

LIMIT THEOREMS FOR INTEGRAL FUNCTIONALS OF HERMITE-DRIVEN PROCESSES

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ABSTRACT. Consider a moving average process X of the form $X(t) = \int_{-\infty}^t \varphi(t-u) dZ_u$, $t \geq 0$, where Z is a (non Gaussian) Hermite process of order $q \geq 2$ and $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$ is sufficiently integrable. This paper investigates the fluctuations, as $T \rightarrow \infty$, of integral functionals of the form $t \mapsto \int_0^{Tt} P(X(s)) ds$, in the case where P is any given polynomial function. It extends a study initiated in [21], where only the quadratic case $P(x) = x^2$ and the convergence in the sense of finite-dimensional distributions were considered.

1. INTRODUCTION

Hermite processes occur naturally when we consider limits of partial sums associated with long-range dependent stationary series. They have become increasingly popular in the recent literature, see for example the book [16] by Pipiras and Taqqu, in particular section 4.11, which contains bibliographical notes on their history and recent developments. They form a family of stochastic processes, indexed by an integer $q \geq 1$ and a self-similarity index $H \in (\frac{1}{2}, 1)$, called the Hurst parameter, that contains the fractional Brownian motion ($q = 1$) and the Rosenblatt process ($q = 2$) as particular cases. We refer the reader to Section 2.2 and the references therein for a precise definition of the Hermite processes. Of primary importance in the sequel is the parameter H_0 , given in terms of H and q by

$$(1) \quad H_0 = 1 - \frac{1-H}{q} \in (1 - \frac{1}{2q}, 1).$$

The goal of the present paper is to investigate the fluctuations, as $T \rightarrow \infty$, of the family of stochastic processes

$$(2) \quad t \mapsto \int_0^{Tt} P(X(s)) ds, \quad t \in [0, 1] \text{ (say),}$$

in the case where $P(x)$ is a polynomial function and X is a moving average process of the form

$$(3) \quad X(t) = \int_{-\infty}^t \varphi(t-u) dZ_u, \quad t \geq 0,$$

2000 *Mathematics Subject Classification.* 60F05, 60G22, 60H05, 60H07.

Key words and phrases. Hermite processes, chaotic decomposition, fractional Brownian motion (fBm), multiple Wiener-Itô integrals.

with Z a Hermite process and $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$ a sufficiently integrable function. We note that integral functionals such as (2) are often encountered in the context of statistical estimation, see e.g. [21] for a concrete example.

Let us first consider the case where $q = 1$, that is to say the case where Z is the fractional Brownian motion. Note that this is the only case where Z is Gaussian, making the study *a priori* much simpler and more affordable. By linearity and passage to the limit, the process X is also Gaussian. Moreover, it is stationary, since the quantity $\mathbb{E}[X(t)X(s)] =: \rho(t-s)$ only depends on $t-s$. For simplicity and without loss of generality, assume that $\rho(0) = 1$, that is, $X(t)$ has variance 1 for any t . As is well-known since the eighties (see [5, 8, 19]), the fluctuations of (2) heavily depends on the centered Hermite rank of P , defined as the integer $d \geq 1$ such that P decomposes in the form

$$(4) \quad P = \mathbb{E}[P(X(0))] + \sum_{k=d}^{\infty} a_k H_k,$$

with H_k the k th Hermite polynomials and $a_d \neq 0$. (Note that the sum (4) is actually finite, since P is a polynomial, so that $\#\{k : a_k \neq 0\} < \infty$.)

The first result of this paper concerns the fractional Brownian motion. Even if it does not follow directly from the well-known results of Breuer-Major [5], Dobrushin-Major [8] and Taqqu [19], the limits obtained are somehow expected. In particular, the threshold $H = 1 - \frac{1}{2d}$ is well known to specialists. However, the proof of this result is not straightforward, and requires several estimations which are interesting in themselves.

Theorem 1. *Let Z be a fractional Brownian motion of Hurst index $H \in (\frac{1}{2}, 1)$, and let $\varphi \in L^1(\mathbb{R}_+) \cap L^{\frac{1}{H}}(\mathbb{R}_+)$. Consider the moving average process \bar{X} defined by (3) and assume without loss of generality that $\text{Var}(X(0)) = 1$ (if not, it suffices to multiply φ by a constant). Finally, let $P(x) = \sum_{n=0}^N a_n x^n$ be a real-valued polynomial function, and let $d \geq 1$ denotes its centered Hermite rank.*

(1) *If $d \geq 2$ and $H \in (\frac{1}{2}, 1 - \frac{1}{2d})$ then*

$$(5) \quad T^{-\frac{1}{2}} \left\{ \int_0^{Tt} (P(X(s)) - \mathbb{E}[P(X(s))]) ds \right\}_{t \in [0,1]}$$

converges in distribution in $C([0,1])$ to a standard Brownian motion W , up to some multiplicative constant C_1 which is explicit and depends only on φ , P and H .

(2) *If $H \in (1 - \frac{1}{2d}, 1)$ then*

$$(6) \quad T^{d(1-H)-1} \left\{ \int_0^{Tt} (P(X(s)) - \mathbb{E}[P(X(s))]) ds \right\}_{t \in [0,1]}$$

converges in distribution in $C([0,1])$ to a Hermite process of index d and Hurst parameter $1 - d(1-H)$, up to some multiplicative constant C_2 which is explicit and depends only on φ , P and H .

Now, let us consider the non-Gaussian case, that is, the case where $q \geq 2$. As we will see, the situation is completely different, both in the results obtained (rather unexpected) and in the methods used (very different from the Gaussian case). Let $L > 0$. We define \mathcal{S}_L to be the set of bounded functions $l : \mathbb{R}_+ \rightarrow \mathbb{R}$ such that $y^L l(y) \rightarrow 0$ as $y \rightarrow \infty$. We observe that $\mathcal{S}_L \subset L^1(\mathbb{R}_+) \cap L^{\frac{1}{H}}(\mathbb{R}_+)$ for any $L > 1$. We can now state the following result.

Theorem 2. *Let Z be a Hermite process of order $q \geq 2$ and Hurst parameter $H \in (\frac{1}{2}, 1)$, and let $\varphi \in \mathcal{S}_L$ for some $L > 1$. Recall H_0 from (1) and consider the moving average process X defined by (3). Finally, let $P(x) = \sum_{n=0}^N a_n x^n$ be a real-valued polynomial function. Then, one and only one of the following two situations takes place at $T \rightarrow \infty$:*

- (i) *If q is odd and if $a_n \neq 0$ for at least one odd $n \in \{1, \dots, N\}$, then*

$$T^{-H_0} \left\{ \int_0^{Tt} (P(X(s)) - \mathbb{E}[P(X(s))]) ds \right\}_{t \in [0,1]}$$

converges in distribution in $C([0, 1])$ to a fractional Brownian motion of parameter $H_1 := H_0$, up to some multiplicative constant K_1 which is explicit and depends only on φ , P , q and H , see Remark 4.

- (ii) *If q is even, or if q is odd and $a_n = 0$ for all odd $n \in \{1, \dots, N\}$, then*

$$T^{1-2H_0} \left\{ \int_0^{Tt} (P(X(s)) - \mathbb{E}[P(X(s))]) ds \right\}_{t \in [0,1]}$$

converges in distribution in $C([0, 1])$ to a Rosenblatt process of Hurst parameter $H_2 := 2H_0 - 1$, up to some multiplicative constant K_2 which is explicit and depends only on φ , P , q and H , see Remark 4.

Remark 3. Whether in Theorem 1 or Theorem 2, the multiplicative constants appearing in the limit can be all given explicitly by following the respective proofs. For example, the constant K_1 and K_2 of Theorem 2 are given by the following intricate expressions:

$$K_1 = \sum_{n=3, n \text{ odd}}^N a_n c_{H,q}^n K_{\varphi,n,1}$$

$$K_2 = \sum_{n=2}^N a_n c_{H,q}^n K_{\varphi,n,2} + a_1 \mathbb{I}_{\{q=2\}} \int_{\mathbb{R}_+} \varphi(\nu) d\nu$$

with

$$K_{\varphi,n,i} = \sum_{\alpha \in A_{n,q}, nq-2|\alpha|=i} \frac{C_{\alpha} K_{\varphi,\alpha,H_0}}{c_{H,i}^{\alpha}}, \quad i = 1, 2,$$

where the sets and constants in the previous formula are defined in Sections 2, 3 and 4.

Remark 4. Note that, unlike the case of a fractional Brownian motion X , where the limit depends on the Hermite rank of the polynomial P , here the Hermite rank of P plays no role and the limit depends on the parity of the non-vanishing coefficients of P . This is not really surprising in our *non-Gaussian* context, since the Hermite rank of P is defined by means of its decomposition into Hermite polynomials, and these latter polynomials only have good probabilistic properties when evaluated in *Gaussian* random variables.

We note that our Theorem 2 contains as a very particular case the main result of [21], which corresponds to the choice $P(x) = x^2$ and thus situation (ii). Moreover, let us emphasize that our Theorem 2 not only studies the convergence of finite-dimensional distributions as in [21], but also provides a *functional* result.

Because the employed method is new, let us sketch the main steps of the proof of Theorem 2, by using the classical notation of the Malliavin calculus (see Section 2 for any unexplained definition or result); in particular we write $I_p^B(h)$ to indicate the p th multiple Wiener-Itô integral of kernel h with respect to the standard (two-sided) Brownian motion B .

