Robust Multidimensional Pricing: Separation without Regret

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We study a robust monopoly pricing problem with a minimax regret objective, where a seller endeavors to sell multiple goods to a single buyer, only knowing that the buyer's values for the goods range over a rectangular uncertainty set. We interpret this pricing problem as a zero-sum game between the seller, who chooses a selling mechanism, and a fictitious adversary or 'nature', who chooses the buyer's values from within the uncertainty set. Using duality techniques rooted in robust optimization, we prove that this game admits a Nash equilibrium in mixed strategies that can be computed in closed form. The Nash strategy of the seller is a *randomized posted price mechanism* under which the goods are sold *separately*, while the Nash strategy of nature is a distribution on the uncertainty set under which the buyer's values are *comonotonic*. We further show that the restriction of the pricing problem to deterministic mechanisms is solved by a *deteministic posted price mechanism* under which the goods are sold *separately*.

1. Introduction

We address the fundamental question of how much money one should charge for new products when there is only minimal information about the buyers' willingness to pay. More precisely, we study a robust monopoly pricing problem, where a seller ("she") endeavors to sell multiple indivisible goods to a single buyer ("he"). The buyer assigns each good a private value, which reflects the maximum amount of money he would be willing to pay for this good. The value assigned to a bundle (*i.e.,* a

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set of multiple goods) equals the sum of the included goods' values. The seller perceives the overall value profile (*i.e.*, the list of values for *all* goods) as an uncertain parameter that is only known to range over a rectangular uncertainty set spanned by the origin and a vector of non-negative upper bounds. This set-based uncertainty model is appropriate in the absence of any trustworthy distributional information or when the acquisition of such information—*e.g.*, via market research or by observations of buyer behavior in prior sales—would be overly expensive or time-consuming.

We assume that the seller aims to design a mechanism for liquidating the goods with the goal to minimize her worst-case regret. The regret of a mechanism is defined as the difference between the hypothetical revenues that could have been realized under full knowledge of the buyer's value profile and the actual revenues generated by the mechanism. The worst-case regret is obtained by maximizing the realized regret across all possible value profiles in the uncertainty set. The minimax regret criterion was introduced by Savage (1951) and captures the idea that decision makers have a low tolerance for missing out on opportunities to earn revenue. It is less pessimistic than the ordinary maxmin criterion commonly used in robust optimization, which seeks mechanisms that generate maximum revenues under the worst possible value profile in the uncertainty set.

The family of possible selling mechanism is vast. For example, the seller could set individual posted prices for different bundles and ask the buyer to self-select his preferred price-bundle-pair. More generally, the seller could offer the buyer a menu of lotteries for winning the goods with different probabilities, set an individual price (or participation fee) for each lottery, and ask the buyer to self-select his preferred price-lottery-pair.

The classical mechanism design literature models the buyer's value profile as a random vector that is governed by a known probability distribution. If there is only one good, it is well known that setting a deterministic posted price (a take-it-or-leave-it offer) maximizes the seller's expected revenues (Myerson (1981), Riley and Zeckhauser (1983)). Moreover, the optimal posted price can be calculated analytically. In the presence of multiple goods, on the other hand, the expected revenue maximizing mechanism is notoriously difficult to characterize and compute. Even if the

buyer's values are independent across the goods and his utility function is quasilinear and additively separable, offering discounts on bundles and using randomized allocation rules can yield strictly higher expected revenues than selling the goods separately (see, *e.g.*, Manelli and Vincent (2006) or Thanassoulis (2004)). Daskalakis et al. (2014) show that, under standard complexity theoretic assumptions, the multidimensional mechanism design problem with expected revenue objective admits no expected polynomial-time solution algorithm even in unrealistically simple settings where the buyer's values are independently distributed on two rational numbers with rational probabilities. Thus, the generic multidimensional mechanism design problem is severely intractable. Nevertheless, closed-form solutions are available for special probability distributions and/or for small numbers of goods (see, *e.g.*, Bhargava (2013), Daskalakis et al. (2013), Giannakopoulos and Koutsoupias (2014) and Daskalakis et al. (2017)). Moreover, under the restrictive assumption that the buyer's values are independent, simple mechanisms (such as selling the goods separately or as a single bundle) provide constant-factor approximations to the expected revenue of the unknown optimal mechanism (see, *e.g.*, Hart and Nisan (2017a) or Li and Yao (2013)). If the buyer's values are correlated, the optimal mechanism becomes even hard to approximate. Indeed, Hart and Nisan (2017b) show that the optimal mechanism for selling more than one good may involve a menu of infinitely many price-lottery-pairs and that no deterministic mechanism can guarantee to extract any positive fraction of the optimal expected revenue. This implies that the seller can be significantly worse off by setting deterministic posted prices for the bundles instead of implementing an optimal mechanism. Note that this inapproximability result holds in spite of the quasilinearity and additive separability of the buyer's utility function.

Modeling the uncertainty in the buyer's value profile through a crisp distribution not only compromises the problem's computational tractability but is also difficult to justify in situations when the demand is poorly understood. A recent stream of literature thus investigates the impact of distributional uncertainty or *ambiguity* on pricing problems. Most existing studies focus on the single-item case and assume that the seller aims to maximize her worst-case expected revenues

in view of all distributions in some *ambiguity set* (see, *e.g.*, Bergemann and Schlag (2011), Carrasco et al. (2018) and Pınar and Kızılkale (2017)). By definition, the ambiguity set contains all distributions that are consistent with the seller's information about the buyer's value profile such as its support, its mean or certain higher-order moments. The maxmin expected revenue criterion results in non-trivial mechanisms only if the seller knows more about the value distribution than just its support. Otherwise, the worst-case expectation reduces to the worst-case realization of the revenue, in which case the underlying pricing problem becomes too conservative to be practically useful. In fact, if the lowest possible value the buyer assigns to any good is zero, then it would be optimal for the seller to keep all goods for herself. This observation prompts Bergemann and Schlag (2008) to study a single-item pricing problem with minimax regret objective. Assuming that there is only support information, they show that the seller's worst-case regret is minimized by setting a *randomized* posted price, whose distribution can be calculated in closed form. In addition, they also identify the best *deterministic* posted price under the minimax regret criterion. The multi-item pricing problem under ambiguity is perceived as challenging and has therefore received only limited attention in the literature. As a notable exception, Carroll (2017) explicitly characterizes the optimal mechanism of a screening problem with maxmin expected revenue objective, where the marginal distributions of the agent's multidimensional type are precisely known to the principal, while their dependence structure (or copula) remains uncertain. In the special case of monopoly pricing when only the marginal distributions of the buyer's values are known, Carroll (2017) shows that the seller does not benefit from bundling and that it is optimal to post a deterministic price for each good separately. Gravin and Lu (2018) show that this separation result continues to hold even if the buyer has a budget for his total payment.

This paper contributes to the rapidly expanding literature on mechanism design from the perspective of mathematical optimization (see, *e.g.*, Vohra (2012), Bichler (2017), Lopomo et al. (2011) or Fanzeres et al. (2019)) and endeavors to further our understanding of multi-item pricing under extreme ambiguity. Specifically, we postulate that the buyer's value profile may follow any distribution on a given rectangular uncertainty set. This assumption leads to pricing problems that can

be analyzed with methods from modern robust optimization (see, *e.g.*, Ben-Tal et al. (2009) or Bertsimas et al. (2011)). If the seller aims to maximize her worst-case revenue, for example, then the robust pricing problem can be interpreted as a robust auction design problem with a single bidder. Bandi and Bertsimas (2014) describe an efficient numerical solution to this problem for any number of bidders. For a single bidder, however, the optimal mechanism collapses to the trivial mechanism under which the seller keeps all items for herself. To mitigate the conservatism of robust pricing, we assume here that the seller minimizes her worst-case regret. In contrast to the single-item pricing model by Bergemann and Schlag (2008), which optimizes over all randomized posted prices, we formulate the robust multi-item pricing problem as an explicit mechanism design problem that searches over all incentive compatible and individually rational allocation and payment rules. While one can show that the two formulations are essentially equivalent in the single-item case (all single-item mechanisms with a right-continuous allocation rule give rise to a randomized posted price), only the explicit formulation advocated here easily generalizes to multiple items. Indeed, randomized posted price mechanisms for multiple goods would already be cumbersome to characterize; they would require a specification of separate (possibly correlated) posted prices for all possible bundles. Moreover, in order to evaluate the seller's worst-case regret, one would have to anticipate the buyer's preferred bundle for each realization of the posted prices and compute the expectation of an implicitly defined piecewise linear function with exponentially many pieces, which seems excruciating.

We highlight the following main contributions of this paper.

1. We formulate the multi-item pricing problem with minimax regret objective as an adaptive robust optimization problem. While such problems are generically NP-hard (Guslitser 2002, Theorem 3.5) and typically only solved approximately using linear decision rules, we show that the pricing problem at hand admits an explicit analytical solution in piecewise logarithmic decision rules. This solution is obtained by leveraging duality techniques rooted in robust optimization, and it represents a randomized mechanism under which the goods are sold *separately*.

2. The robust pricing problem can be interpreted as a zero-sum game between the seller and a fictitious adversary or 'nature', who chooses the buyer's value profile in the uncertainty set with the aim to inflict maximum damage. We demonstrate that this game admits a Nash equilibrium in mixed strategies, which can be computed in closed form. The Nash strategy of the seller coincides with the optimal randomized mechanism, while the Nash strategy of nature is a (non-discrete) distribution on the uncertainty set under which the components of the value profile are comonotonic.

3. We study a restriction of the robust pricing problem that optimizes only over deterministic mechanisms, which is essentially equivalent to searching over all posted price mechanisms. We solve this problem analytically and show that the different goods are again sold *separately* at optimality.

4. We demonstrate that the single-item pricing theory by Bergemann and Schlag (2008) emerges as a special case of the proposed multi-item pricing theory.

The remainder of this paper is structured as follows. Section 2 reviews key microeconomic concepts and formulates the robust multi-item pricing problem as an abstract mechanism design problem. By using duality techniques from robust optimization, Section 3 solves the general pricing problem in closed form and shows that a separable randomized posted price mechanism is optimal. Section 4 solves a restriction of the pricing problem that optimizes only over deterministic mechanisms and shows that a separable deterministic posted price mechanism is optimal.

Notation. For any $\mathcal{S} \subseteq \mathcal{J} = \{1, \ldots, J\}$, the vector $\mathbf{1}_{\mathcal{S}} \in \mathbb{R}^J$ is defined through $(\mathbf{1}_{\mathcal{S}})_j = 1$ if $j \in \mathcal{S}$; $= 0$ if $j \in \mathcal{J} \backslash \mathcal{S}$. We use $\mathbf{1}$ as a shorthand for $\mathbf{1}_{\mathcal{J}}$. Similarly, for any $\boldsymbol{v} \in \mathbb{R}^J$, the vector $\boldsymbol{v}^+ \in \mathbb{R}^J$ is defined through $(\boldsymbol{v}^+)_j = \max\{v_j, 0\}$, $j \in \mathcal{J}$. For a logical expression $\mathcal{E}$, we define $\mathbb{1}_{\mathcal{E}} = 1$ if $\mathcal{E}$ is true; $= 0$ otherwise. For any Borel set $\mathcal{A} \subseteq \mathbb{R}^I$, $\Delta(\mathcal{A})$ represents the family of all probability distributions on $\mathcal{A}$. The set of all bounded Borel-measurable functions from a Borel set $\mathcal{D} \subseteq \mathbb{R}^J$ to a Borel set $\mathcal{R} \subseteq \mathbb{R}^J$ is denoted by $\mathcal{L}(\mathcal{D}, \mathcal{R})$. Random variables are designated by tildes (*e.g.*, $\tilde{\boldsymbol{v}}$), and their realizations are denoted by the same symbols without tildes (*e.g.*, $\boldsymbol{v}$).

2. Problem Formulation and Preliminaries

We consider the problem of designing a mechanism for selling $J \in \mathbb{N}$ different items to a single buyer. The set of items is denoted by $\mathcal{J} = \{1, 2, \ldots, J\}$. The buyer assigns each item $j \in \mathcal{J}$ a

value v_j that reflects his willingness to pay. While the buyer has full knowledge of his value profile $\boldsymbol{v} = (v_1, \ldots, v_J)^\top$, the seller perceives $\boldsymbol{v}$ as an uncertain parameter. Specifically, we assume that the seller only knows an upper bound $\overline{v}_j$ on the buyer's value v_j for each $j \in \mathcal{J}$. However, she has no information about the distribution of $\boldsymbol{v}$ or suspects that any available information is not trustworthy. In the following, we denote by $\mathcal{V} = \times_{j \in \mathcal{J}} [0, \overline{v}_j]$ the uncertainty set of the buyer's value profiles. The seller incurs a cost $c_j \in \mathbb{R}_+$ for supplying item j to the buyer. This cost may capture the expenses for producing and/or delivering the item. We denote by $\boldsymbol{c} = (c_1, \ldots, c_J)^\top$ the seller's cost vector. Without loss of generality, we assume that $\boldsymbol{c} < \overline{\boldsymbol{v}} = (\overline{v}_1, \ldots, \overline{v}_J)^\top$. This assumption is justified more rigorously in Remark 2 of Section 3.

The sale proceeds as follows. First, the seller announces a mechanism $(\mathcal{B}, \boldsymbol{q}, m)$ that consists of a set $\mathcal{B}$ of *messages* available to the buyer, an *allocation rule* $\boldsymbol{q} = (q_1, \ldots, q_J)^\top : \mathcal{B} \to [0, 1]^J$ and a *payment rule* $m : \mathcal{B} \to \mathbb{R}$. Next, the buyer transmits a message $\boldsymbol{b} \in \mathcal{B}$ to the seller. Depending on this message, the seller then allocates item j to the buyer with probability $q_j(\boldsymbol{b})$ for each $j \in \mathcal{J}$ in return for a total payment equal to $m(\boldsymbol{b})$. If the goods are divisible, we can alternatively interpret $q_j(\boldsymbol{v})$ as the proportion of item j acquired by the buyer.

We assume that the buyer is risk-neutral with respect to the randomness of the allocation rule. Thus, the buyer's expected utility coincides with the expected profit $\boldsymbol{q}(\boldsymbol{b})^\top \boldsymbol{v} - m(\boldsymbol{b})$ and is additively separable with respect to the items. Given a mechanism, the buyer selects his message strategically depending on his value profile $\boldsymbol{v}$ so as to maximize his utility, *i.e.*, he reports $\boldsymbol{b}^\star(\boldsymbol{v}) \in \arg\max_{\boldsymbol{b} \in \mathcal{B}} \boldsymbol{q}(\boldsymbol{b})^\top \boldsymbol{v} - m(\boldsymbol{b})$.

A mechanism $(\mathcal{B}, \boldsymbol{q}, m)$ is called *direct* if the set $\mathcal{B}$ of messages coincides with the set $\mathcal{V}$ of value profiles. We henceforth use the shorthand $(\boldsymbol{q}, m)$ to denote any direct mechanism $(\mathcal{V}, \boldsymbol{q}, m)$ because there is no freedom in specifying the set of messages. A direct mechanism $(\boldsymbol{q}, m)$ is called *incentive compatible* if the buyer's optimal strategy is to truthfully report his value profile $\boldsymbol{v}$.

