

# A Minimal Contrast Estimator for the Linear Fractional Stable Motion

Mathias Mørck Ljungdahl\*      Mark Podolskij†

March 26, 2020

## Abstract

In this paper we present an estimator for the three-dimensional parameter  $(\sigma, \alpha, H)$  of the linear fractional stable motion, where  $H$  represents the self-similarity parameter, and  $(\sigma, \alpha)$  are the scaling and stability parameters of the driving symmetric Lévy process  $L$ . Our approach is based upon a minimal contrast method associated with the empirical characteristic function combined with a ratio type estimator for the self-similarity parameter  $H$ . The main result investigates the strong consistency and weak limit theorems for the resulting estimator. Furthermore, we propose several ideas to obtain feasible confidence regions in various parameter settings. Our work is mainly related to [17, 19], in which parameter estimation for the linear fractional stable motion and related Lévy moving average processes has been studied.

*Keywords:* linear fractional processes, Lévy processes, limit theorems, parametric estimation, bootstrap, subsampling, self-similarity, low frequency.

*AMS 2010 subject classifications:* Primary 60G22, 62F12, 62E20; secondary 60E07, 60F05, 60G10.

## 1 Introduction

During the last sixty years fractional stochastic processes have received a great deal of attention in probability, statistics and integration theory. One of the most prominent examples of a fractional model is the (scaled) fractional Brownian motion (fBm), which gained a lot of popularity in science since the pioneering work of Mandelbrot and van Ness [18]. The scaled fBm is the unique zero mean Gaussian process with stationary increments and self-similarity property. As a building block in stochastic models it found numerous applications in natural and social sciences such as physics, biology and economics. From the statistical perspective the scaled fBm is fully determined by its scale parameter  $\sigma > 0$  and the self-similarity parameter (or Hurst index)  $H \in (0, 1)$ . Nowadays, the estimation of

---

\*Department of Mathematics, Aarhus University, ljungdahl@math.au.dk.

†Department of Mathematics, Aarhus University, mpodolskij@math.au.dk.

$(\sigma, H)$  is a well understood problem. We refer to [11] for efficient estimation of the Hurst parameter  $H$  in the low frequency setting, and to [6, 9, 15] for the estimation of  $(\sigma, H)$  in the high frequency setting, among many others. In more recent papers [2, 16] statistical inference for the multifractional Brownian motion has been investigated, which accounts for the time varying nature of the Hurst parameter.

This paper focuses on another extension of the fBm, the *linear fractional stable motion* (lfsm). The lfsm  $(X_t)_{t \geq 0}$  is a three-parameter statistical model defined by

$$X_t = \int_{\mathbb{R}} \{(t-s)_+^{H-1/\alpha} - (-s)_+^{H-1/\alpha}\} dL_s, \quad (1.1)$$

where  $x_+ = x \vee 0$  denotes the positive part and we set  $x_+^a = 0$  for all  $a \in \mathbb{R}$ ,  $x \leq 0$ . Here  $(L_t)_{t \in \mathbb{R}}$  is a symmetric  $\alpha$ -stable Lévy process with  $\alpha \in (0, 2)$  and scale parameter  $\sigma > 0$ , and  $H \in (0, 1)$  represents the Hurst parameter. In some sense the lfsm is a non-Gaussian analogue of fBm. The process  $(X_t)_{t \geq 0}$  has symmetric  $\alpha$ -stable marginals, stationary increments and it is self-similar with parameter  $H$ . It is well known that the process  $X$  has continuous paths when  $H - 1/\alpha > 0$ , see e.g. [5]. We remark that the class of stationary increments self-similar processes becomes much larger if we drop the Gaussianity assumption, cf. [21, 24], but the lfsm is one of its most famous representatives due to the ergodicity property. Linear fractional stable motions are often used in natural sciences, e.g. in physics or Internet traffic, where the process under consideration exhibits stationarity and self-similarity along with heavy tailed marginals, see e.g. [13] for the context of Ethernet and solar flare modelling.

The limit theory for statistics of lfsm, which is indispensable for the estimation of the parameter  $\xi = (\sigma, \alpha, H)$ , turns out to be of a quite complex nature. First central limit theorems for partial sums of bounded functions of Lévy moving average processes, which in particular include the lfsm, have been discussed in [22] and later extended in [23] to certain unbounded functions. In a more recent work [4] the authors presented a rather complete asymptotic theory for power variations of stationary increments Lévy moving average processes. Finally, the results of [4] have been extended to general functions in [3], who demonstrated that the weak limit theory crucially depends on the *Appell rank* of the given function and the parameters of the model (all functions considered in this paper have Appell rank 2). More specifically, they obtained three different asymptotic regimes, a normal and two stable ones, depending on the particular setting. It is this phase transition that depends on the parameter  $(\alpha, H)$  which makes the statistical inference for lfsm a rather complicated matter.

Since the probabilistic theory for functionals of lfsm was not well understood until the recent work [3, 4], the statistical literature on estimation of lfsm is rather scarce. The articles [1, 23] investigate the asymptotic theory for a wavelet-based estimator of  $H$  when  $\alpha \in (1, 2)$ . In [4, 27] the authors use power variation statistics to obtain an estimator of  $H$ , but this method also requires the a priori knowledge of the lower bound for the stability parameter  $\alpha$ . The work [12] suggested to use negative power variations to get a consistent estimator of  $H$ , which applies for any  $\alpha \in (0, 2)$ , but this article does not contain a central limit theorem for this method. The paper [19] was the first instance, where estimation of the full parameter  $\xi = (\sigma, \alpha, H)$  has been studied in low and high frequency settings.

Their idea is based upon the identification of  $\xi$  through power variation statistics and the empirical characteristic function evaluated at two different values.

In this paper we aim at extending the approach of [19] by determining the asymptotic theory for the minimal contrast estimator of the parameter  $\xi = (\sigma, \alpha, H)$ , which is based upon the comparison of the empirical characteristic function with its theoretical counterpart under low frequency sampling. Indeed, the choice of the two evaluation points for the empirical characteristic function in [19] is rather ad hoc and we will show in the empirical study that the minimal contrast estimator exhibits better finite sample properties and robustness in various settings. Similarly to [19], we will show that the weak limit theory for our estimator has a normal and a stable regime, and the asymptotic distribution depends on the interplay between the parameters  $\alpha$  and  $H$ . At this stage we remark that the minimal contrast approach has been investigated in [17] in the context of certain Lévy moving average models, which do not include the lfsm or its associated noise process, but only in the asymptotically normal regime. Another important contribution of our paper is the subsampling procedure, which provides confidence regions for the parameters of the model irrespectively of the unknown asymptotic regime.

The article is organized as follows. In Section 2 we introduce the necessary notation and formulate a new weak limit result related to [19], which is central to their parameter estimation for the linear fractional stable motion. The aforementioned theorem will be our starting point, where the aim is to extend the convergence of the finite dimensional distributions to convergence of integral functionals appearing in the minimal contrast method. Section 3 introduces the estimator and presents the main results of strong consistency and asymptotic distribution. Section 4 is devoted to a simulation study, which tests the finite sample performance of the minimal contrast estimator. We also discuss the parametric bootstrap and the subsampling method that are used to construct feasible confidence regions for the true parameters of the model. All proofs are collected in Section 5 and all larger tables are in Section 6.

## 2 Notation and recent results

We start out with introducing the main notation and statistics of interest. We consider low frequency observations  $X_1, X_2, \dots, X_n$  from the lfsm  $(X_t)_{t \geq 0}$  introduced in (1.1). We denote by  $\Delta_{i,k}^r X$  ( $i, k, r \in \mathbb{N}$ ) the  $k$ th order increment of  $X$  at stage  $i$  and rate  $r$ , i.e.

$$\Delta_{i,k}^r X = \sum_{j=0}^k (-1)^j \binom{k}{j} X_{i-rj}, \quad i \geq rk.$$

The order  $k$  plays a crucial role in determining the asymptotic regime for statistics that we introduce below. We let the function  $h_{k,r} : \mathbb{R} \rightarrow \mathbb{R}$  denote the  $k$ th order increment at rate  $r$  of the kernel in (1.1), specifically

$$h_{k,r}(x) = \sum_{j=0}^k (-1)^j \binom{k}{j} (x - rj)_+^{H-1/\alpha}, \quad x \in \mathbb{R}.$$

We note that  $\Delta_{i,k}^r X = \int_{\mathbb{R}} h_{k,r}(i-s) dL_s$ . For less cumbersome notation we drop the index  $r$  if  $r = 1$ , so  $\Delta_{i,k} X := \Delta_{i,k}^1 X$  and  $h_k = h_{k,1}$ . Throughout this paper we write  $\theta = (\sigma, \alpha)$  and  $\xi = (\sigma, \alpha, H)$ . The main probabilistic tools are statistics of the type

$$V_n(f, k, r) = \frac{1}{n} \sum_{i=rk}^n f(\Delta_{i,k}^r X),$$

where  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a Borel function satisfying  $\mathbb{E}[|f(\Delta_{rk,k}^r X)|] < \infty$ . We will specifically focus on two classes of functions, namely  $f_p(x) = |x|^p$  for  $p \in (-1, 1)$  and  $\delta_t(x) = \cos(tx)$  for  $t \geq 0$ . They correspond to power variation statistics and the real part of the empirical characteristic function respectively, and we use the notation

$$\varphi_n(t) = V_n(\delta_t, k, 1) \quad \text{and} \quad \psi_n(r) = V_n(f_p, k, r). \quad (2.2)$$

We note that Birkhoff's ergodic theorem implies the almost sure convergence

$$\varphi_n(t) \xrightarrow{\text{a.s.}} \varphi_\xi(t) := \exp(-|\sigma \|h_k\|_\alpha t|^\alpha) \quad \text{where} \quad \|h_k\|_\alpha^\alpha := \int_{\mathbb{R}} |h_k(x)|^\alpha dx. \quad (2.3)$$

An important coefficient in our context is

$$\beta = 1 + \alpha(k - H). \quad (2.4)$$

The rate of convergence and the asymptotic distribution of statistics defined at (2.2) crucially depend on whether the condition  $k > H + 1/\alpha$  is satisfied or not. Hence, we define the normalised versions of our statistics as

$$W_n^1(r) = \sqrt{n}(\psi_n(r) - r^H m_{p,k}), \quad W_n^2(t) = \sqrt{n}(\varphi_n(t) - \varphi_\xi(t)) \quad \text{when } k > H + 1/\alpha$$

and

$$S_n^1(r) = n^{1-1/\beta}(\psi_n(r) - r^H m_{p,k}), \quad S_n^2(t) = n^{1-1/\beta}(\varphi_n(t) - \varphi_\xi(t)) \quad \text{when } k < H + 1/\alpha.$$

Here  $m_{p,k} = \mathbb{E}[|\Delta_{k,k} X|^p]$ , which is finite for any  $p \in (-1, \alpha)$ , and  $\varphi_\xi$  is given at (2.3). Note that  $\mathbb{E}[|\Delta_{rk,k}^r X|^p] = r^H m_{p,k}$ , which explains the centring of  $W^1$  and  $S^1$ .

It turns out that the finite dimensional limit of the statistics  $(W_n^1, W_n^2)$  is Gaussian while the corresponding limit of  $(S_n^1, S_n^2)$  is  $\beta$ -stable (see Theorem 2.1 below). We now introduce several notations to describe the limiting distribution. We start with the Gaussian case. For random variables  $X = \int_{\mathbb{R}} g(s) dL_s$  and  $Y = \int_{\mathbb{R}} h(s) dL_s$  with  $\|g\|_\alpha, \|h\|_\alpha < \infty$  we define a dependence measure  $U_{g,h} : \mathbb{R}^2 \rightarrow \mathbb{R}$  as

$$\begin{aligned} U_{g,h}(u, v) &= \mathbb{E}[\exp(i(uX + vY))] - \mathbb{E}[\exp(iuX)]\mathbb{E}[\exp(ivY)] \\ &= \exp(-\sigma^\alpha \|ug + vh\|_\alpha^\alpha) - \exp(-\sigma^\alpha (\|ug\|_\alpha^\alpha + \|vh\|_\alpha^\alpha)). \end{aligned} \quad (2.5)$$

Next, for  $p \in (-1, 1) \setminus \{0\}$ , we introduce the constant

$$a_p := \begin{cases} \int_{\mathbb{R}} (1 - \cos(y)) |y|^{-1-p} dy : & p \in (0, 1) \\ \sqrt{2\pi} \Gamma(-p/2) / 2^{p+1/2} \Gamma((p+1)/2) : & p \in (-1, 0), \end{cases} \quad (2.6)$$

where  $\Gamma$  denotes the Gamma function. Now, for each  $t \in \mathbb{R}_+$ , we set

$$\begin{aligned}\Sigma_{11}(g, h) &= a_p^{-2} \int_{\mathbb{R}^2} |xy|^{-1-p} U_{g,h}(x, y) \, dx \, dy, \\ \Sigma_{12}(g, h; t) &= a_p^{-1} \int_{\mathbb{R}^2} |y|^{-1-p} U_{g,h}(x, t) \, dx\end{aligned}\tag{2.7}$$

whenever the above integrals are finite. As it has been shown in [19], the following identities hold for any  $p \in (-1/2, 1/2) \setminus \{0\}$  with  $p < \alpha/2$  and  $t \in \mathbb{R}_+$ :

$$\text{Cov}(|X|^p, |Y|^p) = \Sigma_{11}(g, h) \quad \text{and} \quad \text{Cov}(|X|^p, \exp(itY)) = \Sigma_{12}(g, h; t).$$

Obviously, the quantities  $\Sigma_{11}(g, h)$  and  $\Sigma_{12}(g, h; t)$  will appear in the asymptotic covariance kernel of the vector  $(\psi_n(r), \varphi_n(t))$  in the normal regime.