(Step 1) In Section 3, we represent the moving average process X as a q th multiple Wiener-Itô integral with respect to B :

$$X(t) = c_{H,q} I_q^B(g(t, \cdot)),$$

where $c_{H,q}$ is an explicit constant and the kernel $g(t, \cdot)$ is given by

$$(7) \quad g(t, \xi_1, \dots, \xi_q) = \int_{-\infty}^t \varphi(t-v) \prod_{j=1}^q (v - \xi_j)_+^{H_0 - \frac{3}{2}} dv,$$

for $\xi_1, \dots, \xi_q \in \mathbb{R}$, $t \geq 0$. Thanks to this representation, we compute in Lemma 5 the chaotic expansion of the n th power of $X(t)$ for any $n \geq 2$ and $t > 0$, and obtain an expression of the form

$$X^n(t) = c_{H,q}^n \sum_{\alpha \in A_{n,q}} C_\alpha I_{nq-2|\alpha|}^B(\otimes_\alpha(g(t, \cdot), \dots, g(t, \cdot))),$$

where we have used the novel notation $\otimes_\alpha(g(t, \cdot), \dots, g(t, \cdot))$ to indicate iterated contractions whose precise definition is given in Section 2.1, and where C_α are combinatorial constants and the sum runs over a family $A_{n,q}$ of suitable multi-indices $\alpha = (\alpha_{ij}, 1 \leq i < j \leq n)$. As an immediate consequence, we deduce that our quantity of interest can be decomposed as follows:

$$(8) \quad \int_0^{Tt} (P(X(s)) - \mathbb{E}[P(X(s))]) ds = a_0 \int_0^{Tt} X(s) ds + \sum_{n=2}^N a_n c_{H,q}^n \sum_{\alpha \in A_{n,q}, nq-2|\alpha| \geq 1} C_\alpha \int_0^{Tt} I_{nq-2|\alpha|}^B(\otimes_\alpha(g(s, \cdot), \dots, g(s, \cdot))) ds.$$

(Step 2) In Proposition 6, we compute an explicit expression for the iterated contractions $\otimes_\alpha(g(t, \cdot), \dots, g(t, \cdot))$ appearing in the right-hand side of (8), by using that g is given by (7).

To ease the description of the remaining steps, let us now set

$$(9) \quad F_{n,q,\alpha,T}(t) = \int_0^{Tt} I_{nq-2|\alpha|}^B(\otimes_\alpha(g(s, \cdot), \dots, g(s, \cdot))) ds.$$

(Step 3) As $T \rightarrow \infty$, we show in Proposition 7 that, if $nq - 2|\alpha| < \frac{q}{1-H}$, then $T^{-1+(1-H_0)(nq-2|\alpha|)} F_{n,q,\alpha,T}(t)$ converges in distribution to a Hermite process (whose order and Hurst index are specified) up to some multiplicative constant. Similarly, we prove in Proposition 9 that, if $nq - 2|\alpha| \geq 3$, then $T^{\alpha_0} F_{n,q,\alpha,T}(t)$ is tight and converges in $L^2(\Omega)$ to zero, where α_0 is given in (28).

(Step 4) By putting together the results obtained in the previous steps, the two convergences stated in Theorem 2 follow immediately.

To illustrate a possible use of our results, we study in Section 6 an extension of the classical fractional Ornstein-Uhlenbeck process (see, e.g., Cheridito *et al* [7]) to the case where the driving process is more generally a Hermite process. To the best of our knowledge, there is very little literature devoted to this mathematical object, only [11, 17].

The rest of the paper is organized as follows. Section 2 presents some basic results about multiple Wiener-Itô integrals and Hermite processes, as well as some other facts that are used throughout the paper. Section 3 contains preliminary results. The proof of Theorem 1 (resp. Theorem 2) is given in Section 5 (resp. Section 4). In Section 6, we provide a complete asymptotic study of the Hermite-Ornstein-Uhlenbeck process, by means of Theorems 1 and 2 and of an extension of Birkhoff's ergodic Theorem. Finally, Section 7 contains two technical results: a power counting theorem and a version of the Hardy-Littlewood inequality, which both play an important role in the proof of our main theorems.

2. PRELIMINARIES ON MULTIPLE WIENER-ITÔ INTEGRALS AND HERMITE PROCESSES

2.1. Multiple Wiener-Itô integrals and a product formula. A function $f : \mathbb{R}^p \rightarrow \mathbb{R}$ is said to be *symmetric* if the following relation holds for all permutation $\sigma \in \mathfrak{S}(p)$:

$$f(t_1, \dots, t_p) = f(t_{\sigma(1)}, \dots, t_{\sigma(p)}), \quad t_1, \dots, t_p \in \mathbb{R}.$$

The subset of $L^2(\mathbb{R}^p)$ composed of symmetric functions is denoted by $L_s^2(\mathbb{R}^p)$.

Let $B = \{B(t)\}_{t \in \mathbb{R}}$ be a two-sided Brownian motion. For any given $f \in L_s^2(\mathbb{R}^p)$ we consider the *multiple Wiener-Itô integral* of f with respect

to B , denoted by

$$I_p^B(f) = \int_{\mathbb{R}^p} f(t_1, \dots, t_p) dB(t_1) \cdots dB(t_p).$$

This stochastic integral satisfies $\mathbb{E}[I_p^B(f)] = 0$ and

$$\mathbb{E}[I_p^B(f)I_q^B(g)] = \mathbf{1}_{\{p=q\}} p! \langle f, g \rangle_{L^2(\mathbb{R}^p)}$$

for $f \in L_s^2(\mathbb{R}^p)$ and $g \in L_s^2(\mathbb{R}^q)$, see [10] and [13] for precise definitions and further details.

It will be convenient in this paper to deal with multiple Wiener-Itô integrals of possibly nonsymmetric functions. If $f \in L^2(\mathbb{R}^p)$, we put $I_p^B(f) = I_p^B(\tilde{f})$, where \tilde{f} denotes the symmetrization of f , that is,

$$\tilde{f}(x_1, \dots, x_p) = \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}(p)} f(x_{\sigma(1)}, \dots, x_{\sigma(p)}).$$

We will need the expansion as a sum of multiple Wiener-Itô integrals for a product of the form

$$\prod_{k=1}^n I_q^B(h_k),$$

where $q \geq 2$ is fixed and the functions h_k belong to $L_s^2(\mathbb{R}^q)$ for $k = 1, \dots, n$. In order to present this extension of the product formula and to define the relevant contractions between the functions h_i and h_j that will naturally appear, we introduce some further notation. Let $A_{n,q}$ be the set of multi-indices $\alpha = (\alpha_{ij}, 1 \leq i < j \leq n)$ such that, for each $k = 1, \dots, n$,

$$\sum_{1 \leq i < j \leq n} \alpha_{ij} \mathbf{1}_{k \in \{i,j\}} \leq q.$$

Set $|\alpha| = \sum_{1 \leq i < j \leq n} \alpha_{ij}$,

$$\beta_k^0 = q - \sum_{1 \leq i < j \leq n} \alpha_{ij} \mathbf{1}_{k \in \{i,j\}}, \quad 1 \leq k \leq n$$

and

$$(10) \quad m := m(\alpha) = \sum_{k=1}^n \beta_k^0 = nq - 2|\alpha|.$$

For each $1 \leq i < j \leq n$, the integer α_{ij} will represent the number of variables in h_i which are contracted with h_j whereas, for each $k = 1, \dots, n$, the integer β_k^0 is the number of variables in h_k which are not contracted. We will also write $\beta_k = \sum_{j=1}^k \beta_j^0$ for $k = 1, \dots, n$ and $\beta_0 = 0$. Finally, we set

$$(11) \quad C_\alpha = \frac{q!^n}{\prod_{k=1}^n \beta_k^0! \prod_{1 \leq i < j \leq n} \alpha_{ij}!}.$$

With these preliminaries, for any element $\alpha \in A_{n,q}$ we can define the contraction $\otimes_\alpha(h_1, \dots, h_n)$ as the function of $nq - 2|\alpha|$ variables obtained

by contracting α_{ij} variables between h_i and h_j for each couple of indices $1 \leq i < j \leq n$. Define the collection $(u^{i,j})_{1 \leq i, j \leq n, i \neq j}$ in the following way:

$$u^{i,j} = \alpha_{\min(i,j), \max(i,j)}.$$

We then have

$$\begin{aligned} & \otimes_{\alpha} (h_1, \dots, h_n)(\xi_1, \dots, \xi_{nq-2|\alpha|}) \\ (12) \quad &= \int_{\mathbb{R}^{|\alpha|}} \prod_{k=1}^n h_k(s_1^{k,1}, \dots, s_{u^{k,1}}^{k,1}, \dots, s_1^{k,n}, \dots, s_{u^{k,n}}^{k,n}, \xi_{1+\beta_{k-1}}, \dots, \xi_{\beta_k}) \\ & \quad \times \prod_{1 \leq i < j \leq n} ds_1^{i,j} \dots ds_{u^{i,j}}^{i,j} \end{aligned}$$

When $n = 2$, α has only one component $\alpha_{1,2}$ and $\otimes_{\alpha}(h_1, h_2) = h_1 \otimes_{\alpha_{1,2}} h_2$ is the usual contraction of $\alpha_{1,2}$ indices between h_1 and h_2 . Notice that the function $\otimes_{\alpha}(h_1, \dots, h_n)$ is not necessarily symmetric.

Then, we have the following result.

Lemma 5. *Let $n, q \geq 2$ be some integers and let $h_i \in L_s^2(\mathbb{R}^q)$ for $i = 1, \dots, n$. We have*

$$(13) \quad \prod_{k=1}^n I_q^B(h_k) = \sum_{\alpha \in A_{n,q}} C_{\alpha} I_{nq-2|\alpha|}^B(\otimes_{\alpha}(h_1, \dots, h_n)).$$

Proof. The product formula for multiple stochastic integrals (see, for instance, [14, Theorem 6.1.1], or formula (2.1) in [3] for $n = 2$) says that

$$(14) \quad \prod_{k=1}^n I_q^B(h_k) = \sum_{\mathcal{P}, \psi} I_{\beta_1^0 + \dots + \beta_n^0}^B \left((\otimes_{k=1}^n h_k)_{\mathcal{P}, \psi} \right),$$

where \mathcal{P} denotes the set of all partitions $\{1, \dots, q\} = J_i \cup (\cup_{k=1, \dots, n, k \neq i} I_{ik})$, where for any $i, j = 1, \dots, n$, I_{ij} and I_{ji} have the same cardinality α_{ij} , ψ_{ij} is a bijection between I_{ij} and I_{ji} and $\beta_k^0 = |J_k|$. Moreover, $(\otimes_{k=1}^n h_k)_{\mathcal{P}, \psi}$ denotes the contraction of the indexes ℓ and $\psi_{ij}(\ell)$ for any $\ell \in I_{ij}$ and any $i, j = 1, \dots, n$. Then, formula (13) follows from (14), by just counting the number of partitions, which is

$$\prod_{k=1}^n \frac{q!}{\prod_{i \text{ or } j \neq k} \alpha_{ij}! \beta_k^0!}$$

and multiplying by the number of bijections, which is $\prod_{1 \leq i < j \leq n} \alpha_{ij}!$. \square

Notice that when $n = 2$, formula (13) reduces to the well-known formula for the product of two multiple integrals. That is, for any two symmetric functions $f \in L_s^2(\mathbb{R}^p)$ and $g \in L_s^2(\mathbb{R}^q)$ we have

$$I_p^B(f) I_q^B(g) = \sum_{r=0}^{\min(p,q)} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}^B(f \otimes_r g).$$

where, for $0 \leq r \leq \min(p, q)$, $f \otimes_r g \in L^2(\mathbb{R}^{p+q-2r})$ denotes the contraction of r coordinates between f and g .

