A Berry–Esseén theorem for partial sums
of functionals of heavy-tailed moving averages

Andreas Basse-O'Connor[*], Mark Podolskij[†], Christoph Thäle[‡]

Abstract

In this paper we obtain Berry–Esseén bounds on partial sums of functionals of heavy-tailed moving averages, including the linear fractional stable noise, stable fractional ARIMA processes and stable Ornstein–Uhlenbeck processes. Our rates are obtained for the Wasserstein and Kolmogorov distances, and depend strongly on the interplay between the memory of the process, which is controlled by a parameter α, and its tail-index, which is controlled by a parameter β. In fact, we obtain the classical $1/\sqrt{n}$ rate of convergence when the tails are not too heavy and the memory is not too strong, more precisely, when $\alpha\beta > 3$ or $\alpha\beta > 4$ in the case of Wasserstein and Kolmogorov distance, respectively.

Our quantitative bounds rely on a new second-order Poincaré inequality on the Poisson space, which we derive through a combination of Stein's method and Malliavin calculus. This inequality improves and generalizes a result by Last, Peccati, Schulte [*Probab. Theory Relat. Fields* **165** (2016)].

1 Introduction

1.1 Overview

The main goal of this paper is to characterize the convergence rates associated with asymptotic normality of a class of statistics of Lévy moving averages. For processes with finite fourth moments, Theorem 8.2 in [16] obtains rates for a class of specific examples. Its proof relies on second-order Poincaré inequalities on the Poisson space [16, Theorem 1.1–1.2], which in turn are based on the celebrated Malliavin-Stein method. The main novelty of our results and methodology is the study of convergence rates for processes having heavy

[*]Department of Mathematics, Aarhus University, Denmark, E-mail: basse@math.au.dk

[†]Department of Mathematics, Aarhus University, Denmark, E-mail: mpodolskij@math.au.dk

[‡]Faculty of Mathematics, Ruhr University Bochum, Germany, E-mail: christoph.thaele@rub.de

tails and strong memory, as e.g. the linear fractional stable noise or fractional stable ARIMA processes. In fact, in our setting the upper bounds in the second-order Poincaré inequalities obtained in [16] may converge to infinity after the application of our standard estimate (4.4) to them. As a consequence, we develop a new modified second-order Poincaré inequality on the Poisson space, which allows us to efficiently bound Wasserstein and Kolmogorov distances associated with normal approximation of a class of statistics of Lévy moving averages. The improved bounds are important in their own right as they may prove to be useful in other contexts, where the considered stochastic process exhibits heavy tails and strong memory.

1.2 Background

The Berry–Esseén theorem gives a quantitative bound for the convergence rate in the classical central limit theorem. To be more precise, let $(X_i)_{i \in \mathbb{N}}$ be a sequence of independent and identically distributed (i.i.d.) random variables with mean zero, variance one and finite third moment, and set $V_n = n^{-1/2} \sum_{i=1}^n X_i$ and $Z \sim \mathcal{N}(0,1)$. Then, the central limit theorem says that $V_n \xrightarrow{d} Z$ as $n \to \infty$, where $\xrightarrow{d}$ denotes convergence in distribution. A natural next question is to ask for quantitative bounds between V_n and Z, that is, how far is V_n from Z in a certain sense. An answer to this question is provided by the Berry–Esseén theorem, which states that

$$d_K(V_n, Z) := \sup_{x \in \mathbb{R}} \left| \mathbb{P}(V_n \le x) - \mathbb{P}(Z \le x) \right| \le C n^{-1/2}, \tag{1.1}$$

where d_K denotes the Kolmogorov metric between two random variables and where C is a constant depending on the third moment of the underlying random variables. The Berry–Esseén bound (1.1) is optimal in the sense that there exist random variables as above such that $d_K(V_n, Z)$ is bounded from below by a constant times $n^{-1/2}$, see e.g. [10, (5.26.2)].

The situation when the summands $(X_i)_{i \in \mathbb{N}}$ are dependent is much more complicated, compared to the classical i.i.d. setting. One of the most important models in this situation is the fractional Gaussian noise, which we will describe in the following. For $H \in (0, 1)$, the fractional Brownian motion is the unique centered Gaussian process $(Y_t)_{t \in \mathbb{R}}$ with covariance function

$$\operatorname{cov}(Y_t, Y_u) = \frac{1}{2}\Big(|t|^{2H} + |u|^{2H} - |t-u|^{2H}\Big), \qquad \text{for all } t, u \in \mathbb{R}.$$

The fractional Gaussian noise $(X_n)_{n \in \mathbb{Z}}$ is the corresponding increment process $X_n = Y_n - Y_{n-1}$. Let

$$V_n = \frac{1}{\sqrt{n}} \sum_{j=1}^n \left(X_j^2 - 1 \right) \qquad \text{and} \qquad v_n = \sqrt{\operatorname{var}(V_n)}.$$

For $H < 3/4$, we have that $v_n^{-1} V_n \xrightarrow{d} Z$ as $n \to \infty$. The first Berry–Esseén bound for the fractional Gaussian noise was obtained in Theorem 4.1 of [22] and reads as

$$d_K\Big(v_n^{-1} V_n, Z\Big) \le C \begin{cases} n^{-1/2} & \text{if } H \in (0, 1/2], \\ n^{2H-3/2} & \text{if } H \in (1/2, 3/4). \end{cases} \tag{1.2}$$

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In (1.2) we observe the phenomenon that for strong memory in X (i.e. $H \in (1/2, 3/4)$), we get a slower rate of convergence. Furthermore, when $H > 3/4$, the memory in X is so strong that V_n after proper normalization converge to the Rosenblatt random variable in distribution, and hence has a non-Gaussian fluctuation, see e.g. Theorem 7.4.1 of [23].

1.3   Heavy-tailed moving averages

Let us now describe our results in more detail. We consider a two-sided Lévy process $L = (L_t)_{t \in \mathbb{R}}$ with no Gaussian component, $L_0 = 0$ a.s. and Lévy measure ν, that is, for all $\theta \in \mathbb{R}$, the characteristic function of L_1 is given by

$$\mathbb{E}[e^{i\theta L_1}] = \exp\left(\int_{\mathbb{R}} \left(e^{i\theta x} - 1 - i\theta x \chi(x) \right) \nu(dx) + ib\theta \right), \tag{1.3}$$

where $b \in \mathbb{R}$, and χ is a truncation function, i.e. a bounded measurable function such that $\chi(x) = 1 + o(|x|)$ as $x \to 0$ and $\chi(x) = O(|x|^{-1})$ as $|x| \to \infty$. We assume that the Lévy measure ν has a density κ satisfying

$$\kappa(x) \leq C|x|^{-1-\beta} \qquad \text{for all } x \in \mathbb{R} \setminus \{0\}, \tag{1.4}$$

for $\beta \in (0, 2)$ and a constant $C > 0$. We consider a Lévy moving average of the form

$$X_t = \int_{-\infty}^{t} g(t - s) \, dL_s, \qquad t \in \mathbb{R}, \tag{1.5}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a measurable function such that the integral exists, see [33] for sufficient conditions. Lévy moving averages are stationary infinitely divisible processes, and are often used to model long-range dependence and heavy tails. When the Lévy process L is symmetric, i.e. when $-L_1$ equals L_1 in distribution, a sufficient condition for X to be well-defined is that $\int_{\mathbb{R}} |g(s)|^\beta \, ds < \infty$, due to assumption (1.4). Throughout the paper we will assume that the kernel function g satisfies

$$|g(x)| \leq K\left(x^\gamma \mathbf{1}_{\{0 < x < 1\}} + x^{-\alpha} \mathbf{1}_{\{x \geq 1\}} \right), \qquad \text{for all } x > 0, \tag{1.6}$$

for some finite constants $K > 0$, $\alpha > 0$ and $\gamma \in \mathbb{R}$. We refer to Subsections 1.4.1–1.4.4 for four important examples in this setting. The statistics of interest are the partial sum functionals V_n given by

$$V_n = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \left(f(X_t) - \mathbb{E}[f(X_1)] \right), \tag{1.7}$$

based on a measurable function $f : \mathbb{R} \to \mathbb{R}$ with $\mathbb{E}[|f(X_1)|] < \infty$. Typical examples, which are important in statistics, are the empirical characteristic functions ($f : x \mapsto e^{i\theta x}$, where $\theta \in \mathbb{R}$), the empirical distribution functions ($f : x \mapsto \mathbf{1}_{(-\infty, t]}(x)$, where $t \in \mathbb{R}$), and power variations ($f : x \mapsto |x|^p$, where $p > 0$). For example, in a recent paper [21] the empirical characteristic function has been successfully employed to estimate the parameters of a linear fractional stable motion observed at high or low frequency.

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The major breakthrough on establishing central limit theorems for V_n was achieved in the paper Hsing [11, Theorem 1], and was extended in [30, 31, 21, 2, 1], whereas non-central limit theorems for V_n are established in [1, 2, 39, 40]. From these results, it follows that if (X_t) is given by (1.5) with L being a β-stable Lévy process and the kernel function g satisfying (1.6) with $\gamma \geq 0$ and $\alpha\beta > 2$ we have that

$$V_n \overset{d}{\longrightarrow} Z \sim \mathcal{N}(0, v^2) \qquad \text{with} \qquad v^2 = \sum_{j \in \mathbb{Z}} \text{cov}\left(f(X_0), f(X_j)\right) \in [0, \infty),$$

for all bounded and measurable $f : \mathbb{R} \to \mathbb{R}$, cf. [30, Theorem 2.1]. For $\alpha\beta < 2$, V_n has a non-Gaussian fluctuation and a different scaling rate, see e.g. [2, Theorem 1.2], and hence we will only consider the case $\alpha\beta > 2$.

1.4 Main results

To present our main result let $C_b^2(\mathbb{R})$ denote the space of twice continuously differentiable functions such that f, f' and f'' are bounded. Our result reads as follows:

Theorem 1.1. *Let $(X_t)_{t \in \mathbb{R}}$ be a Lévy moving average given by (1.5), satisfying (1.4) for some $0 < \beta < 2$, and (1.6) with $\alpha\beta > 2$ and $\gamma > -1/\beta$. Let V_n be the corresponding partial sums of functionals, given by (1.7), based on $f \in C_b^2(\mathbb{R})$. Also, let $Z \sim \mathcal{N}(0, 1)$ be a standard Gaussian random variable, and set $v_n = \sqrt{\text{var}(V_n)}$ for all $n \in \mathbb{N}$. Then, $v_n \to v$, where $v \geq 0$ is given by*

$$v^2 = \sum_{j \in \mathbb{Z}} \text{cov}\left(f(X_0), f(X_j)\right) \tag{1.8}$$

and the series (1.8) converges absolutely. Suppose that $v > 0$. Then, $v_n^{-1} V_n \overset{d}{\longrightarrow} \mathcal{N}(0, 1)$ as $n \to \infty$. Moreover, for each $n \in \mathbb{N}$,

$$d_W\left(v_n^{-1} V_n, Z\right) \leq C \begin{cases} n^{-1/2} & \text{if } \alpha\beta > 3, \\ n^{-1/2}\log(n) & \text{if } \alpha\beta = 3, \\ n^{(2-\alpha\beta)/2} & \text{if } 2 < \alpha\beta < 3, \end{cases} \tag{1.9}$$

and

$$d_K\left(v_n^{-1} V_n, Z\right) \leq C \begin{cases} n^{-1/2} & \text{if } \alpha\beta > 4, \\ n^{-1/2}\log(n) & \text{if } \alpha\beta = 4, \\ n^{(2-\alpha\beta)/4} & 2 < \alpha\beta < 4, \end{cases} \tag{1.10}$$

where $C > 0$ is a constant that does not depend on n and d_W denotes the Wasserstein distance.

Remark 1.2. In the following we will make a few remarks on Theorem 1.1.

1. The bounds on the Wasserstein and Kolmogorov distances to the normal distribution, depend on the interplay between memory of X, which is controlled by α, and the

tail-index of X, which is controlled by β. In fact, we obtain the classical $1/\sqrt{n}$ rate of convergence when the tails are not too heavy and the memory is not too strong, more precisely, when $\alpha\beta > 3$ or $\alpha\beta > 4$ in the case of Wasserstein and Kolmogorov distance, respectively. We conjecture that our bounds are optimal in this case. We note also that all rates in Theorem 1.1 converge to zero.

2. Our main objective is to obtain the quantitative bounds (1.9) and (1.10). However, we obtain the limit theorem $V_n \xrightarrow{d} \mathcal{N}(0, v^2)$ as a by-product, which is new whenever L is not stable.

3. Our proof of Theorem 1.1 relies on new second-order Poincaré inequalities, which provide general bounds between Poisson functionals and Gaussian random variables in the Wasserstein and Kolmogorov distances, see Section 3. We believe that these inequalities, which are improvements of those obtained in [16], are of independent interest. Our new bounds are, in particular, important in the regime of strong heavy tails combined with strong memory, i.e. $\alpha\beta \in (2, 3)$. In this setting, and applying the estimate (4.4) on $D_z V_n$ (which we will used throughout the paper), the upper bound from [16, Theorems 1.1 and 1.2] diverges to infinity, and hence gives no information, whereas our new bounds converge to zero.