Now, we introduce the necessary notations for the stable case. First, we define the functions

$$\begin{aligned}\Phi_r^1(x) &:= a_p^{-1} \int_{\mathbb{R}} (1 - \cos(ux)) \exp(-|\sigma \|h_{k,r}\|_{\alpha} |u|^{\alpha}) |u|^{-1-p} \, du, \quad r \in \mathbb{N}, \\ \Phi_t^2(x) &:= (\cos(tx) - 1) \exp(-|\sigma \|h_k\|_{\alpha} t^{\alpha}), \quad t \geq 0,\end{aligned}\tag{2.8}$$

and set

$$q_{H,\alpha,k} := \prod_{i=0}^{k-1} (H - 1/\alpha - i).$$

Next, we introduce the functions  $\kappa_1 : \mathbb{N} \rightarrow \mathbb{R}_+$  and  $\kappa_2 : \mathbb{R} \rightarrow \mathbb{R}_-$  via

$$\kappa_1(r) := \frac{\alpha}{\beta} \int_0^{\infty} \Phi_r^1(q_{H,\alpha,k} z) z^{-1-\alpha/\beta} \, dz, \quad \kappa_2(t) := \frac{\alpha}{\beta} \int_0^{\infty} \Phi_t^2(q_{H,\alpha,k} z) z^{-1-\alpha/\beta} \, dz.\tag{2.9}$$

In the final step we will need to define two Lévy measures  $\nu_1$  on  $(\mathbb{R}_+)^2$  and  $\nu_2$  on  $\mathbb{R}_+$  that are necessary to determine the asymptotic distribution of  $(S_n^1, S_n^2)$ . Let us denote by  $\nu$  the Lévy measure of the symmetric  $\alpha$ -stable Lévy motion  $L$  (i.e.  $\nu(dx) = c(\sigma) |x|^{-1-\alpha} dx$ ) and define the mappings  $\tau_1 : \mathbb{R} \rightarrow (\mathbb{R}_+)^2$  and  $\tau_2 : \mathbb{R} \rightarrow \mathbb{R}_+$  via

$$\tau_1(x) = |x|^{\alpha/\beta} (\kappa_1(1), \kappa_1(2)), \quad \tau_2(x) = |x|^{\alpha/\beta}.$$

Then, for any Borel sets  $A_1 \subseteq (\mathbb{R}_+)^2$  and  $A_2 \subseteq \mathbb{R}_+$  bounded away from  $(0,0)$  and  $0$ , respectively, we introduce

$$\nu_l(A_l) := \nu(\tau_l^{-1}(A_l)), \quad l = 1, 2.\tag{2.10}$$

In the weak limit theorem below we write  $Z^n \xrightarrow{\mathcal{L}\text{-f}} Z$  to denote the convergence of finite dimensional distributions, i.e. the convergence in distribution

$$(Z_{t_1}^n, \dots, Z_{t_d}^n) \xrightarrow{d} (Z_{t_1}, \dots, Z_{t_d})$$

for any  $d \in \mathbb{N}$  and  $t_j \in \mathbb{R}_+$ . The following theorem is key for statistical applications.

**Theorem 2.1.** *Assume that either  $p \in (-1/2, 0)$  or  $p \in (0, 1/2)$  together with  $p < \alpha/2$ .*

(i) *If  $k > H + 1/\alpha$  then as  $n \rightarrow \infty$*

$$(W_n^1(1), W_n^1(2), W_n^2(t)) \xrightarrow{\mathcal{L}\text{-f}} (W_1^1, W_2^1, W_t),$$

*where  $W^1 = (W_1^1, W_2^1)$  is a centered 2-dimensional normal distribution and  $(W_t)_{t \geq 0}$  is a centred Gaussian process with*

$$\begin{aligned} \text{Cov}(W_j^1, W_{j'}^1) &= \sum_{l \in \mathbb{Z}} \Sigma_{11}(h_{k,j}, h_{k,j'}(\cdot + l)) & j, j' = 1, 2, \\ \text{Cov}(W_j^1, W_t) &= \sum_{l \in \mathbb{Z}} \Sigma_{12}(h_{k,j}, h_k(\cdot + l); t) & j = 1, 2, t \in \mathbb{R}_+, \\ \text{Cov}(W_s, W_t) &= \frac{1}{2} \sum_{l \in \mathbb{Z}} (U_{h_k, h_k(\cdot + l)}(s, t) + U_{h_k, -h_k(\cdot + l)}(s, t)) & s, t \in \mathbb{R}, \end{aligned}$$

*where the quantity  $\Sigma_{ij}$  has been introduced at (2.7). Moreover, the Gaussian process  $W$  exhibits a modification (denoted again by  $W$ ), which is locally Hölder continuous of any order smaller than  $\alpha/4$ .*

(ii) *If  $k < H + 1/\alpha$  then as  $n \rightarrow \infty$*

$$(S_n^1(1), S_n^1(2), S_n^2(t)) \xrightarrow{\mathcal{L}\text{-f}} (S_1^1, S_2^1, \kappa_2(t)S).$$

*where  $S^1 = (S_1^1, S_2^1)$  is a  $\beta$ -stable random vector with Lévy measure  $\nu_1$  independent of the totally right skewed  $\beta$ -stable random variable  $S$  with Lévy measure  $\nu_2$ , and  $\nu_1, \nu_2$  have been defined in (2.10).*

The finite dimensional asymptotic distribution demonstrated in Theorem 2.1 is a direct consequence of [19, Theorem 2.2], which even contains a more general multivariate result. However, the smoothness property of the limiting Gaussian process  $W$  and the particular form of the limit of  $S_n^2$  have not been investigated in [19]. Both properties are crucial for the statistical analysis of the minimal contrast estimator.

We observe that from a statistical perspective it is more favourable to use Theorem 2.1(i) to estimate the parameter  $\xi = (\sigma, \alpha, H)$ , since the convergence rate  $\sqrt{n}$  in (i) is faster than the rate  $n^{1-1/\beta}$  in (ii). However, the phase transition happens at the point  $k = H + 1/\alpha$ , which depends on unknown parameters  $\alpha$  and  $H$ . This poses major difficulties in statistical applications and we will address this issue in the forthcoming discussion.

### 3 Main results

In this section we describe our minimal contrast approach and present the corresponding asymptotic theory. Before stating our main result we define a power variation based estimator of the parameter  $H \in (0, 1)$ . Since the increments of the process  $(X_t)_{t \geq 0}$  are

strongly ergodic (cf. [8]), we deduce by Birkhoff's ergodic theorem the almost sure convergence

$$\psi_n(r) = \frac{1}{n} \sum_{i=rk}^n |\Delta_{i,k}^r X|^p \xrightarrow{\text{a.s.}} \mathbb{E}[|\Delta_{rk,k}^r X|^p] = r^{pH} m_{p,k}$$

for any  $p \in (-1, \alpha)$ . In particular, we have that

$$R_n(p, k) := \frac{\psi_n(2)}{\psi_n(1)} \xrightarrow{\text{a.s.}} 2^{pH},$$

consequently yielding a consistent estimator  $H_n(p, k)$  of  $H$  as

$$H_n(p, k) = \frac{1}{p} \log_2(R_n(p, k)) \xrightarrow{\text{a.s.}} H \quad (3.11)$$

for any  $p \in (-1, \alpha)$ . The idea of using negative powers  $p \in (-1, 0)$  to estimate  $H$ , which has been proposed in [12] and applied in [19], has the obvious advantage that it does not require knowledge of the parameter  $\alpha$ . From Theorem 2.1(i) and the  $\delta$ -method applied to the function  $v_p(x, y) = \frac{1}{p}(\log_2(x) - \log_2(y))$  we immediately deduce the convergence

$$(\sqrt{n}(H_n(p, k) - H), W_n^2(t)) \xrightarrow{\mathcal{L}\text{-f}} (M_1, W_t) \quad (3.12)$$

for  $k > H + 1/\alpha$ , where  $M_1$  is a centred Gaussian random variable. Similarly, when  $k < H + 1/\alpha$ , we deduce the convergence

$$(n^{1-1/\beta}(H_n(p, k) - H), S_n^2(t)) \xrightarrow{\mathcal{L}\text{-f}} (M_2, \kappa_2(t)S) \quad (3.13)$$

from Theorem 2.1(ii).

We will now introduce the minimal contrast estimator of the parameter  $\theta = (\sigma, \alpha)$ . Let  $w \in \mathcal{L}^1(\mathbb{R}_+)$  denote a positive weight function. Define for  $r > 1$  the norm

$$\|h\|_{w,r} = \left( \int_0^\infty |h(t)|^r w(t) dt \right)^{1/r},$$

where  $h : \mathbb{R}_+ \rightarrow \mathbb{R}$  is a Borel function. Denote by  $\mathcal{L}_w^r(\mathbb{R}_+)$  the space of functions  $h$  with  $\|h\|_{w,r} < \infty$ . Let  $(\theta_0, H_0) = (\sigma_0, \alpha_0, H_0) \in (0, \infty) \times (0, 2) \times (0, 1)$  be the true parameter of the model (1.1) and consider an open neighbourhood  $\Theta_0 \subseteq (0, \infty) \times (0, 2)$  around  $\theta_0$  bounded away from  $(0, 0)$ . Define the map  $F : \mathcal{L}_w^2(\mathbb{R}_+) \times (0, 1) \times \Theta_0 \rightarrow \mathbb{R}$  as

$$F(\varphi, H, \theta) = \|\varphi - \varphi_{\theta,H}\|_{w,2}^2, \quad (3.14)$$

where  $\varphi_{\theta,H}$  is the limit introduced at (2.3). We define the minimal contrast estimator  $\theta_n$  of  $\theta_0$  as

$$\theta_n \in \operatorname{argmin}_{\theta \in \Theta_0} F(\varphi_n, H_n(p, k), \theta). \quad (3.15)$$

We remark that by e.g. [26, Theorem 2.17] it is possible to choose  $\theta_n$  universally measurable with respect to the underlying probability space. The joint estimator of  $(\theta_0, H_0)$  is then given as

$$\xi_n = (\theta_n, H_n(p, k))^\top.$$

Below we denote by  $\nabla_\theta$  the gradient with respect to the parameter  $\theta$  and similarly by  $\nabla_\theta^2$  the Hessian. We further write  $\partial_z f$  for the partial derivative of  $f$  with respect to  $z \in \{\sigma, \alpha, H\}$ . The main theoretical result of this paper is the following theorem.

**Theorem 3.1.** *Suppose  $\xi_0 = (\theta_0, H_0)$  is the true parameter of the linear fractional stable motion  $(X_t)_{t \geq 0}$  at (1.1) and that the weight function  $w \in \mathcal{L}^1(\mathbb{R}_+)$  is continuous.*

(i)  $\xi_n \xrightarrow{\text{a.s.}} \xi_0$  as  $n \rightarrow \infty$ .

(ii) If  $k > H + 1/\alpha$  then

$$\sqrt{n}(\xi_n - \xi_0) \xrightarrow{d} \left[ -2\nabla_{\theta}^2 F(\varphi_{\xi_0}, \xi_0)^{-1} \left( \int_0^{\infty} W_t \nabla_{\theta} \varphi_{\xi_0}(t) w(t) dt + \partial_H \nabla_{\theta} F(\varphi_{\xi_0}, \xi_0) M_1 \right), M_1 \right]^{\top}.$$

(iii) If  $k < H + 1/\alpha$  then

$$n^{1-1/\beta}(\xi_n - \xi_0) \xrightarrow{d} \left[ -2\nabla_{\theta}^2 F(\varphi_{\xi_0}, \xi_0)^{-1} \left( S \int_0^{\infty} \kappa_2(t) \nabla_{\theta} \varphi_{\xi_0}(t) w(t) dt + \partial_H \nabla_{\theta} F(\varphi_{\xi_0}, \xi_0) M_2 \right), M_2 \right]^{\top}.$$

In principle, the statement of Theorem 3.1 follows from (3.12), (3.13) and an application of the implicit function theorem. For general infinite dimensional functionals of our statistics we would usually need to show tightness of the process  $(W_n^2(t))_{t \geq 0}$  (or  $(S_n^2(t))_{t \geq 0}$ ), which is by far not a trivial issue. However, in the particular setting of integral functionals, it suffices to show a weaker condition that is displayed in Proposition 5.6. Indeed, this is the key step of the proof.

## 4 Simulations, parametric bootstrap and subsampling

The theoretical results of Theorem 3.1 are far from easy to apply in practice. There are a number of issues, which need to be addressed. First of all, since the parameters  $H_0$  and  $\alpha_0$  are unknown, we do not know whether we are in the regime of Theorem 3.1(ii) or (iii). Furthermore, even if we could determine whether the condition  $k > H_0 + 1/\alpha_0$  holds or not, the exact computation or a reliable numerical simulation of the quantities defined in (2.7) and (2.8) seems to be out of reach. Below we will propose two methods to overcome these problems. In the setting where the lfsm  $(X_t)_{t \geq 0}$  is continuous, which corresponds to the condition  $H_0 - 1/\alpha_0 > 0$ , we will see that it suffices to choose  $k = 2$  to end up in the normal regime of Theorem 3.1(ii). The confidence regions are then constructed using the parametric bootstrap approach. In the general setting we propose a novel subsampling method which, in some sense, automatically adapts to the unknown limiting distribution.

For comparison reasons we include the estimation of the parameter  $H_0$  using  $H_n(p, k)$  defined at (3.11), even though its properties have already been studied in [19]. Moreover, we pick our weight function in the class of Gaussian kernels:

$$w_{\nu}(t) = \exp\left(-\frac{t^2}{2\nu^2}\right) \quad (t, \nu > 0).$$

In particular we can use Gauss-Hermite quadrature, see [25], to estimate the integral

$$\|\varphi_n - \varphi_\xi\|_{w,2}^2 = \int_0^\infty (\varphi_n(t) - \varphi_\xi(t))^2 w_\nu(t) dt.$$

This procedure is based on a number of weights, unless otherwise stated we pick 12 weights. We mentioned that it is possible to choose other weight functions and standard numerical procedures exists for these. We restrict our simulation study to three different values of  $\nu \in \{0.05, 0.1, 1\}$  for the bootstrap method in Section 4.2. Additionally, while the theoretical characteristic function  $\varphi_\xi$  has an explicit form it depends on the norm  $\|h_k\|_\alpha$  which is not readily computable, hence needs to be approximated.

To produce the simulation study we generate observations from the lfsm using the recent R package `r1fsm`, which implements an algorithm based on [28]; this package already includes an implementation of the minimal contrast estimator. To compute the estimator a minimisation

$$\operatorname{argmin}_{\sigma,\alpha} \int_0^\infty (\varphi_n(t) - \varphi_{\sigma,\alpha,H_n(p,k)}(t))^2 w_\nu(t) dt$$

has to be carried out. For this purpose we use [20], which in particular entails picking a starting point for the algorithm. For  $\sigma$  no immediate choice exists, so we simply pick  $\sigma = 2$ , while  $\alpha = 1$  seems obvious.

