{Ker}(\beta \circ \alpha)$ where β is the adjoint of α with respect to the standard pairing given by the cup product on $W(\mathbb{K}/\mathcal{O}_{\mathbb{K}})$ and on $V(\mathbb{K}/\mathcal{O}_{\mathbb{K}})$. Indeed, explicitly we have (see [Dia91]) that the map:

$$\beta \circ \alpha : W(\mathbb{K}/\mathcal{O}_{\mathbb{K}})^{\oplus 2} \to W(\mathbb{K}/\mathcal{O}_{\mathbb{K}})^{\oplus 2}$$

is given by the matrix

$$\begin{pmatrix} l+1 & l^{k-2}T_l S_l^{-1} \\ T_l & l^{k-2}(l+1) \end{pmatrix}$$

acting on $W(\mathbb{K}/\mathcal{O}_{\mathbb{K}})^{\oplus 2}$. So in order to show that V_f is contained in $\Omega = \mathrm{Ker}(\beta \circ \alpha)$ we will prove explicitly that if $y \in V_f$ then $(\beta \circ \alpha)(y) = 0$. Now, since Lemma 1.2.1 holds, we have an isomorphism (by definition) between $\mathrm{ad}^{\epsilon}_{k,\chi}(W_f)$ and V_f, so any $y \in V_f$ can be written as $(x, -\epsilon l^{2-k} R_l x)$ for a unique $x \in W_f$. Finally,

$$(\beta \circ \alpha)(y) = \begin{pmatrix} l+1 & l^{k-2}T_l S_l^{-1} \\ T_l & l^{k-2}(l+1) \end{pmatrix} \begin{pmatrix} x \\ -\epsilon l^{2-k} R_l x \end{pmatrix} = \begin{pmatrix} ((l+1) - \epsilon {R_l}^{-1} T_l)x \\ (T_l - \epsilon(l+1)R_l)x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where the last step holds because R_l is an automorphism and the raising level condition holds, i.e. $T_l - \epsilon(l+1)R_l \in \mathrm{Ann}_{\mathbb{T}}(W_f) = I = \mathrm{Ker}(f)$. This implies that g factors through $\mathbb{T}^{l\text{-new}}_{N,l}$ and the proof is complete.

Remark 1.1.6. Diamond (see Thm. 2 in [Dia91]) proved that if f is a classical newform of level $N \geq 5$, weight $k \geq 2$ and character χ, which satisfies the level raising condition $(\mathrm{LRC})^2$ modulo π^r at a prime l (as usual, not dividing pN) then there exist a family $\{g_i\}_{i=1}^{M}$, for some positive integer M, of l-newforms of level lN such that $f \equiv g_i \bmod \pi^{d_i}$ and $\sum_{i=1}^{M} d_i \geq r$. Using Theorem 1.1.7 it is possible to deduce more information about the coefficients d_i whose existence was predicted by Diamond. Indeed, for every i, the congruence $f \equiv g_i \bmod \pi^{d_i}$ can be seen as a level raising congruence of cuspidal eigenforms modulo π^{d_i}, and as such we deduce that $(\mathrm{LRC})_{\epsilon}$ has to be satisfied modulo π^{d_i} for some $\epsilon \in \{\pm 1\}$. Then we have an upper bound for the coefficients d_i:

$$\max_i\{d_i\} \leq \max\{v_\pi(a_l(f) - (l+1)l^{\frac{k-2}{2}}\zeta);\ v_\pi(a_l(f) + (l+1)l^{\frac{k-2}{2}}\zeta)\}$$

where, as before, $\zeta \in \mathcal{O}_{\mathbb{K}}$ satisfies $\zeta^2 = \chi(l)$.

1.1.5 A result in commutative algebra

Let p be an odd prime and let $\mathbb{K}$ be a finite extension of $\mathbb{Q}_p$. Let H be a finitely generated, free $\mathcal{O}_{\mathbb{K}}$-module, and let $R \subseteq \mathrm{End}_{\mathcal{O}_{\mathbb{K}}}(H)$ be a commutative $\mathcal{O}_{\mathbb{K}}$-subalgebra.
Consider the $\mathcal{O}_{\mathbb{K}}$-module $W := H \otimes_{\mathcal{O}_{\mathbb{K}}} \mathbb{K}/\mathcal{O}_{\mathbb{K}}$. The ring R naturally acts on W via the left action on H, hence W has a natural structure of R-module. Assume that H is a (finite) free R-module of rank s. The following result is probably well known but we were not able to find it in the literature so we will prove it.

Lemma 1.1.9. *Let h be an element of the set $Hom_{\mathcal{O}_{\mathbb{K}}\text{-}Alg}(R, \mathcal{O}_{\mathbb{K}}/(\pi^r))$ an let $I = Ker(h)$ and $\mathcal{M}$ be the unique maximal ideal containing I. Assume that the localization $R_{\mathcal{M}}$ is a Gorenstein ring, i.e. $Hom_{\mathcal{O}_{\mathbb{K}}}(R_{\mathcal{M}}, \mathcal{O}_{\mathbb{K}})$ is a free $R_{\mathcal{M}}$-module of rank 1.*
Then $W[I] = \{w \in W : Iw = 0\}$ is a faithful R/I-module, or equivalently $Ann_R(W[I]) = I$.

Remark 1.1.7. We can apply the above lemma in the context of cuspidal eigenforms modulo prime powers. Namely H is the parabolic cohomology group that we denoted by $W(\mathcal{O}_{\mathbb{K}})$, R is the Hecke algebra $\mathbb{T}_N$ and $h = f$, which gives us the faithfulness of $W[I] = W_f$ as R/I-module. A study of W in weight 2 (in terms of Jacobians attached to modular curves) can be found in an article of Tilouine (see Thm. 3.4 and cor. (1), [Til97]).

Proof. Since R is commutative, the inclusions of ideals $(\pi^r) \subseteq I \subset \mathcal{M}$ induce a chain of inclusions of R-modules:

$$W[\mathcal{M}] \subseteq W[I] \subseteq W[\pi^r].$$

Let $\mathrm{Ta}_\pi(W) = \varprojlim W[\pi^n]$ be the π-adic Tate module associated to W. Since $W[\pi^r]$ and $H/\pi^r H$ are isomorphic as R-modules and the action of R on both modules is compatible with the transition maps of the projective sytems, we have a canonical isomorphism of R-modules $\mathrm{Ta}_\pi(W) \cong H$. We define $\mathcal{Z}$ as the localization of $\mathrm{Ta}_\pi(W)$ at $\mathcal{M}$.
We have that $\mathcal{Z}$ is a finitely generated, free $R_{\mathcal{M}}$-module of rank s. The localized ring $R_{\mathcal{M}}$ is Gorenstein, or equivalently there is an isomorphism of $R_{\mathcal{M}}$-modules $R_{\mathcal{M}} \cong \mathrm{Hom}_{\mathcal{O}_{\mathbb{K}}}(R_{\mathcal{M}}, \mathcal{O}_{\mathbb{K}})$.
As usual, let $W[\pi^r]_{\mathcal{M}}$ be the $R_{\mathcal{M}}$-module obtained by localization of $W[\pi^r]$ at $\mathcal{M}$. We have a chain of isomorphisms of $R_{\mathcal{M}}$-modules:

$$W[\pi^r]_{\mathcal{M}} \cong \mathcal{Z}/(\pi^r)\mathcal{Z} \cong \left(R_{\mathcal{M}}/(\pi^r)R_{\mathcal{M}}\right)^{\oplus s} \cong \mathrm{Hom}_{\mathcal{O}_{\mathbb{K}}}\left(R_{\mathcal{M}}/(\pi^r)R_{\mathcal{M}}, \mathcal{O}_{\mathbb{K}}/(\pi^r)\right)^{\oplus s},$$

where the last isomorphism of $R_{\mathcal{M}}$-modules holds because of the extension of scalars property for homomorphism modules (see Prop. 10, chpt. 1, [Bou89]). The inclusion of ideals $(\pi^r) \subseteq I$ and the exactness of the localization functor induce the isomorphism of $R_{\mathcal{M}}$-modules:

$$W[I]_{\mathcal{M}} \cong \mathrm{Hom}_{\mathcal{O}_{\mathbb{K}}}\left(R_{\mathcal{M}}/(\pi^r)R_{\mathcal{M}}, \mathcal{O}_{\mathbb{K}}/(\pi^r)\right)[I]^{\oplus s}.$$

Moreover, the action of $R_{\mathcal{M}}$ on $W[I]_{\mathcal{M}}$ factors through the quotient $R_{\mathcal{M}}/IR_{\mathcal{M}}$, and so the above isomorphism is an isomorphism of $R_{\mathcal{M}}/IR_{\mathcal{M}}$-modules.

We observe that, since $\pi^r \in I$, there exists an ideal $\bar{I}$ of $R/(\pi^r)R$ such that:

$$ {}^{R/(\pi^r)}\!\big/_{\bar{I}} \cong {}^{R}\!\big/_{I}. $$

The $\mathcal{O}_{\mathbb{K}}$-algebra homomorphism h is surjective and it factors through the quotient by its kernel, so R/I is isomorphic (as a ring) to $\mathcal{O}_{\mathbb{K}}/(\pi^r)$ which is a local ring, hence R/I is a local ring as well. By the exactness of the localization functor we have a ring isomorphism $R_{\mathcal{M}}/IR_{\mathcal{M}} \cong R/I$.
It follows that we have an isomorphism of R/I-modules:

$$ \begin{aligned} W[I] \cong W[I]_{\mathcal{M}} &\cong \operatorname{Hom}_{\mathcal{O}_{\mathbb{K}}}\left(\left({}^{R_{\mathcal{M}}}\!\big/_{(\pi^r)R_{\mathcal{M}}} \right)\Big/_{\bar{I}R_{\mathcal{M}}}, {}^{\mathcal{O}_{\mathbb{K}}}\!\big/_{(\pi^r)} \right)^{\oplus s} \\ &\cong \operatorname{Hom}_{\mathcal{O}_{\mathbb{K}}}\left({}^{R_{\mathcal{M}}}\!\big/_{IR_{\mathcal{M}}}, {}^{\mathcal{O}_{\mathbb{K}}}\!\big/_{(\pi^r)} \right)^{\oplus s} \\ &\cong \operatorname{Hom}_{\mathcal{O}_{\mathbb{K}}}\left({}^{R}\!\big/_{I}, {}^{\mathcal{O}_{\mathbb{K}}}\!\big/_{(\pi^r)} \right)^{\oplus s}, \end{aligned} $$

so $W[I]$ is a faithful R/I-module and the proof is complete.

□

Now, we want to identify a cuspidal eigenform modulo π^r of generic level N (not divisible by p) with a unique class of Hecke submodules of $W[\pi^r]$. In order to do so we will make use of lemma 1.1.7. We will again state the result in the context of commutative algebra. We keep the same setting as the previous lemma. If J is an ideal of R which is contained in a unique maximal ideal, we will denote such maximal ideal as $\mathcal{M}_J$. We define the set:

$$ \mathfrak{B} := \left\{ M \subseteq W[\pi^r] \ \ R\text{-submodule} : \ {}^{R}\!\big/_{\operatorname{Ann}_R(M)} \cong {}^{\mathcal{O}_{\mathbb{K}}}\!\big/_{(\pi^r)};\ R_{\mathcal{M}_{\operatorname{Ann}_R(M)}} \text{ is Gorenstein} \right\}. $$

Let $\sim$ be the equivalence relation on $\mathfrak{B}$ given by: $M \sim N$ if and only if $\operatorname{Ann}_R(N) = \operatorname{Ann}_R(M)$. We have the following lemma:

Lemma 1.1.10. *Define the map between sets as follows:*

$$ \begin{aligned} \varphi : \mathfrak{A} := \left\{ h \in Hom_{\mathcal{O}_{\mathbb{K}}\text{-}Alg}\left(R, {}^{\mathcal{O}_{\mathbb{K}}}\!\big/_{(\pi^r)} \right) : R_{\mathcal{M}_{Ker(h)}} \textit{ is Gorenstein} \right\} &\longrightarrow {}^{\mathfrak{B}}\!\big/_{\sim} \\ h &\longmapsto (W[Ker(h)])_{\sim} \end{aligned} $$

where the symbol $(\cdot)_{\sim}$ denotes the class under the equivalence $\sim$.
Then φ is a bijection.

Proof. First we prove the injectivity of φ. Take h_1 and h_2 in $\mathfrak{A}$ and assume that $\varphi(h_1) = \varphi(h_2)$. Then by definition we have that $(W[\operatorname{Ker}(h_1)])_{\sim} = (W[\operatorname{Ker}(h_2)])_{\sim}$ and so $\operatorname{Ann}_R(W[\operatorname{Ker}(h_1)]) = \operatorname{Ann}_R(W[\operatorname{Ker}(h_2)])$. By lemma 1.1.9, we deduce that $\operatorname{Ker}(h_1) = \operatorname{Ker}(h_2)$ and hence the injectivity follows.
For the surjectivity, let $(M)_{\sim} \in \mathfrak{B}$ and denote by J_M the annihilator $\operatorname{Ann}_R(M)$. Then the natural projection $h_M : R \to R/J_M$ satisfies $h_M \in \mathfrak{A}$ and $\varphi(h_M) = (W[J_M])_{\sim}$. Again by lemma 1.1.9, $\operatorname{Ann}_R(W[J_M]) = J_M$ and so $(W[J_M])_{\sim} = (M)_{\sim}$. This completes the proof. □

Remark 1.1.8. The above lemma formalizes the idea behind the proof of the main result of the article. Applying it to $H = W(\mathcal{O}_{\mathbb{K}})$, and $R = \mathbb{T}_N$ gives us the bijective correspondence between cuspidal eigenforms modulo prime powers of level N and certain classes of Hecke modules related to the first cohomology group involved in the classical Eichler-Shimura isomorphism.