2.2. Hermite processes. Fix $q \geq 1$ and $H \in (\frac{1}{2}, 1)$. The Hermite process of index q and Hurst parameter H can be represented by means of a multiple Wiener-Itô integral with respect to B as follows, see e.g. [9]:

$$(15) \quad Z^{H,q}(t) = c_{H,q} \int_{\mathbb{R}^q} \int_{[0,t]} \prod_{j=1}^q (s - x_j)_+^{H_0 - \frac{3}{2}} ds dB(x_1) \cdots dB(x_q), \quad t \in \mathbb{R}.$$

Here, $x_+ = \max\{x, 0\}$, the constant $c_{H,q}$ is chosen to ensure that $\text{Var}(Z^{H,q}(1)) = 1$, and

$$H_0 = 1 - \frac{1-H}{q} \in (1 - \frac{1}{2q}, 1).$$

Note that $Z^{H,q}$ is self-similar of index H . When $q = 1$, the process $Z^{H,1}$ is Gaussian and is nothing but the fractional Brownian motion with Hurst parameter H . For $q \geq 2$, the processes $Z^{H,q}$ are no longer Gaussian: they belong to the q th Wiener chaos. The process $Z^{H,2}$ is known as the Rosenblatt process.

Let $|\mathcal{H}|$ be the following class of functions:

$$|\mathcal{H}| = \left\{ f : \mathbb{R} \rightarrow \mathbb{R} \mid \int_{\mathbb{R}} \int_{\mathbb{R}} |f(u)||f(v)||u-v|^{2H-2} dudv < \infty \right\}.$$

Maejima and Tudor [9] proved that the stochastic integral $\int_{\mathbb{R}} f(u) dZ^{H,q}(u)$ with respect to the Hermite process $Z^{H,q}$ is well defined when f belongs to $|\mathcal{H}|$. Moreover, for any order $q \geq 1$, index $H \in (\frac{1}{2}, 1)$ and function $f \in |\mathcal{H}|$,

$$(16) \quad \begin{aligned} & \int_{\mathbb{R}} f(u) dZ^{H,q}(u) \\ &= c_{H,q} \int_{\mathbb{R}^q} \left(\int_{\mathbb{R}} f(u) \prod_{j=1}^q (u - \xi_j)_+^{H_0 - \frac{3}{2}} du \right) dB(\xi_1) \cdots dB(\xi_q). \end{aligned}$$

As a consequence of the Hardy-Littlewood-Sobolev inequality featured in [1], we observe that $L^1(\mathbb{R}) \cap L^{\frac{1}{H}}(\mathbb{R}) \subset |\mathcal{H}|$.

3. CHAOTIC DECOMPOSITION OF $\int_0^{Tt} P(X(s)) ds$

Assume $\varphi \in |\mathcal{H}|$ and $q \geq 1$. Using (16) and bearing in mind the notation and results from Section 2, it is immediate that X can be written as

$$(17) \quad X(t) = c_{H,q} I_q^B(g(t, \cdot)),$$

where $g(t, \cdot)$ is given by

$$(18) \quad g(t, \xi_1, \dots, \xi_q) = \int_{-\infty}^t \varphi(t-v) \prod_{j=1}^q (v - \xi_j)_+^{H_0 - 3/2} dv,$$

and $c_{H,q}$ is defined as in (15).

3.1. Computing the chaotic expansion of $X(t)^n$ when $n \geq 2$. Let us denote by $A_{n,q}^0$ the set of elements $\alpha \in A_{n,q}$ such that $nq - 2|\alpha| = 0$ and $A_{n,q}^1$ will be the set of elements $\alpha \in A_{n,q}$ such that $nq - 2|\alpha| \geq 1$. Notice that when nq is odd, $A_{n,q}^0$ is empty. Using (13), we obtain the following formula for the expectation of the n th power ($n \geq 2$) of X given by (3):

$$(19) \quad \mathbb{E}[X(t)^n] = (c_{H,q})^n \sum_{\alpha \in A_{n,q}^0} C_\alpha I_{nq-2|\alpha|}^B(\otimes_\alpha(g(t, \cdot), \dots, g(t, \cdot))).$$

We observe in particular that $\mathbb{E}[X(t)^n] = 0$ whenever nq is odd. From (13) and (19), we deduce for $n \geq 2$ that

$$(20) \quad X(t)^n - \mathbb{E}[X(t)^n] = (c_{H,q})^n \sum_{\alpha \in A_{n,q}^1} C_\alpha I_{nq-2|\alpha|}^B(\otimes_\alpha(g(t, \cdot), \dots, g(t, \cdot))).$$

To clarify this formula, let us write down detailed a expression in the cases $n = 2$ and $n = 3$. When $n = 2$, the right-hand side of (20) is

$$(c_{H,q})^2 \sum_{r=0}^{q-1} r! \binom{q}{r}^2 I_{2q-2r}^B(g(t, \cdot) \otimes_r g(t, \cdot)),$$

because α has just one component $\alpha_{1,2} =: r$ and condition $\alpha \in A_{n,q}^1$ means $0 \leq r \leq q - 1$. For $n = 3$, we have

$$A_{3,q} = \{(\alpha_{1,2}, \alpha_{1,3}, \alpha_{2,3}) : \alpha_{1,2} + \alpha_{1,3} \leq q, \alpha_{1,2} + \alpha_{2,3} \leq q, \alpha_{1,3} + \alpha_{2,3} \leq q\}$$

and the right-hand side of (20) is

$$(c_{H,q})^3 \sum_{\alpha \in A_{3,q}: 3q-2|\alpha| \geq 1} C_\alpha I_{3q-2|\alpha|}^B(\otimes_\alpha(g(t, \cdot), g(t, \cdot), g(t, \cdot))),$$

where

$$C_\alpha = \frac{(q!)^3}{\alpha_{1,2}! \alpha_{1,3}! \alpha_{2,3}! (q - \alpha_{1,2} - \alpha_{1,3})! (q - \alpha_{1,2} - \alpha_{2,3})! (q - \alpha_{1,3} - \alpha_{2,3})!}.$$

In this case, the contraction $\otimes_\alpha(g(t, \cdot), g(t, \cdot), g(t, \cdot))$ is the function of $3q - 2|\alpha|$ variables defined by

$$\int_{\mathbb{R}^{|\alpha|}} g(\bullet, s, u) g(\star, s, v) g(\circ, u, v) ds du dv,$$

with $s = (s_1, \dots, s_{\alpha_{1,2}})$, $u = (u_1, \dots, u_{\alpha_{1,3}})$ and $v = (v_1, \dots, v_{\alpha_{2,3}})$.

From (20) we obtain

$$(21) \quad \int_0^{Tt} (P(X(s)) - \mathbb{E}[P(X(s))]) ds = a_1 \int_0^{Tt} X(s) ds + \sum_{n=2}^N a_n (c_{H,q})^n \sum_{\alpha \in A_{n,q}^1} C_\alpha \int_0^{Tt} I_{nq-2|\alpha|}^B(\otimes_\alpha(g(s, \cdot), \dots, g(s, \cdot))) ds.$$

3.2. Expressing the iterated contractions of g . We now compute an explicit expression for the iterated contractions appearing in (21).

Proposition 6. *Fix $n \geq 2$, $q \geq 1$ and $\alpha \in A_{n,q}$. We have*

$$\begin{aligned} \otimes_{\alpha} (g(t, \cdot), \dots, g(t, \cdot))(\xi) &= \beta(H_0 - \frac{1}{2}, 2 - 2H_0)^{|\alpha|} \\ &\times \int_{(-\infty, t]^n} dv_1 \dots dv_n \prod_{k=1}^n \varphi(t - v_k) \prod_{1 \leq i < j \leq n} |v_i - v_j|^{(2H_0 - 2)\alpha_{ij}} \\ &\times \prod_{k=1}^n \prod_{\ell=1+\beta_{k-1}}^{\beta_k} (v_k - \xi_{\ell})_+^{H_0 - \frac{3}{2}}, \end{aligned}$$

with the convention $\beta_0 = 0$.

Proof. The proof is a straightforward consequence of the following identity

$$(22) \quad \int_{\mathbb{R}} (v - \xi)_+^{H_0 - 3/2} (w - \xi)_+^{H_0 - 3/2} d\xi = \beta(H_0 - \frac{1}{2}, 2 - 2H_0) |v - w|^{(2H_0 - 2)},$$

whose proof is elementary, see e.g. [4]. \square

4. PROOF OF THEOREM 2

We are now ready to prove Theorem 2. To do so, we will mostly rely on the forthcoming Proposition 7, which might be a result of independent interest by itself, and which studies the asymptotic behavior of $F_{n,q,\alpha,T}^B$ given by (9). We will denote by f.d.d. the convergence in law of the finite-dimensional distributions of a given process. Notice that the hypothesis on φ is a bit weaker than the one in the main theorem, the fact that $\varphi \in \mathcal{S}_L$ being required in the forthcoming Proposition 9.