#### 4.1 Empirical bias and variance

In this section we will check the bias and variance performance of our minimal contrast estimator. First, we consider the empirical bias and standard deviation, which are summarized in Tables 5 and 6 for  $n = 1000$  and Tables 7 and 8 for  $n = 10000$ . These are based on Monte Carlo simulation with at least 1000 repetitions. We fix the parameters  $k = 2$ ,  $p = -0.4$ ,  $\nu = 0.1$  and  $\sigma_0 = 0.3$  and perform the estimation procedure for various values of  $\alpha$  and  $H$ .

At this stage we recall that due to Theorem 3.1(iii) we obtain a slower rate of convergence when  $H_0 + 1/\alpha_0 > 2$ ; the stable regime is indicated in bold in Tables 5–8. This explains a rather bad estimation performance for  $\alpha_0 = 0.4$ . The effect is specifically pronounced for the parameter  $\sigma$ , which has the worst performance when  $\alpha_0 = 0.4$ . This observation is in line with the findings of [19], who concluded that the scale parameter  $\sigma$  is the hardest to estimate in practice. Also the starting point  $\sigma = 2$  of the minimisation algorithm, which is not close to  $\sigma_0 = 0.3$ , might have a negative effect on the performance. The estimation performance for values  $\alpha_0 > 0.4$  is quite satisfactory for all parameters, improving from  $n = 1000$  to  $n = 10000$ . We remark the superior performance of our method around the value  $\alpha = 1$ , which is explained by the fact that  $\alpha = 1$  is the starting point of the minimisation procedure.

For comparison, we display the bias and standard deviation of our estimator for  $k = 1$  based on  $n = 10000$  observations in Tables 9 and 10, where the stable regime is again highlighted in bold. We see a better finite sample performance for  $\alpha_0 = 0.4$ , but in most other cases we observe a larger bias and standard deviation compared to  $k = 2$ . This is explained by slower rates of convergence in the setting of the stable regime and  $k = 1$ .

We will now compare the minimal contrast estimator with the estimator proposed in [19]. To recall the latter estimator we observe the following identities due to (2.3):

$$\sigma = \frac{(-\log \varphi_\xi(t_1; k))^{1/\alpha}}{t_1 \|h_k\|_\alpha}, \quad \alpha = \frac{\log|\log \varphi_\xi(t_2; k)| - \log|\log \varphi_\xi(t_1; k)|}{\log t_2 - \log t_1}, \quad \xi = (\sigma, \alpha, H)$$

for fixed values  $0 < t_1 < t_2$ . Since  $h_k$  depends on  $\alpha$  and  $H$  we immediately obtain a function  $G : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  such that

$$(\sigma, \alpha) = G(\varphi_\xi(t_1; k), \varphi_\xi(t_2; k), H), \quad \xi = (\sigma, \alpha, H).$$

Hence an estimator for  $(\sigma, \alpha, H)$  is obtained by insertion of the empirical characteristic function and  $H_n(p; k)$  from (3.11):

$$(\tilde{\sigma}_{\text{low}}, \tilde{\alpha}_{\text{low}}, H_n(p; k)) = G(\varphi_n(t_1; k), \varphi_n(t_2; k), H_n(p; k)). \quad (4.16)$$

The first comparison of the estimators is between Tables 7 and 8 for the minimal contrast estimator with Tables 11 and 12, all of which are based on at least 1000 Monte Carlo repetitions and the same parameter choices. We see that the minimal contrast estimator outperforms the old estimator for values  $\alpha \leq 0.8$ , where the latter is completely unreliable in most cases. But for larger values of  $\alpha$  the old estimator might be slightly better. However, the results for the estimation of the scale parameter  $\sigma$  in the case  $\alpha = 0.4$  and  $H_0 \in \{0.2, 0.4\}$  are hard to interpret, since  $\tilde{\sigma}_{\text{low}}$  often delivers values that are indistinguishable from 0 yielding a small variance and relatively small bias for  $\sigma_0 = 0.3$ .

Another point is the instability of the old estimator. Table 13, which is based on at least 1200 simulations, shows the rate at which the estimators fail to return a value  $\alpha \in (0, 2)$ . In this regard the minimal contrast estimator is far superior in most cases and actually in the case  $k = 1$  the estimator almost never fails, although we dispense with the simulation results. We remark that in theory the minimal contrast estimator should never return  $\alpha$ 's not in the interval  $[0, 2]$ . However, we apply the minimisation procedure from [20] that does not allow constrained optimisation. One could instead use e.g. the procedure from [7]; we choose the first method as it is hailed as very robust.

An additional advantage of the minimal contrast estimator is that it allows incorporation of a priori knowledge of the parameters  $\sigma$  and  $\alpha$ , using the weight function but also the starting point for the minimisation algorithm.

## 4.2 Bootstrap inference in the continuous case

In this section we only consider the continuous case, which corresponds to the setting  $H_0 - 1/\alpha_0 > 0$ . In this setup  $H_0 \in (1/2, 1)$  and  $\alpha_0 \in (1, 2)$  must hold. We can in particular choose  $k = 2$  to ensure that Theorem 3.1(ii) applies, thus yielding the faster convergence rate  $\sqrt{n}$ . We are interested in obtaining feasible confidence regions for all parameters, but, as we mentioned earlier, the computation or reliable numerical approximation of the asymptotic variance in Theorem 3.1(ii) is out of reach. Instead we propose the following parametric bootstrap procedure to estimate the confidence regions for the true parameters.

- (1) Compute the minimal contrast  $\xi_n$  estimator for given observations  $X_1, \dots, X_n$ .
- (2) Generate new samples  $X_1^j, \dots, X_n^j$  for  $j = 1, \dots, N$  using the parameter  $\xi_n$ .
- (3) Compute new estimators  $\xi_n^{(j)}$  from the samples generated in (2) for each  $j = 1, \dots, N$ .
- (4) Calculate the empirical variance  $\hat{\Sigma}_n$  of  $\xi_n$  based on the estimators  $\xi_n^{(1)}, \dots, \xi_n^{(N)}$ .
- (5) For each parameter construct 95%-confidence regions based on the relation

$$\sqrt{n}(\xi_n - \xi_0) \approx \mathcal{N}(0, n\hat{\Sigma}_n).$$

To test this we repeated the above procedure for 200 Monte Carlo simulations with  $N = 200$ . Tables 1 and 2 report acceptance rates for  $\sqrt{n}(\xi_n - \xi_0)$  for an approximate 95% confidence interval. We observe a good performance for all estimators with the exception of  $n = 1000$  for  $\sigma$  with  $\nu = 0.05, 0.1$  and  $\alpha$  for  $\nu = 0.05$  in Table 1. The estimator is fairly stable under changes in  $\nu$  in this parameter regime, but it should be mentioned that a smaller  $\nu$ -value does lead to a larger failure rate, we dispense with the numerics.

**Table 1.** Acceptance rates for the true parameter  $(\sigma_0, \alpha_0, H_0) = (0.3, 1.8, 0.8)$  and power  $p = 0.4$ .

$\nu = 0.1$				$\nu = 1$				$\nu = 0.05$			
$n$	$\sigma$	$\alpha$	$H$	$n$	$\sigma$	$\alpha$	$H$	$n$	$\sigma$	$\alpha$	$H$
1000	69.2	95.8	96.3	1000	100	95.5	94.8	1000	81.7	44.2	99.5
2500	96.6	98.5	95.6	2500	100	96.3	93.1	2500	99.6	97.9	97.9
5000	99.0	99.5	96.1	5000	100	93.5	93.5	5000	100	99.5	96.8

**Table 2.** Acceptance rates for the true parameter  $(\sigma_0, \alpha_0, H_0) = (0.3, 1.3, 0.8)$  and power  $p = 0.4$ .

$\nu = 0.1$				$\nu = 1$				$\nu = 0.05$			
$n$	$\sigma$	$\alpha$	$H$	$n$	$\sigma$	$\alpha$	$H$	$n$	$\sigma$	$\alpha$	$H$
1000	87.1	95.4	93.1	1000	93.9	95.7	93.9	1000	91.9	98.6	95.9
2500	90.1	91.6	97.5	2500	95.7	96.7	94.3	2500	89.8	96.6	95.1
5000	91.6	91.6	95.1	5000	93.5	93.1	93.5	5000	91.4	94.3	97.1

### 4.3 Subsampling method in the general case

In contrast to the continuous setting  $H_0 - 1/\alpha_0 > 0$ , there exists no a priori choice of  $k$  in the general case, which ensures the asymptotically normal regime of Theorem 3.1(ii). This problem was tackled in [19] via the following two stage approach. In the first step they obtained a preliminary estimator  $\alpha_n^0(t_1, t_2)$  of  $\alpha_0$  using (4.16) for  $k = 1$ ,  $t_1 = 1$  and  $t_2 = 2$ . In the second step they defined the random number

$$\hat{k} = 2 + \lfloor \alpha_n^0(t_1, t_2)^{-1} \rfloor, \quad (4.17)$$

and computed the estimator  $(\tilde{\sigma}_{\text{low}}, \tilde{\alpha}_{\text{low}}, H_n(p, k))$  based on  $k = \hat{k}$ . They showed that the resulting estimator is  $\sqrt{n}$ -consistent and derived the associated weak limit theory. However, this approach does not completely solve the original problem, since they obtained four different convergence regimes according to whether  $1 > H_0 + 1/\alpha_0$  or not, and whether  $\alpha_0^{-1} \in \mathbb{N}$  or not.

Nevertheless, we apply their idea to propose a new subsampling method to determine feasible confidence regions for the parameters of the model. For our procedure it is crucial that the convergence rate is known explicitly and the weak convergence of the involved statistics is insured. We proceed as follows:

- (1) Given observations  $X_1, \dots, X_n$  compute  $\hat{k}$  from (4.17) and construct the minimal contrast estimator  $\xi_n = (\sigma_n, \alpha_n, H_n(p, \hat{k}))$ .
- (2) Split  $X_1, \dots, X_n$  into  $L$  groups such that the  $l$ th group contains the data  $(X_{(l-1)n/L+j})_{j=1}^{n/L}$  ( $n/L$  is assumed to be an integer). For each  $l = 1, \dots, L$  calculate  $\hat{k}_l$  from (4.17).
- (3) For each  $l = 1, \dots, L$  construct the minimal contrast estimators  $(\sigma_n^{(l)}, \alpha_n^{(l)})$  and  $H_n^{(l)}(p, \hat{k}_l)$  based on the  $l$ th group. For the estimation of  $(\sigma, \alpha)$  use  $H_n(p, \hat{k})$  from (1) as plug-in.
- (4) Compute the 97.5% and 2.5% quantiles for each of the distribution functions

$$\frac{1}{L} \sum_{l=1}^L \mathbf{1}\{\sqrt{\frac{n}{L}}(\sigma_n^{(l)} - \sigma_n) \leq x\}, \quad \frac{1}{L} \sum_{l=1}^L \mathbf{1}\{\sqrt{\frac{n}{L}}(\alpha_n^{(l)} - \alpha_n) \leq x\}, \quad \frac{1}{L} \sum_{l=1}^L \mathbf{1}\{\sqrt{\frac{n}{L}}(H_n^{(l)}(p, \hat{k}_l) - H_n(p, \hat{k})) \leq x\}.$$

Let us explain the intuition behind the proposed subsampling procedure. First of all, similarly to the theory developed in [19], the minimal contrast estimator  $\xi_n$  obtained through a two step method described in the beginning of the section leads to four different limit regimes for  $\sqrt{n}(\xi_n - \xi_0)$  (although we leave out the theoretical derivation here). Using this knowledge we may conclude that, for each  $l = 1, \dots, L$ ,  $\sqrt{n/L}(\xi_n^{(l)} - \xi_0)$  has the same (unknown) asymptotic distribution as the statistic  $\sqrt{n}(\xi_n - \xi_0)$  as long as  $n/L \rightarrow \infty$ . Since the true parameter  $\xi_0$  is unknown, we use its approximation  $\xi_n$ , which has a much faster rate of convergence than  $\sqrt{n/L}$  when  $L \rightarrow \infty$ . Finally, the statistics constructed on different blocks are asymptotically independent, which follows along the lines of the proofs in [19]. Hence, the law of large numbers implies that the proposed subsampling statistics converge to the unknown true asymptotic distributions when  $L \rightarrow \infty$  and  $n/L \rightarrow \infty$ .

In Tables 3 and 4 we report the empirical 95%-confidence regions for the parameters of the model using the subsampling approach. We perform 500 Monte Carlo simulations and choose  $n = 12.5 \times L^2$ .

Table 3 shows a satisfactory performance for all estimators, while the results of Table 4 are quite unreliable for the parameters  $\sigma$  and  $\alpha$ . The reason for the latter finding is the suboptimal finite sample performance of the estimators in the case of  $(\alpha, H) = (1.8, 0.8)$ , which is displayed in Tables 5 and 6.

We conclude this section by remarking the rather satisfactory performance of our estimator in the continuous setting  $H_0 - 1/\alpha_0 > 0$ . On the other hand, when using the

**Table 3.** Acceptance rates for the true parameter  $(\sigma_0, \alpha_0, H_0) = (0.3, 0.8, 0.8)$ . Here  $p = -0.4$  and  $\nu = 0.1$ .

$L$	$n/L$	$\sigma$	$\alpha$	$H$
80	1000	90.65	94.39	89.72
100	1250	87.72	94.24	89.25

**Table 4.** Acceptance rates for the true parameter  $(\sigma_0, \alpha_0, H_0) = (0.3, 1.8, 0.8)$ . Here  $p = -0.4$  and  $\nu = 0.1$ .

$L$	$n/L$	$\sigma$	$\alpha$	$H$
80	1000	60.70	67.31	92.19
100	1250	68.88	74.92	94.56

subsampling method in the general setting, a further careful tuning seems to be required. In particular, the choice of the weight function  $w$  and the group number  $L$  plays an important role in estimator's performance. We leave this study for future research.

## 5 Proofs

We denote by  $C$  a finite, positive constant which may differ from line to line. Moreover, any important dependence on other constants warrants a subscript. To simplify notations we set  $H_n = H_n(p, k)$ .