In the following we will apply Theorem 1.1 to the four important examples: linear fractional stable noises, fractional Lévy noises, stable fractional ARIMA processes, and stable Ornstein–Uhlenbeck processes. Throughout we will fix the notation used in Theorem 1.1, that is, V_n is given in (1.7) with $f \in C_b^2(\mathbb{R})$, $v_n = \sqrt{\mathrm{var}(V_n)}$, v^2 given in (1.8) satisfies $v^2 > 0$, and $Z \sim \mathcal{N}(0, 1)$ is a standard Gaussian random variable.

1.4.1 Linear fractional stable noises

Our first example concerns the linear fractional stable noise. To define this process we let L be a β-stable Lévy process with $\beta \in (0, 2)$, and

$$X_t = Y_t - Y_{t-1} \qquad \text{where} \qquad Y_t = \int_{-\infty}^t \left\{ (t-s)_+^{H-1/\beta} - (-s)_+^{H-1/\beta} \right\} dL_s, \qquad (1.11)$$

where $H \in (0, 1)$. For $\beta = 1$ we assume furthermore that L is symmetric, that is, L_1 equals $-L_1$ in distribution. The linear fractional stable motion $(Y_t)_{t \in \mathbb{R}}$ has stationary increments and is self-similar with index H, and can be viewed as a heavy-tailed extension of the fractional Brownian motion, see [37] for more details. In this setting we deduce that $\alpha = 1 - H + 1/\beta$ and the condition $\alpha\beta > 2$ translates to $\beta \in (1, 2)$, $0 < H < 1 - 1/\beta$. Since $\beta > 1$ we never have $\alpha\beta \geq 3$.

Corollary 1.3. *Let $(X_t)_{t \in \mathbb{R}}$ be the linear fractional stable noise defined as in (1.11). For $\beta \in (1, 2)$ and $0 < H < 1 - 1/\beta$ we have that*

$$d_W(v_n^{-1} V_n, Z) \leq C n^{(1+\beta(H-1))/2} \qquad \text{and} \qquad d_K(v_n^{-1} V_n, Z) \leq C n^{(1+\beta(H-1))/4},$$

where $C > 0$ is a constant not depending on n.

1.4.2 Linear fractional Lévy noise

In the following we will consider the case of a linear fractional Lévy noise, which has higher moments compared to the linear fractional stable noise. Let L be a mean zero Lévy process with a Lévy density κ satisfying $\kappa(x) \leq |x|^{-1-\zeta}$ for all $x \in [-1, 1]$, where $\zeta \in [0, 2)$, and $\kappa(x) \leq |x|^{-3}$ for all $x \in \mathbb{R}$ with $|x| > 1$. The assumptions on L are e.g. satisfied for tempered stable Lévy processes (with uniform tilting), cf. [34], and ensures that L has finite r-moments for all $r \in (0, 2)$, and that the Blumenthal–Getoor index β_{BG} of L satisfies $\beta_{BG} \leq \zeta$. Let (X_t) be given by

$$X_t = Y_t - Y_{t-1}, \qquad \text{where} \qquad Y_t = \int_{-\infty}^{t} \left\{ (t-s)_+^{-\rho} - (-s)_+^{-\rho} \right\} dL_s, \qquad (1.12)$$

where $\rho \in (0, 1/\zeta)$. We use the convention $1/0 := \infty$. The above assumptions ensures that both X and Y are well-defined stochastic processes. See [32, Section 2.6.8] or [20] for more details. The assumptions of Theorem 1.1 are satisfied for $\alpha = \rho + 1$ and all $\beta \in [\zeta, 2)$, and hence we obtain the following corollary:

Corollary 1.4. *Let (X_t) be the linear fractional Lévy noise defined in (1.12) with $\zeta \in [0, 2)$ and $\rho \in (0, 1/\zeta)$. For all $\epsilon > 0$ we have that*

$$d_W(v_n^{-1} V_n, Z) \leq C \begin{cases} n^{-1/2} & \text{if } \rho > \frac{1}{2}, \\ n^{-\rho+\epsilon} & \text{if } \rho \leq \frac{1}{2}, \end{cases}$$

and

$$d_K(v_n^{-1} V_n, Z) \leq C \begin{cases} n^{-1/2} & \text{if } \rho > 1, \\ n^{-\rho/2+\epsilon} & \text{if } \rho \leq 1. \end{cases}$$

1.4.3 Stable fractional ARIMA processes

In the following we will consider the stable fractional ARIMA process. To this end, we let $p, q \in \mathbb{N}$, and Φ_p and Θ_q be polynomials with real coefficients on the form

$$\Phi_p(z) = 1 - \phi_1 z - \cdots - \phi_p z^p, \qquad \text{and} \qquad \Theta_q(z) = 1 + \theta_1 z + \cdots + \theta_q z^q,$$

where we assume that Φ_p and Θ_q do not have common roots, and that Φ_p has no roots in the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$. The stable fractional ARIMA(p, d, q) process $(X_n)_{n \in \mathbb{N}}$ is the solution to the equation

$$\Phi_p(B) X_n = \Theta_q(B)(1 - B)^{-d} \epsilon_n \qquad (1.13)$$

where $(\epsilon_n)_{n \in \mathbb{N}}$ are independent and identically symmetric β-stable random variables with $\beta \in (0, 2)$, B denotes the backshift operator, and $d \in \mathbb{R} \setminus \mathbb{Z}$. The equation should be understood as in [12, Section 2]. For $d < 1 - 1/\beta$, there exists a unique solution $(X_n)_{n \in \mathbb{N}}$ to (1.13), and it is a discrete moving average of the form

$$X_n = \sum_{j=-\infty}^{n} b_{n-j} \epsilon_j$$

for a certain sequence $(b_j)_{j \in \mathbb{N}}$ with $b_j \sim c_0 j^{d-1}$ as $j \to \infty$, where c_0 denotes a positive constant, cf. Theorem 2.1 of [12]. Notice that the process $(X_n)_{n \in \mathbb{N}}$ can be written in distribution as

$$X_n = \int_{-\infty}^{n} g(n-s) \, dL_s$$

with $g(x) = \sum_{j \geq 0} b_j \mathbf{1}_{[j,j+1)}(x)$ and L being a symmetric β-stable Lévy process. Here $\alpha = 1 - d$ and by Theorem 1.1 we obtain the following result.

Corollary 1.5. *Let $(X_n)_{n \in \mathbb{N}}$ be the stable fractional $ARIMA(p,d,q)$ process given by (1.13), with $\beta \in (0,2)$ and $d \in \mathbb{R} \setminus \mathbb{Z}$ with $d < 1 - 2/\beta$. Then,*

$$d_W\left(v_n^{-1} V_n, Z\right) \leq C \begin{cases} n^{-1/2} & \text{if } d < 1 - 3/\beta, \\ n^{-1/2} \log(n) & \text{if } d = 1 - 3/\beta, \\ n^{1-(1-d)\beta/2} & \text{if } d \in (1 - 3/\beta, 1 - 2/\beta), \end{cases}$$

and

$$d_K\left(v_n^{-1} V_n, Z\right) \leq C \begin{cases} n^{-1/2} & \text{if } d < 1 - 4/\beta, \\ n^{-1/2} \log(n) & \text{if } d = 1 - 4/\beta, \\ n^{(1-(1-d)\beta/2)/2} & \text{if } d \in (1 - 4/\beta, 1 - 2/\beta), \end{cases}$$

where $C > 0$ is a constant not depending on n.

1.4.4 Stable Ornstein–Uhlenbeck processes

In our last example we will consider a stable Ornstein–Uhlenbeck process $(X_t)_{t \in \mathbb{R}}$, given by

$$X_t = \int_{-\infty}^{t} e^{-\lambda(t-s)} \, dL_s, \tag{1.14}$$

where L denotes a β-stable Lévy process with $\beta \in (0,2)$, and $\lambda > 0$ is a finite constant. In this case $\alpha > 0$ can be chosen arbitrarily large and we obtain the following result.

Corollary 1.6. *Let $(X_t)_{t \in \mathbb{R}}$ be a stable Ornstein–Uhlenbeck process given by (1.14). Then*

$$d_W(v_n^{-1} V_n, Z) \leq C n^{-1/2} \qquad \text{and} \qquad d_K(v_n^{-1} V_n, Z) \leq C n^{-1/2},$$

where $C > 0$ is a constant not depending on n.

1.5 Structure of the paper

The paper is structured as follows. Section 2 presents a related result and some discussions. Basic notions of Malliavin calculus on Poisson spaces and the new bounds for the Wasserstein and Kolmogorov distances are demonstrated in Section 3. In Section 4 we prove Theorem 1.1 based on the general bounds obtained in Section 3.

2 Related literature and discussion

Normal approximation of non-linear functionals of Poisson processes defined on general state spaces has become a topic of increasing interest during the last years. In particular, quantitative bounds for normal approximations were obtained by combining Malliavin calculus on the Poisson space with Stein's method. The resulting bounds have successfully been applied in various contexts such as stochastic geometry (see, e.g., [7, 13, 14, 16, 19, 36]), the theory of U-statistics (see, e.g., [7, 8, 9, 36]), non-parametric Bayesian survival analysis (see [26, 27]) or statistics of spherical point fields (see, e.g., [3, 4]). We refer the reader also to [24], which contains a representative collection of survey articles.

The first quantitative bounds for asymptotically normal functionals of Lévy moving averages have been derived in [16]. We briefly introduce their framework, but phrase their results in an equivalent way through Lévy processes instead of Poisson random measures. Let $(X_t)_{t\in\mathbb{R}}$ denote a Lévy moving average of the form (1.5) where L is centered. Assume that the Lévy measure ν satisfies the condition

$$\int_{\mathbb{R}} |y|^j \, \nu(dy) < \infty \qquad \text{for } j \in \{1, 2\}, \tag{2.1}$$

which implies that $(L_t)_{t\in\mathbb{R}}$ is of locally bounded variation with a finite second moment. Suppose that the kernel function g satisfies the condition

$$\int_{\mathbb{R}} |g(x)| + g(x)^2 \, dx < \infty. \tag{2.2}$$

The functional under consideration is defined by

$$F_T = \int_{[0,T]} f(X_t) \, dt, \qquad T > 0,$$

which can be interpreted as the continuous version of the statistic V_n. We now state Theorem 8.2 of [16] in the case of $p = 0$.

Theorem 2.1. *[16, Theorem 8.2] Suppose that conditions (2.1) and (2.2) hold. Assume that*

$$\text{var}(F_T) \geq cT, \qquad T \geq t_0$$

with $c, t_0 > 0$. Furthermore, suppose that $f \in C_b^2(\mathbb{R})$. Assume that $\int_{\mathbb{R}} |y|^4 \, \nu(dy) < \infty$ and $\mathbb{E}[|X_1|^4] < \infty$. Finally, assume that

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |g(y-x)g(y)| \, dy \right)^4 dx < \infty.$$

Let Z be a standard Gaussian random variable. Then there exists a constant $C > 0$ such that for all $T \geq t_0$,

$$d\left(\frac{F_T - \mathbb{E}[F_T]}{\sqrt{\text{var}(F_T)}}, Z \right) \leq \frac{C}{\sqrt{T}},$$

for $d = d_W$ and $d = d_K$.

While Theorem 2.1 (and its proof) relies heavily on a finite fourth moment, Theorem 1.1 works for infinite variance models, as e.g. stable processes. The heavy tails of these processes force us to introduce the improved version of the bounds in [16, Theorems 1.1 and 1.2] in the next section, and they are also responsible for slower rates of convergence in Theorem 1.1 compared to Theorem 2.1.

3　New bounds for normal approximation on Poisson spaces

The aim of this section is to introduce new bounds on the Wasserstein and the Kolmogorov distances between a Poisson functional and a standard Gaussian random variable. Although similar bounds were previously derived in [16, 25], they are not sufficient in certain settings. For this reason, we shall provide an improved version, which is adapted to our needs. Since such a bound might be useful in other contexts as well, we formulate and prove it in a general set-up, which is specialised later in this paper.

3.1　Poisson spaces and Malliavin calculus

We recall some basic notions of Malliavin calculus on Poisson spaces and refer the reader to [17, 24] for further background information. We fix an underlying probability space $(\Omega, \mathcal{A}, \mathbb{P})$, let $(\mathbb{X}, \mathcal{X})$ be a measurable space and λ be a σ-finite measure on $\mathbb{X}$ (in the applications we consider here $\mathbb{X}$ will be of the form $\mathbb{R} \times \mathbb{R}$). By η we denote a Poisson process on $\mathbb{X}$ with intensity measure λ, see [17] for a formal definition and an explicit construction of such a process. We often consider η as random element in the space of integer-valued σ-finite measures on $\mathbb{X}$, denoted by $\mathbf{N}$, which is equipped with the σ-algebra generated by all evaluations $\mu \to \mu(A)$ for $A \in \mathcal{X}$. A real-valued random variable F is called a Poisson functional if there exists a measurable function $\phi : \mathbf{N} \to \mathbb{R}$ such that $\mathbb{P}$-almost surely $F = \phi(\eta)$. We let L^2_η denote the space of all square-integrable Poisson functionals $F = \phi(\eta)$.

