

Small Scale CLTs for the Nodal Length of Monochromatic Waves

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Abstract

We consider the nodal length $L(\lambda)$ of the restriction to a ball of radius r_λ of a *Gaussian pullback monochromatic random wave* of parameter $\lambda > 0$ associated with a Riemann surface (\mathcal{M}, g) without conjugate points. Our main result is that, if r_λ grows slower than $(\log \lambda)^{1/25}$, then (as $\lambda \rightarrow \infty$) the length $L(\lambda)$ verifies a Central Limit Theorem with the same scaling as Berry's random wave model – as established in Nourdin, Peccati and Rossi (2019). Taking advantage of some powerful extensions of an estimate by Bérard (1986) due to Keeler (2019), our techniques are mainly based on a novel intrinsic bound on the coupling of smooth Gaussian fields, that is of independent interest, and moreover allow us to improve some estimates for the nodal length asymptotic variance of pullback random waves in Canzani and Hanin (2016). In order to demonstrate the flexibility of our approach, we also provide an application to phase transitions for the nodal length of arithmetic random waves on shrinking balls of the 2-torus.

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1 Introduction

Let (\mathcal{M}, g) be a compact, smooth, Riemannian surface without boundary, and denote by ϕ_λ the Gaussian **monochromatic random wave** on \mathcal{M} with parameter $\lambda > 0$ (see §1.1 for precise definitions). The aim of the present paper is to study the local behaviour of the **nodal set** of ϕ_λ , when $\lambda \rightarrow \infty$ and ϕ_λ is restricted to a ball whose radius converges to zero as a function of λ .

Our main result, stated in Theorem 1.5 below, is that if \mathcal{M} has *no conjugate points* and $r_\lambda = o((\log \lambda)^{1/25})$, then the **nodal length** of the *pullback wave* $\phi_\lambda^{x_0}$ associated with ϕ_λ at a point $x_0 \in \mathcal{M}$ and restricted to a ball on the tangent space $T_{x_0}\mathcal{M}$ of radius r_λ , verifies a Central Limit Theorem (CLT) with exactly the same asymptotic behaviour of mean and variance as for **Berry's random wave model** on \mathbb{R}^2 — see [Ber77, Ber02, NPR19].

Our techniques are based on three main tools: (i) a quantitative extension of an estimate by Bérard [Bera77] (see also [Bon17]) due to Keeler [Kee19], yielding an explicit bound on the rate of convergence of covariance functions of pullback waves on manifolds without conjugate points (see Theorem 2.1), (ii) some new explicit estimates on the **coupling of smooth Gaussian fields** in C^k topologies (see Theorem 2.2), and (iii) an application of a

mixed Kac–Rice formula in order to control the discrepancy of nodal lengths associated with coupled random functions.

In particular, our way of exploiting the estimates at Point (ii) above will be to explicitly couple, in the C^1 topology, the pullback wave $\phi_\lambda^{x_0}$ with a copy of Berry’s random field, in such a way that the CLT for the pullback nodal length can be directly inferred from the main result of [NPR19]. We stress that the field $\phi_\lambda^{x_0}$ is, in general, *not* stationary: it follows that – in order to couple $\phi_\lambda^{x_0}$ with Berry’s random waves – we cannot take advantage of the techniques recently developed in [BM19], that only apply to the coupling of stationary fields, via the optimal pairing of spectral measures with respect to quadratic transport distances. We also refer to Remark 2.3 below for a brief comparison with a coupling technique outlined in [Sod12, Section 3.1.1].

It is also important to notice that our application of Kac–Rice at Point (iii) will allow us to deduce an exact asymptotic relation for the variance of the nodal length of $\phi_\lambda^{x_0}$. As we will see in more detail below, exact asymptotic characterisations for the mean and variance of nodal lengths (and, a fortiori, second order results like central and non-central limit theorems) are typically available only for exactly solvable models, like e.g. **random spherical harmonics** [Bera85, Wig10, MRW20], **arithmetic random waves** [RW08, KKW13, MPRW16, DNPR19, Cam19, PR18, BM19, BMW18], or the already quoted Berry’s planar waves [Ber02, NPR19]. To the best of our knowledge, our Theorem 1.5 is the first exact second order result for nodal lengths of monochromatic waves holding for such a general class of random fields.

As explained below, the results of the present paper provide a counterpart to the laws of large numbers proved by Canzani and Hanin in [CH16a, Theorem 1], and also yield an explicit quantitative answer to a problem left open in [NPR19, Section 1.4.1]. One should notice that the condition $r_\lambda = o((\log \lambda)^{1/25})$ is much more restrictive than the requirement $r_\lambda = o(\lambda)$ that is sufficient for the main results of [CH16a] to hold: this is due to the fact that the rate $o((\log \lambda)^{1/25})$ is the *maximal* one for which we can effectively couple the pullback wave of ϕ_λ with Berry’s planar field in such a way that the difference between the corresponding normalized nodal lengths converge to zero in L^2 when $\lambda \rightarrow \infty$. While it is clear that the exponent $1/25$ is in part an artefact of some analytical inequalities that are applied in our proofs and could in principle be improved (see e.g. our use of Sobolev embedding in Section 4.2), the logarithmic dependence on λ is a consequence of [Kee19] (see Theorem 2.1) refining a deep result by Bérard [Bera77], and cannot easily be dispensed with – see Section 2.1.

In order to demonstrate the flexibility of our approach, we also provide an application to arithmetic random waves on the flat torus $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ – which in principle do not enter the above described framework of pullback random waves – yielding a small scale CLT that is related to a conjecture of Benatar, Marinucci and Wigman, see [BMW18, §2.2]. In this case, we cannot directly apply the refinement of the estimate by Bérard [Bera77] (see also [Bon17]) due to Keeler [Kee19], and rely indeed on a direct argument based on a classical arithmetic estimate from [KK77] — the idea of using such an estimate for coupling arithmetic random waves with Berry’s model already appears in [BM19]; see also [S20].

We observe that similar problems for random spherical harmonics were studied by A. P. Todino in [Tod18]. In such a paper, the author proves a CLT for the nodal length of random spherical harmonics in shrinking caps via a specific argument (a reduction principle). Our coupling techniques could be applied also in this framework, plausibly at the cost of worse

estimates (in terms of conditions on the radius of the shrinking spherical cap) than those in [Tod18], because of the full generality of our approach.

We will now present a more detailed discussion of our main findings. In what follows, every random object is defined on a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with \mathbb{E} and \mathbf{Var} denoting respectively expectation and variance with respect to \mathbb{P} . Given two positive sequences $\{a_n\}, \{b_n\}$, we write $a_n \sim b_n$ if $a_n/b_n \rightarrow 1$.

1.1 Overview and main results

Let (\mathcal{M}, g) be a compact, smooth, Riemannian surface without boundary, and denote by Δ_g the associated Laplace-Beltrami operator. We write $\{f_j : j \geq 0\}$ to indicate an orthonormal basis of $L^2(\mathcal{M})$ composed of eigenfunctions of Δ_g , that is,

$$\Delta_g f_j + \lambda_j^2 f_j = 0, \quad j \geq 0,$$

where the corresponding eigenvalues are such that $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_j \nearrow \infty$. Following [Zel09, CH16a], we define the **monochromatic random wave** of parameter $\lambda > 0$ on \mathcal{M} to be the Gaussian random field on the manifold

$$\phi_\lambda(x) := \frac{1}{\sqrt{\dim(H_\lambda)}} \sum_{\lambda_j \in [\lambda, \lambda+1]} a_j f_j(x), \quad x \in \mathcal{M}, \quad (1.1)$$

where the a_j are independent and identically distributed (i.i.d.) standard Gaussian random variables, and

$$H_\lambda := \bigoplus_{\lambda_j \in [\lambda, \lambda+1]} \text{Ker}(\Delta_g + \lambda_j^2 \text{Id}),$$

with the symbol Id denoting the identity operator. The Gaussian field ϕ_λ is centred by construction, and its covariance kernel is given by

$$K_\lambda(x, y) := \text{Cov}(\phi_\lambda(x), \phi_\lambda(y)) = \frac{1}{\dim(H_\lambda)} \sum_{\lambda_j \in [\lambda, \lambda+1]} f_j(x) f_j(y), \quad x, y \in \mathcal{M}. \quad (1.2)$$

“Short window” random waves such as ϕ_λ in (1.1) (for manifolds of arbitrary dimension) were first introduced by Zelditch [Zel09] as general approximate models of random Gaussian Laplace eigenfunctions defined on manifolds not necessarily having spectral multiplicities; see e.g., [CH16a, BW18, NS16] and the references therein for further discussions.

Our aim in this paper is to study the local behaviour of the **nodal set** of ϕ_λ , as $\lambda \rightarrow \infty$, restricted to balls of decreasing radius. Our main tool in order to accomplish this task is the notion of a “pullback” random wave that we will study at **points of isotropic scaling**. In order to introduce these notions, we adopt the standard notation $J_0(r)$, $r \geq 0$, to indicate the **Bessel function of the first kind** with index 0, given by

$$J_0(r) := \int_{S^1} e^{i\langle u, z \rangle} \frac{dz}{2\pi},$$

where $\frac{dz}{2\pi}$ is the uniform probability measure on the unit circle, and $u \in \mathbb{R}^2$ is any point such that $\|u\| = r$.

Fix $x_0 \in \mathcal{M}$, and consider the tangent space $T_{x_0}\mathcal{M}$ to the manifold at x_0 : we define the **pullback random wave** associated with ϕ_λ at x_0 as the Gaussian random field on $T_{x_0}\mathcal{M}$ given by

$$\phi_\lambda^{x_0}(u) := \phi_\lambda\left(\exp_{x_0}\left(\frac{u}{\lambda}\right)\right), \quad u \in T_{x_0}\mathcal{M},$$

where $\exp_{x_0} : T_{x_0}\mathcal{M} \rightarrow \mathcal{M}$ is the exponential map at x_0 . The planar field $\phi_\lambda^{x_0}$ is trivially centered and Gaussian and, by virtue of (1.2), its covariance kernel $K_\lambda^{x_0}$ is given by

$$K_\lambda^{x_0}(u, v) = K_\lambda\left(\exp_{x_0}\left(\frac{u}{\lambda}\right), \exp_{x_0}\left(\frac{v}{\lambda}\right)\right), \quad u, v \in T_{x_0}\mathcal{M}.$$

A direct inspection of the above covariance kernel immediately shows that $\phi_\lambda^{x_0}$ is of class C^∞ with probability one.

Definition 1.1 (See [CH16a]). We say that $x_0 \in \mathcal{M}$ is a point of **isotropic scaling** if, for every positive function $\lambda \mapsto r_\lambda$ such that $r_\lambda = o(\lambda)$, as $\lambda \rightarrow \infty$, one has that

$$\sup_{u, v \in \mathbb{B}(r_\lambda)} \left| \partial^\alpha \partial^\beta \left\{ K_\lambda^{x_0}(u, v) - (2\pi)J_0(\|u - v\|_{g_{x_0}}) \right\} \right| \rightarrow 0, \quad \lambda \rightarrow \infty, \quad (1.3)$$

where $\alpha, \beta \in \mathbb{N}^2$ are multi-indices labeling partial derivatives with respect to u and v , respectively, $\|\cdot\|_{g_{x_0}}$ is the norm on $T_{x_0}\mathcal{M}$ induced by g , and $\mathbb{B}(r_\lambda)$ is the corresponding ball of radius r_λ centred at the origin.

Remark 1.2. (a) Sufficient conditions for a point x_0 to be of isotropic scaling are discussed e.g. in [CH16a, Section 2.5], building on the findings [CH16b]. In particular, [CH16b, Theorem 1] implies that a sufficient condition for $x_0 \in \mathcal{M}$ to be of isotropic scaling is that the set

$$\mathcal{L}_{x_0, x_0} := \{\xi \in S_{x_0}\mathcal{M} : \exists t > 0 \text{ s.t. } \exp_{x_0}(t\xi) = x_0\}$$

has volume 0 in $T_{x_0}\mathcal{M}$, where $S_{x_0}\mathcal{M}$ denotes the unit sphere in $T_{x_0}\mathcal{M}$ with respect to the norm $\|\cdot\|_{g_{x_0}}$. For every compact smooth manifold \mathcal{M} and for every $x_0 \in \mathcal{M}$, the property $|\mathcal{L}_{x_0, x_0}| = 0$ is generic in the space of all Riemannian metrics [SZ02, Lemma 6.1]. It is also known that the condition $|\mathcal{L}_{x_0, x_0}| = 0$ holds for every $x_0 \in \mathcal{M}$ whenever \mathcal{M} has no conjugate points (and, in particular, when \mathcal{M} is negatively curved).

(b) Relation (1.3) implies that, for every $u, v \in T_{x_0}\mathcal{M}$ and every multi-indices α, β , the two-dimensional Gaussian field $\{(\partial^\alpha \phi_\lambda^{x_0}(u), \partial^\beta \phi_\lambda^{x_0}(u)) : u \in T_{x_0}\mathcal{M}\}$ converges in the sense of finite-dimensional distributions to

$$\{\sqrt{2\pi}(\partial^\alpha \phi_\infty^{x_0}(u), \partial^\beta \phi_\infty^{x_0}(u)) : u \in T_{x_0}\mathcal{M}\},$$

where $\phi_\infty^{x_0}$ is the centered Gaussian field on $T_x\mathcal{M}$ with covariance

$$\mathbb{E}[\phi_\infty^{x_0}(u)\phi_\infty^{x_0}(v)] = J_0(\|u - v\|_{g_{x_0}}).$$

One can easily check that, with probability one, $\phi_\infty^{x_0}$ is an eigenfunction with eigenvalue 1 of the Laplace operator on $T_{x_0}\mathcal{M}$ associated with the metric g_{x_0} .