### 5.1 Proof of Theorem 2.1(i)

As we mentioned earlier, the convergence of finite dimensional distributions has been shown in [19, Theorem 2.2], and thus we only need to prove the smoothness property of the limit  $W$ . We recall the definition of the quantity  $U_{g,h}$  at (2.5) and start with the following lemma.

**Lemma 5.1 ([23, Eqs. (3.4)–(3.6)]).** *Let  $g, h \in \mathcal{L}^\alpha(\mathbb{R})$ . Then for any  $u, v \in \mathbb{R}$*

$$\begin{aligned}
 |U_{g,h}(u, v)| &\leq 2|uv|^{\alpha/2} \int_0^\infty |g(x)h(x)|^{\alpha/2} dx \\
 &\quad \times \exp\left(-2|uv|^{\alpha/2} (\|g\|_\alpha^\alpha \|h\|_\alpha^\alpha - \int_0^\infty |g(x)h(x)|^{\alpha/2} dx)\right), \\
 |U_{g,h}(u, v)| &\leq 2|uv|^{\alpha/2} \int_0^\infty |g(x)h(x)|^{\alpha/2} dx \\
 &\quad \times \exp\left(-(\|ug\|_\alpha^{\alpha/2} - \|vh\|_\alpha^{\alpha/2})^2\right).
 \end{aligned}$$

*In particular, it holds that  $|U_{g,h}(u, v)| \leq 2|uv|^{\alpha/2} \int_0^\infty |g(x)h(x)|^{\alpha/2} dx$ .*

Next, we define for each  $l \in \mathbb{Z}$

$$\rho_l = \int_0^\infty |h_k(x)h_k(x+l)|^{\alpha/2} dx$$

and recall the following lemma from [19].

**Lemma 5.2 ([19, Lemma 6.2]).** *If  $k > H + 1/\alpha$  and  $l > k$  then*

$$\rho_l \leq l^{\alpha(H-k-1)/2}.$$

To prove that the process  $W$  is locally Hölder continuous of any order smaller than  $\alpha/4$ , we use Kolmogorov's criterion. Since  $W$  is a Gaussian process it suffices to prove that for each  $T > 0$  there exists a constant  $C_T \geq 0$  such that

$$\mathbb{E}[(W_t - W_s)^2] \leq C_T |t - s|^{\alpha/2} \quad \text{for all } s, t \in [0, T]. \quad (5.18)$$

This is performed in a similar fashion as in [17, Section 4.1]. First, we reduce the problem. Using  $\cos(tx) = (\exp(itx) + \exp(-itx))/2$  and the symmetry of the distribution of  $X$ , we observe the identity

$$\text{Cov}(\cos(t\Delta_{j,k}X), \cos(s\Delta_{j+l,k}X)) = \frac{1}{2} \left( U_{h_k, -h_k(l+\cdot)}(t, s) + U_{h_k, h_k(l+\cdot)}(t, s) \right).$$

In the following we focus on the first term in the above decomposition (the second term is treated similarly). More specifically, we will show the inequality (5.18) for the quantity  $\bar{r}(t, s)$ , which is given as

$$\begin{aligned} \bar{r}(t, s) &= \sum_{l \in \mathbb{Z}} \bar{r}_l(t, s) \quad \text{where} \\ \bar{r}_l(t, s) &= U_{h_k, -h_k(l+\cdot)}(t, s) \\ &= \exp(-\|th_k - sh_k(l+\cdot)\|_\alpha^\alpha) - \exp(-(t^\alpha + s^\alpha)\|h_k\|_\alpha^\alpha). \end{aligned}$$

Moreover, since  $\bar{r}(t, t) + \bar{r}(s, s) - 2\bar{r}(t, s) \leq |\bar{r}(t, t) - \bar{r}(t, s)| + |\bar{r}(s, s) - \bar{r}(t, s)|$  it is by symmetry enough to prove that

$$|\bar{r}(t, t) - \bar{r}(t, s)| \leq C_T |t - s|^{\alpha/2} \quad \text{for all } s, t \in [0, T].$$

For  $l \in \mathbb{Z}$  decompose now as follows:

$$\begin{aligned} \bar{r}_l(t, t) - \bar{r}_l(t, s) &= \exp(-2t^\alpha \|h_k\|_\alpha^\alpha) \left[ \exp(-\|t(h_k - h_k(\cdot + l))\|_\alpha^\alpha + 2t^\alpha \|h_k\|_\alpha^\alpha) - 1 \right] \\ &\quad - \exp(-(t^\alpha + s^\alpha)\|h_k\|_\alpha^\alpha) \left[ \exp(-\|th_k - sh_k(l+\cdot)\|_\alpha^\alpha + (t^\alpha + s^\alpha)\|h_k\|_\alpha^\alpha) - 1 \right] \\ &= \left[ \exp(-2t^\alpha \|h_k\|_\alpha^\alpha) - \exp(-(t^\alpha + s^\alpha)\|h_k\|_\alpha^\alpha) \right] \\ &\quad \times \left[ \exp(-\|t(h_k - h_k(l+\cdot))\|_\alpha^\alpha + 2t^\alpha \|h_k\|_\alpha^\alpha) - 1 \right] + \exp(-(t^\alpha + s^\alpha)\|h_k\|_\alpha^\alpha) \\ &\quad \times \left[ \exp(-\|t(h_k - h_k(l+\cdot))\|_\alpha^\alpha + 2t^\alpha \|h_k\|_\alpha^\alpha) - \exp(-\|th_k - sh_k(l+\cdot)\|_\alpha^\alpha + (t^\alpha + s^\alpha)\|h_k\|_\alpha^\alpha) \right] \\ &=: \bar{r}_l^{(1)}(t, s) + \bar{r}_l^{(2)}(t, s). \end{aligned}$$

Applying the second inequality of Lemma 5.1 and the mean value theorem we deduce the estimate:

$$|\bar{r}^{(1)}(t, s)| \leq C_T \rho_l |t^\alpha - s^\alpha| \leq C_T \rho_l |t - s|^{\alpha/2} \quad \text{for all } s, t \in [0, T]. \quad (5.19)$$

Again by the mean value theorem we find that

$$|\bar{r}^{(2)}(t, s)| \leq C_T \left| \|th_k - sh_k(l + \cdot)\|_\alpha^\alpha - \|t(h_k - h_k(l + \cdot))\|_\alpha^\alpha + (t^\alpha - s^\alpha) \|h_k\|_\alpha^\alpha \right|.$$

The last term can be rewritten as

$$\begin{aligned} & \|th_k - sh_k(l + \cdot)\|_\alpha^\alpha - \|t(h_k - h_k(l + \cdot))\|_\alpha^\alpha + (t^\alpha - s^\alpha) \|h_k\|_\alpha^\alpha \\ &= \int_0^\infty |th_k(x) - sh_k(x + l)|^\alpha - |t(h_k(x) - h_k(x + l))|^\alpha + (t^\alpha - s^\alpha) |h_k(x + l)|^\alpha dx. \end{aligned}$$

As  $\alpha \in (0, 2)$  we have  $|x^\alpha - y^\alpha| \leq |x^2 - y^2|^{\alpha/2}$  for all  $x, y \geq 0$ . In particular

$$\begin{aligned} & \left| |th_k(x) - sh_k(x + l)|^\alpha - |t(h_k(x) - h_k(x + l))|^\alpha \right| \leq C_T |t - s|^{\alpha/2} \\ & \quad \times (|h_k(x + l)|^\alpha + |h_k(x)h_k(x + l)|^{\alpha/2}). \end{aligned}$$

It then follows that

$$|\bar{r}_l^{(2)}(t, s)| \leq C_T |t - s|^{\alpha/2} (\rho_l + \mu_l) \quad \text{for all } s, t \in [0, T] \quad (5.20)$$

where  $\mu_l$  is the quantity defined as

$$\mu_l = \int_0^\infty |h_k(x + l)|^\alpha dx.$$

It remains to prove that

$$\sum_{l \in \mathbb{Z}} \rho_l < \infty \quad \text{and} \quad \sum_{l \in \mathbb{Z}} \mu_l < \infty. \quad (5.21)$$

The first claim is a direct consequence of Lemma 5.2. The second convergence can equivalently be formulated as

$$\sum_{l=1}^\infty l \int_l^{l+1} |h_k(x)|^\alpha dx < \infty.$$

Recall that  $|h_k(x)| \leq C|x|^{H-1/\alpha-k}$  for large  $x$ , hence

$$\sum_{l=1}^\infty l \int_l^{l+1} |h_k(x)|^\alpha dx \leq C \sum_{l=1}^\infty l \int_l^{l+1} x^{\alpha(H-k)-1} dx \leq C \sum_{l=1}^\infty l^{\alpha(H-k)} < \infty,$$

where we used the assumption  $k > H + 1/\alpha$ . Combining (5.19) and (5.20) with (5.21) we can conclude (5.18), and hence the proof of Theorem 2.1(i) is complete.

## 5.2 Proof of Theorem 2.1(ii)

We recall that the asymptotic distribution of the vector  $(S_n^1(1), S_n^1(2))$  and its asymptotic independence of  $S_n^2(t)$  have been shown in [19, Theorem 2.2]. Hence, we only need to determine the functional form of the limit of the statistic  $S_n^2(t)$ .

In the following we will recall a number of estimates and decompositions from [19, Theorem 2.2], which will be also helpful in the proof of Theorem 3.1(iii). We start out with a series of estimates on the function  $\Phi_t^2$  given at (2.8), but for a general scale parameter  $\eta > 0$ . Let  $\Phi_{t,\eta}$  denote the function

$$\Phi_{t,\eta}(x) = \mathbb{E}[\cos(t(Y + x))] - \mathbb{E}[\cos(tY)] \quad x \in \mathbb{R},$$

where  $Y$  is an SaS distributed random variable with scale parameter  $\eta$ . We obviously have the representation

$$\Phi_{t,\eta}(x) = (\cos(xt) - 1) \exp(-|\eta t|^\alpha). \quad (5.22)$$

The next lemma gives some estimates on the function  $\Phi_{t,\eta}$ .

**Lemma 5.3.** *For  $\eta > 0$  set  $g_\eta(t) = \exp(-|\eta t|^\alpha)$  and let  $\Phi_{t,\eta}^{(v)}(x)$  denote the  $v$ th derivative at  $x \in \mathbb{R}$ . Then there exists a constant  $C > 0$  such that for all  $t \geq 0$  it holds that*

$$(i) \quad |\Phi_{t,\eta}^{(v)}(x)| \leq C t^v g_\eta(t) \text{ for all } x \in \mathbb{R} \text{ and } v \in \{0, 1, 2\}.$$

$$(ii) \quad |\Phi_{t,\eta}(x)| \leq g_\eta(t)(1 \wedge |xt|^2).$$

$$(iii) \quad |\Phi_{t,\eta}(x) - \Phi_{t,\eta}(y)| \leq t^2 g_\eta(t)((1 \wedge |x| + 1 \wedge |y|)|x - y| \mathbf{1}_{\{|x-y| \leq 1\}} + \mathbf{1}_{\{|x-y| > 1\}}).$$

(iv) *For any  $x, y > 0$  and  $a \in \mathbb{R}$  then*

$$F(a, x, y) := \left| \int_0^y \int_0^x \Phi_{t,\eta}^{(v)}(a + u + v) \, du \, dv \right| \leq C g_\eta(t)(t + 1)^2(1 \wedge x)(1 \wedge y).$$

**Proof.** (i): This follows directly from (5.22).

(ii): This is straightforward using the standard inequality  $1 - \cos(y) \leq y^2$ .

(iii): (i) implies that  $|\Phi_{t,\eta}^{(1)}(x)| \leq t^2 g_\eta(t)(1 \wedge |x|)$  and note that

$$|\Phi_{t,\eta}(x) - \Phi_{t,\eta}(y)| = \left| \int_y^x \Phi_{t,\eta}^{(1)}(u) \, du \right|.$$

If  $|x - y| > 1$  we simply bound the latter by  $t^2 g_\eta(t)$ . If  $|x - y| \leq 1$ , then by the mean value theorem there exists a number  $s$  with  $|x - s| \leq |x - y|$  such that

$$\left| \int_y^x \Phi_{t,\eta}^{(1)}(u) \, du \right| = |\Phi_{t,\eta}^{(1)}(s)| |x - y|.$$

Observe then

$$|\Phi_{t,\eta}^{(1)}(s)| \leq t^2 g_\eta(t)(1 \wedge |s|) \leq t^2 g_\eta(t)(1 \wedge (|x| + |y|)).$$

This completes the proof of the inequality.

(iv): Let  $a$ ,  $x$  and  $y$  be given. Observe that

$$\begin{aligned} \int_0^y \int_0^x \Phi_{t,\eta}^{(2)}(a+u+v) du dv &= \int_0^y \Phi_{t,\eta}^{(1)}(a+x+v) - \Phi_{t,\eta}^{(1)}(a+v) dv \\ &= \Phi_{t,\eta}(a+x+y) - \Phi_{t,\eta}(a+y) - (\Phi_{t,\eta}(a+x) - \Phi_{t,\eta}(a)). \end{aligned}$$

The last equality implies that  $F(a, x, y) \leq Cg_\eta(t)$ . The first equality implies that  $F(a, x, y) \leq Cg_\eta(t)ty$ . Reversing the order of integration we get a similar expression as the first equality with  $x$  replaced by  $y$ . Hence,  $F(a, x, y) \leq Cg_\eta(t)tx$ . Lastly, using (i) on the first integral yields  $F(a, x, y) \leq Cg_\eta(t)t^2xy$ . Splitting into the four cases completes the proof.  $\square$

We will consider the asymptotic decomposition of the statistic  $S_n^2(t)$  given in [3, Section 5] (see also [4, 19]). We set

$$S_n^2(t) = n^{-1/\beta} \sum_{i=k}^n (\cos(t\Delta_{i,k}X) - \varphi_\xi(t)) =: n^{-1/\beta} \sum_{i=k}^n V_i(t).$$

Define for each  $s \geq 0$  we define the  $\sigma$ -algebras

$$\mathcal{G}_s = \sigma(L_v - L_u : v, u \leq s) \quad \text{and} \quad \mathcal{G}_s^1 = \sigma(L_v - L_u : s \leq v, u \leq s+1).$$

We also set for all  $n \geq k$ ,  $i \in \{k, \dots, n\}$  and  $t \geq 0$

$$R_i(t) = \sum_{j=1}^{\infty} \zeta_{i,j}(t) \quad \text{and} \quad Q_i(t) = \sum_{j=1}^{\infty} \mathbb{E}[V_i(t) \mid \mathcal{G}_{i-j}^1],$$

where

$$\zeta_{i,j}(t) = \mathbb{E}[V_i(t) \mid \mathcal{G}_{i-j+1}] - \mathbb{E}[V_i(t) \mid \mathcal{G}_{i-j}] - \mathbb{E}[V_i(t) \mid \mathcal{G}_{i-j}^1].$$

Then the following decomposition holds:

$$S_n^2(t) = n^{-1/\beta} \sum_{i=k}^n R_i(t) + \left( n^{-1/\beta} \sum_{i=k}^n Q_i(t) - \bar{S}_n(t) \right) + \bar{S}_n(t), \quad (5.23)$$

where

$$\begin{aligned} \bar{S}_n(t) &= n^{-1/\beta} \sum_{i=k}^n (\bar{\Phi}_t(L_{i+1} - L_i) - \mathbb{E}[\bar{\Phi}_t(L_{i+1} - L_i)]), \\ \bar{\Phi}_t(x) &:= \sum_{i=1}^{\infty} \Phi_t^2(h_k(i)x). \end{aligned} \quad (5.24)$$

It turns out that the first two terms in (5.23) are negligible while  $\bar{S}_n$  is the dominating term. More specifically, we can use similar arguments as in [4, Eq. (5.22)] and deduce the following proposition from Lemma 5.3.