1.2 Examples

All the following computations were made using MAGMA (see [BCP97]) and the LMFDB database (see [LMFDB]).
We will keep the same notation as before, i.e. $k \geq 2$ and $N \geq 5$ are integers, p is an odd prime and $\mathbb{K}$ is a sufficiently big finite extension of $\mathbb{Q}_p$.
In this section we will present examples related to level raising, underlining the difference between full level raising and partial level raising and connecting Theorem 1.1.7 and Diamond's Theorem 1.2.1, which we now state completely:

Theorem 1.2.1. *Let f be a newform of level N, weight k, character χ and coefficients in $\mathbb{K}$ satisfying the level raising condition $(LRC)^2$ at a prime l modulo π^r, i.e. $a_l(f)^2 - (l+1)^2 l^{k-2}\chi(l) \equiv 0 \ mod\ \pi^r$. Assume that $p \nmid lN\varphi(N)(k-2)!$.*
Then there exists a finite family of positive integers d_i and distinct l-newforms g_i of level lN (i.e. with respect to the congruence subgroup $\Gamma_1(N) \cap \Gamma_0(l)$) such that:

$$(i)\ \ f \equiv g_i \ mod\ \pi^{d_i}$$
$$(ii)\ \ \sum_i d_i \geq r$$

Let f be a cuspidal eigenform satisfying the hypothesis of the above theorem. We will say that f satisfies the full level raising condition modulo π^r at a prime l if, applying Theorem 1.2.1, we have that there exists an index i and a l-newform g_i such that $f \equiv g_i \bmod \pi^r$, i.e. $d_i = r$.

1.2.1 Full level raising modulo prime powers

Consider the complex vector space $S_4(\Gamma_0(22))^{\text{new}}$, it has dimension 3. Let f be the newform whose q-expansion is $f(q) = q - 2q^2 - 7q^3 + 4q^4 - 19q^5 + 14q^6 \ldots$, it has $\mathbb{Q}$ as coefficient field.
For the primes $l = 5$ and $p = 7$, the form f satisfies the following level raising conditions:

$$\begin{aligned}
(\text{LRC})^2 :&\ \ a_5(f)^2 - (5+1)^2 5^2 \equiv 0 \bmod 7^2,\\
(\text{LRC})_+ :&\ \ a_5(f) - (5+1)5 \equiv 0 \bmod 7^2,\\
(\text{LRC})_- :&\ \ a_5(f) + (5+1)5 \not\equiv 0 \bmod 7,
\end{aligned}$$

indeed note that, by Lemma 1.1.6, these conditions are equivalent since p does not divide $l+1$.
In accordance with Diamond's result, we find a newform $g \in S_4(\Gamma_0(110))^{\text{new}}$ such that f

is congruent to g modulo 7^2, i.e. $a_q(f) \equiv a_q(g) \bmod 7^2$ for all primes $q \neq 2, 5, 7, 11$.
Now, denote by $\mathbb{T}_{110}$ the Hecke algebra of level $\Gamma_0(110)$. As predicted by our result (see Thm.1.1.7) there is a cuspidal eigenform modulo 7^2 of level 110, say $h : \mathbb{T}_{110} \to \mathbb{Z}_7/7^2\mathbb{Z}_7$, such that h as $\mathbb{Z}_7$-algebra homomorphism factors through the $\mathbb{Z}_7$-algebra $\mathbb{T}_{110}^{7\text{-new}}$ (because in particular, it factors through $\mathbb{T}_{110}^{\text{new}}$). As ring homomorphism, the eigenform h lifts in characteristic 0, in particular it is nothing else than the reduction of the classical newform g modulo 7^2.
Here is a list of other examples of full level raising, everything is in accordance with Theorem 1.1.7 and Theorem 1.2.1:

(1) Let $f(q) = q - 2q^2 - 5q^3 - 4q^4 + \ldots$ be the unique newform in $S_4(\Gamma_0(23))^{\text{new}}$ with rational coefficients. Then f satisfies the level raising condition $(\text{LRC})^2$ and $(\text{LRC})_-$ modulo 5^3 at $l = 13$.

(2) Let $f(q) = q - 5q^2 - 7q^3 + 17q^4 - 7q^5 + \ldots$ be the unique newform in $S_4(\Gamma_0(13))^{\text{new}}$ with rational coefficients. Let $l = 23$ and $p = 3$, then f satisfies at l the conditions $(\text{LRC})^2$ modulo 3^5, $(\text{LRC})_-$ modulo 3 and $(\text{LRC})_+$ modulo 3^4. Note that the conditions are not equivalent since $v_p(l + 1) = 1$. It is an example of full level raising not with respect of the condition $(\text{LRC})^2$, but with respect to $(\text{LRC})_+$. There is indeed a congruence modulo 3^4 between f and a 23-newform of level 299. In particular, this example shows why the conditions $(\text{LRC})_\epsilon$ for $\epsilon \in \{\pm 1\}$ are the level raising conditions that need to be considered in the prime powers case.

1.2.2 Partial level raising modulo prime powers

As before, consider the 3-dimensional complex vector space $S_4(\Gamma_0(22))^{\text{new}}$. Let $f \in S_4(\Gamma_0(22))^{\text{new}}$ be the newform whose q-expansion is $f(q) = q + 2q^2 + q^3 + 4q^4 - 3q^5 + 2q^6 \ldots$, it has $\mathbb{Q}$ as coefficient field.
It turns out that for the primes $l = 5$ and $p = 3$ the form f satisfies the following non-equivalent level raising conditions:

$$\begin{aligned}
(\text{LRC})^2 :&\quad a_5(f)^2 - (5+1)^2 5^2 \equiv 0 \bmod 3^4, \\
(\text{LRC})_+ :&\quad a_5(f) - (5+1)5 \equiv 0 \bmod 3, \\
(\text{LRC})_- :&\quad a_5(f) + (5+1)5 \equiv 0 \bmod 3^3.
\end{aligned}$$

In accordance with Diamond's theorem (see Thm.2 in [Dia91]), we find that there exists a family of 5-newforms $\{g_i : i = 1, 2, 3, 4, 5\}$ such that $g_i \equiv f \bmod 3$ if $i \neq 5$ and $g_5 \equiv f \bmod 3^2$. More precisely, g_1, g_2, g_3, g_4 are newforms of level 110 and g_5 is a newform of level 10:

$$
\begin{aligned}
g_1(q) &= q + 2q^2 + q^3 + 4q^4 + 5q^5 + 2q^6 + 23q^7 + 8q^8 \ldots, & g_1 &\equiv f \bmod 3 \\
g_2(q) &= q + 2q^2 + 7q^3 + 4q^4 - 5q^5 + 14q^6 + 1q^7 + 8q^8 \ldots, & g_2 &\equiv f \bmod 3 \\
g_3(q) &= q + 2q^2 - 8q^3 + 4q^4 - 5q^5 - 16q^6 + 26q^7 + 8q^8 \ldots, & g_3 &\equiv f \bmod 3^2 \\
g_4(q) &= q - 2q^2 + 4q^3 + 4q^4 + 5q^5 - 8q^6 + 20q^7 - 8q^8 \ldots, & g_4 &\equiv f \bmod 3 \\
g_5(q) &= q + 2q^2 - 8q^3 + 4q^4 + 5q^5 - 16q^6 - 4q^7 + 8q^8 \ldots, & g_5 &\equiv f \bmod 3
\end{aligned}
$$

This gives an example of a strict inequality in the relation $\sum_i d_i \geq r$ where in this case $r = 4$ and $d_i = 1$ for $i \neq 5$ and $d_5 = 2$. Note that there are no congruences modulo 3^4 as predicted.

Concerning the more general case of cuspidal eigenforms modulo prime powers, everything behaves according to our theorem 1.1.7. In particular, the level raising condition $(\mathrm{LRC})_-$, which holds modulo 3, gives rise to a cuspidal eigenform modulo 3 which is 5-new, we call it $h_1 : \mathbb{T}_{110} \to \mathbb{Z}_3/3\mathbb{Z}_3$ such that $f \equiv h_1 \bmod 3$ and the level raising condition $(\mathrm{LRC})_+$, which holds modulo 3^3, gives rise to a cuspidal eigenform modulo 3^3 which is 5-new, say $h_2 : \mathbb{T}_{110} \to \mathbb{Z}_3/3^3\mathbb{Z}_3$ such that $f \equiv h_2 \bmod 3^3$.

Note that, as predicted by the Deligne-Serre's lifting lemma, the h_1 lifts in characteristic 0 as $\mathbb{Z}_3$-algebra homomorphism while h_2 does not since there are no congruence modulo 3^3 between f and classical 5-newforms of level 110. Hence, this gives a counterexample for the full level raising even when the level raising condition $(\mathrm{LRC})^2$ is replaced by the condition $(\mathrm{LRC})_\epsilon$ for some $\epsilon \in \{\pm 1\}$. Nevertheless, it is interesting to observe that in this case the cuspidal eigenform modulo 3^3 that we called h_2 and which does not lift in characteristic 0 (as a ring homomorphism) can be recovered by a linear combination of the classical 5-newforms of Diamond's theorem. More specifically, we have that $h_2 \equiv f + 3g_2 + 5g_3 + 18g_4 \bmod 3^3$. The cusp form $f + 3g_2 + 5g_3 + 18g_4$ is not an eigenform in characteristic zero but it becomes an eigenform when reduced modulo 3^3.

Here is another example of partial level raising, everything is in accordance with Theorem 1.1.7 and Theorem 1.2.1:

Let $f(q) = q + 2q^2 + 4q^4 - 6q^5 - 16q^7 + \ldots$ be the unique newform in $S_4(\Gamma_0(18))^{\mathrm{new}}$. Then f satisfies the following non-equivalent level raising conditions for $p = 5$ at $l = 29$:

$$
\begin{aligned}
(\mathrm{LRC})^2 :&\quad a_{29}(f)^2 - (29+1)^2 29^2 \equiv 0 \bmod 5^3, \\
(\mathrm{LRC})_+ :&\quad a_{29}(f) - (29+1)29 \equiv 0 \bmod 5^2, \\
(\mathrm{LRC})_- :&\quad a_{29}(f) + (29+1)29 \equiv 0 \bmod 5.
\end{aligned}
$$

It is an example of partial level raising because there are no congruences modulo 5^2 between f and 29-newforms of level 522.

Chapter 2

On reductions of crystalline representations modulo prime powers

2.1 Wach modules and crystalline representations

Let p be an odd prime and let $\mathbb{E} \subseteq \overline{\mathbb{Q}}_p$ be a finite extension of $\mathbb{Q}_p$. We denote by $\mathcal{O}_\mathbb{E}$ the ring of integers of $\mathbb{E}$, by $\pi_\mathbb{E}$ a uniformizer, by $k_\mathbb{E}$ the residue field and denote by e the ramification index of $\mathbb{E}$ over $\mathbb{Q}_p$. Let Γ be a group isomorphic to $\mathbb{Z}_p^\times$ via a map $\chi : \Gamma \to \mathbb{Z}_p^\times$. Fix once and for all a topological generator of Γ (which is procyclic as $p \neq 2$), say γ. For the sake of completeness, we will briefly recall the construction of some of the rings of Fontaine necessary for introducing the (φ, Γ)-modules and the theorem characterizing their relation with certain Galois representations of $G_{\mathbb{Q}_p}$.
Let $\{\epsilon^{(n)}\}_{n\geq 1} \subset \bar{\mathbb{Q}}_p$ be a system of roots of unity such that:

$$
\begin{aligned}
&(1)\ \ \epsilon^{(1)} \neq 1,\\
&(2)\ \ \epsilon^{(n)} \in \mu_{p^n} \subset \overline{\mathbb{Q}}_p,\\
&(3)\ \ (\epsilon^{(n+1)})^p = \epsilon^{(n)}.
\end{aligned}
$$

One can think of $\epsilon := (\epsilon^{(1)}, \epsilon^{(2)}, \dots)$ as an element of Fontaine's ring $\mathcal{E} = \varprojlim \mathbb{C}_p$ (with projective limit maps given by the Frobenius maps $z \mapsto z^p$) where $\mathbb{C}_p$ is the p-adic completion of $\overline{\mathbb{Q}}_p$. It is well known that $\mathcal{E}$ is an algebraically closed field of characteristic p. Now, consider the fields $\mathbb{Q}_p^{(n)} := \mathbb{Q}_p(\epsilon^{(n)})$ and define $\mathbb{Q}_p^{(\infty)} = \cup_{n\geq 1}\mathbb{Q}_p^{(n)}$. Denote by $H_{\mathbb{Q}_p}$ the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^{(\infty)})$.
The p-adic cyclotomic character ω gives the exact sequence:

$$1 \longrightarrow H_{\mathbb{Q}_p} \longrightarrow G_{\mathbb{Q}_p} \xrightarrow{\omega} \Gamma \cong \mathbb{Z}_p^\times \longrightarrow 1.$$

Consider the field $\mathbb{F} = \mathbb{F}_p((\epsilon - 1))$ inside $\mathcal{E}$. Let $\mathcal{A}_{\mathbb{Q}_p}$ be the p-adic completion of $\mathbb{Z}_p[[x]][\frac{1}{x}]$; it is a complete discrete valuation ring whose residue field can be identified with $\mathbb{F}$ (one can identify x with a suitable Teichmuller lift of $\epsilon - 1$). Let $\mathcal{A}$ be the p-adic completion of the strict henselization $\mathcal{A}_{\mathbb{Q}_p}^{\mathrm{sh}}$ of $\mathcal{A}_{\mathbb{Q}_p}$ inside $\tilde{A} := W(\mathcal{E})$. Note that $\mathcal{A}_{\mathbb{Q}_p}^{\mathrm{sh}}$ can be identified with the ring of integers of the maximal unramified extension of the field $\mathcal{A}_{\mathbb{Q}_p}[\frac{1}{p}]$ inside