(c) (*Convention on the choice of coordinates*) Since in this paper we are only interested in second order results for a *fixed* $x_0 \in \mathcal{M}$ of isotropic scaling, we will always (tacitly) choose coordinates around x_0 in such a way that $g_{x_0} = \text{Id}$, and we will write $\|\cdot\|_{g_{x_0}} =$

$\|\cdot\|$ in order to simplify the notation. In this way, the field $\phi_\infty^{x_0}$ at item (b) becomes universal (in the sense that it does not depend on \mathcal{M}) and can be identified with **Berry's Random Wave Model** on $\mathbb{R}^2 \simeq T_{x_0}\mathcal{M}$. Such a field is defined as the unique (in distribution) centred real-valued random field $b = \{b(u) : u \in \mathbb{R}^2\}$ such that b is an eigenfunction of the Laplace operator Δ on \mathbb{R}^2 with eigenvalue 1, and b is *isotropic*, that is, the distribution of b is invariant with respect to rigid motions of the plane. It can be proved that these requirements immediately imply that, necessarily,

$$\mathbb{E}[b(u)b(v)] = J_0(\|u - v\|); \quad (1.4)$$

see [NPR19] for details. We observe that, when $g_x = \text{Id}$, condition (1.3) implies that, in the parlance of [NS16], the ensemble $\{\phi_\lambda^x\}$ has *translation invariant local limits*.

As anticipated, the principal focus in our paper is the (random) **nodal set**

$$(\phi_\lambda^{x_0})^{-1}(0) := \{u \in T_{x_0}\mathcal{M} : \phi_\lambda^{x_0}(u) = 0\}$$

which is a.s. a smooth curve [CH16a]. For every $\lambda, r > 0$ and $x_0 \in \mathcal{M}$, we set

$$L(\phi_\lambda^{x_0}; r) := \mathcal{H}^1((\phi_\lambda^{x_0})^{-1}(0) \cap B_r), \quad (1.5)$$

where \mathcal{H}^1 indicates the one-dimensional Hausdorff measure on $T_{x_0}\mathcal{M} \simeq \mathbb{R}^2$ and B_r is the closed ball of radius r centred at the origin; in other words, the random variable $L(\phi_\lambda^{x_0}; r)$ represents the length of the restriction of the nodal set of $\phi_\lambda^{x_0}$ to B_r . Similarly, writing $b = \{b(u) : u \in \mathbb{R}^2\}$ for the Berry's random wave model defined in Remark 1.2-(c) (in particular, formula (1.4)), we write

$$L(b; r) := \mathcal{H}^1(b^{-1}(0) \cap B_r). \quad (1.6)$$

Note that $\text{Vol}(B_r) = \pi r^2$. Our first statement is taken from [CH16a], and contains a 'universal' law of large numbers for the nodal lengths $L(\phi_\lambda^x; r)$ (observe that the convergence in L^2 at (1.8) below is not stated in [CH16a, Theorem 1], but it is rather an immediate consequence of the arguments in the proof).

Theorem 1.3 (Special case of Theorem 1 in [CH16a]). *Let the above notation prevail and let x_0 be a point of isotropic scaling.*

(1) *For every fixed $r > 0$, as $\lambda \rightarrow \infty$, one has that*

$$L(\phi_\lambda^{x_0}; r) \xrightarrow{\text{law}} L(b; r), \quad (1.7)$$

where, here and for the rest of the paper, the symbol $\xrightarrow{\text{law}}$ indicates convergence in distribution of random variables.

(2) *If the function $\lambda \mapsto r_\lambda$ is such that $r_\lambda = o(\lambda)$ as $\lambda \rightarrow \infty$, then*

$$\mathbb{E} \left[\left(\frac{L(\phi_\lambda^{x_0}; r_\lambda)}{r_\lambda^2} - \frac{\pi}{2\sqrt{2}} \right)^2 \right] \rightarrow 0. \quad (1.8)$$

In particular, as $\lambda \rightarrow \infty$,

$$\mathbb{E}[L(\phi_\lambda^{x_0}; r_\lambda)] \sim \frac{\pi}{2\sqrt{2}} r_\lambda^2, \quad \text{and} \quad \text{Var}(L(\phi_\lambda^{x_0}; r_\lambda)) = o(r_\lambda^4). \quad (1.9)$$

In view of (1.8), the next logical step is to address the following question: *as $\lambda \rightarrow \infty$, what is the nature of the fluctuations of $X_\lambda := L(\phi_\lambda^{x_0}; r_\lambda)/r_\lambda^2$, around the limit $\frac{\pi}{2\sqrt{2}}$? In particular, does a properly normalised version of X_λ verify a CLT?*

Plainly, answering such a question would require one to establish some non trivial lower bound for the function

$$\lambda \mapsto \mathbf{Var}(L(\phi_\lambda^{x_0}; r_\lambda)), \quad \lambda \rightarrow \infty,$$

and, in a generic setting like the one of Theorem 1.3, such a task seems to be largely outside the scope of existing techniques – see e.g. the discussion around Theorem 1 in [CH16a].

As anticipated, the main idea developed in the present paper is that, in the case of surfaces without conjugate points and if one considers mappings $\lambda \mapsto r_\lambda$ that diverge to infinity at a rate which is *considerably slower* than λ , then one can deduce precise informations about the fluctuations of $L(\phi_\lambda^{x_0}; r_\lambda)$ from the following central limit theorem involving Berry’s planar waves.

Theorem 1.4 (See [Ber02] and [NPR19]). *As $r \rightarrow \infty$, one has that*

$$\mathbb{E}[L(b; r)] = \frac{\pi}{2\sqrt{2}}r^2, \quad \text{and} \quad \mathbf{Var}(L(b; r)) \sim \frac{r^2 \log r}{256}. \quad (1.10)$$

Moreover,

$$\frac{L(b; r) - \mathbb{E}[L(b; r)]}{\mathbf{Var}(L(b; r))^{1/2}} \xrightarrow{\text{law}} Z \sim \mathcal{N}(0, 1), \quad (1.11)$$

where $\mathcal{N}(0, 1)$ indicates the one-dimensional Gaussian distribution with mean zero and variance 1.

The main achievement of our work is the following small scale second order result, establishing exact estimates for mean and variances, as well as a CLT, for pullback random waves associated with manifolds having no conjugate points. As already recalled, this also answers a question left open in [NPR19, Section 1.4.1].

Theorem 1.5 (Small scale CLT for pullback random waves). *Let the above notation prevail, and assume that (\mathcal{M}, g) is a compact, smooth, Riemannian surface without boundary and without conjugate points. Then, for every $x_0 \in \mathcal{M}$ and every function $\lambda \mapsto r_\lambda$ such that $r_\lambda \rightarrow \infty$ and, as $\lambda \rightarrow \infty$,*

$$\frac{r_\lambda^{25}}{(\log r_\lambda)^4} = o(\log \lambda) \quad (1.12)$$

one has that

$$\mathbb{E}[L(\phi_\lambda^{x_0}; r_\lambda)] \sim \frac{\pi r_\lambda^2}{2\sqrt{2}}, \quad \mathbf{Var}(L(\phi_\lambda^{x_0}; r_\lambda)) \sim \frac{r_\lambda^2 \log r_\lambda}{256}, \quad (1.13)$$

and

$$\frac{L(\phi_\lambda^{x_0}; r_\lambda) - \mathbb{E}[L(\phi_\lambda^{x_0}; r_\lambda)]}{\mathbf{Var}(L(\phi_\lambda^{x_0}; r_\lambda))^{1/2}} \xrightarrow{\text{law}} Z \sim \mathcal{N}(0, 1). \quad (1.14)$$

Remark 1.6. (a) In the regime (1.12), the second relation in (1.13) largely improves the estimate $\mathbf{Var}(L(\phi_\lambda^{x_0}; r_\lambda)) = o(r_\lambda^4)$, which is valid for generic $r_\lambda = o(\lambda)$ – see Theorem 1.3. Note that (1.12) is implied by $r_\lambda = o((\log \lambda)^{1/25})$.

- (b) Our proof of Theorem 1.5 will be achieved along the following route: **(i)** taking advantage of the new bound for the rate of convergence to zero in (1.3) in the case of manifolds without conjugate points [Kee19] (see Theorem 2.1), **(ii)** using the quantitative information at Point **(i)** in order to build a coupling of $\phi_\lambda^{x_0}$ and Berry’s planar wave b in such a way that, as $\lambda \rightarrow \infty$, the difference $\phi_\lambda^{x_0} - \sqrt{2\pi} b$ converges to zero (say, in $L^1(\mathbb{P})$) in the C^1 topology of B_{r_λ} , when $r_\lambda \rightarrow \infty$ sufficiently slow (see Theorem 2.2), and **(iii)** applying a ‘mixed Kac–Rice formula’ in order to show that, for $r_\lambda = o((\log \lambda)^{1/25})$ and for the coupling of $\phi_\lambda^{x_0}$ and b at Point **(ii)** one has actually that, as $\lambda \rightarrow \infty$,

$$\left| \frac{L(b; r_\lambda) - \mathbb{E}[L(b; r_\lambda)]}{\mathbf{Var}(L(b; r_\lambda))^{1/2}} - \frac{L(\phi_\lambda^{x_0}; r_\lambda) - \mathbb{E}[L(\phi_\lambda^{x_0}; r_\lambda)]}{\mathbf{Var}(L(b; r_\lambda))^{1/2}} \right| \rightarrow 0 \text{ in } L^2(\mathbb{P});$$

see Section 3.

- (c) Assume that, for some coupling of $\phi_\lambda^{x_0}$ and b one has that $\phi_\lambda^{x_0} - \sqrt{2\pi} b$ converges to zero in probability, with respect to the C^1 topology of B_{r_λ} , with $r_\lambda \rightarrow \infty$, that is: for every $\epsilon > 0$,

$$\mathbb{P} \left[\max_{\alpha: |\alpha| \leq 1} \sup_{z \in B_{r_\lambda}} |\partial^\alpha \phi_\lambda^{x_0}(z) - \partial^\alpha \sqrt{2\pi} b| > \epsilon \right] \rightarrow 0, \quad \lambda \rightarrow \infty.$$

Then, in general, it is *not* possible to conclude that the difference $L(b; r_\lambda) - L(\phi_\lambda^{x_0}; r_\lambda)$ also converges to zero in probability (this is in contrast with the case of a fixed radius ball – see e.g. [APP18]). This observation explains the necessity of Step **(iii)** in the strategy outlined at the previous item.

Remark 1.7. It is worth noting that our ‘coupling’ approach to limit theorems and variance estimates can be in principle applied to the case of parametric Gaussian ensembles on manifolds, that are locally converging to translation invariant Gaussian fields – see e.g. the general framework outlined in [NS16, Section 1.2] – as soon as limit theorems for nodal lengths (or more general functionals) associated with the latter are known. However, deducing variance asymptotics and limit theorems for local geometric functionals of generic stationary fields, similar to Theorem 1.4, would require a remarkable amount of technical work and novel ideas, and will be investigated elsewhere.

1.2 The case of arithmetic random waves

The framework of pullback random waves described in the previous section does not encompass the case of some exactly solvable models of Gaussian Laplace eigenfunctions defined on manifolds having spectral multiplicities, such as the model of arithmetic random waves [RW08, KKW13, MPRW16, PR18] and random spherical harmonics [Bera85, Wig10, MRW20]. The techniques developed in this paper can nonetheless be suitably extended in order to deal with specific models of this type. The aim of this subsection (and of Section 5 below, containing the proof of our main Theorem 1.9) is to state and prove a small scale CLT for arithmetic random waves restricted to fast shrinking ball. As discussed below, such a theorem represents a counterpart to the main findings in [BMW18], and corroborates a conjecture stated therein [BMW18, §2.2].

1.2.1 Definitions and reminders on global results

It is well-known that the eigenvalues of the Laplace operator on the flat 2-torus \mathbb{T}^2 are of the form $-E_n$, where $E_n := 4\pi^2 n$ and

$$n \in S := \{n \in \mathbb{Z} : n = a^2 + b^2, a, b \in \mathbb{Z}\}$$

is the set of integers that can be represented as the sum of two squares. For $n \in S$, denote by Λ_n the set of frequencies

$$\Lambda_n = \{\xi \in \mathbb{Z}^2 : \|\xi\| = \sqrt{n}\}$$

and by \mathcal{N}_n the cardinality of Λ_n (that is, \mathcal{N}_n is the multiplicity of the eigenspace corresponding to $-E_n$). For $n \in S$, consider the probability measure μ_n induced by Λ_n on the unit circle \mathbb{S}^1 :

$$\mu_n = \frac{1}{\mathcal{N}_n} \sum_{\xi \in \Lambda_n} \delta_{\xi/\sqrt{n}}.$$

Following [RW08], for $n \in S$, the toral random eigenfunction T_n (or **arithmetic random wave** of order n) is defined as the centered Gaussian field on the torus with the following covariance function: for $x, y \in \mathbb{T}^2$,

$$\text{Cov}(T_n(x), T_n(y)) = \frac{1}{\mathcal{N}_n} \sum_{\xi \in \Lambda_n} e^{i2\pi(\xi, x-y)} = \int_{\mathbb{S}^1} e^{i2\pi\sqrt{n}\langle \theta, x-y \rangle} d\mu_n(\theta). \quad (1.15)$$

It is easily checked that, with probability one, $\Delta T_n = -4\pi^2 n T_n$, that is, T_n is an eigenfunction of Δ with eigenvalue $-E_n$. As discussed in [KKW13], there exists a density-1 subsequence $\{n_j : j \geq 1\} \in S$ such that, as $j \rightarrow +\infty$,

$$\mu_{n_j} \Rightarrow \frac{dz}{2\pi},$$

where $\frac{dz}{2\pi}$ denotes as before the uniform probability measure on the unit circle, and \Rightarrow stands for weak convergence. For this subsequence, for $x, y \in \mathbb{T}^2$,

$$\text{Cov}(T_{n_j}(x/2\pi\sqrt{n_j}), T_{n_j}(y/2\pi\sqrt{n_j})) \rightarrow \int_{\mathbb{S}^1} e^{i\langle z, x-y \rangle} \frac{dz}{2\pi} = J_0(\|x-y\|),$$

i.e. the scaling limit of T_{n_j} is Berry's RWM.