It is well known that each $F \in L^2_\eta$ admits a chaotic decomposition

$$F = \mathbb{E}F + \sum_{m=1}^{\infty} I_m(f_m), \tag{3.1}$$

where I_m denotes the mth order integral with respect to the compensated (signed) measure $\hat{\eta} = \eta - \lambda$ and $f_m : \mathbb{X}^m \to \mathbb{R}$ are symmetric functions with $f_m \in L^2(\lambda^m)$. Here, for a measure μ on $\mathbb{X}$ and $k \in \mathbb{N}$ we write $L^2(\mu^k)$ for the space of functions $f : \mathbb{X}^k \to \mathbb{R}$, which are square integrable with respect to the k-fold product measure of μ, and $\| \cdot \|_{L^2(\mu^k)}$ for the corresponding L^2-norm. For $z \in \mathbb{X}$ and a Poisson functional $F = \phi(\eta) \in L^2_\eta$ we denote by

$$D_z F = \phi(\eta + \delta_z) - \phi(\eta)$$

the Malliavin derivative of F in direction z, also known as the difference operator. Here, δ_z stands for the Dirac measure at $z \in \mathbb{X}$. We can consider DF as a function on $\Omega \times \mathbb{X}$ acting as $(\omega, z) \mapsto D_z F(\omega)$. If DF is square integrable with respect to the product measure $\mathbb{P} \otimes \lambda$

we shall write $DF \in L^2(\mathbb{P} \otimes \lambda)$ in what follows. Finally, let us define the second-order Malliavin derivative of F with respect to two points $z_1, z_2 \in \mathbb{X}$ by putting

$$D^2_{z_1,z_2}F := D_{z_1}(D_{z_2}F) = \phi(\eta + \delta_{z_1} + \delta_{z_2}) - \phi(\eta + \delta_{z_1}) - \phi(\delta_{z_2}) + \phi(\eta)$$

(note that this definition is symmetric in z_1 and z_2).

The Kabanov-Skorohod integral δ maps random functions u from $L^2_\eta(\mathbb{P} \otimes \lambda)$ to random variables in L^2_η. To introduce the definition of the operator δ, let

$$u(z) = \sum_{m=0}^{\infty} I_m\left(h_m(z, \cdot)\right), \qquad z \in \mathbb{X},$$

denote the chaos expansion of $u(z)$, where $h_m : \mathbb{X}^{m+1} \to \mathbb{R}$ are measurable functions. The domain $\operatorname{dom} \delta$ of δ consists of all random functions u that satisfy the condition

$$\sum_{m=0}^{\infty} (m+1)! \|\tilde{h}_m\|^2_{L^2(\lambda^{m+1})} < \infty,$$

where $\tilde{h}_m$ denotes the symmetrisation of the function h_m. For $u \in \operatorname{dom} \delta$ the Kabanov-Skorohod integral is defined by

$$\delta(u) = \sum_{m=0}^{\infty} I_{m+1}(\tilde{h}_m).$$

Finally, we introduce the Ornstein-Uhlenbeck generator L and its (pseudo) inverse L^{-1}. The domain $\operatorname{dom} L$ of L consists of all elements $F \in L^2_\eta$ with chaotic decomposition (3.1) that additionally satisfy the condition

$$\sum_{m=1}^{\infty} m^2 m! \|f_m\|^2_{L^2(\lambda^m)} < \infty.$$

For $F \in \operatorname{dom} L$ with chaotic decomposition (3.1) we define

$$LF = -\sum_{m=1}^{\infty} m I_m(f_m), \qquad L^{-1}F = -\sum_{m=1}^{\infty} \frac{1}{m} I_m(f_m).$$

The important relationships between the introduced operators can be summarised as follows:

(i) $LL^{-1}F = F$,   (if F is a centred Poisson functional)

(ii) $LF = -\delta DF$,   (if $F \in \operatorname{dom} L$)

(iii) $\mathbb{E}[F\delta(u)] = \mathbb{E} \int (D_z F)\, u(z)\, \lambda(dz)$,   (integration-by-parts)

where $u \in \operatorname{dom} \delta$. We refer to [24] or [25] for a more detailed exposition.

3.2 Wasserstein distance

In this subsection we derive quantitative bounds for the Wasserstein distance. We recall that the Wasserstein distance between two random variables F and G is defined by

$$d_W(F, Z) := \sup_{h \in \text{Lip}(1)} \left| \mathbb{E}[h(F)] - \mathbb{E}[h(G)] \right|,$$

where the supremum is running over all Lipschitz functions $h : \mathbb{R} \to \mathbb{R}$ with a Lipschitz constant less than or equal to 1. To formulate the next result we introduce the three quantities

$$\gamma_1^2 := 4 \int \mathbb{E}[(D_{z_1}F)^2(D_{z_2}F)^2]^{1/2} \, \mathbb{E}[(D_{z_1,z_3}^2 F)^2(D_{z_2,z_3}^2 F)^2]^{1/2} \, \lambda^3(d(z_1, z_2, z_3)), \quad (3.2)$$

$$\gamma_2^2 := \int \mathbb{E}[(D_{z_1,z_3}^2 F)^2(D_{z_2,z_3}^2 F)^2] \, \lambda^3(d(z_1, z_2, z_3)), \tag{3.3}$$

$$\gamma_3 := \int \mathbb{E}[|D_z F|^3]^{1/3} \, \mathbb{E}[\min(\sqrt{8}|D_z F|^{3/2}, |D_z F|^3)]^{2/3} \, \lambda(dz) \tag{3.4}$$

(although γ_1, γ_2 and γ_3 depend on the Poisson functional F, we suppress this dependency in our notation for simplicity). The theorem below is an improved version of the second-order Poincaré inequality for Poisson functionals from [16, Theorem 1.1], where the main difference stems from the term γ_3.

Theorem 3.1. *Let $F \in L_\eta^2$ be a Poisson functional, which satisfies $DF \in L^2(\mathbb{P} \otimes \lambda)$, $\mathbb{E}[F] = 0$ and $\mathbb{E}[F^2] = 1$. Further, let $Z \sim \mathcal{N}(0,1)$ be a standard Gaussian random variable. Then,*

$$d_W(F, Z) \leq \gamma_1 + \gamma_2 + \gamma_3.$$

Remark 3.2. Our bound γ_3 improves the corresponding quantity in [16, Theorem 1.1], which one can obtain by replacing the term $\mathbb{E}[\min(\sqrt{8}|D_z F|^{3/2}, |D_z F|^3)]$ in γ_3 through $\mathbb{E}[|D_z F|^3]$. It turns out that our improvement is absolutely crucial as the quantity introduced in [16, Theorem 1.1] converges to infinity in our setting.

In our framework the main problem appears when the term $|D_z V_n|$ becomes large, say, larger than 1. Such an event has a relatively high weight under the measure λ. On the contrary, in the setting of Theorem 2.1 large values of $|D_z V_n|$ have a very low probability under higher moment conditions on the Lévy measure imposed in Theorem 2.1. Hence, the main improvement of the original bound introduced in [16, Theorem 1.1] stems from carefully distinguishing between large and small values of $|D_z V_n|$. $\square$

Apart from the application of Malliavin calculus another ingredient on which the proof of Theorem 3.1 is build is Stein's method for normal approximation (we refer to [5] for a general account). The starting point is the following observation: a random variable Z has a standard Gaussian distribution $\mathcal{N}(0,1)$ if and only if

$$\mathbb{E}[f'(Z) - Zf(Z)] = 0$$

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for every continuous and piecewise continuously differentiable function f with $\mathbb{E}[|f'(Z)|] < \infty$. This characterization together with the definition of the Wasserstein distance motivates to consider for given Borel function $h : \mathbb{R} \to \mathbb{R}$ with $\mathbb{E}[|h(Z)|] < \infty$ the first-order differential equation

$$h(x) - \mathbb{E}[h(Z)] = f'(x) - xf(x), \qquad x \in \mathbb{R}, \tag{3.5}$$

also known as the Stein equation for normal approximation. A solution to (3.5) is an absolutely continuous function $f : \mathbb{R} \to \mathbb{R}$ such that there exists a version of the derivative f' satisfying (3.5) for all $x \in \mathbb{R}$. From [5] it is known that for given $h \in \mathrm{Lip}(1)$ the (unique bounded) solution $f = f_h$ of this equation satisfies $f \in C^1$ with f' absolutely continuous and

$$\|f\|_\infty \leq 2, \qquad \|f'\|_\infty \leq 1, \qquad \text{and} \qquad \|f''\|_\infty \leq 2,$$

where we write $\| \cdot \|_\infty$ for the supremum norm. Defining $\mathcal{F}_W$ to be the class of functions satisfying these two constraints, replacing x by F in the Stein equation and taking expectations on both sides we arrive at

$$d_W(F, Z) \leq \sup_{f \in \mathcal{F}_W} \big| \mathbb{E}[f'(F) - Ff(F)] \big|.$$

Starting with this estimate, we can now present the proof of Theorem 3.1.

Proof of Theorem 3.1. Let $f \in \mathcal{F}_W$ and fix $a, b \in \mathbb{R}$. Then using the bound for f' we observe that

$$|f(b) - f(a) - f'(a)(b - a)| \leq |f(b) - f(a)| + |f'(a)||b - a| \leq 2\|f'\|_\infty |b - a| \leq 2|b - a|.$$

Similarly, using the bound for f'' we have, by Taylor approximation,

$$|f(b) - f(a) - f'(a)(b - a)| \leq \frac{\|f''\|_\infty}{2}(b - a)^2 \leq (b - a)^2$$

and so

$$|f(b) - f(a) - f'(a)(b - a)| \leq \min(2|b - a|, (b - a)^2).$$

Next, using the definition of the Malliavin derivative and a Taylor expansion of f around F we see that

$$D_z f(F) = f'(F)(D_z F) + R(D_z F), \qquad z \in \mathbb{X},$$

where in view of the above considerations the remainder term $R(\cdot)$ satisfies the estimate $|R(y)| \leq \min(2|y|, y^2)$ for all $y \in \mathbb{R}$. Applying this together with the three relations for the Malliavin operators presented in the previous subsection, we see that

$$\mathbb{E}[Ff(F)] = \mathbb{E}[LL^{-1}Ff(F)] = -\mathbb{E}[\delta(DL^{-1}F)\,f(F)] = \mathbb{E}\int (D_z f(F))(-D_z L^{-1}F)\,\lambda(dz)$$

$$= \mathbb{E}\Big[f'(F)\int (D_z F)(-D_z L^{-1}F)\,\lambda(dz)\Big] + \mathbb{E}\int R(D_z F)(-D_z L^{-1}F)\,\lambda(dz).$$

As a consequence,

$$
\begin{aligned}
|\mathbb{E}[f'(F) - Ff(F)]| &\leq \left| \mathbb{E}\left[f'(F)\left(1 - \int (D_z F)(-D_z L^{-1} F)\,\lambda(dz) \right) \right] \right| \\
&\quad + \mathbb{E} \int \min(2|D_z F|, (D_z F)^2)\, |D_z L^{-1} F|\,\lambda(dz) \\
&\leq \mathbb{E}\left[\left| 1 - \int (D_z F)(-D_z L^{-1} F)\,\lambda(dz) \right| \right] \\
&\quad + \mathbb{E} \int \min(2|D_z F|, (D_z F)^2)\, |D_z L^{-1} F|\,\lambda(dz),
\end{aligned}
$$

where we used that $\|f'\|_\infty \leq 1$. Next, we apply [16, Proposition 4.1] to conclude that

$$
\mathbb{E}\left[\left| 1 - \int (D_z F)(-D_z L^{-1} F)\,\lambda(dz) \right| \right] \leq \gamma_1 + \gamma_2. \tag{3.6}
$$

For the term γ_3 we use Hölder's inequality with Hölder conjugates 3 and $3/2$. Together with [16, Lemma 3.4], which in our situation says that $\mathbb{E}[|D_z L^{-1} F|^3] \leq \mathbb{E}[|D_z F|^3]$ for all $z \in \mathbb{X}$, this leads to

$$
\begin{aligned}
&\mathbb{E} \int \min(2|D_z F|, (D_z F)^2)\, |D_z L^{-1} F|\,\lambda(dz) \\
&\leq \int (\mathbb{E}[|D_z L^{-1} F|^3])^{1/3} (\mathbb{E}[\min(2|D_z F|, (D_z F)^2)^{3/2}])^{2/3}\,\lambda(dz) \\
&\leq \int (\mathbb{E}[|D_z F|^3])^{1/3} (\mathbb{E}[\min(\sqrt{8}|D_z F|^{3/2}, |D_z F|^3)])^{2/3}\,\lambda(dz) = \gamma_3.
\end{aligned}
$$

The proof is thus complete. $\qquad\qquad\square$

3.3 Kolmogorov distance

Now we turn our attention to the Kolmogorov distance between the two random variables F and Z, which is defined as

$$
d_K(F, Z) = \sup_{x \in \mathbb{R}} |\mathbb{P}(F \leq x) - \mathbb{P}(Z \leq x)|.
$$

Let $f = f_x$ be the bounded solution of the Stein's equation associated with the function $1_{(-\infty, x]}$ for a given $x \in \mathbb{R}$. It is well known that this solution satisfies the inequalities

$$
\|f\|_\infty \leq \frac{\sqrt{2\pi}}{4} \qquad \text{and} \qquad \|f'\|_\infty \leq 1 \tag{3.7}
$$

(we interpret f' as the left-sided derivative at the point x, where f is not differentiable). Hence, with $\mathcal{F}_K$ denoting the class of all absolutely continuous functions satisfying (3.7) we have

$$
d_K(F, Z) \leq \sup_{f \in \mathcal{F}_K} |\mathbb{E}[f'(F) - Ff(F)]|
$$

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where $Z \sim \mathcal{N}(0,1)$. To obtain modified bounds for the Kolmogorov distance we introduce an arbitrary measurable function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $0 \leq \varphi \leq 1$. We may simply use $\varphi = 1_{[-1,1]}$ or a smooth function with compact support. As discussed in Remark 3.2 we will use the function φ to distinguish between large and small values of the quantity $|D_z V_n|$.