DEFINITION 1 (INCENTIVE COMPATIBILITY). A direct mechanism $(\boldsymbol{q}, m)$ is incentive compatible if

$$\boldsymbol{q}(\boldsymbol{v})^\top \boldsymbol{v} - m(\boldsymbol{v}) \geq \boldsymbol{q}(\boldsymbol{w})^\top \boldsymbol{v} - m(\boldsymbol{w}) \quad \forall \boldsymbol{v}, \boldsymbol{w} \in \mathcal{V}. \tag{IC}$$

The incentive compatibility constraint (IC) formalizes the requirement that reporting the true values $\boldsymbol{v}$ should result in the highest expected utility to the buyer. By virtue of the celebrated Revelation Principle due to Myerson (1981), the seller can restrict attention to incentive compatible direct mechanisms without loss of generality. The intuition behind the Revelation Principle is as follows. Given any mechanism $(\mathcal{B}, \boldsymbol{q}, m)$, the seller can construct an equivalent incentive compatible direct mechanism $(\boldsymbol{q}', m')$ by asking the buyer to report his true value profile, allocating the items according to the rule $\boldsymbol{q}'(\boldsymbol{v}) = \boldsymbol{q}(\boldsymbol{b}^\star(\boldsymbol{v}))$ and charging a payment $m'(\boldsymbol{v}) = m(\boldsymbol{b}^\star(\boldsymbol{v}))$ as if the buyer had reported his optimal message $\boldsymbol{b}^\star(\boldsymbol{v})$ for the original mechanism $(\mathcal{B}, \boldsymbol{q}, m)$. Thus, the buyer has no incentive to misreport his value profile. Moreover, the ex-post outcomes are identical under the mechanisms $(\boldsymbol{q}', m')$ and $(\mathcal{B}, \boldsymbol{q}, m)$.

The buyer will participate in the sale only if his utility is non-negative. In order to prevent the buyer from walking away, the seller should thus focus on *individually rational* mechanisms.

DEFINITION 2 (INDIVIDUAL RATIONALITY). A direct mechanism $(\boldsymbol{q}, m)$ is individually rational if

$$\boldsymbol{q}(\boldsymbol{v})^\top \boldsymbol{v} - m(\boldsymbol{v}) \geq 0 \quad \forall \boldsymbol{v} \in \mathcal{V}. \tag{IR}$$

The individual rationality constraint (IR) guarantees that the buyer's utility under truthful reporting is non-negative irrespective of his value profile $\boldsymbol{v}$. The seller can restrict attention to individually rational mechanisms without loss of generality. Indeed, assume that $(\boldsymbol{q}, m)$ is an incentive compatible mechanism that results in a negative utility for the buyer and thus to cancellation of the sale under some value profiles. In this case, the seller can construct an equivalent individually rational mechanism $(\boldsymbol{q}', m')$ defined through $\boldsymbol{q}' = \boldsymbol{q}\, \mathbb{1}_{\boldsymbol{q}(\boldsymbol{v})^\top \boldsymbol{v} - m(\boldsymbol{v}) \geq 0}$ and $m'(\boldsymbol{v}) = m(\boldsymbol{v})\, \mathbb{1}_{\boldsymbol{q}(\boldsymbol{v})^\top \boldsymbol{v} - m(\boldsymbol{v}) \geq 0}$ for all $\boldsymbol{v} \in \mathcal{V}$, which sets the allocation probabilities and the payment to zero whenever the original mechanism would result in a cancellation of the sale. This mechanism is still incentive compatible

because

$$\sup_{\boldsymbol{w}\in\mathcal{V}} \boldsymbol{q}'(\boldsymbol{w})^\top \boldsymbol{v} - m'(\boldsymbol{w}) = \sup_{\boldsymbol{w}\in\mathcal{V}} \left[\boldsymbol{q}(\boldsymbol{w})^\top \boldsymbol{v} - m(\boldsymbol{w})\right] \mathbb{1}_{\boldsymbol{q}(\boldsymbol{w})^\top \boldsymbol{w} - m(\boldsymbol{w}) \geq 0}$$

$$\leq \sup_{\boldsymbol{w}\in\mathcal{V}} \left[\boldsymbol{q}(\boldsymbol{w})^\top \boldsymbol{v} - m(\boldsymbol{w})\right]^+ \mathbb{1}_{\boldsymbol{q}(\boldsymbol{w})^\top \boldsymbol{w} - m(\boldsymbol{w}) \geq 0}$$

$$\leq \sup_{\boldsymbol{w}\in\mathcal{V}} \left[\boldsymbol{q}(\boldsymbol{w})^\top \boldsymbol{v} - m(\boldsymbol{w})\right]^+$$

$$= \left[\boldsymbol{q}(\boldsymbol{v})^\top \boldsymbol{v} - m(\boldsymbol{v})\right]^+ = \boldsymbol{q}'(\boldsymbol{v})^\top \boldsymbol{v} - m'(\boldsymbol{v}).$$

Moreover, the ex-post outcomes are identical under the mechanisms $(\boldsymbol{q}', m')$ and $(\boldsymbol{q}, m)$ when correctly accounting for the walk-away option.

Throughout the rest of the paper, without loss of generality, we focus only on direct mechanisms that are both incentive compatible and individually rational.

The seller's ex-post regret is defined as the difference between the maximum profit that could have been realized under complete information about $\boldsymbol{v}$ and the expected profit $m(\boldsymbol{v}) - \boldsymbol{q}(\boldsymbol{v})^\top \boldsymbol{c}$ earned with the mechanism $(\boldsymbol{q}, m)$. If the seller was fully aware of the buyer's willingness to pay, she would sell item j at price v_j whenever $v_j \geq c_j$ and would keep the item otherwise. The maximum profit under complete information can thus be expressed as $\boldsymbol{1}^\top (\boldsymbol{v} - \boldsymbol{c})^+$, while the ex-post regret equals $\boldsymbol{1}^\top (\boldsymbol{v} - \boldsymbol{c})^+ - (m(\boldsymbol{v}) - \boldsymbol{q}(\boldsymbol{v})^\top \boldsymbol{c})$. The worst-case regret is obtained by maximizing the ex-post regret over all value profiles $\boldsymbol{v} \in \mathcal{V}$.

Throughout this paper we assume that the seller aims to design an incentive compatible and individually rational mechanism that minimizes the worst-case regret. This mechanism design problem can be formalized as follows.

$$z^\star = \inf_{\boldsymbol{q}, m} \ \sup_{\boldsymbol{v}\in\mathcal{V}} \ \boldsymbol{1}^\top (\boldsymbol{v} - \boldsymbol{c})^+ - \left(m(\boldsymbol{v}) - \boldsymbol{q}(\boldsymbol{v})^\top \boldsymbol{c}\right)$$

$$\text{s.t.} \quad \boldsymbol{q} \in \mathcal{L}(\mathcal{V}, \mathbb{R}_+^J), \ m \in \mathcal{L}(\mathcal{V}, \mathbb{R})$$

$$\boldsymbol{q}(\boldsymbol{v})^\top \boldsymbol{v} - m(\boldsymbol{v}) \geq \boldsymbol{q}(\boldsymbol{w})^\top \boldsymbol{v} - m(\boldsymbol{w}) \ \ \forall \boldsymbol{v}, \boldsymbol{w} \in \mathcal{V} \qquad (\mathcal{MDP})$$

$$\boldsymbol{q}(\boldsymbol{v})^\top \boldsymbol{v} - m(\boldsymbol{v}) \geq 0 \ \ \forall \boldsymbol{v} \in \mathcal{V}$$

$$\boldsymbol{q}(\boldsymbol{v}) \leq \boldsymbol{1} \ \ \forall \boldsymbol{v} \in \mathcal{V}$$

The last constraint in $(\mathcal{MDP})$ ensures that each item is sold at most once. In the following, we use will the shorthand $\mathcal{X}$ to denote the set of all mechanisms feasible in $(\mathcal{MDP})$.

REMARK 1. Problem $(\mathcal{MDP})$ can be interpreted as a zero-sum game between the seller, who chooses the mechanism $(\boldsymbol{q}, m)$, and some fictitious adversary or nature, who chooses the buyer's value profile $\boldsymbol{v}$ with the goal to inflict maximum damage to the seller. As we allow for randomized allocation rules, the seller plays a mixed strategy and thus solves a convex minimization problem. Nature, on the other hand, chooses a pure strategy from within the uncertainty set $\mathcal{V}$ but solves a *non*-convex maximization problem. This problem can be convexified by allowing nature to play a mixed strategy $\mathbb{P} \in \Delta(\mathcal{V})$, thereby replacing the non-convex inner maximization problem in $(\mathcal{MDP})$ with an infinite-dimensional linear program.

$$z^{\star} = \inf_{(\boldsymbol{q},m)\in\mathcal{X}} \ \sup_{\mathbb{P}\in\Delta(\mathcal{V})} \ \mathbb{E}_{\mathbb{P}}\left[\mathbf{1}^{\top}(\tilde{\boldsymbol{v}}-\boldsymbol{c})^{+} - \left(m(\tilde{\boldsymbol{v}})-\boldsymbol{q}(\tilde{\boldsymbol{v}})^{\top}\boldsymbol{c}\right)\right] \tag{1}$$

Problem (1) is clearly equivalent to $(\mathcal{MDP})$ because $\Delta(\mathcal{V})$ contains all Dirac point measures supported on $\mathcal{V}$. Using this formulation, we will show below that the game between the seller and nature admits a Nash equilibrium in mixed strategies. $\quad\square$

The set of mechanisms feasible in $(\mathcal{MDP})$ is vast. Posted price mechanisms sell different bundles of the items at fixed posted prices. They range among the most popular selling mechanisms.

DEFINITION 3 (POSTED PRICE MECHANISM). A mechanism $(\boldsymbol{q}, m)$ is called a *posted price mechanism* if there exists a vector of posted prices $\boldsymbol{p} \in \mathbb{R}^{2^{J}}$ such that $\boldsymbol{q}(\boldsymbol{v}) = \mathbf{1}_{s(\boldsymbol{v})}$ and $m(\boldsymbol{v}) = p_{s(\boldsymbol{v})}$, where $s \in \mathcal{L}(\mathcal{V}, 2^{\mathcal{J}})$ represents a *bundle allocation rule* that satisfies $s(\boldsymbol{v}) \in \arg\max_{\mathcal{S}\subseteq\mathcal{J}} \mathbf{1}_{\mathcal{S}}^{\top}\boldsymbol{v} - p_{\mathcal{S}}$ for all $\boldsymbol{v} \in \mathcal{V}$.

The set-valued bundle allocation rule s maps any value profile $\boldsymbol{v} \in \mathcal{V}$ to a bundle $\mathcal{S} \subseteq \mathcal{J}$ that maximizes the utility of the buyer of type $\boldsymbol{v}$. It is almost uniquely determined by the vector of posted prices $\boldsymbol{p}$, but ties are broken at the discretion of the seller in cases when multiple bundles are optimal. One can prove that any posted price mechanism induced by $\boldsymbol{p}$ is individually rational if and only if $p_{\mathcal{S}} \leq 0$ for at least one bundle $\mathcal{S} \subseteq \mathcal{J}$. Any posted price mechanism satisfying this condition is feasible in $(\mathcal{MDP})$. Intuitively, the seller may implement a posted price mechanism by setting an individual price for each bundle and let the buyer choose the bundle that maximizes his
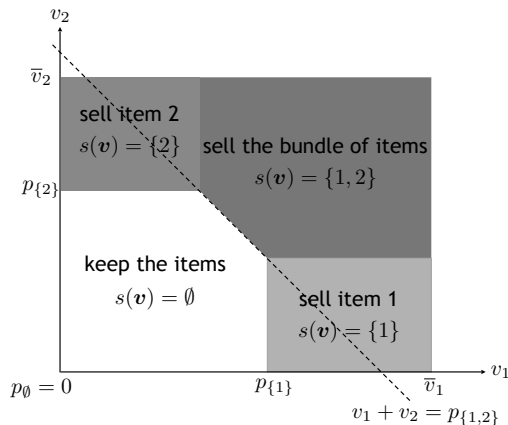
Figure 1   Visualization of the bundle allocation rule corresponding to a posted price mechanism with $\boldsymbol{p} =$ $[p_\emptyset, p_{\{1\}}, p_{\{2\}}, p_{\{1,2\}}]^\top$ and $p_\emptyset = 0$. If the buyer's value v_1 for item 1 exceeds $p_{\{1\}}$ but his value v_2 for item 2 falls short of $p_{\{1,2\}} - p_{\{1\}}$, then he receives only item 1 at price $p_{\{1\}}$. Similarly, if $v_1 + v_2 \geq p_{\{1,2\}}$, $v_1 \geq p_{\{1,2\}} - p_{\{2\}}$ and $v_2 \geq p_{\{1,2\}} - p_{\{1\}}$, then the buyer acquires the bundle $\{1,2\}$ at price $p_{\{1,2\}}$, etc.

utility. Figure 1 visualizes the allocation rule of a posted price mechanism for two items. A more in-depth discussion of posted price mechanisms is relegated to Section 4.

In the following sections, we will derive optimal randomized and deterministic mechanisms in closed form. Many of the subsequent results will depend on a well-know equivalent characterization of incentive compatible mechanisms for selling a single item due to Myerson and Satterthwaite (1983). Note that, for $J = 1$, the allocation rule reduces to a scalar function denoted by q.

PROPOSITION 1. *For $J = 1$, a mechanism (q, m) is incentive compatible if and only if*

(i) $q(v)$ is non-decreasing in $v \in \mathcal{V}$,

(ii) $m(v) = m(0) + q(v)v - \displaystyle\int_0^v q(x)\, \mathrm{d}x \quad \forall v \in \mathcal{V}$.

3. Optimal Mechanism

One particularly simple policy for the seller would be to sell each item individually via a separable mechanism that ignores the possibility of bundling.

DEFINITION 4 (SEPARABILITY). A mechanism $(\boldsymbol{q}, m)$ is called *separable* if there exists $\hat{q}_j \in \mathcal{L}([0, \overline{v}_j], [0,1])$ and $\hat{m}_j \in \mathcal{L}([0, \overline{v}_j], \mathbb{R})$ for all $j \in \mathcal{J}$ such that $\boldsymbol{q}(\boldsymbol{v}) = [\hat{q}_1(v_1), \ldots, \hat{q}_J(v_J)]^\top$ and $m(\boldsymbol{v}) = \sum_{j \in \mathcal{J}} \hat{m}_j(v_j)$ for all $\boldsymbol{v} \in \mathcal{V}$.

Under a separable mechanism, the sale of each item j is processed according to a separate single-item mechanism $(\hat{q}_j, \hat{m}_j)$ that depends exclusively on the value v_j.

In the remainder of this section we will investigate the separable mechanism $(\boldsymbol{q}^\star, m^\star)$ with corresponding single-item mechanisms $(\hat{q}_j, \hat{m}_j)$, $j \in \mathcal{J}$, defined through

$$(\hat{q}_j(v_j), \hat{m}_j(v_j)) = \begin{cases} \left(1 + \log\left(\frac{v_j - c_j}{\overline{v}_j - c_j}\right), v_j - \frac{(\overline{v}_j - c_j)}{e} + c_j \log\left(\frac{v_j - c_j}{\overline{v}_j - c_j}\right)\right) & \text{if } \frac{v_j - c_j}{\overline{v}_j - c_j} \geq \frac{1}{e}, \\ (0, 0) & \text{otherwise,} \end{cases} \tag{2}$$

for all $v_j \in [0, \overline{v}_j]$.

LEMMA 1. *The separable mechanism* $(\boldsymbol{q}^\star, m^\star)$ *defined via* (2) *is feasible in* $(\mathcal{MDP})$ *and attains an objective value of* $\frac{1}{e} \cdot \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c})$.

Proof. The proof is divided into two parts. We first show that the mechanism $(\boldsymbol{q}^\star, m^\star)$ is feasible in $(\mathcal{MDP})$ (Step 1), and then we calculate its objective value (Step 2).

Step 1: By the construction of $(\boldsymbol{q}^\star, m^\star)$ in (2), each component of $\boldsymbol{q}^\star(\boldsymbol{v})$ is non-negative and bounded above by 1 for all $\boldsymbol{v} \in \mathcal{V}$ because $-1 \leq \log\left(\frac{v_j - c_j}{\overline{v}_j - c_j}\right) \leq 0$ for all $v_j \leq \overline{v}_j$ with $\frac{v_j - c_j}{\overline{v}_j - c_j} \geq \frac{1}{e}$. Thus, we have $\boldsymbol{0} \leq \boldsymbol{q}^\star(\boldsymbol{v}) \leq \boldsymbol{1}$ for all $\boldsymbol{v} \in \mathcal{V}$.