Proposition 7. *Fix $n \geq 2$, $q \geq 1$ and $\alpha \in A_{n,q}$. Assume the function φ belongs to $L^1(\mathbb{R}_+) \cap L^{\frac{1}{H}}(\mathbb{R}_+)$, recall H_0 from (1) and let m be defined as in (10). Finally, assume that $2m < \frac{q}{1-H}$ (which is automatically satisfied when $m = 1$ or $m = 2$). Then, as $T \rightarrow \infty$,*

$$(23) \quad (T^{-1+(1-H_0)m} F_{n,q,\alpha,T}(t))_{t \in [0,1]} \xrightarrow{f.d.d.} \left(\frac{C_{\alpha} K_{\varphi,\alpha,H_0}}{c_{H(m),m}} Z^{H(m),m}(t) \right)_{t \in [0,1]},$$

where $Z^{H(m),m}$ denotes the m th Hermite process of Hurst index $H(m) = 1 - \frac{m}{q}(1 - H)$ and the constants C_{α} and K_{φ,α,H_0} are defined in (11) and (24), respectively. Furthermore, $\{T^{-1+(1-H_0)m}(F_{n,q,\alpha,T}(t))_{t \in [0,1]}, T > 0\}$ is tight in $C([0, 1])$.

Remark 8. Note that for $m_1 < m_2$ the chaos of order m_1 dominates the chaos of order m_2 .

Proof of Proposition 7. Let $n \geq 2$, $q \geq 1$ and $\alpha \in A_{n,q}$.

Step 1: We will first show the convergence (23). We will make several change of variables in order to transform the expression of $F_{n,q,\alpha,T}(t)$. By means of an application of stochastic's Fubini's theorem, we can write

$$F_{n,q,\alpha,T}(t) = C_\alpha \int_{\mathbb{R}^m} \Psi_T(\xi_1, \dots, \xi_m) dB(\xi_1) \cdots dB(\xi_m),$$

where

$$\begin{aligned} \Psi_T(\xi_1, \dots, \xi_m) &:= T^{-1+m(1-H_0)} \int_0^{Tt} ds \int_{(-\infty, s]^n} dv_1 \cdots dv_n \prod_{k=1}^n \varphi(s - v_k) \\ &\quad \times \prod_{1 \leq i < j \leq n} |v_i - v_j|^{(2H_0-2)\alpha_{ij}} \prod_{k=1}^n \prod_{\ell=1+\beta_{k-1}}^{\beta_k} (v_k - \xi_\ell)_+^{H_0 - \frac{3}{2}}. \end{aligned}$$

Using the change of variables $s \rightarrow Ts$ and $v_k \rightarrow Ts - v_k$, $1 \leq k \leq n$, we obtain

$$\begin{aligned} \Psi_T(\xi_1, \dots, \xi_m) &:= T^{m(1-H_0)} \int_0^t ds \int_{(-\infty, Ts]^n} dv_1 \cdots dv_n \prod_{k=1}^n \varphi(Ts - v_k) \\ &\quad \times \prod_{1 \leq i < j \leq n} |v_i - v_j|^{(2H_0-2)\alpha_{ij}} \prod_{k=1}^n \prod_{\ell=1+\beta_{k-1}}^{\beta_k} (v_k - \xi_\ell)_+^{H_0 - \frac{3}{2}} \\ &= T^{-\frac{m}{2}} \int_0^t ds \int_{[0, \infty)^n} dv_1 \cdots dv_n \prod_{k=1}^n \varphi(v_k) \\ &\quad \times \prod_{1 \leq i < j \leq n} |v_i - v_j|^{(2H_0-2)\alpha_{ij}} \prod_{k=1}^n \prod_{\ell=1+\beta_{k-1}}^{\beta_k} \left(s - \frac{v_k}{T} - \frac{\xi_\ell}{T}\right)_+^{H_0 - \frac{3}{2}}. \end{aligned}$$

By the scaling property of the Brownian motion, the processes

$$(F_{n,q,\alpha,T}(t))_{\alpha \in A_{n,q}, 2 \leq n \leq N, t \in [0,1]}$$

and

$$(\widehat{F}_{n,q,\alpha,T}(t))_{\alpha \in A_{n,q}, 2 \leq n \leq N, t \in [0,1]}$$

have the same probability distribution, where

$$\widehat{F}_{n,q,\alpha,T}(t) = C_\alpha \int_{\mathbb{R}^m} \widehat{\Psi}_T(\xi_1, \dots, \xi_m) dB(\xi_1) \cdots dB(\xi_m)$$

$$\begin{aligned} \widehat{\Psi}_T(\xi_1, \dots, \xi_m) &:= \int_0^t ds \int_{(-\infty, 0]^n} dv_1 \cdots dv_n \prod_{k=1}^n \varphi(v_k) \\ &\quad \times \prod_{1 \leq i < j \leq n} |v_i - v_j|^{(2H_0-2)\alpha_{ij}} \prod_{k=1}^n \prod_{\ell=1+\beta_{k-1}}^{\beta_k} \left(s - \frac{v_k}{T} - \xi_\ell\right)_+^{H_0 - \frac{3}{2}}. \end{aligned}$$

Set

$$\widehat{\Psi}(\xi_1, \dots, \xi_m) := K_{\varphi, \alpha, H_0} \int_0^t ds \prod_{\ell=1}^m (s - \xi_\ell)_+^{H_0 - \frac{3}{2}},$$

where

$$(24) \quad K_{\varphi, \alpha, H_0} = \int_{\mathbb{R}_+^n} dv_1 \cdots dv_n \prod_{k=1}^n \varphi(v_k) \prod_{1 \leq i < j \leq n} |v_i - v_j|^{(2H_0 - 2)\alpha_{ij}}.$$

Notice that, by Lemma 15, K_{φ, α, H_0} is well defined. We claim that

$$(25) \quad \lim_{T \rightarrow \infty} \widehat{\Psi}_T = \widehat{\Psi},$$

where the convergence holds in $L^2(\mathbb{R}^m)$. This will imply the convergence in $L^2(\Omega)$ of $\widehat{F}_{n, q, \alpha, T}(t)$, as $T \rightarrow \infty$ to a Hermite process of order m , multiplied by the constant $C_\alpha K_{\varphi, \alpha, H_0}$.

Proof of (25): It suffices to show that the inner products $\langle \widehat{\Psi}_T, \widehat{\Psi}_T \rangle_{L^2(\mathbb{R}^m)}$ and $\langle \widehat{\Psi}_T, \widehat{\Psi} \rangle_{L^2(\mathbb{R}^m)}$ converge, as $T \rightarrow \infty$, to

$$\|\widehat{\Psi}\|_{L^2(\mathbb{R}^m)}^2 = K_{\varphi, \alpha, H_0}^2 \beta(H_0 - \frac{1}{2}, 2 - 2H_0)^m \int_{[0, t]^2} ds ds' |s - s'|^{(2H_0 - 2)m},$$

which is finite because $m < \frac{1}{2(1-H_0)} = \frac{q}{2(1-H)}$. We will show the convergence of $\langle \widehat{\Psi}_T, \widehat{\Psi}_T \rangle_{L^2(\mathbb{R}^m)}$ and the second term can be handled by the same arguments. We have

$$\begin{aligned} \|\widehat{\Psi}_T\|_{L^2(\mathbb{R}^m)}^2 &= \int_{[0, t]^2} ds ds' \int_{\mathbb{R}_+^{2n}} dv_1 \cdots dv_n dv'_1 \cdots dv'_n \\ &\quad \times \prod_{k=1}^n \varphi(v_k) \varphi(v'_k) \prod_{1 \leq i < j \leq n} |v_i - v_j|^{(2H_0 - 2)\alpha_{ij}} |v'_i - v'_j|^{(2H_0 - 2)\alpha_{ij}} \\ &\quad \times \prod_{k=1}^n \beta(H_0 - \frac{1}{2}, 2 - 2H_0)^{\beta_k} |s - s' - \frac{v_k - v'_k}{T}|^{(2H_0 - 2)\beta_k}. \end{aligned}$$

Let us first show that given $w_k \in \mathbb{R}$, $1 \leq k \leq n$,

$$(26) \quad \lim_{T \rightarrow \infty} \int_{[0, t]^2} ds ds' \prod_{k=1}^n |s - s' - \frac{w_k}{T}|^{(2H_0 - 2)\beta_k} = \int_{[0, t]^2} ds ds' |s - s'|^{(2H_0 - 2)m}$$

and, moreover,

$$(27) \quad \sup_{w_k \in \mathbb{R}, 1 \leq k \leq n} \int_{[0, t]^2} ds ds' \prod_{k=1}^n |s - s' - w_k|^{(2H_0 - 2)\beta_k} < \infty.$$

By the dominated convergence theorem and using Lemma 15, (26) and (27) imply (25).

To show (26), choose ϵ such that $|w_k|/T < \epsilon$, $1 \leq k \leq n$, for T large enough (depending on the fixed w_k 's). Then, we can write

$$\begin{aligned} & \int_{[0,t]^2} ds ds' \left| \prod_{k=1}^n \left| s - s' - \frac{w_k}{T} \right|^{(2H_0-2)\beta_k} - |s - s'|^{(2H_0-2)m} \right| \\ & \leq t \int_{|\xi| > 2\epsilon} d\xi \left| \prod_{k=1}^n \left| \xi - \frac{w_k}{T} \right|^{(2H_0-2)\beta_k} - |\xi|^{(2H_0-2)m} \right| \\ & \quad + 2t \sup_{|w_k| < \epsilon} \int_{|\xi| \leq 2\epsilon} d\xi \prod_{k=1}^n |\xi - w_k|^{(2H_0-2)\beta_k} \\ & := B_1 + B_2. \end{aligned}$$

The term B_1 tends to zero as $T \rightarrow \infty$, for each $\epsilon > 0$. On the other hand, the term B_2 tends to zero as $\epsilon \rightarrow 0$. Indeed,

$$B_2 = 2t\epsilon^{(2H_0-2)m+1} \sup_{|w_k| < 1} \int_{|\xi| \leq 2} d\xi \prod_{k=1}^n |\xi - w_k|^{(2H_0-2)\beta_k}.$$

Note that the above supremum is finite because the function $(w_1, \dots, w_k) \rightarrow \int_{|\xi| \leq 2} d\xi \prod_{k=1}^n |\xi - w_k|^{(2H_0-2)\beta_k}$ is continuous.