$\tilde{\mathcal{A}}[\frac{1}{p}]$.
The Galois group $G_{\mathbb{Q}_p}$ acts on $\mathcal{E}$ by acting on $\mathbb{C}_p$ and by functoriality on the projective limit. By functoriality of the Witt vectors, the group $G_{\mathbb{Q}_p}$ also acts on $\tilde{\mathcal{A}} = W(\mathcal{E})$ and we have that $\mathcal{A}$ is $G_{\mathbb{Q}_p}$-stable. It is also true that $\mathcal{A}^{H_{\mathbb{Q}_p}} = \mathcal{A}_{\mathbb{Q}_p}$.
Now, we define $\mathcal{A}_{\mathbb{E}}$ as $\mathcal{A}_{\mathbb{Q}_p} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathbb{E}}$, by a result of Dee (see prop. 2.2.2 in [Dee01]) we have that $\mathcal{A}_{\mathbb{E}}$ is isomorphic to the $\pi_{\mathbb{E}}$-adic completion of $\mathcal{O}_{\mathbb{E}}[[x]][\frac{1}{x}]$. One can think of $\mathcal{A}_{\mathbb{E}}$ inside the ring $\mathcal{A}^{(\mathcal{O}_{\mathbb{E}})} := \mathcal{A} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathbb{E}}$. The ring $\mathcal{A}_{\mathbb{E}}$ inherits a $G_{\mathbb{Q}_p}$-action via the natural action of $G_{\mathbb{Q}_p}$ on $\mathcal{A}$ and trivial action on $\mathcal{O}_{\mathbb{E}}$. It follows that $(\mathcal{A}^{(\mathcal{O}_{\mathbb{E}})})^{H_{\mathbb{Q}_p}} = \mathcal{A}_{\mathbb{E}}$ and so $\mathcal{A}_{\mathbb{E}}$ has a structure of Γ-module.
Hence, the ring $\mathcal{A}_{\mathbb{E}}$ has a natural $\mathcal{O}_{\mathbb{E}}$-linear action of Γ and a $\mathcal{O}_{\mathbb{E}}$-linear Frobenius endomorphism φ given by the following expressions:

$$\varphi(f(x)) = f((1+x)^p - 1) \qquad \text{for all } f(x) \in \mathcal{A}_{\mathbb{E}},$$
$$\eta(f(x)) = f((1+x)^{\chi(\eta)} - 1) \quad \text{for all } f(x) \in \mathcal{A}_{\mathbb{E}}, \text{ for all } \eta \in \Gamma.$$

Finally, we can recall the following:

Definition 2.1.1. *An étale (φ, Γ)-module D over $\mathcal{O}_{\mathbb{E}}$ is an $\mathcal{A}_{\mathbb{E}}$-module of finite type endowed with a semilinear Froebenius map φ such that $\varphi(D)$ generates D as $\mathcal{A}_{\mathbb{E}}$-module (this is the étale property) and a semilinear continuous action of Γ which commutes with φ.*
The category of étale (φ, Γ)-module over $\mathcal{O}_{\mathbb{E}}$ will be denoted by $\mathbf{Mod}^{ét}_{(\varphi,\Gamma)}(\mathcal{O}_{\mathbb{E}})$.

Denote by $\mathbf{Rep}_{\mathcal{O}_{\mathbb{E}}}(G_{\mathbb{Q}_p})$ the category of $\mathcal{O}_{\mathbb{E}}$-linear representations of $G_{\mathbb{Q}_p}$, i.e. the category of $\mathcal{O}_{\mathbb{E}}$-modules of finite type with a continuous $\mathcal{O}_{\mathbb{E}}$-linear action of $G_{\mathbb{Q}_p}$.
By a theorem of Fontaine (see A.3.4 in [Fon90]) and its generalization by Dee (see 2.2 in [Dee01]) we have the following:

Theorem 2.1.1. *There exists a natural isomorphism*

$$\mathfrak{D} : \mathbf{Rep}_{\mathcal{O}_{\mathbb{E}}}(G_{\mathbb{Q}_p}) \to \mathbf{Mod}^{ét}_{(\varphi,\Gamma)}(\mathcal{O}_{\mathbb{E}})$$

given by $\mathfrak{D}(T) = (\mathcal{A}^{(\mathcal{O}_{\mathbb{E}})} \otimes_{\mathcal{O}_{\mathbb{E}}} T)^{H_{\mathbb{Q}_p}}$, where $\mathcal{A}^{(\mathcal{O}_{\mathbb{E}})} := \mathcal{A} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathbb{E}}$.
A quasi-inverse functor, which is a natural isomorphism as well, is given by:

$$\mathfrak{T} : \mathbf{Mod}^{ét}_{(\varphi,\Gamma)}(\mathcal{O}_{\mathbb{E}}) \to \mathbf{Rep}_{\mathcal{O}_{\mathbb{E}}}(G_{\mathbb{Q}_p})$$

given by $\mathfrak{T}(D) = (\mathcal{A}^{(\mathcal{O}_{\mathbb{E}})} \otimes_{\mathcal{A}_{\mathbb{E}}} D)^{\varphi=1}$.

Remark 2.1.1. Note that the equivalence of categories given by the above theorem preserves the objects killed by a fixed power of a chosen uniformizer. This essentially follows from the exactness of the functor $\mathfrak{D}$ (see prop. 2.1.9 in [Dee01]) and so $(\pi_{\mathbb{E}}^n) \cdot \mathfrak{D}(T) = \mathfrak{D}((\pi_{\mathbb{E}}^n) \cdot T)$ in the category $\mathbf{Mod}^{ét}_{(\varphi,\Gamma)}(\mathcal{O}_{\mathbb{E}})$. Same goes for the quasi-inverse functor $\mathfrak{T}$ (see prop. 2.1.24 in [Dee01]).

Remark 2.1.2. Let $\mathbf{Mod}^{ét,tors}_{(\varphi,\Gamma)}(\mathcal{O}_{\mathbb{E}})$ and $\mathbf{Mod}^{ét,free}_{(\varphi,\Gamma)}(\mathcal{O}_{\mathbb{E}})$ be respectively the categories of torsion and free (as $\mathcal{A}_{\mathbb{E}}$-modules) étale (φ, Γ)-modules. There is a notion of Tate dual for such étale (φ,Γ)-modules. Let $\mathcal{B}_{\mathbb{E}}$ be $\mathcal{A}_{\mathbb{E}}[\frac{1}{p}]$ and define the Tate dual as follows (see sec. I.2 in [Col10]):

$$\text{if } D \in \mathbf{Mod}^{\text{ét,tors}}_{(\varphi,\Gamma)}(\mathcal{O}_{\mathbb{E}}) \text{ then } D^* := \mathrm{Hom}_{\mathcal{A}_{\mathbb{E}}}\Big(D, \mathcal{B}_{\mathbb{E}}/\mathcal{A}_{\mathbb{E}}\frac{dx}{1+x}\Big)$$
$$\text{if } D \in \mathbf{Mod}^{\text{ét,free}}_{(\varphi,\Gamma)}(\mathcal{O}_{\mathbb{E}}) \text{ then } D^* := \mathrm{Hom}_{\mathcal{A}_{\mathbb{E}}}\Big(D, \mathcal{A}_{\mathbb{E}}\frac{dx}{1+x}\Big)$$

where $\mathcal{B}_{\mathbb{E}}/\mathcal{A}_{\mathbb{E}}\frac{dx}{1+x}$ is the inductive limit of $p^{-n}\mathcal{A}_{\mathbb{E}}/\mathcal{A}_{\mathbb{E}}\frac{dx}{1+x}$ in the category $\mathbf{Mod}^{\text{ét,tors}}_{(\varphi,\Gamma)}(\mathcal{O}_{\mathbb{E}})$ and the structure of (φ,Γ)-module is given by:

$$\gamma\Big(\frac{dx}{1+x}\Big) = \chi(\gamma)\frac{dx}{1+x} \text{ and } \varphi\Big(\frac{dx}{1+x}\Big) = \frac{dx}{1+x}.$$

Note the important fact that these two notions of dual are compatible in the following sense: if $D \in \mathbf{Mod}^{\text{ét,free}}_{(\varphi,\Gamma)}(\mathcal{O}_{\mathbb{E}})$ then for all $n \in \frac{1}{e}(\mathbb{Z}_{\geq 1})$ we have that $D^*/p^nD^* \cong (D/p^nD)^*$ (see prop I.2.5 in [Col10]).

The étale (φ,Γ)-modules corresponding to crystalline representations can be described more explicitly via the theory of Wach modules developed by Berger and Wach. We will briefly recall the definition of Wach modules and their main properties. First, we define $\mathcal{A}^+_{\mathbb{E}} = \mathcal{O}_{\mathbb{E}}[[x]]$ inside $\mathcal{A}_{\mathbb{E}}$. It inherits naturally the actions of φ and Γ by restriction from $\mathcal{A}_{\mathbb{E}}$. Note also that $\mathcal{A}_{\mathbb{E}}$ is obtained by taking $\pi_{\mathbb{E}}$-adic completion of the localization $\mathcal{O}_{\mathbb{E}}[[x]][1/x]$ of $\mathcal{A}^+_{\mathbb{E}} = \mathcal{O}_{\mathbb{E}}[[x]]$ at the multiplicative set $\{1, x, x^2, \dots\}$. Moreover, since $\mathcal{A}_{\mathbb{E}}$ is Noetherian, we have that $\mathcal{A}_{\mathbb{E}}$ is flat as $\mathcal{A}^+_{\mathbb{E}}$-module since localization and completion preserve such property. Following Berger (see [Ber12]), we have the following

Definition 2.1.2. *A Wach module of height $h \geq 1$ is a free $\mathcal{A}^+_{\mathbb{E}}$-module of finite rank endowed with commutative $\mathcal{A}^+_{\mathbb{E}}$-semilinear actions of a Frobenius map φ and of the group Γ such that:*

(1) $D(N) := \mathcal{A}_{\mathbb{E}} \otimes_{\mathcal{A}^+_{\mathbb{E}}} N \in \boldsymbol{Mod}^{\acute{e}t}_{(\varphi,\Gamma)}(\mathcal{O}_{\mathbb{E}})$,
(2) *Γ acts trivially on N/xN,*
(3) *$N/\varphi^*(N)$ is killed by Q^h,*

where $\varphi^*(N)$ denotes the $\mathcal{A}^+_{\mathbb{E}}$-module generated by $\varphi(N)$, and $Q = \frac{(1+x)^p-1}{x} \in \mathcal{A}^+_{\mathbb{E}}$.

We recall that the Wach modules are the right linear algebra objects to specialize Fontaine's equivalence to crystalline representations. Indeed, we have the following (see Prop. 1.1 in [Ber12]):

Proposition 2.1.2. *Let N be a Wach module of height h. The $\mathbb{E}$-linear representation $\mathbb{E} \otimes_{\mathcal{O}_{\mathbb{E}}} \mathfrak{T}(\mathcal{A}_{\mathbb{E}} \otimes_{\mathcal{A}^+_{\mathbb{E}}} N)$ of $G_{\mathbb{Q}_p}$ is crystalline with Hodge-Tate weights in the interval $[-h;0]$; and*

$$D_{cris}(\mathbb{E} \otimes_{\mathcal{O}_{\mathbb{E}}} \mathfrak{T}(\mathcal{A}_{\mathbb{E}} \otimes_{\mathcal{A}^+_{\mathbb{E}}} N)) \cong \mathbb{E} \otimes_{\mathcal{O}_{\mathbb{E}}} N/xN \quad \text{as } \varphi\text{-modules.}$$

Moreover, all crystalline representations with Hodge-Tate weights in $[-h;0]$ arise in this way.

2.2 Continuity of the Wach modules

Berger proved (see sec. III.4 or Thm. 2 in [Ber04]) that the there exists an equivalence of categories between rational Wach modules (over $\mathbb{E} \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{A}_{\mathbb{E}}^{+}$) and $G_{\mathbb{Q}_p}$ crystalline representations. As a consequence, Berger proved that if we fix a $\mathbb{E}$-linear representation V of $G_{\mathbb{Q}_p}$ and denote by D its corresponding étale (φ, Γ)-module via Fontaine's functor there is an inclusion preserving bijection between lattices inside V and Wach modules (over $\mathcal{A}_{\mathbb{E}}^{+}$) contained in D and which are $\mathcal{A}_{\mathbb{E}}^{+}$-lattices. Denote by $\mathfrak{N}$ such bijective map that associates to each $G_{\mathbb{Q}_p}$-stable lattice of a crystalline representation its corresponding Wach module.
In this section, we will prove that in some natural sense p-adically close Wach modules will correspond (via $\mathfrak{N}$) to p-adically close $\mathcal{O}_{\mathbb{E}}$-linear representations sitting in crystalline representation and viceversa. We will start by clarifying what we mean by p-adically close Wach modules. Given two Wach modules N_1 and N_2 we say that N_1 and N_2 are congruent modulo some prime power, i.e. $N_1 \equiv N_2 \mod \pi_{\mathbb{E}}^{m}$ for some $m \in \mathbb{Z}_{\geq 1}$, if there exists a $\mathcal{A}_{\mathbb{E}}^{+}$-module isomorphism between $N_1 \otimes_{\mathcal{A}_{\mathbb{E}}^{+}} \mathcal{A}_{\mathbb{E}}^{+}/(\pi_{\mathbb{E}}^{m})$ and $N_2 \otimes_{\mathcal{A}_{\mathbb{E}}^{+}} \mathcal{A}_{\mathbb{E}}^{+}/(\pi_{\mathbb{E}}^{m})$ which is (φ, Γ)-equivariant.
Note that, essentially by definition, we have that $N_1 \equiv N_2 \mod \pi_{\mathbb{E}}^{m}$ if and only if there exist basis of N_1 and N_2 as $\mathcal{A}_{\mathbb{E}}^{+}$-modules such that, after defining $P_1 = \mathrm{Mat}(\varphi|_{N_1})$, $P_2 = \mathrm{Mat}(\varphi|_{N_2})$, $G_1 = \mathrm{Mat}(\gamma|_{N_1})$, $G_2 = \mathrm{Mat}(\gamma|_{N_2})$ (note that $P_1, P_2, G_1, G_2 \in \mathrm{M}_{d\times d}(\mathcal{A}_{\mathbb{E}}^{+})$ where $d = \mathrm{rank}_{\mathcal{A}_{\mathbb{E}}^{+}}(N_i)$ for $i = 1, 2$) it follows that:

$$\begin{cases} P_1 \equiv P_2 \mod \pi_{\mathbb{E}}^{m}, \\ G_1 \equiv G_2 \mod \pi_{\mathbb{E}}^{m}. \end{cases}$$