Next, it is easy to see that $(\boldsymbol{q}^\star, m^\star)$ inherits incentive compatibility and individual rationality from the single-item mechanisms $(\hat{q}_j, \hat{m}_j)$, $j \in \mathcal{J}$. By Proposition 1, $(\hat{q}_j, \hat{m}_j)$ is incentive compatible if and only if the allocation rule $\hat{q}_j(v_j)$ is non-decreasing in v_j, which follows immediately from (2), while the payment rule satisfies

$$\hat{m}_j(v_j) = \hat{m}_j(0) + \hat{q}_j(v_j) v_j - \int_0^{v_j} \hat{q}_j(x) \, \mathrm{d}x \quad \forall v_j \in [0, \overline{v}_j].$$

This equality trivially holds when $\frac{v_j - c_j}{\overline{v}_j - c_j} < \frac{1}{e}$, in which case both sides reduce to 0. Moreover, for $\frac{v_j - c_j}{\overline{v}_j - c_j} \geq \frac{1}{e}$, the right-hand side of the above equation simplifies to

$$\begin{aligned} \hat{q}_j(v_j) v_j - \int_0^{v_j} \hat{q}_j(x) \, \mathrm{d}x &= \left(1 + \log\left(\frac{v_j - c_j}{\overline{v}_j - c_j}\right)\right) v_j - \int_{\frac{\overline{v}_j - c_j}{e} + c_j}^{v_j} 1 + \log\left(\frac{x - c_j}{\overline{v}_j - c_j}\right) \mathrm{d}x \\ &= \left(1 + \log\left(\frac{v_j - c_j}{\overline{v}_j - c_j}\right)\right) v_j - \left[(x - c_j) \log\left(\frac{x - c_j}{\overline{v}_j - c_j}\right)\right]_{\frac{\overline{v}_j - c_j}{e} + c_j}^{v_j} \\ &= v_j + c_j \log\left(\frac{v_j - c_j}{\overline{v}_j - c_j}\right) + \left(\frac{\overline{v}_j - c_j}{e}\right) \log\left(\frac{1}{e}\right), \end{aligned}$$

which manifestly equals $\hat{m}_j(v_j)$. Hence, the mechanism $(\hat{q}_j, \hat{m}_j)$ is incentive compatible as it satisfies both conditions of Proposition 1.

Finally, the mechanism $(\hat{q}_j, \hat{m}_j)$ is also individually rational because

$$\hat{q}_j(v_j)v_j - \hat{m}_j(v_j) = \int_0^{v_j} \hat{q}_j(x)\,\mathrm{d}x - \hat{m}_j(0) \geq 0,$$

where the equality follows again from Proposition 1, while the inequality holds because $\hat{m}_j(0) = 0$ and $\hat{q}_j$ is non-negative. This concludes Step 1.

Step 2: Thanks to its separability, the objective function value of $(q^\star, m^\star)$ can be expressed as

$$\sup_{v \in \mathcal{V}} \mathbf{1}^\top (v - c)^+ - \left(m^\star(v) - q^\star(v)^\top c\right) = \sum_{j \in \mathcal{J}} \sup_{0 \leq v_j \leq \overline{v}_j} r_j(v_j), \tag{3}$$

where $r_j(v_j) = (v_j - c_j)^+ - \hat{m}_j(v_j) + c_j\hat{q}_j(v_j)$. For each $j \in \mathcal{J}$, the supremum of $r_j(v_j)$ can be determined by distinguishing two cases as in (2). If $\frac{v_j - c_j}{\overline{v}_j - c_j} < \frac{1}{e}$, then both $\hat{q}_j(v_j)$ and $\hat{m}_j(v_j)$ vanish, and v_j belongs to the interval $[0, c_j + \frac{\overline{v}_j - c_j}{e})$. The supremum of $r_j(v_j) = (v_j - c_j)^+$ over this interval is attained at the interval's right boundary and amounts to $\frac{1}{e} \cdot (\overline{v}_j - c_j)$. If $\frac{v_j - c_j}{\overline{v}_j - c_j} \geq \frac{1}{e}$, on the other hand, we find

$$r_j(v_j) = (v_j - c_j) - \left(v_j - \frac{\overline{v}_j - c_j}{e} + c_j \log\left(\frac{v_j - c_j}{\overline{v}_j - c_j}\right)\right) + c_j\left(1 + \log\left(\frac{v_j - c_j}{\overline{v}_j - c_j}\right)\right)$$

$$= \frac{1}{e} \cdot (\overline{v}_j - c_j).$$

In summary, we have that $r_j(v_j) \leq \frac{1}{e} \cdot (\overline{v}_j - c_j)$ for all $v_j \in [0, \overline{v}_j]$ and that this inequality is tight for all $v_j \in [c_j + \frac{\overline{v}_j - c_j}{e}, \overline{v}_j]$. Thus the objective value (3) of the mechanism $(q^\star, m^\star)$ simplifies to $\frac{1}{e} \cdot \mathbf{1}^\top (\overline{v} - c)$. This observation completes the proof. $\square$

Next, we will show that the mechanism $(q^\star, m^\star)$ is not only feasible but also optimal in $(\mathcal{MDP})$. To this end, consider the following discrete approximation of $(\mathcal{MDP})$.

$$\begin{aligned}
z_n^\star = \inf \quad & \sup_{v \in \mathcal{V}_n} \mathbf{1}^\top (v - c)^+ - \left(m(v) - q(v)^\top c\right) \\
\text{s.t.} \quad & q \in \mathcal{L}(\mathcal{V}_n, \mathbb{R}_+^J), \, m \in \mathcal{L}(\mathcal{V}_n, \mathbb{R}) \\
& q(v)^\top v - m(v) \geq 0 && \forall v \in \mathcal{V}_n \\
& q(v)^\top v - m(v) \geq q(w)^\top v - m(w) && \forall v, w \in \mathcal{V}_n \\
& q(v) \leq \mathbf{1} && \forall v \in \mathcal{V}_n
\end{aligned} \tag{4}$$

Problem (4) differs from ($\mathcal{MDP}$) only in that it involves a discrete uncertainty set $\mathcal{V}_n = \times_{j \in \mathcal{J}} \mathcal{V}_{n,j}$, where $\mathcal{V}_{n,j} = \left\{ c_j, \frac{1}{n}(\overline{v}_j - c_j) + c_j, \frac{2}{n}(\overline{v}_j - c_j) + c_j, \ldots, \overline{v}_j \right\}$ represents a uniform one-dimensional grid with $n+1$ discretization points for some $n \in \mathbb{N}$. Note also that any (scalar) function defined on $\mathcal{V}_n$ corresponds to an $(n+1)^J$-dimensional vector.

LEMMA 2. *For any $n \in \mathbb{N}$, we have $z_n^\star \leq z^\star$.*

Proof. By construction it is clear that $\mathcal{V}_n \subseteq \mathcal{V}$. Thus, the objective function of (4) is majorized by that of ($\mathcal{MDP}$) uniformly across all $\boldsymbol{q}$ and m, and the feasible set of (4) contains that of ($\mathcal{MDP}$) as it relaxes all constraints associated with value profiles $\boldsymbol{v} \in \mathcal{V} \setminus \mathcal{V}_n$. The optimal value of (4) is therefore non-inferior to that of ($\mathcal{MDP}$). □

As its objective function is convex and piecewise linear, problem (4) can be reformulated as an equivalent finite linear program of the form

$$
\begin{aligned}
z_n^\star = \ \inf \quad & r \\[1mm]
\text{s.t.} \quad & \boldsymbol{q} \in \mathcal{L}(\mathcal{V}_n, \mathbb{R}_+^J),\, m \in \mathcal{L}(\mathcal{V}_n, \mathbb{R}),\, r \in \mathbb{R} \\[1mm]
& r \geq \mathbf{1}^\top (\boldsymbol{v} - \boldsymbol{c})^+ - \big(m(\boldsymbol{v}) - \boldsymbol{q}(\boldsymbol{v})^\top \boldsymbol{c}\big) \quad && \forall \boldsymbol{v} \in \mathcal{V}_n \\[1mm]
& \boldsymbol{q}(\boldsymbol{v})^\top \boldsymbol{v} - m(\boldsymbol{v}) \geq 0 \quad && \forall \boldsymbol{v} \in \mathcal{V}_n \\[1mm]
& \boldsymbol{q}(\boldsymbol{v})^\top \boldsymbol{v} - m(\boldsymbol{v}) \geq \boldsymbol{q}(\boldsymbol{w})^\top \boldsymbol{v} - m(\boldsymbol{w}) \quad && \forall \boldsymbol{v}, \boldsymbol{w} \in \mathcal{V}_n \\[1mm]
& \boldsymbol{q}(\boldsymbol{v}) \leq \mathbf{1} \quad && \forall \boldsymbol{v} \in \mathcal{V}_n,
\end{aligned}
\tag{5}
$$

where r represents an auxiliary epigraphical variable.

The linear program dual to (5) is given by

$$
\begin{aligned}
z_n^\star = \ \sup \quad & \sum_{j \in \mathcal{J}} \sum_{\boldsymbol{v} \in \mathcal{V}_n} \alpha(\boldsymbol{v})(v_j - c_j)^+ - \sum_{j \in \mathcal{J}} \sum_{\boldsymbol{v} \in \mathcal{V}_n} \lambda_j(\boldsymbol{v}) \\[1mm]
\text{s.t.} \quad & \alpha, \beta \in \mathcal{L}(\mathcal{V}_n, \mathbb{R}_+),\, \gamma \in \mathcal{L}(\mathcal{V}_n \times \mathcal{V}_n, \mathbb{R}_+),\, \boldsymbol{\lambda} \in \mathcal{L}(\mathcal{V}_n, \mathbb{R}_+^J) \\[1mm]
& \sum_{\boldsymbol{v} \in \mathcal{V}_n} \alpha(\boldsymbol{v}) = 1 \\[1mm]
& \beta(\boldsymbol{v}) + \sum_{\boldsymbol{w} \in \mathcal{V}_n} \gamma(\boldsymbol{v}, \boldsymbol{w}) - \sum_{\boldsymbol{w} \in \mathcal{V}_n} \gamma(\boldsymbol{w}, \boldsymbol{v}) = \alpha(\boldsymbol{v}) \quad && \forall \boldsymbol{v} \in \mathcal{V}_n \\[1mm]
& \lambda_j(\boldsymbol{v}) + \alpha(\boldsymbol{v}) c_j \geq \beta(\boldsymbol{v}) v_j + \sum_{\boldsymbol{w} \in \mathcal{V}_n} \gamma(\boldsymbol{v}, \boldsymbol{w}) v_j - \sum_{\boldsymbol{w} \in \mathcal{V}_n} \gamma(\boldsymbol{w}, \boldsymbol{v}) w_j \quad && \forall j \in \mathcal{J},\, \forall \boldsymbol{v} \in \mathcal{V}_n,
\end{aligned}
\tag{6}
$$

where α represents the dual variable of the epigraphical constraint, β and γ are the dual variables of the individual rationality and incentive compatibility constraints, respectively, and $\boldsymbol{\lambda}$ collects the dual variables of the upper probability bounds in (5). Strong duality holds because the trivial mechanism that sets $\boldsymbol{q}(\boldsymbol{v}) = \boldsymbol{0}$ and $m(\boldsymbol{v}) = 0$ for all $\boldsymbol{v} \in \mathcal{V}$ is feasible in (5) for every $r \geq \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c})$.

Since the linear program (6) seeks to make the decision variables $\lambda_j(\boldsymbol{v})$ as small as possible while ensuring that they remain non-negative and satisfy the last constraint of (6), it is clear that

$$
\begin{aligned}
\lambda_j(\boldsymbol{v}) &= \left(\beta(\boldsymbol{v}) v_j + \sum_{\boldsymbol{w} \in \mathcal{V}_n} \gamma(\boldsymbol{v}, \boldsymbol{w}) v_j - \sum_{\boldsymbol{w} \in \mathcal{V}_n} \gamma(\boldsymbol{w}, \boldsymbol{v}) w_j - \alpha(\boldsymbol{v}) c_j \right)^+ \\
&= \left(\alpha(\boldsymbol{v})(v_j - c_j) + \sum_{\boldsymbol{w} \in \mathcal{V}_n} \gamma(\boldsymbol{w}, \boldsymbol{v})(v_j - w_j) \right)^+ \qquad \forall j \in \mathcal{J}, \forall \boldsymbol{v} \in \mathcal{V}_n
\end{aligned}
$$

at optimality, where the second equality exploits the second equality constraint in (6) to eliminate $\beta(\boldsymbol{v})$. By substituting the above expression for $\lambda_j(\boldsymbol{v})$ into the objective function of problem (6), we then obtain the following equivalent non-linear program in the decision variables α and γ.

$$
\begin{aligned}
z_n^\star = \sup \quad & \sum_{j \in \mathcal{J}} \sum_{\boldsymbol{v} \in \mathcal{V}_n} \alpha(\boldsymbol{v})(v_j - c_j)^+ - \left(\alpha(\boldsymbol{v})(v_j - c_j) + \sum_{\boldsymbol{w} \in \mathcal{V}_n} \gamma(\boldsymbol{w}, \boldsymbol{v})(v_j - w_j) \right)^+ \\
\text{s.t.} \quad & \alpha \in \mathcal{L}(\mathcal{V}_n, \mathbb{R}_+), \ \gamma \in \mathcal{L}(\mathcal{V}_n \times \mathcal{V}_n, \mathbb{R}_+) \\
& \sum_{\boldsymbol{v} \in \mathcal{V}_n} \alpha(\boldsymbol{v}) = 1 \\
& \alpha(\boldsymbol{v}) + \sum_{\boldsymbol{w} \in \mathcal{V}_n} \gamma(\boldsymbol{w}, \boldsymbol{v}) - \sum_{\boldsymbol{w} \in \mathcal{V}_n} \gamma(\boldsymbol{v}, \boldsymbol{w}) \geq 0 \ \ \forall \boldsymbol{v} \in \mathcal{V}_n
\end{aligned}
\tag{7}
$$

Note that, by construction, the optimal objective value of (7) is still equal to $z_n^\star$.

Lemma 3 below constructs a feasible solution for problem (7) that asymptotically attains the objective value $\frac{1}{e} \cdot \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c})$ as n tends to infinity. This will allow us later to conclude that the separable mechanism $(\boldsymbol{q}^\star, m^\star)$ defined via (2) is indeed optimal in $(\mathcal{MDP})$.

LEMMA 3. *We have* $\liminf\limits_{n \to \infty} z_n^\star \geq \frac{1}{e} \cdot \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c})$.

Proof. For any $n \in \mathbb{N}$ satisfying $n > e(1+e)$, we define

$$
\alpha_n(\boldsymbol{v}) = \begin{cases} \frac{n}{ek(k+1)} & \text{if } \exists k \in \left\{ \lfloor \frac{n}{e} \rfloor, \ldots, n-1 \right\} \text{ with } \boldsymbol{v} - \boldsymbol{c} = \frac{k}{n}(\overline{\boldsymbol{v}} - \boldsymbol{c}), \\ 1 - \sum_{k=\lfloor \frac{n}{e} \rfloor}^{n-1} \frac{n}{ek(k+1)} & \text{if } \boldsymbol{v} = \overline{\boldsymbol{v}}, \\ 0 & \text{otherwise}, \end{cases}
$$

and

$$
\gamma_n(\boldsymbol{w}, \boldsymbol{v}) = \begin{cases} \frac{(n-e-e^2)n}{e(n-e)(k+1)} & \text{if } \exists k \in \left\{ \lfloor \frac{n}{e} \rfloor, \ldots, n-1 \right\} \text{ with } \boldsymbol{v} - \boldsymbol{c} = \frac{k}{n}(\overline{\boldsymbol{v}} - \boldsymbol{c}) \text{ and } \boldsymbol{w} = \boldsymbol{v} + \frac{1}{n}(\overline{\boldsymbol{v}} - \boldsymbol{c}), \\ 0 & \text{otherwise}. \end{cases}
$$

We will show that (α_n, γ_n) is feasible in (7) (Step 1) and yields a lower bound on $z_n^\star$ (Step 2). The claim then follows by showing that this lower bound converges to $\frac{1}{e} \cdot \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c})$ as n tends to infinity (Step 3).