Property (27) follows immediately from the fact that the function

$$(w_1, \dots, w_k) \rightarrow \int_{[0,t]^2} ds ds' \prod_{k=1}^n |s - s' - w_k|^{(2H_0-2)\beta_k}$$

is continuous and vanishes as $|(w_1, \dots, w_k)|$ tends to infinity.

We have $H_0 = 1 - \frac{1-H}{q} = 1 - \frac{1-H(m)}{m}$ with $H(m)$ as above. As a result, we obtain the convergence of the finite-dimensional distributions of $T^{-1+(1-H_0)m} F_{n,q,\alpha,T}(t)$ to those the m th Hermite process $Z^{H(m),m}$ multiplied by the constant $\frac{C_\alpha K_{\varphi,\alpha,H_0}}{c_{H(m),m}}$.

Step 2: Tightness. Fix $0 \leq s < t \leq 1$. To check that tightness holds in $C([0,1])$, let us compute the squared $L^2(\Omega)$ -norm

$$\Phi_T := T^{-1+(1-H_0)m} \mathbb{E}(|F_{n,q,\alpha,T}(t) - F_{n,q,\alpha,T}(s)|^2).$$

Proceeding as in the first step of the proof, we obtain

$$\begin{aligned} \Psi_T &= \mathbb{E} \left(\left| \int_{\mathbb{R}^m} dB(\xi_1) \cdots dB(\xi_m) \int_s^t du \int_{\mathbb{R}_+^n} dv_1 \cdots dv_n \prod_{k=1}^n \varphi(v_k) \right. \right. \\ & \quad \times \left. \left. \prod_{1 \leq i < j \leq n} |v_i - v_j|^{(2H_0-2)\alpha_{ij}} \prod_{k=1}^n \prod_{\ell=1+\beta_{k-1}}^{\beta_k} \left(u - \frac{v_k}{T} - \xi_\ell \right)_+^{H_0 - \frac{3}{2}} \right|^2 \right) \\ & \leq m! \int_{\mathbb{R}^m} d\xi_1 \cdots d\xi_m \left| \int_s^t du \int_{\mathbb{R}_+^n} dv_1 \cdots dv_n \prod_{k=1}^n \varphi(v_k) \right. \end{aligned}$$

$$\times \prod_{1 \leq i < j \leq n} |v_i - v_j|^{(2H_0-2)\alpha_{ij}} \prod_{k=1}^n \prod_{\ell=1+\beta_{k-1}}^{\beta_k} \left(u - \frac{v_k}{T} - \xi_\ell \right)_+^{H_0 - \frac{3}{2}} \Big|^2.$$

Using (22) yields

$$\begin{aligned} \Psi_T &\leq m! \int_{[s,t]^2} dud u' \int_{\mathbb{R}_+^{2n}} dv_1 \cdots dv_n dv'_1 \cdots dv'_n \\ &\quad \times \prod_{k=1}^n \varphi(v_k) \varphi(v'_k) \prod_{1 \leq i < j \leq n} |v_i - v_j|^{(2H_0-2)\alpha_{ij}} |v'_i - v'_j|^{(2H_0-2)\alpha_{ij}} \\ &\quad \times \prod_{k=1}^n \beta(H_0 - \frac{1}{2}, 2 - 2H_0)^{\beta_k} \left| u - u' - \frac{v_k - v'_k}{T} \right|^{(2H_0-2)\beta_k} \\ &\leq m!(t-s) \int_{-1}^1 d\xi \int_{\mathbb{R}_+^{2n}} dv_1 \cdots dv_n dv'_1 \cdots dv'_n \\ &\quad \times \prod_{k=1}^n \varphi(v_k) \varphi(v'_k) \prod_{1 \leq i < j \leq n} |v_i - v_j|^{(2H_0-2)\alpha_{ij}} |v'_i - v'_j|^{(2H_0-2)\alpha_{ij}} \\ &\quad \times \prod_{k=1}^n \beta(H_0 - \frac{1}{2}, 2 - 2H_0)^{\beta_k} \left| \xi - \frac{v_k - v'_k}{T} \right|^{(2H_0-2)\beta_k} \\ &\leq C(t-s). \end{aligned}$$

Then the equivalence of all $L^p(\Omega)$ -norms, $p \geq 2$, on a fixed Wiener chaos, also known as the hypercontractivity property, allows us to conclude the proof of the tightness. \square

We will make use of the notation

$$(28) \quad \alpha_0 = (1 - 2H_0) \mathbf{1}_{\{nq \text{ is even}\}} - H_0 \mathbf{1}_{\{nq \text{ is odd}\}}.$$

Proposition 9. *Fix $n, q \geq 2$ and $\alpha \in A_{n,q}$, assume that the function φ belongs to \mathcal{S}_L for some $L > 1$ and that $m \geq 3$. Then for any $t \in [0, 1]$, $T^{\alpha_0} F_{n,q,\alpha,T}(t)$ converge in $L^2(\Omega)$ to zero as $T \rightarrow \infty$; furthermore, the family $\{(F_{n,q,\alpha,T}(t))_{t \in [0,1], T > 0}\}$ is tight in $C([0, 1])$.*

Proof. If $(2H_0-2)m > -1$, by Proposition 7, we know that $T^{-1+m(1-H_0)} F_{n,q,\alpha,T}(t)$ converges to zero in $L^2(\Omega)$ as $T \rightarrow \infty$. This implies the convergence to zero in $L^2(\Omega)$ as $T \rightarrow \infty$ of $T^{-\alpha_0} F_{n,q,\alpha,T}(t)$ because $-1 + m(1-H_0) > \alpha_0$. We should then concentrate on the case $(2H_0-2)m \leq -1$. Once again, we shall divide the proof in two steps:

Step 1: Let us first prove the convergence in $L^2(\Omega)$. Fix $\alpha \in A_{n,q}$. We are going to show that

$$\lim_{T \rightarrow \infty} T^{2\alpha_0} \mathbb{E} (|F_{n,q,\alpha,T}(t)|^2) = 0.$$

We know that

$$T^{2\alpha_0} \mathbb{E} (|F_{n,q,\alpha,T}(t)|^2) = T^{2\alpha_0} m! \times \left\| \int_{[0,Tt]} ds \otimes_{\alpha} (g(s, \cdot), \dots, g(s, \cdot)) \right\|_{L^2(\mathbb{R}^m)}^2.$$

In view of the expression for the contractions obtained in Proposition 7, it suffices to show that

$$\begin{aligned} & \lim_{T \rightarrow \infty} T^{2\alpha_0} \int_{[0,Tt]^2} ds ds' \int_{\mathbb{R}^m} \int_{(-\infty, s]^n} \int_{(-\infty, s']^n} dv_1 \cdots dv_n dv'_1 \cdots dv'_n d\xi_1 \cdots d\xi_m \\ & \times \prod_{k=1}^n \varphi(s - v_k) \varphi(s' - v'_k) \prod_{1 \leq i < j \leq n} |v_i - v_j|^{(2H_0-2)\alpha_{ij}} |v'_i - v'_j|^{(2H_0-2)\alpha_{ij}} \\ & \times \prod_{k=1}^n \prod_{\ell=1+\beta_{j-1}}^{\beta_j} (v_k - \xi_{\ell})_+^{H_0 - \frac{3}{2}} \prod_{\ell=1+\beta_{j-1}}^{\beta_j} (v'_k - \xi_{\ell})_+^{H_0 - \frac{3}{2}} = 0. \end{aligned}$$

Integrating in the variables ξ 's and using (22), it remains to show that

$$\begin{aligned} & \lim_{T \rightarrow \infty} T^{2\alpha_0} \int_{[0,Tt]^2} ds ds' \int_{(-\infty, s]^n} \int_{(-\infty, s']^n} dv_1 \cdots dv_n dv'_1 \cdots dv'_n \\ & \times \prod_{k=1}^n \varphi(s - v_k) \varphi(s' - v'_k) \prod_{1 \leq i < j \leq n} |v_i - v_j|^{(2H_0-2)\alpha_{ij}} |v'_i - v'_j|^{(2H_0-2)\alpha_{ij}} \\ & \times \prod_{k=1}^n |v_k - v'_k|^{(2H_0-2)\beta_k} = 0. \end{aligned}$$

Set

$$\begin{aligned} \Phi_T & := T^{2\alpha_0} \int_{[0,Tt]^2} ds ds' \int_{(-\infty, s]^n} \int_{(-\infty, s']^n} dv_1 \cdots dv_n dv'_1 \cdots dv'_n \\ & \times \prod_{k=1}^n \varphi(s - v_k) \varphi(s' - v'_k) \prod_{1 \leq i < j \leq n} |v_i - v_j|^{(2H_0-2)\alpha_{ij}} |v'_i - v'_j|^{(2H_0-2)\alpha_{ij}} \\ & \times \prod_{k=1}^n |v_k - v'_k|^{(2H_0-2)\beta_k}. \end{aligned}$$

Making the change of variables $w_k = s - v_k$, $w'_k = s' - v'_k$ for $k = 1, \dots, n$, yields

$$\begin{aligned} \Phi_T & = T^{2\alpha_0} \int_{[0,Tt]^2} ds ds' \int_{\mathbb{R}_+^n} \int_{[0,\infty)^n} dw_1 \cdots dw_n dw'_1 \cdots dw'_n \\ & \times \prod_{k=1}^n \varphi(w_k) \varphi(w'_k) \prod_{1 \leq i < j \leq n} |w_i - w_j|^{(2H_0-2)\alpha_{ij}} |w'_i - w'_j|^{(2H_0-2)\alpha_{ij}} \\ & \times \prod_{k=1}^n |s - s' - w_k + w'_k|^{(2H_0-2)\beta_k}. \end{aligned}$$

Now we use Fubini's theorem and make the change of variables $s - s' = \xi$ to obtain

$$\begin{aligned} \Phi_T &= tT^{2\alpha_0+1} \int_{\mathbb{R}_+^{2n}} dw_1 \cdots dw_n dw'_1 \cdots dw'_n \\ &\times \prod_{k=1}^n |\varphi(w_k)\varphi(w'_k)| \prod_{1 \leq i < j \leq n} |w_i - w_j|^{(2H_0-2)\alpha_{ij}} |w'_i - w'_j|^{(2H_0-2)\alpha_{ij}} \\ &\times \int_{-tT}^{tT} d\xi \prod_{k=1}^n |\xi - w_k + w'_k|^{(2H_0-2)\beta_k}. \end{aligned}$$

We shall distinguish again two subcases:

Case $(2H_0 - 2)m < -1$: Notice that the exponent $2\alpha_0 + 1$ is negative:

(i) If nq is even, then $\alpha_0 = 1 - 2H_0$ and

$$2\alpha_0 + 1 = 3 - 4H_0 < 0$$

because $H_0 > \frac{3}{4}$.