We have the following continuity result (this is Thm. IV.1.1 in [Ber04]):

Proposition 2.2.1. *Let T_1 and T_2 be two Galois stable lattices inside respectively two crystalline $\mathbb{E}$-linear representation V_1 and V_2 of Hodge-Tate weights inside $[-r; 0]$, and assume there is an $n \in \frac{1}{e}(\mathbb{Z}_{\geq 1})$ with $n \geq \alpha(r)$ such that $T_1 \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{O}_{\mathbb{E}}/(p^n) \cong T_2 \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{O}_{\mathbb{E}}/(p^n)$ as $G_{\mathbb{Q}_p}$-modules, then $\mathfrak{N}(T_1) \equiv \mathfrak{N}(T_2) \mod p^{n-\alpha(r)}$.*

Now, if N is a Wach module denote by $\mathfrak{T}(N) := \mathfrak{T}(D(N))$ the $\mathcal{O}_{\mathbb{E}}$-linear representation of $G_{\mathbb{Q}_p}$ attached to N. We recall that $\mathfrak{T}(N) \otimes_{\mathcal{O}_{\mathbb{E}}} \mathbb{E}$ is a crystalline representation.
We are interested in the following result:

Proposition 2.2.2. *Let N_1 and N_2 be two Wach modules (over $\mathcal{O}_{\mathbb{E}}$) with the same rank as $\mathcal{A}_{\mathbb{E}}^{+}$-modules. Assume that $N_1 \equiv N_2 \mod \pi_{\mathbb{E}}^{n}$ for some $n \in \mathbb{Z}_{\geq 1}$. Then $\mathfrak{T}(N_1) \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{O}_{\mathbb{E}}/(\pi_{\mathbb{E}}^{n}) \cong \mathfrak{T}(N_2) \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{O}_{\mathbb{E}}/(\pi_{\mathbb{E}}^{n})$ as $G_{\mathbb{Q}_p}$-modules.*

Proof. Consider the étale (φ, Γ)-modules $D(N_i) = N_i \otimes_{\mathcal{A}_{\mathbb{E}}^{+}} \mathcal{A}_{\mathbb{E}}$ for $i = 1, 2$. Since $\mathcal{A}_{\mathbb{E}}$ is flat as $\mathcal{A}_{\mathbb{E}}^{+}$-module (module structure given by inclusion) we have the following chain of isomorphisms of torsion étale (φ, Γ)-modules:

$$D(N_1)/\pi_{\mathbb{E}}^{n} D(N_1) \cong N_1/\pi_{\mathbb{E}}^{n} N_1 \otimes_{\mathcal{A}_{\mathbb{E}}^{+}} \mathcal{A}_{\mathbb{E}} \cong N_2/\pi_{\mathbb{E}}^{n} N_2 \otimes_{\mathcal{A}_{\mathbb{E}}^{+}} \mathcal{A}_{\mathbb{E}} \cong D(N_2)/\pi_{\mathbb{E}}^{n} D(N_2).$$

Now, the claim follows just by applying Fontaine's functor $\mathfrak{T}$, which is exact (see Theorem 2.1.1 and see Remark 2.1.1). □

2.3 Local constancy with respect to the trace

In this section we are going to prove an explicit local constancy result with respect to the trace (i.e. the weight k will be fixed) for reductions modulo prime powers of representations of the type V_{k,a_p}.

2.3.1 Some linear algebra of Wach modules

As explained in the previous section, a congruence between Wach modules (modulo some prime power) can be translated into a congruence (modulo the same prime power) between systems of matrices representing the (φ, Γ) actions on the Wach modules involved. In this section, we will see how to p-adically deform a Wach module into another one via linear algebra means for the systems of matrices associated to the (φ, Γ)-module structure.
As long as it will be possible, we will keep the same notation as Berger (see [Ber12]). We recall that p is an odd prime and $\mathbb{E}$ is a finite extension of $\mathbb{Q}_p$ with ramification index e. Let v be the normalized p-adic valuation (i.e. $v(p) = 1$). Let $r \geq 1$ be a integer and define $\alpha(r) := \sum_{j=1}^{r} v(1 - \chi(\gamma)^j)$ where we recall that γ is a fixed topological generator of the pro-cyclic group Γ. The constant $\alpha(r)$ has also an explicit description given by $\alpha(r) = \sum_{n \geq 1} \lfloor \frac{r}{p^{n-1}(p-1)} \rfloor$ for a proper choice of γ (see [Ber12]).
We start by recalling two useful results in linear algebra (see Lemma 2.1 in [Ber12]):

Lemma 2.3.1. *If $P_0 \in M_2(\mathcal{O}_{\mathbb{E}})$ is a matrix with eigenvalues $\lambda \neq \mu$, and if $\delta = \lambda - \mu$, then there exists $Y \in M_2(\mathcal{O}_{\mathbb{E}})$ such that $Y^{-1} \in \delta^{-1} M_2(\mathcal{O}_{\mathbb{E}})$ and $Y^{-1} P_0 Y = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$.*

and the following corollary (see cor. 2.2 in [Ber12]):

Corollary 2.3.2. *If $\alpha \in \frac{1}{e}\mathbb{Z}_{\geq 0}$ and $\epsilon \in \mathcal{O}_{\mathbb{E}}$ are such that $v(\epsilon) \geq 2v(\delta) + \alpha$, then there exists $H_0 \in p^{\alpha} M_2(\mathcal{O}_{\mathbb{E}})$ such that $det(Id + H_0) = 1$ and $Tr(H_0 P_0) = \epsilon$.*

The corollary above represents the starting point to deform a Wach module into another one. Given the matrix P_0, it gives a p-adically small matrix H_0 such that the product $H_0 P_0$ will have a prescribed p-adically small trace. In practice, this will be applied when P_0 is obtained by the action of φ on $D_{\mathrm{cris}}(V_{k,a_p})$ for some $k \geq 2$ and $a_p \in \mathfrak{m}_{\mathbb{E}}$.
The idea behind the next results is to show how H_0 gives rise to a deformation of a whole system of matrices (attached to a Wach module) to a p-adically close one preserving the characterizing linear algebra properties of the action on φ and Γ on the Wach module.
We have the following result (it is a little generalization of Prop. 2.3 in [Ber12]):

Proposition 2.3.3. *Let $m \in \frac{1}{e}\mathbb{Z}_{\geq 0}$ and $d \in \mathbb{Z}_{\geq 2}$. If $G \in Id + xM_d(\mathcal{O}_{\mathbb{E}}[[x]])$ and $k \geq 2$ and $H_0 \in p^{\alpha(k-1)+m} M_d(\mathcal{O}_{\mathbb{E}})$, then there exists $H \in p^m M_d(\mathcal{O}_{\mathbb{E}}[[x]])$ such that $H(0) = H_0$ and $HG \equiv G\gamma(H)$ mod x^k.*

Proof. As $G \in \mathrm{Id} + xM_d(\mathcal{O}_{\mathbb{E}}[[x]])$, we can write $G = \mathrm{Id} + xG_1 + x^2 G_2 + \dots$ where $G_i \in \mathrm{M}_d(\mathcal{O}_{\mathbb{E}})$ for all $i \in \mathbb{Z}_{\geq 1}$. We prove that for any positive integer r, there exists an $H_r \in p^{\alpha(k-1)-\alpha(r)+m} M_d(\mathcal{O}_{\mathbb{E}})$ such that if we define $H = H_0 + xH_1 + x^2 H_2 + \cdots + x^{k-1} H_{k-1}$ we have that $HG \equiv G\gamma(H)$ mod x^k.

We start from $r = 1$, then since $\gamma(H) = H_0 + \gamma(x)H_1 + \gamma(x^2)H_2 + \dots \gamma(x^{k-1})H_{k-1}$ and for all $w \in \mathbb{Z}_{\geq 1}$ we have $\gamma(x^w) = ((1+x)^{\chi(\gamma)} - 1)^w$, we deduce that we can define H_1 such that $(1 - \chi(\gamma))H_1 = G_1H_0 - H_0G_1$. Since by hypothesis $H_0 \equiv 0 \bmod p^{\alpha(k-1)+m}$ and by definition $\alpha(1) = v_p(1 - \chi(\gamma))$, we deduce that $H_1 \in p^{\alpha(k-1)-\alpha(1)+m}M_d(\mathcal{O}_{\mathbb{E}})$.
Using now the same argument, one can actually see how H_r is uniquely determined by $H_0, H_1, \dots H_{r-1}$ and moreover $(1 - \chi(\gamma)^r)H_r \in p^{\alpha(k-1)-\alpha(r-1)+m}M_d(\mathcal{O}_{\mathbb{E}})$, note that $\alpha(r) = \alpha(k-1) - \alpha(r-1)$. To be precise, we prove this by induction on $r \leq k-1$. The first case $r = 1$ is proven above, now assume the case $r - 1$, we are going to prove the statement for r.
It is straightforward to prove that, expanding the expression $HG \equiv G\gamma(H) \mod x^k$, the following identity holds:

$$(1 - \chi(\gamma)^r)H_r = \sum_{h=0}^{r-1} \Big(\sum_{n=0}^{h} \gamma(x^n)_h H_n \Big) G_{r-h} - \sum_{i=0}^{r-1} H_i G_{r-i}, \qquad \text{for } r \leq k-1$$

where $\gamma(x^n)_h$ is the h-th coefficient (i.e. coefficient of x^h) of the polynomial $\gamma(x^n)$. Note now that the map $\alpha(n)$ is non-decreasing as n grows. Hence, by inductive hypothesis, we deduce that for any i such that $0 \leq i \leq r-1$ we have $H_i \equiv 0 \mod (p^{\alpha(k-1)-\alpha(r-1)+m})$. Since $v(1 - \chi(\gamma)^r) + \alpha(r-1) = \alpha(r)$, we deduce that $H_r \equiv 0 \mod (p^{\alpha(k)-\alpha(r)+m})$. This concludes the proof. □

The following result completes the linear algebra deformation process of a system of matrices that will represent a (φ, Γ)-action on Wach modules:

Proposition 2.3.4. *Let $m \in \frac{1}{e}\mathbb{Z}_{\geq 0}$ and $d \in \mathbb{Z}_{\geq 2}$. Let $G \in Id + xM_d(\mathcal{O}_{\mathbb{E}}[[x]])$ and $P \in M_d(\mathcal{O}_{\mathbb{E}}[[x]])$ satisfy $P\varphi(G) = G\gamma(P)$ and $det(P) = Q^{k-1}$ where $Q = \frac{(1+x)^p - 1}{x}$.*
If $H_0 \in p^{\alpha(k-1)+m}M_d(\mathcal{O}_{\mathbb{E}})$, then there exist $G' \in Id + xM_d(\mathcal{O}_{\mathbb{E}}[[x]])$ and $H \in p^m M_d(\mathcal{O}_{\mathbb{E}}[[x]])$ such that:

(1) $H(0) = H_0$;
(2) $P'\varphi(G') = G'\gamma(P')$, *where* $P' = (Id + H)P$;
(3) $P \equiv P'$ *mod* p^m;
(4) $G \equiv G'$ *mod* p^m.