**Proposition 5.4.** *For any  $\varepsilon > 0$  there exists a constant  $C > 0$  such that for all  $n \in \mathbb{N}$*

$$\sup_{t \geq 0} \mathbb{E} \left[ \left( n^{-1/\beta} \sum_{i=k}^n R_i(t) \right)^2 \right] \leq Cn^{2(2-\beta-1/\beta)+\varepsilon}.$$

Using the inequality  $2 - x - 1/x < 0$  for all  $x > 1$  on  $\beta > 1$  it follows by picking  $\varepsilon > 0$  small enough that the first term in (5.23) is asymptotically negligible. Decomposing the second term and using arguments as in the equations (5.30), (5.31) and (5.38) in [4] we obtain the following result.

**Proposition 5.5.** *For any  $\varepsilon > 0$  there exist an  $r > 1$ , an  $r' > \beta \vee r$  and a constant  $C > 0$  such that for all  $n$  in  $\mathbb{N}$*

$$\sup_{t \geq 0} \mathbb{E} \left[ \left| n^{-1/\beta} \sum_{i=k}^n Q_i(t) - \overline{S}_n(t) \right|^r \right] \leq C (n^{r(\varepsilon+2-\beta-1/\beta)} + n^{\frac{r}{r'}(1-r'/\beta)})$$

Using again the inequality  $2 - x - 1/x < 0$  for all  $x > 1$ , it follows immediately that the second term in (5.23) is asymptotically negligible. Hence,  $\overline{S}_n(t)$  is asymptotically equivalent to the statistic  $S_n^2(t)$ , and it suffices to analyse its finite dimensional distribution.

Consider  $t_1, \dots, t_d \in \mathbb{R}_+$ . We will now recall the limiting distribution of the vector  $(\overline{S}_n(t_1), \dots, \overline{S}_n(t_d))$ . Observing the definition (5.24), we deduce the uniform convergence

$$\sup_{t \geq 0} | |x|^{-\alpha/\beta} \overline{\Phi}_t(x) - \kappa_2(t) | \rightarrow 0 \quad \text{as } x \rightarrow -\infty, \quad (5.25)$$

where  $\kappa_2$  has been introduced at (2.9). Indeed by substituting  $u = (|x|/z)^{\alpha/\beta}$  we have that

$$\begin{aligned} & \sup_{t \geq 0} | |x|^{-\alpha/\beta} \overline{\Phi}_t(x) - \kappa_2(t) | \\ &= \sup_{t \geq 0} \left| |x|^{-\alpha/\beta} \int_0^\infty \Phi_t^2(h_k(\lfloor u \rfloor + 1)|x|) du - \int_0^\infty \Phi_t^2(q_{H,\alpha,k}z) z^{-1-\alpha/\beta} du \right| \\ &= \frac{\alpha}{\beta} \sup_{t \geq 0} \left| \int_0^\infty \Phi_t^2(h_k(\lfloor (|x|/z)^{\alpha/\beta} \rfloor + 1)|x|) z^{-1-\alpha/\beta} du - \int_0^\infty \Phi_t^2(q_{H,\alpha,k}z) z^{-1-\alpha/\beta} du \right| \\ &\leq \frac{\alpha}{\beta} \int_0^\infty \sup_{t \geq 0} | \Phi_t^2(h_k(\lfloor (|x|/z)^{\alpha/\beta} \rfloor + 1)x) - \Phi_t^2(q_{H,\alpha,k}z) | z^{-1-\alpha/\beta} du. \end{aligned}$$

By Lemma 5.3(iii) the integrand vanishes pointwise in  $z$  as  $x \rightarrow -\infty$  due to the asymptotics

$$h_k(x) \sim q_{H,\alpha,k} x^{-\beta/\alpha} \quad \text{as } x \rightarrow \infty. \quad (5.26)$$

Due to Lebesgue's dominated convergence theorem it is enough to bound the integrand uniformly in  $x < -1$ . By the triangle inequality it is enough to treat each  $\Phi_t^2$ -term separately. For the first term Lemma 5.3(ii) implies that

$$\sup_{t \geq 0} | \Phi_t^2(h_k(\lfloor (|x|/z)^{\alpha/\beta} \rfloor + 1)|x|) | z^{-1-\alpha/\beta} \leq C (1 \wedge |h_k(\lfloor (|x|/z)^{\alpha/\beta} \rfloor + 1)x|^2) z^{-1-\alpha/\beta}.$$

For large  $z$ , say  $z > 1$ , the latter is bounded by the integrable function  $z^{-1-\alpha/\beta} \mathbf{1}_{\{z > 1\}}$ . For  $z \in (0, 1]$  we deduce by (5.26)

$$|h_k(\lfloor (|x|/z)^{\alpha/\beta} \rfloor + 1)x|^2 z^{-1-\alpha/\beta} \leq C x^2 (\lfloor (|x|/z)^{\alpha/\beta} \rfloor + 1)^{-2\beta/\alpha} z^{-1-\alpha/\beta} \leq C z^{1-\alpha/\beta},$$

where we used that  $(|x|/z)^{\alpha/\beta} \leq \lfloor (|x|/z)^{\alpha/\beta} \rfloor + 1$ . Recalling (2.4), we deduce that  $\alpha/\beta < 2$  and an integrable bound is obtained. The second  $\Phi_t^2$ -term is treated similarly. Hence, we have (5.25).

Define the map  $\tau_{t_1, \dots, t_d} : \mathbb{R} \rightarrow (\mathbb{R}_-)^d$  as

$$\tau_{t_1, \dots, t_d}(x) = |x|^{\alpha/\beta}(\kappa_2(t_1), \dots, \kappa_2(t_d)).$$

Following [3, Lemma 6.6] the limit of the vector  $(\bar{S}_n(t_1), \dots, \bar{S}_n(t_d))$  is determined by the Lévy measure

$$\nu_{t_1, \dots, t_d}(A) := \nu(\tau_{t_1, \dots, t_d}^{-1}(A)),$$

where  $A \subseteq (\mathbb{R}_-)^d$  is a Borel set and  $\nu$  is the Lévy measure of  $L$ . But  $\nu_{t_1, \dots, t_d}$  is also the Lévy measure of the vector  $(\kappa_2(t_1), \dots, \kappa_2(t_d))S$ , where the random variable  $S$  has been introduced in Theorem 2.1(ii). This implies the desired result.

### 5.3 Finite dimensional convergence and integral functionals

Let  $(Y^n)_{n \geq 1}$  and  $Y$  be stochastic processes indexed by  $\mathbb{R}_+$  with paths in  $\mathcal{L}^1(\mathbb{R}_+)$ . We will give simple sufficient conditions for when the implication

$$Y^n \xrightarrow{\mathcal{L}\text{-f}} Y \quad \Longrightarrow \quad \int_0^\infty Y_u^n \, du \xrightarrow{d} \int_0^\infty Y_u \, du \quad (5.27)$$

holds true. Such a result is obviously required to obtain Theorem 3.1 from Theorem 2.1. Before we state these conditions we remark that the question has already been studied in the literature. As an example Theorem 22 in Appendix I of [14] gives two sufficient conditions for (5.27) to hold, but the second condition is a Hölder type criteria, which is not easily verifiable in our setting. Moreover, the theorem only deals with integration over bounded sets. The article [10] studies this question in general, but the conditions of e.g. Lemma 1 therein are too abstract even though we are in the case of a finite measure (the one induced by the weight function  $w$ ). What can be deduced from [10] is that some kind of uniform integrability (with respect to the product measure) is sufficient for (5.27).

To formulate the lemma, define for each  $n, m, l \in \mathbb{N}$  the intermediate random variables

$$X_{n,m,l} = \int_0^l Y_{\lfloor um \rfloor / m}^n \, du \quad \text{and} \quad X_{n,l} = \int_0^l Y_u^n \, du.$$

**Proposition 5.6.** *Suppose that  $(Y^n)_{n \geq 1}$  and  $Y$  are continuous stochastic processes and assume that the following conditions hold:*

$$\lim_{l \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_l^\infty \mathbb{E}[|Y_u^n|] \, du = 0, \quad \lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}(|X_{n,m,l} - X_{n,l}| \geq \varepsilon) = 0, \quad (5.28)$$

for all  $l, \varepsilon > 0$ . Then (5.27) holds.

**Proof.** Note that for each  $n, m, l \in \mathbb{N}$  we have the decomposition

$$\int_0^\infty Y_u^n \, du = X_{n,m,l} + (X_{n,l} - X_{n,m,l}) + \int_l^\infty Y_u^n \, du.$$

As  $Y^n \xrightarrow{\mathcal{L}\text{-f}} Y$  we deduce the weak convergence

$$X_{n,m,l} \xrightarrow[n \rightarrow \infty]{d} Y_{m,l} := \int_0^l Y_{\lfloor um \rfloor / m} du \quad \text{for each } m \in \mathbb{N}.$$

The continuity of  $Y$  implies immediately that  $Y_{m,l} \xrightarrow{\text{a.s.}} \int_0^l Y_u du$  as  $m \rightarrow \infty$ . The assumptions in (5.28) then imply the convergence  $\int_0^\infty Y_u^n du \xrightarrow{d} \int_0^\infty Y_u du$ .  $\square$

#### 5.4 Proof of Theorem 3.1(i) and (ii)

The strong consistency result of Theorem 3.1(i) is an immediate consequence of (3.11) and  $\|\varphi_n - \varphi_{\xi_0}\|_{w,2} \xrightarrow{\text{a.s.}} 0$ , where the latter follows from (2.3) and the dominated convergence theorem. Hence, we are left to proving Theorem 3.1(ii).

Recall the definition of the function  $F : \mathcal{L}_w^2(\mathbb{R}_+) \times (0,1) \times \Theta_0 \rightarrow \mathbb{R}$  at (3.14), where  $\Theta_0 \subseteq (0,\infty) \times (0,2)$  is an open neighbourhood of  $(\sigma_0, \alpha_0)$  bounded away from  $(0,0)$ . Now, the minimal contrast estimator at (3.15) can be obtained using the criteria

$$\nabla_\theta F(\psi, H, \theta) = 0,$$

which is satisfied at  $(\varphi_{\xi_0}, \xi_0)$ . Denote by  $\zeta(\psi, H)$  an element of  $(0,1) \times \Theta_0$  such that

$$\nabla_\theta F(\psi, H, \zeta(\psi, H)) = 0.$$

To determine the derivative of  $\zeta$  we will need the infinite dimensional implicit function theorem, which we briefly repeat.

Consider three Banach spaces  $(E_i, \|\cdot\|_i)$ ,  $i = 1, 2, 3$ , and open subsets  $U_i \subseteq E_i$ ,  $i = 1, 2$ . Let  $f : U_1 \times U_2 \rightarrow E_3$  be a Fréchet differentiable map. For a point  $(e_1, e_2) \in U_1 \times U_2$  and a direction  $(h_1, h_2) \in E_1 \times E_2$  we denote by  $D_{e_1, e_2}^k f(h_k)$ ,  $k = 1, 2$ , the Fréchet derivative of  $f$  at the point  $(e_1, e_2)$  in the direction  $h_k \in E_k$ . Assume that  $(e_1^0, e_2^0) \in U_1 \times U_2$  satisfies the equation  $f(e_1^0, e_2^0) = 0$  and that the map  $D_{e_1^0, e_2^0}^2 f : E_2 \rightarrow E_3$  is continuous and invertible. Then there exists open sets  $V_1 \subseteq U_1$  and  $V_2 \subseteq U_2$  with  $(e_1^0, e_2^0) \in V_1 \times V_2$  and a bijective function  $G : V_1 \rightarrow V_2$  such that

$$f(e_1, e_2) = 0 \quad \iff \quad G(e_1) = e_2.$$

Moreover,  $G$  is Fréchet differentiable with derivative

$$D_{e_1} G(h) = -(D_{e_1, G(e_1)}^2 f)^{-1} (D_{e_1, G(e_1)}^1 f(h)). \quad (5.29)$$

We will adapt this to our setup, which corresponds to  $U_1 = \mathcal{L}_w^r(\mathbb{R}_+) \times (0,1)$ , for a  $r > 1$ ,  $E_2 = \Theta_0$ ,  $E_3 = \mathbb{R}$  and  $f = \nabla_\theta F$ . A straightforward calculation shows that the map  $\theta \mapsto \nabla_\theta^2 F(\varphi, H, \theta)$  is differentiable with derivative at  $(\varphi_{\xi_0}, \xi_0)$  represented by the Hessian

$$D_{\varphi_{\xi_0}, \xi_0}^2 \nabla_\theta F = \nabla_\theta^2 F(\varphi_{\xi_0}, \xi_0) = 2 \left( \int_0^\infty \partial_{\theta_i} \varphi_{\xi_0}(t) \partial_{\theta_j} \varphi_{\xi_0}(t) w(t) dt \right)_{i,j=1,2}.$$

The linear independence of the maps  $\partial_{\theta_1} \varphi_{\xi_0}$  and  $\partial_{\theta_2} \varphi_{\xi_0}$  immediately shows the invertibility of the Hessian. Moreover, standard theory for convergence in  $\mathcal{L}^r(\mathbb{R}_+)$ ,  $r > 1$ , shows that the map  $(\varphi, H) \mapsto \nabla_\theta^2 F(\varphi, H, \cdot)$  is continuous, which is needed to assert that  $\nabla_\theta F$  is  $C^1$ .