Let us now set $\mathcal{L}_n := \text{length}(T_n^{-1}(0))$. The expected nodal length was computed in [RW08] to be equal to

$$\mathbb{E}[\mathcal{L}_n] = \frac{1}{2\sqrt{2}} E_n,$$

while in [KKW13] it is shown that, as $\mathcal{N}_n \rightarrow +\infty$, the variance of \mathcal{L}_n satisfies the following exact relation

$$\text{Var}(\mathcal{L}_n) \sim \frac{1 + \widehat{\mu}_n(4)^2}{512} \frac{E_n}{\mathcal{N}_n^2},$$

where $\widehat{\mu}_n(4)$ denotes the fourth Fourier coefficients of μ_n . In order to have an asymptotic law for the variance, one should select a subsequence $\{n_j\}$ of energy levels such that (i) $\mathcal{N}_{n_j} \rightarrow +\infty$ and (ii) $|\widehat{\mu}_{n_j}(4)| \rightarrow \eta$, for some $\eta \in [0, 1]$. Note that for each $\eta \in [0, 1]$, there exists a subsequence $\{n_j\}$ such that both (i) and (ii) hold (see [KKW13, KW17]). For such

subsequences, the asymptotic distribution of the nodal length was shown to be non-Gaussian in [MPRW16]:

$$\frac{\mathcal{L}_{n_j} - \mathbb{E}[\mathcal{L}_{n_j}]}{\sqrt{\text{Var}(\mathcal{L}_{n_j})}} \xrightarrow{\text{law}} \frac{1}{2\sqrt{1+\eta^2}}(2 - (1-\eta)Z_1^2 - (1+\eta)Z_2^2), \quad (1.16)$$

where Z_1 and Z_2 are i.i.d. standard Gaussian random variables. A complete quantitative version (in Wasserstein distance) of (1.16) is given in [PR18].

1.2.2 Phase transitions for nodal lengths on shrinking balls

The following remarkable statement is extrapolated from [BMW18, Theorem 1.1] and contains a characterization of the fluctuation of the nodal length of arithmetic random waves above the Planck scale. For every $n \in S$ and every $r > 0$, we set

$$\mathcal{L}_n(r) := \text{length}\left(T_n^{-1}(0) \cap B_{r/\sqrt{n}}\right),$$

where B_r denotes the ball of radius r centred at the origin. Note that, in general

$$\mathbb{E}\left[\text{length}\left(T_n^{-1}(0) \cap B_r\right)\right] = \frac{\pi r^2}{2\sqrt{2}}E_n.$$

Theorem 1.8 (Special Case of Theorem 1.1 in [BMW18]). *For every $\gamma \in (0, 1/2)$, there exists a density one sequence $\{n_j\} \subset S$ verifying the following properties:*

1. *as $n_j \rightarrow \infty$, one has that $\mathcal{N}_{n_j} \rightarrow \infty$ and μ_{n_j} converges weakly to the uniform measure on \mathbb{S}^1 ;*
2. *as $n_j \rightarrow \infty$,*

$$\text{Var}(\mathcal{L}_{n_j}(n_j^\gamma)) \sim \frac{1 + \widehat{\mu}_{n_j}(4)^2}{512} \frac{E_{n_j}}{\mathcal{N}_{n_j}^2} \times \{\pi(n_j^{\gamma-1/2})^2\}^2;$$

3. *as $n_j \rightarrow \infty$,*

$$\frac{\mathcal{L}_{n_j} - \mathbb{E}[\mathcal{L}_{n_j}]}{\sqrt{\text{Var}(\mathcal{L}_{n_j})}} \xrightarrow{\text{law}} 1 - \frac{Z_1^2 + Z_2^2}{2},$$

where Z_1, Z_2 are two independent standard Gaussian random variables.

In [BMW18, Section 2.2] a general conjecture is stated concerning nodal lengths of arithmetic random waves, containing in particular the following

Conjecture. *There exists $A_0 > 0$ such that (i) the conclusion of Theorem 1.8 continues to hold if one replaces the sequence n^γ with any sequence $\alpha_n \geq (\log n)^C$, for any $C > A_0$, and (ii) the conclusion of Theorem 1.8 fails to hold if one replaces the sequence n^γ with any sequence $\alpha_n = (\log n)^C$, for any $C < A_0$.*

The following statement is the main result of this section and shows that, if such an A_0 exists, then necessarily $A_0 \geq \frac{1}{18}(\log \pi - \log 2) = 0.02508\dots$.

Theorem 1.9. *Fix $\rho < \frac{1}{2}(\log \pi - \log 2) = 0.225791\dots$. Then there exists a density one sequence $\{n_j\} \subset S$ such that, as $n_j \rightarrow \infty$,*

1. *$\mathcal{N}_{n_j} \rightarrow \infty$ and μ_{n_j} converges weakly to the uniform measure on \mathbb{S}^1 ;*

2. for every sequence $n \mapsto \alpha_n$ such that $\alpha_n = O((\log n)^{\rho/9})$,

$$\mathbf{Var}(\mathcal{L}_{n_j}(\alpha_{n_j})) \sim \frac{1}{n_j} \frac{\alpha_{n_j}^2 \log \alpha_{n_j}}{256}$$

and

$$\frac{\mathcal{L}_{n_j}(\alpha_{n_j}) - \mathbb{E}[\mathcal{L}_{n_j}(\alpha_{n_j})]}{\sqrt{\mathbf{Var}(\mathcal{L}_{n_j}(\alpha_{n_j}))}} \xrightarrow{\text{law}} Z,$$

where Z denotes as before a standard Gaussian random variable.

Remark 1.10. (a) The conclusion of Theorem 1.9 is implicitly based on a more general estimate, yielding that if $L(n)$ denotes the nodal length of the rescaled random wave $x \mapsto T_n(x/2\pi\sqrt{n})$ on the ball with radius α_n and $L'(n)$ denotes the nodal length of Berry's random wave on the same ball, then one can couple each $L(n_j)$ and $L'(n_j)$ in such a way that

$$\mathbb{E}[(L(n_j) - L'(n_j))^2] = O\left(\frac{\alpha_{n_j}^5}{(\log n_j)^{\rho/3}}\right).$$

(b) While circulating an earlier draft of the present paper, it was brought to our attention that comparable lower bounds on the constant A_0 have been independently obtained in [S20, Theorem 1.4], by combining coupling techniques from [BM19] with explicit estimates of the nodal length of perturbed random fields.

1.3 Plan

In §2 we first recall the new result by Keeler on wave equation theory on compact manifolds without conjugate points, which improves some estimates by Bérard, and then we state our result on coupling of Gaussian fields. In §3 we prove our main theorem, dealing first with an application of a mixed Kac–Rice formula in order to control the discrepancy of nodal lengths associated with coupled random functions. Some technical lemmas are collected in §4. Finally in §5, we prove our main result on the phase transition for nodal lengths of arithmetic random waves.

2 Some estimates

2.1 Explicit rates of convergence on manifolds without conjugate points

As written in Remark 1.2, if (\mathcal{M}, g) has no conjugate points, then every $x_0 \in \mathcal{M}$ is of isotropic scaling and (1.3) holds at any point. However in order to prove our main result, we need an explicit rate in (1.3).

In [Bera77], the author solves this question for the on-diagonal case, i.e., for $u = v$ (using the same notations as in (1.3)) and when no derivatives are involved. The recent result by Keeler in [Kee19] is a breakthrough in this direction, indeed he greatly improved the error in Weyl's law on manifolds without conjugate points considering also the case of off-diagonal terms and derivatives of all order.

The following theorem is Corollary 1.1 in [Kee19] and completely answers the question addressed just above, giving a (logarithmic) rate for (1.3) in full generality.

Theorem 2.1 (Corollary 1.1 in [Kee19]). *Let (\mathcal{M}, g) be a smooth, compact, Riemannian manifold of dimension two without conjugate points, then as $\lambda \rightarrow +\infty$, for any multi-indices $\alpha, \beta \in \mathbb{N}^2$*

$$\sup_{u, v \in \mathbb{B}(r_\lambda)} \left| \partial^\alpha \partial^\beta \left\{ K_\lambda^{x_0}(u, v) - (2\pi) J_0(\|u - v\|) \right\} \right| = O\left(\frac{1}{\log \lambda}\right),$$

whenever $r_\lambda = O\left(\sqrt{\frac{\lambda}{\log \lambda}}\right)$. Here the implicit constant in the O -notation depends on the choice of $x_0 \in \mathcal{M}$ and r_λ , and on the order of differentiation.

Note that $\mathbb{B}(r_\lambda)$ corresponds to a shrinking ball of radius $\frac{r_\lambda}{\lambda} = O\left(\frac{1}{\sqrt{\lambda \log \lambda}}\right)$ on \mathcal{M} .

2.2 Coupling of smooth Gaussian fields

We now state our main results about the coupling of smooth Gaussian fields on subsets of \mathbb{R}^d . Since the present paper only involves infinitely differentiable random fields, we will uniquely focus on the case of covariance functions of class $\mathcal{C}^{\infty, \infty}$; it is a standard task to adapt our findings to the case of covariance functions of class $\mathcal{C}^{k, k}$, for some finite integer k . Proofs are deferred to Section 4.1 and Section 4.2.

For the rest of the section, fix an integer $d \geq 1$. For every $R \geq 1$, we denote as before by B_R the open ball centered at the origin and with radius R , and write $|B_R|$ for the volume of B_R . For integers $p, q \geq 1$, we denote by $\mathbb{W}^{p, q}(B_R)$ and $\mathcal{C}_b^p(B_R)$, respectively, the Sobolev space of B_R with indices p, q , and the Banach space of continuous functions on B_R having partial derivatives of order $\leq p$ that are uniformly continuous on B_R .

In what follows, we shall consider two real-valued covariance kernels

$$K : (x, y) \mapsto K(x, y), \quad \text{and} \quad C : (x, y) \mapsto C(x, y)$$

defined on $\mathbb{R}^d \times \mathbb{R}^d$; we assume that C is of class $\mathcal{C}^{\infty, \infty}$ (C is infinitely continuously differentiable in each variable x and y), and K is of class $\mathcal{C}_b^{\infty, \infty}$ (K is infinitely continuously differentiable in each variable x and y , with bounded derivatives of every order). Standard results (see e.g. [AT07, Section 1.4] or [NS16, Appendix A.9]) imply that, if X is a centered Gaussian field on \mathbb{R}^d with covariance given by K or C , then X admits a modification that is of class \mathcal{C}^∞ with probability one; in what follows, we will uniquely (tacitly) consider such a modification.

Given multi-indices α, β , we introduce the shorthand notation

$$K_{\alpha\beta}(x, y) := \partial^\alpha \partial^\beta K(x, y), \quad x, y \in \mathbb{R}^d,$$

and we define analogously the kernel $C_{\alpha\beta}$. For every integer $M = 0, 1, 2, \dots$, we will write

$$S(M) := \{\alpha : \alpha \text{ is a multi-index s.t. } |\alpha| \leq M\}. \quad (2.17)$$

The following statement contains a coupling result for Gaussian fields belonging to Sobolev spaces, and is one of the main tools exploited in the sections to follow.

Theorem 2.2. *Let the above notation and assumptions prevail, fix $M = 0, 1, \dots$ and $R \geq 1$, and write*

$$\eta = \eta(M, R) := \max_{\alpha, \beta \in S(M)} \sup_{x, y \in B_R} \left| K_{\alpha\beta}(x, y) - C_{\alpha\beta}(x, y) \right|. \quad (2.18)$$

Then, on some probability space $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$, there exists a centred two-dimensional Gaussian field

$$\{(X_0(z), Y_0(z)) : z \in B_R\}$$

such that X_0 has covariance K , Y_0 has covariance C and

$$\mathbb{E}_0 \left[\|X_0 - Y_0\|_{\mathbb{W}^{M,2}(B_R)}^2 \right] \leq A \left\{ \eta |B_R| + \sqrt{\eta} |B_R|^{1/2} R^{\frac{3d+1}{2}} \right\}, \quad (2.19)$$

where $A = A(M, d)$ is an absolute finite constant independent of R . If moreover $M > j := \lfloor \frac{d}{2} \rfloor + 1$, then one has the additional estimate

$$\mathbb{E}_0 \left[\|X_0 - Y_0\|_{\mathcal{C}_b^{M-j}(B_R)}^2 \right] \leq A' R^{2M-d} \left\{ \eta |B_R| + \sqrt{\eta} |B_R|^{1/2} R^{\frac{3d+1}{2}} \right\}, \quad (2.20)$$

where $A' = A'(M, d) \in (0, \infty)$ is independent of R .

The estimate (2.20) is deduced by combining (2.19) with a version of the Sobolev embedding theorem for open sets of \mathbb{R}^d such as the one stated in [DD12, Theorem 2.7.2] — see Section 4.2 for a detailed proof.