Proof. After applying the previous proposition for the existence of the matrix H which satisfies $H(0) = H_0$, the existence of G' follows directly from Prop. 2.4 in [Ber12]. Hence the claims (1) and (2) hold.
The claim (3) is clear since $H \in p^m M_d(\mathcal{O}_{\mathbb{E}}[[x]])$ implies that $P \equiv P' \bmod p^m$. What it is left to prove is (4), i.e. $G \equiv G' \bmod p^m$. In order to prove this, we need to look at how the matrix G' is defined.
The matrix G' is constructed as an x-adic limit inside $\mathrm{Id} + x\mathrm{M}_d(\mathcal{O}_{\mathbb{E}}[[x]])$ (note that we are dealing with non-commutative rings). Define $G'_k := G$ and observe that it satisfies by construction $G'_k - P'\varphi(G'_k)\gamma(P')^{-1} = x^k R_k$ for some $R_k \in M_d(\mathcal{O}_{\mathbb{E}}[[x]])$. Then define G' as the x-adic limit of G'_j, for $j \geq k$, which satisfies $G'_{j+1} = G'_j + x^j S_j$ for some $S_j \in M_d(\mathcal{O}_{\mathbb{E}})$,

and $G'_j - P'\varphi(G'_j)\gamma(P')^{-1} = x^j R_j$ where $R_j \in M_d(\mathcal{O}_{\mathbb{E}}[[x]])$.
We will prove by induction that:

$$(i)\ \ R_j \equiv 0 \bmod p^m,$$
$$(ii)\ \ G'_j \equiv G \bmod p^m.$$

First the case $j = k$: since $H \equiv 0 \bmod p^m$ then $P \equiv P' \bmod p^m$ which implies that $R_k \equiv 0 \bmod p^m$, and by construction $G'_k = G$ so the first case of induction is done. Now assume $j \geq k$ and that the above claims hold for j, we will prove them for $j+1$.
We have that there exists $S_j \in M_d(\mathcal{O}_{\mathbb{E}})$ such that:

$$\begin{aligned} G'_{j+1} - P'\varphi(G'_{j+1})\gamma(P')^{-1} &= G'_j + x^j S_j - P'\varphi(G'_j)\gamma(P')^{-1} - P'x^jQ^jS_j\gamma(P')^{-1} = \\ &= x^j(R_j + S_j - Q^{j-k+1}P'S_jQ^{k-1})\gamma(P')^{-1}) \in x^{j+1}M_d(\mathcal{O}_{\mathbb{E}}[[x]]). \end{aligned}$$

Note that we used that φ acts trivially on $\mathrm{M}_d(\mathcal{O}_{\mathbb{E}})$ and that $\varphi(x^j) = x^jQ^j$ for all $j \in \mathbb{Z}_{\geq 1}$.
We want to prove that there exists $S_j \in p^m M_d(\mathcal{O}_{\mathbb{E}})$ such that:

$$R_j + S_j - Q^{j-k+1}P'S_jQ^{k-1}\gamma(P')^{-1} \in xM_d(\mathcal{O}_{\mathbb{E}}).$$

Evaluating the above expression at $x = 0$, the claim is equivalent to prove that there exists $S_j \in p^m M_d(\mathcal{O}_{\mathbb{E}})$ such that:

$$S_j - p^{j-k+1}P'(0)S_jp^{k-1}(\gamma(P')^{-1})(0) = -R_j(0).$$

Now, since $R_j \equiv 0 \bmod p^m$, we have that $R_j(0) \equiv 0 \bmod p^m$. It is clear that the map $S \mapsto S - p^{j-k+1}P'(0)Sp^{k-1}(\gamma(P')^{-1})(0)$ gives a bijection of $M_d(\mathcal{O}_{\mathbb{E}})$. Moreover, it is also clear that it is a bijection on $p^m M_d(\mathcal{O}_{\mathbb{E}})$. As $R_j(0) \equiv 0 \bmod p^m$, we have the existence of $S_j \equiv 0 \bmod p^m$ such that the above relations are satisfied. By inductive hypothesis $R_j \equiv 0 \bmod p^m$, so $R_{j+1} \equiv 0 \bmod p^m$. Since $S_j \equiv 0 \bmod p^m$ implies that $G'_{j+1} = G'_j + x^jS_j \equiv G \bmod p^m$. This concludes the proof. ☐

2.3.2 Local constancy modulo prime powers with respect to a_p

Let $k \geq 2$ be a positive integer and let $a_p \in \mathfrak{m}_{\mathbb{E}}$. In this section, we will apply the continuity properties of the Wach modules to prove local constancy results modulo prime powers when we fix the weight k and we let the trace of the crystalline Frobenius a_p vary p-adically.
The main result of this section is the following (this is a generalization of Thm. A of [Ber12]):

Theorem 2.3.5. *Let $a_p, a'_p \in \mathfrak{m}_{\mathbb{E}}$ and $k \geq 2$ be an integer. Let $m \in \frac{1}{e}(\mathbb{Z}_{\geq 1})$ such that $v(a_p - a'_p) \geq 2 \cdot v(a_p) + \alpha(k-1) + m$, then for every $G_{\mathbb{Q}_p}$-stable lattice T_{k,a_p} inside V_{k,a_p} there exists a $G_{\mathbb{Q}_p}$-stable lattice T_{k,a'_p} inside V_{k,a'_p} such that*

$$T_{k,a_p} \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{O}_{\mathbb{E}}/(p^m) \cong T_{k,a'_p} \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{O}_{\mathbb{E}}/(p^m) \text{ as } G_{\mathbb{Q}_p}\text{-modules.}$$

Proof. Consider the $G_{\mathbb{Q}_p}$-representation $V^*_{k,a_p} = \mathrm{Hom}_{\mathbb{E}}(V_{k,a_p}, \mathbb{E})$; it is crystalline and it has Hodge-Tate weights 0 and $-(k-1)$. Let $T^*_{k,a_p} := \mathrm{Hom}_{\mathcal{O}_{\mathbb{E}}}(T_{k,a_p}, \mathcal{O}_{\mathbb{E}})$ be the $G_{\mathbb{Q}_p}$-stable lattice in V^*_{k,a_p}, dual of T_{k,a_p}. By a result of Berger (see prop. III.4.2 and III.4.4 in [Ber04]), it is possible to attach to T^*_{k,a_p} a Wach module N_{k,a_p} of height $k-1$. Fixing a basis of N_{k,a_p} as $\mathcal{A}^+_{\mathbb{E}}$-module (we recall that in our notation $\mathcal{A}^+_{\mathbb{E}} = \mathcal{O}_{\mathbb{E}}[[x]]$), the actions of φ and γ on N_{k,a_p} can be respectively represented by the matrices $P \in \mathrm{Mat}_2(\mathcal{A}^+_{\mathbb{E}})$ and $Q \in \mathrm{Id} + x\mathrm{Mat}_2(\mathcal{A}^+_{\mathbb{E}})$. Note that since the actions of φ and Γ commute (in a semi-linear sense) we have that $P\varphi(G) = G\gamma(P)$. We recall also that the matrix $P(0)$ has characteristic polynomial $T^2 - a_pT + p^{k-1}$. Now, let $a'_p \in \mathcal{O}_{\mathbb{E}}$ be as in the hypothesis, i.e. it satisfies $v(a_p - a'_p) \geq 2 \cdot v(a_p) + \alpha(k-1) + m$ for some $m \in \frac{1}{e}(\mathbb{Z}_{\geq 1})$. Applying in sequence the results in section 4.1, we deduce the existence of two matrices P' and G' which give rise to a Wach module N' and such that $P \equiv P' \mod p^m$ and $G \equiv G' \mod p^m$. Since by construction $P' = (\mathrm{Id} + H)P$, we have that evaluating at $x = 0$ we deduce that the characteristic polynomial of $P'(0)$ is $T^2 - a'_pT + p^{k-1}$ (note that $\mathrm{Trace}(H(0)P(0)) = a'_p - a_p$ and $\mathrm{Det}(\mathrm{Id} + H(0)) = 1$). By a result of Berger (see prop. 1.2 in [Ber12]), we can deduce that $N' = N_{k,a'_p}$, or equivalently $V_{k,a'_p} = \mathfrak{T}(N') \otimes_{\mathcal{O}_{\mathbb{E}}} \mathbb{E}$ and $D_{\mathrm{cris}}(V^*_{k,a'_p}) = N'/xN' \otimes_{\mathcal{O}_{\mathbb{E}}} \mathbb{E}$. Since $P \equiv P' \mod p^m$ and $G \equiv G' \mod p^m$, we have that $N_{k,a_p} \equiv N_{k,a'_p} \mod p^m$. As a consequence of Remark 2.1.1, we have that $\mathfrak{D}(N_k, a_p) \equiv \mathfrak{D}(N_{k,a'_p}) \mod p^m$, i.e.

$$\mathfrak{D}(N_k, a_p) \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{O}_{\mathbb{E}}/(p^m) \cong \mathfrak{D}(N_{k,a'_p}) \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{O}_{\mathbb{E}}/(p^m) \text{ in the category } \mathbf{Mod}^{\text{ét}}_{(\varphi,\Gamma)}(\mathcal{O}_{\mathbb{E}}).$$

Hence, we define $T_{k,a'_p} := \mathfrak{T}(\mathfrak{D}(N_{k,a'_p})^*)$ which is a $G_{\mathbb{Q}_p}$-stable lattice in V_{k,a'_p} that satisfies (since Fontaine's functor $\mathfrak{T}$ is compatible with duals):

$$T_{k,a_p} \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{O}_{\mathbb{E}}/(p^m) \cong T_{k,a'_p} \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{O}_{\mathbb{E}}/(p^m) \text{ as } G_{\mathbb{Q}_p}\text{-modules.}$$

Indeed, since $\mathfrak{D}(N_{k,a_p}) \equiv \mathfrak{D}(N_{k,a'_p}) \mod p^m$, by exactness of Fontaine's functor $\mathfrak{T}$ we can deduce that $T^*_{k,a_p} \equiv \mathfrak{T}(\mathfrak{D}(N_{k,a'_p})) \mod p^m$. This completes the proof of the theorem. □

2.3.3 Converse of local constancy with respect to the trace

Via the continuity properties of the Wach modules, it is also possible to find an explicit necessary condition for the existence of local constancy phenomena modulo prime powers. In precise terms, let $k \geq 2$ be an integer and let $a_p, a'_p \in \mathfrak{m}_{\mathbb{E}}$; then we have the following:

Proposition 2.3.6. *Let $m \in \frac{1}{e}(\mathbb{Z}_{\geq 1})$ and assume $m \geq \alpha(k-1)$. If $V_{k,a_p} \equiv V_{k,a'_p} \mod p^m$, then $v(a_p - a'_p) \geq m - \alpha(k-1)$.*

Proof. This is a straightforward application of Berger's Proposition 2.2.1. Indeed, we have that $D_{\mathrm{cris}}(V^*_{k,a_p}) = N_{k,a_p}/xN_{k,a_p} \otimes \mathbb{E}$ and $D_{\mathrm{cris}}(V^*_{k,a'_p}) = N_{k,a'_p}/xN_{k,a'_p} \otimes \mathbb{E}$ for some Wach modules N_{k,a_p} and N_{k,a'_p} corresponding respectively to the $G_{\mathbb{Q}_p}$-stable lattices in V_{k,a_p} and V_{k,a'_p} that are congruent modulo p^m. By proposition 3.1, we have that $N_{k,a_p} \equiv N_{k,a'_p} \mod p^{m-\alpha(k-1)}$ and looking at the characteristic polynomials of φ acting on $N_{k,a_p}/xN_{k,a_p}$ and $N_{k,a'_p}/xN_{k,a_p}$ the claim follows. □

2.4 Local constancy with respect to the weight

In this section, we are going to prove a local constancy result for reductions modulo prime powers once we fix the trace of the crystalline Frobenius a_p and we let the weight k vary. In order to simplify the notation, we will say that two $\mathbb{E}$-linear representations V and V' of $G_{\mathbb{Q}_p} := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ are congruent modulo some prime power (i.e. $V \equiv V' \mod \pi_{\mathbb{E}}^n$ for some $n \in \mathbb{Z}_{\geq 1}$) if there exist $G_{\mathbb{Q}_p}$-stable lattices $T \subset V$ and $T' \subset V'$ such that we have an isomorphism

$$T \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{O}_{\mathbb{E}}/(\pi^n) \cong T' \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{O}_{\mathbb{E}}/(\pi^n) \quad \text{of } G_{\mathbb{Q}_p}\text{-modules.}$$

Note that the above definition requires a bit of attention when used as it clearly doesn't define an equivalence relation (in general, it is not transitive).
The main result of this section is the following:

Theorem 2.4.1. *Let p be an odd prime. Let $a_p \in m_{\mathbb{E}} - \{0\}$ for some finite extension $\mathbb{E}/\mathbb{Q}_p$. Let $k \geq 2$ be an integer and $m \in \frac{1}{e}(\mathbb{Z}_{\geq 1})$ be fixed. Assume that*

$$k \geq (3v(a_p) + m) \cdot \Big(1 - \frac{p}{(p-1)^2}\Big)^{-1} + 1. \qquad (*)$$

There exists an integer $r = r(k, a_p) \geq 1$ such that if $k' - k \in p^{r+m}(p-1)\mathbb{Z}_{\geq 0}$ then $V_{k,a_p} \equiv V_{k',a_p} \mod p^m$

Remark 2.4.1. As it will be clear from the proof below, the condition in the hypothesis is not optimal in the sense that it can be replaced by the weaker condition given by the system:

$$\begin{cases} k \geq 3v(a_p) + \alpha(k-1) + 1 + m, \\ k' \geq 3v(a_p) + \alpha(k'-1) + 1 + m. \end{cases}$$

as in Berger's result (see Thm. B in [Ber12]) when $m = 1/e$. For the sake of simplicity, we just assume the stronger condition $(*)$ which has the advantage that it is explicit in the weight, doesn't depend on the function α and automatically holds for k' if it holds for k assuming $k' \geq k$.
The condition $(*)$ in the theorem can be deduced directly from the above conditions in the system by noticing that $\alpha(k-1) = \sum_{n\geq 1} \left\lfloor \frac{k-1}{p^{n-1}(p-1)} \right\rfloor$ satisfies the inequality $\alpha(k-1) \leq \frac{(k-1)p}{(p-1)^2}$.

Remark 2.4.2. Note that Theorem 2.4.1 and Theorem 2.3.5 can be applied in sequence (i.e. one can first deform the weight and then deform the trace) in order to have a local constancy result in which both the trace and the weight vary. This is possible because Theorem 2.3.5 is independent of the starting chosen lattice inside V_{k,a_p}. Note that the order in which the theorems can be applied in sequence cannot be switched, as it is always necessary to keep track of the lattices involved in the congruences.

Remark 2.4.3. It could be interesting to consider the question of finding explicitly a radius for the local constancy in the weight. Partial results have been obtained by Bhattacharya (see [Bha18]). As already pointed out in the introduction, the above theorem can be seen as a converse (in a special crystalline case) of a non-published theorem of Winterberger, proven by Berger and Colmez as a consequence of a continuity property of the Sen periods (see [BC08]). Such result provides a connection between the local constancy radius of our theorem and the constant $c(2, \mathbb{Q}_p)$ of Berger and Colmez. To be precise, combining Theorem 2.4.1 above and Cor. 7.1.2 in [BC08] we get the upper bound on the radius for the local constancy in weight $p^{-(r+m)} \leq p^{-(\lfloor \frac{m}{2} \rfloor - c(2,\mathbb{Q}_p))}$.