Step 1: From the definitions of α_n and γ_n, it is easy to verify that $\sum_{\boldsymbol{v} \in \mathcal{V}_n} \alpha_n(\boldsymbol{v}) = 1$, while $\alpha_n(\boldsymbol{v}) \geq 0$ for all $\boldsymbol{v} \in \mathcal{V}_n \setminus \{\overline{\boldsymbol{v}}\}$ and $\gamma_n(\boldsymbol{w}, \boldsymbol{v}) \geq 0$ for all $\boldsymbol{w}, \boldsymbol{v} \in \mathcal{V}$. In addition, we observe that

$$
\begin{aligned}
\alpha_n(\overline{\boldsymbol{v}}) &= 1 - \sum_{k=\lfloor \frac{n}{e} \rfloor}^{n-1} \frac{n}{ek(k+1)} = 1 - \frac{n}{e} \left(\sum_{k=\lfloor \frac{n}{e} \rfloor}^{n-1} \frac{1}{k} - \frac{1}{k+1} \right) \\
&= 1 - \frac{n}{e} \left(\frac{1}{\lfloor \frac{n}{e} \rfloor} - \frac{1}{n} \right) > 1 - \frac{n}{n-e} + \frac{1}{e},
\end{aligned} \tag{8}
$$

where the third equality follows from the cancellation of all intermediate terms within the telescoping series, and the inequality holds because $\lfloor \frac{n}{e} \rfloor > \frac{n}{e} - 1$. The assumption $n > e(1+e)$ further ensures that

$$
1 - \frac{n}{n-e} + \frac{1}{e} = -\frac{e}{n-e} + \frac{1}{e} > 0.
$$

Hence, we may conclude that $\alpha_n(\overline{\boldsymbol{v}}) > 0$. To show that (α_n, γ_n) is feasible in (7), it thus remains to show that $\alpha_n(\boldsymbol{v}) + \sum_{\boldsymbol{w} \in \mathcal{V}_n} \gamma_n(\boldsymbol{w}, \boldsymbol{v}) - \sum_{\boldsymbol{w} \in \mathcal{V}_n} \gamma_n(\boldsymbol{v}, \boldsymbol{w}) \geq 0$ for all $\boldsymbol{v} \in \mathcal{V}_n$. By the definitions of α_n and γ_n, it suffices to show that this inequality holds when there exists an integer $k \in \left\{ \lfloor \frac{n}{e} \rfloor, \ldots, n \right\}$ with $\boldsymbol{v} - \boldsymbol{c} = \frac{k}{n}(\overline{\boldsymbol{v}} - \boldsymbol{c})$. Otherwise, the left-hand side of the inequality trivially evaluates to 0. When $k = \lfloor \frac{n}{e} \rfloor$, we have $\gamma_n(\boldsymbol{v}, \boldsymbol{w}) = 0$ for all $\boldsymbol{w} \in \mathcal{V}_n$, and the postulated inequality indeed holds because

$\alpha_n(\boldsymbol{v}) \geq 0$ and $\gamma_n(\boldsymbol{w}, \boldsymbol{v}) \geq 0$ for all $\boldsymbol{w} \in \mathcal{V}_n$. On the other hand, when $k = n$ or, equivalently, when $\boldsymbol{v} = \overline{\boldsymbol{v}}$, we have $\gamma_n(\boldsymbol{w}, \overline{\boldsymbol{v}}) = 0$ for all $\boldsymbol{w} \in \mathcal{V}_n$ and $\gamma_n(\overline{\boldsymbol{v}}, \boldsymbol{w}) = 0$ for all $\boldsymbol{w} \in \mathcal{V}_n \setminus \{\overline{\boldsymbol{v}} - (\overline{\boldsymbol{v}} - \boldsymbol{c})/n\}$. Hence

$$\alpha_n(\overline{\boldsymbol{v}}) + \sum_{\boldsymbol{w} \in \mathcal{V}_n} \gamma_n(\boldsymbol{w}, \overline{\boldsymbol{v}}) - \sum_{\boldsymbol{w} \in \mathcal{V}_n} \gamma_n(\overline{\boldsymbol{v}}, \boldsymbol{w}) = \alpha_n(\overline{\boldsymbol{v}}) - \gamma_n(\overline{\boldsymbol{v}}, \overline{\boldsymbol{v}} - (\overline{\boldsymbol{v}} - \boldsymbol{c})/n)$$

$$> \left(1 - \frac{n}{n-e} + \frac{1}{e}\right) - \frac{n - e - e^2}{e(n-e)} = 0,$$

where the inequality follows from (8). Finally, when $\lfloor \frac{n}{e} \rfloor + 1 \leq k \leq n - 1$, we have

$$\alpha_n(\boldsymbol{v}) + \sum_{\boldsymbol{w} \in \mathcal{V}_n} \gamma_n(\boldsymbol{w}, \boldsymbol{v}) - \sum_{\boldsymbol{w} \in \mathcal{V}_n} \gamma_n(\boldsymbol{v}, \boldsymbol{w}) = \alpha_n(\boldsymbol{v}) + \gamma_n(\boldsymbol{v} + (\overline{\boldsymbol{v}} - \boldsymbol{c})/n, \boldsymbol{v}) - \gamma_n(\boldsymbol{v}, \boldsymbol{v} - (\overline{\boldsymbol{v}} - \boldsymbol{c})/n)$$

$$= \frac{n}{ek(k+1)} + \frac{(n - e - e^2)n}{e(n-e)}\left(\frac{1}{k+1} - \frac{1}{k}\right)$$

$$= \frac{n}{ek(k+1)} - \frac{(n - e - e^2)n}{ek(k+1)(n-e)} = \frac{ne}{k(k+1)(n-e)} > 0.$$

We may therefore conclude that (α_n, γ_n) is feasible in (7) as postulated.

Step 2: The objective function value of (α_n, γ_n) in the non-linear program (7) can be expressed as $z_n^+ - z_n^-$, where

$$z_n^+ = \sum_{j \in \mathcal{J}} \sum_{\boldsymbol{v} \in \mathcal{V}_n} \alpha_n(\boldsymbol{v})(v_j - c_j)^+,$$

$$z_n^- = \left(\sum_{j \in \mathcal{J}} \sum_{\boldsymbol{v} \in \mathcal{V}_n} \alpha_n(\boldsymbol{v})(v_j - c_j) + \sum_{\boldsymbol{w} \in \mathcal{V}_n} (v_j - w_j)\gamma_n(\boldsymbol{w}, \boldsymbol{v})\right)^+.$$

By construction, $\alpha_n(\boldsymbol{v})$ is non-zero if and only if there exists an integer $k \in \left\{\lfloor \frac{n}{e} \rfloor, \ldots, n\right\}$ with $\boldsymbol{v} - \boldsymbol{c} = \frac{k}{n}(\overline{\boldsymbol{v}} - \boldsymbol{c})$. Therefore, z_n^+ can be reformulated as

$$z_n^+ = \sum_{j \in \mathcal{J}} \sum_{k=\lfloor \frac{n}{e} \rfloor}^{n-1} \frac{n}{ek(k+1)} \cdot \frac{k}{n}(\overline{v}_j - c_j)^+ + \sum_{j \in \mathcal{J}} \alpha_n(\overline{\boldsymbol{v}})(\overline{v}_j - c_j)^+$$

$$= \sum_{j \in \mathcal{J}} \sum_{k=\lfloor \frac{n}{e} \rfloor}^{n-1} \frac{1}{e(k+1)} \cdot (\overline{v}_j - c_j) + \sum_{j \in \mathcal{J}} \alpha_n(\overline{\boldsymbol{v}})(\overline{v}_j - c_j),$$

where the second equality follows from our standing assumption that $\boldsymbol{c} < \overline{\boldsymbol{v}}$.

In order to reformulate z_n^-, we first observe that

$$\alpha_n(\boldsymbol{v})(v_j - c_j) + \sum_{\boldsymbol{w} \in \mathcal{V}_n} (v_j - w_j)\gamma_n(\boldsymbol{w}, \boldsymbol{v}) = \begin{cases} \alpha_n(\boldsymbol{v})(v_j - c_j) - \frac{1}{n}(\overline{v}_j - c_j)\gamma_n\left(\boldsymbol{v} + \frac{1}{n}(\overline{\boldsymbol{v}} - \boldsymbol{c}), \boldsymbol{v}\right) \\ \qquad\qquad\qquad \text{if } \boldsymbol{v} \in \mathcal{V}_n \setminus \{\overline{\boldsymbol{v}}\}, \\ \\ \alpha_n(\overline{\boldsymbol{v}})(\overline{v}_j - c_j) \quad \text{if } \boldsymbol{v} = \overline{\boldsymbol{v}}, \end{cases}$$

where the equality follows from the definition of γ_n, which implies that $\gamma_n(\boldsymbol{w}, \boldsymbol{v}) = 0$ unless $\boldsymbol{w} = \boldsymbol{v} + \frac{1}{n}(\overline{\boldsymbol{v}} - \boldsymbol{c}) \in \mathcal{V}_n$. By the definitions of α_n and γ_n, the right-hand side of the above equality trivially vanishes if $\boldsymbol{v} \neq \overline{\boldsymbol{v}}$ and there is no $k \in \{\lfloor \frac{n}{e} \rfloor, \ldots, n-1\}$ with $\boldsymbol{v} - \boldsymbol{c} = \frac{k}{n}(\overline{\boldsymbol{v}} - \boldsymbol{c})$. Moreover, if $\boldsymbol{v} \neq \overline{\boldsymbol{v}}$ and there exists $k \in \{\lfloor \frac{n}{e} \rfloor, \ldots, n-1\}$ with $\boldsymbol{v} - \boldsymbol{c} = \frac{k}{n}(\overline{\boldsymbol{v}} - \boldsymbol{c})$, then the right-hand side evaluates to

$$
\frac{n}{ek(k+1)} \cdot \frac{k}{n}(\overline{v}_j - c_j) - \frac{1}{n}(\overline{v}_j - c_j) \cdot \frac{(n-e-e^2)n}{e(n-e)(k+1)} = \frac{1}{e(k+1)}(\overline{v}_j - c_j)\left(1 - \frac{n-e-e^2}{n-e}\right)
$$
$$
= \frac{e}{(n-e)(k+1)}(\overline{v}_j - c_j).
$$

This observation together with the definition of z_n^- implies that

$$
z_n^- = \left(\frac{e}{(n-e)}\sum_{j \in \mathcal{J}}\sum_{k=\lfloor \frac{n}{e} \rfloor}^{n-1}\frac{1}{k+1}(\overline{v}_j - c_j) + \sum_{j \in \mathcal{J}}\alpha_n(\overline{\boldsymbol{v}})(\overline{v}_j - c_j)\right)^+
$$
$$
= \frac{e}{(n-e)}\sum_{j \in \mathcal{J}}\sum_{k=\lfloor \frac{n}{e} \rfloor}^{n-1}\frac{1}{k+1}(\overline{v}_j - c_j) + \sum_{j \in \mathcal{J}}\alpha_n(\overline{\boldsymbol{v}})(\overline{v}_j - c_j),
$$

where the second equality holds because $\boldsymbol{c} < \overline{\boldsymbol{v}}$ by assumption.

Step 3: Our insights from Steps 1 and 2 imply that

$$
z_n^\star \geq z_n^+ - z_n^- = \left(\frac{1}{e} - \frac{e}{n-e}\right)\sum_{j \in \mathcal{J}}\sum_{k=\lfloor \frac{n}{e} \rfloor}^{n-1}\frac{1}{k+1}\cdot(\overline{v}_j - c_j) = \left(\frac{1}{e} - \frac{e}{n-e}\right)\cdot\mathbf{1}^\top(\overline{\boldsymbol{v}} - \boldsymbol{c})\sum_{k=\lfloor \frac{n}{e} \rfloor+1}^{n}\frac{1}{k},
$$

where the inequality follows from the feasibility of (α_n, γ_n) in (7) established in Step 1, the first equality exploits the explicit formulas for z_n^+ and z_n^- derived in Step 2, and the last equality is due to the index shift $k \leftarrow k+1$. We may thus conclude that

$$
\liminf_{n \to \infty} z_n^\star \geq \frac{1}{e}\cdot\mathbf{1}^\top(\overline{\boldsymbol{v}} - \boldsymbol{c})\lim_{n \to \infty}\sum_{k=\lfloor \frac{n}{e} \rfloor+1}^{n}\frac{1}{k} = \frac{1}{e}\cdot\mathbf{1}^\top(\overline{\boldsymbol{v}} - \boldsymbol{c}).
$$

Here, the last equality follows from the observation that $\sum_{k=\lfloor \frac{n}{e} \rfloor+1}^{n}\frac{1}{k}$ constitutes a difference of two Harmonic series. Using Theorem 2.2.1 by Lagarias (2013), its limit can thus be calculated in closed form as

$$
\lim_{n \to \infty}\sum_{k=\lfloor \frac{n}{e} \rfloor+1}^{n}\frac{1}{k} = \lim_{n \to \infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \sum_{k=1}^{\lfloor \frac{n}{e} \rfloor}\frac{1}{k}\right) = \lim_{n \to \infty}\log(n) - \log\left(\left\lfloor \frac{n}{e} \right\rfloor\right) = 1.
$$

Thus, the claim follows. $\square$

We are now ready to prove that the separable mechanism $(\boldsymbol{q}^\star, m^\star)$ is indeed optimal.

THEOREM 1. *The separable mechanism $(\boldsymbol{q}^\star, m^\star)$ defined through* (2) *is optimal in* $(\mathcal{MDP})$. *The optimal value of* $(\mathcal{MDP})$ *is given by* $\frac{1}{e} \cdot \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c})$.