Remark 2.3. There is an alternate procedure for coupling smooth Gaussian processes via a discretization procedure, as described in [Sod12, Section 3.1.1]. Such a procedure consists in the following steps: (i) for some parameter $\alpha > 0$, choose an α -net contained in B_R , (ii) build an optimal coupling of ϕ_λ^x and $\sqrt{2\pi}b$ on the α -net fixed above (using some optimal criterion for coupling of Gaussian vectors – see e.g. [OP82]), (iii) extend the finite coupling at Step (ii) by using an additional collection of independent Gaussian random variables, (iv) compute a bound on the $\mathcal{C}_b^1(B_R)$ distance between the coupled fields by using some a priori estimates on their \mathcal{C}^2 norms, combined e.g. with some version of the Kolmogorov-Landau inequality on a finite domain (such as e.g. an appropriate tensorization of [Che93, Theorem 3.5]), and optimize in α . While preparing our work, we actually pursued such a strategy in full detail, and managed to obtain a bound analogous to the content of Theorem 2.2, but where the right-hand side of (2.20) is replaced by the quantity

$$A \sqrt{\eta^{1/(d+1)} R^{(3d+1)/(d+1)} (\log R)^{(2d+1)/(d+1)}}. \quad (2.21)$$

Using such a bound in our proof yields a version of Theorem 1.5 where condition (1.12) is replaced by the slightly stronger requirement that

$$\frac{r_\lambda^{28}}{(\log r_\lambda)^7} = o(\log \lambda) \quad (2.22)$$

We also mention that, with respect to the methods developed in the present paper, the approach of [Sod12] has the advantage of allowing one to deal directly with random fields of class \mathcal{C}^2 . It is also reasonable to expect that (2.21) might perform better than (2.20) for large values of the dimensional parameter d .

3 Proof of Theorem 1.5

3.1 Preparation

Fix $x \in \mathcal{M}$. For the rest of the Section we write $K(x, y) = K(x - y) = (2\pi)J_0(\|x - y\|)$ and $C_\lambda(x, y) = K_\lambda^x(x, y)$. Fix $r_\lambda = o((\log \lambda)^{1/25})$. By virtue of Theorem 2.1, we know that

$$\eta_\lambda := \max_{\alpha, \beta \in S(3)} \sup_{x, y \in B_{r_\lambda}} \left| \partial^\alpha \partial^\beta \{K(x - y) - C_\lambda(x, y)\} \right| = O\left(\frac{1}{\log \lambda}\right), \quad (3.23)$$

where the notation $\partial^\alpha \partial^\beta K(x-y)$ indicates that the operator ∂^α acts on the variable x , and ∂^β on the variable y . According to Theorem 2.2, as applied to the case $d = 2$ and $M = 3$, for every $\lambda > 0$ there exists a jointly Gaussian coupling (Y_λ, X) of ϕ_λ^x and $\sqrt{2\pi} b$ such that

$$\mathbb{E} \left[\|X - Y_\lambda\|_{C_b^1(B_{r_\lambda})}^2 \right] \leq A \sqrt{\frac{r_\lambda^{17}}{\log \lambda}} =: a(\lambda) \rightarrow 0, \quad (3.24)$$

where the constant A is independent of λ (we stress that the probability space $(\Omega_\lambda, \mathcal{F}_\lambda, \mathbb{P}_\lambda)$ on which the coupling is defined depends in general on λ , but we will omit such a dependence for the sake of readability). For every $\lambda > 0$ and every $x, y \in B_R$ we introduce the following notation for mixed covariances: for every multi-indices α, β

$$M_{\alpha, \beta}^\lambda(x, y) := \mathbb{E}[\partial^\alpha X(x) \partial^\beta Y_\lambda(y)] \quad (3.25)$$

and

$$\zeta_\lambda := \max_{\alpha, \beta \in S(3)} \sup_{x, y \in B_{r_\lambda}} \left| \partial^\alpha \partial^\beta K(x-y) - M_{\alpha, \beta}^\lambda(x, y) \right|. \quad (3.26)$$

The previous discussion implies that, for every $\alpha, \beta \in S(3)$, there exists a finite constant B , independent of λ , such that, for every $\alpha, \beta \in S(3)$ and every $x, y \in B_{r_\lambda}$,

$$\left| M_{\alpha, \beta}^\lambda(x, y) - \partial^\alpha \partial^\beta K(x, y) \right| \leq \mathbb{E}[|\partial^\alpha X(x)| \cdot |\partial^\beta Y_\lambda(y) - \partial^\beta X(y)|] \leq B \sqrt{a(\lambda)}, \quad (3.27)$$

where $a(\lambda)$ is defined in (3.24). Now we adopt a strategy close to the one pursued in [NPR19, Section 7.1], which is in turn inspired by [ORW08, RW08]. We fix a large parameter $N > 0$ (independent of λ , and whose value will be clarified later). We denote by Q_0 the square $[0, 1/N]^2$ and denote by $Q_{\mathbf{z}}$ the translation of Q_0 in the direction \mathbf{z}/N , where $\mathbf{z} \in \mathbb{Z}^2$. We write \mathcal{Q} for the collection of all $Q_{\mathbf{z}}$ and, for every $\lambda > 0$ we set $\mathcal{Q}_\lambda := \{Q_{\mathbf{z}} : Q_{\mathbf{z}} \cap B_{r_\lambda} \neq \emptyset\}$, in such a way that $|\mathcal{Q}_\lambda| = O(r_\lambda^2)$, as $\lambda \rightarrow \infty$, where the constant implicitly involved in such a relation only depends on the choice of N . Fix a small number $\epsilon > 0$.

Definition 3.1. *We say that two cubes $Q_{\mathbf{x}}$ and $Q_{\mathbf{y}}$ are singular if there exists $(x, y) \in Q_{\mathbf{x}} \times Q_{\mathbf{y}}$ such that, for some $\alpha, \beta \in S(1)$, $K_{\alpha\beta}(x, y) > \epsilon$.*

We will need the following technical lemma.

Lemma 3.2. (i) *It is possible to choose N large enough in order to have the following property: if $K_{\alpha\beta}(x-y) > \epsilon$ for some $\alpha, \beta \in S(1)$ and some $(x, y) \in Q_{\mathbf{x}} \times Q_{\mathbf{y}}$, then $K_{\alpha\beta}(a-b) > \epsilon/2$ for every $(a, b) \in Q_{\mathbf{x}} \times Q_{\mathbf{y}}$.*

(ii) *For $\lambda > 0$, and $Q \in \mathcal{Q}_\lambda$, write*

$$L(\phi_\lambda^x; Q) := \mathcal{H}^1((\phi_\lambda^x)^{-1}(0) \cap Q). \quad (3.28)$$

Then, for r_λ as above there exists a finite constant D , independent of λ , such that

$$\sup_{Q \in \mathcal{Q}_\lambda} \mathbb{E}[L(\phi_\lambda^x; Q)^2] \leq D, \quad \lambda > 0.$$

Proof. Since the proof of Point (i) is essentially the same of [NPR19, Lemma 7.3], we omit it.

Throughout all the proof of Point (ii), we set $\sigma_\lambda(x) = \sqrt{C_\lambda(x, x)}$. Moreover, we use the convention that $c_i > 0$, $i = 1, 2, \dots$, always denotes a constant that is **independent** of λ .

The proof of Point (ii) is now divided into several steps. Let us fix $Q \in \mathcal{Q}_\lambda$.

Step 1. We claim that $|K(x, y) - 2\pi + \pi\|x - y\|^2| \leq c_1\|x - y\|^3$ for all $x, y \in \mathbb{R}^2$. Indeed, fix $x \in \mathbb{R}^2$, and write $\widehat{K}_x(y) = K(x, y) = \mathbb{E}[X(x)X(y)]$. By definition we have $\widehat{K}_x(x) = \mathbb{E}[X(x)^2] = 2\pi$ and $\nabla\widehat{K}_x(x) = \mathbb{E}[X(x)\nabla X(x)] = 0$. Thus, by a classical Taylor expansion:

$$\widehat{K}_x(y) = 2\pi + \langle (\text{Hess } \widehat{K}_x)(x)(y - x), y - x \rangle + O(\|x - y\|^3), \quad (3.29)$$

where the big O is uniform with respect to x , because the third partial derivatives of K_x are uniformly bounded with respect to x . On the other hand, using that $K(x, y) = (2\pi)J_0(\|x - y\|)$, one can easily compute that $(\text{Hess } \widehat{K}_x)(x) = -\pi I_2$ for all $x \in \mathbb{R}^2$ (with I_2 the 2×2 identity matrix). Plugging this into (3.29), we get the announced inequality, that is,

$$K(x, y) = \widehat{K}_x(y) = 2\pi - \pi\|x - y\|^2 + O(\|x - y\|^3).$$

Step 2. If λ is large enough so that $\frac{1}{2}\sqrt{2\pi} \leq \sigma_\lambda(x) \leq \frac{3}{2}\sqrt{2\pi}$ for all $x \in Q$, we claim that

$$\left| \frac{C_\lambda(x, y)}{\sigma_\lambda(x)\sigma_\lambda(y)} - 1 + \frac{1}{2}\|x - y\|^2 \right| \leq c_2\eta_\lambda\|x - y\|^2 + c_3\|x - y\|^3 \quad (3.30)$$

for all $x, y \in Q$. Indeed, fix $x \in \mathbb{R}^2$, and write $\widehat{C}_{x,\lambda}(y) = \frac{C_\lambda(x, y)}{\sigma_\lambda(x)\sigma_\lambda(y)}$. We have $\widehat{C}_{x,\lambda}(x) = 1$ and $\nabla\widehat{C}_{x,\lambda}(x) = 0$. As a consequence, using in particular that the third partial derivatives of $\widehat{C}_{x,\lambda}$ are equal to that of $\frac{\widehat{K}_x}{2\pi}$ plus a remainder bounded by $O(\eta_\lambda)$, we can write that

$$|\widehat{C}_{x,\lambda}(y) - 1 - \langle (\text{Hess } \widehat{C}_{x,\lambda})(x)(y - x), y - x \rangle| \leq (c_4 + \eta_\lambda)\|x - y\|^3 \leq c_5\|x - y\|^3.$$

On the other hand, we have

$$(\text{Hess } \widehat{C}_{x,\lambda})(x) = -\frac{1}{2}I_2 + ((\text{Hess } \widehat{C}_{x,\lambda})(x) - \frac{1}{2\pi}(\text{Hess } \widehat{K}_x)(x))$$

and the second term of the right-hand side is bounded by η_λ for any $x \in Q$. The desired conclusion follows.

Step 3. According to Kac–Rice, one has

$$\begin{aligned} & \mathbb{E}[L(\phi_\lambda^x; Q)^2] \\ &= \int_{Q \times Q} \mathbb{E}[\|\nabla Y_\lambda(x)\| \|\nabla Y_\lambda(y)\| | Y_\lambda(x) = Y_\lambda(y) = 0] p_{(Y_\lambda(x), Y_\lambda(y))}(0, 0) dx dy. \end{aligned} \quad (3.31)$$

Using classical linear regression for Gaussian vectors, one can write

$$\nabla Y_\lambda(x) = a_{x,y}Y_\lambda(x) + b_{x,y}Y_\lambda(y) + Z_{\lambda,x,y},$$

with $a_{x,y}, b_{x,y}$ two deterministic vectors of \mathbb{R}^2 and $Z_{\lambda,x,y}$ a Gaussian vector independent of $Y_\lambda(x)$ and $Y_\lambda(y)$. As a consequence,

$$\begin{aligned} \mathbb{E}[\|\nabla Y_\lambda(x)\|^2 | Y_\lambda(x) = Y_\lambda(y) = 0] &= \mathbb{E}[\|Z_{\lambda,x,y}\|^2] \\ &\leq \mathbb{E}[\|\nabla Y_\lambda(x)\|^2] \leq \sup_{x \in Q} \mathbb{E}[\|\nabla Y_\lambda(x)\|^2]. \end{aligned}$$

Similarly $\mathbb{E}[\|\nabla Y_\lambda(y)\|^2 | Y_\lambda(x) = Y_\lambda(y) = 0] \leq \sup_{x \in Q} \mathbb{E}[\|\nabla Y_\lambda(x)\|^2]$ implying, by Cauchy-Schwarz, that

$$\mathbb{E}[\|\nabla Y_\lambda(x)\| \|\nabla Y_\lambda(y)\| | Y_\lambda(x) = Y_\lambda(y) = 0] \leq \sup_{x \in Q} \mathbb{E}[\|\nabla Y_\lambda(x)\|^2] \leq c_6, \quad (3.32)$$

the last bound being due to the fact that, for any $x \in Q$:

$$\begin{aligned}\mathbb{E}[\|\nabla Y_\lambda(x)\|^2] &= \mathbb{E}[\|\nabla X(x)\|^2] + \partial_{(1,0)}^2(C_\lambda - K)(x, x) + \partial_{(0,1)}^2(C_\lambda - K)(x, x) \\ &\leq c_7 + \eta_\lambda \leq c_8.\end{aligned}$$

Step 4. Assume that λ is large enough so that $\frac{1}{2}\sqrt{2\pi} \leq \sigma_\lambda(x) \leq \frac{3}{2}\sqrt{2\pi}$ for all $x \in Q$. Using (3.30) we have, for any $x, y \in Q$

$$\begin{aligned}p_{(Y_\lambda(x), Y_\lambda(y))}(0, 0) &= \frac{1}{2\pi\sigma_\lambda(x)\sigma_\lambda(y)\sqrt{1 - \frac{C_\lambda(x, y)^2}{\sigma_\lambda(x)^2\sigma_\lambda(y)^2}}} \leq \frac{c_9}{\sqrt{1 - \frac{C_\lambda(x, y)^2}{\sigma_\lambda(x)^2\sigma_\lambda(y)^2}}} \\ &\leq \frac{c_9}{\sqrt{\left|\frac{1}{2}\|x - y\|^2 - \left|\frac{C_\lambda(x, y)^2}{\sigma_\lambda(x)^2\sigma_\lambda(y)^2} - 1 + \frac{1}{2}\|x - y\|^2\right|\right|}} \\ &\leq \frac{c_{10}}{\|x - y\|\sqrt{|1 - c_{11}\eta_\lambda - c_{12}\|x - y\||}}.\end{aligned}$$