The determination of the remaining derivative  $D_{\varphi,\xi}^1 \nabla_{\theta} F$  for a point  $(\varphi, \xi) \in \mathcal{L}_w^r(\mathbb{R}_+) \times (0, 1) \times \Theta_0$  is slightly more involved. It is given by its two components  $D^1 = (D^{1,1}, D^{1,2})$  corresponding to the partial derivatives. Indeed,  $D_{\varphi,\xi}^{1,1} \nabla_{\theta} F$  is the derivative with respect to the functional coordinate  $\varphi \in \mathcal{L}_w^r(\mathbb{R}_+)$  and  $D_{\varphi,\xi}^{1,2} \nabla_{\theta} F$  the derivative with respect to the Hurst parameter  $H \in (0, 1)$ , where  $\xi = (H, \alpha, \sigma)$ . It is easily seen that

$$D_{\varphi,\xi}^{1,1} \nabla_{\theta} F(h) = D_{\xi}^{1,1} \nabla_{\theta} F(h) = -2 \int_0^{\infty} h(t) \nabla_{\theta} \varphi_{\xi}(t) w(t) dt, \quad h \in \mathcal{L}_w^r(\mathbb{R}_+).$$

An application of Hölder's inequality proves the continuity of the linear map  $\xi \mapsto D_{\xi}^{1,1} \nabla_{\theta} F$ . The second partial derivative at  $(\varphi, \xi)$  is the linear map represented by the two dimensional vector

$$D_{\varphi,\xi}^{1,2} \nabla_{\theta} F = 2 \int_0^{\infty} \partial_H \varphi_{\xi}(t) \nabla_{\theta} \varphi_{\xi}(t) w(t) dt - 2 \int_0^{\infty} (\varphi(t) - \varphi_{\xi}(t)) \partial_H \nabla_{\theta} \varphi_{\xi}(t) w(t) dt.$$

Evaluated at the point  $(\varphi_{\xi_0}, \xi_0) = (\varphi_{\xi_0}, G(\varphi_{\xi_0}, H_0))$  yields the simpler expression:

$$D_{\varphi_{\xi_0}, \xi_0}^{1,2} \nabla_{\theta} F = 2 \int_0^{\infty} \partial_H \varphi_{\xi_0}(t) \nabla_{\theta} \varphi_{\xi_0}(t) w(t) dt.$$

Suppose we are in the case  $k > H + 1/\alpha$  then we may pick  $r = 2$  in the discussion above. By Fréchet differentiability it follows that

$$\begin{aligned} \sqrt{n}(\xi_n - \xi_0) &= \sqrt{n}(G(\varphi_n, H_n) - G(\varphi_{\xi_0}, H_0)) \\ &= D_{\varphi_{\xi_0}, \xi_0} G(\sqrt{n}(\varphi_n - \varphi_{\xi_0}), \sqrt{n}(H_n - H_0)) \\ &\quad + \sqrt{n}(\|\varphi_n - \varphi_{\xi_0}\|_{w,2} + |H_n - H_0|) R(\varphi_n - \varphi_{\xi_0}, H_n - H_0), \end{aligned} \tag{5.30}$$

where the remainder term  $R$  satisfies that  $R(\varphi_n - \varphi_{\xi_0}, H_n - H_0) \xrightarrow{\text{a.s.}} 0$  as  $\|\varphi_n - \varphi_{\xi_0}\|_{w,2} + |H_n - H_0| \xrightarrow{\text{a.s.}} 0$ . Recall now the derivative of  $G$  at (5.29). In order to show Theorem 3.1(ii) it suffices to prove the convergences

$$\begin{aligned} \sqrt{n}(\|\varphi_n - \varphi_{\xi_0}\|_{w,2} + |H_n - H_0|) &\xrightarrow{d} \|W\|_{w,2} + |M_1|, \\ \sqrt{n} \int_0^{\infty} (\varphi_n(t) - \varphi_{\xi_0}(t)) \nabla_{\theta} \varphi_{\xi_0}(t) w(t) dt &\xrightarrow{d} \int_0^{\infty} W_t \nabla_{\theta} \varphi_{\xi_0}(t) w(t) dt, \end{aligned} \tag{5.31}$$

where  $W = (W_t)_{t \geq 0}$  has been introduced in Theorem 2.1(i).

We will only consider the second convergence at (5.31) since the first is shown similarly (see also [17, page 14]). For the conditions (5.28) it suffices to find a constant  $C > 0$  such that

$$\sup_{n \in \mathbb{N}, t \geq 0} \text{Var}(W_n^2(t)) \leq C < \infty. \tag{5.32}$$

The identity  $\Delta_{i,k} X = \int_{\mathbb{R}} h_k(i-s) dL_s$  together with stationarity of the increments  $\{\Delta_{i,k} X \mid i \geq k\}$  shows that

$$|\text{Cov}(W_n^2(s), W_n^2(t))| \leq \frac{1}{2} \sum_{l \in \mathbb{Z}} |U_{h_k, h_k(l+\cdot)}(s, t) + U_{h_k, -h_k(l+\cdot)}(s, t)|. \tag{5.33}$$

Indeed, split the series at (5.33) into three terms. For  $l = 0$  it follows from (2.5) that

$$U_{h_k, h_k}(t, t) + U_{h_k, -h_k}(t, t) = 1 + 2 \exp(-|2t\sigma \|h_k\|_\alpha^\alpha) - 2 \exp(-2|\sigma t \|h_k\|_\alpha^\alpha), \quad (5.34)$$

which is obviously uniformly bounded in  $t \geq 0$ . For  $l \neq 0$  with  $|l| \leq k$  the first inequality of Lemma 5.1 implies that

$$\begin{aligned} \sum_{l \in \mathbb{Z}: |l| \leq k} |U_{h_k, h_k(l+\cdot)}(t, t) + U_{h_k, -h_k(l+\cdot)}(t, t)| &\leq 2t^\alpha \sum_{l \in \mathbb{Z}: |l| \leq k} \rho_l \exp(-2t^\alpha (\|h_k\|_\alpha^\alpha - \rho_l)) \\ &\leq Ct^\alpha \exp(-2t^\alpha (\|h_k\|_\alpha^\alpha - \max_{|l| \leq k} \rho_l)). \end{aligned} \quad (5.35)$$

Now by Cauchy-Schwarz inequality  $\rho_l < \|h_k\|_\alpha^\alpha$  for all  $l$ , and we obtain a uniform bound in  $t \geq 0$ . By Lemmas 5.1 and 5.2 there exist constants  $C, K > 0$  such that

$$\begin{aligned} \sum_{|l| > k} |U_{h_k, h_k(l+\cdot)}(t, t) + U_{h_k, -h_k(l+\cdot)}(t, t)| \\ \leq 2^\alpha t^\alpha \exp(-2t^\alpha (\|h_k\|_\alpha^\alpha - \sup_{|l| > k} \rho_l)) \sum_{|l| > k} |l|^{(\alpha(H-k)-1)/2} \\ \leq Ct^\alpha \exp(-Kt^\alpha), \end{aligned} \quad (5.36)$$

where we used the assumption  $k > H + 1/\alpha$  and that  $\rho_l \rightarrow 0$  for  $|l| \rightarrow \infty$  by Lemma 5.2. Combining (5.34), (5.35) and (5.36) we can conclude (5.32). This completes the proof of Theorem 3.1(ii).

## 5.5 Proof of Theorem 3.1(iii)

As in the proof of Theorem 3.1(ii) we obtain the decomposition (5.30), where the convergence rate  $\sqrt{n}$  is replaced by  $n^{1-1/\beta}$ . Furthermore, as in (5.31), it suffices to prove that for some  $r \in (1, 2)$  then as  $n \rightarrow \infty$

$$\begin{aligned} n^{1-1/\beta} (\|\varphi_n - \varphi_{\xi_0}\|_{w,r} + |H_n - H_0|) &\xrightarrow{d} \|\kappa_2 S\|_{w,r} + |M_2|, \\ n^{1-1/\beta} \int_0^\infty (\varphi_n(t) - \varphi_{\xi_0}(t)) \nabla_\theta \varphi_{\xi_0}(t) w(t) dt &\xrightarrow{d} S \int_0^\infty \kappa_2(t) \nabla_\theta \varphi_{\xi_0}(t) w(t) dt. \end{aligned}$$

As before in the Gaussian case it is enough to provide uniform bounds (in  $n$  and  $t$ ) on the moments in order to use Proposition 5.6.

Recall that the dominating term in (5.23) is given by

$$\overline{S}_n(t) = n^{-1/\beta} \sum_{i=k}^n (\overline{\Phi}_t(L_{i+1} - L_i) - \mathbb{E}[\overline{\Phi}_t(L_{i+1} - L_i)]).$$

Inspired by the classical case of i.i.d. random variables, each in the domain of attraction of a stable distribution, we shall prove the following result.

**Proposition 5.7.** *For any  $r \in (0, \beta)$  we have that*

$$\sup_{n \in \mathbb{N}, t \geq 0} \mathbb{E}[|\overline{S}_n(t)|^r] < \infty.$$

**Proof.** By Jensen's inequality it suffices to consider  $r > 1$  (indeed  $\beta \in (1, 2)$ ). Recall the relation  $\Phi_{t,\sigma\|h_k\|_\alpha}(x) = \Phi_t^2(x) = \exp(-|\sigma t\|h_k\|_\alpha|^\alpha)(\cos(tx) - 1)$  together with

$$\bar{\Phi}_t(x) = \sum_{i=1}^{\infty} \Phi_t^2(h_k(i)x).$$

Note that for all  $x$  in some bounded set by Lemma 5.3(ii):

$$\begin{aligned} \sup_{t \geq 0} |\bar{\Phi}_t(x)| &\leq \sup_{t \geq 0} \exp(-|\sigma\|h_k\|_\alpha t|^\alpha) \sum_{i=1}^{\infty} (1 \wedge (|xth_k(i)|)^2) \\ &\leq \sup_{t \geq 0} \exp(-|\sigma\|h_k\|_\alpha t|^\alpha) (t+1)^2 (|x|+1)^2 \sum_{i=1}^{\infty} (1 \wedge |h_k(i)|^2) \leq C < \infty. \end{aligned}$$

By (5.25) there exists  $x_0 < -1$  such that for all  $t > 0$

$$\left| \frac{|\bar{\Phi}_t(x)|}{|x|^q} - |\kappa_2(t)| \right| \leq 1 \quad \text{for all } x < x_0, \quad (5.37)$$

where  $q = \alpha/\beta$ ,  $\beta = 1 + \alpha(k - H) \in (1, 2)$  and  $\kappa_2(t) = Kt^q \exp(-|t\|h_k\|_\alpha\sigma|^\alpha)$  with  $K < 0$ . For shorter notation we write  $D_i = L_{i+1} - L_i$  to denote the  $i$ th increment of  $L$ . Since  $\mathbb{E}[\bar{\Phi}_t(D_1)]$  is bounded in  $t$  we may replace  $\bar{\Phi}_t(x)$  with  $\bar{\Phi}_t(x) - \mathbb{E}[\bar{\Phi}_t(D_1)]$  in (5.37) if  $x_0$  is chosen large enough. Define for each  $t \geq 0$ ,  $n \in \mathbb{N}$  and  $i \in \{k, \dots, n\}$

$$\begin{aligned} Y_{n,t,i} &= (\bar{\Phi}_t(D_i) - \mathbb{E}[\bar{\Phi}_t(D_i)]) \mathbf{1}_{\{|\bar{\Phi}_t(D_i) - \mathbb{E}[\bar{\Phi}_t(D_i)]| \leq n^{1/\beta}\}}, \\ Z_{n,t,i} &= (\bar{\Phi}_t(D_i) - \mathbb{E}[\bar{\Phi}_t(D_i)]) \mathbf{1}_{\{|\bar{\Phi}_t(D_i) - \mathbb{E}[\bar{\Phi}_t(D_i)]| > n^{1/\beta}\}}. \end{aligned}$$

We have the decomposition

$$\begin{aligned} T_{n,t} &:= \sum_{i=k}^n (\bar{\Phi}_t(D_i) - \mathbb{E}[\bar{\Phi}_t(D_i)]) = \sum_{i=k}^n (Y_{n,t,i} - \mathbb{E}[Y_{n,t,i}]) + \sum_{i=k}^n (Z_{n,t,i} - \mathbb{E}[Z_{n,t,i}]) \\ &=: T_{n,t,1} + T_{n,t,2}. \end{aligned}$$

The proposition then asserts that

$$\sup_{n \in \mathbb{N}, t \geq 0} \mathbb{E}[|n^{-1/\beta} T_{n,t}|^r] < \infty \quad \text{for all } r \in (1, \beta).$$

To prove this we observe that

$$\mathbb{E}[|n^{-1/\beta} T_{n,t}|^r] \leq C_r (\mathbb{E}[|n^{-1/\beta} T_{n,t,1}|^r] + \mathbb{E}[|n^{-1/\beta} T_{n,t,2}|^r]).$$

For the first term we obtain the inequality

$$\mathbb{E}[|n^{-1/\beta} T_{n,t,1}|^r] \leq \mathbb{E}[|n^{-1/\beta} T_{n,t,1}|^2]^{r/2} \leq C_r (n^{1-2/\beta} \mathbb{E}[|Y_{n,t,k}|^2])^{r/2}.$$