In order to prove the theorem, the idea is to make use of Kedlaya's theory of (φ, Γ)-modules of slope zero over the Robba ring (see [Ked04]) and to realize the representations V_{k,a_p} (for k suff. big) as trianguline representations in the sense of Colmez (see [Col08]). A theorem of Colmez will then ensure us that locally such representations vary in a continuous way, in the sense that they come in analytic families. We will make this precise in the next section. We refer the reader to [Ber11] for a nice summary on the theory of trianguline representations and its applications in arithmetic geometry.
Let $\mathcal{R}_{\mathbb{E}}$ be the Robba ring with coefficients in $\mathbb{E}$ and for any multiplicative character $\delta : \mathbb{Q}_p^\times \to \mathbb{E}^\times$, denote by $\mathcal{R}_{\mathbb{E}}(\delta) := \mathcal{R}_{\mathbb{E}} e_\delta$ the (φ, Γ)-module (in the sense of Kedlaya, see [Ked04]) of rank one obtained by defining the actions $\varphi(e_\delta) = \delta(p) e_\delta$ and $\gamma(e_\delta) = \delta(\chi(\gamma)) e_\delta$ for all $\gamma \in \Gamma$, where χ denotes the chosen fixed isomorphism between Γ and $\mathbb{Z}_p^\times$.
Colmez (see Thm. 0.2 in [Col08]) proved that all (φ, Γ)-modules of rank one arise as $\mathcal{R}_{\mathbb{E}}(\delta)$ for a unique multiplicative character δ; moreover, if δ_1, $\delta_2 : \mathbb{Q}_p^\times \to \mathbb{E}^\times$ are multiplicative characters then $\mathrm{Ext}^1(\mathcal{R}_{\mathbb{E}}(\delta_1), \mathcal{R}_{\mathbb{E}}(\delta_2))$ is an $\mathbb{E}$-vector space of dimension 1 unless $\delta_1 \delta_2^{-1}$ is of the form x^{-i} for some integer $i \geq 0$, or $|x| x^i$ for some integer $i \geq 1$; in both cases, the dimension over $\mathbb{E}$ is two and the attached projective space is isomorphic to $\mathbb{P}^1(\mathbb{E})$; here x denotes the identity character of $\mathbb{Q}_p^\times$.
Hence, where the extension is not unique (up to isomorphism), one will need to specify the corresponding parameter in $\mathbb{P}^1(\mathbb{E})$ usually called L-invariant and denoted as $\mathfrak{L}$. The corresponding Galois representation will be denoted by $V(\delta_1, \delta_2, \mathfrak{L})$. For an extensive discussion about $\mathfrak{L}$-invariant, we refer the reader to the original article of Colmez (see sec. 4.5 in [Col08]).
Each trianguline representation $V(\delta_1, \delta_2)$ corresponds (up to considering blow-up in case $\delta_1 \delta_2^{-1} = x^{-i}$ or $|x| x^i$; see [Col08]) to the point $(\delta_1, \delta_2) \in \mathfrak{X} \times \mathfrak{X}$ where $\mathfrak{X}$ is isomorphic (non-canonically, as there are choices involved) to the $\mathbb{Q}_p$-rigid analytic space $\mu(\mathbb{Q}_p) \times \mathbb{G}_m^{\mathrm{rig}} \times \mathbb{B}^1(1,1)^-_{\mathbb{Q}_p}$, which parametrizes multiplicative characters $\mathbb{Q}_p^\times$ with values in the multiplicative group of some finite extension of $\mathbb{Q}_p$.
From now on, we denote by $\mathbb{B}^1(a,r)^+_{\mathbb{Q}_p}$ the closed affinoid rigid $\mathbb{Q}_p$-ball centered in a and with radius r. The expression $\mathbb{B}^1(a,r)^-_{\mathbb{Q}_p}$ will instead denote the open rigid $\mathbb{Q}_p$-ball as a $\mathbb{Q}_p$-rigid analytic space. If we are working with an algebraic closure $\overline{\mathbb{Q}}_p$, then the expression $\mathbb{B}^1(a,r)^+$ will simply denote the standard p-adic ball centered in a with radius r.

2.4.1 Proof of Theorem 2.4.1

Let k' be an integer satisfying $k' - k \in (p-1)\mathbb{Z}_{\geq 0}$. The claim is to prove that if k' and k are sufficiently p-adically close then the corresponding representations are isomorphic

modulo a prescribed prime power.
The assumption $(*)$ on the weight k allow us, applying Theorem 2.3.5, to deduce that:

$$V_{k,a_p+\frac{p^{k-1}}{a_p}} \equiv V_{k,a_p} \mod p^m,$$
$$V_{k',a_p+\frac{p^{k'-1}}{a_p}} \equiv V_{k',a_p} \mod p^m.$$

Indeed, note that assumption $(*)$ implies that $k-1 > 2v(a_p)$ and hence in this specific case, the Theorem 2.3.5 can be applied both starting from V_{k,a_p} or starting from $V_{k,a_p+\frac{p^{k-1}}{a_p}}$ (same goes for k') and hence this gives us a strong control over the lattices involved in the congruences. Therefore, this first step reduces the claim to prove that if k' and k are sufficiently p-adically close (in the weight space) then we have the congruence $V_{k,a_p+\frac{p^{k-1}}{a_p}} \equiv V_{k',a_p+\frac{p^{k'-1}}{a_p}} \mod p^m$.
The following proposition of Colmez (see prop. 3.1 in [Ber12] or see sec. 4.5 in [Col08]) allow us to realize the above representations as trianguline representations:

Proposition 2.4.2. *If $z \in \mathfrak{m}_{\mathbb{E}}$ is a root of $z^2 - a_p z + p^{k-1}$ which satisfies $v(z) < k-1$, then we have that $V(\mu_z, \mu_{\frac{1}{z}}\chi^{1-k}, \infty) = V^*_{k,a_p}$.*
Here $\mu_z : \mathbb{Q}_p^\times \to \mathbb{E}^\times$ is the character which satisfies $\mu_z(p) = z$ and $\mu_z(\mathbb{Z}_p^\times) = 1$ and $\chi : \mathbb{Q}_p^\times \to \mathbb{E}^\times$ is the character which satisfies $\chi(p) = 1$ and $\chi(y) = y$ for all $y \in \mathbb{Z}_p^\times$.

Hence, if $k'-1 \geq k-1 > v(a_p)$, we have that the crystalline representations $V_{k,a_p+\frac{p^{k-1}}{a_p}}$ and $V_{k',a_p+\frac{p^{k'-1}}{a_p}}$ coincide respectively with the trianguline representations $V(\mu_{a_p}, \delta_{k,a_p}, \infty)$ and $V(\mu_{a_p}, \delta_{k',a_p}, \infty)$, where $\delta_{k,a_p} := \mu_{\frac{1}{a_p}}\chi^{1-k}$ and $\delta_{k',a_p} = \mu_{\frac{1}{a_p}}\chi^{1-k'}$. Since the $\mathfrak{L}$-invariant is going to be ∞ for all the trianguline representations involved, we will drop the notation $V(\cdot,\cdot,\infty)$ writing simply $V(\cdot,\cdot)$.
The trianguline representations $V(\mu_{a_p}, \delta_{k,a_p})$ and $V(\mu_{a_p}, \delta_{k',a_p})$ define two $\mathbb{E}$-points, respectively $u_{k,a_p} = (\mu_{a_p}, \delta_{k,a_p})$ and $u_{k',a_p} = (\mu_{a_p}, \delta_{k',a_p})$, on the rigid analytic space $\mathfrak{X} \times \mathfrak{X}$ (see [Col08]) parametrizing couples of multiplicative characters of $\mathbb{Q}_p^\times$ with values in $\mathbb{L}^\times$ where $\mathbb{L}$ is some finite extension of $\mathbb{Q}_p$.
The following basic lemma represents the first step for constructing an explicit 1-parameter family of trianguline representations inside $\mathfrak{X}^2 = (\mu(\mathbb{Q}_p) \times \mathbb{G}_m^{\mathrm{rig}} \times \mathbb{B}^1(1,1)^-_{\mathbb{Q}_p})^2$ interpolating $V(\mu_{a_p}, \delta_{k,a_p})$ and $V(\mu_{a_p}, \delta_{k',a_p})$ when k and k' will be sufficiently p-adically close (in the weight space):

Lemma 2.4.3. *Let $\alpha \in 1 + p\mathbb{Z}_p$, then we have that*

$$\psi_\alpha : \mathbb{B}^1(0,1)^+_{\mathbb{Q}_p} \longrightarrow \mathbb{B}^1(1, |\alpha - 1|)^+_{\mathbb{Q}_p}$$
$$[s] \mapsto [exp_p(s \cdot log_p(\alpha))]$$

is an isomorphism in the category of $\mathbb{Q}_p$-rigid analytic spaces. Here $[s]$ denotes the maximal ideal of $\mathbb{Q}_p\langle T\rangle$ corresponding to the element $s \in \overline{\mathbb{Z}}_p$ and the analogue for $\psi_\alpha([s])$.

Proof. First, we will clarify that ψ_α is a well defined map for every $\alpha \in 1 + p\mathbb{Z}_p$. For all $s \in \overline{\mathbb{Z}}_p$, we have that ψ_α converges when evaluated in s since $|s \cdot \log_p(\alpha)| \leq |\alpha - 1| \leq$

p^{-1}. Moreover since $\psi_\alpha(s) \in \mathbb{Q}_p(s)$, we have that the map ψ_α is Galois equivariant, i.e. $\psi_\alpha(\sigma(s)) = \sigma(\psi_\alpha(s))$ for every $\sigma \in G_{\mathbb{Q}_p}$. Note also that we can find the explicit expression $\psi_\alpha(s) = \alpha^s = (1 + (\alpha - 1))^s = \sum_{n\geq 0} \binom{s}{n}(\alpha - 1)^n$ which converges for every $s \in \overline{\mathbb{Z}}_p$. This allow us to define ψ_α on the set of $G_{\mathbb{Q}_p}$-orbits of $\overline{\mathbb{Z}}_p$ which can be identified set-theoretically with $\mathbb{B}^1(0, 1)^+_{\mathbb{Q}_p}$. Proving that ψ_α is a morphism of $\mathbb{Q}_p$-rigid analytic spaces (affinoid spaces in this case) boils down to show that the induced map on the corresponding affinoid algebras is a morphism. Let $\mathcal{O}_{0,1}$ and $\mathcal{O}_{1,|\alpha-1|}$ denote respectively the $\mathbb{Q}_p$-affinoid algebras attached to the $\mathbb{Q}_p$-affinoid spaces $\mathbb{B}^1(0, 1)^+_{\mathbb{Q}_p}$ and $\mathbb{B}^1(1, |\alpha-1|)^+_{\mathbb{Q}_p}$. We have that the associated map $\psi^*_\alpha : \mathcal{O}_{1,|\alpha-1|} \to \mathcal{O}_{0,1}$ is given by $\psi^*_\alpha(f) = f \circ \psi_\alpha$ for all $f \in \mathcal{O}_{1,|\alpha-1|} \cong \mathbb{Q}_p\langle \frac{T-1}{\alpha-1} \rangle$.
In order to show that ψ^*_α is a morphism of affinoid algebras, since it is given by the pull-back, it is sufficient to show that it is a well-defined map, in the sense that $\psi^*_\alpha(f)$ belongs to $\mathcal{O}_{0,1} \cong \mathbb{Q}_p\langle T\rangle$; or in other words, it is a converging series. Indeed, the problem is that, in general, composition of p-adic analytic functions is not analytic. The convergence property will be deduced by the following convergence criterion (see Thm. 4.3.3 in [Gou97]):

Theorem 2.4.4. *Let $f(X) = \sum a_n X^n$ and $g(X) = \sum b_n X^n$ be formal power series in $\overline{\mathbb{Q}}_p[[X]]$ with $g(0) = 0$, and let $h(X) = f(g(X))$ be their formal composition.*
Suppose that:

(i) $g(x)$ converges,

(ii) $f(g(x))$ converges (i.e. plugging the number to which $g(x)$ converges into $f(X)$ gives a convergent series),

(iii) for every $n \in \mathbb{Z}_{\geq 0}$, we have $|b_n x^n| \leq |g(x)|$ (i.e. no term of the series converging to $g(x)$ is bigger than the sum).

Then $h(x)$ also converges, and $f(g(x)) = h(x)$.