(ii) If nq is odd, then $\alpha_0 = -H_0$ and

$$2\alpha_0 + 1 = 1 - 2H_0 < 0.$$

Therefore, in order to show that $\lim_{T \rightarrow \infty} \Phi_T = 0$, it suffices to check that

$$\begin{aligned} J &:= \int_{\mathbb{R}^{2n}} dw_1 \cdots dw_n dw'_1 \cdots dw'_n \prod_{k=1}^n |\varphi(w_k)\varphi(w'_k)| \\ &\times \prod_{1 \leq i < j \leq n} |w_i - w_j|^{(2H_0-2)\alpha_{ij}} |w'_i - w'_j|^{(2H_0-2)\alpha_{ij}} \\ (29) \quad &\times \int_{\mathbb{R}} d\xi \prod_{k=1}^n |\xi - w_k + w'_k|^{(2H_0-2)\beta_k} < \infty, \end{aligned}$$

where, by convention $\varphi(w) = 0$ if $w < 0$. We will apply the Power Counting Theorem 14 to prove that this integral is finite. We consider functions on \mathbb{R}^{2n+1} with variables $\{(w_k)_{k \leq n}, (w'_k)_{k \leq n}, \xi\}$. The set of linear functions is

$$\begin{aligned} T &= \{\omega_k, \omega'_k, 1 \leq k \leq n\} \cup \{w_i - w_j, w'_i - w'_j, 1 \leq i < j \leq n\} \\ &\cup \{\xi - w_k + w'_k, 1 \leq k \leq n\}. \end{aligned}$$

The corresponding exponents (μ_M, ν_M) for each $M \in T$ are $(0, -L)$ for the linear functions w_k and w'_k (taking into account that $\varphi \in \mathcal{S}_L$), $(2H_0 - 2)\alpha_{ij}$ for each function of the form $w_i - w_j$ or $w'_i - w'_j$ and $(2H_0 - 2)\beta_k$ for each function of the form $\xi - w_k + w'_k$.

Then $J < \infty$, provided conditions (a) and (b) are satisfied.

• *Verification of (b)*: Let $W \subset T$ be a linearly independent proper subset of T , and

$$d_\infty = 2n + 1 - \dim(\text{Span}(W)) + \sum_{M \in T \setminus (\text{Span}(W) \cap T)} \nu_M.$$

Let S be the following subset of T : $S = \{w_k, w'_k, 1 \leq k \leq n\}$. Let $e = \text{Card}(S \cap \text{Span}(W))$. Consider the following two cases:

- (i) There exists $k \leq n$ such that $\xi - w_k + w'_k \in \text{Span}(W) \cap T$. Then $\dim(\text{Span}(W)) \geq e + 1$. As a consequence,

$$d_\infty \leq 2n + 1 - (e + 1) - (2n - e)L < 0,$$

because $L > 1$ and in this case, we should have $e < 2n$ because W is a proper subset of T .

- (ii) Otherwise,

$$d_\infty \leq 2n + 1 - e - (2n - e)L + (2H_0 - 2)m < 0,$$

because $L > 1$ and $(2H_0 - 2)m < -1$.

• *Verification of (a)*: A direct verification would require to solve a seemingly difficult combinatorial problem. We can simply remark that

$$\begin{aligned} & \int_{[-1,1]^{2n}} dw_1 \cdots dw_n dw'_1 \cdots dw'_n \\ & \quad \times \prod_{1 \leq i < j \leq n} |w_i - w_j|^{(2H_0-2)\alpha_{ij}} |w'_i - w'_j|^{(2H_0-2)\alpha_{ij}} \\ & \quad \times \int_{-1}^1 d\xi \prod_{k=1}^n |\xi - w_k + w'_k|^{(2H_0-2)\beta_k} \\ & = m! \frac{1}{\beta(H_0 - \frac{1}{2}, 2 - 2H_0)^{|\alpha|}} \mathbb{E} \left[\left(\int_0^1 I_{nq-2|\alpha|}^B(f(s, \cdot), \dots, f(s, \cdot)) \right)^2 \right] < \infty \end{aligned}$$

where $f(s, \xi_1, \dots, \xi_q) = \int_{-\infty}^{+\infty} \mathbb{I}_{[-1,1]}(s - v) \prod_{j=1}^q (v - \xi_j)_+^{H_0-3/2} dv$. Since $\varphi \in \mathcal{S}_L$, φ is bounded on $[-1, 1]$. This implies that (a) is verified by the converse side of the Power Counting Theorem.

Case $(2H_0 - 2)m = -1$: In this case, we can apply Hölder and Jensen inequalities to Φ_T in order to get

$$\Phi_T \leq T^{2\alpha_0+1} A^{\frac{\epsilon}{1+\epsilon}} B^{\frac{1}{1+\epsilon}},$$

with $2\alpha_0 + 1 < 0$, $A = \left(\int_{\mathbb{R}} |\varphi(w)| dw \right)^{2n}$ and

$$\begin{aligned} B &= \int_{\mathbb{R}^{2n}} dw_1 \cdots dw_n dw'_1 \cdots dw'_n \prod_{k=1}^n |\varphi(w_k) \varphi(w'_k)| \\ & \quad \times \prod_{1 \leq i < j \leq n} |w_i - w_j|^{(2H'_0-2)\alpha_{ij}} |w'_i - w'_j|^{(2H'_0-2)\alpha_{ij}} \\ & \quad \times \int_{\mathbb{R}} d\xi \prod_{k=1}^n |\xi - w_k + w'_k|^{(2H'_0-2)\beta_k}, \end{aligned}$$

where $H'_0 = H_0(1 + \epsilon) - \epsilon$. If ϵ is small enough, H'_0 can still be expressed as $1 - \frac{1-H'}{q}$ for some $\frac{1}{2} < H' < H$. Moreover, in this case $(2H'_0 - 2)m < -1$

so we are exactly in the situation of the previous case, and the integral B is finite.

Step 2: Using the same arguments as previously and the hypercontractivity property, we deduce that there exists a constant $K > 0$ such that for all $0 \leq s < t \leq 1$,

$$\mathbb{E} (|F_{n,q,\alpha,T}(t) - F_{n,q,\alpha,T}(s)|^4) \leq K|t - s|^2,$$

which proves the tightness in $C([0, 1])$. □

It remains to study what happens when $n = 1$. The proof of Proposition 10 is very similar to that of Proposition 7 (although much simpler) and details are left to the reader.

Proposition 10. *Fix $q \geq 1$ and assume the function φ belongs to $L^1(\mathbb{R}_+) \cap L^{\frac{1}{H}}(\mathbb{R}_+)$. Then the finite-dimensional distributions of the process*

$$(30) \quad G_T(t) := T^{q(1-H_0)-1} \int_0^{Tt} ds I_q^B(g(s, \cdot)), \quad t \in [0, 1],$$

where $g(s, \cdot)$ is defined in (18), converge in law to those of a q th Hermite process of Hurst parameter $1 - q(1 - H_0)$ multiplied by the constant $c_{H_0,q}^{-1} \int_0^\infty \varphi(w) dw$, and the family $\{(G_T(t))_{t \in [0,1]}, T > 0\}$ is tight in $C([0, 1])$.

We are now ready to make the proof of Theorem 2.

Proof of Theorem 2. It suffices to consider the decomposition (21) and to apply the results shown in Propositions 7 and 9. □

5. PROOF OF THEOREM 1

Let Z be a fractional Brownian motion of Hurst index $H \in (\frac{1}{2}, 1)$, and let $\varphi \in L^1(\mathbb{R}_+) \cap L^{\frac{1}{H}}(\mathbb{R}_+)$. Consider the moving average process X defined by

$$X(t) = \int_{-\infty}^t \varphi(t - u) dZ_u, \quad t \geq 0,$$

which is easily checked to be a stationary centered Gaussian process. Denote by $\rho : \mathbb{R} \rightarrow \mathbb{R}$ the correlation function of X , that is, $\rho(t - s) = \mathbb{E}[X(t)X(s)]$, $s, t \geq 0$. By multiplying the function φ by a constant if necessary, we can assume without loss of generality that $\rho(0) = 1 (= \text{Var}(X(t))$ for all t). Let $P(x) = \sum_{n=0}^N a_n x^n$ be a real-valued polynomial function, and let d denotes its centered Hermite rank.

5.1. Proof of (6). In this section, we assume that $d \geq 1$ and that $H \in (1 - \frac{1}{2d}, 1)$, and our goal is to show that

$$T^{d(1-H)-1} \left\{ \int_0^{Tt} (P(X(s)) - \mathbb{E}[P(X(s))]) ds \right\}_{t \in [0,1]}$$

converges in distribution in $C([0,1])$ to a Hermite process of index d and Hurst parameter $1 - d(1 - H)$, up to some multiplicative constant C_2 . Since P has centered Hermite rank d , it can be rewritten as

$$P(x) = \mathbb{E}[P(X(s))] + \sum_{l=d}^N b_l H_l(x),$$

for some $b_d, \dots, b_N \in \mathbb{R}$, with $b_d \neq 0$ and H_l the l th Hermite polynomial. As a result, we have

$$\int_0^{Tt} (P(X(s)) - \mathbb{E}[P(X(s))]) ds = \sum_{l=d}^N b_l (c_{H,1})^l \int_0^{Tt} I_l^B(g(s, \cdot)^{\otimes l}) ds,$$

and the desired conclusion follows thanks to Propositions 7 and 10.