Proof. By Lemma 1, $(\boldsymbol{q}^\star, m^\star)$ is feasible in $(\mathcal{MDP})$. Moreover, its objective value satisfies

$$\frac{1}{e} \cdot \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c}) \geq z^\star \geq \liminf_{n \to \infty} z_n^\star \geq \frac{1}{e} \cdot \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c}),$$

where the three inequalities follow from Lemma 1, Lemma 2 and Lemma 3, respectively. This implies that $z^\star = \frac{1}{e} \cdot \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c})$, and thus $(\boldsymbol{q}^\star, m^\star)$ is optimal in $(\mathcal{MDP})$. $\square$

REMARK 2. All results of this section remain valid if the assumption that $\boldsymbol{c} < \overline{\boldsymbol{v}}$ is relaxed. To see this, denote by $\mathcal{J}^-$ the set of items with $c_j \geq \overline{v}_j > 0$, and construct a separable mechanism $(\boldsymbol{q}^\star, m^\star)$ where the underlying single-item mechanism $(\hat{q}_j, \hat{m}_j)$ is given by (2) for all items $j \in \mathcal{J} \setminus \mathcal{J}^-$ and equals the trivial mechanism $(0, 0)$ for all items $j \in \mathcal{J}^-$. One can then show that $(\boldsymbol{q}^\star, m^\star)$ is optimal in $(\mathcal{MDP})$ and attains a worst-case regret of $\frac{1}{e} \cdot \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c})^+$. The proof of this result critically relies on the rectangularity of the uncertainty set $\mathcal{V} = \times_{j \in \mathcal{J}} [0, \overline{v}_j]$. Details are omitted for brevity. $\square$

The discretized linear program (4) and its dual (6) not only enable us to solve the mechanism design problem $(\mathcal{MDP})$ but also allow us to construct a Nash equilibrium for the game between the seller and nature described in Remark 1. To see this, for any $n > e(1 + e)$ we define $\mathbb{P}_n \in \Delta(\mathcal{V})$ as the discrete distribution that assigns probability $\alpha_n(\boldsymbol{v})$ to any $\boldsymbol{v} \in \mathcal{V}_n$, where $\alpha_n(\boldsymbol{v})$ is defined as in the proof of Lemma 3. Note that $\mathbb{P}_n$ is normalized because $\sum_{\boldsymbol{v} \in \mathcal{V}_n} \alpha_n(\boldsymbol{v}) = 1$. Moreover, we define $\mathbb{P}^\star \in \Delta(\mathcal{V})$ via the relations

$$\mathbb{P}^\star(\tilde{\boldsymbol{v}} \leq \boldsymbol{v}) = \begin{cases} \min_{j \in \mathcal{J}} \left(1 - \frac{1}{e} \left(\frac{\overline{v}_j - c_j}{v_j - c_j} \right) \right)^+ & \text{if } \boldsymbol{v} \in \mathcal{V} \setminus \{\overline{\boldsymbol{v}}\}, \\ 1 & \text{if } \boldsymbol{v} = \overline{\boldsymbol{v}}, \end{cases} \tag{9}$$

$\boldsymbol{v} \in \mathcal{V}$, which fully characterize the cumulative distribution function of $\mathbb{P}^\star$. If we define the marginal distributions $\mathbb{P}_j^\star \in \Delta([0, \overline{v}_j])$, $j \in \mathcal{J}$, through $\mathbb{P}_j^\star(\tilde{v}_j \leq v_j) = \left(1 - \frac{1}{e} \left(\frac{\overline{v}_j - c_j}{v_j - c_j} \right) \right)^+$ if $v_j \in [0, \overline{v}_j)$; $= 1$ if $v_j = \overline{v}_j$, then (9) simplifies to $\mathbb{P}^\star(\tilde{\boldsymbol{v}} \leq \boldsymbol{v}) = \min_{j \in \mathcal{J}} \mathbb{P}_j^\star(\tilde{v}_j \leq v_j)$. This reveals that $\mathbb{P}^\star$ is the unique comonotone distribution with marginals $\mathbb{P}_j^\star$, $j \in \mathcal{J}$. Moreover, it is easy to verify that the support of $\mathbb{P}^\star$ is confined to the line segment $\{\boldsymbol{c} + s(\overline{\boldsymbol{v}} - \boldsymbol{c}) : s \in [\frac{1}{e}, 1]\}$. We are now ready to prove that $\mathbb{P}^\star$ can be viewed as the limit of the discrete distributions $\mathbb{P}_n$.

LEMMA 4. *The discrete distributions $\mathbb{P}_n$ converge weakly to $\mathbb{P}^\star$.*

Proof. For any $n > e(1 + e)$, by construction of $\mathbb{P}_n$, we have

$$
\mathbb{P}_n\left(\tilde{\boldsymbol{v}} \leq \boldsymbol{v}\right) = \begin{cases} \displaystyle\sum_{k=\lfloor n/e \rfloor}^{\left\lfloor n\left(\min_{j \in \mathcal{J}} \frac{v_j - c_j}{\overline{v}_j - c_j}\right)\right\rfloor} \dfrac{n}{ek(k+1)} & \text{if } \boldsymbol{v} \in \mathcal{V} \setminus \{\overline{\boldsymbol{v}}\}, \\[4mm] 1 & \text{if } \boldsymbol{v} = \overline{\boldsymbol{v}}. \end{cases}
$$

As $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$, the sum in the above expression can be viewed as a telescoping series. For any $\boldsymbol{v} \in \mathcal{V} \setminus \{\overline{\boldsymbol{v}}\}$, we thus have

$$
\mathbb{P}_n\left(\tilde{\boldsymbol{v}} \leq \boldsymbol{v}\right) = \frac{n}{e}\left(\frac{1}{\lfloor n/e \rfloor} - \frac{1}{\left\lfloor n\left(\min_{j \in \mathcal{J}} \frac{v_j - c_j}{\overline{v}_j - c_j}\right)\right\rfloor + 1}\right)^+,
$$

which in turn implies that

$$
\lim_{n \to \infty} \mathbb{P}_n\left(\tilde{\boldsymbol{v}} \leq \boldsymbol{v}\right) = \frac{1}{e}\left(e - \frac{1}{\left(\min_{j \in \mathcal{J}} \frac{v_j - c_j}{\overline{v}_j - c_j}\right)}\right)^+ = \min_{j \in \mathcal{J}}\left(1 - \frac{1}{e}\left(\frac{\overline{v}_j - c_j}{v_j - c_j}\right)\right)^+ = \mathbb{P}^\star\left(\tilde{\boldsymbol{v}} \leq \boldsymbol{v}\right).
$$

Finally, we note that $\lim_{n \to \infty} \mathbb{P}_n\left(\tilde{\boldsymbol{v}} \leq \overline{\boldsymbol{v}}\right) = 1 = \mathbb{P}^\star\left(\tilde{\boldsymbol{v}} \leq \overline{\boldsymbol{v}}\right)$. Thus, the claim follows. $\square$

In Section 2 we have argued that problem $(\mathcal{MDP})$ can be interpreted as a zero-sum game between the seller, who chooses a mechanism $(\boldsymbol{q}, m) \in \mathcal{X}$, and nature, who chooses a probability distribution $\mathbb{P} \in \Delta(\mathcal{V})$ over the buyer's value profiles; see Remark 1. We can now show that the distribution $\mathbb{P}^\star$, which was extracted from the (discretized) dual mechanism design problem (6), actually represents nature's Nash strategy. To simplify the subsequent discussion, we denote by

$$
z(\boldsymbol{q}, m; \mathbb{P}) = \mathbb{E}_{\mathbb{P}}\left[\mathbf{1}^\top(\tilde{\boldsymbol{v}} - \boldsymbol{c})^+ - (m(\tilde{\boldsymbol{v}}) - \boldsymbol{q}(\tilde{\boldsymbol{v}})^\top \boldsymbol{c})\right]
$$

the expected regret of the mechanism $(\boldsymbol{q}, m)$ under the probability distribution $\mathbb{P}$.

THEOREM 2. *The separable mechanism $(\boldsymbol{q}^\star, m^\star)$ defined in (2) and the comonotone probability distribution $\mathbb{P}^\star$ defined in (9) satisfy the saddle point condition*

$$
\max_{\mathbb{P} \in \Delta(\mathcal{V})} z(\boldsymbol{q}^\star, m^\star; \mathbb{P}) \leq z(\boldsymbol{q}^\star, m^\star; \mathbb{P}^\star) \leq \min_{(\boldsymbol{q}, m) \in \mathcal{X}} z(\boldsymbol{q}, m; \mathbb{P}^\star). \tag{10}
$$

Theorem 2 implies that $(\boldsymbol{q}^\star, m^\star)$ and $\mathbb{P}^\star$ form a Nash equilibrium of the game between the seller and nature. Indeed, the first inequality in (10) implies that $\mathbb{P}^\star$ is a best response to the mechanism $(\boldsymbol{q}^\star, m^\star)$, while the second inequality in (10) implies that $(\boldsymbol{q}^\star, m^\star)$ is a best response to the probability distribution $\mathbb{P}^\star$.

Proof of Theorem 2. We first show that $\mathbb{P}^\star$ solves the maximization problem on the left-hand side of (10) (Step 1), and then we prove that $(\boldsymbol{q}^\star, m^\star)$ solves the minimization problem on the right-hand side of (10) (Step 2).

Step 1: Fix the separable mechanism $(\boldsymbol{q}^\star, m^\star)$ and an arbitrary distribution $\mathbb{P} \in \Delta(\mathcal{V})$. Then, the expected regret $z(\boldsymbol{q}^\star, m^\star; \mathbb{P})$ admits the upper bound

$$
\begin{aligned}
z(\boldsymbol{q}^\star, m^\star; \mathbb{P}) &= \mathbb{E}_{\mathbb{P}}\left[\sum_{j \in \mathcal{J}} (\tilde{v}_j - c_j)^+ - \hat{m}_j(\tilde{v}_j) + \hat{q}_j(\tilde{v}_j)c_j\right] \\
&= \mathbb{E}_{\mathbb{P}}\left[\sum_{j \in \mathcal{J}} (\tilde{v}_j - c_j)^+ + \mathbb{1}_{\tilde{v}_j \geq c_j + \frac{1}{e}(\overline{v}_j - c_j)}\left(-\tilde{v}_j + \frac{\overline{v}_j - c_j}{e} - c_j\log\left(\frac{\tilde{v}_j - c_j}{\overline{v}_j - c_j}\right) + c_j + c_j\log\left(\frac{\tilde{v}_j - c_j}{\overline{v}_j - c_j}\right)\right)\right] \\
&= \mathbb{E}_{\mathbb{P}}\left[\sum_{j \in \mathcal{J}} (\tilde{v}_j - c_j)^+ + \mathbb{1}_{\tilde{v}_j \geq c_j + \frac{1}{e}(\overline{v}_j - c_j)}\left(-\tilde{v}_j + \frac{\overline{v}_j - c_j}{e} + c_j\right)\right] \\
&= \mathbb{E}_{\mathbb{P}}\left[\sum_{j \in \mathcal{J}} (\tilde{v}_j - c_j)^+ - \left(\tilde{v}_j - c_j - \frac{1}{e}(\overline{v}_j - c_j)\right)^+\right] \\
&\leq \mathbb{E}_{\mathbb{P}}\left[\sum_{j \in \mathcal{J}} \frac{1}{e}(\overline{v}_j - c_j)\right] = \frac{1}{e} \cdot \mathbf{1}^\top(\overline{\boldsymbol{v}} - \boldsymbol{c}),
\end{aligned}
$$

where the first and the second equalities follow from definitions of $(\boldsymbol{q}^\star, m^\star)$ and $(\hat{q}_j, \hat{m}_j)$, $j \in \mathcal{J}$, respectively, while the inequality holds due to the sub-additivity of the maximum operator and the assumption that $\overline{\boldsymbol{v}} > \boldsymbol{c}$. As the support of $\mathbb{P}^\star$ is a subset of the rectangle $\times_{j \in \mathcal{J}}[c_j + \frac{1}{e}(\overline{v}_j - c_j), \overline{v}_j]$, the inequality is tight for $\mathbb{P}^\star$. Thus, $\mathbb{P}^\star$ solves nature's maximization problem in (10).

Step 2: Fix now the distribution $\mathbb{P}^\star$, and consider a relaxation of the minimization problem on the right-hand side of (10) that enforces the incentive compatibility and individual rationality constraints only on the line segment $\mathcal{V}^s = \{\boldsymbol{c} + s(\overline{\boldsymbol{v}} - \boldsymbol{c}) : s \in [0, 1]\}$ containing the support of $\mathbb{P}^\star$. Defining $\mathbb{P}^s \in \Delta([0, 1])$ through $\mathbb{P}^s(\tilde{s} \leq s) = \left(1 - \frac{1}{es}\right)^+$ if $s \in [0, 1)$; $= 1$ if $s = 1$, this relaxation can

be reformulated as

$$
\inf \quad \mathbb{E}_{\mathbb{P}^s} \left[\tilde{s} \cdot \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c}) - m^s(\tilde{s}) + \boldsymbol{q}^s(\tilde{s})^\top \boldsymbol{c} \right]
$$

$$
\text{s.t.} \quad \boldsymbol{q}^s \in \mathcal{L}([0,1],[0,1]^J),\ m^s \in \mathcal{L}([0,1],\mathbb{R})
$$

$$
\boldsymbol{q}^s(s)^\top (\boldsymbol{c} + s(\overline{\boldsymbol{v}} - \boldsymbol{c})) - m^s(s) \geq \boldsymbol{q}^s(t)^\top (\boldsymbol{c} + s(\overline{\boldsymbol{v}} - \boldsymbol{c})) - m^s(t) \quad \forall s,t \in [0,1] \tag{11a}
$$

$$
\boldsymbol{q}^s(s)^\top (\boldsymbol{c} + s(\overline{\boldsymbol{v}} - \boldsymbol{c})) - m^s(s) \geq 0 \quad \forall s \in [0,1].
$$

Indeed, note that any mechanism $(\boldsymbol{q}, m)$ feasible on the right-hand side of (10) induces a pair of functions $(\boldsymbol{q}^s, m^s)$ feasible in (11a) with the same objective value, where $\boldsymbol{q}^s(s) = \boldsymbol{q}(\boldsymbol{c} + s(\overline{\boldsymbol{v}} - \boldsymbol{c}))$ and $m^s(s) = m(\boldsymbol{c} + s(\overline{\boldsymbol{v}} - \boldsymbol{c}))$ for all $s \in [0,1]$. Note also that the two constraints in (11a) are easily recognized as the incentive compatibility and individual rationality constraints restricted to the line segment $\mathcal{V}^s$, respectively. Using the variable substitution $f(s) \leftarrow \boldsymbol{q}^s(s)^\top (\overline{\boldsymbol{v}} - \boldsymbol{c})$ and $g(s) \leftarrow m^s(s) - \boldsymbol{q}^s(s)^\top \boldsymbol{c}$, problem (11a) can be reformulated as

$$
\inf \quad \mathbb{E}_{\mathbb{P}^s} \left[\tilde{s} \cdot \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c}) - g(\tilde{s}) \right]
$$

$$
\text{s.t.} \quad f \in \mathcal{L}([0,1],[0,\mathbf{1}^\top(\overline{\boldsymbol{v}} - \boldsymbol{c})]),\ g \in \mathcal{L}([0,1],\mathbb{R})
$$

$$
sf(s) - g(s) \geq sf(t) - g(t) \quad \forall s,t \in [0,1] \tag{11b}
$$

$$
sf(s) - g(s) \geq 0 \quad \forall s \in [0,1].
$$

Normalizing f and g by the positive constant $\mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c})$ further simplifies problem (11b) to

$$
\inf \quad \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c}) \cdot \mathbb{E}_{\mathbb{P}^s} \left[\tilde{s} - g(\tilde{s}) \right]
$$

$$
\text{s.t.} \quad f \in \mathcal{L}([0,1],[0,1]),\ g \in \mathcal{L}([0,1],\mathbb{R})
$$

$$
sf(s) - g(s) \geq sf(t) - g(t) \quad \forall s,t \in [0,1] \tag{11c}
$$

$$
sf(s) - g(s) \geq 0 \quad \forall s \in [0,1].
$$

Note that minimizing $\mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c}) \cdot \mathbb{E}_{\mathbb{P}^s} \left[\tilde{s} - g(\tilde{s}) \right]$ is tantamount to maximizing $\mathbb{E}_{\mathbb{P}^s} \left[g(\tilde{s}) \right]$. This reveals that problem (11c) is equivalent to a single-item pricing problem with the objective of maximizing expected revenues under the probability distribution $\mathbb{P}^s$, where the cost of procuring the item vanishes. Note that the auxiliary variables f and g in (11c) are naturally interpreted as univariate allocation and payment rules, respectively.