To formulate the analogue of Theorem 3.1 for the Kolmogorov distance we introduce the quantities

$$\overline{\gamma}_3 := 2 \int \mathbb{E}[(D_z F)^2 (1 - \varphi(D_z F))^2]^{1/2} \mathbb{E}[|D_z F|^2]^{1/2} \, \lambda(dz), \tag{3.8}$$

$$\gamma_4 := \left(\frac{1}{2} (\mathbb{E}F^4)^{1/4} + \frac{\sqrt{2\pi}}{8} \right) \int \mathbb{E}[(D_z F)^4 \varphi(D_z F)^2]^{1/2} \mathbb{E}[|D_z F|^4]^{1/4} \, \lambda(dz),$$

$$\gamma_5 := \sqrt{\int \mathbb{E}[\varphi(D_z F)^2 (D_z F)^4]^{1/2} \mathbb{E}[(D_z F)^4]^{1/2} \, \lambda(dz)},$$

$$\gamma_6^2 := 3 \int \left(\mathbb{E}[(D_{z_2}(\varphi(D_{z_1} F) D_{z_1} F))^4]^{1/2} \mathbb{E}[|D_{z_1} F|^4]^{1/2} \right.$$

$$+ \mathbb{E}[\varphi(D_{z_1} F)^2 (D_{z_1} F)^4]^{1/2} \mathbb{E}[|D_{z_2,z_1}^2 F|^4]^{1/2}$$

$$\left. + \mathbb{E}[(D_{z_2}(\varphi(D_{z_1} F) D_{z_1} F))^4]^{1/2} \mathbb{E}[|D_{z_2,z_1}^2 F|^4]^{1/2} \right) \lambda^2(d(z_1, z_2)) \tag{3.9}$$

(again, we suppress in our notation the dependency on the Poisson functional F).

We are now prepared to present our general estimate for the Kolmogorov distance between a Poisson functional and a standard Gaussian random variable.

Theorem 3.3. *Let* $F \in L_\eta^2$ *be a Poisson functional which satisfies* $DF \in L^2(\mathbb{P} \otimes \lambda)$, $\mathbb{E}[F] = 0$ *and* $\mathbb{E}[F^2] = 1$. *Further, let* $Z \sim \mathcal{N}(0,1)$ *be a standard Gaussian random variable. Then,*

$$d_K(F, Z) \leq \gamma_1 + \gamma_2 + \overline{\gamma}_3 + \gamma_4 + \gamma_5 + \gamma_6.$$

Remark 3.4. Theorem 3.3 improves the original bounds from [16, Theorem 1.2], which correspond to the choice $\varphi = 1$. We recall once again that the bound from [16, Theorem 1.2] converges to infinity in our setting, and thus the improvement in Theorem 3.3 is absolutely crucial for the proof of Theorem 1.1. □

Proof. As in the proof of Theorem 3.1 we need to bound the term

$$\mathbb{E}[f'(F) - F f(F)] = \mathbb{E}\left[f'(F)\left(1 - \int (D_z F)(-D_z L^{-1} F) \, \lambda(dz) \right) \right]$$

$$- \mathbb{E}\left[\int (-D_z L^{-1} F) \int_0^{D_z F} \{ f'(F + t) - f'(F) \} \, dt \, \lambda(dz) \right],$$

where the decomposition is derived in [9, Equation (3.4)]. We have already seen in (3.6) that the inequality

$$\left| \mathbb{E}\left[f'(F)\left(1 - \int (D_z F)(-D_z L^{-1} F)\, \lambda(dz) \right) \right] \right| \leq \gamma_1 + \gamma_2 \tag{3.10}$$

holds, where we have used that $\|f\|_\infty \leq 1$. To treat the second term of the decomposition we need to distinguish whether $D_z F$ takes small or large values. In particular, we have that

$$\left| \mathbb{E}\left[\int (-D_z L^{-1} F) \int_0^{D_z F} \{ f'(F+t) - f'(F) \}\, dt\, \lambda(dz) \right] \right| \tag{3.11}$$

$$\leq 2\mathbb{E} \int |D_z F|(1 - \varphi(D_z F))|D_z L^{-1} F|\, \lambda(dz)$$

$$+ \mathbb{E}\left[\int |D_z L^{-1} F| \varphi(D_z F) \left| \int_0^{D_z F} \{ f'(F+t) - f'(F) \}\, dt \right|\, \lambda(dz) \right].$$

Now, repeating the methods from the proof of [9, Theorem 3.1] (see pages 7–9 therein) for the last term in (3.11), we conclude that

$$\left| \mathbb{E}\left[\int (-D_z L^{-1} F) \int_0^{D_z F} \{ f'(F+t) - f'(F) \}\, dt\, \lambda(dz) \right] \right|$$

$$\leq 2\mathbb{E} \int |D_z F|(1 - \varphi(D_z F))|D_z L^{-1} F|\, \lambda(dz)$$

$$+ \frac{\sqrt{2\pi}}{8} \mathbb{E} \int \varphi(D_z F)(D_z F)^2 |D_z L^{-1} F|\, \lambda(dz)$$

$$+ \frac{1}{2} \mathbb{E} \int \varphi(D_z F)(D_z F)^2 |F \times D_z L^{-1} F|\, \lambda(dz)$$

$$+ \sup_{x \in \mathbb{R}} \mathbb{E} \int \varphi(D_z F)(D_z F) D_z (1_{\{F>x\}})|D_z L^{-1} F|\, \lambda(dz). \tag{3.12}$$

In the next step, we need to apply the ideas from [16] to the new bound at (3.12). We obtain that

$$2\mathbb{E} \int |D_z F|(1 - \varphi(D_z F))|D_z L^{-1} F|\, \lambda(dz)$$

$$\leq 2 \int \mathbb{E}[(D_z F)^2 (1 - \varphi(D_z F))^2]^{1/2} \mathbb{E}[|D_z L^{-1} F|^2]^{1/2} \lambda(dz)$$

$$\leq 2 \int \mathbb{E}[(D_z F)^2 (1 - \varphi(D_z F))^2]^{1/2} \mathbb{E}[|D_z F|^2]^{1/2} \lambda(dz) = \overline{\gamma}_3.$$

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Similarly, we have that

$$\frac{\sqrt{2\pi}}{8}\mathbb{E}\int \varphi(D_z F)(D_z F)^2 |D_z L^{-1}F|\,\lambda(dz)$$

$$\leq \frac{\sqrt{2\pi}}{8}\int \mathbb{E}[(D_z F)^4 \varphi(D_z F)^2]^{1/2}\mathbb{E}[|D_z F|^2]^{1/2}\,\lambda(dz) =: \gamma_4^{(1)},$$

and

$$\frac{1}{2}\mathbb{E}\int \varphi(D_z F)(D_z F)^2 |F \times D_z L^{-1}F|\,\lambda(dz)$$

$$\leq \frac{1}{2}(\mathbb{E}F^4)^{1/4}\int \mathbb{E}[(D_z F)^4 \varphi(D_z F)^2]^{1/2}\mathbb{E}[|D_z F|^4]^{1/4}\,\lambda(dz) =: \gamma_4^{(2)}.$$

Moreover, we note that

$$\gamma_4^{(1)} + \gamma_4^{(2)} \leq \gamma_4.$$

Next, we treat the last term in (3.12). We set $g(z) = \varphi(D_z F)(D_z F)|D_z L^{-1}F|$ and observe the inequality

$$\mathbb{E}\int D(1_{\{F>x\}})g(z)\,\lambda(dz) = \mathbb{E}[1_{\{F>x\}}\delta(g)] \leq \mathbb{E}[\delta^2(g)]^{1/2}$$

is valid, where we recall that $\delta(g)$ stands for the Kabanov-Skorohod integral of g. Furthermore, we have the Kabanov-Skorohod isometric formula

$$\mathbb{E}[\delta^2(g)] = \mathbb{E}\int g^2(z)\,\lambda(dz) + \mathbb{E}\int\int (D_y g(z))^2\lambda(dy)\,\lambda(dz) =: A_1 + A_2,$$

see [15, Theorem 5]. Applying the Cauchy-Schwarz inequality we deduce that

$$A_1 \leq \int \mathbb{E}[\varphi(D_z F)^2(D_z F)^4]^{1/2}\mathbb{E}[(D_z F)^4]^{1/2}\,\lambda(dz) = \gamma_5^2.$$

For the last term A_2 we conclude, using the inequality

$$|D_y(GH)| \leq |HD_y G| + |GD_y H| + |D_y H D_y G|,$$

that

$$A_2 \leq 3\int \Big((D_{z_2}(\varphi(D_{z_1}F)D_{z_1}F))^2 |D_{z_1}L^{-1}F|^2 + \varphi(D_{z_1}F)(D_{z_1}F)^2|D^2_{z_2,z_1}L^{-1}F|^2$$

$$+ (D_{z_2}(\varphi(D_{z_1}F)D_{z_1}F))^2 |D^2_{z_2,z_1}L^{-1}F|^2 \Big)\,\lambda^2(d(z_1,z_2))$$

$$\leq 3\int \Big(\mathbb{E}[(D_{z_2}(\varphi(D_{z_1}F)D_{z_1}F))^4]^{1/2}\mathbb{E}[|D_{z_1}F|^4]^{1/2}$$

$$+ \mathbb{E}[\varphi(D_{z_1}F)^2(D_{z_1}F)^4]^{1/2}\mathbb{E}[|D^2_{z_2,z_1}F|^4]^{1/2}$$

$$+ \mathbb{E}[(D_{z_2}(\varphi(D_{z_1}F)D_{z_1}F))^4]^{1/2}\mathbb{E}[|D^2_{z_2,z_1}F|^4]^{1/2} \Big)\,\lambda^2(d(z_1,z_2))$$

$$= \gamma_6^2.$$

Combining (3.10) and (3.12), we conclude the assertion of Theorem 3.3. $\qquad\square$

4   Proof of Theorem 1.1

All positive constants, which do not depend on n, are denoted by C although they may change from occasion to occasion. Furthermore, we assume without loss of generality that $K = 1$ in condition (1.6). We extend the definition of the kernel g to the whole real line by setting $g(x) = 0$ for $x \leq 0$. To apply Theorems 3.1 and 3.3 it will be useful for us to represent the process $(X_t)_{t \in \mathbb{R}}$, in (1.5), in terms of an integral with respect to a Poisson random measure. Namely, if η denotes a Poisson random measure on $\mathbb{R}^2$ constructed from L, see e.g. [38, Theorem 19.2], then η has intensity measure λ given by

$$\lambda(ds, dy) = ds\, \nu(dy),$$

where ν is the Lévy measure of L, defined in (1.3). We can re-write $X_t = \int_{-\infty}^{t} g(t - s)\, dL_s$ as

$$X_t = \int_{\mathbb{R}^2} g(t-s)x \left(\eta(ds, dx) - \chi(g(t-s)x)\, ds\, \nu(dx) \right) + \tilde{\theta} \tag{4.1}$$

where the integral is defined as in [35, p. 3236], and

$$\tilde{\theta} = \int_{\mathbb{R}} \left[g(s)\theta + \int_{\mathbb{R}} \left(\chi(xg(s)) - g(s)\chi(x) \right) \nu(dx) \right] ds. \tag{4.2}$$

The integrals in (4.1) and (4.2) exist since X_t is well-defined, cf. [33, Theorem 2.7]. By (4.1) it follows that X_t is a Poisson functional for all $t \in \mathbb{R}$, i.e. there exists a measurable mapping $\phi = \phi_t : \mathbf{N} \to \mathbb{R}$ such that $X_t = \phi(\eta)$. Throughout this section we will repeatedly use that for any measurable positive function $f : \mathbb{R}^2 \to \mathbb{R}_+$ we have that

$$\int_{\mathbb{R}^2} f(z)\, \lambda(dz) \leq C \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(s, x)\, ds \right) |x|^{-1-\beta}\, dx,$$

which follows by assumption (1.4) on ν. Here and below, C will denote a strictly positive and finite constant whose value might change from occasion to occasion.

4.1   Preliminary estimates

We let $z = (x, s) \in \mathbb{R}^2$, $z_j = (x_j, s_j) \in \mathbb{R}^2$ for $j \in \{1, 2, 3\}$, and $f \in C_b^2(\mathbb{R})$. By the mean-value theorem and (4.1), we have that

$$|D_z f(X_j)| = |f(X_j + xg(j - s)) - f(X_j)| \leq C \min(1, |xg(j - s)|), \tag{4.3}$$

since f and f' are bounded. By (4.3) we obtain that

$$|D_z V_n| \leq \frac{C}{\sqrt{n}} \sum_{t=1}^{n} \min(1, |xg(t - s)|) =: A_n(z). \tag{4.4}$$

Furthermore,

$$D_{z_1, z_2}^2 V_n = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \left\{ f\left(x_1 g(t - s_1) + x_2 g(t - s_2) + X_t \right) \right.$$

$$\left. - f\left(x_1 g(t - s_1) + X_t \right) - f\left(x_2 g(t - s_2) + X_t \right) + f(X_t) \right\}.$$

Again, by applying the mean-value theorem and using that f, f' and f'' are bounded we obtain the estimate

$$|D^2_{z_1,z_2} V_n| \leq \frac{C}{\sqrt{n}} \sum_{t=1}^{n} \min(1, |x_1 g(t - s_1)|) \min(1, |x_2 g(t - s_2)|) =: A_n(z_1, z_2). \quad (4.5)$$

Notice that $A_n(z_1, z_2) \leq C \min(A_n(z_1), A_n(z_2))$.