As a consequence, if λ is large enough to ensure that $1 - c_{11}\eta_\lambda \geq \frac{1}{2}$ then, for any $x, y \in Q$ satisfying $\|x - y\| \leq \frac{1}{4c_{12}}$, one has

$$p_{(Y_\lambda(x), Y_\lambda(y))}(0, 0) \leq \frac{c_{13}}{\|x - y\|}.$$

Step 5. Now, let us deal with the opposite situation where $\|x - y\| \geq \frac{1}{4c_{12}}$. For any $u, v \in [\frac{1}{2}\sqrt{2\pi}, \frac{3}{2}\sqrt{2\pi}]$, we can write

$$|2\pi - uv| = \left| \frac{v^2(2\pi - u^2) + 2\pi(2\pi - v^2)}{2\pi + uv} \right| \leq \frac{9}{4}|2\pi - u^2| + |2\pi - v^2|. \quad (3.33)$$

Observe that $|\sigma_\lambda(x)^2 - 2\pi| \leq \eta_\lambda$ for all $x \in Q$ (this is an immediate fact, since $K(x, x) = 2\pi$). Thus, we deduce from (3.33) that

$$|\sigma_\lambda(x)\sigma_\lambda(y) - 2\pi| \leq \frac{13}{4}\eta_\lambda \quad \text{for all } x, y \in Q.$$

This implies in turn that

$$\begin{aligned}&1 - \frac{C_\lambda(x, y)^2}{\sigma_\lambda(x)^2\sigma_\lambda(y)^2} \\ &= 1 - \left\{ \frac{K(x, y)}{2\pi} - \frac{K(x, y) - C_\lambda(x, y)}{\sigma_\lambda(x)\sigma_\lambda(y)} - \frac{K(x, y)}{2\pi\sigma_\lambda(x)\sigma_\lambda(y)} (\sigma_\lambda(x)\sigma_\lambda(y) - 2\pi) \right\}^2 \\ &\geq 1 - \left\{ \frac{|K(x, y)|}{2\pi} + \frac{|K(x, y) - C_\lambda(x, y)|}{\sigma_\lambda(x)\sigma_\lambda(y)} + \frac{|K(x, y)|}{2\pi\sigma_\lambda(x)\sigma_\lambda(y)} |\sigma_\lambda(x)\sigma_\lambda(y) - 2\pi| \right\}^2 \\ &\geq 1 - \left\{ \frac{|K(x, y)|}{2\pi} + \frac{2}{\pi}\eta_\lambda + \frac{2}{\pi}|\sigma_\lambda(x)\sigma_\lambda(y) - 2\pi| \right\}^2 \geq 1 - \left\{ \frac{|K(x, y)|}{2\pi} + \frac{17}{2\pi}\eta_\lambda \right\}^2 \\ &= 1 - \frac{K(x, y)^2}{4\pi^2} - \frac{17}{2\pi^2}\eta_\lambda|K(x, y)| - \frac{289}{4\pi^2}\eta_\lambda^2 \\ &\geq 1 - \frac{K(x, y)^2}{4\pi^2} - c_{14}\eta_\lambda.\end{aligned}$$

But $\inf_{x,y \in \mathbb{R}^2: \|x-y\| \geq \frac{1}{4c_{12}}} \left(1 - \frac{K(x,y)^2}{4\pi^2}\right) := m > 0$. Hence, for $\lambda > 0$ so that $c_{14}\eta_\lambda \leq \frac{m}{2}$, one has

$$1 - \frac{C_\lambda(x,y)^2}{\sigma_\lambda(x)^2 \sigma_\lambda(y)^2} \geq c_{15} \quad \text{for all } x, y \in Q \text{ such that } \|x-y\| \geq \frac{1}{4c_{12}},$$

implying in turn that

$$p_{(Y_\lambda(x), Y_\lambda(y))}(0,0) \leq c_{16}$$

for all $x, y \in Q$ satisfying $\|x-y\| \geq \frac{1}{4c_{12}}$.

Step 6. By merging the conclusions of steps 4 and 5, we arrive at

$$p_{(Y_\lambda(x), Y_\lambda(y))}(0,0) \leq c_{17} \left(\frac{1}{\|x-y\|} \mathbf{1}_{\{\|x-y\| \leq \frac{1}{4c_{12}}\}} + \mathbf{1}_{\{\|x-y\| \geq \frac{1}{4c_{12}}\}} \right).$$

Plugging this and (3.32) into (3.31) leads to

$$\begin{aligned} \mathbb{E}[L(\phi_\lambda^x, Q)^2] &\leq c_{17} \int_{Q \times Q} \left(\frac{1}{\|x-y\|} \mathbf{1}_{\{\|x-y\| \leq \frac{1}{4c_{12}}\}} + \mathbf{1}_{\{\|x-y\| \geq \frac{1}{4c_{12}}\}} \right) dx dy \\ &= c_{17} \int_Q dx \int_{-x+Q} dz \left(\frac{1}{\|z\|} \mathbf{1}_{\{\|z\| \leq \frac{1}{4c_{12}}\}} + \mathbf{1}_{\{\|z\| \geq \frac{1}{4c_{12}}\}} \right) \\ &\leq c_{17} \int_Q dx \int_{B(0, \frac{1}{4c_{12}})} \frac{dz}{\|z\|} + c_{17} \text{Leb}(Q)^2 = c_{18} \text{Leb}(Q) + c_{17} \text{Leb}(Q)^2, \end{aligned}$$

which automatically implies the desired conclusion. \square

From now on, N is fixed in such a way that the property at Point (i) is verified.

Lemma 3.3. *As $\lambda \rightarrow \infty$, the number of singular pairs $(Q, Q') \in \mathcal{Q}_\lambda \times \mathcal{Q}_\lambda$ is $o(r_\lambda^2 \log r_\lambda)$.*

Proof. For every fixed $Q \in \mathcal{Q}_\lambda$, write $Z_\lambda(Q)$ for the number of cubes in \mathcal{Q}_λ that are singular to Q . Then, for an absolute constant A uniquely depending on N, ϵ , one has that

$$Z_\lambda(Q) \leq A \max_{\alpha, \beta \in S(1)} \int_{B_{2r_\lambda} - B_{2r_\lambda}} |K_{\alpha\beta}(x)|^6 dx \leq A \max_{\alpha, \beta \in S(1)} \int_{B_{4r_\lambda}} |K_{\alpha\beta}(x)|^6 dx.$$

Applying [NPR19, Lemma 7.6], via an appropriate change of variables, yields that

$$\int_{B_{4r_\lambda}} |K_{\alpha\beta}(x)|^6 dx = o(\log r_\lambda),$$

and the desired conclusion follows immediately. \square

Remark 3.4. *We observe that uniformly in $x, y \in B_{r_\lambda}$ one has that*

$$\mathbb{E} [\|\nabla X(x)\| \|\nabla X(y)\| \mid X(x) = 0 = X(y)] \quad \text{and} \quad p_{(Y_\lambda(x), Y_\lambda(y))}(0,0)$$

are finite. This follows from the argument leading to relation (3.32) for the former and step 5 of Point (ii) of Lemma 3.2 for the latter.

In the sequel we will denote by $\Sigma := \Sigma(x, y)$, for $x, y \in \mathbb{R}^d$ the covariance matrix of the random vector $(\nabla X(x), \nabla X(y), X(x), X(y))$. We further define the submatrices

$$\begin{cases} \Sigma_{11}(x, y) & := \text{Cov}((\nabla X(x), \nabla X(y))), \\ \Sigma_{22}(x, y) & := \text{Cov}((X(x), X(y))), \\ \Sigma_{12}(x, y) & := \mathbb{E}((\nabla X(x), \nabla X(y))(X(x), X(y))^t), \\ \Sigma_{21}(x, y) & := \Sigma_{12}^t(x, y) \end{cases}$$

where A^t denotes the transpose of the matrix. For the random vector $(\nabla Y_\lambda, Y_\lambda)(x)$, we analogously define $\Sigma^{(\lambda)}$.

Remark 3.5. *The Gaussian vector $(\nabla X(x), \nabla X(y))$, $x, y \in \mathbb{R}^d$, conditionally on the event $\{(X(x), X(y)) = (0, 0)\}$, is distributed as a mean zero Gaussian vector with covariance matrix*

$$\tilde{\Sigma} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}.$$

More explicitly for the case $d = 2$, $\tilde{\Sigma}$ equals

$$\begin{pmatrix} \frac{1}{2} - (K_{x_1})^2/\rho & -K_{x_1}K_{x_2}/\rho & K_{x_1y_1} + KK_{x_1}K_{y_1}/\rho & K_{x_1y_2} + KK_{x_1}K_{y_2}/\rho \\ -K_{x_1}K_{x_2}/\rho & \frac{1}{2} - (K_{x_2})^2/\rho & K_{x_2y_1} + KK_{x_2}K_{y_1}/\rho & K_{x_2y_2} + KK_{x_2}K_{y_2}/\rho \\ K_{x_1y_1} + KK_{x_1}K_{y_1}/\rho & K_{x_2y_1} + KK_{x_2}K_{y_1}/\rho & \frac{1}{2} - (K_{y_1})^2/\rho & -K_{y_1}K_{y_2}/\rho \\ K_{x_1y_2} + KK_{x_1}K_{y_2}/\rho & K_{x_2y_2} + KK_{x_2}K_{y_2}/\rho & -K_{y_1}K_{y_2}/\rho & \frac{1}{2} - (K_{y_2})^2/\rho \end{pmatrix}$$

where $K_{x_i} := \partial_{x_i}K(x, y)$, $K_{y_i} := \partial_{y_i}K(x, y)$, $K_{x_iy_j} := \partial_{x_i}\partial_{y_j}K(x, y)$ for $i, j = 1, 2$ and $\rho := \rho(x, y) := \det(\Sigma_{22})$.

Furthermore, we introduce the following functions:

$$\begin{aligned} F_0(x, y) &:= \mathbb{E} [\|\nabla X(x)\| \|\nabla X(y)\| \mid X(x) = 0 = X(y)] p_{(X(x), X(y))}(0, 0); \\ F_\lambda(x, y) &:= \mathbb{E} [\|\nabla Y_\lambda(x)\| \|\nabla Y_\lambda(y)\| \mid Y_\lambda(x) = 0 = Y_\lambda(y)] p_{(Y_\lambda(x), Y_\lambda(y))}(0, 0); \\ G_\lambda(x, y) &:= \mathbb{E} [\|\nabla X(x)\| \|\nabla Y_\lambda(y)\| \mid X(x) = 0 = Y_\lambda(y)] p_{(X(x), Y_\lambda(y))}(0, 0) \end{aligned}$$

and also

$$\begin{aligned} H_0(x, y) &:= \mathbb{E} [\|\nabla X(x)\| \mid X(x) = 0] \mathbb{E} [\|\nabla X(y)\| \mid X(y) = 0] p_{X(x)}(0)p_{X(y)}(0); \\ H_\lambda(x, y) &:= \mathbb{E} [\|\nabla Y_\lambda(x)\| \mid Y_\lambda(x) = 0] \mathbb{E} [\|\nabla Y_\lambda(y)\| \mid Y_\lambda(y) = 0] p_{Y_\lambda(x)}(0)p_{Y_\lambda(y)}(0); \\ L_\lambda(x, y) &:= \mathbb{E} [\|\nabla X(x)\| \mid X(x) = 0] \mathbb{E} [\|\nabla Y_\lambda(y)\| \mid Y_\lambda(y) = 0] p_{X(x)}(0)p_{Y_\lambda(y)}(0). \end{aligned}$$

We collect some facts in a first lemma, namely Lemma 3.6 below. In a second lemma, Lemma 3.7 below, we combine those facts together with Kac–Rice formulae to obtain useful bounds on the difference between the nodal lengths of two Gaussian random fields.

Lemma 3.6. *Assume that $\eta_\lambda \rightarrow 0$, as $\lambda \rightarrow \infty$. Then*

i) *there exists a $t_0 > 0$ such that*

$$\inf_{x, y: \|x-y\| > t_0} \det(\tilde{\Sigma}(x, y)) > 0 \tag{3.34}$$

holds;

ii) for $\alpha \in \mathbb{R}^{2d}$, A a positive definite matrix denote by $f(\alpha, A)$ the density of the multivariate normal distribution. We have that

$$|f(\alpha, \tilde{\Sigma}(x, y)) - f(\alpha, \tilde{\Sigma}^{(\lambda)}(x, y))| \leq P(\alpha) f(\alpha, \hat{\Sigma}(x, y)) \eta_\lambda$$

where $\hat{\Sigma}(x, y) \in \text{Conv}(\{\Sigma(x, y), \tilde{\Sigma}(x, y)\})$ and $P(\alpha)$ is a polynomial in α .

Proof.

i) Using Theorem II in [Ger31], it follows that each eigenvalue of $\tilde{\Sigma}(x, y)$ lies in an open ball around some diagonal element $\tilde{\Sigma}(i, i)$, $1 \leq i \leq d$, where the radius is given by $R_i := \sum_{j \neq i} |\tilde{\Sigma}(i, j)|$.