5.2. Proof of (5). In this section, we assume that $d \geq 2$ and that $H \in (\frac{1}{2}, 1 - \frac{1}{2d})$, and our goal is to show that

$$T^{-\frac{1}{2}} \left\{ \int_0^{Tt} (P(X(s)) - \mathbb{E}[P(X(s))]) ds \right\}_{t \in [0,1]}$$

converges in distribution in $C([0,1])$ to a standard Brownian motion W , up to some multiplicative constant C_1 . To do so, we will rely on the Breuer-Major theorem, which asserts that the desired conclusion holds as soon as

$$(31) \quad \int_{\mathbb{R}} |\rho(s)|^d ds < \infty,$$

where $\rho(s) = \mathbb{E}[X(s)X(0)]$ (see, e.g., [6] for a continuous version of the Breuer-Major theorem).

The rest of this section is devoted to checking that (31) holds true. Let us first compute ρ :

$$\begin{aligned} \rho(t-s) &= \mathbb{E}[X(t)X(s)] \\ &= H(2H-1) \iint_{\mathbb{R}^2} \varphi(t-v) \mathbf{1}_{(-\infty, t]}(v) \varphi(s-u) \mathbf{1}_{(-\infty, s]}(u) |v-u|^{2H-2} dudv \\ &= H(2H-1) \iint_{\mathbb{R}^2} \varphi(u) \varphi(v) |t-s-v+u|^{2H-2} dudv, \end{aligned}$$

with the convention that $\varphi(u) = 0$ if $u < 0$. This allows us to write

$$\rho(s) = c_H [\tilde{\varphi} * (I^{2H-1} \varphi)](s),$$

where $\tilde{\varphi}(u) = \varphi(-u)$, I^{2H-1} is the fractional integral operator of order $2H-1$ and c_H is a constant depending on H . As a consequence, applying

Young's inequality and Hardy-Littlewood's inequality (see [18, Theorem 1]) yields

$$\|\rho\|_{L^d(\mathbb{R})} \leq c_H \|\varphi\|_{L^p(\mathbb{R})} \|I^{2H-1}\varphi\|_{L^q(\mathbb{R})} \leq c_{H,p} \|\varphi\|_{L^p(\mathbb{R})}^2,$$

where $\frac{1}{d} = \frac{1}{p} + \frac{1}{q} - 1$ and $\frac{1}{q} = \frac{1}{p} - (2H - 1)$. This implies $p = (H + \frac{1}{2d})^{-1}$ and we have $\|\varphi\|_{L^p(\mathbb{R})} < \infty$, because $p \in (1, \frac{1}{H})$ and $\varphi \in L^1(\mathbb{R}) \cap L^{\frac{1}{H}}(\mathbb{R})$. The proof of (5) is complete. \square

6. THE STATIONARY HERMITE-ORNSTEIN-UHLENBECK PROCESS

We dedicate this section to the study of the extension of the Ornstein-Uhlenbeck process to the case where the driving process is a Hermite process. To our knowledge, there is not much literature about this object. Among the few existing references, we mention [17] and [11]. The special case in which the driving process is a fractional Brownian motion has been, in contrast, well studied, see for instance [7]. In what follows, we will prove a first-order ergodic theorem for the stationary Hermite-Ornstein-Uhlenbeck process. Then, we will use Theorem 2 to study its second order fluctuations.

Let $\alpha > 0$. Consider the function $\varphi(s) = e^{-\alpha s} \mathbb{1}_{s>0}$ and let $Z^{H,q}$ be a Hermite process of order $q \geq 1$ and Hurst index $H > \frac{1}{2}$. Then $\varphi \in S_L$ for all $L > 0$, and we can define the stationary Hermite-Ornstein-Uhlenbeck process as:

$$(32) \quad (U_t)_{t \geq 0} = \int_{-\infty}^t \varphi(t-s) dZ_s^{H,q}.$$

As its name suggests, this process is strongly stationary, that is, for any $h > 0$ the processes $(U_t)_{t \geq 0}$ and $(U_{t+h})_{t \geq 0}$ have the same finite-dimensional distributions. We then state the following general ergodic type result.

Proposition 11. *Let $(u_t)_{t \geq 0}$ be a real valued process of the form $u_t = I_q^B(f_t)$, where $f_t \in L_s^2(\mathbb{R}^q)$ for each $t \geq 0$. Assume that u is strongly stationary, has integrable sample paths and satisfies, for each $1 \leq r \leq q$,*

$$\|f_0 \otimes_r f_s\|_{L^2(\mathbb{R}^{2q-2r})} \xrightarrow{s \rightarrow \infty} 0.$$

Then, for all measurable function such that $\mathbb{E}[|f(u_0)|] < +\infty$,

$$\frac{1}{T} \int_0^T f(u_s) ds \xrightarrow[T \rightarrow \infty]{a.s.} \mathbb{E}[f(u_0)].$$

Proof. According to Theorem 1.3 in [12], the process u is strongly mixing if for all $t > 0$ and $1 \leq r \leq q$, the following convergence holds

$$\|f_t \otimes_r f_{t+s}\|_{L^2(\mathbb{R}^{2q-2r})} \xrightarrow{s \rightarrow \infty} 0.$$

Taking into account that u is strongly stationary, we can write

$$\|f_t \otimes_r f_{t+s}\|_{L^2(\mathbb{R}^{2q-2r})} = \|f_0 \otimes_r f_s\|_{L^2(\mathbb{R}^{2q-2r})},$$

and the conclusion follows immediately from Birkhoff's continuous ergodic theorem. \square

We can now particularize to the Hermite-Ornstein-Uhlenbeck process.

Theorem 12. *Let U be the Hermite-Ornstein-Uhlenbeck process defined by (32). Let f be a measurable function such that $|f(x)| \leq \exp(|x|^\gamma)$ for some $\gamma < \frac{2}{q}$. Then,*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(U_s) ds = \mathbb{E}[f(U_0)] \quad a.s.$$

Proof. We shall prove that the process U verifies the conditions of Proposition 11. We have $U_t = I_q^B(f_t)$ with

$$f_t(x_1, \dots, x_q) = c_{H,q} \mathbb{I}_{[-\infty, t]^q}(x_1, \dots, x_q) \int_{x_1 \vee \dots \vee x_q}^t e^{-\alpha(t-u)} \prod_{i=1}^q (u - x_i)^{H_0 - \frac{3}{2}} du.$$

Step 1. Let us first show the mixing condition, that is

$$\lim_{s \rightarrow \infty} \|f_0 \otimes_r f_s\|_{L^2(\mathbb{R}^{2q-2r})} = 0$$

for all $r \in \{1, \dots, q\}$. We can write

$$\begin{aligned} & f_0 \otimes_r f_s(y_1, \dots, y_{2q-2r}) \\ &= c_{H,q}^2 \int_{(-\infty, 0]^r} \left(\int_{x_1 \vee \dots \vee x_r \vee y_1 \vee \dots \vee y_{q-r}}^0 e^{\alpha u} \prod_{i=1}^r \prod_{j=1}^{q-r} (u - x_i)^{H_0 - \frac{3}{2}} (u - y_j)^{H_0 - \frac{3}{2}} du \right) \\ & \quad \times \left(\int_{x_1 \vee \dots \vee x_r \vee y_{q-r+1} \vee \dots \vee y_{2q-2r}}^s e^{-\alpha(s-u)} \right. \\ & \quad \times \left. \prod_{i=1}^r \prod_{j=q-r+1}^{2q-2r} (u - x_i)^{H_0 - \frac{3}{2}} (u - y_j)^{H_0 - \frac{3}{2}} du \right) dx_1 \cdots dx_r \\ &= c_{H,q}^2 \int_{y_1 \vee \dots \vee y_{q-r}}^0 e^{\alpha u} \int_{y_{q-r+1} \vee \dots \vee y_{2q-2r}}^s e^{-\alpha(s-v)} \\ & \quad \times \left(\int_{(-\infty, u \wedge v]} (u - x)^{H_0 - \frac{3}{2}} (v - x)^{H_0 - \frac{3}{2}} dx \right)^r \\ & \quad \times \prod_{j=1}^{q-r} \prod_{l=q-r+1}^{2q-2r} (u - y_j)^{H_0 - \frac{3}{2}} (v - y_l)^{H_0 - \frac{3}{2}} dv du \\ &= c_{H,q}^2 \beta(H_0 - \frac{1}{2}, 2 - 2H_0)^r \int_{y_1 \vee \dots \vee y_{q-r}}^0 e^{\alpha u} \int_{y_{q-r+1} \vee \dots \vee y_{2q-2r}}^s e^{-\alpha(s-v)} |u - v|^{r(2H_0-2)} \\ & \quad \times \prod_{j=1}^{q-r} \prod_{l=q-r+1}^{2q-2r} (u - y_j)^{H_0 - \frac{3}{2}} (v - y_l)^{H_0 - \frac{3}{2}} dv du, \end{aligned}$$

where we used again the identity (22). We then have

$$\|f_0 \otimes_r f_s\|_{L^2(\mathbb{R}^{2q-2r})}^2 = c_{H,q}^4 \beta(H_0 - \frac{1}{2}, 2 - 2H_0)^{2q}$$

$$\begin{aligned}
& \times \int_{(-\infty, 0]^2} \int_{(-\infty, s]^2} e^{\alpha(u+u_1)} e^{-\alpha(2s-(v+v_1))} |u - u_1|^{(q-r)(2H_0-2)} \\
& \times |v - v_1|^{(q-r)(2H_0-2)} |u - v|^{r(2H_0-2)} |u_1 - v_1|^{r(2H_0-2)} dv_1 dv du_1 du \\
& \leq c_{H,q}^4 \beta(H_0 - \frac{1}{2}, 2 - 2H_0)^{2q} A_0 A_s R_s^2,
\end{aligned}$$

with

$$A_x = \left(\int_{(-\infty, x]^2} e^{-q\alpha(2x-(u+u_1))} |u - u_1|^{q(2H_0-2)} du du_1 \right)^{\frac{1}{a}}$$

and

$$R_s = \left(\int_{-\infty}^0 \int_{-\infty}^s e^{-q\alpha(s-(u+v))} |u - v|^{q(2H_0-2)} dv du \right)^{\frac{1}{b}},$$

where we used the Hölder inequality with $a = \frac{q}{q-r}$, $b = \frac{q}{r}$. Making the change of variable $x - u = v$, $x - u_1 = v_1$, we obtain

$$\begin{aligned}
& \int_{(-\infty, x]^2} e^{-q\alpha(2x-(u+u_1))} |u - u_1|^{q(2H_0-2)} du du_1 \\
& = \int_{[0, \infty)^2} e^{-q\alpha(v+v_1)} |v - v_1|^{q(2H_0-2)} du du_1 < \infty.
\end{aligned}$$

On the other hand, we have $q(2H_0 - 2) = 2H - 2$, so

$$R_s^b = \text{Cov}(U_0^H, U_s^H)$$

where U^H is a stationary Ornstein Uhlenbeck process driven by a fractional Brownian motion of index H (and with $\alpha^H = q\alpha$). According to [7, Lemma 2.2], one has $R_s^b = O_{s \rightarrow \infty}(s^{-H})$, implying in turn that $\lim_{s \rightarrow \infty} R_s = 0$ and concluding the proof of the mixing condition.