We define the quantity

$$\rho_k := \int_{\mathbb{R}} |g(x)g(x + k)|^{\beta/2} \, dx, \qquad k \in \mathbb{Z}. \quad (4.6)$$

We will show later that the terms γ_1 and γ_2 appearing in Theorems 3.1 and 3.3 are both bounded by $Cn^{-1/2}$ solely under the condition $\sum_{k=1}^{\infty} \rho_k < \infty$, cf. Lemma 4.1, while the other terms require the stronger assumption (1.6).

Lemma 4.1. *With ρ_k given in (4.6) we have that $\rho_k \leq Ck^{-\alpha\beta/2}$ for all $k \geq 1$. Furthermore,*

$$\int_{\mathbb{R}} \Big(\prod_{i=1}^{4} |g(t_i - s)| \Big)^{\beta/4} \, ds \leq C|t_2 - t_1|^{-\alpha\beta/4}|t_3 - t_1|^{-\alpha\beta/4}|t_4 - t_1|^{-\alpha\beta/4}, \quad (4.7)$$

for all $t_1, \ldots, t_4 \geq 1$, where we use the convention $0^{-r} := 1$ for all $r > 0$.

Proof. From the substitution $u = k^{-1}s$ we obtain

$$\rho_k \leq \int_0^1 s^{\gamma\beta/2}(k + s)^{-\alpha\beta/2} \, ds + \int_1^{\infty} s^{-\alpha\beta}(k + s)^{-\alpha\beta/2} \, ds$$

$$= k^{1+\gamma\beta/2-\alpha\beta/2} \int_0^{1/k} u^{\gamma\beta/2}(1 + u)^{-\alpha\beta/2} \, du + k^{1-\alpha\beta} \int_{1/k}^{\infty} u^{-\alpha\beta/2}(1 + u)^{-\alpha\beta/2} \, du$$

$$\leq Ck^{-\alpha\beta/2}. \quad (4.8)$$

Moreover, by the same procedure as in (4.8) and using succesive substitutions we obtain the bound (4.7). □

Lemma 4.2. *The series v^2 defined in Theorem 1.1 converges absolutely, and $v_n^2 \to v^2$ as $n \to \infty$.*

Proof. In the following we will show that

$$\sum_{j=1}^{\infty} |\mathrm{cov}(f(X_j), f(X_0))| < \infty. \quad (4.9)$$

To prove (4.9) we use the covariance identity from Theorem 5.1 in [18] to get

$$\mathrm{cov}(f(X_j), f(X_0)) = \mathbb{E}\Big[\int_0^1 \int \mathbb{E}[D_z f(X_j) \mid \mathcal{G}_u] \mathbb{E}[D_z f(X_0) \mid \mathcal{G}_u] \lambda(dz) \, du \Big] \quad (4.10)$$

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where $(\mathcal{G}_u)_{u \in [0,1]}$ are certain σ-algebras (which will not be important for us). As noticed in [18, Proof of Theorem 1.4] we may always assume that we are in the setting of [18, Theorem 1.5]. By Cauchy–Schwarz inequality and the contractive properties of conditional expectation it follows from (4.10) that

$$
\begin{aligned}
\left| \mathrm{cov}(f(X_j), f(X_0)) \right| &\leq \int_0^1 \int \mathbb{E}\left[\left| \mathbb{E}[D_z f(X_j) \mid \mathcal{G}_u] \mathbb{E}[D_z f(X_0) \mid \mathcal{G}_u] \right| \right] \lambda(dz)\, du \\
&\leq \int \mathbb{E}[|D_z f(X_j)|^2]^{1/2} \mathbb{E}[|D_z f(X_0)|^2]^{1/2} \lambda(dz) \\
&\leq C \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \min(|x^2 g(j-s)g(-s)|, 1)\, \nu(dx) \right) ds \\
&\leq C \int_{\mathbb{R}} |g(j-s)g(-s)|^{\beta/2}\, ds \\
&= C\rho_j \leq Cj^{-\alpha\beta/2}
\end{aligned}
\tag{4.11}
$$

where we have used (4.3) in the third inequality, the equality follows by the definition of ρ_j, and the last inequality follows by Lemma 4.1. Since $\alpha\beta > 2$, (4.11) implies (4.9).

By (4.9), the series v^2 converges absolutely. Moreover, the stationarity of $(X_j)_{j \in \mathbb{N}}$ implies that

$$
\begin{aligned}
v_n^2 = \mathbb{E}[V_n^2] &= n^{-1} \sum_{j,i=1}^n \mathrm{cov}(f(X_j), f(X_i)) \\
&= \mathrm{var}(f(X_0)) + 2 \sum_{j=1}^n (1 - j/n)\mathrm{cov}(f(X_0), f(X_j)) \\
&\to \mathrm{var}(f(X_0)) + 2 \sum_{j=1}^\infty \mathrm{cov}(f(X_0), f(X_j)) = v^2 \qquad \text{as } n \to \infty,
\end{aligned}
$$

where the convergence follows by Lebesgue's dominated convergence theorem together with (4.9). $\qquad \square$

4.2 Bounding the Wasserstein distance

Since $v_n^2 = \mathrm{var}(V_n) \to v^2$ as $n \to \infty$, cf. Lemma 4.2, and $v > 0$ by assumption, we note that v_n is bounded away from zero. Let $\gamma_1, \gamma_2, \gamma_3$ be defined in (3.2), (3.3) and (3.4) with $F = V_n$. Then, by Theorem 3.1 and using that v_n is bounded away from 0, we have that

$$
d_W(V_n/v_n, Z) \leq C(\gamma_1 + \gamma_2 + \gamma_3).
\tag{4.12}
$$

Using the estimates (4.4) and (4.5) we will now compute bounds for the quantities γ_1, γ_2 and γ_3 appearing in the bound for the Wasserstein distance in Theorem 3.1. Notice that the right hand sides in both estimates (4.4) and (4.5) are deterministic, so the expectations in the definitions of $\gamma_1, \gamma_2, \gamma_3$ can be omitted. We start with the term γ_1.

Lemma 4.3. *There exists a constant $C > 0$ such that $\gamma_1 \leq Cn^{-1/2}$.*

Proof. To estimate

$$\gamma_1^2 = 4 \int \mathbb{E}[(D_{z_1} V_n)^2 (D_{z_2} V_n)^2]^{1/2} \, \mathbb{E}[(D_{z_1,z_3}^2 V_n)^2 (D_{z_2,z_3}^2 V_n)^2]^{1/2} \lambda^3(d(z_1, z_2, z_3)),$$

we deduce the following inequality by (4.4) and (4.5):

$$\mathbb{E}[(D_{z_1} V_n)^2 (D_{z_2} V_n)^2]^{1/2} \mathbb{E}[(D_{z_1,z_3}^2 V_n)^2 (D_{z_2,z_3}^2 V_n)^2]^{1/2}$$

$$\leq \frac{C}{n^2} \sum_{t_1,\dots,t_4=1}^{n} \Big\{ \min(1, x_1^2 |g(t_1 - s_1) g(t_3 - s_1)|) \min(1, x_2^2 |g(t_2 - s_2) g(t_4 - s_2)|)$$

$$\times \min(1, x_3^2 |g(t_3 - s_3) g(t_4 - s_3)|) \Big\}.$$

By the substitution $w_i^2 = x_i^2 y_i$ for $i \in \{1, 2, 3\}$ we see that for any $y_1, y_2, y_3 > 0$ it holds that

$$\int_{\mathbb{R}^3} \min(1, x_1^2 y_1) \min(1, x_2^2 y_2) \min(1, x_3^2 y_3) |x_1 x_2 x_3|^{-1-\beta} \, dx_1 \, dx_2 \, dx_3 = C y_1^{\beta/2} y_2^{\beta/2} y_3^{\beta/2}.$$

Indeed, we have that

$$\int_{\mathbb{R}^3} \min(1, x_1^2) \min(1, x_2^2) \min(1, x_3^2) |x_1 x_2 x_3|^{-1-\beta} \, dx_1 \, dx_2 \, x_3$$

$$= \Big(\int_{\mathbb{R}} \min(1, x^2) |x|^{-1-\beta} \, dx \Big)^3 < \infty,$$

since $\beta \in (0, 2)$. Therefore, we deduce the estimate

$$\gamma_1^2 \leq C \int \mathbb{E}[(D_{z_1} V_n)^2 (D_{z_2} V_n)^2]^{1/2} \mathbb{E}[(D_{z_1,z_3}^2 V_n)^2 (D_{z_2,z_3}^2 V_n)^2]^{1/2} \lambda^3(dz_1, dz_2, dz_3)$$

$$\leq \frac{C}{n^2} \sum_{t_1,\dots,t_4=1}^{n} \Big\{ \int_{\mathbb{R}} |g(t_1 - s_1) g(t_3 - s_1)|^{\beta/2} \, ds_1 \int_{\mathbb{R}} |g(t_2 - s_2) g(t_4 - s_2)|^{\beta/2} \, ds_2$$

$$\times \int_{\mathbb{R}} |g(t_3 - s_3) g(t_4 - s_3)|^{\beta/2} \, ds_3 \Big\}$$

$$= \frac{C}{n^2} \sum_{t_1,\dots,t_4=1}^{n} \rho_{t_1-t_3} \rho_{t_2-t_4} \rho_{t_3-t_4} \leq \frac{C}{n} \sum_{v_1,v_2,v_3=-n}^{n} \rho_{v_1} \rho_{v_2} \rho_{v_3} \leq \frac{C}{n} \Big(\sum_{k=0}^{\infty} \rho_k \Big)^3 \qquad (4.13)$$

where the first equality follows by substitution, the next inequality follows by the change of variables $v_1 = t_1 - t_3$, $v_2 = t_2 - t_4$, $v_3 = t_3 - t_4$, and the last inequality follows from the symmetry $\rho_{-k} = \rho_k$. By Lemma 4.1, we have that $\sum_{k=0}^{\infty} \rho_k < \infty$, and hence, (4.13) completes the proof of the estimate $\gamma_1 \leq C n^{-1/2}$. $\qquad \square$

Using a similar reasoning, we can also bound the term γ_2.

Lemma 4.4. *There exists a constant $C > 0$ such that $\gamma_2 \leq Cn^{-1/2}$.*

Proof. Recall that

$$\gamma_2^2 = \int \mathbb{E}[(D_{z_1,z_3}^2 V_n)^2 (D_{z_2,z_3}^2 V_n)^2]\, \lambda^3(d(z_1, z_2, z_3)).$$

By the inequality (4.5) we immediately conclude that

$$\mathbb{E}[(D_{z_1,z_3}^2 V_n)^2 (D_{z_2,z_3}^2 V_n)^2] \leq \frac{C}{n^2} \sum_{t_1,\dots,t_4=1}^{n} \Big\{ \min(1, x_1^2 |g(t_1 - s_1)g(t_2 - s_1)|)$$

$$\times \min(1, x_2^2 |g(t_3 - s_2)g(t_4 - s_2)|) \min(1, x_3^4 |g(t_1 - s_3)g(t_2 - s_3)g(t_3 - s_3)g(t_4 - s_3)|) \Big\}.$$

As in the proof of Lemma 4.3 a substitution shows that for any $y_1, y_2, y_3 > 0$,

$$\int_{\mathbb{R}^3} \min(1, x_1^2 y_1) \min(1, x_2^4 y_2) \min(1, x_3^2 y_3) |x_1 x_2 x_3|^{-1-\beta}\, dx_1\, dx_2\, dx_3 = C y_1^{\beta/2} y_2^{\beta/4} y_3^{\beta/2}.$$

Therefore, we have the estimate

$$\gamma_2^2 = \int \mathbb{E}[(D_{z_1,z_3}^2 F)^2 (D_{z_2,z_3}^2 F)^2] \lambda^3(dz_1, dz_2, dz_3)$$

$$\leq \frac{C}{n^2} \sum_{t_1,\dots,t_4=1}^{n} \Big\{ \int_{\mathbb{R}} |g(t_1 - s_1)g(t_2 - s_1)|^{\beta/2} ds_1 \int_{\mathbb{R}} |g(t_3 - s_2)g(t_4 - s_2)|^{\beta/2} ds_2$$

$$\times \int_{\mathbb{R}} |g(t_1 - s_3)g(t_2 - s_3)g(t_3 - s_3)g(t_4 - s_3)|^{\beta/4} ds_3 \Big\}. \qquad (4.14)$$

Now, the inequality $|xy| \leq x^2 + y^2$, valid for all $x, y \in \mathbb{R}$, implies

$$\int_{\mathbb{R}} |g(t_1 - s_3)g(t_2 - s_3)g(t_3 - s_3)g(t_4 - s_3)|^{\beta/4} ds_3$$

$$\leq \int_{\mathbb{R}} |g(t_1 - s_3)g(t_3 - s_3)|^{\beta/2} ds_3 + \int_{\mathbb{R}} |g(t_2 - s_3)g(t_4 - s_3)|^{\beta/2} ds_3,$$

which by (4.14), shows that

$$\gamma_2^2 \leq \frac{C}{n} \Big(\sum_{k \geq 0} \rho_k \Big)^3, \qquad (4.15)$$

by the same arguments as in the proof of Lemma 4.3. Since $\sum_{k \geq 0} \rho_k < \infty$, cf. Lemma 4.1, the estimate $\gamma_2 \leq Cn^{-1/2}$ follows from (4.15). $\qquad \square$

The final term γ_3 in the bound for the Wasserstein distance is more subtle. It is this term, which decays slower than $n^{-1/2}$ for certain parameter regimes.