By straightforward computations we obtain the relations:

$$\rho(x, y) := \det(\Sigma_{22}(x, y)) = 1 - J_0^2(\|x - y\|), \quad K_{x_i} = -J_1(\|x - y\|) \frac{x_i - y_i}{\|x - y\|}$$

and $K_{x_i} = -K_{y_i}$, $1 \leq i \leq d$. Furthermore $K_{x_i y_j}$ equals

$$\frac{1}{2} (J_0(\|x - y\|) - J_2(\|x - y\|)) \frac{(x_i - y_i)(x_j - y_j)}{\|x - y\|^2} - J_1(\|x - y\|) \frac{(x_i - y_i)(x_j - y_j)}{\|x - y\|^3}.$$

For every $n \geq 0$, it is well known that $J_n(z) \rightarrow 0$, as $z \rightarrow \infty$. Hence, it follows that any eigenvalue of $\tilde{\Sigma}(x, y)$ is strictly positive, whenever $\|x - y\|$ is large enough.

ii) Recall that the formula for the density of a multivariate normal random vector is

$$\frac{1}{((2\pi)^{2d} \det(A))^{1/2}} \exp\left(\frac{-\alpha^t \text{adj}(A) \alpha}{2 \det(A)}\right)$$

where $\alpha \in \mathbb{R}^{2d}$ and $\text{adj}(\cdot)$ is the adjugate matrix, i.e., the transpose of its cofactor matrix. Further let

$$g_{kl}(A) := -\frac{1}{2} \partial_{a_{kl}} \det(A) \tag{3.35}$$

$$h_{kl}(\alpha, A) := -\frac{1}{2} \left\{ \alpha^t (\det(A) \partial_{a_{kl}} \text{adj}(A) - \text{adj}(A) \partial_{a_{kl}} \det(A)) \alpha \right\} \tag{3.36}$$

where $\partial_{a_{kl}} A$, $1 \leq k, l \leq 2d$ is the partial derivative w.r.t. the $A(k, l)$ -th element. Straightforward calculations show that $\partial_{a_{kl}} f(\alpha, A)$ equals

$$f(\alpha, A) \left(\frac{g_{kl}(A)}{\det(A)} + \frac{h_{kl}(\alpha, A)}{\det(A)^2} \right) \tag{3.37}$$

By the mean value inequality, taking derivatives w.r.t. the matrix entries, one has for all $\alpha \in \mathbb{R}^{2d}$ for some $c \in]0, 1[$ that

$$|f(\alpha, \tilde{\Sigma}) - f(\alpha, \tilde{\Sigma}^{(\lambda)})| \leq |\nabla f(\alpha, c\tilde{\Sigma} + (1-c)\tilde{\Sigma}^{(\lambda)})| \|\tilde{\Sigma} - \tilde{\Sigma}^{(\lambda)}\|_{op}.$$

Since the operator norm is bounded by the Hilbert–Schmidt norm, one infers $\|\tilde{\Sigma} - \tilde{\Sigma}^{(\lambda)}\|_{op} \leq C_1 \eta_\lambda$ for some constant C_1 independent of α . By continuity of the determinant and the uniform convergence of the covariance matrices, one has for λ large enough that

$$\det(\tilde{\Sigma})/2 \leq \det(c\tilde{\Sigma} + (1-c)\tilde{\Sigma}^{(\lambda)}) \leq 2 \det(\tilde{\Sigma}).$$

Combining this with Point (i), we see that $\det(c\tilde{\Sigma} + (1-c)\tilde{\Sigma}^{(\lambda)})^k$, $k = -1, -2$ are bounded by a constant. Furthermore, since Bessel functions are bounded and determinants are monomials, it follows that $g_{kl}(c\tilde{\Sigma} + (1-c)\tilde{\Sigma}^{(\lambda)})$ is bounded by a constant times η_λ and $h_{kl}(\alpha, c\tilde{\Sigma} + (1-c)\tilde{\Sigma}^{(\lambda)})$ is bounded by η_λ times a polynomial in α .

□

Recall the definition of η_λ in (3.23) and the definition of ζ_λ in (3.26).

Lemma 3.7. *For every $x, y \in B_{r_\lambda}$, such that $|x - y| > t_0$, with t_0 as found in Lemma 3.6, it holds that*

$$\begin{cases} |F_0(x, y) - F_\lambda(x, y)|, |H_0(x, y) - H_\lambda(x, y)| &= O(\eta_\lambda); \\ |F_0(x, y) - G_\lambda(x, y)|, |H_0(x, y) - L_\lambda(x, y)| &= O(\zeta_\lambda); \end{cases}$$

as $\lambda \rightarrow \infty$, where the constants depend only on t_0 and the dimension.

Proof. We only prove the first relation $|F_0(x, y) - F_\lambda(x, y)| = O(\eta_\lambda)$, since the others are proven in an analogous way.

By Remark 3.4 and the triangle inequality, it then suffices to bound

$$|p_{(X(x), X(y))}(0, 0) - p_{(Y_\lambda(x), Y_\lambda(y))}(0, 0)| \quad (3.38)$$

and

$$\begin{aligned} & \left| \mathbb{E} [\|\nabla X(x)\| \|\nabla X(y)\| \mid X(x) = 0 = X(y)] \right. \\ & \left. - \mathbb{E} [\|\nabla Y_\lambda(x)\| \|\nabla Y_\lambda(y)\| \mid Y_\lambda(x) = 0 = Y_\lambda(y)] \right| \end{aligned} \quad (3.39)$$

We will now bound (3.38). By continuity of the determinant and the uniform convergence of the covariance matrices, one has for λ large enough that

$$\frac{1}{2} \sqrt{\det(\Sigma_{22}(x, y))} \leq \sqrt{\det(\Sigma_{22}^{(\lambda)}(x, y))} \leq \frac{3}{2} \sqrt{\det(\Sigma_{22}(x, y))}.$$

Applying the bound: $|1/\sqrt{x} - 1/\sqrt{y}| \leq |y - x|((\sqrt{y} + \sqrt{x})\sqrt{xy})^{-1}$ together with Point (i) of Lemma 3.6, the continuity of the determinant and the boundedness of Bessel functions we find that

$$\left| \det(\Sigma_{22}(x, y))^{-1/2} - \det(\Sigma_{22}^{(\lambda)}(x, y))^{-1/2} \right| = O(\eta_\lambda)$$

as $\lambda \rightarrow \infty$. Again, from the continuity of the determinant we infer that (3.38) is smaller than $\gamma_1 \eta_\lambda$, for some constant γ_1 .

Next we turn to (3.39). By definition $\mathbb{E} [\|\nabla X(x)\| \|\nabla X(y)\| \mid X(x) = 0 = X(y)]$ equals

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\left(\sum_{i=1}^d \alpha_i^2 \right)^{1/2} \left(\sum_{i=d+1}^{2d} \alpha_i^2 \right)^{1/2}}{\left((2\pi)^{2d} \det(\tilde{\Sigma}(x, y)) \right)^{1/2}} \exp \left(\frac{-\alpha^t \text{adj}(\tilde{\Sigma}(x, y)) \alpha}{2 \det(\tilde{\Sigma}(x, y))} \right) d\alpha$$

where $\text{adj}(\cdot)$ is the adjugate matrix, i.e., the transpose of its cofactor matrix. Now $\mathbb{E} [\|\nabla Y_\lambda(x)\| \|\nabla Y_\lambda(y)\| \mid Y_\lambda(x) = 0 = Y_\lambda(y)]$ satisfies a similar identity with $\tilde{\Sigma}^{(\lambda)}(x, y)$ instead of $\tilde{\Sigma}(x, y)$.

From Point (ii) of Lemma 3.6 we infer (3.39) is less than

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \left(\sum_{i=1}^d \alpha_i^2 \right)^{1/2} \left(\sum_{i=d+1}^{2d} \alpha_i^2 \right)^{1/2} P(\alpha) f(\alpha, \hat{\Sigma}(x, y)) \eta_\lambda d\alpha$$

where $\hat{\Sigma} := c\tilde{\Sigma} + (1-c)\tilde{\Sigma}^{(\lambda)}$, for some $c \in]0, 1[$. Now the matrix $\hat{\Sigma}$ is symmetric and positive definite. Since every moment of a multivariate normal is, up to a constant, bounded by its second moment, one has that

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \left(\sum_{i=1}^d \alpha_i^2 \right)^{1/2} \left(\sum_{i=d+1}^{2d} \alpha_i^2 \right)^{1/2} f(\alpha, \hat{\Sigma}(x, y)) P(\alpha) d\alpha$$

is a polynomial in $\hat{\Sigma}(x, y)$. Because Bessel functions are bounded, the above integral is bounded. Hence (3.39) is $O(\eta_\lambda)$, for λ large enough. \square

We eventually point out the following elementary fact: if a sequence of two dimensional random vectors $\{(U_n, V_n) : n \geq 1\}$ is such that

$$\mathbb{E}[(U_n - V_n)^2] \rightarrow 0 \quad \text{and} \quad \mathbb{E}[U_n^2] \rightarrow 1, \quad (3.40)$$

then $\mathbb{E}[V_n^2] \rightarrow 1$.

3.2 The proof

For any subset $A \subset \mathbb{R}^2$ and for $W = X, Y_\lambda$ ($\lambda > 0$) we write

$$L(W; A) := \mathcal{H}^1(W^{-1}(0) \cap A),$$

with the simplified notation $L(W; r) := L(W; B_r)$. The first assertion in (1.13) already appears in (1.9). In view of the elementary fact recalled in (3.40), the second assertion in (1.13) will follow immediately, once we prove that, as $\lambda \rightarrow \infty$,

$$\mathbf{V}(\lambda) := \mathbf{Var} [(L(X; r_\lambda) - L(Y_\lambda; r_\lambda))] = o(r_\lambda^2 \log r_\lambda). \quad (3.41)$$

For every $\lambda > 0$ denote by \mathcal{B}_λ the collection of all subsets of \mathbb{R}^2 having the form $Q \cap B_{r_\lambda}$, with $Q \in \mathcal{Q}_\lambda$. A pair $(B, B') \in \mathcal{B}_\lambda \times \mathcal{B}_\lambda$ is *singular* if the underlying pair of squares (Q, Q') is. Our starting point is the obvious decomposition

$$\mathbf{V}(\lambda) = \mathbf{V}_s(\lambda) + \mathbf{V}_{ns}(\lambda),$$

where

$$\mathbf{V}_s(\lambda) := \sum_{(B, B') \in \mathcal{B}_\lambda \times \mathcal{B}_\lambda \text{ singular}} \text{Cov} \left\{ (L(X; B) - L(Y_\lambda; B)), (L(X; B') - L(Y_\lambda; B')) \right\},$$

and

$$\mathbf{V}_{ns}(\lambda) := \sum_{(B, B') \in \mathcal{B}_\lambda \times \mathcal{B}_\lambda \text{ non singular}} \text{Cov} \left\{ (L(X; B) - L(Y_\lambda; B)), (L(X; B') - L(Y_\lambda; B')) \right\}.$$

Since $\mathbb{E}[L(X, B)^2] \leq \mathbb{E}[L(X, Q_0)^2]$ (by stationarity) and $\mathbb{E}[L(Y_\lambda, B)^2] \leq D$ (by virtue of Lemma 3.2-(i)), applying Cauchy-Schwarz and the triangle inequality yields that

$$\mathbf{V}_s(\lambda) = O\left(\left| \{(B, B') \in \mathcal{B}_\lambda \times \mathcal{B}_\lambda \text{ singular}\} \right|\right),$$

and therefore $\mathbf{V}_s(\lambda) = o(r_\lambda^2 \log r_\lambda)$, in view of Lemma 3.3. We now decompose $\mathbf{V}_{ns}(\lambda)$ as

$$\begin{aligned} \mathbf{V}_{ns}(\lambda) &= \sum_{(B, B') \text{ non singular}} [\text{Cov}\{L(X; B), L(X; B')\} + \text{Cov}\{L(Y_\lambda; B), L(Y_\lambda; B')\} \\ &\quad - 2\text{Cov}\{L(X; B), L(Y_\lambda; B')\}] \\ &=: \mathbf{V}_1(\lambda) + \mathbf{V}_2(\lambda) - 2\mathbf{V}_3(\lambda). \end{aligned}$$

Now call $P_\lambda \subset B_{r_\lambda} \times B_{r_\lambda}$ the union of all cartesian product of the type $B \times B'$ such that (B, B') is not singular. Exploiting the definition of non-singular pairs together with property (3.27), for λ large enough we can represent each of the quantities $\mathbf{V}_i(\lambda)$, $i = 1, 2, 3$, by means of the Kac–Rice formula (or some slight variation of it — see [AW09, Chapter 6]), as follows

$$\begin{aligned} \mathbf{V}_1(\lambda) &= \int_{P_\lambda} \mathbb{E}[\|\nabla X(x)\| \|\nabla X(y)\| \mid X(x) = X(y) = 0] p_{(X(x), X(y))}(0, 0) dx dy \\ &\quad - \int_{P_\lambda} \mathbb{E}[\|\nabla X(x)\| \mid X(x) = 0] \mathbb{E}[\|\nabla X(y)\| \mid X(y) = 0] p_{X(x)}(0) p_{X(y)}(0) dx dy \\ &=: \int_{P_\lambda} H_1(x, y) dx dy; \\ \mathbf{V}_2(\lambda) &= \int_{P_\lambda} \mathbb{E}[\|\nabla Y_\lambda(x)\| \|\nabla Y_\lambda(y)\| \mid Y_\lambda(x) = Y_\lambda(y) = 0] p_{(Y_\lambda(x), Y_\lambda(y))}(0, 0) dx dy \\ &\quad - \int_{P_\lambda} \mathbb{E}[\|\nabla Y_\lambda(x)\| \mid Y_\lambda(x) = 0] \mathbb{E}[\|\nabla Y_\lambda(y)\| \mid Y_\lambda(y) = 0] p_{Y_\lambda(x)}(0) p_{Y_\lambda(y)}(0) dx dy \\ &=: \int_{P_\lambda} H_2(x, y) dx dy; \\ \mathbf{V}_3(\lambda) &= \int_{P_\lambda} \mathbb{E}[\|\nabla X(x)\| \|\nabla Y_\lambda(y)\| \mid X(x) = Y_\lambda(y) = 0] p_{(X(x), Y_\lambda(y))}(0, 0) dx dy \\ &\quad - \int_{P_\lambda} \mathbb{E}[\|\nabla X(x)\| \mid X(x) = 0] \mathbb{E}[\|\nabla Y_\lambda(y)\| \mid Y_\lambda(y) = 0] p_{X(x)}(0) p_{Y_\lambda(y)}(0) dx dy \\ &=: \int_{P_\lambda} H_3(x, y) dx dy, \end{aligned}$$

where $p_U(0)$ denotes the density of U in 0, and $p_{(U, V)}(0, 0)$ denotes the density of (U, V) in $(0, 0)$. Exploiting the fact that each P_λ only involves non-singular pairs, we wish now to prove that, for every $1 \leq j \neq k \leq 3$, and for some constant C independent of λ ,

$$\int_{P_\lambda} |H_j(x, y) - H_k(x, y)| dx dy \leq C \sqrt{a(\lambda)} |P_\lambda| = O\left(\left[\frac{r_\lambda^{17}}{\log \lambda}\right]^{1/4} r_\lambda^4\right). \quad (3.42)$$

Once such a relation is established, the proof will be finished, since from condition (1.12) we deduce that (3.42) $= o(r_\lambda^2 \log(r_\lambda))$, and by virtue of the triangle inequality it holds that

$$|V_{ns}(\lambda)| \leq 2|V_1(\lambda) - V_3(\lambda)| + |V_1(\lambda) - V_2(\lambda)|.$$

The validity of (3.42) for every j, k is immediately deduced from Lemma 3.7, as well as (3.23) and (3.27), taking into account the definition (3.24).