It is well-known that the maximum expected revenue under $\mathbb{P}^{\mathrm{s}}$ is given by $\max_{p\in\mathbb{R}} p(1-\mathbb{P}^{\mathrm{s}}(\tilde{s}\leq p))$; see Myerson (1981) or Riley and Zeckhauser (1983). By the definition of $\mathbb{P}^s$, we have

$$p(1-\mathbb{P}^{\mathrm{s}}(\tilde{s}\leq p)) = \begin{cases} p & \text{if } p < \frac{1}{e}, \\[2mm] \frac{1}{e} & \text{if } p \in [\frac{1}{e}, 1), \\[2mm] 0 & \text{otherwise}, \end{cases}$$

and thus $\max_{p\in\mathbb{R}} p(1-\mathbb{P}^{\mathrm{s}}(\tilde{s}\leq p)) = \frac{1}{e}$. Moreover, a direct calculation shows that $\mathbb{E}_{\mathbb{P}^{\mathrm{s}}}[\tilde{s}] = \frac{2}{e}$. The optimal value of (11c) therefore amounts to $\frac{1}{e}\cdot\mathbf{1}^\top(\overline{\boldsymbol{v}}-\boldsymbol{c})$. As (11c) was obtained by relaxing the mechanism design problem on the right-hand side of (10) and as $z(\boldsymbol{q}^\star,m^\star;\mathbb{P}^\star) = \frac{1}{e}\cdot\mathbf{1}^\top(\overline{\boldsymbol{v}}-\boldsymbol{c})$ by our reasoning in Step 1, we may thus conclude that $(\boldsymbol{q}^\star,m^\star)$ solves the seller's minimization problem in (10). $\quad\square$

We are now ready to elucidate how our results relate to those by Bergemann and Schlag (2008), who investigate a *single-item* pricing problem with worst-case regret objective that minimizes over all randomized posted price mechanisms encoded by univariate distributions $\mathbb{Q}\in\Delta([0,\overline{v}])$. Under any such mechanism, the seller draws a random price $\tilde{p}$ from $\mathbb{Q}$ and sells the good to the buyer at price $\tilde{p}$ whenever $\tilde{p}$ is smaller or equal to the buyer's value $\tilde{v}$. The best randomized posted price mechanism can thus be found by solving the worst-case regret minimization problem

$$\inf_{\mathbb{Q}\in\Delta([0,\overline{v}])} \sup_{\mathbb{P}\in\Delta([0,\overline{v}])} \int_0^{\overline{v}}\int_0^{\overline{v}} (v-c)^+ - \mathbb{1}_{p\leq v}(p-c)\,\mathrm{d}\mathbb{Q}(p)\,\mathrm{d}\mathbb{P}(v), \tag{12}$$

which can again be viewed as a zero-sum game akin to (1). In the following, denote by $(q^\star,m^\star)$ the mechanism defined via (2) for $J=1$, and define the univariate distribution $\mathbb{Q}^\star\in\Delta([0,\overline{v}])$ through $\mathbb{Q}^\star(\tilde{p}\leq p) = q^\star(p)$ for all $p\in[0,\overline{v}]$. Moreover, denote by $\mathbb{P}^\star\in\Delta([0,\overline{v}])$ the distribution defined in (9) for $J=1$. Bergemann and Schlag (2008) show that $\mathbb{Q}^\star$ and $\mathbb{P}^\star$ form a Nash equilibrium for problem (12) and that the optimal value of (12) evaluates to $\frac{1}{e}\cdot(\overline{v}-c)$. Our Theorems 1 and 2 thus encompass the single-item pricing theory by Bergemann and Schlag (2008) as a special case. More specifically, one can show that there is a one-to-one correspondence between the single-item mechanisms (q,m) feasible in $(\mathcal{MDP})$ that involve a *right-continuous* allocation rule and

the randomized posted price mechanisms $\mathbb{Q}$ feasible in (12) that satisfy $\mathbb{Q}(\tilde{p} \leq p) = q(p)$ for all $p \in [0, \overline{v}]$. The analysis of multi-item pricing problems portrayed in this section critically relies on our representation of the selling mechanisms in terms of generic allocation and payment rules that are subject to explicit incentive compatibility and individual rationality constraints. In contrast, randomized posted price mechanisms for multiple items are difficult to characterize because they require a separate posted price for each of the exponentially many bundles $\mathcal{S} \in 2^{\mathcal{J}}$. Moreover, each realization of the posted prices leads to a different tessellation of the uncertainty set $\mathcal{V}$ into 2^J polytopes (see Figure 1 for a visualization when $J = 2$), and the revenue of the seller depends on the particular polytope that accommodates the uncertain value profile $\tilde{\boldsymbol{v}}$. In order to evaluate the seller's expected revenue, one would thus have to compute the probabilities of exponentially many (random) polytopes with respect to $\mathbb{P}$ and integrate a (random) weighted sum of these probabilities with respect to $\mathbb{Q}$, which seems excruciating.

When $J = 1$, the optimal randomized posted price mechanism $\mathbb{Q}^\star$ offers distinct implementational advantages over the optimal single-item mechanism $(q^\star, m^\star)$ even though the agents' expected utilities are identical under both mechanisms irrespective of the value distribution $\mathbb{P}$. Specifically, under the randomized posted price mechanism the buyer only needs to make a payment if he actually receives the good. In contrast, under the optimal single-item mechanism the seller offers the buyer a lottery to win the good with probability $q^\star(v)$, and the payment $m^\star(v)$ can be interpreted as a participation fee that is due upfront. It could thus happen that the buyer ends up making a payment without obtaining the good. On the other hand, buyers who are lucky to win the good under the optimal single-item mechanism incur a lower cost than under the randomized posted price mechanism. We conclude that the randomized posted price mechanism is more likely to be accepted in practice because the prospect of making a payment without any reward is likely to disconcert potential buyers.

When $J > 1$, we have shown here that the optimal multi-item mechanism $(\boldsymbol{q}^\star, m^\star)$ is separable. Therefore, it can still easily be implemented as a randomized posted price mechanism without

the need to specify separate posted prices for all possible bundles. Instead, one only needs one randomized posted price per item, and the buyer only has to compare his value for a particular item with the respective posted price.

4. Optimal Deterministic Mechanism

Some agents may feel uncomfortable about randomized selling mechanisms. Thus, we study now a *deterministic* variant of problem $(\mathcal{MDP})$, where the allocation rule $\boldsymbol{q}$ must be chosen from $\mathcal{L}(\mathcal{V}, \{0,1\}^J)$. This means that the items are assigned to the buyer either with probability 0 or 1.

Every deterministic allocation rule $\boldsymbol{q} \in \mathcal{L}(\mathcal{V}, \{0,1\}^J)$ induces a unique *bundle allocation rule* $s \in \mathcal{L}(\mathcal{V}, 2^{\mathcal{J}})$ defined through $s(\boldsymbol{v}) = \{j \in \mathcal{J} : q_j(\boldsymbol{v}) = 1\}$ for all $\boldsymbol{v} \in \mathcal{V}$. Thus, $s(\boldsymbol{v}) \subseteq \mathcal{J}$ represents the bundle acquired by a buyer with value profile $\boldsymbol{v}$. Moreover, the incentive compatibility constraint implies that $m(\boldsymbol{v}) = m(\boldsymbol{w})$ whenever $s(\boldsymbol{v}) = s(\boldsymbol{w})$. This means that the seller receives the same payment from all buyers who acquire the same bundle. Thus, for every $\mathcal{S} \subseteq \mathcal{J}$, there exists a bundle price $p_{\mathcal{S}}$, and the payment rule must satisfy $m(\boldsymbol{v}) = p_{s(\boldsymbol{v})}$. This observation allows us to reformulate the deterministic mechanism design problem over $\boldsymbol{q} \in \mathcal{L}(\mathcal{V}, \{0,1\}^J)$ and $m \in \mathcal{L}(\mathcal{V}, \mathbb{R})$ as an equivalent problem over all bundle allocation rules $s \in \mathcal{L}(\mathcal{V}, 2^{\mathcal{J}})$ and vectors of posted prices $\boldsymbol{p} \in \mathbb{R}^{2^J}$. The deterministic pricing problem thus simplifies to

$$
\begin{aligned}
z_{\mathrm{d}}^{\star} = \ \inf \quad & \sup_{\boldsymbol{v} \in \mathcal{V}} \ \mathbf{1}^{\top}(\boldsymbol{v} - \boldsymbol{c})^{+} - (p_{s(\boldsymbol{v})} - \mathbf{1}_{s(\boldsymbol{v})}^{\top}\boldsymbol{c}) \\
\text{s.t.} \quad & s \in \mathcal{L}(\mathcal{V}, 2^{\mathcal{J}}), \ \boldsymbol{p} \in \mathbb{R}^{2^J} \\
& \mathbf{1}_{s(\boldsymbol{v})}^{\top}\boldsymbol{v} - p_{s(\boldsymbol{v})} \geq \mathbf{1}_{s(\boldsymbol{w})}^{\top}\boldsymbol{v} - p_{s(\boldsymbol{w})} \qquad \forall \boldsymbol{v}, \boldsymbol{w} \in \mathcal{V} \\
& \mathbf{1}_{s(\boldsymbol{v})}^{\top}\boldsymbol{v} - p_{s(\boldsymbol{v})} \geq 0 \qquad\qquad\qquad\quad \forall \boldsymbol{v} \in \mathcal{V},
\end{aligned}
\tag{13}
$$

where the constraints ensure incentive compatibility and individual rationality, respectively.

We can now prove that optimizing over all deterministic mechanisms is equivalent to optimizing over all (deterministic) posted price mechanisms in the sense of Definition 3 and that the price for the empty bundle must vanish at optimality.

LEMMA 5. *For any $(s, \boldsymbol{p}) \in \mathcal{L}(\mathcal{V}, 2^{\mathcal{J}}) \times \mathbb{R}^{2^J}$ feasible in (13), there exists $\hat{\boldsymbol{p}} \in \mathbb{R}^{2^J}$ such that $(s, \hat{\boldsymbol{p}})$ is also feasible in (13), attains a weakly lower objective value than $(s, \boldsymbol{p})$, and satisfies*

(i) $s(v) \in \arg\max_{\mathcal{S} \subseteq \mathcal{J}} \mathbf{1}_{\mathcal{S}}^\top v - \hat{p}_{\mathcal{S}}$ *for all* $v \in \mathcal{V}$,

(ii) $\hat{p}_\emptyset = 0$.

Condition (i) ensures that the buyer acquires a bundle that maximizes his utility, which in turn implies that the mechanism induced by $\hat{p}$ represents a posted price mechanism in the sense of Definition 3. Condition (ii) eliminates the arbitrage opportunity that would allow the buyer to earn free money when acquiring no item.

Proof of Lemma 5. We first show that for every deterministic mechanism there exists an equally desirable one that satisfies condition (i) (Step 1). Next, we prove that for every mechanism satisfying condition (i) there exists a weakly preferable one that satisfies both conditions (i) and (ii) (Step 2).

Step 1: For a fixed bundle allocation rule $s \in \mathcal{L}(\mathcal{V} \times 2^\mathcal{J})$, we define $\text{range}(s) = \{s(v) : v \in \mathcal{V}\}$ as the collection of all purchasable bundles. Since the posted prices $p_{\mathcal{S}}$, $\mathcal{S} \notin \text{range}(s)$, do not enter the deterministic pricing problem (13) at all, we can assign arbitrary values to them. In particular, we may introduce a new posted price vector $\hat{p}$ with $\hat{p}_{\mathcal{S}} = p_{\mathcal{S}}$ for all $\mathcal{S} \in \text{range}(s)$ and $\hat{p}_{\mathcal{S}} = \mathbf{1}_{\mathcal{S}}^\top \overline{v}$ for all $\mathcal{S} \notin \text{range}(s)$. By construction, $(s, \hat{p})$ is feasible in (13) and attains the same objective value as (s, p). Moreover, we have

$$s(v) \in \arg\max_{\mathcal{S} \in \text{range}(s)} \mathbf{1}_{\mathcal{S}}^\top v - p_{\mathcal{S}} \subseteq \arg\max_{\mathcal{S} \subseteq \mathcal{J}} \mathbf{1}_{\mathcal{S}}^\top v - \hat{p}_{\mathcal{S}} \quad \forall v \in \mathcal{V},$$

where the first inclusion follows from the incentive compatibility of (s, p) as implied by the first constraint in (13), while the second inclusion follows from the individual rationality of (s, p) and the definition of $\hat{p}$, which ensure that $\max_{\mathcal{S} \in \text{range}(s)} \mathbf{1}_{\mathcal{S}}^\top v - p_{\mathcal{S}} \geq 0$ and that $\mathbf{1}_{\mathcal{S}}^\top v - \hat{p}_{\mathcal{S}} = \mathbf{1}_{\mathcal{S}}^\top v - \mathbf{1}_{\mathcal{S}}^\top \overline{v} \leq 0$ for all $\mathcal{S} \notin \text{range}(s)$, respectively. Hence, $(s, \hat{p})$ satisfies condition (i).

Step 2: By the insights gained in Step 1, we may assume without loss of generality that (s, p) satisfies condition (i). Next, set $\delta = \inf_{v \in \mathcal{V}} \mathbf{1}_{s(v)}^\top v - p_{s(v)}$ and note that $\delta \geq 0$ because of individual rationality. We may now introduce a new posted price vector $\hat{p} = p + \delta \mathbf{1}$, which increases the price of each bundle by δ. It is easy to verify that $(s, \hat{p})$ remains incentive compatible and individually rational and still satisfies condition (i). Moreover, $(s, \hat{p})$ incurs a weakly lower regret than (s, p). It

remains to be shown that $\hat{p}_\emptyset = 0$. To this end, denote by $\{\boldsymbol{v}_k\}_{k\in\mathbb{N}}$ a sequence of value profiles that asymptotically attain the infimal buyer utility δ under the mechanism $(s, \boldsymbol{p})$. Thus, we have

$$\lim_{k\to\infty} \mathbf{1}_{s(\boldsymbol{v}_k)}^\top \boldsymbol{v}_k - p_{s(\boldsymbol{v}_k)} = \delta,$$

which in turn implies

$$0 = \lim_{k\to\infty} \mathbf{1}_{s(\boldsymbol{v}_k)}^\top \boldsymbol{v}_k - p_{s(\boldsymbol{v}_k)} - \delta = \lim_{k\to\infty} \mathbf{1}_{s(\boldsymbol{v}_k)}^\top \boldsymbol{v}_k - \hat{p}_{s(\boldsymbol{v}_k)} \geq \lim_{k\to\infty} \mathbf{1}_\emptyset^\top \boldsymbol{v}_k - \hat{p}_\emptyset. = -\hat{p}_\emptyset$$

Hence, $\hat{p}_\emptyset \geq 0$. Finally, if $\emptyset \in \text{range}(s)$, the individual rationality constraint implies $-\hat{p}_\emptyset \geq 0$, and if $\emptyset \notin \text{range}(s)$, then we are free to set $\hat{p}_\emptyset = \mathbf{1}_\emptyset^\top \overline{\boldsymbol{v}} = 0$ without compromising condition (i) (see Step 1). In both cases, we have $\hat{p}_\emptyset = 0$. $\quad\square$

LEMMA 6. *Any posted price mechanism $(s, \boldsymbol{p}) \in \mathcal{L}(\mathcal{V}, 2^{\mathcal{J}}) \times \mathbb{R}^{2^J}$ that satisfies conditions (i) and (ii) from Lemma 5 is both incentive compatible and individually rational, and thus it is feasible in* (13).