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Lemma 4.5. *There exists a constant $C > 0$ such that*

$$\gamma_3 \leq C \begin{cases} n^{-1/2} & \text{if } \alpha\beta > 3, \\ n^{-1/2}\log(n) & \text{if } \alpha\beta = 3, \\ n^{(2-\alpha\beta)/2} & \text{if } 2 < \alpha\beta < 3. \end{cases}$$

Proof. Recalling the inequality (4.4), we have that

$$\gamma_3 \leq C \int_{\mathbb{R}^2} \min\left(|A_n(x,s)|^2, |A_n(x,s)|^3\right) \lambda(dx, ds). \tag{4.16}$$

From the inequality (4.16), Lemma 4.5 follows from the result of Lemma 4.6 below. $\square$

Proof of Theorem 1.1 for the Wasserstein distance. The Wasserstein bound (1.9) is a direct consequence of Lemmas 4.3, 4.4 and 4.5 and the second-order Poincaré inequality (4.12)

$$d_W\left(V_n/v_n, Z\right) \leq C(\gamma_1 + \gamma_2 + \gamma_3) \leq C \begin{cases} n^{-1/2} & \text{if } \alpha\beta > 3, \\ n^{-1/2}\log(n) & \text{if } \alpha\beta = 3, \\ n^{(2-\alpha\beta)/2} & \text{if } 2 < \alpha\beta < 3. \end{cases}$$

$\square$

The following bound used in the proof of Lemma 4.5 is stated separately as a lemma, since we will also use it in the proof of upper bound for the Kolmogorov distance.

Lemma 4.6. *Let $p \in [0,2]$ and $q > 2$. There exists a finite constant C such that*

$$\int_{\mathbb{R}^2} \min\left(|A_n(z)|^p, |A_n(z)|^q\right) \lambda(dz) \leq C \begin{cases} n^{1-q/2} & \text{if } \alpha\beta > q, \\ n^{1-q/2}\log(n) & \text{if } \alpha\beta = q, \\ n^{(2-\alpha\beta)/2} & \text{if } 2 < \alpha\beta < q. \end{cases}$$

Proof of Lemma 4.6. To obtain the upper bound for the right hand side we need to decompose the integral into different parts according to whether $|x| \in (0,1)$, $|x| \in [1, n^\alpha]$ or $|x| \in (n^\alpha, \infty)$. Using the symmetry in x this means that

$$\int_{\mathbb{R}^2} \min\left(|A_n(x,s)|^p, |A_n(x,s)|^q\right) \lambda(ds, dx)$$

$$= 2 \left(\int_0^1 \int_{\mathbb{R}} \min\left(|A_n(x,s)|^p, |A_n(x,s)|^q\right) \lambda(ds, dx) \right.$$

$$+ \int_1^{n^\alpha} \int_{\mathbb{R}} \min\left(|A_n(x,s)|^p, |A_n(x,s)|^q\right) \lambda(ds, dx)$$

$$\left. + \int_{n^\alpha}^{\infty} \int_{\mathbb{R}} \min\left(|A_n(x,s)|^p, |A_n(x,s)|^q\right) \lambda(ds, dx) \right) =: I_1 + I_2 + I_3.$$

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We start by bounding the term I_1. For $x > 0$ and $s \in [0, n] \setminus \mathbb{N}$ we have that

$$\sum_{t=1}^{n} \min(1, |xg(t-s)|) \leq \min(1, |xg(1+[s]-s)|) + \sum_{t=[s]+2}^{n} |xg(t-s)|$$

$$\leq \min(1, x(1+[s]-s)^\gamma) + x \sum_{t=[s]+2}^{n} (t-s)^{-\alpha}$$

$$\leq \min(1, x(1+[s]-s)^\gamma) + Cx =: f_1(s, x) + f_2(s, x),$$

where we used $g(u) = 0$ for all $u < 0$ in the first inequality and assumption (1.6) on g in the second inequality. The third inequality follows from the fact that $\alpha > 1$, which is implied by the assumptions $\alpha > 2/\beta$ and $\beta < 2$. For $\gamma < 0$ we have

$$\int_0^n |f_1(s, x)|^q \, ds = n \int_0^1 |\min(1, xs^\gamma)|^q \, ds$$

$$= n \left(x^q \int_{x^{-1/\gamma}}^1 s^{q\gamma} \, ds + \int_0^{x^{-1/\gamma}} 1 \, ds \right) \leq Cnx^{-1/\gamma},$$

where the first equality follows by substitution. For $\gamma \geq 0$, we have the simple estimate $\int_0^n |f_1(s, x)|^q ds \leq Cnx^q$. Similarly, we have that $\int_0^n |f_2(s, x)|^q \, ds \leq Cnx^q$. By combining the above estimates we obtain that

$$\int_0^1 x^{-1-\beta} \left(\int_0^n |A_n(x, s)|^q \, ds \right) dx$$

$$\leq Cn^{1-q/2} \int_0^1 x^{-1-\beta} (x^{-1/\gamma} \mathbf{1}_{\{\gamma < 0\}} + x^q) \, dx \leq Cn^{1-q/2}, \tag{4.17}$$

where the last inequality follows from the assumption $\gamma > -1/\beta$.

For $s \in (-\infty, 0)$ we use the assumption (1.6) on g to obtain

$$\sum_{t=1}^{n} \min(1, |xg(t-s)|) \leq Cx \sum_{t=1}^{n} (t-s)^{-\alpha} \leq Cx \left((1-s)^{1-\alpha} - (n-s)^{1-\alpha} \right).$$

For $\alpha > 1 + 1/q$ we have that

$$\int_{-\infty}^0 |(1-s)^{1-\alpha} - (n-s)^{1-\alpha}|^q \, ds \leq \int_{-\infty}^0 (1-s)^{q(1-\alpha)} \, ds \leq C,$$

and for $\alpha < 1 + 1/q$ we have that

$$\int_{-\infty}^0 |(1-s)^{1-\alpha} - (n-s)^{1-\alpha}|^q \, ds \leq \int_1^\infty |u^{1-\alpha} - (u+n)^{1-\alpha}|^q \, du$$

$$= n^{q(1-\alpha)+1} \int_{n^{-1}}^\infty |v^{1-\alpha} - (v+1)^{1-\alpha}|^q \, dv$$

$$\leq Cn^{q(1-\alpha)+1} \left(\int_{n^{-1}}^1 v^{q(1-\alpha)} \, dv + \int_1^\infty v^{-q\alpha} \, dv \right) \leq Cn^{q(1-\alpha)+1}, \tag{4.18}$$

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where we used $1 < \alpha < 1 + 1/q$ in the last inequality. The above estimates imply for $\alpha \neq 1 + 1/q$ that

$$\int_0^1 x^{-1-\beta} \left(\int_{-\infty}^0 |A_n(x,s)|^q \, ds \right) dx \tag{4.19}$$

$$\leq Cn^{-q/2} \int_0^1 x^{q-1-\beta} \, dx \int_{-\infty}^0 |(1-s)^{1-\alpha} - (n-s)^{1-\alpha}|^q \, ds$$

$$\leq C(n^{-q/2} + n^{1-q\alpha+q/2}) \leq Cn^{1-q/2}, \tag{4.20}$$

where the last inequality follows since $\alpha > 1$. For $\alpha = 1+1/q$, assumption (1.6) is satisfied for $\tilde{\alpha} = \alpha - \epsilon$ for all $\epsilon > 0$ small enough. Hence, by (4.18) used with $\tilde{\alpha}$ we obtain that (4.19) is bounded by $Cn^{1-q/2}$ by choosing ϵ small enough.

The assumption that $g(x) = 0$ for all $x < 0$, implies that $A_n(x,s) = 0$ for all $s > n$, and hence (4.17) and (4.20) show that

$$I_1 \leq Cn^{1-q/2}. \tag{4.21}$$

Next, we treat the term I_3. For the integral

$$\int_{n^\alpha}^\infty \int_{-\infty}^{-n} \min\left(|A_n(x,s)|^p, |A_n(x,s)|^q\right) \lambda(ds, dx)$$

we need to distinguish different cases, namely $s \leq -x^{1/\alpha}$, $-x^{1/\alpha} < s \leq n - x^{1/\alpha} - 1$ and $n - x^{1/\alpha} - 1 < s \leq n$.

We start with the case $s \leq -x^{1/\alpha}$. Note that

$$|A_n(x,s)| \leq xn^{-1/2} \sum_{t=1}^n |g(t-s)| \leq Cx\sqrt{n}(-s)^{-\alpha} \tag{4.22}$$

and observe that $x\sqrt{n}(-s)^{-\alpha} > 1$ if and only if $s > -x^{1/\alpha}n^{1/(2\alpha)}$. We obtain the inequality

$$\int_{-x^{1/\alpha}n^{1/(2\alpha)}}^{-x^{1/\alpha}} (-s)^{-\alpha p} \, ds \leq Cx^{(1-\alpha p)/\alpha} \left(1 + n^{(1-\alpha p)/(2\alpha)}\right).$$

On the other hand, we have that

$$\int_{-\infty}^{-x^{1/\alpha}n^{1/(2\alpha)}} (-s)^{-\alpha q} \, ds \leq Cx^{(1-\alpha q)/\alpha}n^{(1-\alpha q)/(2\alpha)}.$$

By (4.22) we thus conclude that

$$\int_{n^\alpha}^\infty \int_{-\infty}^{-x^{1/\alpha}} \min\left(|A_n(x,s)|^p, |A_n(x,s)|^q\right) \lambda(ds, dx)$$

$$\leq C \int_{n^\alpha}^\infty x^{-1-\beta} \left(x^p n^{p/2} \int_{-x^{1/\alpha}n^{1/(2\alpha)}}^{-x^{1/\alpha}} (-s)^{-\alpha p} \, ds + x^q n^{q/2} \int_{-\infty}^{-x^{1/\alpha}n^{1/(2\alpha)}} (-s)^{-\alpha q} \, ds \right) dx$$

$$\leq C \left(n^{1-\alpha\beta+1/(2\alpha)} + n^{1-\alpha\beta+p/2} \right).$$

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For $x > 1$ and $-x^{1/\alpha} < s \leq n - x^{1/\alpha} - 1$ we obtain

$$\sum_{t=1}^{n} \min(1, |xg(t-s)|) = \sum_{t=1}^{[s+x^{1/\alpha}]} 1 + \sum_{t=[s+x^{1/\alpha}]+1}^{n} x(t-s)^{-\alpha}$$

$$\leq C\left((s + x^{1/\alpha}) + x\left((x^{1/\alpha})^{1-\alpha} - (n-s)^{1-\alpha}\right)\right). \tag{4.23}$$

The substitution $v = x^{-1/\alpha}(n-s)$ yields

$$\int_{-x^{1/\alpha}}^{n-x^{1/\alpha}-1} \left| x\left((x^{1/\alpha})^{1-\alpha} - (n-s)^{1-\alpha}\right) \right|^p ds$$

$$= x^{(p+1)/\alpha} \int_{1+x^{-1/\alpha}}^{1+nx^{-1/\alpha}} |1 - v^{1-\alpha}|^2 \, dv \leq Cnx^{p/\alpha}, \tag{4.24}$$

and

$$\int_{-x^{1/\alpha}}^{n-x^{1/\alpha}-1} |s + x^{1/\alpha}|^p \, ds \leq \int_0^n u^p \, du = \frac{1}{p+1} n^{p+1}. \tag{4.25}$$

From (4.23), (4.24) and (4.25) we deduce that

$$\int_{n^\alpha}^{\infty} x^{-1-\beta} \left(\int_{-x^{1/\alpha}}^{n-x^{1/\alpha}-1} |A_n(x,s)|^p \, ds \right) dx$$

$$\leq Cn^{-p/2}\left(n^{p+1} \int_{n^\alpha}^{\infty} x^{-1-\beta} \, dx + n \int_{n^\alpha}^{\infty} x^{-1-\beta+p/\alpha} \, dx \right)$$

$$\leq Cn^{1-\alpha\beta+p/2},$$

where we used the assumption $\alpha\beta > 2$ in the second inequality.