Indeed, in our case, it is enough to prove that for all $n \in \mathbb{Z}_{\geq 0}$ we have

$$|c_n s^n| \leq |\psi_\alpha(s)| \qquad \text{where } \psi_\alpha(s) = \exp_p(s \cdot \log_p(\alpha)) = \sum_{n\geq 0} c_n s^n \quad \text{and} \quad c_n := \frac{(\log_p(\alpha))^n}{n!}.$$

We have that $|\psi_\alpha(s)| = |\exp_p(s \cdot \log_p(\alpha))| = 1$ for all $s \in \mathbb{B}^1(0, 1)^+$, hence since $|s| \leq 1$, it is sufficient to prove that $|c_n| \leq 1$.
Since $\alpha \in 1 + p\mathbb{Z}_p$ and since the p-adic logarithm is an isometry we have that $|\log_p(\alpha)|^n = |\alpha - 1|^n \leq p^{-n}$. We also recall from classical p-adic analysis that $v_p(n!) < \frac{n}{p-1}$ or in other words $|n!| > p^{-\frac{n}{p-1}}$. It follows at once that

$$|c_n| = \frac{|\log_p(\alpha)|^n}{|n!|} < \frac{p^{-n}}{p^{-\frac{n}{p-1}}} < 1.$$

Note that the inverse $\psi^{-1}_\alpha : \mathbb{B}^1(1, |\alpha - 1|)^+_{\mathbb{Q}_p} \longrightarrow \mathbb{B}^1(0, 1)^+_{\mathbb{Q}_p}$ sends $[t]$ to $\left[\frac{\log_p(t)}{\log_p(\alpha)}\right]$ and it is, via the same argument, a well-defined morphism of $\mathbb{Q}_p$-affinoid spaces. This concludes the proof. □

We will make use of the map ψ_α just defined to construct a family of points (i.e. trianguline representations) on $\mathfrak{X}^2$ which will pass through u_{1-k} (i.e. the representation $V(\mu_{a_p}, \mu_{\frac{1}{a_p}}\chi^{1-k})$).
For each $s \in \overline{\mathbb{Z}}_p$, we define a multiplicative character of $\mathbb{Q}_p^\times$ as follows:

$$\begin{aligned} \delta_{k,a_p}^{(s)} : \mathbb{Q}_p^\times &\longrightarrow \mathbb{E}(s)^\times \\ x &\longmapsto \delta_{k,a_p}^{(s)}(x) := \mu_{\frac{1}{a_p}}(x) \cdot \omega(x)^{1-k} \cdot \psi_{\langle x \rangle}(s), \end{aligned}$$

where $\mathbb{E}(s)$ is the finite extension obtained from $\mathbb{E}$ by adding s; and where $x = p^{v_p(x)}\omega(x)\langle x \rangle$ is the unique decomposition given by a fixed isomorphism $\mathbb{Q}_p^\times \cong p^{\mathbb{Z}} \times \mu(\mathbb{Q}_p) \times 1 + p\mathbb{Z}_p$. Note that $\psi_{\langle x \rangle}(s)$ is an element in $\mathbb{Q}_p(s)$ since $\binom{s}{n}(\langle x \rangle - 1)^n \in \mathbb{Q}_p(s)$ for any $n \in \mathbb{Z}_{\geq 1}$.
Finally, we are ready to apply this in the context of rigid analytic spaces, indeed we will define a 1-dimensional p-adic family of points in $\mathfrak{X}^2$ through which we will control the "p-adic" distance between u_{1-k} and $u_{1-k'}$.
We define $\mathcal{Z}$ to be the $\mathbb{Q}_p$-affinoid spaces given by $\{\mu_{a_p}\} \times \{1/a_p\} \times \{\zeta_p^{k-1}\} \times \mathbb{B}^1(0,1)^+_{\mathbb{Q}_p}$; here $\{\mu_{a_p}\}$ denotes the singleton corresponding to the character μ_{a_p} on $\mathfrak{X}$, the singleton $\{1/a_p\}$ corresponds to the $\mathbb{E}$-point $1/a_p$ in $\mathbb{G}_m^{\text{rig}}$ and the singleton $\{\zeta_p^{k-1}\}$ corresponds to the point ζ_{p-1}^{k-1} in $\mu(\mathbb{Q}_p)$.
By a little abuse of notation, we will still denote a point in $\mathcal{Z}$ by s for the corresponding point $s \in \mathbb{B}^1(0,1)^+_{\mathbb{Q}_p}$. Now, we define the injective map:

$$\begin{aligned} \Phi : \mathcal{Z} &\longrightarrow \mathfrak{X}^2 \\ s &\longmapsto \Phi(s) := (\mu_{a_p}, \delta_{k,a_p}^{(s)}) \end{aligned}$$

and note that if $k' \in \mathbb{Z}_{\geq 2}$ satisfies $k' - k \in (p-1)\mathbb{Z}_{\geq 0}$ we have, by construction, that $\Phi(1-k) = u_{k,a_p}$ and $\Phi(1-k') = u_{k',a_p}$ since $\delta_{k,a_p}^{(1-k)} = \mu_{\frac{1}{a_p}}\chi^{1-k}$ and $\delta_{k,a_p}^{(1-k')} = \mu_{\frac{1}{a_p}}\chi^{1-k'}$.

Proposition 2.4.5. *The map $\Phi : \mathcal{Z} \to \mathfrak{X}^2$ is a rigid analytic closed immersion.*

Proof. In order to see that Φ is a morphism of $\mathbb{Q}_p$-rigid analytic spaces it is sufficient to observe that, decomposing $\mathfrak{X}^2$ as $\mathfrak{X} \times \mu(\mathbb{Q}_p) \times \mathbb{G}_m^{\text{rig}} \times \mathbb{B}^1(1,1)^-_{\mathbb{Q}_p}$, the map Φ is a product of constant morphisms and ψ_{1+p}:

$$\begin{aligned} \Phi : \mathcal{Z} &\longrightarrow \mathfrak{X}^2 = \mathfrak{X} \times \mu(\mathbb{Q}_p) \times \mathbb{G}_m^{\text{rig}} \times \mathbb{B}^1(1,1)^-_{\mathbb{Q}_p} \\ s &\longmapsto (\mu_{a_p}, \delta_{k,a_p}^{(s)}) = (\mu_{a_p}, \delta_{k,a_p}^{(s)}(\zeta_{p-1}), \delta_{k,a_p}^{(s)}(p), \delta_{k,a_p}^{(s)}(1+p)) \\ &\qquad = (\mu_{a_p}, [\zeta_{p-1}^{1-k}], [a_p^{-1}], \psi_{1+p}(s)). \end{aligned}$$

The universal property of fiber products (in the category of rigid analytic spaces) allows us to reduce the claim to prove that the composition of Φ with the projection on the last factor of $\mathfrak{X}^2$, which is exactly ψ_{1+p}, belongs to $\text{Mor}(\mathcal{Z}, \mathbb{B}^1(1,1)^-_{\mathbb{Q}_p})$. This follows at once from Lemma 2.4.3. Moreover, the image is an affinoid subdomain of $\mathfrak{X}^2$ making Φ a closed immersion. □

Now, the heart of the proof is that the representations attached to points of $\mathfrak{X}^2$ vary locally in a continuous way. In precise terms, this is the following result of Colmez and Chenevier (see Prop. 5.2 in [Col08] and its generalization Prop. 3.9 in [Che13], see also Prop. 3.2 in [Ber12]):

Theorem 2.4.6. *Let $\delta_1, \delta_2 : \mathbb{Q}_p^\times \to \mathbb{E}^\times$ be two characters such that $\delta_1\delta_2^{-1} \neq x^{-i}$ for some $i \geq 0$, where x denotes the identity character of $\mathbb{Q}_p^\times$. Let $u = (\delta_1, \delta_2)$ be the corresponding point in $\mathfrak{X}^2$. Then there exists a open neighborhood $\mathfrak{U}$ of u and a finite, free $\mathcal{O}_{\mathfrak{U}}$-module $\mathbb{V}$ of rank 2 with an action of $G_{\mathbb{Q}_p} := Gal(\bar{\mathbb{Q}}_p, \mathbb{Q}_p)$ such that $\mathbb{V}(\tilde{u}) = V(\delta_1(\tilde{u}), \delta_2(\tilde{u}))$ for every $\tilde{u} \in \mathfrak{U}$.*

As we are interested in points inside $\mathfrak{U} \subset \mathfrak{X}^2$ which is an open neighborhood of u_{1-k}, we will first prove that if k' is sufficiently p-adically close to k (close as points in the weight space) then also $\Phi(1-k') = u_{1-k'}$ will lie in $\mathfrak{U}$.
Without loss of generality, as $\mathfrak{U}$ is an admissible open we can assume (up to restriction) that $(\mathfrak{U}, \mathcal{O}_{\mathfrak{U}})$ is an affinoid space. Since Φ is a morphism of rigid analytic spaces, it is in particular continuous for the G-topology, hence $\Phi^{-1}(\mathfrak{U})$ is an admissible open of the affinoid space $\mathcal{Z}$.
In particular, we can deduce that there exists a minimal $r \in \frac{1}{e}(\mathbb{Z}_{\geq 1})$ such that the affinoid subdomain $\mathcal{Z}_r := \{\mu_{a_p}\} \times \{[a_p^{-1}]\} \times \{[\zeta_p^{k-1}]\} \times \mathbb{B}^1(1-k, p^{-r})^+_{\mathbb{Q}_p}$ of $\mathcal{Z}$ is contained in $\Phi^{-1}(\mathfrak{U})$. As usual, we identify the algebra of funcions $\mathcal{O}_{\mathcal{Z}_r}$ of the $\mathbb{Q}_p$-affinoid space $\mathcal{Z}_r$ with $\mathbb{E} \otimes \mathbb{Q}_p\langle \frac{T-(1-k)}{p^r} \rangle$. By restricting the morphism Φ to $\mathcal{Z}_r$, we get the morphism of $\mathbb{Q}_p$-affinoid spaces:

$$\Phi : \mathcal{Z}_r \to \mathfrak{U}$$

and as usual, we denote its associated morphism of $\mathbb{Q}_p$-affinoid algebras by $\Phi^* : \mathcal{O}_{\mathfrak{U}} \to \mathcal{O}_{\mathcal{Z}_r}$.
Now, observe that if we fix a $\bar{u} \in \mathfrak{U}$ with field of definition $\mathbb{L}_{\bar{u}}$, it induces a $\mathbb{Q}_p$-Banach spaces morphism given by the evaluation $\mathrm{ev}_{\bar{u}} : \mathcal{O}_{\mathfrak{U}} \to \mathbb{L}_{\bar{u}}$. Consider now the finite, free $\mathcal{O}_{\mathfrak{U}}$-module $\mathbb{V}$ of rank 2 considered by Colmez. The ring homomorphism $\mathrm{ev}_{\bar{u}}$ induces on $\mathbb{L}_{\bar{u}}$ a structure of $\mathcal{O}_{\mathfrak{U}}$-module, hence we define $\mathbb{V}(\bar{u}) := \mathbb{V} \otimes_{\mathcal{O}_{\mathfrak{U}}} \mathbb{L}_{\bar{u}}$. By Chenevier's and Colmez's Theorem 2.4.6, we have that $\mathbb{V}(\bar{u}) = V(\delta_1(\bar{u}), \delta_2(\bar{u}))$ where $\bar{u} := (\delta_1(\bar{u}), \delta_2(\bar{u})) \in \mathfrak{X}^2$. In particular, we note that when $\bar{u} = u_{1-k}$, then $\mathbb{L}_{u_{1-k}} = \mathbb{E}$ and $\mathbb{V}(u_{1-k}) = V(\mu_{a_p}, \delta_{k,a_p})$. Clearly the analogue statement holds for $k' \in \mathcal{Z}_r$ such that $k' - k \in (p-1)\mathbb{Z}_{\geq 0}$.
The idea is now to pull back the analytic family of representations given by Colmez in order to create a new analytic family parametrized by points in $\mathcal{Z}_r$ which has the advantage that will depend only on one parameter. The notion of analytic family of representations parametrized by an affinoid space is used in the sense of Berger and Colmez (see [BC08]) but one could also have approached the problem from the point of view of analytic family of (φ, Γ)-modules over variations of the Robba rings in the sense of Bellaïche (see [Bel12]) considering the existence of a fully faithful functor which connects the two categories (see sec. 3 in [Bel12]).
Coming back to the closed immersion $\Phi : \mathcal{Z}_r \to \mathfrak{U}$ of $\mathbb{Q}_p$-affinoid spaces, we have that the induced map $\Phi^* : \mathcal{O}_{\mathfrak{U}} \to \mathcal{O}_{\mathcal{Z}_r}$ is given by the pull-back, i.e. $\Phi^*(f) = f \circ \Phi$ for all $f \in \mathcal{O}_{\mathfrak{U}}$. The ring homomorphism Φ^* gives to $\mathcal{O}_{\mathcal{Z}_r}$ the structure of $\mathcal{O}_{\mathfrak{U}}$-module and so we can define $\mathbb{V}_{\mathbb{B}^{1,+}} := \mathbb{V} \otimes_{\mathcal{O}_{\mathfrak{U}}} \mathcal{O}_{\mathcal{Z}_r}$, it is a finite, free $\mathcal{O}_{\mathcal{Z}_r}$-module of rank 2 with a continuous $G_{\mathbb{Q}_p}$-action (given by the action of the Galois group on $\mathbb{V}$). In particular, for all $s \in \mathcal{Z}_r$

we have by definition that $\mathbb{V}_{\mathcal{Z}_r}(s) \cong \mathbb{V}(\Phi(s))$.
Now, in order to deal wth reductions we first need to identify an integral analytic family of lattices. First, we recall that there is a notion of integral model for affinoid algebras. We define $\mathcal{O}_{\mathfrak{U}}^\circ := \{g \in \mathcal{O}_{\mathfrak{U}} : \ |g|_{\sup} \leq 1\}$. It is a model for $\mathcal{O}_{\mathfrak{U}}$, i.e. it is a $\mathbb{Z}_p$-subalgebra of $\mathcal{O}_{\mathfrak{U}}$ topologically of finite type (i.e. it is a quotient of $\mathbb{Z}_p\langle x_1, \ldots, x_n\rangle$ for some integer $n \geq 1$) and such that $\mathcal{O}_{\mathfrak{U}}^\circ[\frac{1}{p}] = \mathcal{O}_{\mathfrak{U}}$.
The existence of an integral analytic family of representations inside $\mathbb{V}$ follows from the following lemma:

Lemma 2.4.7. *Let G be a profinite group, let A be a $\mathbb{Q}_p$-affinoid algebra and let V be a finite, free A-module endowed with an A-linear, continuous action of G. Let A° be a model for A. Then there exists a finite, free A°-module V_0 inside V such that $V_0 \otimes_{A^\circ} A = V$ and which is stable under the continuous action of G.*

Proof. Let W be a finite and free A°-module such that $W \otimes_{A^\circ} A = V$. Since A° is open inside A, and since V is a topological finite direct sum of copies of A we have that W is open inside V. The action of G can be represented by a continuous map $G \times W \to V$. Since W is open inside V, then the subgroup $H_W \subset G$ stabilizing W is an open subgroup of G. Since G is profinite, we have that H_W is of finite index. Let $\{h_i\}_i$ be a finite set of representatives for the left H_W-cosets in G. Hence, defining V_0 as $\sum_i h_i W$ we have that V_0 is a G-stable, finite and free A°-module such that $V_0 \otimes_{A^\circ} A = V$, this completes the proof. $\square$