Proof. Condition (i) implies that

$$\mathbf{1}_{s(\boldsymbol{v})}^\top \boldsymbol{v} - p_{s(\boldsymbol{v})} = \max_{\mathcal{S} \subseteq \mathcal{J}} \mathbf{1}_{\mathcal{S}}^\top \boldsymbol{v} - p_{\mathcal{S}} \geq \mathbf{1}_{s(\boldsymbol{w})}^\top \boldsymbol{v} - p_{s(\boldsymbol{w})} \quad \forall \boldsymbol{v}, \boldsymbol{w} \in \mathcal{V}$$

and is thus a sufficient condition for the incentive compatibility constraint in (13). By condition (i), we further have

$$\mathbf{1}_{s(\boldsymbol{v})}^\top \boldsymbol{v} - p_{s(\boldsymbol{v})} \geq \mathbf{1}_\emptyset^\top \boldsymbol{v} - p_\emptyset = 0,$$

where the equality follows from condition (ii). Thus, condition (ii) can be viewed as a sufficient condition for the individual rationality constraint in (13). $\quad\square$

Lemma 5 implies that conditions (i) and (ii) may be appended as constraints to problem (13) without increasing its optimal value, while Lemma 6 shows that the incentive compatibility and individual rationality constraints are redundant in the resulting optimization problem and may thus be eliminated. Thus, the deterministic mechanism design problem (13) is equivalent to

$$z_{\mathrm{d}}^\star = \inf_{} \ \sup_{\boldsymbol{v} \in \mathcal{V}} \ \mathbf{1}^\top (\boldsymbol{v} - \boldsymbol{c})^+ - (p_{s(\boldsymbol{v})} - \mathbf{1}_{s(\boldsymbol{v})}^\top \boldsymbol{c})$$

$$\text{s.t.} \quad s \in \mathcal{L}(\mathcal{V}, 2^{\mathcal{J}}), \ \boldsymbol{p} \in \mathcal{P} \qquad\qquad (\mathcal{DMDP})$$

$$s(\boldsymbol{v}) \in \arg\max_{\mathcal{S} \subseteq \mathcal{J}} \mathbf{1}_{\mathcal{S}}^\top \boldsymbol{v} - p_{\mathcal{S}} \quad \forall \boldsymbol{v} \in \mathcal{V},$$

where $\mathcal{P} = \{\boldsymbol{p} \in \mathbb{R}^{2^J} : p_\emptyset = 0\}$ will henceforth be referred to as the set of admissible posted prices.

Problem $(\mathcal{DMDP})$ is easily recognized as the worst-case regret minimization problem over all posted price mechanisms that set the price of the empty bundle to zero; see also Definition 3.

REMARK 3. Problem $(\mathcal{DMDP})$ can be interpreted as a two-stage robust bilevel program, where the seller acts as the leader, and the buyer acts as the follower. Indeed, the leader chooses a posted price vector $\boldsymbol{p}$ before $\boldsymbol{v}$ is revealed with the aim to minimize her worst-case regret, while the follower chooses the bundle $s(\boldsymbol{v})$ after $\boldsymbol{v}$ is revealed with the aim to maximize his utility. More precisely, problem $(\mathcal{DMDP})$ constitutes an *optimistic* bilevel program because ties are broken at the discretion of the leader whenever the follower's problem admits multiple optimal solutions. This optimistic bilevel program essentially models an *indirect* implementation of a posted price mechanism, whereby the buyer first reports his value profile $\boldsymbol{v}$ to the seller, and the seller then picks a utility-maximizing bundle on behalf of the buyer. □

In order to solve the deterministic mechanism design problem $(\mathcal{DMDP})$, we introduce the sets

$$\mathcal{V}_\mathcal{S}(\boldsymbol{p}, \delta) = \left\{ \boldsymbol{v} \in \mathcal{V} : \mathbf{1}_\mathcal{S}^\top \boldsymbol{v} - p_\mathcal{S} \geq \mathbf{1}_{\mathcal{S}'}^\top \boldsymbol{v} - p_{\mathcal{S}'} + \delta \quad \forall \mathcal{S}' \subseteq \mathcal{J} : \mathcal{S}' \neq \mathcal{S} \right\}$$

parameterized by $\boldsymbol{p} \in \mathcal{P}$, $\mathcal{S} \subseteq \mathcal{J}$ and $\delta \geq 0$. By definition, $\mathcal{V}_\mathcal{S}(\boldsymbol{p}, 0)$ is the set of all value profiles under which the buyer weakly prefers bundle $\mathcal{S}$ to any other bundle $\mathcal{S}'$, and the interior of $\mathcal{V}_\mathcal{S}(\boldsymbol{p}, 0)$ contains those value profiles under which the buyer strictly prefers $\mathcal{S}$. Note that the polytopes $\{\mathcal{V}_\mathcal{S}(\boldsymbol{p}, 0)\}_{\mathcal{S} \subseteq \mathcal{J}}$ have disjoint interiors but may have overlapping boundaries. Thus, for any fixed $\boldsymbol{p} \in \mathcal{P}$, the bundle allocation rule $s(\boldsymbol{v})$ is uniquely determined *almost* everywhere.

We now define an auxiliary problem parameterized by $\delta \geq 0$.

$$z'_\mathrm{d}(\delta) = \inf_{\boldsymbol{p} \in \mathcal{P}} \max_{\mathcal{S} \subseteq \mathcal{J}} \max_{\boldsymbol{v} \in \mathcal{V}_\mathcal{S}(\boldsymbol{p}, \delta)} \mathbf{1}^\top (\boldsymbol{v} - \boldsymbol{c})^+ - (p_\mathcal{S} - \mathbf{1}_\mathcal{S}^\top \boldsymbol{c}) \tag{14}$$

REMARK 4. Problem (14) with $\delta = 0$ admits again an intuitive interpretation as a two-stage robust bilevel program, where the leader chooses a posted price vector $\boldsymbol{p}$ that minimizes her worst-case regret, anticipating that the follower will choose a bundle $\mathcal{S}$ that maximizes his utility. Indeed, note

that the regret of a particular value profile $\boldsymbol{v}$ in (14) is always evaluated under the bundle $\mathcal{S}$ that maximizes the utility of the buyer of type $\boldsymbol{v}$. More precisely, problem (14) with $\delta = 0$ constitutes a *pessimistic* bilevel program because ties are broken at the discretion of the follower, while the leader hedges against the most adverse of the follower's optimal solutions. This pessimistic bilevel program essentially models a *direct* implementation of a posted price mechanism, whereby the buyer picks a utility-maximizing bundle on his own. $\square$

In the remainder of this section we will demonstrate that the optimistic bilevel program $(\mathcal{DMDP})$ is in fact equivalent to the pessimistic bilevel program (14) with $\delta = 0$, which implies that it is immaterial whether a posted price mechanism is implemented in a direct or an indirect fashion. To prove this equivalence, we will also study the auxiliary problem (14) with $\delta > 0$, which lacks an intuitive physical interpretation.

LEMMA 7. *We have $z'_{\mathrm{d}}(\delta) \leq z^\star_{\mathrm{d}} \leq z'_{\mathrm{d}}(0)$ for all $\delta > 0$.*

Proof. Problem $(\mathcal{DMDP})$ is equivalent to

$$z^\star_{\mathrm{d}} = \inf \quad \sup_{\mathcal{S} \subseteq \mathcal{J}} \sup_{\boldsymbol{v}:s(\boldsymbol{v})=\mathcal{S}} \mathbf{1}^\top (\boldsymbol{v} - \boldsymbol{c})^+ - (p_{\mathcal{S}} - \mathbf{1}_{\mathcal{S}}^\top \boldsymbol{c})$$

$$\text{s.t.} \quad s \in \mathcal{L}(\mathcal{V}, 2^{\mathcal{J}}), \, \boldsymbol{p} \in \mathcal{P}$$

$$s(\boldsymbol{v}) \in \arg\max_{\mathcal{S} \subseteq \mathcal{J}} \mathbf{1}_{\mathcal{S}}^\top \boldsymbol{v} - p_{\mathcal{S}} \quad \forall \boldsymbol{v} \in \mathcal{V},$$

where the maximization over all value profiles $\boldsymbol{v}$ is decomposed into a maximization over all bundles $\mathcal{S}$ and a subsequent maximization over all value profiles $\boldsymbol{v}$ under which bundle $\mathcal{S}$ is acquired. Assume first that $\delta > 0$. As $\mathcal{V}_{\mathcal{S}}(\boldsymbol{p}, \delta)$ only contains value profiles under which the buyer strictly prefers bundle $\mathcal{S}$ to any other bundle, we have $\mathcal{V}_{\mathcal{S}}(\boldsymbol{p}, \delta) \subseteq \{v : s(\boldsymbol{v}) = \mathcal{S}\}$. This inclusion holds for all admissible $(\boldsymbol{p}, s)$, and thus we find $z^\star_d \geq z'_d(\delta)$. Similarly, for $\delta = 0$, the inequality $z^\star_d \leq z'_d(0)$ follows from the observation that $\mathcal{V}_{\mathcal{S}}(\boldsymbol{p}, 0) \supseteq \{v : s(\boldsymbol{v}) = \mathcal{S}\}$ for all admissible $(\boldsymbol{p}, s)$. $\square$

In the following we prove that the posted price vector $\boldsymbol{p}^\star \in \mathbb{R}^{2^J}$ defined through

$$p_{\mathcal{S}}^\star = \frac{1}{2} \cdot \mathbf{1}_{\mathcal{S}}^\top (\overline{\boldsymbol{v}} + \boldsymbol{c}) \quad \forall \mathcal{S} \subseteq \mathcal{J} \tag{15}$$

is optimal in $(\mathcal{DMDP})$. Note that under this pricing scheme, each item $j \in \mathcal{J}$ is assigned a price $\frac{1}{2}(\overline{v}_j + c_j)$, while the price of each bundle $\mathcal{S} \subseteq \mathcal{J}$ is obtained by summing up the prices of all items in the bundle. Thus, the items are priced separately, and there are no discounts for bundles. Note also that $\boldsymbol{p}^\star$ is feasible in $(\mathcal{DMDP})$ as $p_\emptyset^\star = 0$.

LEMMA 8. *We have* $z_{\mathrm{d}}'(0) \leq \frac{1}{2} \cdot \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c})$.

Proof. Since $\boldsymbol{p}^\star \in \mathcal{P}$, it suffices to show that

$$\max_{\boldsymbol{v} \in \mathcal{V}_\mathcal{S}(\boldsymbol{p}^\star, 0)} \mathbf{1}^\top (\boldsymbol{v} - \boldsymbol{c})^+ - (p_\mathcal{S}^\star - \mathbf{1}_\mathcal{S}^\top \boldsymbol{c}) \leq \frac{1}{2} \cdot \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c}) \tag{16}$$

for all $\mathcal{S} \subseteq \mathcal{J}$. Consider a fixed bundle $\mathcal{S} \subseteq \mathcal{J}$. Note that (16) trivially holds if $\mathcal{V}_\mathcal{S}(\boldsymbol{p}^\star, 0)$ is empty, in which case the left-hand side evaluates to $-\infty$. Suppose now that $\mathcal{V}_\mathcal{S}(\boldsymbol{p}^\star, 0) \neq \emptyset$ and recall that, for any $\boldsymbol{v} \in \mathcal{V}_\mathcal{S}(\boldsymbol{p}^\star, 0)$, the buyer weakly prefers bundle $\mathcal{S}$ over any larger bundle $\mathcal{S}' \supseteq \mathcal{S}$. Formally, for any $\boldsymbol{v} \in \mathcal{V}_\mathcal{S}(\boldsymbol{p}^\star, 0)$ and $\mathcal{S}' \supseteq \mathcal{S}$, we have

$$\mathbf{1}_\mathcal{S}^\top \boldsymbol{v} - p_\mathcal{S}^\star \geq \mathbf{1}_{\mathcal{S}'}^\top \boldsymbol{v} - p_{\mathcal{S}'}^\star \iff \mathbf{1}_\mathcal{S}^\top \boldsymbol{v} - \frac{1}{2} \cdot \mathbf{1}_\mathcal{S}^\top (\overline{\boldsymbol{v}} + \boldsymbol{c}) \geq \mathbf{1}_{\mathcal{S}'}^\top \boldsymbol{v} - \frac{1}{2} \cdot \mathbf{1}_{\mathcal{S}'}^\top (\overline{\boldsymbol{v}} + \boldsymbol{c})$$

$$\iff \frac{1}{2} \cdot \mathbf{1}_{\mathcal{S}' \setminus \mathcal{S}}^\top (\overline{\boldsymbol{v}} + \boldsymbol{c}) \geq \mathbf{1}_{\mathcal{S}' \setminus \mathcal{S}}^\top \boldsymbol{v}.$$

For any $j \notin \mathcal{S}$, evaluating the last inequality at $\mathcal{S}' = \mathcal{S} \cup \{j\}$ shows that $v_j \leq \frac{1}{2}(\overline{v}_j + c_j)$ for all $\boldsymbol{v} \in \mathcal{V}_\mathcal{S}(\boldsymbol{p}^\star, 0)$. For any $j \in \mathcal{S}$, on the other hand, we have the trivial upper bound $v_j \leq \overline{v}_j$ for all $\boldsymbol{v} \in \mathcal{V}_\mathcal{S}(\boldsymbol{p}^\star, 0)$. These uniform upper bounds on the components of $\boldsymbol{v}$ imply that

$$\mathbf{1}^\top (\boldsymbol{v} - \boldsymbol{c})^+ = \sum_{j \in \mathcal{S}} (v_j - c_j)^+ + \sum_{j \in \mathcal{J} \setminus \mathcal{S}} (v_j - c_j)^+ \leq \sum_{j \in \mathcal{S}} (\overline{v}_j - c_j)^+ + \sum_{j \in \mathcal{J} \setminus \mathcal{S}} \left(\frac{1}{2}(\overline{v}_j + c_j) - c_j \right)^+$$

$$= \sum_{j \in \mathcal{S}} (\overline{v}_j - c_j)^+ + \frac{1}{2} \sum_{j \in \mathcal{J} \setminus \mathcal{S}} (\overline{v}_j - c_j)^+$$

$$= \frac{1}{2} (2 \cdot \mathbf{1}_\mathcal{S} + \mathbf{1}_{\mathcal{J} \setminus \mathcal{S}})^\top (\overline{\boldsymbol{v}} - \boldsymbol{c})$$

$$= \frac{1}{2} (\mathbf{1} + \mathbf{1}_\mathcal{S})^\top (\overline{\boldsymbol{v}} - \boldsymbol{c}),$$

for all $\boldsymbol{v} \in \mathcal{V}_\mathcal{S}(\boldsymbol{p}^\star, 0)$, where the second equality follows from the standing assumption that $\boldsymbol{c} < \overline{\boldsymbol{v}}$. Thus, we find

$$\max_{\boldsymbol{v} \in \mathcal{V}_\mathcal{S}(\boldsymbol{p}^\star, 0)} \mathbf{1}^\top (\boldsymbol{v} - \boldsymbol{c})^+ - (p_\mathcal{S}^\star - \mathbf{1}_\mathcal{S}^\top \boldsymbol{c}) \leq \frac{1}{2}(\mathbf{1} + \mathbf{1}_\mathcal{S})^\top (\overline{\boldsymbol{v}} - \boldsymbol{c}) - (p_\mathcal{S}^\star - \mathbf{1}_\mathcal{S}^\top \boldsymbol{c}) = \frac{1}{2} \cdot \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c}),$$

where the equality follows from the definition of $\boldsymbol{p}^\star$. Thus, (16) holds for all $\mathcal{S} \subseteq \mathcal{J}$. This observation completes the proof. $\square$

REMARK 5. It is possible to prove that the inequality (16) holds in fact as an equality. Specifically, one can show that $\boldsymbol{v}^\star \in \mathbb{R}^J$ defined through $v_j^\star = \overline{v}_j$ if $j \in \mathcal{S}$; $= \frac{1}{2}(\overline{v}_j + c_j)$ otherwise, solves the maximization problem on the left-hand side of (16). However, this stronger statement does not help to prove that $\boldsymbol{p}^\star$ is optimal in $(\mathcal{DMDP})$. $\square$

LEMMA 9. *We have $z_{\mathrm{d}}'(\delta) \geq \frac{1}{2} \cdot \mathbf{1}^\top(\overline{\boldsymbol{v}} - \boldsymbol{c}) - J\delta$ for all $\delta \in (0, \frac{1}{2}\min_{j \in \mathcal{J}} \overline{v}_j - c_j]$.*

Note that the interval $(0, \frac{1}{2}\min_{j \in \mathcal{J}} \overline{v}_j - c_j]$ is non-empty because of the assumption that $\overline{\boldsymbol{v}} > \boldsymbol{c}$.