Finally, for the last case $n - x^{1/\alpha} - 1 < s \leq n$ we have

$$\sum_{t=1}^{n} \min(1, |xg(t-s)|) \leq n,$$

which leads to

$$\int_{n^\alpha}^{\infty} x^{-1-\beta} \left(\int_{n-x^{1/\alpha}-1}^{n} |A_n(x,s)|^p \, ds \right) dx \leq Cn^{p/2} \int_{n^\alpha}^{\infty} x^{-1-\beta+1/\alpha} \, dx \leq Cn^{1-\alpha\beta+p/2}.$$

Summarizing, we arrive at the bound

$$I_3 \leq C\left(n^{1-\alpha\beta+1/(2\alpha)} + n^{1-\alpha\beta+p/2} \right). \tag{4.26}$$

Next, we will bound the term I_2 as follows:

$$I_2 \leq C\left\{ \int_1^{n^\alpha} x^{-1-\beta} \left(\int_{-n}^{n} \min\left(|A_n(x,s)|^p, |A_n(x,s)|^q\right) ds \right) dx \right.$$

$$\left. + \int_1^{n^\alpha} x^{-1-\beta} \left(\int_{-\infty}^{-n} \min\left(|A_n(x,s)|^p, |A_n(x,s)|^q\right) ds \right) dx \right\} = J_1 + J_2,$$

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where we recall that $A_n(x,s) = 0$ for $s > n$. To estimate J_1 we have for $s \in \mathbb{R}$ and $x > 1$ that

$$\sum_{t=1}^{n} \min(1, |xg(t-s)|) \leq \sum_{t=[s]+1}^{[s+x^{1/\alpha}]} 1 + \sum_{t=[s+x^{1/\alpha}]+1}^{n} x(t-s)^{-\alpha}$$

$$\leq C \begin{cases} x^{1/\alpha} + x\left((x^{1/\alpha})^{1-\alpha} - (n-s)^{1-\alpha}\right) & \text{if } s + x^{1/\alpha} \leq n, \\ n - s & \text{if } s + x^{1/\alpha} > n, \end{cases}$$

$$\leq Cx^{1/\alpha}. \tag{4.27}$$

We note that $x^{1/\alpha}n^{-1/2} \leq 1$ if and only if $x \leq n^{\alpha/2}$, and write J_1 as $J_1 = J_1' + J_1''$. Note that

$$J_1' := \int_1^{n^{\alpha/2}} x^{-1-\beta}\left(\int_{-n}^{n} \min\left(|A_n(x,s)|^p, |A_n(x,s)|^q\right) ds\right) dx$$

$$\leq C \int_1^{n^{\alpha/2}} \left(\int_{-n}^{n} |x^{1/\alpha}n^{-1/2}|^q ds\right) x^{-1-\beta} dx \leq n^{1-q/2} \int_1^{n^{\alpha/2}} x^{-1-\beta+q/\alpha} dx$$

$$\leq C \begin{cases} n^{1-q/2} & \text{if } \alpha\beta > q \\ n^{1-q/2}\log(n) & \text{if } \alpha\beta = q \\ n^{(2-\alpha\beta)/2} & \text{if } 2 < \alpha\beta < q, \end{cases} \tag{4.28}$$

where we have used (4.27) in the first inequality. Furthermore,

$$J_1'' := \int_{n^{\alpha/2}}^{n^{\alpha}} x^{-1-\beta}\left(\int_{-n}^{n} \min\left(|A_n(x,s)|^p, |A_n(x,s)|^q\right) ds\right) dx$$

$$\leq C \int_{n^{\alpha/2}}^{n^{\alpha}} \left(\int_{-n}^{n} |x^{1/\alpha}n^{-1/2}|^p ds\right) x^{-1-\beta} dx \leq Cn^{1-p/2} \int_{n^{\alpha/2}}^{n^{\alpha}} x^{-1-\beta+p/\alpha} dx$$

$$= Cn^{1-\alpha\beta+p/2} \int_{n^{-1/2}}^{1} v^{p-1-\alpha\beta} dv \leq Cn^{(2-\alpha\beta)/2}, \tag{4.29}$$

where we have applied (4.27) in the first inequality, and the substitution $v = n^{-1}x^{1/\alpha}$ in the second equality.

To estimate J_2, again we need to distinguish several cases. We recall the inequality (4.22) and the statement below it, and notice that $-x^{1/\alpha}n^{1/(2\alpha)} > -n$ if and only if $x < n^{\alpha-1/2}$. We obtain the estimate

$$\int_1^{n^{\alpha-1/2}} x^{-1-\beta}\left(\int_{-\infty}^{-n} \min\left(|A_n(x,s)|^p, |A_n(x,s)|^q\right) ds\right) dx$$

$$\leq Cn^{1-q\alpha+q/2} \int_1^{n^{\alpha-1/2}} x^{-1-\beta+q} dx \leq Cn^{1-\alpha\beta+\beta/2}.$$

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Recalling again the inequality (4.22) we deduce that

$$\int_{n^{\alpha-1/2}}^{n^\alpha} x^{-1-\beta} \left(\int_{-\infty}^{-x^{1/\alpha} n^{1/(2\alpha)}} \min\left(|A_n(x,s)|^p, |A_n(x,s)|^q\right) ds \right) dx$$

$$\leq C n^{1/(2\alpha)} \int_{n^{\alpha-1/2}}^{n^\alpha} x^{-1-\beta+1/\alpha} dx \leq C n^{1-\alpha\beta+\beta/2}.$$

Finally, we also get for $p \neq \beta$ that

$$\int_{n^{\alpha-1/2}}^{n^\alpha} x^{-1-\beta} \left(\int_{-x^{1/\alpha} n^{1/(2\alpha)}}^{-n} \min\left(|A_n(x,s)|^p, |A_n(x,s)|^q\right) ds \right) dx \qquad (4.30)$$

$$\leq C n^{1-\alpha p+p/2} \int_{n^{\alpha-1/2}}^{n^\alpha} x^{-1-\beta+p} dx \leq C \left(n^{1-\alpha\beta+\beta/2} + n^{1-\alpha\beta+p/2} \right).$$

Next, we summarise our findings. Since $\alpha > 1$ we have that $1 - \alpha\beta + \beta/2 < (2 - \alpha\beta)/2$. On the other hand, $\alpha\beta > 2 \geq p$ implies the inequality $1 - \alpha\beta + p/2 < (2 - \alpha\beta)/2$ (when $p = \beta$ an additional $\log n$ factor appears in (4.30), but both rates are still dominated by $n^{(2-\alpha\beta)/2}$). Thus, we conclude from (4.28) and (4.29) that

$$I_2 \leq C \begin{cases} n^{1-q/2} & \text{if } \alpha\beta > q, \\ n^{1-q/2} \log(n) & \text{if } \alpha\beta = q, \\ n^{(2-\alpha\beta)/2} & \text{if } 2 < \alpha\beta < q. \end{cases}$$

Due to (4.21) and (4.26) we obtain the desired assertion since $1 - \alpha\beta + 1/(2\alpha) < (2 - \alpha\beta)/2$. $\qquad\square$

4.3   Bounding the Kolmogorov distance

We let $\gamma_1, \gamma_2, \overline{\gamma}_3, \gamma_4, \gamma_5$ and γ_6 be as defined in (3.2), (3.3) and (3.8)–(3.9) with $F = V_n$. By our second-order Poincaré inequality Theorem 3.3, and using the fact that v_n is bounded away from zero we have that

$$d_K (V_n/v_n, Z) \leq C(\gamma_1 + \gamma_2 + \overline{\gamma}_3 + \gamma_4 + \gamma_5 + \gamma_6). \qquad (4.31)$$

In this subsection we will bound the terms $\overline{\gamma}_3$, γ_4, γ_5 and γ_6 to obtain the bound for the Kolmogorov distance in Theorem 1.1. Throughout the proof we consider a continuously differentiable function $\varphi : \mathbb{R} \to \mathbb{R}$ with bounded derivative, whose support $\text{supp}(\varphi)$ is contained in the interval $[-2, 2]$, which satisfies $\varphi(x) = 1$ for $x \in [-1, 1]$ and is such that $\|\varphi\|_\infty = 1$. In particular, this ensures that $\|\varphi'\|_\infty < \infty$. We start with the term $\overline{\gamma}_3$, which we handle as γ_3 in Lemma 4.5.

Lemma 4.7. *There exists a constant $C > 0$ such that*

$$\overline{\gamma}_3 \leq C \begin{cases} n^{-1} & \text{if } \alpha\beta > 4 \\ n^{-1} \log(n) & \text{if } \alpha\beta = 4 \\ n^{(2-\alpha\beta)/2} & \text{if } 2 < \alpha\beta < 4. \end{cases} \qquad (4.32)$$

Proof. Applying the inequality (4.4) we conclude that

$$\overline{\gamma}_3 \leq C \int_{\mathbb{R}^2} \mathbb{E}[(D_z V_n)^2 \mathbf{1}_{\{|D_z V_n|>1\}}]^{1/2} \mathbb{E}[(D_z V_n)^2]^{1/2} \, \lambda(dz)$$

$$\leq C \int_{\mathbb{R}^2} A_n(x,s)^2 \mathbf{1}_{\{|A_n(x,s)|>1\}} \, \lambda(dx,ds)$$

$$\leq C \int_{\mathbb{R}^2} \min\left(A_n(x,s)^2, A_n(x,s)^4\right) \lambda(dx,ds),$$

which together with Lemma 4.6 implies (4.32). $\qquad\square$

Lemma 4.8. *There exists a finite constant C such that*

$$\int A_n(z)^2 \, \lambda(dz) \leq C, \tag{4.33}$$

and

$$\int A_n(z)^4 \, \lambda(dz) \leq C \begin{cases} n^{-1} & \text{if } \alpha\beta > 4, \\ n^{-1}(\log(n))^3 & \text{if } \alpha\beta = 4, \\ n^{2-\frac{3}{4}\alpha\beta} & \text{if } 2 < \alpha\beta < 4. \end{cases} \tag{4.34}$$

Proof of Lemma 4.8. To show (4.33) we proceed as follows:

$$\int A_n(z)^2 \, \lambda(dz)$$

$$= \frac{1}{n} \sum_{t_1,t_2=1}^{n} \int \left(\int \min(1, |xg(t_1-s)|) \min(1, |xg(t_2-s)|)|x|^{-1-\beta} \, dx \right) ds$$

$$\leq \frac{1}{n} \sum_{t_1,t_2,=1}^{n} \int \left(\int \min(1, |x^2 g(t_1-s)g(t_2-s)|)|x|^{-1-\beta} \, dx \right) ds$$

$$\leq \frac{C}{n} \sum_{t_1,t_2,=1}^{n} \int |g(t_1-s)g(t_2-s)|^{\beta/2} \, ds = \frac{C}{n} \sum_{t_1,t_2,=1}^{n} \rho_{t_1-t_2}$$

$$\leq C \sum_{t=0}^{n} \rho_t \leq C \sum_{t=1}^{n} t^{-\alpha\beta/2} \leq C,$$

where the second inequality follows from the substitution $u = x^2 g(t_1-s)g(t_2-s)$, and the last inequality is a consequence of the assumption that $\alpha\beta > 2$.

Our proof of (4.34) relies on the estimate

$$\int A_n(z)^4 \, \lambda(dz) \leq C \frac{1}{n^2} \sum_{t_1,\ldots,t_4=1}^{n} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \prod_{i=1}^{4} \min\{1, |xg(t_i-s)|\} \, |x|^{-1-\beta} \, dx \right) ds. \tag{4.35}$$

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Moreover, we have

$$\int_{\mathbb{R}} \prod_{i=1}^{4} \min\{1, |xg(t_i - s)|\} |x|^{-1-\beta} \, dx \leq \int_{\mathbb{R}} \min\left\{1, x^4 \prod_{i=1}^{4} |g(t_i - s)|\right\} |x|^{-1-\beta} \, dx$$

$$= \frac{1}{4}\left(\prod_{i=1}^{4} |g(t_i - s)|\right)^{\beta/4} \int_{\mathbb{R}} \min\{1, |u|\} |u|^{-1-\beta/4} \, du \leq C\left(\prod_{i=1}^{4} |g(t_i - s)|\right)^{\beta/4}, \quad (4.36)$$

where the equality follows by the substitution $u = x^4 \prod_{i=1}^{4} |g(t_i - s)|$. From the two estimates (4.35) and (4.36) we obtain

$$\int A_n(z)^4 \, \lambda(dz) \leq C\frac{1}{n^2} \sum_{t_1,\dots,t_4=1}^{n} \int_{\mathbb{R}} \left(\prod_{i=1}^{4} |g(t_i - s)|\right)^{\beta/4} ds$$

$$\leq \frac{C}{n}\left(\sum_{r=1}^{n} r^{-\alpha\beta/4}\right)^3 \leq C \begin{cases} n^{-1} & \text{if } \alpha\beta > 4, \\ n^{-1}(\log(n))^3 & \text{if } \alpha\beta = 4, \\ n^{2-\frac{3}{4}\alpha\beta} & \text{if } 2 < \alpha\beta < 4, \end{cases}$$

where the second inequality follows by (4.7). This completes the proof of the lemma. $\square$

Lemma 4.9. *There exists a constant C such that*

$$\gamma_4 \leq C \begin{cases} n^{-1} & \text{if } \alpha\beta > 4, \\ n^{-1}\log(n) & \text{if } \alpha\beta = 4, \\ n^{(2-\alpha\beta)/2} & \text{if } 8/3 \leq \alpha\beta < 4, \\ n^{3-\frac{5}{4}\alpha\beta} & \text{if } 2 < \alpha\beta < 8/3. \end{cases} \quad (4.37)$$

Proof. By our choice of the function φ we have that

$$\gamma_4 \leq C\left((\mathbb{E}V_n^4)^{1/4} + 1\right) \int \mathbb{E}[(D_z V_n)^4 \mathbf{1}_{\{|D_z V_n| \leq 1\}}]^{1/2} \mathbb{E}[|D_z V_n|^4]^{1/4} \, \lambda(dz).$$

The inequality $x^4 \mathbf{1}_{\{|x| \leq 1\}} \leq \min(x^4, x^2)$ implies that

$$\int \mathbb{E}[|D_z V_n|^4 \mathbf{1}_{\{|D_z V_n| \leq 1\}}]^{1/2} \mathbb{E}[|D_z V_n|^4]^{1/4} \, \lambda(dz)$$

$$\leq \int \mathbb{E}[\min(|D_z V_n|^2, |D_z V_n|^4)]^{1/2} \mathbb{E}[|D_z V_n|^4]^{1/4} \, \lambda(dz)$$

$$\leq \int \min(A_n(z)^2, A_n(z)^4) \, \lambda(dz) \leq C \begin{cases} n^{-1} & \text{if } \alpha\beta > 4, \\ n^{-1}\log(n) & \text{if } \alpha\beta = 4, \\ n^{(2-\alpha\beta)/2} & \text{if } 2 < \alpha\beta < 4, \end{cases} \quad (4.38)$$

where the last inequality follows by Lemma 4.6.