4 Proof of technical estimates

4.1 Eigenvalues of Hilbert–Schmidt operators on expanding domains

In the present and following section, we adopt the notation and assumptions of Section 2.2. In particular, we fix an integer $d \geq 1$, and consider two smooth covariance kernels K, C on \mathbb{R}^d , such that K has bounded derivatives of every order; the symbol B_R indicates the open ball with radius R centred at the origin. Our proof of Theorem 2.2 relies on a number of results involving Hilbert–Schmidt operators associated with the class of increasing domains B_R , $R \geq 1$, that we will gather below.

A simple geometric fact. For every $A \subset \mathbb{R}^d$ and every $n \geq 2$, we use the notation

$$\varepsilon_n(A) := \inf \{ \varepsilon > 0 : A \text{ can be covered by } n - 1 \text{ open balls of radius } \varepsilon \text{ centred in } A \}.$$

We will make use of the following elementary geometric fact

Lemma 4.1. *There exist constants $0 < \beta_1 < \beta_2 < \infty$ such that*

$$\frac{\beta_1}{n^{1/d}} \leq \frac{\varepsilon_n(\overline{B_R})}{R} \leq \frac{\beta_2}{n^{1/d}}, \quad \text{for every } n, R \geq 1.$$

Proof. Consider first the case $R = 1$, and denote by Q the hypercube with unit side centred at the origin. It is easily seen that $\varepsilon_n(Q) \leq \beta n^{-1/d}$ for some absolute constant β , and also that $\varepsilon_n(Q) \geq \varepsilon_n(\overline{B_1})$. To conclude the proof of the upper bound, observe that there exists an ε -cover of cardinality $n - 1$ of $\overline{B_1}$ if and only if there exists an $(R\varepsilon)$ -cover of cardinality $n - 1$ of $\overline{B_R}$. The lower bound follows from the following observation: for every $\varepsilon > \varepsilon_n(\overline{B_R})$, we have that

$$\text{Vol}(\overline{B_R}) \leq (n - 1)\text{Vol}(B_\varepsilon).$$

□

A class of Hilbert–Schmidt operators. Given a finite set J , we denote by $L_J^2(B_R)$ the Hilbert space of (equivalence classes of) square-integrable \mathbb{R}^J -valued functions on B_R , endowed with the following inner product: for $f = \{f_j : j \in J\}$, $g = \{g_j : j \in J\} \in L_J^2(B_R)$,

$$\langle f, g \rangle_{L_J^2(B_R)} = \sum_{j \in J} \int_{B_R} f_j(x) g_j(x) dx.$$

Recalling the definition of $S(M)$ given in (2.17), for $M = 0, 1, \dots$ and $R \geq 1$, we define the operator

$$\begin{aligned} K^{(M,R)} &: L_{S(M)}^2(B_R) \rightarrow L_{S(M)}^2(B_R) \\ &: f = \{f_\alpha : \alpha \in S(M)\} \mapsto K^{(M,R)} f = \left\{ (K^{(M,R)} f)_\alpha : \alpha \in S(M) \right\}, \end{aligned} \quad (4.43)$$

with

$$(K^{(M,R)} f)_\alpha(x) := \sum_{\beta \in S(M)} \int_{B_R} K_{\alpha\beta}(x, y) f_\beta(y) dy, \quad x \in B_R.$$

Note that the quantity $(K^{(M,R)} f)_\alpha(x)$ is also unambiguously defined for every x in the complement of $\overline{B_R}$. It is easily checked that, for every choice of M, R as above, $K^{(M,R)}$ is compact, self-adjoint, Hilbert–Schmidt and positive definite.

The eigenvalues of $K^{(M,R)}$ are denoted by

$$\lambda_1^{(M,R)} \geq \lambda_2^{(M,R)} \geq \dots \geq 0.$$

We record the following consequence of the matrix-valued Mercer's theorem (see e.g. [dVUV13, Theorem 4.1]).

Proposition 4.2. *Under the above notation and assumptions, let*

$$\{e^j = \{e_\alpha^j : \alpha \in S(M)\} : j \geq 1\}$$

be an orthonormal basis of $\overline{\text{Im } K^{(M,R)}}$ (that is, of the closure of the image of $K^{(M,R)}$ in $L^2(B_R)$) composed of continuous eigenfunctions of $K^{(M,R)}$. Then, for every $\alpha, \beta \in S(M)$,

$$K_{\alpha\beta}(x, y) = \sum_{j=1}^{\infty} \lambda_j^{(M,R)} e_\alpha^j(x) e_\beta^j(y),$$

where the series converges uniformly on every compact subset of $B_R \times B_R$. This implies that $K^{(M,R)}$ is trace class and

$$\text{Tr } K^{(M,R)} = \sum_{j=1}^{\infty} \lambda_j^{(M,R)} = \sum_{\alpha \in S(M)} \int_{B_R} K_{\alpha\alpha}(x, x) dx. \quad (4.44)$$

Eigenvalue decay. Under the above introduced notation, we will now prove a useful estimate, yielding an upper bound on the decay of the eigenvalues of $K^{(M,R)}$. Observe that bounds on the eigenvalue decay for kernel operators such as $K^{(M,R)}$ are already available in the literature – see e.g. the classical (and somehow definitive) reference [Kuh86], as well as [HK84]. However, such estimates are typically provided for kernels defined on *fixed domains*, whereas the applications developed in the present paper require bounds on eigenvalues where the dependence on the ‘increasing radius’ R appears explicitly. The next lemma, whose proof is partially inspired by the arguments developed in [HK84], shows that the dependence on the radius R is indeed sub-algebraic.

Lemma 4.3 (Eigenvalue decay for kernel operators). *For every $M = 0, 1, \dots$ and every $\ell \geq 0$, there exists a constant $A = A(M, d, \ell) \in (0, \infty)$, exclusively depending on M, d, ℓ , such that*

$$\lambda_n^{(M,R)} \leq A \frac{R^{\ell+1+d}}{n^{\frac{\ell+1}{d}}}, \quad n \geq 1. \quad (4.45)$$

Proof. Since $K^{(M,R)}$ is a positive self-adjoint operator, one has that, by Theorem 2.1 in Chapter II of [GK69]

$$\lambda_n^{(M,R)} = \inf \{ \|K^{(M,R)} - U\|_{op} : \text{rank } U < n \},$$

where the compact notation indicates that the infimum is taken over all linear operators $U : L^2_{S(M)}(B_R) \rightarrow L^2_{S(M)}(B_R)$ having a rank strictly less than n . For every smooth mapping ψ on \mathbb{R}^d every $x_0 \in \mathbb{R}^d$ and every $\ell = 0, 1, \dots$, we denote by

$$x \mapsto P_\ell(\psi, x_0)(x)$$

the Taylor polynomial of order ℓ associated with ψ and centered at x_0 . Observe that $\mapsto P_\ell(\psi, x_0)$ involves a number $r = r(d, \ell)$ of monomials (each centred at x_0), such that the

number r only depends on d and ℓ . Now fix n and take $\varepsilon > \varepsilon_n(\overline{B_R})$. Then, there exists a collection $\mathcal{B} = \{B(x_1, \varepsilon), \dots, B(x_{n-1}, \varepsilon)\}$ of open balls centered in $\overline{B_R}$ and covering $\overline{B_R}$. Let $\phi = \{\phi_1, \dots, \phi_{n-1}\}$ be a partition of unity subordinated to \mathcal{B} . It is an elementary fact (see e.g. [Rud86, Theorem 2.13]) that one can choose ϕ in such a way that each ϕ_j is supported in the ball $B(x_j, \varepsilon)$. Define the operator

$$\begin{aligned} U &: L^2_{S(M)}(B_R) \rightarrow L^2_{S(M)}(B_R) \\ &: f = \{f_\alpha : \alpha \in S(M)\} \mapsto Uf = \{(Uf)_\alpha : \alpha \in S(M)\} \end{aligned}$$

by the relation

$$(Uf)_\alpha(x) := \sum_{j=1}^{n-1} \phi_j(x) P_\ell((K^{(M,R)}f)_\alpha, x_j)(x), \quad x \in B_R,$$

and observe that, in view of the previous considerations, $\text{rank } U \leq |S(M)| r(d, \ell) \times (n-1) := \gamma \times (n-1)$. Also, by inverting derivation and integration, one sees that

$$P_\ell((K^{(M,R)}f)_\alpha, x_j)(x) = \sum_{\beta \in S(M)} \int_{B_R} P_\ell(K_{\alpha\beta}(\cdot, y), x_j)(x) f_\beta(y) dy,$$

from which we infer that, for some absolute constant A and every $x \in B_R$

$$|(K^{(M,R)}f)_\alpha(x) - (Uf)_\alpha(x)| \leq A \sum_{j=1}^{n-1} \phi_j(x) \|x - x_j\|^{\ell+1} \left(\sum_{\beta \in S(M)} \int_{B_R} |f_\beta(y)| dy \right).$$

We stress that the last inequality uses the fact that, by assumption, the derivatives of K are bounded. Now, by virtue of the properties of ϕ one has the estimate $\sum_{j=1}^{n-1} \phi_j(x) \|x - x_j\|^{\ell+1} \leq \varepsilon^{\ell+1}$ (for every $x \in B_R$), and applying Cauchy-Schwarz we finally obtain that

$$\|K^{(M,R)} - U\|_{op} \leq A\varepsilon^{\ell+1} R^d.$$

Letting ε converge to $\varepsilon_n(\overline{B_R})$, applying Lemma 4.1 and using the fact that $\text{rank } U \leq \gamma(n-1)$ (where the integer γ has been defined above, and only depends on M, d, ℓ), we see that (4.45) holds for integers of the form $n = \gamma k$, where $k \geq 1$. The fact that the conclusion is indeed true for every positive integer n follows from standard arguments. \square

Square roots. For every $M = 0, 1, \dots$ and $R \geq 1$, we define $\sqrt{K^{(M,R)}}$ to be the Hilbert-Schmidt operator on $L^2_{S(M)}(B_R)$ defined by the following relations: if $\{e^j : j \geq 1\}$ denotes a continuous orthonormal basis of $\overline{\text{Im } K^{(M,R)}}$, then, for every $h \in L^2_{S(M)}(B_R)$

$$\sqrt{K^{(M,R)}}h := \{(\sqrt{K^{(M,R)}}h)_\alpha : \alpha \in S(M)\},$$

with

$$(\sqrt{K^{(M,R)}}h)_\alpha(x) = \sum_{j=1}^{\infty} \sum_{\beta \in S(M)} \sqrt{\lambda_j^{(M,R)}} e_\alpha^j(x) \int_{B_R} e_\beta^j(y) h(y) dy,$$

with convergence in $L^2(B_R)$. If C denotes the other covariance kernel of class $\mathcal{C}^{\infty, \infty}$ introduced in Section 2.2, we define the operator $C^{(M,R)}$ (for every $M = 0, 1, \dots$ and $R \geq 1$) in the same way as above, and write

$$\gamma_1^{(M,R)} \geq \gamma_2^{(M,R)} \geq \dots \geq 0,$$

to indicate the sequence of its eigenvalues. We also define analogously the square root $\sqrt{C^{(M,R)}}$. The next estimate is crucial for our arguments.

Proposition 4.4. *For K, C as above, for every $M = 0, 1, \dots$ and $R \geq 1$, one has that*

$$\begin{aligned} \|\sqrt{K^{(M,R)}} - \sqrt{C^{(M,R)}}\|_{H.S.}^2 &\leq |\mathrm{Tr} K^{(M,R)} - \mathrm{Tr} C^{(M,R)}| + \\ &2\|K^{(M,R)} - C^{(M,R)}\|_{H.S.}^{1/2} \cdot \mathrm{Tr} \sqrt{K^{(M,R)}} \end{aligned} \quad (4.46)$$

and moreover

$$\mathrm{Tr} \sqrt{K^{(M,R)}} \leq B \cdot R^{\frac{3d+1}{2}}, \quad (4.47)$$

where $B = B(M, d)$ is a finite constant uniquely depending on M, d .