Proof of Lemma 9. Fix an arbitrary $\boldsymbol{p} \in \mathcal{P}$ and $\delta \in (0, \frac{1}{2}\min_{j \in \mathcal{J}} \overline{v}_j - c_j]$. The claim will follow if we can show that

$$\max_{\mathcal{S} \subseteq \mathcal{J}} \max_{\boldsymbol{v} \in \mathcal{V}_{\mathcal{S}}(\boldsymbol{p}, \delta)} \mathbf{1}^\top (\boldsymbol{v} - \boldsymbol{c})^+ - (p_{\mathcal{S}} - \mathbf{1}_{\mathcal{S}}^\top \boldsymbol{c}) \geq \frac{1}{2} \cdot \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c}) - J\delta. \tag{17}$$

To this end, we define the perturbation vector $\boldsymbol{\epsilon} = \boldsymbol{p} - \boldsymbol{p}^\star$ and prove (17) separately for the cases $\boldsymbol{\epsilon} \not\geq \mathbf{0}$ (Step 1) and $\boldsymbol{\epsilon} \geq \mathbf{0}$ (Step 2).

Step 1 ($\boldsymbol{\epsilon} \not\geq \mathbf{0}$): Denote by $\mathcal{S}'$ the bundle with the smallest perturbation, i.e., $\epsilon_{\mathcal{S}'} \leq \epsilon_{\mathcal{S}}$ for all $\mathcal{S} \subseteq \mathcal{J}$. As $\boldsymbol{\epsilon} \not\geq \mathbf{0}$, this implies that $\epsilon_{\mathcal{S}'} < 0$. Next, we define an auxiliary value profile $\hat{\boldsymbol{v}} \in \mathbb{R}^J$ through $\hat{v}_j = \overline{v}_j$ if $j \in \mathcal{S}'$; $= \frac{1}{2}(\overline{v}_j + c_j) - \delta$ otherwise. In order to establish (17), we will prove that $\hat{\boldsymbol{v}} \in \mathcal{V}_{\mathcal{S}'}(\boldsymbol{p}, \delta)$ and that the seller's regret under the value profile $\hat{\boldsymbol{v}}$ strictly exceeds $\frac{1}{2} \cdot \mathbf{1}^\top(\overline{\boldsymbol{v}} - \boldsymbol{c}) - J\delta$. By construction, $\mathbf{0} \leq \hat{\boldsymbol{v}} \leq \overline{\boldsymbol{v}}$ and thus $\hat{\boldsymbol{v}} \in \mathcal{V}$. In order to prove the stronger statement that $\hat{\boldsymbol{v}} \in \mathcal{V}_{\mathcal{S}'}(\boldsymbol{p}, \delta) \subseteq \mathcal{V}$, we first reformulate the buyer's utility from choosing bundle $\mathcal{S}'$ as

$$\mathbf{1}_{\mathcal{S}'}^\top \hat{\boldsymbol{v}} - p_{\mathcal{S}'} = \mathbf{1}_{\mathcal{S}'}^\top \overline{\boldsymbol{v}} - \left(\frac{1}{2} \cdot \mathbf{1}_{\mathcal{S}'}^\top(\overline{\boldsymbol{v}} + \boldsymbol{c}) + \epsilon_{\mathcal{S}'} \right) = \frac{1}{2} \cdot \mathbf{1}_{\mathcal{S}'}^\top(\overline{\boldsymbol{v}} - \boldsymbol{c}) - \epsilon_{\mathcal{S}'},$$

where the first equality exploits the relation $p_{\mathcal{S}'} = p_{\mathcal{S}'}^\star + \epsilon_{\mathcal{S}'}$, and compare it against his utility from choosing any other bundle $\mathcal{S} \subseteq \mathcal{J}$, which is given by

$$\begin{aligned}
\mathbf{1}_{\mathcal{S}}^\top \hat{\boldsymbol{v}} - p_{\mathcal{S}} &= \left(\mathbf{1}_{\mathcal{S} \cap \mathcal{S}'}^\top \overline{\boldsymbol{v}} + \frac{1}{2} \cdot \mathbf{1}_{\mathcal{S} \setminus \mathcal{S}'}^\top(\overline{\boldsymbol{v}} + \boldsymbol{c}) - |\mathcal{S} \setminus \mathcal{S}'|\delta \right) - \left(\frac{1}{2} \cdot \mathbf{1}_{\mathcal{S}}^\top(\overline{\boldsymbol{v}} + \boldsymbol{c}) + \epsilon_{\mathcal{S}} \right) \\
&= \frac{1}{2} \cdot \mathbf{1}_{\mathcal{S} \cap \mathcal{S}'}^\top(\overline{\boldsymbol{v}} - \boldsymbol{c}) - |\mathcal{S} \setminus \mathcal{S}'|\delta - \epsilon_{\mathcal{S}}.
\end{aligned}$$

As $\boldsymbol{c} < \overline{\boldsymbol{v}}$, $\mathcal{S} \cap \mathcal{S}' \subseteq \mathcal{S}'$ and $\epsilon_{\mathcal{S}'} \leq \epsilon_{\mathcal{S}}$, it becomes evident that the buyer's utility arising from bundle $\mathcal{S}'$ exceeds that from bundle $\mathcal{S}$ by at least $\frac{1}{2} \cdot \mathbf{1}_{\mathcal{S}' \setminus \mathcal{S}}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c}) + |\mathcal{S} \setminus \mathcal{S}'| \delta \geq \delta$. Thus, we have $\hat{\boldsymbol{v}} \in \mathcal{V}_{\mathcal{S}'}(\boldsymbol{p}, \delta)$. Next, the seller's regret under the value profile $\hat{\boldsymbol{v}}$ can be expressed as

$$
\begin{aligned}
\mathbf{1}^\top (\hat{\boldsymbol{v}} - \boldsymbol{c})^+ - (p_{\mathcal{S}'} - \mathbf{1}_{\mathcal{S}'}^\top \boldsymbol{c}) &= \mathbf{1}^\top (\hat{\boldsymbol{v}} - \boldsymbol{c}) - (p_{\mathcal{S}'} - \mathbf{1}_{\mathcal{S}'}^\top \boldsymbol{c}) \\
&= \mathbf{1}_{\mathcal{S}'}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c}) + \mathbf{1}_{\mathcal{J} \setminus \mathcal{S}'}^\top \left(\frac{1}{2} (\overline{\boldsymbol{v}} + \boldsymbol{c}) - \boldsymbol{c} - \delta \cdot \mathbf{1} \right) - (p_{\mathcal{S}'} - \mathbf{1}_{\mathcal{S}'}^\top \boldsymbol{c}) \\
&= \frac{1}{2} \left(2 \cdot \mathbf{1}_{\mathcal{S}'} + \mathbf{1}_{\mathcal{J} \setminus \mathcal{S}'} \right)^\top (\overline{\boldsymbol{v}} - \boldsymbol{c}) - \left(p_{\mathcal{S}'} - \mathbf{1}_{\mathcal{S}'}^\top \boldsymbol{c} \right) - (J - |\mathcal{S}'|) \delta \\
&= \frac{1}{2} \left(\mathbf{1} + \mathbf{1}_{\mathcal{S}'} \right)^\top (\overline{\boldsymbol{v}} - \boldsymbol{c}) - \left(p_{\mathcal{S}'} - \mathbf{1}_{\mathcal{S}'}^\top \boldsymbol{c} \right) - (J - |\mathcal{S}'|) \delta \\
&= \frac{1}{2} \cdot \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c}) - \epsilon_{\mathcal{S}'} - (J - |\mathcal{S}'|) \delta \\
&> \frac{1}{2} \cdot \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c}) - J \delta,
\end{aligned}
$$

where the first equality holds because $\delta \leq \frac{1}{2} \min_{j \in \mathcal{J}} \overline{v}_j - c_j$ while the second and the last equalities follow from the definitions of $\hat{\boldsymbol{v}}$ and $\boldsymbol{p}$, respectively. This concludes Step 1 because

$$
\max_{\mathcal{S} \subseteq \mathcal{J}} \max_{\boldsymbol{v} \in \mathcal{V}_{\mathcal{S}}(\boldsymbol{p}, \delta)} \mathbf{1}^\top (\boldsymbol{v} - \boldsymbol{c})^+ - (p_{\mathcal{S}} - \mathbf{1}_{\mathcal{S}}^\top \boldsymbol{c}) \geq \mathbf{1}^\top (\hat{\boldsymbol{v}} - \boldsymbol{c})^+ - (p_{\mathcal{S}'} - \mathbf{1}_{\mathcal{S}'}^\top \boldsymbol{c}) > \frac{1}{2} \cdot \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c}) - J \delta.
$$

Step 2 ($\boldsymbol{\epsilon} \geq \mathbf{0}$): Recall that $\mathcal{V}_\emptyset(\boldsymbol{p}, \delta) = \left\{ \boldsymbol{v} \in \mathcal{V} : \mathbf{1}_{\mathcal{S}}^\top \boldsymbol{v} + \delta \leq p_{\mathcal{S}}, \forall \mathcal{S} \subseteq \mathcal{J} : \mathcal{S} \neq \emptyset \right\}$. In analogy to Step 1, we set $\hat{\boldsymbol{v}} = \frac{1}{2} \cdot (\overline{\boldsymbol{v}} + \boldsymbol{c}) - \delta \cdot \mathbf{1}$. By construction, $\mathbf{0} \leq \hat{\boldsymbol{v}} \leq \overline{\boldsymbol{v}}$ and thus $\hat{\boldsymbol{v}} \in \mathcal{V}$. Next, we prove the stronger statement that $\hat{\boldsymbol{v}} \in \mathcal{V}_\emptyset(\boldsymbol{p}, \delta)$. Indeed, a direct calculation reveals that

$$
\mathbf{1}_{\mathcal{S}}^\top \hat{\boldsymbol{v}} + \delta = \frac{1}{2} \cdot \mathbf{1}_{\mathcal{S}}^\top (\overline{\boldsymbol{v}} + \boldsymbol{c}) - (|\mathcal{S}| - 1) \delta \leq \frac{1}{2} \cdot \mathbf{1}_{\mathcal{S}}^\top (\overline{\boldsymbol{v}} + \boldsymbol{c}) = p_{\mathcal{S}}^\star \leq p_{\mathcal{S}} \quad \forall \mathcal{S} \neq \emptyset.
$$

Furthermore, the seller's regret under the value profile $\hat{\boldsymbol{v}}$ can be expressed as

$$
\mathbf{1}^\top (\hat{\boldsymbol{v}} - \boldsymbol{c})^+ - (p_\emptyset - \mathbf{1}_\emptyset^\top \boldsymbol{c}) = \mathbf{1}^\top (\hat{\boldsymbol{v}} - \boldsymbol{c})^+ = \frac{1}{2} \cdot \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c}) - J \delta,
$$

where the equality holds because $\delta \leq \frac{1}{2} \min_{j \in \mathcal{J}} \overline{v}_j - c_j$. This concludes Step 2 because

$$
\max_{\mathcal{S} \subseteq \mathcal{J}} \max_{\boldsymbol{v} \in \mathcal{V}_{\mathcal{S}}(\boldsymbol{p}, \delta)} \mathbf{1}^\top (\boldsymbol{v} - \boldsymbol{c})^+ - (p_{\mathcal{S}} - \mathbf{1}_{\mathcal{S}}^\top \boldsymbol{c}) \geq \mathbf{1}^\top (\hat{\boldsymbol{v}} - \boldsymbol{c})^+ - (p_\emptyset - \mathbf{1}_\emptyset^\top \boldsymbol{c}) \geq \frac{1}{2} \cdot \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c}) - J \delta.
$$

Thus, the claim follows. $\quad \square$

THEOREM 3. *Define* $\boldsymbol{p}^\star$ *as in* (15), *and choose any* $s^\star \in \mathcal{L}(\mathcal{V}, 2^{\mathcal{J}})$ *such that* $s^\star(\boldsymbol{v}) \in$ $\arg\max_{\mathcal{S} \subseteq \mathcal{J}} \mathbf{1}_{\mathcal{S}}^\top \boldsymbol{v} - p_{\mathcal{S}}^\star$ *for all* $\boldsymbol{v} \in \mathcal{V}$. *Then, the posted price mechanism* $(s^\star, \boldsymbol{p}^\star)$ *is optimal in* $(\mathcal{DMDP})$, *and the optimal value of* $(\mathcal{DMDP})$ *is given by* $\frac{1}{2} \cdot \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c})$.

Proof. Lemmas 7–9 imply that

$$\frac{1}{2} \cdot \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c}) \geq z_{\mathrm{d}}'(0) \geq z_{\mathrm{d}}^\star \geq \liminf_{\delta \downarrow 0} z_{\mathrm{d}}'(\delta) \geq \frac{1}{2} \cdot \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c}).$$

Hence, the optimal value of $(\mathcal{DMDP})$ equals $z_{\mathrm{d}}^\star = \frac{1}{2} \cdot \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c})$. It is therefore sufficient to show that the worst-case regret of $(s^\star, \boldsymbol{p}^\star)$ is bounded below by $\frac{1}{2} \cdot \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c})$.

Using similar arguments as in the proof of Lemma 7, one can show that the objective value of $(s^\star, \boldsymbol{p}^\star)$ in $(\mathcal{DMDP})$ is bounded below by

$$\max_{\mathcal{S} \subseteq \mathcal{J}} \max_{\boldsymbol{v} \in \mathcal{V}_{\mathcal{S}}(\boldsymbol{p}^\star, \delta)} \mathbf{1}^\top (\boldsymbol{v} - \boldsymbol{c})^+ - (p_{\mathcal{S}}^\star - \mathbf{1}_{\mathcal{S}}^\top \boldsymbol{c}) \geq \frac{1}{2} \cdot \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c}) - J\delta$$

for any sufficiently small $\delta > 0$, where the inequality follows from (17). By driving δ to 0, we may conclude that $(s^\star, \boldsymbol{p}^\star)$ attains the worst-case regret of at least $\frac{1}{2} \cdot \mathbf{1}^\top (\overline{\boldsymbol{v}} - \boldsymbol{c})$, and thus the claim follows. $\square$

Note that any optimal solution $(s^\star, \boldsymbol{p}^\star)$ of $(\mathcal{DMDP})$ gives rise to an optimal solution $(\boldsymbol{q}^\star, m^\star)$ of the original mechanism design problem $(\mathcal{MDP})$ restricted to deterministic allocation rules, where $\boldsymbol{q}^\star(\boldsymbol{v}) = \mathbf{1}_{s^\star(\boldsymbol{v})}$ and $m^\star(\boldsymbol{v}) = p_{s^\star(\boldsymbol{v})}^\star$ for all $\boldsymbol{v} \in \mathcal{V}$.

REMARK 6. By Definition 3, under any posted price mechanism the buyer is forced to purchase one single (possibly empty) bundle. If the seller offers a menu of bundles, however, then this restriction would not apply, and the buyer could acquire multiple bundles. In this case the seller would have to ensure that the price of a bundle $\mathcal{S} \subseteq \mathcal{J}$ satisfies $p_{\mathcal{S}} \leq \sum_{i \in \mathcal{I}} p_{\mathcal{S}_i}$ for all possible partitions $\{\mathcal{S}_i : i \in \mathcal{I}\}$ of $\mathcal{S}$. Put differently, the seller might offer discounts but could never charge markups on bundles. Otherwise, the buyer would never choose a bundle that becomes cheaper when split into disjoint subsets. Even though the deterministic pricing problem $(\mathcal{DMDP})$ principally grants the seller the right to charge markups, the optimal mechanism is separable and does not capitalize on this flexibility. $\square$

Theorems 1 and 3 immediately imply the following corollary.

COROLLARY 1. *The optimal deterministic mechanism provides an $\frac{e}{2}$-approximation to the optimal worst-case regret in $(\mathcal{MDP})$.*

Corollary 1 suggests that a seller who implements the optimal posted price mechanism instead of the optimal randomized mechanism derived in Section 3 increases her worst-case regret by $\frac{e-2}{2} \approx 36\%$.

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