Lemma 4.2 of [16] shows that

$$\mathbb{E}[V_n^4] \leq C \max\left\{\int (\mathbb{E}[(D_z V_n)^4])^{1/2} \lambda(dz), \int \mathbb{E}[(D_z V_n)^4] \, \lambda(dz), 1\right\}. \quad (4.39)$$

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Hence, a combination of (4.39), the inequality $|D_z V_n| \leq A_n(z)$, cf. (4.4), and (4.33)–(4.34) of Lemma 4.8 implies that

$$(\mathbb{E}[V_n^4])^{1/4} \leq C \begin{cases} 1 & \text{if } \alpha\beta \geq 8/3, \\ n^{(2-\frac{3}{4}\alpha\beta)/4} & \text{if } 2 < \alpha\beta < 8/3. \end{cases} \qquad (4.40)$$

The two inequalities (4.38) and (4.40) yield the bound (4.37), which completes the proof of the lemma. □

In the next step we treat the term γ_5.

Lemma 4.10. *There exists a constant $C > 0$ such that*

$$\gamma_5 \leq C \begin{cases} n^{-1/2} & \text{if } \alpha\beta > 4 \\ n^{-1/2} \log(n)^{1/2} & \text{if } \alpha\beta = 4 \\ n^{(2-\alpha\beta)/4} & \text{if } 2 < \alpha\beta < 4 \end{cases}$$

Proof. We use the inequality $x^4 \mathbf{1}_{\{|x|\leq 1\}} \leq \min(x^4, 1)$ to obtain the upper bound

$$\gamma_5^2 \leq \int \mathbb{E}[(D_z V_n)^4 \mathbf{1}_{\{|D_z V_n|\leq 1\}}]^{1/2} \mathbb{E}[(D_z V_n)^4]^{1/2} \, \lambda(dz)$$

$$\leq \int \min\left(A_n(z)^4, A_n(z)^2\right) \lambda(dz).$$

Lemma 4.6 completes the proof. □

Finally, we derive an upper bound for the term γ_6.

Lemma 4.11. *There exists a constant $C > 0$ such that*

$$\gamma_6 \leq C \begin{cases} n^{-1/2} & \text{if } \alpha\beta > 4, \\ n^{-1/2} \log(n) & \text{if } \alpha\beta = 4, \\ n^{(2-\alpha\beta)/4} & \text{if } \frac{8}{3} \leq \alpha\beta < 4, \\ n^{(3-\frac{5}{4}\alpha\beta)/2} & \text{if } 2 < \alpha\beta < \frac{8}{3}. \end{cases} \qquad (4.41)$$

Proof. Since the quantity γ_6^2 consists of three terms, we use the decomposition

$$\gamma_6^2 =: 3\left(\gamma_{6.1}^2 + \gamma_{6.2}^2 + \gamma_{6.3}^2\right),$$

with the terms $\gamma_{6.1}$, $\gamma_{6.2}$ and $\gamma_{6.3}$ given by

$$\gamma_{6.1}^2 := \int \mathbb{E}[(D_{z_2}(\varphi(D_{z_1} V_n) D_{z_1} V_n))^4]^{1/2} \mathbb{E}[|D_{z_1} V_n|^4]^{1/2} \, \lambda^2(d(z_1, z_2)),$$

$$\gamma_{6.2}^2 := \int \mathbb{E}[\varphi(D_{z_1} V_n)^2 (D_{z_1} V_n)^4]^{1/2} \mathbb{E}[|D_{z_2, z_1}^2 V_n|^4]^{1/2} \, \lambda^2(d(z_1, z_2)),$$

$$\gamma_{6.3}^2 := \int \mathbb{E}[(D_{z_2}(\varphi(D_{z_1} V_n) D_{z_1} V_n))^4]^{1/2} \mathbb{E}[|D_{z_2, z_1}^2 V_n|^4]^{1/2} \, \lambda^2(d(z_1, z_2)).$$

We first prove the inequality

$$M := \int \mathbb{E}[(D_{z_1} V_n)^4]^{1/2} \mathbb{E}[(D^2_{z_2,z_1} V_n)^4]^{1/2} \lambda^2(d(z_1, z_2))$$

$$\leq C \begin{cases} n^{-1} & \text{if } \alpha\beta > 4, \\ n^{-1}(\log(n))^2 & \text{if } \alpha\beta = 4, \\ n^{(2-\alpha\beta)/2} & \text{if } 2 < \alpha\beta < 4. \end{cases} \qquad (4.42)$$

Indeed, by applying the estimates (4.4) and (4.5), we conclude that

$$M \leq \int A_n(z_1)^2 A_n(z_1, z_2)^2 \lambda^2(d(z_1, z_2)).$$

Following the same arguments as in the proofs of Lemma 4.3 and Lemma 4.4, we deduce the inequality

$$M \leq \frac{C}{n^2} \sum_{t_1,\ldots,t_4=1}^{n} \left[\int \left(\int \min\left(1, x^4 \prod_{i=1}^{4} |g(t_i - s)|\right) |x|^{-1-\beta} \, dx \right) ds \right.$$

$$\times \int \left(\int \min\left(1, x^2 |g(t_3 - s)g(t_4 - s)|\right) |x|^{-1-\beta} \, dx \right) ds \Bigg]$$

$$\leq \frac{C}{n^2} \sum_{t_1,\ldots,t_4=1}^{n} \left[\int_{\mathbb{R}} |g(t_1 - s)g(t_2 - s)g(t_3 - s)g(t_4 - s)|^{\beta/4} \, ds \right.$$

$$\times \int_{\mathbb{R}} |g(t_3 - s)g(t_4 - s)|^{\beta/2} \, ds \Bigg].$$

Hence, by Lemma 4.1 we have that

$$M \leq \frac{C}{n} \left(\sum_{r=1}^{n} r^{-\alpha\beta/4} \right)^2 \left(\sum_{r=1}^{n} r^{-\frac{3}{4}\alpha\beta} \right) \leq C \begin{cases} n^{-1} & \text{if } \alpha\beta > 4, \\ n^{-1}(\log(n))^2 & \text{if } \alpha\beta = 4, \\ n^{1-\alpha\beta/2} & \text{if } 2 < \alpha\beta < 4, \end{cases}$$

which shows (4.42).

Now, we start deriving the bounds for the terms $\gamma^2_{6.1}$, $\gamma^2_{6.2}$ and $\gamma^2_{6.3}$. First, we consider the quantity $\gamma^2_{6.1}$. We have that

$$|D_{z_2}(\varphi(D_{z_1} V_n))| = \left|\varphi(D_{z_1} V_n + D^2_{z_1,z_2} V_n) - \varphi(D_{z_1} V_n)\right|$$

$$\leq \mathbf{1}_{\{|D_{z_1} V_n| \leq 3\}} C |D^2_{z_1,z_2} V_n| + \mathbf{1}_{\{|D_{z_1} V_n| > 3\}} C \mathbf{1}_{\{|D^2_{z_1,z_2} V_n| > 1\}}, \qquad (4.43)$$

where the inequality follows by using the mean-value theorem and the fact that φ' is bounded for the first term, and for the second term we use that φ is bounded and has support in $[-2, 2]$.

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By (4.43) we obtain the decomposition

$$
\gamma_{6.1}^2 \leq C\Big(\int \mathbb{E}[(\mathbf{1}_{\{|D_{z_1}V_n|\leq 3\}}|D_{z_1,z_2}^2 V_n|D_{z_1}V_n)^4]^{1/2}\mathbb{E}[|D_{z_1}V_n|^4]^{1/2}\, \lambda^2(d(z_1,z_2))
$$

$$
+ \int \mathbb{E}[(\mathbf{1}_{\{|D_{z_1}V_n|>3\}}\mathbf{1}_{\{|D_{z_1,z_2}^2 V_n|>1\}}D_{z_1}V_n)^4]^{1/2}\mathbb{E}[|D_{z_1}V_n|^4]^{1/2}\, \lambda^2(d(z_1,z_2))\Big)
$$

$$
\leq C\Big(M + \int |A_n(z)|^4\, \lambda(dz) \times \int \mathbf{1}_{\{|A_n(z)|>1\}}\, \lambda(dz)\Big), \tag{4.44}
$$

where the second inequality follows by the two estimates $|D_{z_1,z_2}^2 V_n| \leq A_n(z_2)$, cf. (4.5) and the line following it, and $|D_{z_1}V_n| \leq A_n(z_1)$, cf. (4.4). Lemma 4.8 shows that

$$
\int A_n(z)^4\, \lambda(dz) \leq C \begin{cases} n^{-1} & \text{if } \alpha\beta > 4, \\ n^{-1}(\log(n))^3 & \text{if } \alpha\beta = 4, \\ n^{2-\frac{3}{4}\alpha\beta} & \text{if } 2 < \alpha\beta < 4, \end{cases} \tag{4.45}
$$

and Lemma 4.6 implies that

$$
\int 1_{\{A_n(z)>1\}}\, \lambda(dz) \leq \int_{\mathbb{R}^2} \min\big(1, A_n(x,s)^4\big)\, \lambda(dx,ds) \tag{4.46}
$$

$$
\leq C \begin{cases} n^{-1} & \text{if } \alpha\beta > 4, \\ n^{-1}\log(n) & \text{if } \alpha\beta = 4, \\ n^{(2-\alpha\beta)/2} & \text{if } 2 < \alpha\beta < 4. \end{cases}
$$

Hence from (4.44), (4.45), (4.46) and (4.42) we deduce the inequality

$$
\gamma_{6.1}^2 \leq C \begin{cases} n^{-1} & \text{if } \alpha\beta > 4, \\ n^{-1}(\log(n))^2 & \text{if } \alpha\beta = 4, \\ n^{3-\frac{5}{4}\alpha\beta} & \text{if } 2 < \alpha\beta < 4. \end{cases} \tag{4.47}
$$

Since φ is bounded we have that

$$
\gamma_{6.2}^2 = \int \mathbb{E}[\varphi(D_{z_1}V_n)^2(D_{z_1}V_n)^4]^{1/2}\mathbb{E}[|D_{z_2,z_1}^2 V_n|^4]^{1/2}\, \lambda^2(d(z_1,z_2)) \leq CM, \tag{4.48}
$$

and

$$
\gamma_{6.3}^2 = \int \mathbb{E}[(D_{z_2}(\varphi(D_{z_1}V_n)D_{z_1}V_n))^4]^{1/2}\mathbb{E}[|D_{z_2,z_1}^2 V_n|^4]^{1/2}\, \lambda^2(d(z_1,z_2))
$$

$$
\leq C\int \mathbb{E}[(D_{z_1}V_n)^4]^{1/2}\mathbb{E}[|D_{z_2,z_1}^2 V_n|^4]^{1/2}\, \lambda^2(d(z_1,z_2)) = CM. \tag{4.49}
$$

The inequalities (4.47), (4.48), (4.49) and (4.42) now imply (4.41), and the proof of the lemma is complete. $\square$

Proof of Theorem 1.1 for the Kolmogorov distance. We now combine the statements of our second-order Poincaré inequality (4.31), and Lemmas 4.3, 4.4, 4.7, 4.10 and 4.11. For $\alpha\beta \geq 8/3$ we have the inequality

$$d_K(V_n/v_n, Z) \leq C(\gamma_1 + \gamma_2 + \overline{\gamma}_3 + \gamma_4 + \gamma_5 + \gamma_6)$$

$$\leq C \begin{cases} n^{-1/2} & \text{if } \alpha\beta > 4, \\ n^{-1/2}\log(n) & \text{if } \alpha\beta = 4, \\ n^{(2-\alpha\beta)/4} & \text{if } \frac{8}{3} \leq \alpha\beta < 4. \end{cases}$$

On the other hand, for $2 < \alpha\beta < 8/3$ we will use the bound

$$d_K(V_n/v_n, Z) \leq \sqrt{d_W(V_n/v_n, Z)} \leq Cn^{(2-\alpha\beta)/4}.$$

Thus, we obtain the assertion of Theorem 1.1 for the Kolmogorov distance. $\qquad\square$

All our corollaries, i.e., Corollaries 1.3–1.6, follow directly from Theorem 1.1.

Acknowledgements

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