Proof. For the rest of the proof, we use the notation

$$\alpha := |\mathrm{Tr} K^{(M,R)} - \mathrm{Tr} C^{(M,R)}|, \quad \beta := \|K^{(M,R)} - C^{(M,R)}\|_{H.S.}.$$

Then $\beta \geq \|K^{(M,R)} - C^{(M,R)}\|_{op} \geq \|\sqrt{K^{(M,R)}} - \sqrt{C^{(M,R)}}\|_{op}^2$. Since we have compact operators, this follows from the finite dimensional case. For a proof of the latter, see for instance Theorem V.1.9. in [Bha97]. Writing moreover $D := \sqrt{K^{(M,R)}} - \sqrt{C^{(M,R)}}$ and choosing an orthonormal basis $\{e^j : j \geq 1\}$ diagonalizing $K^{(M,R)}$, we infer that

$$\begin{aligned} \|D\|_{H.S.}^2 &= \|\sqrt{K^{(M,R)}}\|_{H.S.}^2 + \|\sqrt{C^{(M,R)}}\|_{H.S.}^2 - 2\langle \sqrt{K^{(M,R)}}, \sqrt{C^{(M,R)}} \rangle_{H.S.} \\ &\leq \alpha + 2 \left| \langle D, \sqrt{K^{(M,R)}} \rangle_{H.S.} \right| \\ &= \alpha + 2 \left| \sum_{j \geq 1} \langle D e^j, \sqrt{K^{(M,R)}} e^j \rangle_{L_{S(M)}^2(B_R)} \right| \\ &\leq \alpha + 2 \|D\|_{op} \sum_{j \geq 1} \|\sqrt{K^{(M,R)}} e^j\|_{L_{S(M)}^2(B_R)} \\ &= \alpha + 2 \|D\|_{op} \mathrm{Tr} \sqrt{K^{(M,R)}}. \end{aligned}$$

The estimate (4.47) follows from (4.45), by selecting $\ell = 2d$. \square

4.2 Proof of Theorem 2.2

Fix $M \geq 0$ and $R \geq 1$, and consider a probability space $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$ supporting an isonormal Gaussian process over $L_{S(M)}^2(B_R)$, written

$$G = \{G(h) : h \in L_{S(M)}^2(B_R)\}.$$

Recall that, by definition, G is a centred Gaussian family, indexed by the elements of $L_{S(M)}^2(B_R)$ and such that

$$\mathbb{E}_0[G(h)G(h')] = \langle h, h' \rangle_{L_{S(M)}^2(B_R)};$$

observe that G exists for every choice of M and R – see [NP12, Proposition 2.2.1]. We denote by $\{e^j : j \geq 1\}$ and $\{g^j : j \geq 1\}$ two orthonormal systems of continuous functions, composed respectively of eigenfunctions of $K^{(M,R)}$ and $C^{(M,R)}$ with non-zero eigenvalues; we assume that e^j has eigenvalue $\lambda_j^{(M,R)}$ and g^j has eigenvalue $\gamma_j^{(M,R)}$. For every $N \geq 1$ and $x \in B_R$, we set

$$\begin{aligned} X^N(x) &:= \sum_{j=1}^N \sqrt{\lambda_j^{(M,R)}} G(f^j) f^j(x), \\ Y^N(x) &:= \sum_{j=1}^N \sqrt{\gamma_j^{(M,R)}} G(g^j) g^j(x). \end{aligned}$$

Now, for every fixed $x \in B_R$ and $\alpha \in S(M)$, the sequence

$$X_\alpha^N(x) = \sum_{j=1}^N \sqrt{\lambda_j^{(M,R)}} G(f^j) f_\alpha^j(x), \quad N \geq 1,$$

is Cauchy in $L^2(\mathbb{P}_0)$ and a.s.- \mathbb{P}_0 (by virtue of Lévy Theorem – see [NP12, p. 23]), since $K_{\alpha\alpha}(x, x) = \sum_{j=1}^\infty \lambda_j^{(M,R)} f_\alpha^j(x)^2$, were we have applied Proposition 4.2. Denote by $\hat{X}_\alpha(x)$ the limit of $X_\alpha^N(x)$. The following facts can be verified by a standard application of the same arguments that lead to the proof of Kolmogorov’s Theorem (see e.g., [NS16, Appendix A.10]):

- the process $\{\hat{X}(x) = \{\hat{X}_\alpha(x) : \alpha \in S(M)\}, x \in B_R\}$ admits a modification $\{X(x) = \{X_\alpha(x) : \alpha \in S(M)\}$ such that X_0 (where 0 indicates the zero multi-index) is an element of C_b^∞ with \mathbb{P}_0 -probability one and, \mathbb{P}_0 -almost surely for every $x \in B_R$, $X_\alpha(x) = \partial^\alpha X_0(x)$, for every $\alpha \in S(M)$;
- for every $\alpha, \beta \in S(M)$ and for every $x, y \in B_R$

$$\mathbb{E}[X_\alpha(x) X_\beta(y)] = K_{\alpha\beta}(x, y);$$

- for every $\alpha \in S(M)$ the real valued Gaussian field X_α^N converges to X_α uniformly on every compact subset of B_R .

We define the process $Y = \{Y_\alpha : \alpha \in S(M)\}$ in a similar way. To conclude, we just observe that, by the isometric properties of G ,

$$\mathbb{E}_0 \left[\|X_0 - Y_0\|_{\mathbb{W}^{M,2}(B_R)}^2 \right] = \|\sqrt{K^{(M,R)}} - \sqrt{C^{(M,R)}}\|_{H.S.}^2,$$

and apply Theorem 2, together with the estimates

$$\begin{aligned} \left| \text{Tr } K^{(M,R)} - \text{Tr } C^{(M,R)} \right| &\leq \sum_{\alpha \in S(M)} \int_{B_R} |K_{\alpha\alpha}(x, x) - C_{\alpha\alpha}(x, x)| dx \leq |S(M)| |B_R| \eta, \\ \|K^{(M,R)} - C^{(M,R)}\|_{H.S.}^{1/2} &= \left\{ \sum_{\alpha, \beta \in S(M)} \int_{B_R} \int_{B_R} (K_{\alpha\beta}(x, y) - C_{\alpha\beta}(x, y))^2 dx dy \right\}^{1/4} \\ &\leq |S(M)|^{1/2} |B_R|^{1/2} \sqrt{\eta}. \end{aligned}$$

Relation (2.20) follows from an application of the following consequence of Sobolev’s embedding theorem on open subsets of \mathbb{R}^d , such as the one stated in [DD12, Theorem 2.7.2].

Lemma 4.5. For every $d, p \geq 1$, every $m > j := \left\lfloor \frac{d}{p} \right\rfloor + 1$ and every $R \geq 1$, the following estimate holds for every $u \in \mathcal{C}_b^\infty(B_R)$:

$$\|u\|_{\mathcal{C}^{m-j}(B_R)} \leq A \cdot R^{m-\frac{d}{p}} \|u\|_{\mathbb{W}^{m,p}(B_R)}, \quad (4.48)$$

for a constant A independent of R and u .

Proof. The inequality is true for $R = 1$ and some absolute constant A , by virtue of Sobolev embedding. On the other hand, since $R \geq 1$,

$$\|u\|_{\mathcal{C}^{m-j}(B_R)} = \max_{|\alpha| \leq m-j} \|\partial^\alpha u\|_{\infty, B_R} = \max_{|\alpha| \leq m-j} \frac{1}{R^{|\alpha|}} \|\partial^\alpha u^R\|_{\infty, B_1} \leq \|u^R\|_{\mathcal{C}^{m-j}(B_1)},$$

where $u^R(x) := u(Rx)$. The conclusion follows from the relation

$$\|u^R\|_{\mathbb{W}^{m,p}(B_1)} \leq R^{m-\frac{d}{p}} \|u\|_{\mathbb{W}^{m,p}(B_R)},$$

which is a consequence of the equality, valid for every $\alpha \in S(M)$

$$\left\{ \int_{B_1} |\partial^\alpha u^R(x)| dx \right\}^{1/p} = R^{|\alpha|-\frac{d}{p}} \left\{ \int_{B_R} |\partial^\alpha u(y)|^p dy \right\}^{1/p},$$

deduced from the change of variables $y = Rx$. □

5 Proof of Theorem 1.9

For every $n \in S$, write

$$\tilde{T}_n(x) := T_n \left(\frac{x}{2\pi\sqrt{n}} \right),$$

in such a way that

$$\tilde{u}_n(x-y) := \mathbb{E} \left[\tilde{T}_n(x) \tilde{T}_n(y) \right] = \int_{\mathbb{S}^1} e^{i\langle x-y, z \rangle} \mu_n(dz).$$

For every $n \in S$, we also denote by $\mu_n^\#$ the probability measure on the interval $[0, 2\pi]$ characterised by the following relation: for every bounded and measurable f ,

$$\int_{\mathbb{S}^1} f(z) \mu_n(dz) = \int_0^{2\pi} f(e^{i\theta}) \mu_n^\#(d\theta);$$

we also use the symbol $\mathbf{u}(dx)$ for the uniform probability measure on $[0, 2\pi]$. We measure the discrepancy between $\mu_n^\#$ and \mathbf{u} in two ways: (i) **Kolmogorov distance**,

$$\mathbf{Kol}(\mu_n^\#, \mathbf{u}) := \sup_{t \in [0, 2\pi]} \left| \mu_n^\#[0, t] - \frac{t}{2\pi} \right|,$$

and (ii) the 1-**Wasserstein distance** \mathbf{W}_1 defined as

$$\mathbf{W}_1(\mu_n^\#, \mathbf{u}) = \sup_{h \in \text{Lip}(1)} \left| \int_0^{2\pi} h(\theta) \mu_n^\#(d\theta) - \int_0^{2\pi} h(\theta) d\theta \right|,$$

where $\text{Lip}(1)$ denotes the class of all real-valued 1-Lipschitz functions. It is a well known fact that

$$\mathbf{W}_1(\mu_n^\#, \mathbf{u}) = \int_0^{2\pi} \left| \mu_n^\#[0, t] - \frac{t}{2\pi} \right| dt,$$

and therefore $\mathbf{W}_1(\mu_n^\#, \mathbf{u}) \leq 2\pi \mathbf{Kol}(\mu_n^\#, \mathbf{u})$.

The conclusion of Theorem 1.9 follows from the next proposition, once we observe that, by a change of variable,

$$\mathcal{L}_n(\alpha_n) \stackrel{\text{law}}{=} \frac{1}{2\pi\sqrt{n}} \times \text{length} \left(\tilde{T}_n^{-1}(0) \cap B_{2\pi\alpha_n} \right).$$

To simplify the discussion, for $r > 0$ we also write

$$\tilde{\mathcal{L}}_n(r) := \text{length} \left(\tilde{T}_n^{-1}(0) \cap B_r \right).$$

Proposition 5.1. *Fix $\rho < \frac{1}{2} \log \frac{\pi}{2}$. Then, there exists a density one sequence $\{n_j\} \subset S$ such that, as $n_j \rightarrow \infty$,*

1. $\mathcal{N}_{n_j} \rightarrow \infty$ and μ_{n_j} converges weakly to the uniform measure on \mathbb{S}^1 ;
2. for every sequence $n \mapsto \alpha_n$ such that $\alpha_n \rightarrow \infty$ and $\alpha_n = o((\log n)^{\rho/9})$,

$$\mathbf{Var}(\tilde{\mathcal{L}}_n(\alpha_{n_j})) \sim \frac{\alpha_{n_j}^2 \log \alpha_{n_j}}{256},$$

and

$$\frac{\tilde{\mathcal{L}}_n(\alpha_{n_j}) - \mathbb{E}[\tilde{\mathcal{L}}_n(\alpha_{n_j})]}{\sqrt{\mathbf{Var}(\tilde{\mathcal{L}}_n(\alpha_{n_j}))}} \stackrel{\text{law}}{\rightarrow} Z \sim \mathcal{N}(0, 1).$$

Proof. Fix $\rho < \frac{1}{2} \log \frac{\pi}{2}$ and $\alpha_n = o((\log n)^{\rho/9})$ as in the statement, and select parameters $0 < \gamma < \alpha := \frac{\log \pi}{\log 2} - 1$ and $0 < \kappa < \beta := \frac{\log 2}{2}$ such that $\gamma\kappa = \rho$. Thanks to the main result in [KK77], we know that there exists a density one sequence $\{n_j\} \subset S$ such that Point 1 in the statement holds, and moreover

- (a) $\mathcal{N}_{n_j} = (\log n_j)^{\beta + \epsilon_{n_j}}$, with $\epsilon_{n_j} \rightarrow 0$, in such a way that $\mathcal{N}_{n_j} \geq (\log n_j)^\kappa$ for n_j sufficiently large (see also [BMW18]);
- (b) $\mathbf{Kol}(\mu_{n_j}^\#, \mathbf{u}) \leq 2\mathcal{N}_{n_j}^{-\gamma}$ and therefore, for n_j large enough,

$$\mathbf{Kol}(\mu_{n_j}^\#, \mathbf{u}) \leq 2 \left(\frac{1}{\log n_j} \right)^\rho.$$

One key observation is that, for arbitrary multi-indices α, β

$$\left| \partial^\alpha \partial^\beta (\tilde{u}_{n_j}(x - y) - J_0(\|x - y\|)) \right| \leq \|x - y\| \mathbf{W}_1(\mu_{n_j}^\#, \mathbf{u}),$$

and consequently, by the previous discussion,

$$\left| \partial^\alpha \partial^\beta (\tilde{u}_{n_j}(x - y) - J_0(\|x - y\|)) \right| \leq 4\pi \|x - y\| \left(\frac{1}{\log n_j} \right)^\rho,$$

when n_j is large enough. This yields in particular that, for some absolute constant C ,

$$\max_{\alpha, \beta \in S(3)} \sup_{x, y \in B_{\alpha n_j}} \left| \partial^\alpha \partial^\beta (\tilde{u}_{n_j}(x - y) - J_0(\|x - y\|)) \right| \leq C \frac{\alpha_{n_j}}{(\log n_j)^\rho} =: \eta(n_j) \rightarrow 0,$$

since $\alpha_n = o((\log n)^\rho)$. The conclusion is obtained by reproducing verbatim the proof of Theorem 1.5 given in Section 3, by replacing r_λ with α_{n_j} , and then η_λ with $\eta(n_j)$ (see (3.23)), as well as the right-hand side of (3.27) with

$$B \frac{\alpha_{n_j}}{(\log n_j)^{\rho/3}},$$

where B is an absolute constant, and we have exploited [BM19, Theorem 5.5] in the case $n = 2$. \square

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