Step 2. We now show the integrability condition $\mathbb{E}[|f(U_0)|] < \infty$. From the results of [7], we have $\mathbb{E}[U_0^2] = \frac{1}{\alpha^{2H}} \Gamma(2H)$. A power series development yields

$$\mathbb{E}[|f(U_0)|] \leq \sum_{k=0}^{\infty} \frac{1}{k!} \mathbb{E}[|U_0|^{\gamma k}],$$

where U_0 is an element of the q th Wiener chaos. By the hypercontractivity property, for all $k \geq \frac{2}{\gamma}$,

$$\mathbb{E}[|U_0|^{\gamma k}] \leq g(k) := (k-1)^{\frac{\gamma q k}{2}} \left(\frac{1}{\alpha^{2H}} \Gamma(2H) \right)^{\frac{\gamma k}{2}}.$$

Stirling formula allows us to write

$$(33) \quad \frac{g(k)}{k!} \sim_{k \rightarrow \infty} \frac{(k-1)^{\frac{\gamma q k}{2}} \left(\frac{1}{\alpha^{2H}} \Gamma(2H) \right)^{\frac{\gamma k}{2}} e^k}{\sqrt{2\pi k}},$$

and the associated series converges if $\gamma q < 2$. \square

The next result analyzes the fluctuations in the ergodic theorem proved in Theorem 12.

Theorem 13. (A) [Case $q = 1$] Let f be in $L^2(\mathbb{R}, \gamma)$ for $\gamma = \mathcal{N}(0, \frac{\Gamma(2H)}{\alpha^{2H}})$. We denote by $(a_i)_{i \geq 0}$ the coefficients of f in its Hermite expansion, and we let d be the centered Hermite rank of f . Then,

- if $\frac{1}{2} < H < 1 - \frac{1}{2d}$,

$$\frac{1}{\sqrt{T}} \int_0^{Tt} (f(U_s) - \mathbb{E}[f(U_0)]) ds \xrightarrow[T \rightarrow \infty]{f.d.d} c_{f,H} W_t,$$

- if $H = 1 - \frac{1}{2d}$,

$$\frac{1}{\sqrt{T \log T}} \int_0^{Tt} (f(U_s) - \mathbb{E}[f(U_0)]) ds \xrightarrow[T \rightarrow \infty]{f.d.d} c_{f,H} W_t,$$

- if $H > 1 - \frac{1}{2d}$,

$$T^{q(1-H)-1} \int_0^{Tt} (f(U_s) - \mathbb{E}[f(U_0)]) ds \xrightarrow[T \rightarrow \infty]{f.d.d} c_{f,H} Z_t^{d,H},$$

where $Z^{d,H}$ is a Hermite process of order d and index $d(H-1)+1$, W is a Brownian motion and

$$(34) \quad c_{f,H} = \begin{cases} \sqrt{\sum_{k \geq d} k! a_k^2 \int_{\mathbb{R}_+} |\rho(s)|^k} & \text{if } H < 1 - \frac{1}{2d} \\ a_d \sqrt{d! \frac{3}{16\alpha^2}} & \text{if } H = 1 - \frac{1}{2d} \\ a_d \sqrt{d! \frac{H^d \Gamma(2H)^d}{\alpha^{2Hd}}} & \text{if } H > 1 - \frac{1}{2d} \end{cases}$$

with

$$\rho(s) = \mathbb{E}[U_s U_0] = \int_{-\infty}^0 \int_{-\infty}^s e^{-\alpha(s-(u+v))} |u-v|^{2H-2} du dv.$$

Moreover, if $f \in L^p(\mathbb{R}, \gamma)$ for some $p > 2$, the previous convergences holds true in the Banach space $C([0, 1])$.

(B) [Case $q > 1$] Let P be a real valued polynomial. Then, the conclusions of Theorem 2 apply to U .

Proof. Except for $H = 1 - \frac{1}{2d}$ in Part A, this is a direct consequence of Theorems 1 and 2. The convergence in the critical case can be checked through easy but tedious computations, by reducing to the case where f is the d th Hermite polynomial. Details are left to the reader. \square

7. APPENDIX

In this section we present two technical lemmas that play an important role along the paper. First, we shall reproduce a very useful result from [20]:

Theorem 14 (Power Counting Theorem). Let $T = \{M_1, \dots, M_K\}$ a set of linear functionals on \mathbb{R}^n , $\{f_1, \dots, f_K\}$ a set of real measurable functions on

\mathbb{R}^n such that there exist real numbers $(a_i, b_i, \mu_i, \nu_i)_{1 \leq i \leq K}$, satisfying for each $i = 1, \dots, K$,

$$\begin{aligned} 0 &< a_i \leq b_i, \\ |f_i(x)| &\leq |x|^{\mu_i} \text{ if } |x| \leq a_i, \\ |f_i(x)| &\leq |x|^{\nu_i} \text{ if } |x| \geq b_i, \\ f_i &\text{ is bounded over } [a_i, b_i]. \end{aligned}$$

For a linearly independent subset of W of T , we write $S_T(W) = \text{Span}(M) \cap T$. We also define

$$\begin{aligned} d_0(W) &= \dim(\text{Span}(W)) + \sum_{i: M_i \in S_T(W)} \mu_i, \\ d_\infty(W) &= n - \dim(\text{Span}(W)) + \sum_{i: M_i \in T \setminus S_T(W)} \nu_i. \end{aligned}$$

Assume $\dim(\text{Span}(T)) = n$. Then, the two conditions (a) : $d_0(W) > 0$ for all linearly independent subsets $W \subset T$, (b) : $d_\infty(W') < 0$ for all linearly independent proper subsets $W' \subset T$, imply

$$(35) \quad \int_{\mathbb{R}^n} \prod_{i=1}^K |f_i(M_i(x))| dx < \infty$$

Moreover, assume that $|f_i(x)| = |x|^{\mu_i}$ if $|x| \leq a_i$, Then

$\int_{[-1,1]^n} \prod_{i=1}^K |f_i(M_i(x))| dx < \infty$, if and only if for any linearly independent subset $W \subset T$ condition (a) holds.

The next lemma is an application of the Hardy-Littlewood-Sobolev inequality,

Lemma 15. Fix $n, q \geq 2$ and $\alpha \in A_{n,q}$. Recall H and H_0 from (1). Assume $\varphi \in L^1(\mathbb{R}) \cap L^{\frac{1}{H}}(\mathbb{R})$ Then

$$\int_{\mathbb{R}^n} \prod_{k=1}^n |\varphi(\eta_k)| \prod_{1 \leq i < j \leq n} |\eta_i - \eta_j|^{(2H_0-2)\alpha_{ij}} d\eta_1 \dots d\eta_n < \infty.$$

Proof. We are going to use the multilinear Hardy-Littlewood-Sobolev inequality, that we recall here for the convenience of the reader (see [1, Theorem 6]): if $f : \mathbb{R} \rightarrow \mathbb{R}$ is a measurable function, if $p \in (1, n)$ and if the $\gamma_{ij} \in (0, 1)$ are such that $\sum_{1 \leq i < j \leq n} \gamma_{ij} = 1 - \frac{1}{p}$, then there exists $c_{p,\gamma} > 0$ such that

$$(36) \quad \int_{\mathbb{R}^n} \prod_{k=1}^n |f(u_k)| \prod_{1 \leq i < j \leq n} |u_i - u_j|^{-\gamma_{ij}} du_1 \dots du_n \leq c_{p,\gamma} \left(\int_{\mathbb{R}} |f(u)|^p du \right)^{\frac{n}{p}}.$$

Set $p = 1/(1 - (1 - H)\frac{2|\alpha|}{nq})$. Since $2|\alpha| \leq nq$, we have that $p > 1$. On the other hand, since $H > \frac{1}{2}$, one has $nH > \frac{n}{2} \geq 1$; this implies that

$(1 - H)\frac{2|\alpha|}{q} < (1 - H)n < n - 1$, that is, $p < n$. Moreover, set $\gamma_{ij} = (2 - 2H_0)\alpha_{ij} = (1 - H)\frac{2\alpha_{ij}}{q} \in (0, 1)$; we have $\sum_{1 \leq i < j \leq n} \gamma_{ij} = 2(1 - H)\frac{|\alpha|}{q} \leq (1 - H)n < n - 1$. We deduce from (36) that

$$\int_{\mathbb{R}^n} \prod_{k=1}^n |\varphi(\eta_k)| \prod_{1 \leq i < j \leq n} |\eta_i - \eta_j|^{(2H_0 - 2)\alpha_{ij}} d\eta_1 \dots d\eta_n \leq c_{p,\gamma} \left(\int_{-\infty}^{\infty} |x(u)|^p du \right)^{\frac{n}{p}}.$$

But $p \in (1, \frac{1}{H})$ and $x \in L^1(\mathbb{R}) \cap L^{\frac{1}{H}}(\mathbb{R})$, so the claim follows. \square

Acknowledgments: We are grateful to Pawel Wolff for making us aware of the multivariate Hardy-Littlewood-Sobolev inequality from [1], and to two referees for their valuable comments and suggestions which helped to improve the presentation of the paper. The works by V. Garino and I. Nourdin have been supported by the FNR OPEN grantAPOGee at Luxembourg University. The work by David Nualart has been supported by the NSF grant DMS-1811181. Majid Salamat would like to thank Ivan Nourdin for his invitation and Luxembourg University for financial support during his visit.

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