Applying the above result when $G = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, $A = \mathcal{O}_{\mathfrak{U}}$, $A^\circ = \mathcal{O}_{\mathfrak{U}}^\circ$ and $V = \mathbb{V}$ allow us to deduce that there exists $\mathbb{T}$ inside $\mathbb{V}$ finite, free $\mathcal{O}_{\mathfrak{U}}^\circ$-module of rank 2, which is $G_{\mathbb{Q}_p}$-stable and such that $\mathbb{T} \otimes_{\mathcal{O}_{\mathfrak{U}}^\circ} \mathcal{O}_{\mathfrak{U}} \cong \mathbb{V}$ as $G_{\mathbb{Q}_p}$-modules.
After defining $\mathbb{V}_{\mathcal{Z}_r}$ as $\mathbb{V}_{\mathcal{O}_{\mathfrak{U}}} \otimes_{\mathcal{O}_{\mathfrak{U}}} \mathcal{O}_{\mathcal{Z}_r}$, consider the model $\mathcal{O}_{\mathcal{Z}_r}^\circ := \{g \in \mathcal{O}_{\mathcal{Z}_r} : |g|_{\sup} \leq 1\}$ inside $\mathcal{O}_{\mathfrak{U}}$. Since every morphism of affinoid algebras is in particular a contraction, we have that via the restriction of Φ to the power bounded elements inside $\mathcal{O}_{\mathfrak{U}}$ we obtain the $\mathbb{Z}_p$-algebra morphism $\Phi : \mathcal{O}_{\mathfrak{U}}^\circ \twoheadrightarrow \mathcal{O}_{\mathcal{Z}_r}^\circ$. This allows to define $\mathbb{T}_{\mathcal{Z}_r} := \mathbb{T} \otimes_{\mathcal{O}_{\mathfrak{U}}^\circ} \mathcal{O}_{\mathcal{Z}_r}^\circ$.
The properties of $\mathbb{T}_{\mathbb{B}_r^{1,+}}$ are summarized in the following:

Lemma 2.4.8. *The $\mathcal{O}_{\mathcal{Z}_r}^\circ$-module $\mathbb{T}_{\mathcal{Z}_r}$ is finite, free of rank 2 submodule of $\mathbb{V}_{\mathcal{Z}_r}$. It has a natural action of the Galois group $G_{\mathbb{Q}_p}$; in particular we have that:*

$$\mathbb{T}_{\mathcal{Z}_r} \otimes_{\mathcal{O}_{\mathcal{Z}_r}^\circ} \mathcal{O}_{\mathcal{Z}_r} \cong \mathbb{V}_{\mathcal{Z}_r}$$

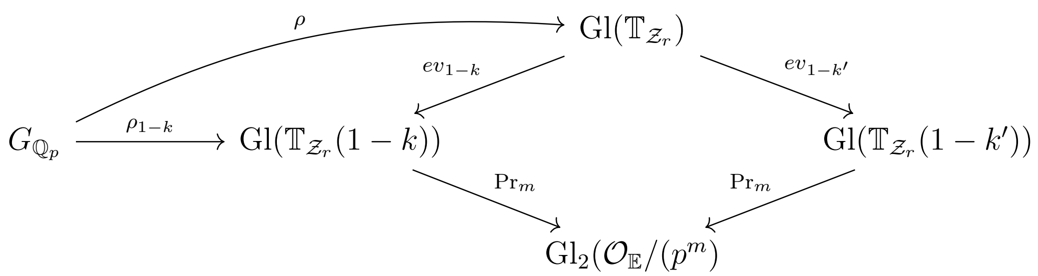
is an isomorphism of $G_{\mathbb{Q}_p}$-modules.

Proof. The only claim that is not clear is the isomorphism of $G_{\mathbb{Q}_p}$-modules. By definition we have an isomorphism $\mathbb{T} \otimes_{\mathcal{O}_{\mathfrak{U}}^\circ} \mathcal{O}_{\mathfrak{U}} \cong \mathbb{V}$ of $G_{\mathbb{Q}_p}$-modules, hence tensorizing by $\mathcal{O}_{\mathcal{Z}_r}$ (considered as $\mathcal{O}_{\mathfrak{U}}$-module via Φ^*) we get the isomorphism $(\mathbb{T} \otimes_{\mathcal{O}_{\mathfrak{U}}^\circ} \mathcal{O}_{\mathfrak{U}}) \otimes_{\mathcal{O}_{\mathfrak{U}}} \mathcal{O}_{\mathcal{Z}_r} \cong \mathbb{V}_{\mathcal{Z}_r}$ of $G_{\mathbb{Q}_p}$-modules.
We have the following chain of isomorphism of $G_{\mathbb{Q}_p}$-modules:

$$(\mathbb{T} \otimes_{\mathcal{O}_{\mathfrak{U}}^\circ} \mathcal{O}_{\mathfrak{U}}) \otimes_{\mathcal{O}_{\mathfrak{U}}} \mathcal{O}_{\mathcal{Z}_r} \cong \mathbb{T} \otimes_{\mathcal{O}_{\mathfrak{U}}^\circ} (\mathcal{O}_{\mathfrak{U}} \otimes_{\mathcal{O}_{\mathfrak{U}}} \mathcal{O}_{\mathcal{Z}_r}) \cong \mathbb{T} \otimes_{\mathcal{O}_{\mathfrak{U}}^\circ} \mathcal{O}_{\mathcal{Z}_r} \cong$$
$$\mathbb{T} \otimes_{\mathcal{O}_{\mathfrak{U}}^\circ} (\mathcal{O}_{\mathcal{Z}_r}^\circ \otimes_{\mathcal{O}_{\mathcal{Z}_r}^\circ} \mathcal{O}_{\mathcal{Z}_r}) \cong (\mathbb{T} \otimes_{\mathcal{O}_{\mathfrak{U}}^\circ} \mathcal{O}_{\mathcal{Z}_r}^\circ) \otimes_{\mathcal{O}_{\mathcal{Z}_r}^\circ} \mathcal{O}_{\mathcal{Z}_r} \cong \mathbb{T}_{\mathcal{Z}_r} \otimes_{\mathcal{O}_{\mathcal{Z}_r}^\circ} \mathcal{O}_{\mathcal{Z}_r},$$

where the isomorphisms are given by the associative property of tensor product for general modules (see Prop. 3.8, chap. 3 in [Bou89]). Hence, the claim follows. □

In terms of Berger's and Colmez's notion of analytic families of p-adic representations (see [BC08]), the two above results essentially grant that the Galois properties of analytic integral subfamilies are preserved via pull-back. Note that now for all $s \in \mathcal{Z}_r$, if we denote by $\mathbb{L}_{u_s}$ the field of definition of the point $\Phi(s) := u_s$ then $\mathbb{T}_{\mathcal{Z}_r}(s) \otimes \mathbb{L}_{u_s} \cong \mathbb{V}_{\mathcal{Z}_r}(s)$ as $G_{\mathbb{Q}_p}$-modules. Hence, we deduce that $\mathbb{T}_{\mathcal{Z}_r}(1-k)$ and $\mathbb{T}_{\mathcal{Z}_r}(1-k')$ are two $G_{\mathbb{Q}_p}$-stable $\mathcal{O}_{\mathbb{E}}$-lattices inside respectively $\mathbb{V}_{\mathcal{Z}_r}(1-k)$ and $\mathbb{V}_{\mathcal{Z}_r}(1-k')$.
Let $\rho : G_{\mathbb{Q}_p} \to \mathrm{Gl}(\mathbb{T}_{\mathcal{Z}_r}) \cong \mathrm{Gl}_2(\mathcal{O}^{\circ}_{\mathcal{Z}_r})$ the Galois representations attached to $\mathbb{T}_{\mathcal{Z}_r}$. For every $s \in \mathcal{Z}_r$, we denote by $\rho_s : G_{\mathbb{Q}_p} \to \mathrm{Gl}(\mathbb{T}_{\mathcal{Z}_r}(s))$ the specialization of ρ at s. This representation correspond to a $G_{\mathbb{Q}_p}$-stable lattice inside the trianguline representation $V(\delta_1(s), \delta_2(s)) \cong \mathbb{V}_{\mathcal{Z}_r}(s) \cong \mathbb{V}(\Phi(s))$. In particular, we have that ρ_{1-k} and $\rho_{1-k'}$ correspond to $G_{\mathbb{Q}_p}$-stable lattices inside respectively the representations $V(\mu_{a_p}, \delta_{k,a_p})$ and $V(\mu_{a_p}, \delta_{k',a_p})$. Moreover, as we have already seen, the representations ρ_s can be obtained from ρ via composition by the evaluation map at s, i.e. we have $\rho_s = \mathrm{ev}_s \circ \rho$ where we keep ev_s as our notation for the induced map on Gl_2 from ev_s.
Now, for a fixed $m \in \mathbb{Z}_{\geq 1}$, we can consider the diagram:



where Pr_m denotes the induced homomorphism on Gl_2 from the natural projection $\mathcal{O}_{\mathbb{E}} \twoheadrightarrow \mathcal{O}_{\mathbb{E}}/(p^m)$.
It is clear that the above diamond in the diagram commutes if and only if for all $f \in \mathcal{O}^{\circ}_{\mathcal{Z}_r}$ we have $f(1-k) - f(1-k') \in (p^m)$ inside $\mathcal{O}_{\mathbb{E}}$.
Hence, we reduced the claim of Theorem 2.4.1 to prove that there exists a positive integer $n = n(k, a_p, m) \geq 1$ such that if $k' - k \in p^n(p-1)\mathbb{Z}_{\geq 0}$ then $|\mathrm{ev}_{1-k'}(f) - \mathrm{ev}_{1-k}(f)| = |f(1-k') - f(1-k)| \leq p^{-m}$.
This follows from the general following (this is just a slight variation of Prop. 7.2.1.1 in [BGR]):

Lemma 2.4.9. *Let $r \geq 0$ be an integer. For every $g \in \overline{\mathbb{Q}}_p\langle \frac{T}{p^r} \rangle := \{\sum_n a_n T^n : a_n p^{rn} \to 0$ as $n \to \infty\}$ and for any $x, y \in \mathbb{B}^1(0, p^{-r})^+$ we have*

$$|g(x) - g(y)| \leq p^r |g|_r |x - y|.$$

Here $|\cdot|_r$ denotes the norm on $\overline{\mathbb{Q}}_p\langle \frac{T}{p^r} \rangle$ given by $|\sum a_n T^n| := max\ |a_n p^{rn}|$ and $|\cdot|$ denotes the usual norm on $\overline{\mathbb{Q}}_p\langle T \rangle$.

Proof. Consider the map $\alpha : \overline{\mathbb{Q}}_p\langle\frac{T}{p^r}\rangle \to \overline{\mathbb{Q}}_p\langle T\rangle$ sending T to p^rT; it is an isometric isomorphism with respect to the norms $|\cdot|_r$ and $|\cdot|$ respectively. Denote by α^* the induced map on maximal spectra, i.e. $\alpha^* : \mathbb{B}^1(0,1)^+ \to \mathbb{B}^1(0,p^{-r})^+$ sending z to p^rz; it is bijective. Let $g \in \overline{\mathbb{Q}}_p\langle\frac{T}{p^r}\rangle$ such that $\alpha(g) = f$.
It is a classical result in the theory of Tate's algebras (see Prop. 7.2.1.1 in [BGR]) that for any $f \in \overline{\mathbb{Q}}_p\langle T\rangle$ and for any $\tilde{x}, \tilde{y} \in \mathbb{B}^1(0,1)^+$:

$$|f(\tilde{x}) - f(\tilde{y})| \leq |f||\tilde{x} - \tilde{y}|.$$

Since α is an isometry and defining $\alpha^*(\tilde{x}) = x$ and $\alpha^*(\tilde{y}) = y$, this is equivalent to say:

$$\begin{aligned}
&|\alpha(g)(\tilde{x}) - \alpha(g)(\tilde{y})| \leq |f| \cdot |\tilde{x} - \tilde{y}| \\
\Longleftrightarrow &|g(\alpha^*(\tilde{x})) - g(\alpha^*(\tilde{y}))| \leq |\alpha(g)| \cdot |\tilde{x} - \tilde{y}| \\
\Longleftrightarrow &|g(x) - g(y)| \leq |g|_r \cdot |\tilde{x} - \tilde{y}| = |g|_r \cdot \left|\frac{x}{p^r} - \frac{y}{p^r}\right| = p^r|g|_r|x - y|.
\end{aligned}$$

Note that we used the fact that $\alpha(g)(\tilde{x}) = g(\alpha^*(\tilde{x}))$ (and the same for y) which is a standard property of affinoid maps (see Lemma 7.1.4.2 in [BGR]). □

Finally, we can complete the proof of Theorem 2.4.1. Indeed, we have that the model $\mathcal{O}^o_{\mathcal{Z}_r}$ is isomorphc to $\mathbb{E}\langle\frac{T}{p^r}\rangle^\circ := \{g \in \mathbb{E}\langle\frac{T}{p^r}\rangle : |g|_{\sup} = |g|_r \leq 1\}$. Hence, for all $g \in \mathbb{E}\langle\frac{T}{p^r}\rangle^\circ$ we have: $|g(x) - g(y)| \leq p^r|x - y| \quad$ for all $x, y \in \overline{\mathbb{Z}}_p$ representing the corresponding maximal ideals in $\mathcal{Z}_r$. Note that we are considering fixed an embedding of $\mathbb{E}$ in $\overline{\mathbb{Q}}_p$.
For any fixed positive integer m such that the hypothesis of the theorem holds, there exists a positive integer n, namely $n = m + r$, such that the representations $\mathbb{T}_{\mathcal{Z}_r}(1-k)$ and $\mathbb{T}_{\mathcal{Z}_r}(1-k')$ are congruent modulo p^m. By the definition of $\mathbb{T}_{\mathcal{Z}_r}$ we deduce that the same is true for $\mathbb{T}(u_{1-k})$ and $\mathbb{T}(u_{1-k'})$. This completes the proof of Theorem 2.4.1.

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