For short notation let  $E_t = \mathbb{E}[\bar{\Phi}_t(D_1)]$ , which is uniformly bounded in  $t \geq 0$ . Additionally let  $p_\alpha$  denote the density of an SoS distribution and recall that  $p_\alpha(x) \leq C(1+|x|)^{-1-\alpha}$  for

all  $x \in \mathbb{R}$ , cf. [29, Theorem 1.1]. We decompose  $\mathbb{E}[|Y_{n,t,k}|^2]$  into two regions corresponding to (5.37):

$$\begin{aligned} n^{1-2/\beta} \mathbb{E}[|Y_{n,t,k}|^2] &= 2n^{1-2/\beta} \int_{x_0}^0 |\bar{\Phi}_t(x) - E_t|^2 \mathbf{1}_{\{|\bar{\Phi}_t(x) - E_t| \leq n^{1/\beta}\}} p_\alpha(x) dx \\ &\quad + 2n^{1-2/\beta} \int_{-\infty}^{x_0} |\bar{\Phi}_t(x) - E_t|^2 \mathbf{1}_{\{|\bar{\Phi}_t(x) - E_t| \leq n^{1/\beta}\}} p_\alpha(x) dx. \end{aligned}$$

The first term vanishes as  $n \rightarrow \infty$  since  $2/\beta > 1$  and the fact that  $|\bar{\Phi}_t(x) - E_t|$  is bounded for all  $t$  and  $x \in (x_0, 0)$ . The second term is further split into two terms:

$$\begin{aligned} n^{1-2/\beta} \int_{-\infty}^{x_0} |\bar{\Phi}_t(x) - E_t|^2 \mathbf{1}_{\{|\bar{\Phi}_t(x) - E_t| \leq n^{1/\beta}\}} p_\alpha(x) dx \\ = n^{1-2/\beta} \int_{-\infty}^{x_0} |\bar{\Phi}_t(x) - E_t|^2 \mathbf{1}_{\{|x|^q < |\bar{\Phi}_t(x) - E_t| \leq n^{1/\beta}\}} p_\alpha(x) dx \\ + n^{1-2/\beta} \int_{-\infty}^{x_0} |\bar{\Phi}_t(x) - E_t|^2 \mathbf{1}_{\{|\bar{\Phi}_t(x) - E_t| \leq n^{1/\beta} \wedge |x|^q\}} p_\alpha(x) dx. \end{aligned}$$

Using (5.37) and the boundedness of  $\kappa_2$  on the first term we have that

$$\begin{aligned} n^{1-2/\beta} \int_{-\infty}^{x_0} |\bar{\Phi}_t(x) - E_t|^2 \mathbf{1}_{\{|x|^q < |\bar{\Phi}_t(x) - E_t| \leq n^{1/\beta}\}} p_\alpha(x) dx \\ \leq n^{1-2/\beta} \int_{-x_0}^{\infty} (|\kappa_2(t)| + 1)^2 x^{2q} \mathbf{1}_{\{x^q \leq n^{1/\beta}\}} p_\alpha(x) dx \\ \leq C_q n^{1-2/\beta} \int_{-x_0}^{n^{1/q\beta}} x^{2q-1-\alpha} dx = C_q n^{1-2/\beta} (1 + n^{(2q-\alpha)/q\beta}) \leq C_q. \end{aligned}$$

The second term contains a similar consideration, indeed

$$\begin{aligned} n^{1-2/\beta} \int_{-\infty}^{x_0} |\bar{\Phi}_t(x) - E_t|^2 \mathbf{1}_{\{|\bar{\Phi}_t(x) - E_t| \leq n^{1/\beta} \wedge |x|^q\}} p_\alpha(x) dx &\leq \int_{-x_0}^{\infty} (n^{1/\beta} \wedge x^q)^2 p_\alpha(x) dx \\ &= n^{1-2/\beta} \int_{-x_0}^{n^{1/q\beta}} x^{2q} p_\alpha(x) dx + n^{1-2/\beta} \int_{n^{1/q\beta}}^{\infty} n^{2/\beta} p_\alpha(x) dx \leq C_q. \end{aligned}$$

In the next step we treat the term  $T_{n,t,2}$ . Note first that by the von Bahr-Esseen inequality we obtain

$$\mathbb{E}[|n^{-1/\beta} T_{n,t,2}|^r] \leq C_r n^{1-r/\beta} \mathbb{E}[|Z_{n,t,k}|^r].$$

Decomposing as above we have that

$$\begin{aligned} n^{1-r/\beta} \mathbb{E}[|Z_{n,t,k}|^r] &= 2n^{1-r/\beta} \int_{x_0}^0 |\bar{\Phi}_t(x) - E_t|^r \mathbf{1}_{\{|\bar{\Phi}_t(x) - E_t| > n^{1/\beta}\}} p_\alpha(x) dx \\ &\quad + 2n^{1-r/\beta} \int_{-\infty}^{x_0} |\bar{\Phi}_t(x) - E_t|^r \mathbf{1}_{\{|\bar{\Phi}_t(x) - E_t| > n^{1/\beta}\}} p_\alpha(x) dx. \end{aligned}$$

For the first term we recall that  $\bar{\Phi}_t(x)$  is bounded uniformly in  $t$  when  $x$  lies in a bounded set, hence  $n^{1/\beta} > |\bar{\Phi}_t(x) - E_t|$  for all sufficiently large  $n$ , independent of  $x \in (0, x_0)$  and  $t \geq 0$ , so the first term is zero for sufficiently large  $n$ .

The last term requires more computations. Due to (5.37) and the fact that  $\kappa_2$  is bounded it follows that

$$\begin{aligned}
& n^{1-r/\beta} \int_{-\infty}^{x_0} |\bar{\Phi}_t(x) - E_t|^r \mathbf{1}_{\{|\bar{\Phi}_t(x) - E_t| > n^{1/\beta}\}} p_\alpha(x) dx \\
& \leq C n^{1-r/\beta} \int_{-\infty}^{x_0} (|\kappa_2(t)| + 1)^r |x|^{rq} \mathbf{1}_{\{(|\kappa_2(t)| + 1)|x|^q > n^{1/\beta}\}} p_\alpha(x) dx \\
& \leq C n^{1-r/\beta} \int_{-\infty}^{x_0} |x|^{rq-1-\alpha} \mathbf{1}_{\{|x| > n^{1/q\beta}/K\}} dx \\
& = C n^{1-r/\beta} \int_{n^{1/\alpha}/K}^{\infty} x^{rq-1-\alpha} dx \leq C,
\end{aligned}$$

where we used that  $rq - \alpha < 0$  since  $r < \beta$ . □

Combining Propositions 5.4–5.7 we finally complete the proof of Theorem 3.1(iii).

## 6 Tables

**Table 5.** Absolute value of the bias based on  $n = 1000$ ,  $k = 2$ ,  $p = -0.4$ ,  $\nu = 0.1$  and  $\sigma_0 = 0.3$  for the minimal contrast estimator.

$H_0$	$\alpha_0$	$\sigma_n$	$\alpha_n$	$H_n(p, k)$
0.2	<b>0.4</b>	6.1153	0.4078	0.2015
	0.6	0.0998	0.0076	0.1818
	0.8	0.0417	0.0018	0.1346
	1	0.0540	0.0000	0.1137
	1.2	0.0451	0.0008	0.0881
	1.4	0.0367	0.0045	0.0702
	1.6	0.0309	0.0207	0.0706
	1.8	0.0117	0.0535	0.0643
0.4	<b>0.4</b>	5.4236	0.3281	0.1040
	<b>0.6</b>	0.0774	0.0035	0.1237
	0.8	0.0472	0.0051	0.0748
	1	0.0322	0.0003	0.0501
	1.2	0.0276	0.0010	0.0351
	1.4	0.0115	0.0090	0.0189
	1.6	0.0072	0.0487	0.0261
	1.8	0.0496	0.1229	0.0193
0.6	<b>0.4</b>	3.6173	0.2007	0.0638
	<b>0.6</b>	0.0493	0.0058	0.0608
	0.8	0.0375	0.0036	0.0402
	1	0.0328	0.0020	0.0235
	1.2	0.0115	0.0161	0.0187
	1.4	0.0067	0.0327	0.0065
	1.6	0.0366	0.0933	0.0112
	1.8	0.1129	0.2501	0.0089
0.8	<b>0.4</b>	1.6906	0.0866	0.0204
	<b>0.6</b>	0.0483	0.0027	0.0404
	<b>0.8</b>	0.0514	0.0025	0.0303
	1	0.0345	0.0027	0.0167
	1.2	0.0053	0.0215	0.0107
	1.4	0.0160	0.0511	0.0057
	1.6	0.0702	0.1425	0.0014
	1.8	0.1724	0.3915	0.0016

**Table 6.** Standard deviation based on  $n = 1000$ ,  $k = 2$ ,  $p = -0.4$ ,  $\nu = 0.1$  and  $\sigma_0 = 0.3$  for the minimal contrast estimator.

$H_0$	$\alpha_0$	$\sigma_n$	$\alpha_n$	$H_n(p, k)$
0.2	<b>0.4</b>	7.4104	0.4585	0.1609
	0.6	0.4289	0.0925	0.1622
	0.8	0.2445	0.0681	0.1468
	1	0.2047	0.0773	0.1355
	1.2	0.1831	0.0923	0.1283
	1.4	0.1733	0.1118	0.1244
	1.6	0.1551	0.1376	0.1216
	1.8	0.1415	0.1461	0.1210
0.4	<b>0.4</b>	7.9313	0.4407	0.1664
	<b>0.6</b>	0.3619	0.0717	0.1723
	0.8	0.2295	0.0686	0.1546
	1	0.1864	0.0869	0.1503
	1.2	0.1698	0.1151	0.1448
	1.4	0.1590	0.1583	0.1384
	1.6	0.1527	0.1964	0.1447
	1.8	0.1255	0.1959	0.1397
0.6	<b>0.4</b>	7.3589	0.3497	0.1923
	<b>0.6</b>	0.2561	0.0652	0.1729
	0.8	0.1948	0.0676	0.1579
	1	0.1817	0.1030	0.1538
	1.2	0.1683	0.1411	0.1406
	1.4	0.1651	0.1974	0.1498
	1.6	0.1559	0.2431	0.1465
	1.8	0.1171	0.2271	0.1410
0.8	<b>0.4</b>	5.0779	0.2183	0.1848
	<b>0.6</b>	0.2539	0.0632	0.1758
	<b>0.8</b>	0.2014	0.0748	0.1523
	1	0.1864	0.1140	0.1546
	1.2	0.1843	0.1765	0.1533
	1.4	0.1749	0.2353	0.1424
	1.6	0.1402	0.2413	0.1417
	1.8	0.1156	0.2342	0.1361

**Table 7.** Absolute value of the bias based on  $n = 10\,000$ ,  $k = 2$ ,  $p = -0.4$ ,  $\nu = 0.1$  and  $\sigma_0 = 0.3$  for the minimal contrast estimator.

$H_0$	$\alpha_0$	$\sigma_n$	$\alpha_n$	$H_n(p, k)$
0.2	<b>0.4</b>	2.2799	0.1415	0.2563
	0.6	0.0106	0.0018	0.1652
	0.8	0.0032	0.0025	0.1134
	1	0.0046	0.0005	0.0848
	1.2	0.0078	0.0012	0.0636
	1.4	0.0110	0.0015	0.0516
	1.6	0.0107	0.0012	0.0413
	1.8	0.0077	0.0010	0.0318
0.4	<b>0.4</b>	0.6629	0.0458	0.1766
	<b>0.6</b>	0.0081	0.0003	0.1022
	0.8	0.0050	0.0017	0.0600
	1	0.0018	0.0010	0.0366
	1.2	0.0052	0.0018	0.0234
	1.4	0.0068	0.0019	0.0170
	1.6	0.0097	0.0065	0.0103
	1.8	0.0124	0.0145	0.0041
0.6	<b>0.4</b>	0.1517	0.0171	0.1130
	<b>0.6</b>	0.0021	0.0010	0.0588
	0.8	0.0067	0.0015	0.0314
	1	0.0057	0.0011	0.0156
	1.2	0.0025	0.0008	0.0070
	1.4	0.0032	0.0010	0.0034
	1.6	0.0081	0.0050	0.0008
	1.8	0.0143	0.0126	0.0017
0.8	<b>0.4</b>	0.0575	0.0060	0.0793
	<b>0.6</b>	0.0053	0.0013	0.0335
	<b>0.8</b>	0.0033	0.0002	0.0131
	1	0.0049	0.0004	0.0065
	1.2	0.0000	0.0006	0.0014
	1.4	0.0030	0.0014	0.0004
	1.6	0.0095	0.0045	0.0016
	1.8	0.0039	0.0035	0.0002

**Table 8.** Standard deviation based on  $n = 10\,000$ ,  $k = 2$ ,  $p = -0.4$ ,  $\nu = 0.1$  and  $\sigma_0 = 0.3$  for the minimal contrast estimator.

$H_0$	$\alpha_0$	$\sigma_n$	$\alpha_n$	$H_n(p, k)$
0.2	<b>0.4</b>	4.4680	0.2759	0.0693
	0.6	0.0983	0.0292	0.0560
	0.8	0.1062	0.0313	0.0510
	1	0.0642	0.0236	0.0499
	1.2	0.0684	0.0354	0.0484
	1.4	0.0699	0.0506	0.0489
	1.6	0.0711	0.0666	0.0479
	1.8	0.0654	0.0678	0.0476
0.4	<b>0.4</b>	2.2370	0.1458	0.0704
	<b>0.6</b>	0.0826	0.0230	0.0628
	0.8	0.0933	0.0305	0.0509
	1	0.0577	0.0309	0.0504
	1.2	0.0574	0.0393	0.0494
	1.4	0.0647	0.0620	0.0465
	1.6	0.0812	0.0959	0.0484
	1.8	0.0825	0.1168	0.0479
0.6	<b>0.4</b>	0.6954	0.0689	0.0713
	<b>0.6</b>	0.1014	0.0226	0.0576
	0.8	0.0756	0.0290	0.0506
	1	0.0589	0.0327	0.0481
	1.2	0.0546	0.0446	0.0480
	1.4	0.0710	0.0749	0.0479
	1.6	0.0955	0.1238	0.0462
	1.8	0.0957	0.1469	0.0469
0.8	<b>0.4</b>	0.3571	0.0510	0.0749
	<b>0.6</b>	0.1290	0.0256	0.0596
	<b>0.8</b>	0.0524	0.0245	0.0515
	1	0.0645	0.0388	0.0497
	1.2	0.0670	0.0593	0.0475
	1.4	0.0845	0.0951	0.0470
	1.6	0.1119	0.1512	0.0482
	1.8	0.0987	0.1621	0.0508

**Table 9.** Absolute value of bias based on  $n = 10\,000$ ,  $k = 1$ ,  $p = -0.4$ ,  $\nu = 0.1$  and  $\sigma_0 = 0.3$ .

$H_0$	$\alpha_0$	$\sigma_n$	$\alpha_n$	$H_n(p, k)$
0.2	<b>0.4</b>	0.0202	0.0177	0.1873
	<b>0.6</b>	0.0671	0.0001	0.1329
	<b>0.8</b>	0.0256	0.0015	0.0966
	<b>1</b>	0.0044	0.0028	0.0745
	<b>1.2</b>	0.0026	0.0018	0.0589
	1.4	0.0097	0.0065	0.0465
	1.6	0.0161	0.0135	0.0376
	1.8	0.0216	0.0270	0.0286
0.4	<b>0.4</b>	0.0656	0.0090	0.0824
	<b>0.6</b>	0.0623	0.0017	0.0564
	<b>0.8</b>	0.0201	0.0020	0.0394
	<b>1</b>	0.0017	0.0037	0.0278
	<b>1.2</b>	0.0076	0.0052	0.0202
	<b>1.4</b>	0.0109	0.0089	0.0128
	<b>1.6</b>	0.0229	0.0271	0.0066
	1.8	0.0177	0.0288	0.0048
0.6	<b>0.4</b>	0.0659	0.0110	0.0109
	<b>0.6</b>	0.0724	0.0018	0.0114
	<b>0.8</b>	0.0254	0.0022	0.0025
	<b>1</b>	0.0050	0.0101	0.0009
	<b>1.2</b>	0.0131	0.0127	0.0002
	<b>1.4</b>	0.0189	0.0194	0.0027
	<b>1.6</b>	0.0083	0.0167	0.0004
	<b>1.8</b>	0.0241	0.0456	0.0014
0.8	<b>0.4</b>	0.0813	0.0116	0.1184
	<b>0.6</b>	0.0937	0.0039	0.0825
	<b>0.8</b>	0.0343	0.0117	0.0499
	<b>1</b>	0.0033	0.0194	0.0258
	<b>1.2</b>	0.0035	0.0120	0.0005
	<b>1.4</b>	0.0012	0.0197	0.0034
	<b>1.6</b>	0.0162	0.0439	0.0054
	<b>1.8</b>	0.0214	0.0571	0.0034

**Table 10.** Standard deviation based on  $n = 10\,000$ ,  $k = 1$ ,  $p = -0.4$ ,  $\nu = 0.1$  and  $\sigma_0 = 0.3$ .

$H_0$	$\alpha_0$	$\sigma_n$	$\alpha_n$	$H_n(p, k)$
0.2	<b>0.4</b>	0.6538	0.0803	0.0655
	<b>0.6</b>	0.0681	0.0247	0.0548
	<b>0.8</b>	0.0599	0.0239	0.0501
	<b>1</b>	0.0693	0.0324	0.0474
	<b>1.2</b>	0.0633	0.0422	0.0459
	1.4	0.0711	0.0598	0.0461
	1.6	0.0835	0.0901	0.0458
	1.8	0.0841	0.1071	0.0437
0.4	<b>0.4</b>	0.3547	0.0625	0.0638
	<b>0.6</b>	0.0769	0.0271	0.0562
	<b>0.8</b>	0.0754	0.0355	0.0494
	<b>1</b>	0.0549	0.0409	0.0473
	<b>1.2</b>	0.0636	0.0560	0.0478
	<b>1.4</b>	0.0751	0.0777	0.0452
	<b>1.6</b>	0.1003	0.1242	0.0459
	1.8	0.0685	0.1096	0.0441
0.6	<b>0.4</b>	0.8005	0.0619	0.0637
	<b>0.6</b>	0.0668	0.0342	0.0544
	<b>0.8</b>	0.0722	0.0512	0.0521
	<b>1</b>	0.0695	0.0575	0.0494
	<b>1.2</b>	0.0772	0.0717	0.0453
	<b>1.4</b>	0.1065	0.1194	0.0458
	<b>1.6</b>	0.0777	0.1138	0.0441
	<b>1.8</b>	0.0648	0.1238	0.0449
0.8	<b>0.4</b>	0.7398	0.0648	0.0660
	<b>0.6</b>	0.1042	0.0448	0.0550
	<b>0.8</b>	0.1182	0.0728	0.0506
	<b>1</b>	0.0845	0.0830	0.0460
	<b>1.2</b>	0.0764	0.0747	0.0444
	<b>1.4</b>	0.0630	0.1042	0.0464
	<b>1.6</b>	0.0700	0.1390	0.0452
	<b>1.8</b>	0.0602	0.1412	0.0419

**Table 11.** Absolute value of bias for  $(\tilde{\sigma}_{\text{low}}, \tilde{\alpha}_{\text{low}})$  for  $n = 10\,000$ ,  $p = -0.4$  and  $\sigma_0 = 0.3$ .

$H_0$	$\alpha_0$	$\tilde{\sigma}_{\text{low}}$	$\tilde{\alpha}_{\text{low}}$
0.2	0.4	0.2872	0.0371
	0.6	0.2860	0.1737
	0.8	0.2725	0.3817
	1	0.2390	0.4889
	1.2	0.0888	0.1818
	1.4	0.0065	0.0044
	1.6	0.0077	0.0080
	1.8	0.0047	0.0013
0.4	0.4	0.2805	0.0364
	0.6	0.2693	0.1606
	0.8	0.2053	0.2938
	1	0.0543	0.1374
	1.2	0.0052	0.0043
	1.4	0.0012	0.0011
	1.6	0.0005	0.0010
	1.8	0.0006	0.0007
0.6	0.4	0.2764	0.0460
	0.6	0.2285	0.1362
	0.8	0.1752	0.2583
	1	0.0077	0.0135
	1.2	0.0020	0.0023
	1.4	0.0001	0.0001
	1.6	0.0002	0.0005
	1.8	0.0006	0.0000
0.8	0.4	0.2688	0.0713
	0.6	0.1389	0.0917
	0.8	0.0096	0.0286
	1	0.0062	0.0073
	1.2	0.0013	0.0000
	1.4	0.0003	0.0023
	1.6	0.0001	0.0010
	1.8	0.0000	0.0020

**Table 12.** Standard deviation of  $(\tilde{\sigma}_{\text{low}}, \tilde{\alpha}_{\text{low}})$  for  $n = 10\,000$ ,  $p = -0.4$  and  $\sigma_0 = 0.3$ .

$H_0$	$\alpha_0$	$\tilde{\sigma}_{\text{low}}$	$\tilde{\alpha}_{\text{low}}$
0.2	0.4	0.0514	0.2249
	0.6	0.0420	0.2195
	0.8	0.0676	0.2181
	1	0.1036	0.2549
	1.2	0.1322	0.2632
	1.4	0.1000	0.2301
	1.6	0.0510	0.1362
	1.8	0.0264	0.0765
0.4	0.4	0.0680	0.2127
	0.6	0.0741	0.2198
	0.8	0.1525	0.2459
	1	0.2039	0.3633
	1.2	0.0528	0.0875
	1.4	0.0236	0.0451
	1.6	0.0151	0.0308
	1.8	0.0112	0.0222
0.6	0.4	0.0734	0.2108
	0.6	0.1522	0.2142
	0.8	0.1790	0.2618
	1	0.1245	0.1896
	1.2	0.0259	0.0371
	1.4	0.0149	0.0283
	1.6	0.0106	0.0252
	1.8	0.0078	0.0204
0.8	0.4	0.0853	0.2041
	0.6	0.2856	0.2329
	0.8	0.2399	0.2664
	1	0.0659	0.0793
	1.2	0.0201	0.0306
	1.4	0.0118	0.0281
	1.6	0.0084	0.0255
	1.8	0.0064	0.0208

**Table 13.** Failure rates for  $n = 10\,000$ ,  $p = -0.4$ ,  $k = 2$ ,  $\nu = 0.1$  and  $\sigma_0 = 0.3$ .

$H_0$	$\alpha_0$	Failure rate (%)	
		$(\tilde{\sigma}_{\text{low}}, \tilde{\alpha}_{\text{low}}, \tilde{H}_{\text{low}})$	$\xi_n$
0.2	0.4	88.17	14.83
	0.6	84.08	0
	0.8	69.67	0
	1	38.58	0
	1.2	0.42	0
	1.4	3.25	0.17
	1.6	3.83	1
	1.8	1.58	2.25
0.4	0.4	86.08	20.17
	0.6	76.75	0.17
	0.8	62.58	0
	1	19.17	0.18
	1.2	0	0
	1.4	0	0.33
	1.6	0	1.42
	1.8	0	7.17
0.6	0.4	87	28.67
	0.6	72.67	0
	0.8	41.75	0
	1	0.08	0
	1.2	0	0
	1.4	0	0.08
	1.6	0	2.42
	1.8	0	10.25
0.8	0.4	89.50	26.33
	0.6	69.33	0.33
	0.8	2.17	0
	1	0.08	0.25
	1.2	0	0
	1.4	0	0.17
	1.6	0	3.92
	1.8	0	11

## References

- [1] A. Ayache and J. Hamonier. Linear fractional stable motion: A wavelet estimator of the  $\alpha$  parameter. *Statistics & Probability Letters*, 82(8):1569–1575, 2012.
- [2] J.-M. Bardet and D. Surgailis. Nonparametric estimation of the local Hurst function of multifractional Gaussian processes. *Stochastic Processes and their Applications*, 123(3):1004–1045, 2013.
- [3] A. Basse-O’Connor, C. Heinrich, and M. Podolskij. On limit theory for functionals of stationary increments Lévy driven moving averages. *Electronic Journal of Probability*, 24(79):1–42, 2019.
- [4] A. Basse-O’Connor, R. Lachièze-Rey, and M. Podolskij. Power variation for a class of Lévy driven moving averages. *Annals of Probability*, 45(6B):4477–4528, 2017.
- [5] A. Benassi, S. Cohen, and J. Istas. On roughness indices for fractional fields. *Bernoulli*, 10(2):357–373, 2004.
- [6] A. Brouste and M. Fukasawa. Local asymptotic normality property for fractional Gaussian noise under high-frequency observations. *Annals of Statistics*, 46(5):2045–2061, 2018.
- [7] R. Byrd, P. Lu, J. Nocedal, and C. Zhu. A limited memory algorithm for bound constrained optimization. *SIAM Journal on Scientific Computing*, 16(5):1190–1208, 1995.
- [8] S. Cambanis, C.D. Hardin, Jr., and A. Weron. Ergodic properties of stationary stable processes. *Stochastic Processes and their Applications*, 24(1):1–18, 1987.
- [9] J.-F. Coeurjolly and J. Istas. Cramèr-Rao bounds for fractional Brownian motions. *Statistics and Probability Letters*, 53(4):435–447, 2001.
- [10] H. Cremers and D. Kadelka. On weak convergence of integral functionals of stochastic processes with applications to processes taking paths in  $L_p^e$ . *Stochastic Processes and their Applications*, 21(2):305–317, 1986.
- [11] R. Dahlhaus. Efficient parameter estimation for self-similar processes. *Annals of Statistics*, 17(4):1749–1766, 1989.
- [12] T.T.N. Dang and J. Istas. Estimation of the Hurst and the stability indices of a H-self-similar stable process. *Electronic Journal of Statistics*, 11(2):4103–4150, 2017.
- [13] D. Grahovac, N.N. Leonenko, and M.S. Taqqu. Scaling properties of the empirical structure function of linear fractional stable motion and estimation of its parameters. *Journal of Statistical Physics*, 158(1):105–119, 2015.
- [14] I.A. Ibragimov and R.Z. Has’minskii. *Statistical Estimation—Asymptotic Theory*, volume 16 of *Stochastic Modelling and Applied Probability*. Springer-Verlag New York, 1981.

- [15] J. Istas and G. Lang. Quadratic variations and estimation of the local Hölder index of a Gaussian process. *Annales de l'Institut Henri Poincaré*, 33(4):407–436, 1997.
- [16] J. Lebovits and M. Podolskij. Estimation of the global regularity of a multifractional Brownian motion. *Electronic Journal of Statistics*, 11(1):78–98, 2016.
- [17] M.M. Ljungdahl and M. Podolskij. A note on parametric estimation of Lévy moving average processes. *Springer proceedings in mathematics & statistics*, 294, 2019.
- [18] B.B. Mandelbrot and J.W.V. Ness. Fractional Brownian motions, fractional noises and applications. *SIAM Review*, 10(4):422–437, 1968.
- [19] S. Mazur, D. Otryakhin, and M. Podolskij. Estimation of the linear fractional stable motion. *Bernoulli*, 2019+.
- [20] J. A. Nelder and R. Mead. A simplex method for function minimization. *The Computer Journal*, 7(4):308–313, 1965.
- [21] V. Pipiras and M.S. Taqqu. The structure of self-similar stable mixed moving averages. *Annals of Probability*, 30(2):898–932, 2002.
- [22] V. Pipiras and M.S. Taqqu. Central limit theorems for partial sums of bounded functionals of infinite-variance moving averages. *Bernoulli*, 9(5):833–855, 2003.
- [23] V. Pipiras, M.S. Taqqu, and P. Abry. Bounds for the covariance of functions of infinite variance stable random variables with applications to central limit theorems and wavelet-based estimation. *Bernoulli*, 13(4):1091–1123, 2007.
- [24] J. Rosinski. On the structure of stationary stable processes. *Annals of Probability*, 23(3):1163–1187, 1995.
- [25] N.M. Steen, G.D. Byrne, and E.M. Gelbard. Gaussian quadratures for the integrals  $\int_0^\infty \exp(-x^2)f(x)dx$  and  $\int_0^b \exp(-x^2)f(x)dx$ . *Mathematics of Computation*, 23(107):661–671, 1969.
- [26] M.B. Stinchcombe and H. White. Some measurability results for extrema of random functions over random sets. *The Review of Economic Studies*, 59(3):495–514, 1992.
- [27] S. Stoev, V. Pipiras, and M. Taqqu. Estimation of the self-similarity parameter in linear fractional stable motion. *Signal Processing*, 82(12):1873–1901, 2002.
- [28] S. Stoev and Murad S. Taqqu. Simulation methods for linear fractional stable motion and FARIMA using the Fast Fourier Transform. *Fractals*, 12(1):95–121, 2004.
- [29] T. Watanabe. Asymptotic estimates of multi-dimensional stable densities and their applications. *Transactions of the American Mathematical Society*, 359(6):2851–2879, 2007.